2.29 Numerical Fluid Mechanics
Fall 2011 – **Lecture 26**

REVIEW Lecture 25:

- Solution of the Navier-Stokes Equations
  - Pressure Correction Methods: i) Solve momentum for a known pressure leading to new velocity, then; ii) Solve Poisson to obtain a corrected pressure and iii) Correct velocity, go to i) for next time-step.
    - A Simple Explicit and Implicit Schemes
      - Nonlinear solvers, Linearized solvers and ADI solvers
    - Implicit Pressure Correction Schemes for steady problems: iterate using
      - Outer iterations:

$$\mathbf{A}^{\mathbf{u}_i^{m*}} \mathbf{u}_i^{m*} = \mathbf{b}_{\mathbf{u}_i^{m*}}^{m-1} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m-1} \text{ but require } \mathbf{A}^{\mathbf{u}_i^{m}} \mathbf{u}_i^{m} = \mathbf{b}_{\mathbf{u}_i^{m}}^{m} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m} \text{ and } \frac{\boldsymbol{\delta}\, \mathbf{u}_i^m}{\boldsymbol{\delta} x_i} = 0 \quad \Rightarrow \quad \frac{\boldsymbol{\delta}}{\boldsymbol{\delta} x_i}\left(\frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m}\right) \approx \frac{\boldsymbol{\delta}}{\boldsymbol{\delta} x_i}\left(\mathbf{A}^{\mathbf{u}_i^{m*}} \mathbf{u}_i^{m} - \mathbf{b}_{\mathbf{u}_i^{m*}}^{m}\right)$$

      - Inner iterations:

$$\mathbf{A}^{\mathbf{u}_i^{m*}} \mathbf{u}_i^{m} = \mathbf{b}_{\mathbf{u}_i^{m*}}^{m} - \frac{\boldsymbol{\delta} p}{\boldsymbol{\delta} x_i}^{m}$$

    - Projection Methods: Non-Incremental and Incremental Schemes
  - Fractional Step Methods:
    - Example using Crank-Nicholson

$$u_i^{n+1} = u_i^n + (C_i + D_i + P_i)\,\Delta t \quad \Rightarrow \quad \boxed{\begin{aligned} u_i^* &= u_i^n + C_i\,\Delta t \\ u_i^{**} &= u_i^* + D_i\,\Delta t \\ u_i^{n+1} &= u_i^{**} + P_i\,\Delta t \end{aligned}}$$

  - Streamfunction-Vorticity Methods: Scheme and boundary conditions

TODAY (Lecture 26): Navier-Stokes Equations and Intro to Finite Elements

- Solution of the Navier-Stokes Equations
  - Pressure Correction / Projection Methods
  - Fractional Step Methods
  - Streamfunction-Vorticity Methods: scheme and boundary conditions
  - Artificial Compressibility Methods: scheme definitions and example
  - Boundary Conditions: Wall/Symmetry and Open boundary conditions
- Finite Element Methods
  - Introduction
  - Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  - General Approach to Finite Elements:
    - Steps in setting-up and solving the discrete FE system
    - Galerkin Examples in 1D and 2D
  - Computational Galerkin Methods for PDE: general case
    - Variations of MWR: summary
    - Finite Elements and their basis functions on local coordinates (1D and 2D)
    - Unstructured grids: isoparametric and triangular elements

References and Reading Assignments

- Chapter 7 on “Incompressible Navier-Stokes equations” of “J. H. Ferziger and M. Peric, *Computational Methods for Fluid Dynamics*. Springer, NY, 3rd edition, 2002”
- Chapter 11 on “Incompressible Navier-Stokes Equations” of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, *Computational Fluid Dynamics for Engineers*. Springer, 2005.
- Chapter 17 on “Incompressible Viscous Flows” of Fletcher, *Computational Techniques for Fluid Dynamics*. Springer, 2003.
- Chapters 31 on “Finite Elements” of “Chapra and Canale, Numerical Methods for Engineers, 2006.”

Artificial Compressibility Methods

- Compressible flow is of great importance (e.g. aerodynamics and turbine engine design)
- Many methods have been developed (e.g. MacCormack, Beam-Warming, etc)
- Can they be used for incompressible flows?
- Main difference between incompressible and compressible NS is the mathematical character of the equations
  - Incompressible eqns: no time derivative in the continuity eqn: $\nabla.\vec{v} = 0$
    - They have a mixed parabolic-elliptic character in time-space
  - Compressible eqns: there is a time-derivative in the continuity equation:
    - They have a hyperbolic character: $$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0$$
    - Allow pressure/sound waves
    - How to use methods for compressible flows in incompressible flows?

