2.29 Numerical Fluid Mechanics
Fall 2011 – **Lecture 27**

REVIEW Lecture 26:

- Solution of the Navier-Stokes Equations
  - Pressure Correction and Projection Methods (review)
  - Fractional Step Methods (Example using Crank-Nicholson)
  - Streamfunction-Vorticity Methods: scheme and boundary conditions
  - Artificial Compressibility Methods:
    - Scheme definitions and example
  - Boundary Conditions:
    - Wall/Symmetry and Open boundary conditions

$$\frac{1}{\beta}\frac{\partial p}{\partial t}+\frac{\partial \rho u_i}{\partial x_i}=0 \quad \begin{cases} \dfrac{p^{n+1}-p^n}{\beta\,\Delta t}+\left[\dfrac{\delta(\rho u_i)}{\delta x_i}\right]^{n+1}=0 \\ \left(\rho u_i\right)^{n+1}\approx\left(\rho u_i^*\right)^{n+1}+\left[\dfrac{\delta(\rho u_i^*)}{\delta\left(\delta p/\delta x_i\right)}\right]^{n+1}\left(\dfrac{\delta p}{\delta x_i}^{n+1}-\dfrac{\delta p}{\delta x_i}^{n}\right) \end{cases}$$

- Finite Element Methods
  - Introduction
  - Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  - General Approach to Finite Elements:
    - Steps in setting-up and solving the discrete FE system
    - Galerkin Examples in 1D and 2D

Project Presentations: Schedule 15 minutes each, including questions

Notes:

i) “Project” Office Hours As needed

ii) Need Draft Titles by Monday Dec 12

iii) Reports latest at noon on Tue Dec 20

- 4 to 25 pages of text, single space, 12ft
- 1 to 10 pages of figures

TODAY (Lecture 27): Finite Elements and Intro to Turbulent Flows

Finite Element Methods

- Introduction, Method of Weighted Residuals: Galerkin, Subdomain and Collocation
- General Approach to Finite Elements:
  - Steps in setting-up and solving the discrete FE system
  - Galerkin Examples in 1D and 2D
- Computational Galerkin Methods for PDE: general case
  - Variations of MWR: summary
  - Isoparametric finite elements and basis functions on local coordinates (1D, 2D, triangular)

Turbulent Flows and their Numerical Modeling

- Properties of Turbulent Flows
  - Stirring and Mixing
  - Energy Cascade and Scales
  - Turbulent Wavenumber Spectrum and Scales
- Numerical Methods for Turbulent Flows: Classification

Note: special recitation today on (Hybrid-) Discontinuous Galerkin Methods

References and Reading Assignments

- Chapter 31 on “Finite Elements” of “Chapra and Canale, Numerical Methods for Engineers, 2006.”
- Lapidus and Pinder, 1982: Numerical solutions of PDEs in Science and Engineering.
- Chapter 5 on “Weighted Residuals Methods” of Fletcher, Computational Techniques for Fluid Dynamics. Springer, 2003.

- Chapter 9 on “Turbulent Flows” of “J. H. Ferziger and M. Peric, *Computational Methods for Fluid Dynamics*. Springer, NY, 3rd edition, 2002”
- Chapter 3 on “Turbulence and its Modelling” of H. Versteeg, W. Malalasekra, An Introduction to Computational Fluid Dynamics: The Finite Volume Method. Prentice Hall, Second Edition.
- Chapter 4 of “I. M. Cohen and P. K. Kundu. Fluid Mechanics. Academic Press, Fourth Edition, 2008”
- Chapter 3 on “Turbulence Models” of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, Computational Fluid Dynamics for Engineers. Springer, 2005.

General Approach to Finite Elements

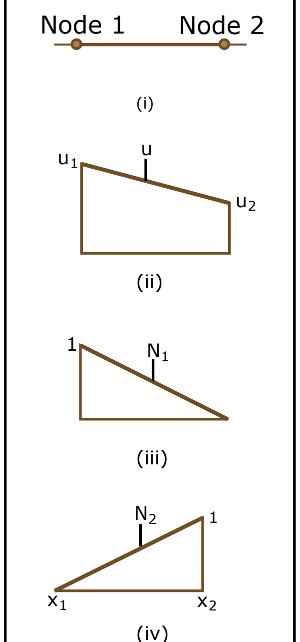
1. Discretization: divide domain into "finite elements"
   – Define nodes (vertex of elements) and nodal lines/planes
2. Set-up Element equations
   i. Choose appropriate basis functions $\phi_i(x)$: $\tilde{u}(x) = \sum_{i=1}^{n} a_i\, \phi_i(x)$
      - 1D Example with Lagrange's polynomials: Interpolating functions $N_i(x)$

$$\tilde{u} = a_0 + a_1\, x = u_1 N_1(x) + u_2 N_2(x) \quad \text{where } \mathrm{N}_1(x) = \frac{x_2 - x}{x_2 - x_1} \text{ and } \mathrm{N}_2(x) = \frac{x - x_1}{x_2 - x_1}$$

      - With this choice, we obtain for example the 2nd order CDS and Trapezoidal rule: $\frac{d\,\tilde{u}}{dx} = a_1 = \frac{u_2 - u_1}{x_2 - x_1}$ and $\int_{x_1}^{x_2} \tilde{u}\, dx = \frac{u_1 + u_2}{2}(x_2 - x_1)$

   ii. Evaluate coefficients of these basis functions by approximating the solution in an optimal way
      - This develops the equations governing the element's dynamics
      - Two main approaches: Method of Weighted Residuals (MWR) or Variational Approach

      $\Rightarrow$ Result: relationships between the unknown coefficients a_i so as to satisfy the PDE in an optimal approximate way



(i) A line element
(ii) The shape function or linear approximation of the line element
(iii) and (iv) Corresponding interpolation functions.

