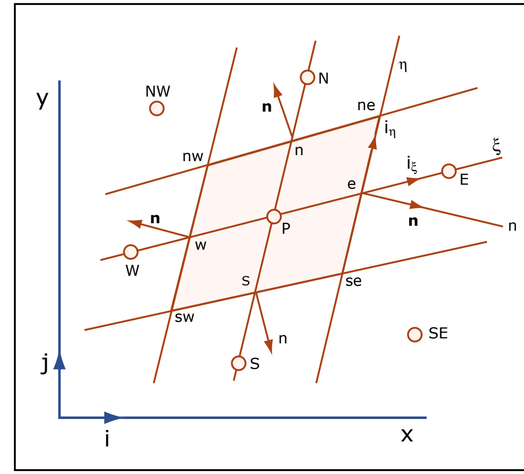
2.29 Numerical Fluid Mechanics
Fall 2011 – **Lecture 24**

REVIEW Lecture 23:



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- Grid Generation
  - Unstructured grid generation
    - Delaunay Triangulation
    - Advancing Front method
- Finite Volume on Complex geometries
  - Computation of convective fluxes:
    - For mid-point rule: $F_e = \int_{S_e} \rho\phi\,(\vec{v}.\vec{n})\,dS \approx f_e S_e = (\rho\phi\vec{v}.\vec{n})_e S_e = \phi_e \dot{m}_e = \phi_e\,\rho_e\,(S_e^x\,u_e + S_e^y\,v_e)$
  - Computation of diffusive fluxes: mid-point rule for complex geometries often used
    - Either use shape function $\phi\,(x, y)$, with mid-point rule: $F_e^d \approx (k\nabla\phi.\vec{n})_e S_e = k_e\left(S_e^x \left.\frac{\partial\phi}{\partial x}\right|_e + S_e^y \left.\frac{\partial\phi}{\partial y}\right|_e \right)$
    - Or compute derivatives at CV centers first, then interpolate to cell faces. Option include either:
      - Gauss Theorem: $\left.\frac{\partial\phi}{\partial x_i}\right|_P \approx \left.\overline{\frac{\partial\phi}{\partial x_i}}\right|_P = \int_{CV} \frac{\partial\phi}{\partial x_i}dV \Big/ dV = \sum_{4\ c\ \text{faces}} \phi_c S_c^{x_i} \Big/ dV$
      - Deferred-correction approach: $F_e^d \approx k_e S_e \frac{\phi_E - \phi_P}{|\mathbf{r}_E - \mathbf{r}_P|} + k_e S_e \left[\overline{\nabla\phi}\big|_e\right]^{\text{old}} \left(\mathbf{n} - \mathbf{i}_\xi\right)$

        where $\left[\overline{\nabla\phi}\big|_e\right]^{\text{old}}$ is interpolated from $\left[\overline{\nabla\phi}\big|_P\right]^{\text{old}}$, e.g. $\left.\overline{\frac{\partial\phi}{\partial x_i}}\right|_P = \sum_{4\ c\ \text{faces}} \phi_c S_c^{x_i} \Big/ dV$
  - Comments on 3D

REVIEW Lecture 23, Cont'd:

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla.(\rho \vec{v}\, \vec{v}) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$

$$\nabla.\vec{v} = 0$$

- Solution of the Navier-Stokes Equations
  - Discretization of the convective and viscous terms
  - Discretization of the pressure term $\quad \tilde{p} = p - \rho \mathbf{g}.\mathbf{r} + \mu \frac{2}{3} \nabla.\mathbf{u} \quad (p\, \vec{e}_i - \rho g_i x_i \vec{e}_i + \frac{2}{3} \mu \frac{\partial u_j}{\partial x_j} \vec{e}_i)$
  - Conservation principles $\quad \int_S -\tilde{p}\, \vec{e}_i.\vec{n} dS$
    - Momentum and Mass
    - Energy

$$\frac{\partial}{\partial t} \int_{CV} \rho \frac{\|\vec{v}\|^2}{2} dV = -\int_{CS} \rho \frac{\|\vec{v}\|^2}{2} (\vec{v}.\vec{n}) dA - \int_{CS} p\, \vec{v}.\vec{n}\, dA + \int_{CS} (\vec{\vec{\varepsilon}}.\vec{v}).\vec{n}\, dA + \int_{CV} \left(-\vec{\vec{\varepsilon}} : \nabla \vec{v} + p \nabla.\vec{v} + \rho \vec{g}.\vec{v} \right) dV$$

  - Choice of Variable Arrangement on the Grid
    - Collocated and Staggered
  - Calculation of the Pressure