Artificial Compressibility Methods, Cont'd

- Most straightforward: Append a time derivative to the continuity equation
  - Since density is constant, adding a time-rate-of-change for ρ not possible
  - Use pressure instead (linked to ρ via an eqn. of state in the general case):

$$\frac{1}{\beta}\frac{\partial p}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0$$

    - where β is an artificial compressibility parameter (dimension of velocity^2)
    - Its value is key to the performance of such methods:
      - The larger/smaller β is, the more/less incompressible the scheme is
      - Large β makes the equation stiff (not well conditioned for time-integration)
    - Methods most useful for solving steady flow problem (at convergence: $\frac{\partial p}{\partial t}=0$) or inner-iterations in dual-time schemes.
  - To solve this new problem, many methods can be used, especially
    - All the time-marching schemes (R-K, multi-steps, etc) that we have seen
    - Finite differences or finite volumes in space
    - Alternating direction method is attractive: one spatial direction at a time

Artificial Compressibility Methods, Cont'd

- Connecting these methods with the previous ones:
  - Consider the intermediate velocity field $(\rho u_i^*)^{n+1}$ obtained from solving momentum with the old pressure
  - It does not satisfy the incompressible continuity equation: $\frac{\delta(\rho u_i^*)^{n+1}}{\delta x_i} \equiv \frac{\partial \rho^*}{\partial t}$
    - There remains an erroneous time rate of change of mass flux

      $\Rightarrow$ method needs to correct for it

- Example of an artificial compressibility scheme
  - Instead of explicit in time, let's use implicit Euler (larger time steps)

$$\frac{p^{n+1} - p^n}{\beta \, \Delta t} + \left[\frac{\delta(\rho u_i)}{\delta x_i} \right]^{n+1} = 0$$

  - Issue: velocity field at $n+1$ is not known
  - One can linearize about the old (intermediate) state and transform the above equation into a Poisson equation for the pressure or pressure correction!

Artificial Compressibility Methods: Example Scheme, Cont'd

- First, expand unknown velocity using Taylor series in pressure derivatives

$$(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta p}\right]^{n+1} (p^{n+1} - p^n) \qquad (p^{*n+1} = p^n)$$

  – Inserting $(\rho u_i)^{n+1}$ in the continuity equation leads an equation for p^{n+1}

$$\frac{p^{n+1} - p^n}{\beta\, \Delta t} + \frac{\delta}{\delta x_i}\left[(\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta p}\right]^{n+1} (p^{n+1} - p^n)\right] = 0$$

  – Then, take the divergence and derive a Poisson-like equation for p^{n+1}

- One could have also used directly:

$$(\rho u_i)^{n+1} \approx (\rho u_i^*)^{n+1} + \left[\frac{\delta(\rho u_i^*)}{\delta\left(\frac{\delta p}{\delta x_i}\right)}\right]^{n+1} \left(\frac{\delta p}{\delta x_i}^{n+1} - \frac{\delta p}{\delta x_i}^{n}\right)$$

  – Then, still take divergence and derive Poisson-like equation

- Ideal value of β is problem dependent

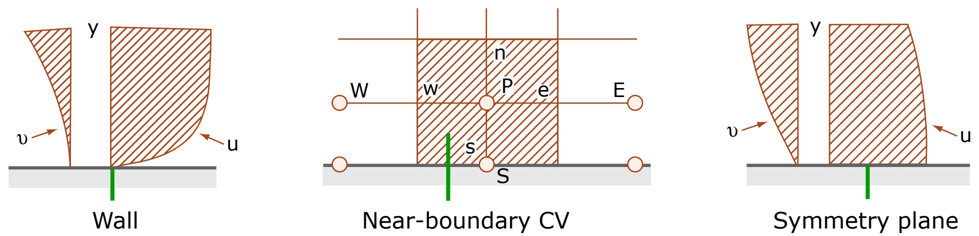
  – The larger the β, the more incompressible. Lowest values of β can be computed by requiring that pressure waves propagate much faster than the flow velocity or vorticity speeds

Numerical Boundary Conditions for N-S eqns.