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General Approach to Finite Elements, Cont'd

2. Set-up Element equations, Cont'd
   - Mathematically, combining i. and ii. gives the element equations: a set of (often linear) algebraic equations for a given element *e:*

$$\mathbf{K}_e \, \mathbf{u}_e = \mathbf{f}_e$$

   where $\mathbf{K}_\mathrm{e}$ is the element property matrix (stiffness matrix in solids), $\mathbf{u}_\mathrm{e}$ the vector of unknowns at the nodes and $\mathbf{f}_\mathrm{e}$ the vector of external forcing

3. Assembly:
   - After the individual element equations are derived, they must be assembled: i.e. impose continuity constraints for contiguous elements
   - This leas to: $\mathbf{K}\,\mathbf{u} = \mathbf{f}$

   where $\mathbf{K}$ is the assemblage property or coefficient matrix, $\mathbf{u}$ and $\mathbf{f}$ the vector of unknowns at the nodes and $\mathbf{f}_\mathrm{e}$ the vector of external forcing

4. Boundary Conditions: Modify " $\mathbf{K}\,\mathbf{u} = \mathbf{f}$ " to account for BCs
5. Solution: use LU, banded, iterative, gradient or other methods
6. Post-processing: compute secondary variables, errors, plot, etc

Galerkin's Method: Simple Example

Differential Equation

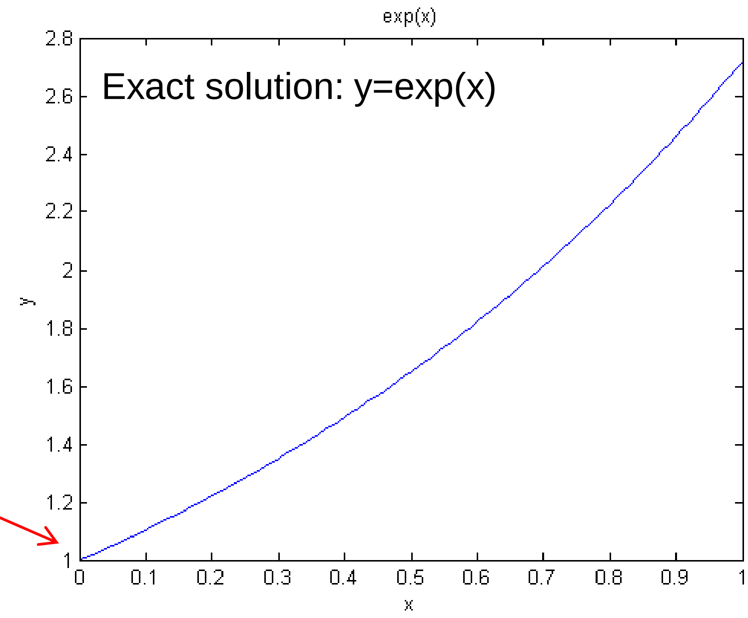
$$\frac{dy}{dx} - y = 0$$

Boundary Conditions

$$y = 1\ , \quad x = 0$$

1. Discretization:

Generic N (here 3) equidistant points along x



2. Element equations:

i. Basis (Shape) Functions:
Power Series

$$\tilde{y} = 1 + \sum_{j=1}^{N} a_j x^j$$

Boundary Condition

Note: this is equivalent to imposing the BC on the full sum

$$\tilde{y} = \sum_{j=0}^{N} a_j x^j$$

$$a_0 = 1$$

In this simple example, a single element is chosen to cover the whole domain ⇒ the element/mass matrix is the full one ($\mathbf{K}=\mathbf{K}_e$)

Galerkin's Method: Simple Example, Cont'd

ii. Optimal coefficients with MWR: set weighted residuals (remainder) to zero

Remainder: $R = \frac{d\tilde{y}}{dx} - \tilde{y}$

$$R = -1 + \sum_{j=1}^{N} a_j (j x^{j-1} - x^j)$$

Galerkin $\Rightarrow$ set remainder orthogonal to each shape function:

Denoting inner products as: $(f,g) = \langle f,g \rangle = \int_0^1 f\, g\, dx$

leads to: $(R, x^{k-1}) = 0, \quad k = 1,\ldots,N$

which then leads to the Algebraic Equations:

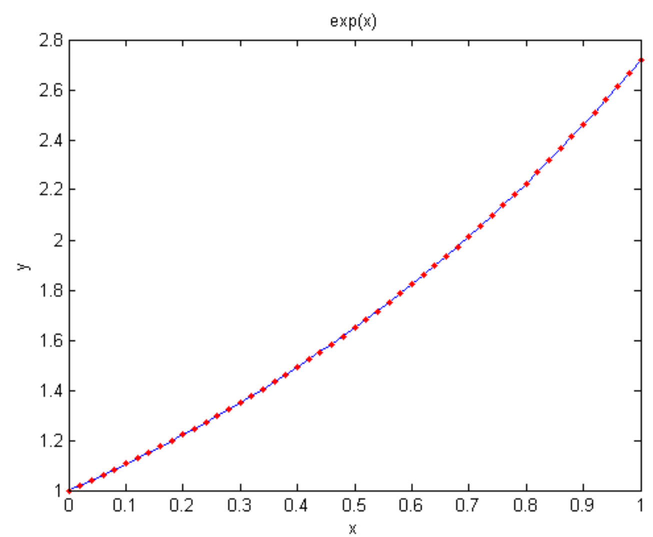
$$\mathbf{Ma} = \mathbf{d}$$

$$d_k = (1, x^{k-1})$$

$$m_{kj} = (j x^{j-1} - x^j, x^{k-1}) = \frac{j}{j+k-1} - \frac{1}{j+k}$$

exp_eq.m

```
N=3;
d=zeros(N,1);
m=zeros(N,N);
for k=1:N
    d(k)=1/k;
    for j=1:N
        m(k,j) = j/(j+k-1)-1/(j+k);
    end
end
a=inv(m)*d;
y=ones(1,n);
for k=1:N
    y=y+a(k)*x.^k
end
```



Galerkin's Method Simple Example, Cont'd

3 - 4. Assembly and boundary conditions:

Already done (element fills whole domain)

```
N=3;
d=zeros(N,1);
m=zeros(N,N);
for k=1:N
    d(k)=1/k;
    for j=1:N
        m(k,j) = j/(j+k-1)-1/(j+k);
    end
end
a=inv(m)*d;
y=ones(1,n);
for k=1:N
    y=y+a(k)*x.^k
end
```

exp_eq.m

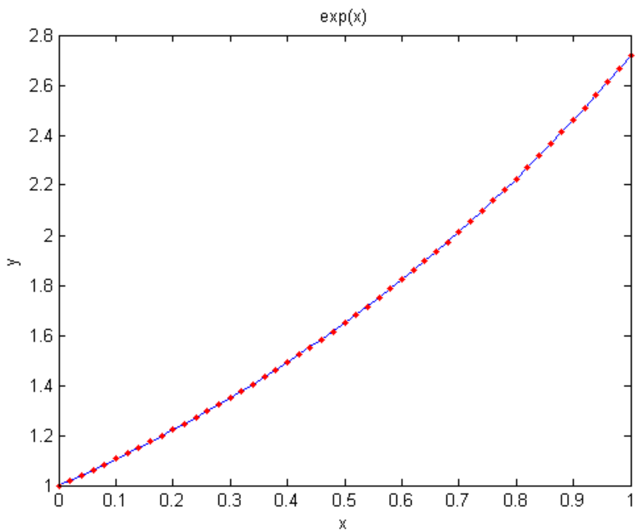
5. Solution: For $N = 3$

$$\mathbf{a}^T = [1.0141, 0.4225, 0.2817];$$

$$\tilde{y} = 1 + 1.0141x + 0.4225x^2 + 0.2817x^3$$

L_2 Error:

$$L_2 = ||y - \tilde{y}||_2 = \sqrt{\sum_{\ell=1}^{L} (y(x_\ell) - \tilde{y}(x_\ell))^2}$$