TODAY (Lecture 24): Numerical Methods for the Navier-Stokes Equations

- Solution of the Navier-Stokes Equations
  - Discretization of the convective and viscous terms
  - Discretization of the pressure term
  - Conservation principles
  - Choice of Variable Arrangement on the Grid
  - Calculation of the Pressure
  - Pressure Correction Methods
    - A Simple Explicit Scheme
    - A Simple Implicit Scheme
      - Nonlinear solvers, Linearized solvers and ADI solvers
    - Implicit Pressure Correction Schemes for steady problems
      - Outer and Inner iterations
    - Projection Methods
      - Non-Incremental and Incremental Schemes
    - Fractional Step Methods:
      - Example using Crank-Nicholson

References and Reading Assignments

- Chapter 7 on "Incompressible Navier-Stokes equations" of "J. H. Ferziger and M. Peric, *Computational Methods for Fluid Dynamics*. Springer, NY, 3rd edition, 2002"
- Chapter 11 on "Incompressible Navier-Stokes Equations" of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, *Computational Fluid Dynamics for Engineers*. Springer, 2005.
- Chapter 17 on "Incompressible Viscous Flows" of Fletcher, *Computational Techniques for Fluid Dynamics*. Springer, 2003.

Conservation Principles for NS: Cont'd
Kinetic Energy Conservation

- Derivation of Kinetic energy equation
  - Take dot product of momentum equation with velocity
  - Integrate over a control volume CV or full volume of domain of interest
  - This gives

$$\frac{\partial}{\partial t}\int_{CV} \rho \frac{\|\vec{v}\|^2}{2} dV = -\int_{CS} \rho \frac{\|\vec{v}\|^2}{2} (\vec{v}.\vec{n}) dA - \int_{CS} p\, \vec{v}.\vec{n}\, dA + \int_{CS} (\vec{\vec{\varepsilon}}.\vec{v}).\vec{n}\, dA + \int_{CV} \left(-\vec{\vec{\varepsilon}} : \nabla\vec{v} + p\,\nabla.\vec{v} + \rho\vec{g}.\vec{v}\right) dV$$

  where $\varepsilon_{ij} = \tau_{ij} + p\delta_{ij}$ is the viscous component of the stress tensor

  - Here, the three RHS terms in the volume integral are zero if the flow is inviscid (term 1 = dissipation), incompressible (term 2) and there are no body forces (term 3)
  - Other terms are surface terms and kinetic energy is conserved in this sense: $\Rightarrow$ discretization on CV should ideally lead to no contribution over the volume

- Some observations
  - Guaranteeing global conservation of the *discrete* kinetic energy is not automatic since the kinetic energy equation is a consequence of the momentum equation.
  - Discrete momentum and kinetic energy conservations cannot be enforced separately: the latter should be a consequence of the former

Conservation Principles for NS, Cont'd

- Some observations, Cont'd
  - If a numerical method is (kinetic) energy conservative, it guarantees that the total (kinetic) energy in the domain does not grow with time (if the energy fluxes at boundaries are null/bounded)
    - This ensures that the velocity at every point in the domain is bounded: important stability property
  - Since kinetic energy conservation is a consequence of momentum conservation, global discrete kinetic energy conservation must be a consequence of the discretized momentum equations
    - It is thus a property of the discretization method and it is not guaranteed
    - One way to ensure it is to impose that the discretization of the pressure gradient and divergence of velocity are "compatible", i.e. lead to discrete energy conservation directly
  - A Poisson equation is often used to compute pressure
    - It is obtained from the divergence of momentum equations, which contains the pressure gradient (see next)
    - Divergence and gradient operators must be such that mass conservation is satisfied (especially for incompressible flows), and ideally also kinetic energy

Conservation Principles for NS, Cont'd

- Some observations, Cont'd
  - Time-differencing method can destroy the energy conservation property (and mass conservation for incompressible fluid)
    - Ideally, it should be automatically satisfied by the numerical scheme
    - Example: Crank-Nickolson
      - Time derivatives are approximated by: $\frac{\rho \Delta V}{\Delta t}(u_i^{n+1} - u_i^n)$
      - If one takes the scalar product of this equation with $u_i^{n+1/2}$, which in C-N is approximated by, $u_i^{n+1/2} = (u_i^{n+1} + u_i^n)/2$

        the result is the change of the kinetic energy equation

$$\frac{\rho \Delta V}{\Delta t}\left[\left(\frac{v^2}{2}\right)^{n+1} - \left(\frac{v^2}{2}\right)^{n}\right] \quad \text{where } v^2 = u_i\, u_i \quad \text{(summation implied)}$$