- At a wall, the no-slip boundary condition applies:
  - Velocity at the wall is the wall velocity (Dirichlet)
  - In some cases, the tangential velocity stays constant along the wall (only for fully-developed), which by continuity, implies no normal viscous stress:

$$\left.\frac{\partial u}{\partial x}\right|_{\text{wall}} = 0 \quad \Rightarrow \quad \left.\frac{\partial v}{\partial y}\right|_{\text{wall}} = 0$$

$$\Rightarrow \quad \tau_{yy} = 2\mu \left.\frac{\partial v}{\partial y}\right|_{\text{wall}} = 0$$



On the boundary conditions at a wall and a symmetry plane

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  - For the shear stress: $F_S^{shear} = \int_{S_S} \tau_{xy}\, dS = \int_{S_S} \mu \frac{\partial u}{\partial y}\, dS \approx \mu_S\, S_S\, \frac{u_P - u_S}{y_P - y_S}$

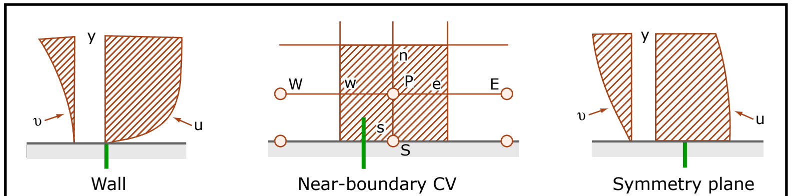
- At a symmetry plane, it is the opposite:
  - Shear stress is null: $\tau_{xy} = \mu \left.\frac{\partial u}{\partial y}\right|_{\text{sym}} = 0 \quad \Rightarrow F_S^{shear} = 0$
  - Normal stress is non-zero:

$$\tau_{yy} = 2\mu \left.\frac{\partial v}{\partial y}\right|_{\text{sym}} \neq 0 \quad \Rightarrow F_S^{normal} = \int_{S_S} \tau_{yy}\, dS = \int_{S_S} 2\mu \frac{\partial v}{\partial y}\, dS \approx 2\mu_S\, S_S\, \frac{v_P - v_S}{y_P - y_S}$$

Numerical Boundary Conditions for N-S eqns, Cont'd

- Wall/Symmetry Pressure BCs for the Momentum equations
  - For the momentum equations with staggered grids, the pressure is not required at boundaries (pressure is computed in the interior in the middle of the CV or FD cell)
  - With collocated arrangements, values at the boundary for p are needed. They can be extrapolated from the interior (may require grid refinement)
- Wall/Symmetry Pressure BCs for the Poisson equation
  - When the mass flux (velocity) is specified at a boundary, this means that:
    - Correction to the mass flux (velocity) at the boundary is also zero
    - This should be implemented in the continuity equation: zero normal-velocity-correction ⇒ often means gradient of the pressure-correction at the boundary is then also zero

(take the dot product of the velocity correction equation with the normal at the bnd)



On the boundary conditions at a wall and a symmetry plane

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Numerical BCs for N-S eqns: Outflow/Outlet Conditions

- Outlet often most problematic since information is advected from the interior to the (open) boundary
- If velocity is extrapolated to the far-away boundary, $\frac{\partial u}{\partial n} = 0 \quad i.e., \quad u_E = u_P$,
  - It may need to be corrected so as to ensure that the mass flux is conserved (same as the flux at the inlet)
  - These corrected BC velocities are then kept fixed for the next iteration. This implies no corrections to the mass flux BC, thus a von Neuman condition for the pressure correction (note that *p* itself is linear along the flow if fully developed).
  - The new interior velocity is then extrapolated to the boundary, etc.
  - To avoid singularities for p (von Neuman at all boundaries for p), one needs to specify p at a one point to be fixed (or impose a fixed mean p)
- If flow is not fully developed: $\frac{\partial u}{\partial n} \neq 0 \quad \Rightarrow \quad \frac{\partial p'}{\partial n} \neq 0 \quad \Rightarrow \text{ e.g. } \frac{\partial^2 u}{\partial n^2} = 0 \quad or \quad \frac{\partial^2 p'}{\partial n^2} = 0$
- If the pressure difference between the inlet and outlet is specified, then the velocities at these boundaries can not be specified.
  - They have to be computed so that the pressure loss is the specified value
  - Can be done again by extrapolation of the boundary velocities from the interior: these extrapolated velocities can be corrected to keep a constant mass flux.
- Much research in OBC in ocean modeling