Comparisons with other Weighted Residual Methods

$$\frac{dy}{dx} - y = 0$$

$$\tilde{y} = 1 + \sum_{j=1}^{N} a_j x^j$$

Least Squares

$$\begin{bmatrix} 1/3 & 1/4 & 1/5 \\ 1/4 & 8/15 & 2/3 \\ 1/5 & 2/3 & 33/35 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/3 \\ 3/4 \end{bmatrix}$$

Subdomain Method

$$\begin{bmatrix} 5/18 & 8/81 & 11/324 \\ 3/18 & 20/81 & 69/324 \\ 1/18 & 26/81 & 163/324 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Galerkin

$$\begin{bmatrix} 1/2 & 2/3 & 3/4 \\ 1/6 & 5/12 & 11/20 \\ 1/12 & 3/10 & 13/30 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$$

Collocation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.75 & 0.625 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Comparisons with other Weighted Residual Methods

Comparison of coefficients for approximate solution of $dy/dx - y = 0$

Scheme \ Coefficient	a_1	a_2	a_3
Least squares	1.0131	0.4255	0.2797
Galerkin	1.0141	0.4225	0.2817
Subdomain	1.0156	0.4219	0.2813
Collocation	1.0000	0.4286	0.2857
Taylor series	1.0000	0.5000	0.1667
Optimal $L_{2,d}$	1.0138	0.4264	0.2781

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Comparison of approximate solutions of $dy/dx - y = 0$

x	Least squares	Galerkin	Subdomain	Collocation	Taylor series	Optimal $L_{2,d}$	Exact
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	1.2219	1.2220	1.2223	1.2194	1.2213	1.2220	1.2214
0.4	1.4912	1.4913	1.4917	1.4869	1.4907	1.4915	1.4918
0.6	1.8214	1.8214	1.8220	1.8160	1.8160	1.8219	1.8221
0.8	2.2260	2.2259	2.2265	2.2206	2.2053	2.2263	2.2255
1.0	2.7183	2.7183	2.7187	2.7143	2.6667	2.7183	2.7183
$\lVert y_a - y \rVert_{2,d}$	0.00105	0.00103	0.00127	0.0094	0.0512	0.00101	

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Galerkin's Method in 2 Dimensions

Differential Equation

$$L(u) = 0$$

Boundary Conditions

$$S(u) = 0$$

Test Function Solution (u_o satisifies BC)

$$\tilde{u} = u_0(x,y) + \sum_{j=1}^{N} a_j \phi_j(x,y)$$

Remainder

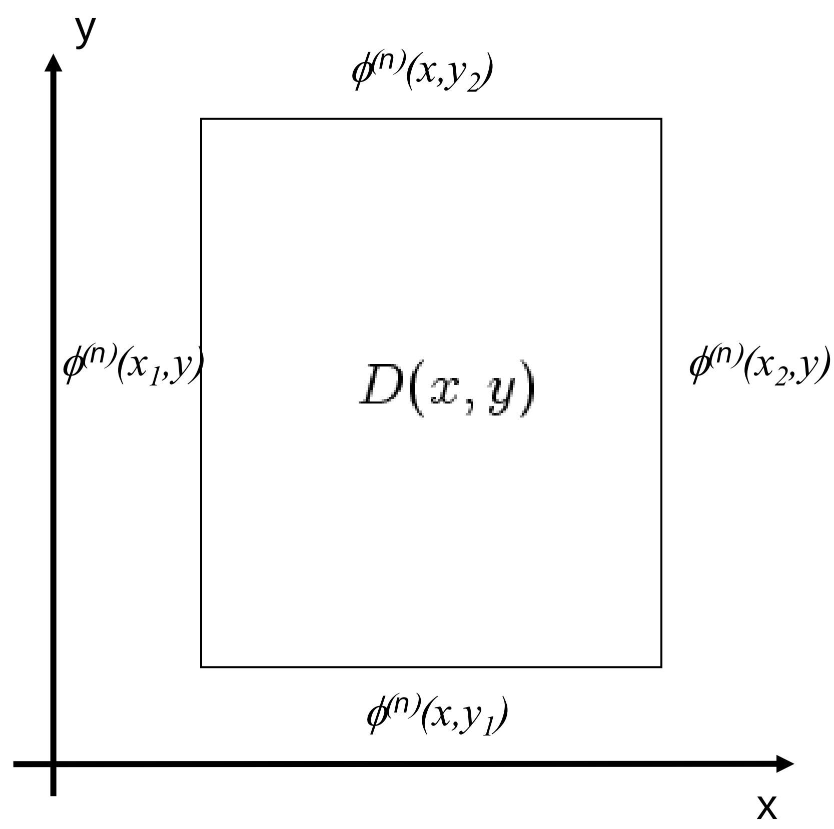
$$R(u_0, a_1, \dots a_N, x, y) = L(\tilde{u}) = L(u_0) + \sum_{j=1}^{N} a_j L(\phi_j(x,y))$$

Inner Product: $(f,g) = \int\int_D f g dx dy$

Galerkin's Method

$$(R, \phi_k) = 0$$

$$\sum_{j=1}^{N} a_j (L(\phi_j), \phi_k) = -(L(u_0), \phi_k)$$



Galerkin's method: 2D Example
Fully-developed Laminar Viscous Flow in Duct

Steady, Very Viscous Fluid Flow in Duct

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\nu\left\{\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right\}$$

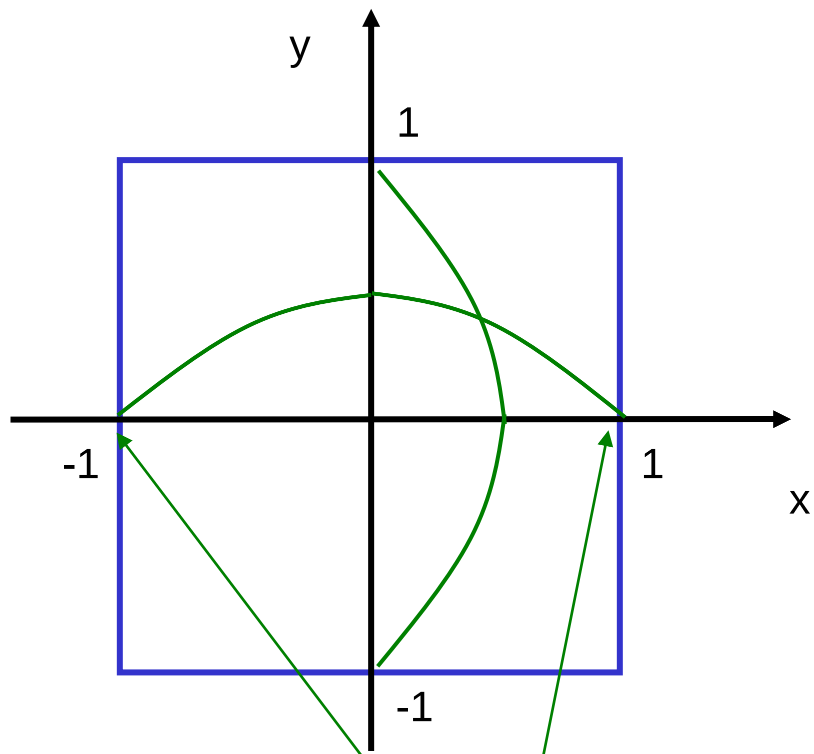
$$\frac{\partial p}{\partial z}=\text{const}$$