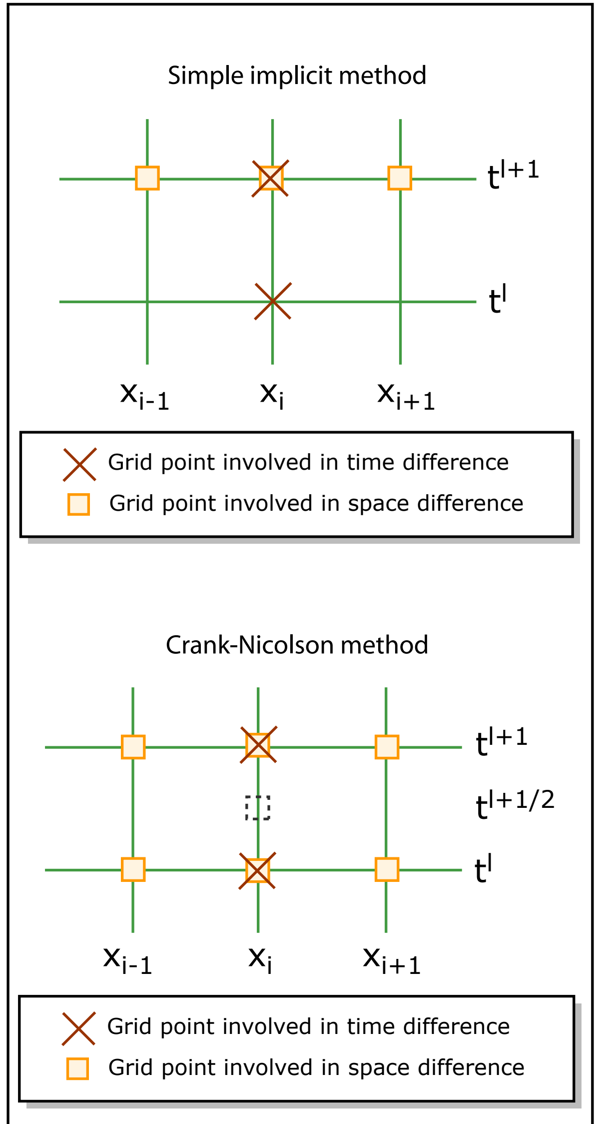
    - With proper choices for the other terms, the C-N scheme is <u>energy conservative</u>

Parabolic PDE: **Implicit Schemes**

(review Lecture 17)

Leads to a system of equations to be solved at each time-step

B-C: Unconditionally stable, 1st order accurate in time, 2nd order in space



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- Evaluates RHS at time $t+1$ instead of time t (for explicit scheme)

Unconditionally stable, 2nd order accurate in time, 2nd order in space

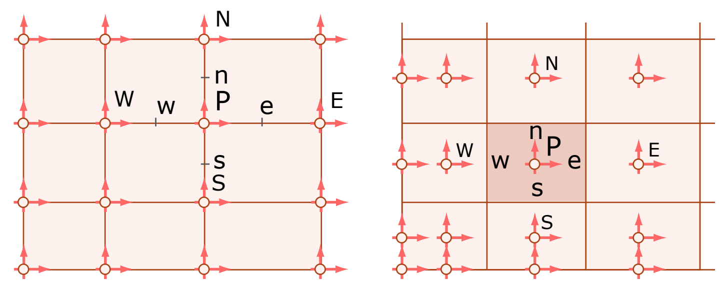
- Time: centered FD, but evaluated at mid-point
- 2nd derivative in space determined at mid-point by averaging at t and $t+1$

Conservation Principles for NS, Cont'd

- Some observations, Cont'd
  - Since momentum and kinetic energy (and mass cons.) are not independent, satisfying all of them is not direct: trial and error in deriving schemes that are conservatives
  - Kinetic energy conservation is particularly important in unsteady flows (e.g. weather, ocean, turbulence, etc)
    - Less important for steady flows
  - Kinetic energy is not the only quantity whose discrete conservation is desirable (and not automatic)
    - Angular momentum is another one
    - Important for flows in rotating machinery, internal combustion engines and any other devices that exhibit strong rotations/swirl
  - If numerical schemes do not conserve these "important" quantities, numerical simulation is likely to get into trouble, even for stable schemes

Choice of Variable Arrangement on the Grid

- Because the Navier-Stokes equations are coupled equations for vector fields, several variants of the arrangement of the computational points/nodes are possible
- Collocated arrangement
  - Obvious choice: store all the variables at the same grid points and use the same grid points or CVs for all variables: Collocated grid