FINITE ELEMENT METHODS: Introduction

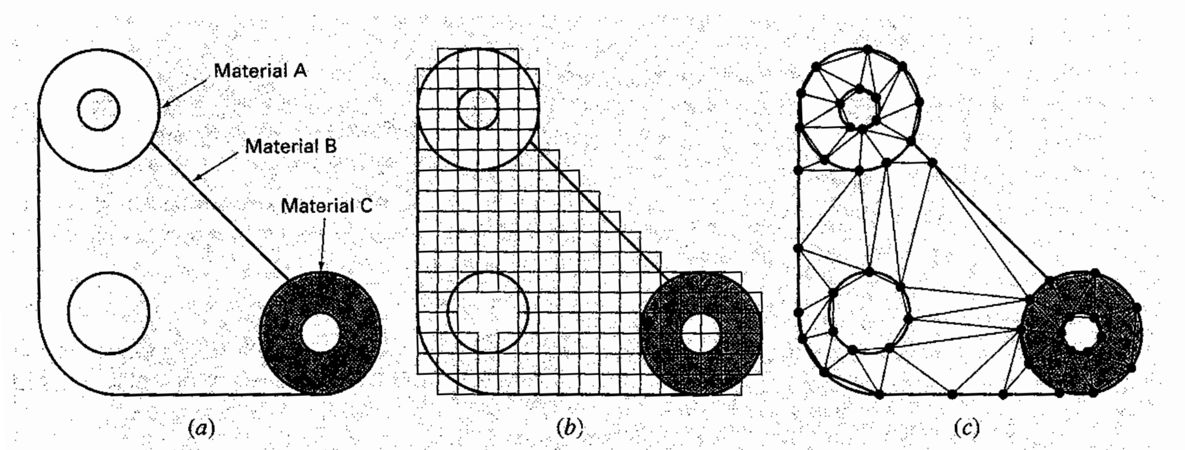
- Finite Difference Methods: based on a discretization of the differential form of the conservation equations
  - Solution domain divided in a grid of discrete points or nodes
  - PDE replaced by finite-divided differences = "point-wise" approximation
  - Harder to apply to complex geometries
- Finite Volume Methods: based on a discretization of the integral forms of the conservation equations:
  - Grid generation: divide domain into set of discrete control volumes (CVs)
  - Discretize integral equation
  - Solve the resultant discrete volume/flux equations
- Finite Element Methods: based on reformulation of PDEs into minimization problem, pre-assuming piecewise shape of solution over finite elements
  - Grid generation: divide the domain into simply shaped regions or "elements"
  - Develop approximate solution of the PDE for each of these elements
  - Link together or assemble these individual element solutions, ensuring some continuity at inter-element boundaries => PDE is satisfied in piecewise fashion

Finite Elements: Introduction, Cont'd

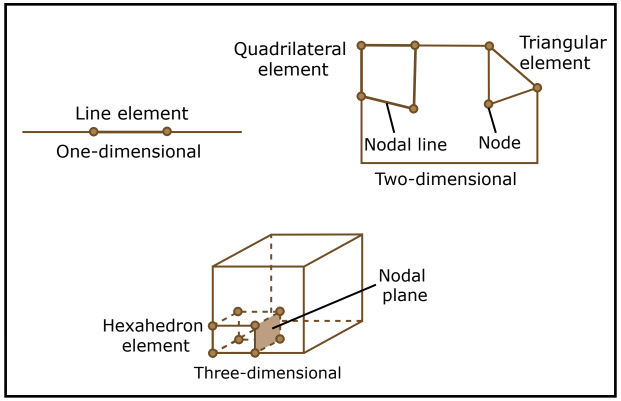
- Originally based on the Direct Stiffness Method (Navier in 1826) and Rayleigh-Ritz, and further developed in its current form in the 1950's (Turner and others)
- Can replace somewhat "ad-hoc" integrations of FV with more rigorous minimization principles
- Originally more difficulties with convection-dominated (fluid) problems, applied to solids with diffusion-dominated properties

Comparison of FD and FE grids

(a) A gasket with irregular geometry and nonhomogeneous composition. (b) Such a system is very difficult to model with a finite-difference approach. This is due to the fact that complicated approximations are required at the boundaries of the system and at the boundaries between regions of differing composition. (c) A finite-element discretization is much better suited for such systems.



