REVIEW Lecture 24:

• Solution of the Navier-Stokes Equations
  
  \[ \frac{\partial \rho \vec{v}}{\partial t} + \nabla . (\rho \vec{v} \cdot \vec{v}) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \]
  
  \[ \nabla . \vec{v} = 0 \]

  – Discretization of the convective and viscous terms
  
  – Discretization of the pressure term
    \[ \tilde{p} = p - \rho g \cdot \mathbf{r} + \mu \frac{2}{3} \nabla . \mathbf{u} \quad (p \tilde{e}_i - \rho g \tilde{e}_i + \frac{2}{3} \mu \frac{\partial u_j}{\partial x_j}) \]

  – Conservation principles
    • Momentum and Mass
    • Energy
      \[ \frac{\partial}{\partial t} \int_{CV} \rho \frac{\|\vec{v}\|^2}{2} dV = -\int_{CS} \rho \frac{\|\vec{v}\|^2}{2} (\tilde{v} \cdot \tilde{n}) dA - \int_{CS} p \tilde{v} \cdot \tilde{n} dA + \int_{CS} (\tilde{e} \cdot \tilde{v}) \tilde{n} dA + \int_{CV} (-\tilde{e} : \nabla \tilde{v} + p \nabla \tilde{v} + \rho \tilde{g} \tilde{v}) dV \]

  – Choice of Variable Arrangement on the Grid
    • Collocated and Staggered

  – Calculation of the Pressure

  \[ \nabla . \nabla p = \nabla^2 p = -\nabla . \left( \frac{\partial \rho \vec{v}}{\partial t} - \nabla . (\rho \nabla \vec{v}) \right) + \nabla . (\mu \nabla^2 \vec{v}) + \nabla . (\rho \vec{g}) = -\nabla . (\nabla . (\rho \vec{v} \cdot \vec{v})) \]

  \[ \Rightarrow \frac{\partial}{\partial x_i} \left( \frac{\partial p}{\partial x_i} \right) = -\frac{\partial}{\partial x_i} \left( \frac{\partial \left( \rho u_i u_j \right)}{\partial x_j} \right) \]
Solution of the Navier-Stokes Equations

- Pressure Correction Methods:
  - i) Solve momentum for a known pressure leading to new velocity, then
  - ii) Solve Poisson to obtain a corrected pressure and
  - iii) Correct velocity, go to i) for next time-step.

- A Simple Explicit Scheme (Poisson for \( P \) at \( t_n \), then mom. for velocity at \( t_{n+1} \))

- A Simple Implicit Scheme

\[
(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left( -\frac{\delta (\rho u_i u_j)}{\delta x_j} + \frac{\delta \tau_{ij}^{n+1}}{\delta x_j} - \frac{\delta p^{n+1}}{\delta x_i} \right) \quad \frac{\delta}{\delta x_i} \left( \frac{\delta p^{n+1}}{\delta x_i} \right) = \frac{\delta}{\delta x_i} \left( -\frac{\delta (\rho u_i u_j)^{n+1}}{\delta x_j} + \frac{\delta \tau_{ij}^{n+1}}{\delta x_j} \right)
\]

- Nonlinear solvers, Linearized solvers and ADI solvers

- Implicit Pressure Correction Schemes for steady problems: iterate using
  - Outer iterations:
    \[
    A^{u^m_{n+1}} u^m_{n+1} = b^m_{u^m_{n+1}} - \frac{\delta p^m_{n+1}}{\delta x_i}
    \]
    but require \( A^{u^m_{n}} u^m_{n} = b^m_{u^m_{n}} - \frac{\delta p^m_{n}}{\delta x_i} \) and \( \frac{\delta u^m_{n}}{\delta x_i} = 0 \) \( \Rightarrow \frac{\delta}{\delta x_i} \left( \frac{\delta p^m_{n}}{\delta x_i} \right) \approx \frac{\delta}{\delta x_i} \left( A^{u^m_{n+1}} u^m_{n+1} - b^m_{u^m_{n+1}} \right) \)
  - Inner iterations:
    \[
    A^{u^m_{n}} u^m_{n} = b^m_{u^m_{n}} - \frac{\delta p^m_{n}}{\delta x_i}
    \]

- Projection Methods: Non-Incremental and Incremental Schemes
TODAY (Lecture 25): Navier-Stokes Equations and Intro to Finite Elements