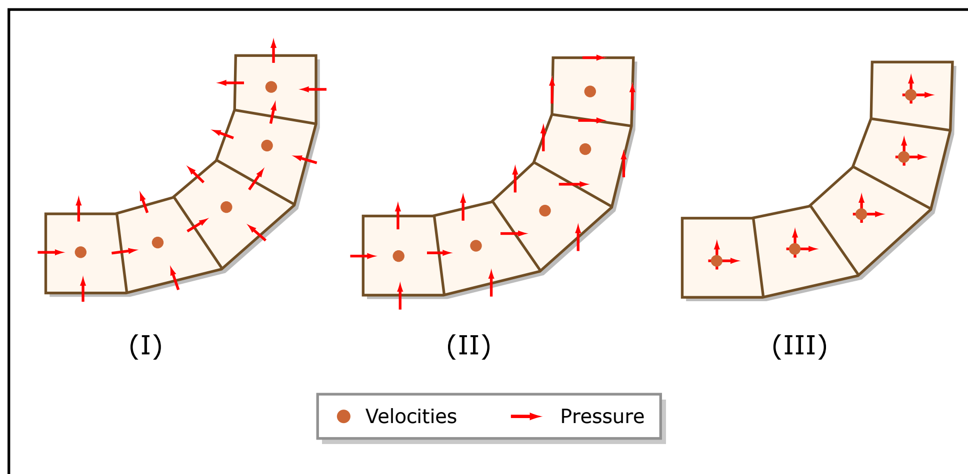
Collocated arrangement of velocity components and pressure on FD and FV grids.

Image by MIT OpenCourseWare.

  - Advantages:
    - All (geometric) coefficients evaluated at the same points
    - Easy to apply to multigrid procedures (collocated refinements of the grid)

Choice of Variable Arrangement on the Grid

- Collocated arrangement: Disadvantages
  - Was out of favor and not used much until the 1980s because of:
    - Occurrence of oscillations in the pressure
    - Difficulties with pressure-velocity coupling
  - However, when non-orthogonal grids started to be used over complex geometries, the situation changed
    - This is because the non-collocated (staggered) approach on non-orthogonal grids is based on grid-oriented components of the (velocity) vectors and tensors.
    - This implies using curvature terms, which are more difficult to treat numerically and can create non-conservative errors
    - Hence, collocated grids became more popular



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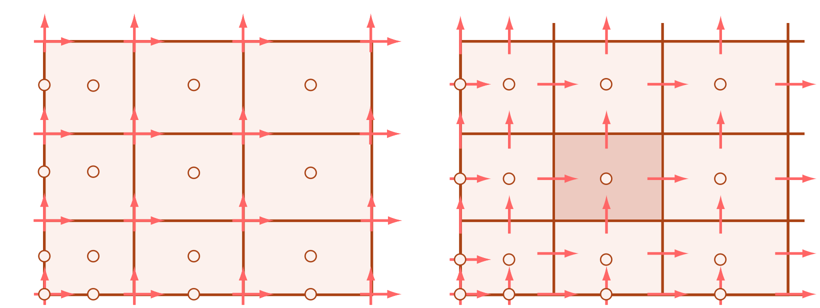
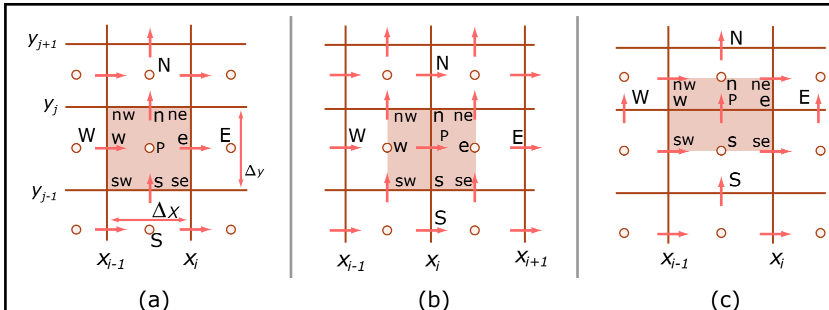
Choice of Variable Arrangement on the Grid

- Staggered arrangements
  - No need for all variables to share the same grid
  - "Staggered" arrangements can be advantageous (couples p and $\mathbf{v}$)
- For example, consider the Cartesian coordinates
  - Advantages of staggered grids
    - Several terms that require interpolation in collocated grids can be evaluated (to 2nd order) without interpolation
    - This applies to the pressure term (located at CV centers) and the diffusion term (first derivative needed at CS centers), when obtained by central differences
    - Can be shown to directly conserve kinetic energy
    - Many variations: partially staggered, etc



Fully and partially staggered arrangements of velocity components and pressure.