Poisson's Equation, Non-dimensional:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=-1$$

Shape/Test Functions

$$\tilde{w}=\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N}a_{ij}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y$$



Shape/Test functions satisfy boundary conditions

4 BCs: No-slip (zero flow) at the walls

Again: element fills the whole domain in this example

Galerkin's Method: Viscous Flow in Duct, Cont'd

Remainder:

$$R = -\left[\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N} a_{ij}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y\left\{\left(i\frac{\pi}{2}\right)^2+\left(j\frac{\pi}{2}\right)^2\right\}-1\right]$$

Inner product:

$$\left(R,\cos k\frac{\pi}{2}x\cos \ell\frac{\pi}{2}y\right)\ ,\ i,j=1,3,5,\ldots$$

Analytical Integration:

$$a_{ij} = \left(\frac{8}{\pi^2}\right)^2\frac{(-1)^{(i+j)/2-1}}{ij(i^2+j^2)}$$

Galerkin Solution:

$$\tilde{w} = \left(\frac{8}{\pi^2}\right)^2\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N}\frac{(-1)^{(i+j)/2-1}}{ij(i^2+j^2)}\cos i\frac{\pi}{2}x\cos j\frac{\pi}{2}y$$

Flow Rate:

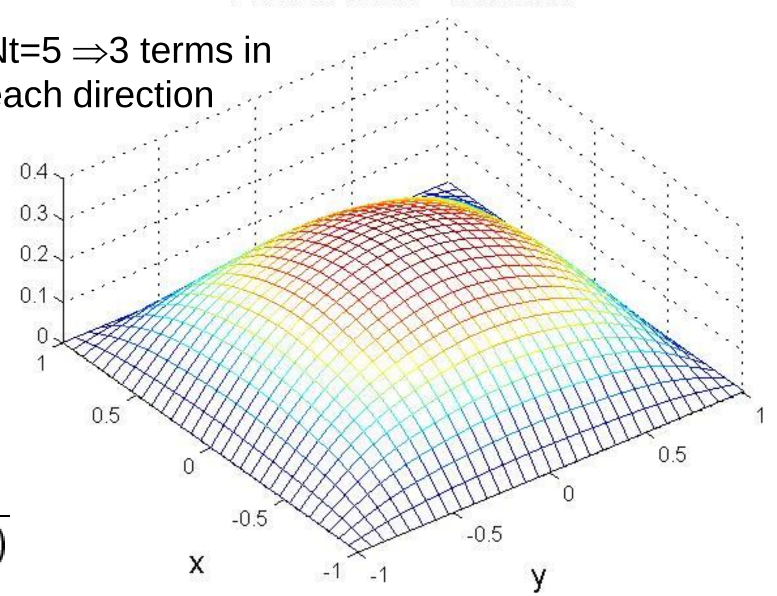
$$\begin{aligned}\dot{q} &= \int_{-1}^{1}\int_{-1}^{1}\tilde{w}(x,y)dxdy\\ &= 2\left(\frac{8}{\pi^2}\right)^3\sum_{i=1,3,5\ldots}^{N}\sum_{j=1,3,5\ldots}^{N}\frac{1}{i^2j^2(i^2+j^2)}\end{aligned}$$

duct_galerkin.m

```
x=[-1:h:1]';
y=[-1:h:1];
n=length(x); m=length(y); w=zeros(n,m);
Nt=5;
for j=1:n
 xx(:,j)=x; yy(j,:)=y;
end
for i=1:2:Nt
    for j=1:2:Nt
        w=w+(8/pi^2)^2*
           (-1)^((i+j)/2-1)/(i*j*(i^2+j^2))
           *cos(i*pi/2*xx).*cos(j*pi/2*yy);
    end
end
```

Flow in Duct - Galerkin

Nt=5 ⇒3 terms in each direction



Computational Galerkin Methods: General Case

Differential Equation: $L(u) = 0$

Residuals

- PDE: $L(\tilde{u}) = R$
- ICs: $I(\tilde{u}) = R_I$
- BCs: $S(\tilde{u}) = R_B$

$$R = 0$$

$$R_B = 0$$

$$R, R_B \neq 0$$

Boundary problem
- PDE satisfied exactly
- Boundary Element Method
  - Panel Method
  - Spectral Methods

Inner problem
- Boundary conditions satisfied exactly
- Finite Element Method
- Spectral Methods

Mixed Problem

Global Test Function:

$$\tilde{u}(\mathbf{x}, t) = u_0(\mathbf{x}, t) + \sum_{j=1}^{N} a_j \phi(\mathbf{x}, t)$$

Time Marching:

$$\tilde{u}(\mathbf{x}, t) = u_0(\mathbf{x}, t) + \sum_{j=1}^{N} a_j(t) \phi(\mathbf{x})$$

Weighted Residuals

$$(R, w_k(\mathbf{x})) = 0, k = 1, \ldots N$$

$$\lim_{N\rightarrow\infty} ||\tilde{u} - u||_2 = 0$$

Different forms of the Methods of Weighted Residuals: Summary

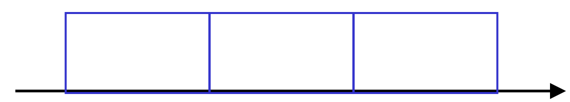
Inner Product $(L(u), w) = 0$

Discrete Form $(f, g) = \sum_{i=1}^{N} f_i g_i$

$k = 1, 2, \ldots, n$

Subdomain Method:

$$w_k = \begin{cases} 1 & \text{in } D_k \\ 0 & \text{outside } D_k \end{cases}$$



$$\int_t \int_{V_k} R(\mathbf{x})\, d\mathbf{x}\, dt = 0$$

Collocation Method:

$$w_k(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_k)$$

$$R(\mathbf{x}_k) = 0$$

$$R(\mathbf{x}_k) = 0$$

Least Squares Method:

$$\left.\begin{array}{l} (\mathrm{R},\mathrm{R}) = \text{minimum} \\ \dfrac{\partial}{\partial a_i}\left(\int_t \int_{V_k} R(\mathbf{x})\, R(\mathbf{x}) d\mathbf{x}\, dt \right) = 0 \end{array}\right\} \Rightarrow \quad w_k = \frac{\partial R}{\partial a_k}$$

In the least-square method, the coefficients are adjusted so as to minimize the integral of the residuals. It amounts to the continuous form of regression.