• Solution of the Navier-Stokes Equations
  – Pressure Correction Methods
    • Implicit Pressure Correction Schemes for steady problems: iterate using Outer iterations and Inner iterations
    • Projection Methods: Non-Incremental and Incremental Schemes
  – Fractional Step Methods:
    • Example using Crank-Nicholson
  – Streamfunction-Vorticity Methods: scheme and boundary conditions
  – Artificial Compressibility Methods: scheme definitions and example
  – Boundary Conditions: Wall/Symmetry and Open boundary conditions

• Finite Element Methods
  – Introduction
  – Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  – General Approach to Finite Elements:
    • Steps in setting-up and solving the discrete FE system
    • Galerkin Examples in 1D and 2D
References and Reading Assignments


• Chapters 31 on “Finite Elements” of “Chapra and Canale, Numerical Methods for Engineers, 2006.”
Implicit Pressure-Correction Methods

- Simple implicit approach based on linearization most useful for unsteady problems
  - It is not accurate for large (time) steps (because the linearization would then lead to a large error)
  - Should not be used for steady problems

- Steady problems are often solved with an implicit method (with pseudo-time), but with large time steps (no need to reproduce the pseudo-time history)
  - The aim is to rapidly converge to the steady solution

- Many steady-state solvers are based on variations of the implicit schemes
  - They use a pressure (or pressure-correction) equation to enforce continuity at each “pseudo-time” steps, also called “outer iteration”
Methods for solving (steady) NS problems: Implicit Pressure-Correction Methods, Cont’d

• For a fully implicit scheme, the steady state momentum equations are:

\[(\rho u_i)^{n+1} - (\rho u_i)^n = 0 \Rightarrow \left(-\frac{\delta (\rho u_i u_j)}{\delta x_j}^{n+1} + \frac{\delta \tau_{ij}}{\delta x_j}^{n+1} - \frac{\delta p}{\delta x_i}^{n+1}\right) = 0\]

• With the discretized matrix notation, the result is a nonlinear algebraic system

\[A^{u_i^{n+1}} u_i^{n+1} = b^{u_i^{n+1}} - \frac{\delta p}{\delta x_i}^{n+1}\]

– The b term in the RHS contains all terms that are explicit (in \(u_i^n\)) or linear in \(u_i^{n+1}\) or that are coefficients function of other variables at \(t_{n+1}\), e.g. temperature

– Pressure is written in symbolic matrix difference form to indicate that any spatial derivatives can be used

– The algebraic system is nonlinear. Again, nonlinear iterative solvers can be used. For steady flows, the tolerance of the convergence of these nonlinear-solver iterations does not need to be as strict as for a true time-marching scheme

– Note the two types of successive iterations:

  • Outer iterations: (over one pseudo-time step) use nonlinear solvers which update the elements of the matrix \(A^{u_i^{n+1}}\) as well as \(u_i^{n+1}\) (uses no or approximate pressure term)

  • Inner Iterations: linear algebra to solve the linearized system with fixed coefficients
Methods for solving (steady) NS problems: Implicit Pressure-Correction Methods, Cont’d

• Outer iteration $m$ (pseudo-time): nonlinear solvers which update the elements of the matrix $A^{u^m*}$ as well as $u^m_i$:

$$A^{u^m*} u^m_i = b^m_{u^m*} - \frac{\delta p^m}{\delta x_i}$$

- The resulting velocities $u^m_i$ do not satisfy continuity (hence the *) since the RHS is obtained from $p^{m-1}$ at the end of the previous outer iteration.
- Hence, $u^m_i$ needs to be corrected. The final $u^m_i$ needs to satisfy

$$A^m u^m_i = b^m_i - \frac{\delta p^m}{\delta x_i}$$

and

$$\frac{\delta}{\delta x_i} \left( \frac{\delta p^m}{\delta x_i} \right) = \frac{\delta}{\delta x_i} \left( A^m u^m_i - b^m_i \right) \approx \frac{\delta}{\delta x_i} \left( A^{u^m*} u_i^m - b^m_{u^m*} \right)$$

• Inner iteration: After solving a Poisson equation for the pressure, the final velocity is calculated using the inner iteration (fixed coefficient $A$)

$$A^{u^m*} u^m_i = b^m_{u^m*} - \frac{\delta p^m}{\delta x_i}$$

• Finally, increase $m$ to $m+1$ and iterate (outer, then inner)

• This scheme is a variation of the previous time-marching schemes:
  - Main difference is that terms in RHS can be explicit or implicit in outer iteration
These schemes that first construct a velocity field that does not satisfy continuity, but then correct it using a pressure gradient are called “projection methods”:

- The divergence producing part of the velocity is “projected out”