Control volumes for a staggered grid for (a) mass conservation and scalar quantities, (b) x-momentum, and (c) y-momentum

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Choice of Variable Arrangement on the Grid

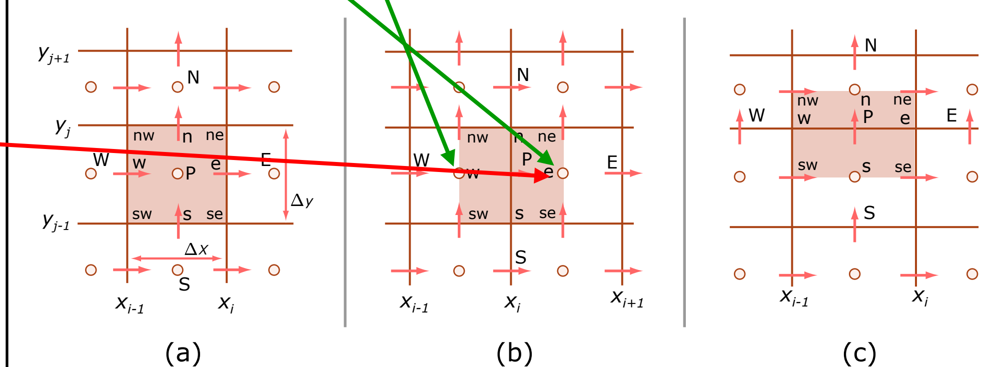
- Staggered arrangements:
  - Example with Cartesian coordinates, Cont'd
    - Terms can be evaluated (to 2nd order) without interpolation
    - This applies to the pressure term (normal at center of CS). For example, along x direction:
      - Each p value on the bnd of the velocity grid is conveniently at the center the "scalar" grid:

$$-\int_{S_e} p\,\mathbf{i}.\mathbf{n}\,dS \approx -p_e S_e + p_w S_w$$

    - Diffusion term (first derivative at CS) obtained by central differences.

For example:

$$(\tau_{xx})_e = 2\left(\mu \frac{\partial u}{\partial x}\right)_e \approx 2\mu \frac{u_E - u_P}{x_E - x_p}$$



Control volumes for a staggered grid for (a) mass conservation and scalar quantities, (b) x-momentum, and (c) y-momentum

Image by MIT OpenCourseWare.

Calculation of the Pressure

- The Navier-Stokes equations do not have an independent equation for pressure
  - But pressure gradient contributes to each of the three momentum equations
  - For incompressible fluids, mass conservation becomes a kinematic constraint on the velocity field: we then have no dynamic equations for both density and pressure
  - For compressible fluids, mass conservation is a dynamic equation for density
    - Pressure can then be computed from density using an equation of state
    - For incompressible flows (or low Mach numbers), density is not a state variable, hence can't be solved for
- For incompressible flows:
  - Momentum equations lead to the velocities $\Rightarrow$
  - Continuity equation should lead to the pressure, but it does not contain pressure! How can p be estimated?

Calculation of the Pressure

- Navier-Stokes, incompressible: $$\frac{\partial \rho \vec{v}}{\partial t} + \nabla.(\rho \vec{v}\ \vec{v}) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g}$$ $$\nabla.\vec{v} = 0$$
- Combine the two conservation eqns to obtain an equation for p
  – Since the cons. of mass has a divergence form, take the divergence of the momentum equation, using cons. of mass:
    - For constant viscosity and density:

$$\nabla.\nabla p = \underline{\nabla^2 p} = -\nabla.\frac{\partial \rho \vec{v}}{\partial t} - \nabla.\left(\nabla.(\rho \vec{v}\ \vec{v})\right) + \nabla.\left(\mu \nabla^2 \vec{v}\right) + \nabla.\left(\rho \vec{g}\right) = \underline{-\nabla.\left(\nabla.(\rho \vec{v}\ \vec{v})\right)}$$

  – This pressure equation is elliptic (Poisson eqn. once velocity is known)
    - It can be solved by methods we have seen earlier for elliptic equations

$$\frac{\partial}{\partial x_i}\left(\frac{\partial p}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}\left(\frac{\partial\left(\rho u_i u_j\right)}{\partial x_j}\right)$$

  – Important Notes
    - Terms inside divergence (derivatives of momentum terms) must be approximated in a form consistent with those of momentum equations. Divergence is that of cons. of mass.
    - Laplacian operator comes from divergence of cons. of mass and gradient in momentum equations: consistency must be maintained, i.e. divergence and gradient operators in the Laplacian should be those of the cons. of mass and of the momentum eqns., respectively
    - Best to derive pressure equation from discretized momentum/continuity equations

Pressure-correction Methods

- First solve the momentum equations to obtain the velocity field for a known pressure
- Then solve the Poisson equation to obtain an updated/corrected pressure field
- Another way: modify the continuity equation so that it becomes hyperbolic (even though it is elliptic)
  - Artificial Compressibility Methods
- Notes:
  - The general pressure-correction method is independent of the discretization chosen for the spatial derivatives $\Rightarrow$ in theory any discretization can be used
  - We keep density in the equations (flows are assumed incompressible, but small density variations are considered)