Method of Moments: $w_k(\mathbf{x}) = x^k \;,\quad k = 0, 1, \ldots, N$

Galerkin: $w_k(\mathbf{x}) = \phi_k(\mathbf{x})$

In Galerkin, weight functions are basis functions: they sum to one at any position in the element. In many cases, Galerkin's method yields the same result as variational methods

How to obtain solution for Nodal Unknowns? Modal Basis vs. Interpolating (Nodal) Basis functions

- $\tilde{u}(x,y) = \sum_{j=1}^{N} \bar{u}_j \; N_j(x,y)$

- $\tilde{u}(x,y) = \sum_{k=1}^{N} a_k \; \phi_k(x,y)$

$$\Rightarrow \quad \bar{u}_j = \sum_{k=1}^{N} a_k \; \phi_k(x_j, y_j)$$

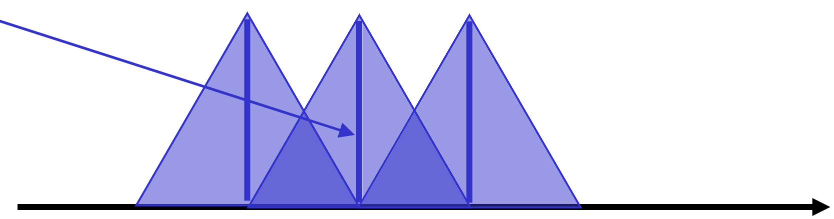
$$\Rightarrow \quad \bar{\mathbf{u}} = \mathbf{\Phi}\,\mathbf{a} \quad \Rightarrow \quad \mathbf{a} = \mathbf{\Phi}^{-1}\,\bar{\mathbf{u}}$$

- $\tilde{u}(x,y) = \sum_{k=1}^{N} \left(\sum_{j=1}^{N} \left(\mathbf{\Phi}^{-1}\right)_{kj} \bar{u}_j \right) \phi_k(x,y)$

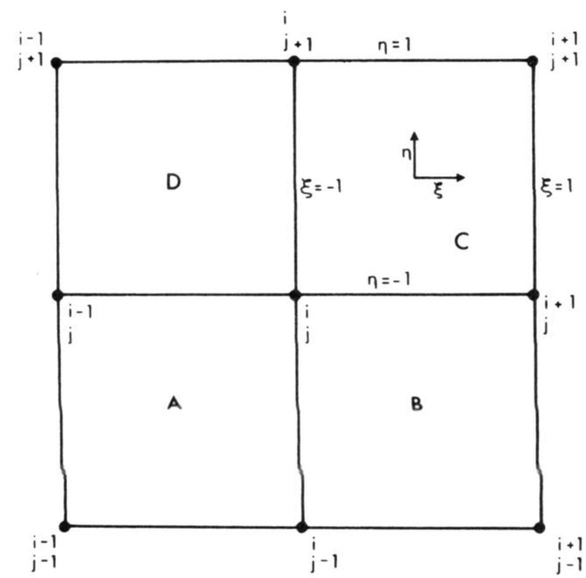
$$= \sum_{j=1}^{N} \bar{u}_j \left(\sum_{k=1}^{N} \left(\mathbf{\Phi}^{-1}\right)_{kj} \phi_k(x,y) \right)$$

$$\Rightarrow \quad N_j(x,y) = \sum_{k=1}^{N} \left(\mathbf{\Phi}^{-1}\right)_{kj} \phi_k(x,y)$$