One of the most common methods of this type are again pressure-correction schemes

- Substitute \( u_i^m = u_i^{m*} + u' \) and \( p^m = p^{m-1} + p' \) in the previous equations

- Variations of these pressure-correction methods include:
  - SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) method:
    - Neglects contributions of \( u' \) in the pressure equation
  - SIMPLEC: approximate \( u' \) in the pressure equation as a function of \( p' \)
  - SIMPLER and PISO methods: iterate to obtain \( u' \)

- There are many other variations of these methods: all are based on outer and inner iterations until convergence at \( m(n+1) \) is achieved.
Non-Incremental (Chorin, 1968):
- No pressure term used in predictor momentum equation
- Correct pressure based on continuity
- Update velocity using corrected pressure in momentum equation

\[
\begin{align*}
\left(\rho u_i^*\right)^{n+1} &= \left(\rho u_i\right)^n + \Delta t \left(-\frac{\delta (\rho u_i u_j)^{n+1}}{\delta x_j} + \frac{\delta \tau_{ij}^{n+1}}{\delta x_j}\right) \ ; \ (\rho u_i^*)^{n+1}\bigg|_{\partial D} = 0 \\
\left(\rho u_i\right)^{n+1} &= \left(\rho u_i^*\right)^{n+1} - \Delta t \frac{\delta p^{n+1}}{\delta x_i} \\
\frac{\delta (\rho u_i)^{n+1}}{\delta x_i} &= 0 \\
\left(\rho u_i\right)^{n+1} &= \left(\rho u_i^*\right)^{n+1} - \Delta t \frac{\delta p^{n+1}}{\delta x_i}
\end{align*}
\]

Note: advection term can be treated:
- implicitly for \(u^*\) at \(n+1\) (need to iterate then), or,
- explicitly (evaluated with \(u\) at \(n\)), as in 2d FV code and many others.
Incremental (Goda, 1979):
- Old pressure term used in predictor momentum equation
- Correct pressure based on continuity: $p^{n+1} = p^n + p'$
- Update velocity using pressure increment in momentum equation

\[
(\rho u_i^*)^{n+1} = (\rho u_i)^n + \Delta t \left( -\frac{\delta (\rho u_i u_j)^{n+1}}{\delta x_j} + \frac{\delta \tau_{ij}^{n+1}}{\delta x_j} + \frac{\delta p^n}{\delta x_i} \right); \quad (\rho u_i^*)^{n+1} \bigg|_{\partial D} = 0
\]

\[
(\rho u_i)^{n+1} = (\rho u_i^*)^{n+1} - \Delta t \frac{\delta (p^{n+1} - p^n)}{\delta x_i}
\]

\[
\frac{\delta (\rho u_i)^{n+1}}{\delta x_i} = 0
\]

\[
(\rho u_i)^{n+1} = (\rho u_i^*)^{n+1} - \Delta t \frac{\delta (p^{n+1} - p^n)}{\delta x_i}
\]

\[
\delta \left( \frac{\delta (p^{n+1} - p^n)}{\delta x_i} \right) = -\frac{1}{\Delta t} \delta \left( (\rho u_i^*)^{n+1} \right); \quad \frac{\delta (p^{n+1} - p^n)}{\delta n} \bigg|_{\partial D} = 0
\]

Notes:
- this scheme assumes $u^* = 0$ in the pressure equation. It is as the SIMPLE method, but without the iterations
- As for the non-incremental, the advection term can be explicit or implicit
Other Methods: Fractional Step Methods

• In the previous methods, pressure is used to:
  – Enforce continuity: it is more a mathematical variable than a physical one
  – Fill the RHS of the momentum eqns. explicitly (predictor step for velocity)

• The fractional step methods (Kim and Moin, 1985) generalize ADI
  – But works on term-by-term (instead of dimension-by-dimension). Hence, does not necessarily use pressure in the predictor step
  – Let’s write the NS equations a in symbolic form:

\[
{u}_{i}^{n+1} = {u}_{i}^{n} + (C_{i} + D_{i} + P_{i}) \Delta t
\]

where \(C_{i}, D_{i}\) and \(P_{i}\) represent the convective, diffusive and pressure terms

  – The equation is readily split into a three-steps method:

\[
{u}_{i}^{*} = {u}_{i}^{n} + C_{i} \Delta t \\
{u}_{i}^{**} = {u}_{i}^{*} + D_{i} \Delta t \\
{u}_{i}^{n+1} = {u}_{i}^{**} + P_{i} \Delta t
\]