A Simple Explicit Time Advancing Scheme

- Simple method to illustrate how the numerical Poisson equation for the pressure is constructed and the role it plays in enforcing continuity
- Specifics of spatial derivative scheme not important, hence, we look at the equation discretized in space, but not in time.
  - Use $\frac{\delta}{\delta x_i}$ to denote spatial derivatives. This gives:

Note: $p = p_{real} - \rho g_i x_i$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial(\rho u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \Rightarrow \frac{\partial \rho u_i}{\partial t} = -\frac{\delta(\rho u_i u_j)}{\delta x_j} - \frac{\delta p}{\delta x_i} + \frac{\delta \tau_{ij}}{\delta x_j} = H_i - \frac{\delta p}{\delta x_i}$$

  - Simplest approach: Forward Euler for time integration, which gives:

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(H_i^n - \frac{\delta p}{\delta x_i}^n \right)$$

    - In general, the new velocity field we obtain at time $n+1$ does not satisfy the continuity equation:

$$\frac{\delta(\rho u_i)^{n+1}}{\delta x_i} = 0$$

A Simple Explicit Time Advancing Scheme

- How can we enforce continuity at $n+1$?
- Take the numerical divergence of the NS equations:

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(H_i^n - \frac{\delta p^n}{\delta x_i} \right) \Rightarrow \boxed{\frac{\delta (\rho u_i)^{n+1}}{\delta x_i} - \frac{\delta (\rho u_i)^n}{\delta x_i} = \Delta t \left[\frac{\delta}{\delta x_i} \left(H_i^n - \frac{\delta p^n}{\delta x_i} \right) \right]}$$

  - The first term is the divergence of the new velocity field, which we want to be zero
  - Second term is zero if continuity was enforced at time step n
  - Third term can be zero or not

- All together, we obtain:

$$\boxed{\frac{\delta}{\delta x_i} \left(\frac{\delta p^n}{\delta x_i} \right) = \frac{\delta H_i^n}{\delta x_i}}$$

  - Note that this includes the divergence operator from the continuity eqn. (outside) and the pressure gradient from the momentum equation (inside)
  - Pressure gradient could be explicit (n) or implicit ($n+1$)

A Simple Explicit Time Advancing Scheme: Summary of the Algorithm

- Start with velocity at time t_n which is divergence free
- Compute RHS of pressure equation at time t_n
- Solve the Poisson equation for the pressure at time t_n
- Compute the velocity field at the new time step using the momentum equation: It will be divergence free
- Continue to next time step

A Simple Implicit Time Advancing Scheme

- Some additional difficulties arise when an implicit method is used to solve the (incompressible) NS equations
- To illustrate, let's first try the simplest: backward/implicit Euler
  - Recall: $$\frac{\partial \rho u_i}{\partial t} = -\frac{\delta(\rho u_i u_j)}{\delta x_j} - \frac{\delta p}{\delta x_i} + \frac{\delta \tau_{ij}}{\delta x_j} = H_i - \frac{\delta p}{\delta x_i}$$
  - Implicit Euler: $$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(H_i^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right) = \Delta t \left(-\frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right)$$
- Difficulties (specifics for incompressible case)

  1) Set numerical divergence of velocity field at new time-step to be zero

  - Take divergence of momentum, assume velocity is divergent at time t_n and demand zero divergence at t_{n+1}. This leads to:

$$\frac{\delta(\rho u_i)^{n+1}}{\delta x_i} - \frac{\delta(\rho u_i)^n}{\delta x_i} = \Delta t \left[\frac{\delta}{\delta x_i} \left(H_i^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right) \right] \Rightarrow \frac{\delta}{\delta x_i} \left(\frac{\delta p}{\delta x_i}^{n+1} \right) = \frac{\delta}{\delta x_i} \left(-\frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} \right)$$

  - Problem: The RHS can not be computed until velocities are known at t_{n+1} (and these velocities can not be computed until p^{n+1} is available)
  - Result: Poisson and momentum equations have to be solved simultaneously

A Simple Implicit Time Advancing Scheme, Cont'd

2) Even if p^{n+1} known, a large system of nonlinear momentum equations must be solved for the velocity field:

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(- \frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right)$$

Three approaches for solution:

- First approach: nonlinear solvers
  - Use velocities at t_n for initial guess of u_i^{n+1} (or use explicit first guess) and then employ a nonlinear solver (Fixed-point, Newton-Raphson or Secant methods) at each time step
  - Nonlinear solver is applied to the nonlinear algebraic equations

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left(- \frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} - \frac{\delta p}{\delta x_i}^{n+1} \right)$$

$$\frac{\delta}{\delta x_i}\left(\frac{\delta p}{\delta x_i}^{n+1} \right) = \frac{\delta}{\delta x_i}\left(- \frac{\delta(\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} \right)$$