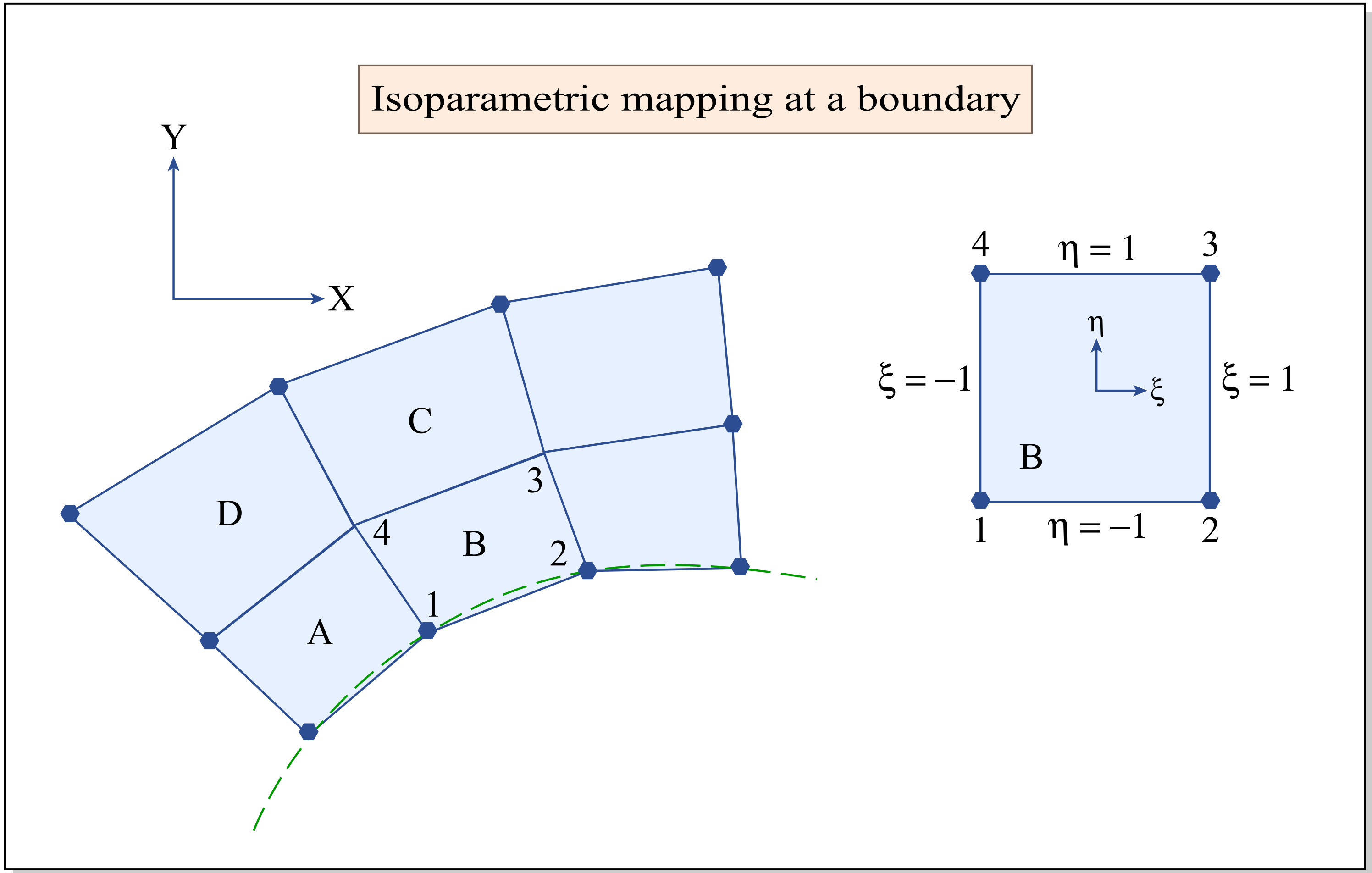
1 Dimension



2 Dimensions



Complex Boundaries
Isoparametric Elements



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Finite Elements
1-dimensional Elements

Trial Function Solution

$$\tilde{u} = \sum_{j=1}^{N} N_j(x)\bar{u}_j$$

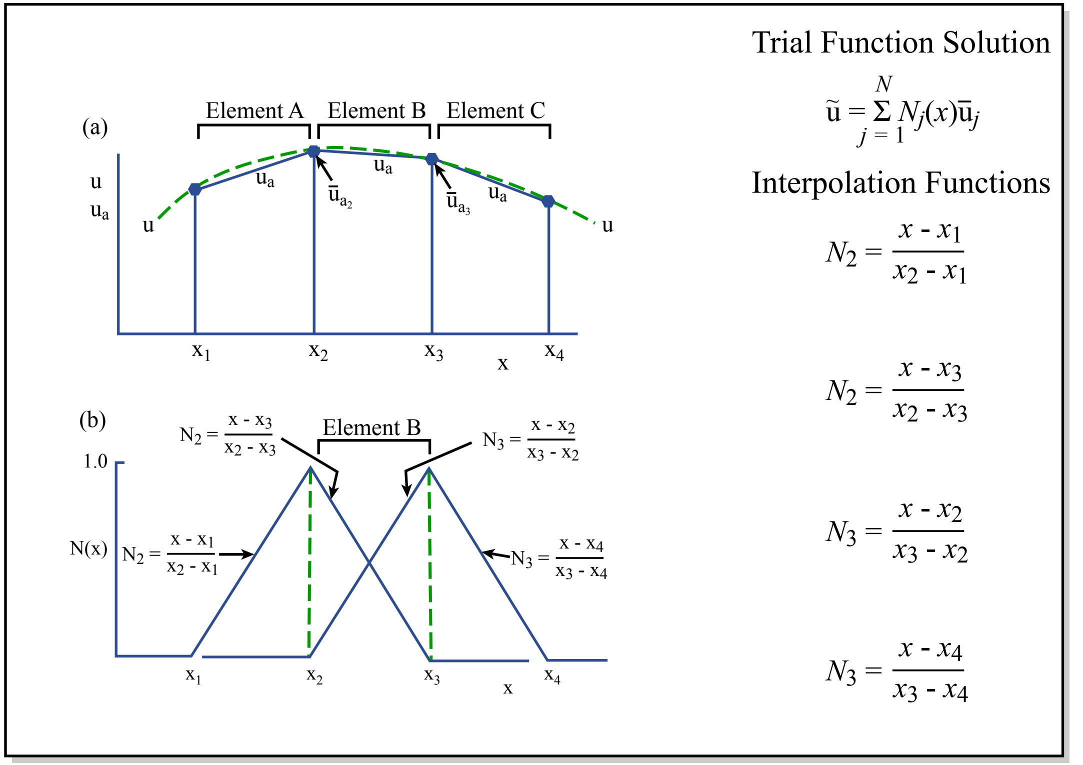
Interpolation (Nodal) Functions

$$N_2 = \frac{x - x_1}{x_2 - x_1}$$

$$N_2 = \frac{x - x_3}{x_2 - x_3}$$

$$N_3 = \frac{x - x_2}{x_3 - x_2}$$

$$N_4 = \frac{x - x_4}{x_3 - x_4}$$



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Finite Elements
1-dimensional Elements

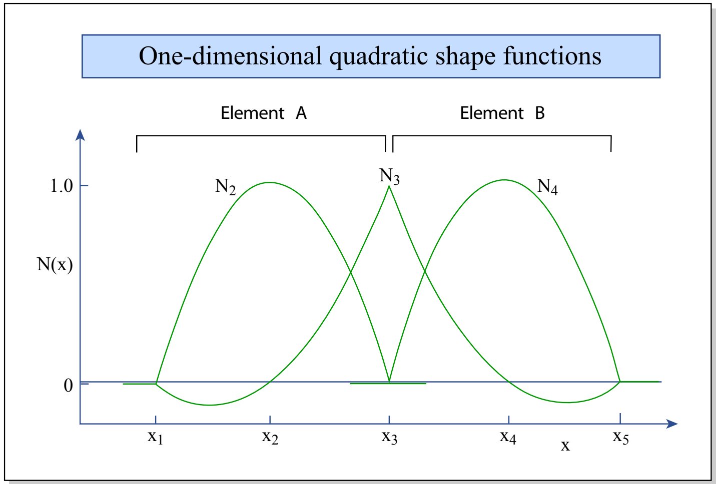
Quadratic Interpolation Functions

$$N_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$$N_3 = \frac{(x - x_4)(x - x_5)}{(x_3 - x_4)(x_3 - x_5)}$$

$$N_4 = \frac{(x - x_3)(x - x_5)}{(x_4 - x_3)(x_4 - x_5)}$$

$$N_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$



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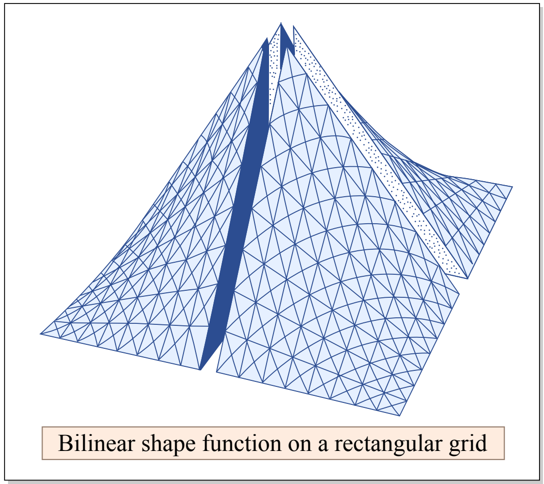
Finite Elements in 1D: Nodal Basis Functions in the Local Coordinate System