Examples of Finite elements



Finite Elements: Introduction, Cont'd

- Classic example: Rayleigh-Ritz / Calculus of variations
  - Finding the solution of $$\frac{\partial^2 u}{\partial x^2} = -f \quad \text{on }]0,1[$$

    is the same as finding u that minimizes $$J(u) = \int_0^1 \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - u\, f\, dx$$
  - R-R approximation:
    - Expand unknown u into shape/trial functions $$u(x) = \sum_{i=1}^{n} a_i\, \phi_i(x)$$

      and find coefficients a_i such that $J(u)$ is minimized
- Finite Elements:
  - As Rayleigh-Ritz but choose trial functions to be piecewise shape function defined over set of elements, with some continuity across elements

Finite Elements: Introduction, Cont'd
Method of Weigthed Residuals

- There are several avenues that lead to the same FE formulation
  - A conceptually simple, yet mathematically rigorous, approach is the Method of Weighted Residuals (MWR)
  - Two special cases of MWR: the Galerkin and Collocation Methods
- In the MWR, the desired function u is replaced by a finite series approximation into shape/basis/interpolation functions:

$$\tilde{u}(x) = \sum_{i=1}^{n} a_i \, \phi_i(x)$$

  - $\phi_i(x)$ chosen such they satisfy the boundary conditions of the problem
  - But, they will not in general satisfy the PDE: $L(u) = f$

    $\Rightarrow$ they lead to a residual: $L(\tilde{u}(x)) - f(x) = R(x) \neq 0$
  - The objective is to select the undetermined coefficients a_i so that this residual is minimized in some sense

Finite Elements: Method of Weigthed Residuals, Cont'd

- One possible choice is to set the integral of the residual to be zero. This only leads to one equation for n unknowns

$\Rightarrow$ Introduce the so-called weighting functions $w_i(x)$ $i=1,2,\ldots, n,$ and set the integral of each of the weighted residuals to zero to yield n independent equations:

$$\int_t \int_0^L R(x)\, w_i(x)\, dx\, dt = 0, \quad i = 1,2,\ldots,n$$

- In 3D, this becomes:

$$\int_t \int_V R(\mathbf{x})\, w_i(\mathbf{x})\, d\mathbf{x}\, dt = 0, \quad i = 1,2,\ldots,n$$

- A variety of FE schemes arise from the definition of the weighting functions and of the choice of the shape functions
  - Galerkin: the weighting functions are chosen to be the shape functions
  - Subdomain method: the weighting function is chosen to be unity in the sub-region over which it is applied
  - Collocation Method: the weighting function is chosen to be a Dirac-delta

Finite Elements: Method of Weigthed Residuals, Cont'd

- Galerkin: $\int_t \int_V R(\mathbf{x})\, \phi_i(\mathbf{x})\, d\mathbf{x}\, dt = 0, \quad i = 1, 2, \ldots, n$
  - Basis functions formally required to be complete set of functions
  - Can be seen as "residual forced to zero by being orthogonal to all basis functions"

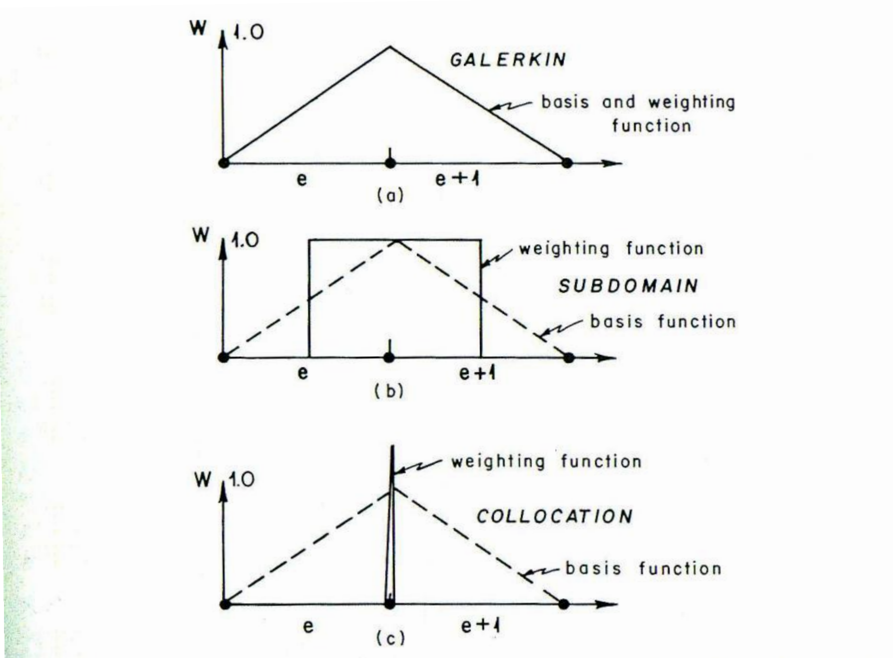


Figure 2.4. Schematic representation of the one-dimensional weighting functions for the Galerkin, subdomain and collocation methods. (It is assumed here that the chapeau function is used as a basis for all methods.)