A Simple Implicit Time Advancing Scheme, Cont'd

– Second approach: linearize the equations about the result at t_n

$$u_i^{n+1} = u_i^n + \Delta u_i \quad \Rightarrow$$

$$u_i^{n+1} u_j^{n+1} = u_i^n u_j^n + u_i^n \Delta u_j + u_j^n \Delta u_i + \Delta u_i \Delta u_j$$

- We'd expect the last term to be of 2nd order in Δt, it can thus be neglected (for example, it would be of same order than a C-N approximation in time).
- Hence, doing the same in the other tems, the (incompressible) momentum equations are then approximated by:

$$(\rho u_i)^{n+1} - (\rho u_i)^n = \rho \Delta u_i = \Delta t \left(-\frac{\delta(\rho u_i u_j)^n}{\delta x_j} - \frac{\delta(\rho u_i^n \Delta u_j)^n}{\delta x_j} - \frac{\delta(\rho \Delta u_i u_j^n)^n}{\delta x_j} + \frac{\delta \tau_{ij}{}^n}{\delta x_j} + \frac{\delta \Delta \tau_{ij}}{\delta x_j} - \frac{\delta p^n}{\delta x_i} - \frac{\delta \Delta p}{\delta x_i} \right)$$

- This linearization takes advantage of the fact that the nonlinear term is only quadratic
- However, a large system still needs to be inverted. Direct solution is not recommended: use an iterative scheme

– A third interesting solution scheme: an Alternate Direction Implicit scheme

Parabolic PDEs: Two spatial dimensions
ADI scheme (Two Half steps in time)

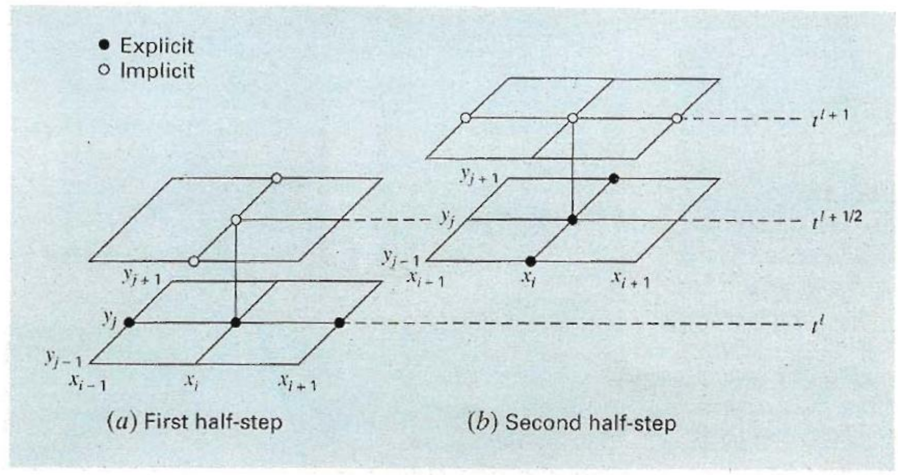


FIGURE 30.10
The two half-steps used in implementing the alternating-direction implicit scheme for solving parabolic equations in two spatial dimensions.

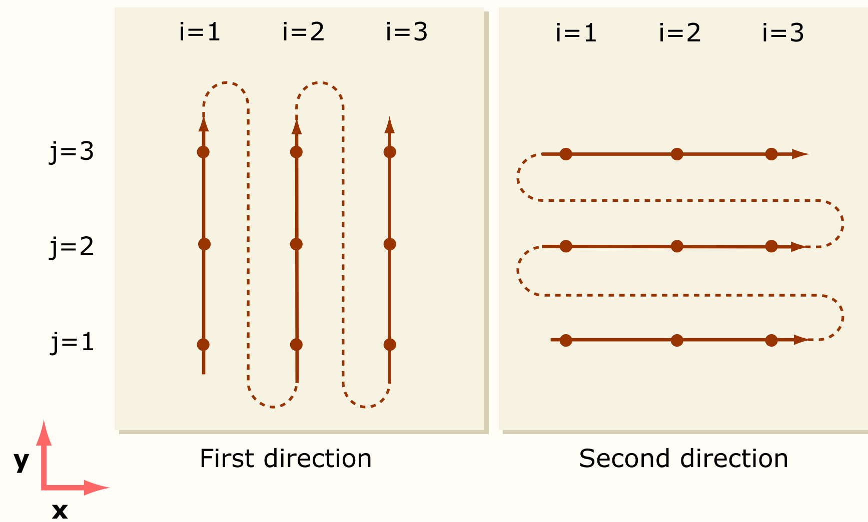
1) From time n to $n+1/2$: Approx. of 2nd order x derivative explicit, y derivative implicit. Hence, tri-diagonal matrix to be solved

$$\frac{T_{i,j}^{n+1/2} - T_{i,j}^{n}}{\Delta t/2} = c^2 \frac{T_{i-1,j}^{n} - 2T_{i,j}^{n} + T_{i+1,j}^{n}}{\Delta x^2} + c^2 \frac{T_{i,j-1}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i,j+1}^{n+1/2}}{\Delta y^2} \qquad \left(O(\Delta x^2 + \Delta y^2)\right)$$