Please see pp. 63-65 in Lapidus, L., and G. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering*. 1st ed. Wiley-Interscience, 1982. [See the selection using Google Books preview]

Finite Elements
2-dimensional Elements



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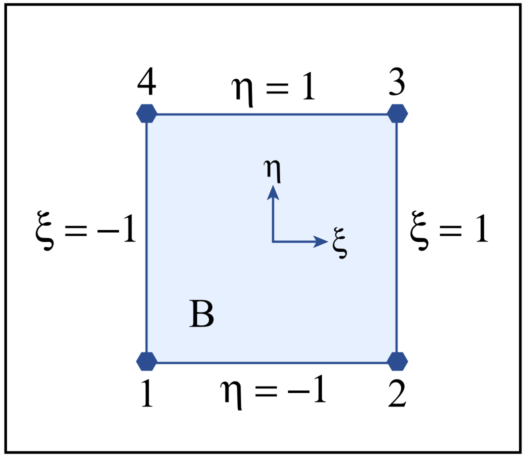
$$\tilde{u} = \sum_{i=1}^{N} \sum_{j=1}^{N} N_{ij}(x)\overline{u}_{ij}$$

$$\tilde{u} = \sum_{\ell=1}^{4} N_{\ell}(\xi, \eta)\overline{u}_{\ell}$$



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Linear Interpolation (Nodal) Functions

$$\begin{aligned} N_1 &= 0.25(1-\xi)(1-\eta) \\ N_2 &= 0.25(1+\xi)(1-\eta) \\ N_3 &= 0.25(1-\xi)(1+\eta) \\ N_4 &= 0.25(1+\xi)(1+\eta) \end{aligned}$$

$$N_\ell = 0.25(1+\xi_\ell \xi)(1+\eta_\ell \eta)$$

Quadratic Interpolation (Nodal) Functions

$$\prod_{r \neq i} \frac{(\xi - \xi_r)(\eta - \eta_r)}{(\xi_i - \xi_r)(\eta_i - \eta_r)}$$

$$\begin{aligned} N_i &= 0.25\xi_i\xi(1+\xi_i\xi)\eta_i\eta(1+\eta_i\eta) \\ N_i &= 0.5(1-\xi^2)\eta_i\eta(1+\eta_i\eta)\ , \ \xi_i = 0 \\ N_i &= 0.5(1-\eta^2)\xi_i\xi(1+\xi_i\xi)\ , \ \eta_i = 0 \\ N_i &= (1-\xi^2)(1-\eta^2) \end{aligned}$$

Two-Dimensional Finite Elements
Example: Flow in Duct, Bilinear Basis functions

Finite Element Solution

$$\tilde{w} = \sum_{j=1}^{N} \overline{w}_j N_j(x, y)$$

$$N_j = 0.25(1 + \xi_j \xi)(1 + \eta_j \eta)$$

$$\left(\frac{\partial^2 \tilde{w}}{\partial x^2}, N_k\right) + \left(\frac{\partial^2 \tilde{w}}{\partial y^2}, N_k\right) = (-1, N_k)$$

Integration by Parts

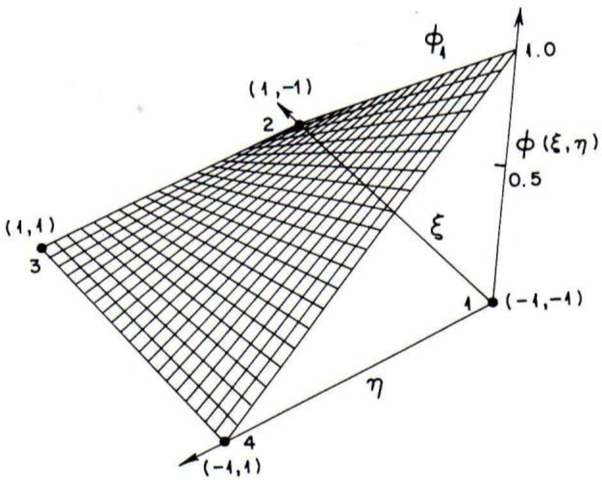
$$\left(\frac{\partial^2 w}{\partial x^2}, N_k\right) \equiv \int_{-1}^{1} \frac{\partial^2 w}{\partial x^2} N_k \, dx = \left[\frac{\partial w}{\partial x} N_k\right]_{-1}^{1} - \int_{-1}^{1} \frac{\partial w}{\partial x} \frac{dN_k}{dx} \, dx$$

$$\left(\frac{\partial^2 \tilde{w}}{\partial x^2}, N_k\right) = -\left(\frac{\partial \tilde{w}}{\partial x}, \frac{\partial N_k}{\partial x}\right) \qquad \text{(for center nodes)}$$

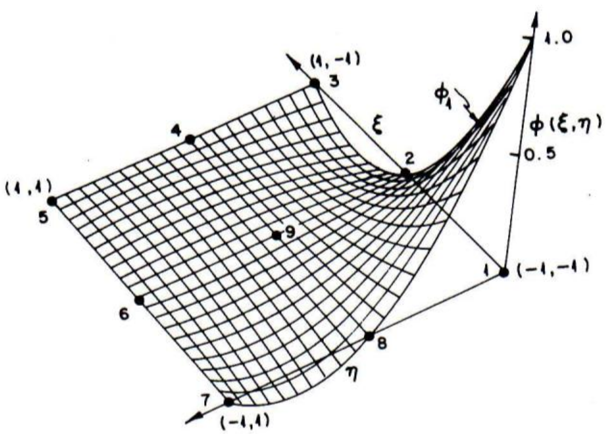
Algebraic Equations for center nodes

$$-\sum_{j=1}^{N} \left(\int_{-1}^{1} \int_{-1}^{1} \frac{\partial N_j}{\partial x} \frac{\partial N_k}{\partial x} + \frac{\partial N_j}{\partial y} \frac{\partial N_k}{\partial y} \, dx dy \right) \overline{w}_j = -\int_{-1}^{1} \int_{-1}^{1} 1 N_k dx dy, \quad k = 1, \ldots N$$

Finite Elements in 2D: Nodal Basis Functions in the Local Coordinate System

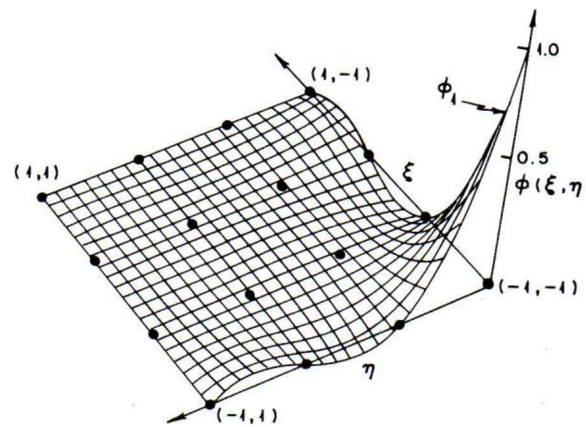


(a)



(b)

Figure 2.11. (*a*) Two-dimensional basis function that is linear along each side. (*b*) Two-dimensional Lagrangian basis function that is quadratic along each side. Note the occurrence of a central node where the basis function must be zero.