  – In the 3\textsuperscript{rd} step, the pressure gradient must satisfy the continuity equation
Fractional Step Methods, Cont’d

• Many variations of Fractional step methods exists
  – Pressure can be a pseudo-pressure (depends on the specific steps, what is in $u_i^{**}, P_i$)
  – Terms can be split further (one coordinate at a time, etc)
  – For the time-marching, Runge-Kutta explicit, direct 2nd order implicit or Crank-Nicholson scheme are used
  – Linearization and ADI is also used
  – Used by Choi and Moin (1994) with central difference in space for direct simulations of turbulence (Direct Navier Stokes, DNS)

• Here, we will describe a scheme similar to that of Choi and Moin, but using Crank-Nicolson
Fractional Step Methods:  
Example based on Crank-Nicholson

- **In the first step**, velocity is advanced using:  
  \[(\rho u_i)^* - (\rho u_i)^n = \Delta t \left( \frac{H(u_i^n) + H(u_i^*)}{2} - \frac{\delta p^n}{\delta x_i} \right)\]

  - Pressure from the previous time-step
  - Convective, viscous and body forces are represented as an average of old and new values (Crank-Nicolson)
  - Nonlinear equations \(\Rightarrow\) iterate, e.g. Newton’s scheme used by Choi et al (1994)

- **Second-step**: Half the pressure gradient term is removed from \(u_i^*\), to lead \(u_i^{**}\)

  \[(\rho u_i)^{**} - (\rho u_i)^* = -\Delta t \left( \frac{1}{2} \frac{\delta p^n}{\delta x_i} \right)\]

- **Final step**: use half of the gradient of the still unknown new pressure

  \[(\rho u_i)^{n+1} - (\rho u_i)^{**} = -\Delta t \left( \frac{1}{2} \frac{\delta p^{n+1}}{\delta x_i} \right)\]

- **New velocity** must satisfy the continuity equation (is divergence free):

  - Taking the divergence of final step:
    \[\frac{\delta}{\delta x_i} \left( \frac{\delta p^{n+1}}{\delta x_i} \right) = 2 \frac{\delta (\rho u_i)^{**}}{\Delta t \delta x_i}\]

  - Once \(p\) is solved for, the final step above gives the new velocities
Fractional Step Methods: Example based on Crank-Nicholson

• Putting all steps together:

\[(\rho u_i)^{n+1} - (\rho u_i)^n = \Delta t \left[ \frac{H(u_i^n) + H(u_i^*)}{2} - \frac{1}{2} \left( \frac{\delta p^n}{\delta x_i} + \frac{\delta p^{n+1}}{\delta x_i} \right) \right]\]

– To represent Crank-Nicolson correctly, \(H(u_i^*)\) should be \(H(u_i^{n+1})\)

– However, we can show that the error is 2\(^{nd}\) order in time and thus consistent with C-N’s truncation error: subtract the first step from the final step, to obtain,

\[(\rho u_i)^{n+1} - (\rho u_i)^* = -\frac{\Delta t}{2} \left( \frac{\delta p^{n+1}}{\delta x_i} - \frac{\delta p^n}{\delta x_i} \right) \approx -\frac{\Delta t^2}{2} \frac{\delta}{\delta x_i} \left( \frac{\delta p}{\delta t} \right)\]

– With this, one also obtains:

\[(\rho u_i)^{n+1} - (\rho u_i)^* = -\frac{\Delta t}{2} \frac{\delta (p^{n+1} - p^n)}{\delta x_i} = -\frac{\Delta t}{2} \frac{\delta (p^*)}{\delta x_i}\]

which is similar to the final step, but has the form of a pressure-correction on \(u_i^*\)

• Fractional steps methods have become rather popular

– Many variations, but all are based on the same principles (illustrated by C-N here)

– Main difference with SIMPLE-type time-marching schemes: SIMPLE schemes solve the pressure and momentum equations several times per time-step in outer iterations
Incompressible Fluid

Vorticity Equation

\[ \tilde{\omega} \equiv \text{curl} \mathbf{V} \equiv \nabla \times \mathbf{V} \]

Navier-Stokes Equation

\[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V} \]

*curl* of Navier-Stokes Equation

\[ \frac{D\tilde{\omega}}{Dt} = (\tilde{\omega} \cdot \nabla) \mathbf{V} + \nu \nabla^2 \tilde{\omega} \]
Streamfunction-Vorticity Methods

- For incompressible, 2D flows with constant fluid properties, NS can be simplified by introducing the streamfunction $\psi$ and vorticity $\omega$ as dependent variables