2) From time $n+1/2$ to $n+1$: Approximation of 2nd order x derivative implicit, y derivative explicit

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n+1/2}}{\Delta t/2} = c^2 \frac{T_{i-1,j}^{n+1} - 2T_{i,j}^{n+1} + T_{i+1,j}^{n+1}}{\Delta x^2} + c^2 \frac{T_{i,j-1}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i,j+1}^{n+1/2}}{\Delta y^2} \qquad \left(O(\Delta x^2 + \Delta y^2)\right)$$

Parabolic PDEs: Two spatial dimensions
ADI scheme (Two Half steps in time)



The ADI method applied along the y direction and x direction. This method only yields tridiagonal equations if applied along the implicit dimension.

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For $\Delta x = \Delta y$:

1) From time n to $n+1/2$:

$$-rT_{i,j-1}^{n+1/2} + 2(1+r)T_{i,j}^{n+1/2} - rT_{i,j+1}^{n+1/2} = rT_{i-1,j}^{n} + 2(1-r)T_{i,j}^{n} + rT_{i+1,j}^{n}$$

2) From time $n+1/2$ to $n+1$:

$$-rT_{i-1,j}^{n+1} + 2(1+r)T_{i,j}^{n+1} - rT_{i+1,j}^{n+1} = rT_{i,j-1}^{n+1/2} + 2(1-r)T_{i,j}^{n+1/2} + rT_{i,j+1}^{n+1/2}$$

A Simple Implicit Time Advancing Scheme, Cont'd

- Alternate Direction Implicit method
  - Split the NS momentum equations into a series of 1D problems, each which is block tri-diagonal. Then, either:
  - ADI nonlinear: iterate for the nonlinear terms, or,
  - ADI with a local linearization:
    - Δp can first be set to zero to obtain a new velocity u_i^* which does not satisfy continuity:

$$\left(\rho u_i^*\right)^{n+1} - \left(\rho u_i\right)^n = \Delta t \left(-\frac{\delta(\rho u_i u_j)^n}{\delta x_j} - \frac{\delta(\rho u_i^n \Delta u_j)^n}{\delta x_j} - \frac{\delta(\rho \Delta u_i u_j^n)^n}{\delta x_j} + \frac{\delta \tau_{ij}{}^n}{\delta x_j} + \frac{\delta \Delta \tau_{ij}}{\delta x_j} - \frac{\delta p^n}{\delta x_i} \right)$$

    - Solve a Poisson equation for the pressure correction. Taking the divergence of:

$$\left(\rho u_i\right)^{n+1} - \left(\rho u_i\right)^n = \Delta t \left(\underbrace{-\frac{\delta(\rho u_i u_j)^n}{\delta x_j} - \frac{\delta(\rho u_i^n \Delta u_j)^n}{\delta x_j} - \frac{\delta(\rho \Delta u_i u_j^n)^n}{\delta x_j} + \frac{\delta \tau_{ij}{}^n}{\delta x_j} + \frac{\delta \Delta \tau_{ij}}{\delta x_j} - \frac{\delta p^n}{\delta x_i}} - \frac{\delta \Delta p}{\delta x_i} \right)$$

$$\Leftrightarrow \quad \left(\rho u_i\right)^{n+1} = \left(\rho u_i^*\right)^{n+1} - \Delta t \frac{\delta \Delta p}{\delta x_i}$$

gives: $$\frac{\delta}{\delta x_i}\left(\frac{\delta \Delta p}{\delta x_i}\right) = \frac{\delta(\rho u_i^*)^{n+1}}{\delta x_i}$$

    - Finally, update the velocity: $$\left(\rho u_i\right)^{n+1} = \left(\rho u_i^*\right)^{n+1} - \Delta t \frac{\delta \Delta p}{\delta x_i}$$

Methods for solving (steady) NS problems: Implicit Pressure-Correction Methods

- Previous implicit approach based on linearization most useful for unsteady problems
  - It is not accurate for large (time) steps (because the linearization would then lead to a large error)
  - Should not be used for steady problems
- Steady problems are often solved with an implicit method (with pseudo-time), but with large time steps (no need to reproduce the pseudo-time history)
  - The aim is to rapidly converge to the steady solution
- Many steady-state solvers are based on variations of the implicit schemes just discussed
  - They use a pressure (or pressure-correction) equation to enforce continuity at each “pseudo-time” steps, also called “outer iteration”

2.29 Numerical Fluid Mechanics
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