- Subdomain method:

$$\int_t \int_{V_i} R(\mathbf{x})\, d\mathbf{x}\, dt = 0, \quad i = 1, 2, \ldots, n$$

  - Non-overlapping domains V_i often set to elements
  - Easy integration, but not as accurate

- Collocation Method: $\int_t \int_V R(\mathbf{x})\, \delta_{x_i}(\mathbf{x})\, d\mathbf{x}\, dt = 0, \quad i = 1, 2, \ldots, n$
  - Mathematically equivalent to say that each residual vanishes at each collocation points $x_i \Rightarrow$ Accuracy strongly depends on locations x_i.
  - Requires no integration.

General Approach to Finite Elements

1. Discretization: divide domain into "finite elements"
   – Define nodes (vertex of elements) and nodal lines/planes

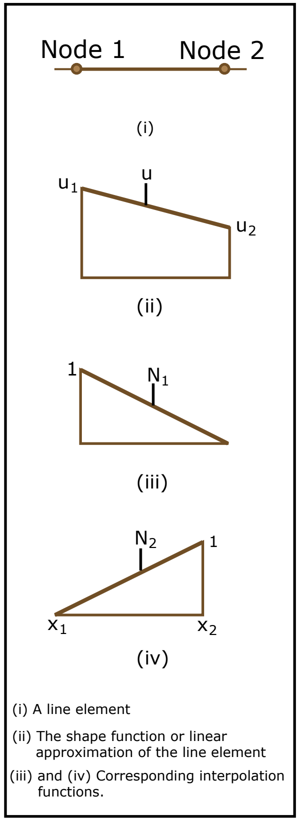
2. Set-up Element equations
   i. Choose appropriate basis functions $\phi_i(x)$: $\tilde{u}(x) = \sum_{i=1}^{n} a_i \, \phi_i(x)$
      - 1D Example with Lagrange's polynomials: Interpolating functions $N_i(x)$

$$\tilde{u} = a_0 + a_1 \, x = u_1 N_1(x) + u_2 N_2(x) \quad \text{where } \mathrm{N}_1(x) = \frac{x_2 - x}{x_2 - x_1} \text{ and } \mathrm{N}_2(x) = \frac{x - x_1}{x_2 - x_1}$$

      - With this choice, we obtain for example the 2nd order CDS and Trapezoidal rule: $\frac{d\tilde{u}}{dx} = a_1 = \frac{u_2 - u_1}{x_2 - x_1}$ and $\int_{x_1}^{x_2} \tilde{u} \, dx = \frac{u_1 + u_2}{2}(x_2 - x_1)$

   ii. Evaluate coefficients of these basis functions by approximating the solution in an optimal way
      - This develops the equations governing the element's dynamics
      - Two main approaches: Method of Weighted Residuals (MWR) or Variational Approach

      $\Rightarrow$ Result: relationships between the unknown coefficients a_i so as to satisfy the PDE in an optimal approximate way



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General Approach to Finite Elements, Cont'd

2. Set-up Element equations, Cont'd
   – Mathematically, combining i. and ii. gives the element equations: a set of (often linear) algebraic equations for a given element *e:*

$$\mathbf{K}_e\,\mathbf{u}_e = \mathbf{f}_e$$

   where $\mathbf{K}_\text{e}$ is the element property matrix (stiffness matrix in solids), $\mathbf{u}_\text{e}$ the vector of unknowns at the nodes and $\mathbf{f}_\text{e}$ the vector of external forcing

3. Assembly:
   – After the individual element equations are derived, they must be assembled: i.e. impose continuity constraints for contiguous elements
   – This leas to: $\mathbf{K}\,\mathbf{u} = \mathbf{f}$

   where $\mathbf{K}$ is the assemblage property or coefficient matrix, $\mathbf{u}$ and $\mathbf{f}$ the vector of unknowns at the nodes and $\mathbf{f}_\text{e}$ the vector of external forcing

4. Boundary Conditions: Modify " $\mathbf{K}\,\mathbf{u} = \mathbf{f}$ " to account for BCs
5. Solution: use LU, banded, iterative, gradient or other methods
6. Post-processing: compute secondary variables, errors, plot, etc

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