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

- Streamlines: constant $\psi$

- Vorticity vector is orthogonal to plane of the 2D flow

- 2D Continuity is automatically satisfied: \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \)

- In 2D, substituting $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ in $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ leads to the kinematic condition:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$$

- The vertical component of the vorticity equation leads to:

$$\rho \frac{\partial \omega}{\partial t} + \rho u \frac{\partial \omega}{\partial x} + \rho v \frac{\partial \omega}{\partial y} = \mu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right)$$
Streamfunction-Vorticity Methods, Cont’d

• Main advantages:
  – Pressure does not appear in either of these questions!
  – NS has been replaced by a set of 2 coupled PDEs
    • Instead of 2 velocities and 1 pressure, we have only two dependent variables

• Explicit solution scheme
  – Given initial velocity field, compute vorticity by differentiation
  – Use this vorticity $\omega^n$ in the RHS of the dynamic equation for vorticity, to obtain $\omega^{n+1}$
  – With $\omega^{n+1}$ the streamfunction $\psi^{n+1}$ can be obtained from the Poisson equation
  – With $\psi^{n+1}$, we can differentiate to obtain the velocity
  – Continue to time $n+2$, and so on

• One issue with this scheme: boundary conditions
Streamfunction-Vorticity Methods, Cont’d

Boundary conditions

• Boundary conditions for $\psi$
  – Solid boundaries are streamlines and require: $\psi = \text{constant}$
  – However, values of $\psi$ at these boundaries can be computed only if velocity field is known

• Boundary conditions for $\omega$
  – Neither vorticity nor its derivatives at the boundaries are known in advance
  – For example, at the wall: “$\omega_{\text{wall}} = -\tau_{\text{wall}} / \mu$” since $\tau_{\text{wall}} = \mu \frac{\partial u}{\partial y}_{\text{wall}}$
    • Vorticity at the wall is proportional to the shear stress, but the shear stress is often what one is trying to compute
  – Boundary values for $\omega$ can be obtained from $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega$,
    • i.e. one-sided differences at the wall: $\frac{\partial^2 \psi}{\partial n^2} = -\omega$
      but this usually converges slowly and can require refinement
  – Discontinuities also occur at corners
Discontinuities also occur at corners for vorticity

- The derivatives \( \frac{\partial v}{\partial x} \) and \( \frac{\partial u}{\partial y} \) are not continuous at A and B
- This means special treatment for

\[
\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\]

e.g. refine the grid at corners

- Vorticity-streamfunction approach useful in 2D, but is now less popular because extension to 3D difficult
  - In 3D, vorticity has 3 components, hence problem becomes as/more expensive as NS
  - Streamfunction is still used in quasi-2D problems, for example, in the ocean or in the atmosphere
Artificial Compressibility Methods

- Compressible flow is of great importance (e.g. aerodynamics and turbine engine design)
- Many methods have been developed (e.g. MacCormack, Beam-Warming, etc)
- Can they be used for incompressible flows?
- Main difference between incompressible and compressible NS is the mathematical character of the equations
  - Incompressible eqns: no time derivative in the continuity eqn: \( \nabla \cdot \bar{v} = 0 \)
    - They have a mixed parabolic-elliptic character in time-space
  - Compressible eqns: there is a time-derivative in the continuity equation:
    - They have a hyperbolic character: \( \frac{\partial \rho}{\partial t} + \nabla.(\rho \bar{v}) = 0 \)
    - Allow pressure/sound waves
    - How to use methods for compressible flows in incompressible flows?
Artificial Compressibility Methods, Cont’d

• Most straightforward: Append a time derivative to the continuity equation
  – Since density is constant, adding a time-rate-of-change for $\rho$ not possible
  – Use pressure instead (linked to $\rho$ via an eqn. of state in the general case):
    \[
    \frac{1}{\beta} \frac{\partial p}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
    \]
    • where $\beta$ is an artificial compressibility parameter (dimension of velocity$^2$)
    • Its value is key to the performance of such methods:
      – The larger/smaller $\beta$ is, the more/less incompressible the scheme is
      – Large $\beta$ makes the equation stiff (not well conditioned for time-integration)
    • Methods most useful for solving steady flow problem (at convergence: $\frac{\partial p}{\partial t} = 0$)
  – To solve this new problem, many methods can be used, especially
    • All the time-marching schemes (R-K, multi-steps, etc) that we have seen
    • Finite differences or finite volumes in space
    • Alternating direction method is attractive: one spatial direction at a time