(c)

Figure 2.11. (*Continued*)

Figure 2.11. (*c*) Two-dimensional Lagrangian basis function that is cubic along each side. Note the occurrence of four interior nodes where the basis function is defined to be zero.

Finite Elements in 2D: Nodal Basis Functions in the Local Coordinate System

Please see table 2.7a, “Basis Functions Formulated Using Quadratic, Cubic, and Hermitian Cubic Polynomials,” in Lapidus, L., and G. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering*. 1st ed. Wiley-Interscience, 1982.

Triangular Coordinates

Linear Polynomial Modal Basis Functions:

$$u(x,y) = a_0 + a_{1,1}\ x + a_{1,2}\ y$$

$$\left.\begin{aligned} u_1(x,y) &= a_0 + a_{1,1}\ x_1 + a_{1,2}\ y_1 \\ u_2(x,y) &= a_0 + a_{1,1}\ x_2 + a_{1,2}\ y_2 \\ u_3(x,y) &= a_0 + a_{1,1}\ x_3 + a_{1,2}\ y_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_{1,1} \\ a_{1,2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

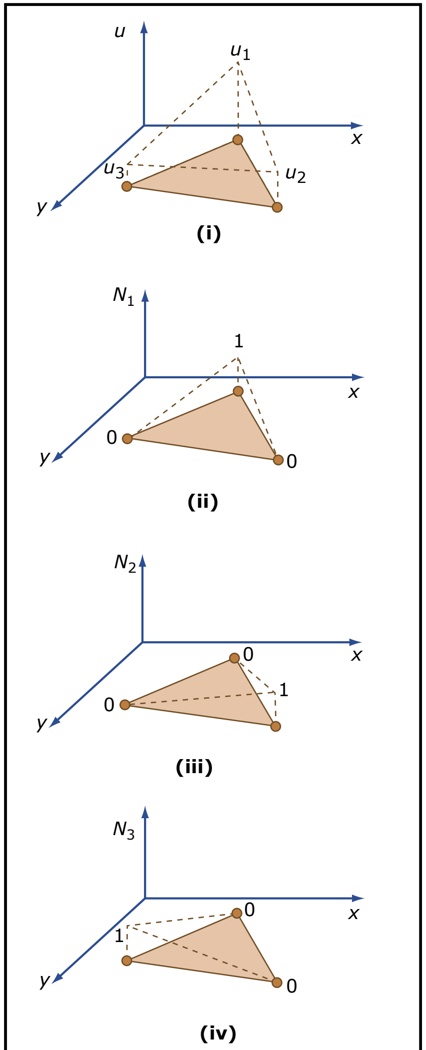
Nodal Basis (Interpolating) Functions:

$$u(x,y) = u_1\ N_1(x,y) + u_2\ N_2(x,y) + u_3\ N_3(x,y)$$

$$N_1(x,y) = \frac{1}{2A_T}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)\ x + (x_3 - x_2)\ y\right]$$

$$N_2(x,y) = \frac{1}{2A_T}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)\ x + (x_1 - x_3)\ y\right]$$

$$N_3(x,y) = \frac{1}{2A_T}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)\ x + (x_2 - x_1)\ y\right]$$



A linear approximation function (i) and its interpolation functions (ii)-(iv).

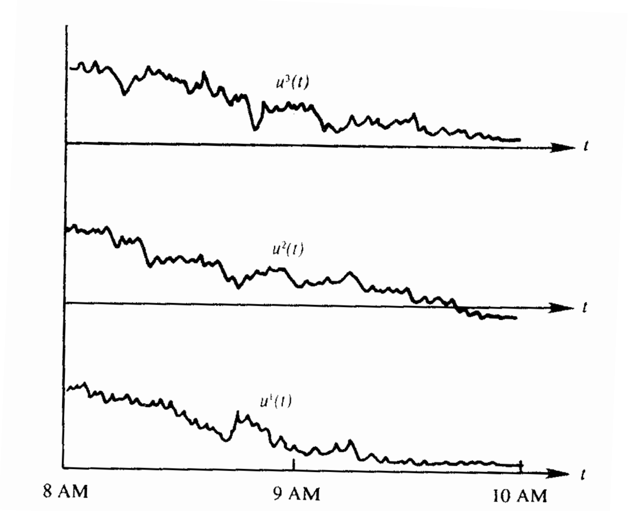
Image by MIT OpenCourseWare.

Turbulent Flows and their Numerical Modeling

- Most real flows are turbulent (at some time and space scales)
- Properties of turbulent flows
  - Highly unsteady: velocity at a point appears random
  - Three-dimensional in space: instantaneous field fluctuates rapidly, in all three dimensions (even if time-averaged or space-averaged field is 2D)

Some Definitions

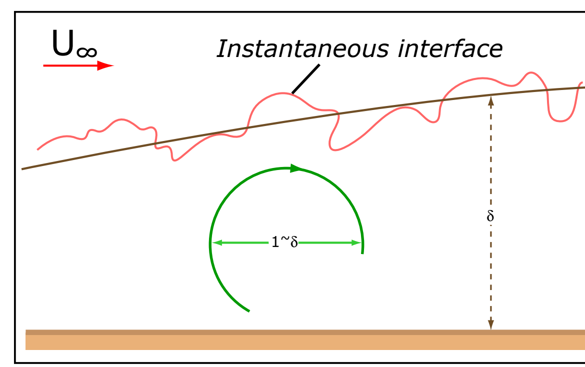
- Ensemble averages: “average of a collection of experiments performed under identical conditions”
- Stationary process: “statistics independent of time”
- For a stationary process, time and ensemble averages are equal



Three turbulent velocity realizations in an atmospheric BL in the morning (Kundu and Cohen, 2008)

Turbulent Flows and their Numerical Modeling

- Properties of turbulent flows, Cont'd
  - Highly nonlinear (e.g. high Re)
  - High vorticity: vortex stretching is one of the main mechanisms to maintain or increase the intensity of turbulence
  - High stirring: turbulence increases rate at which conserved quantities are stirred
    - Stirring: advection process by which conserved quantities of different values are brought in contact (swirl, folding, etc)
    - Mixing: irreversible molecular diffusion (dissipative process). Mixing increases if stirring is large (because stirring leads to large 2nd and higher spatial derivatives).
    - Turbulent diffusion: averaged effects of stirring modeled as "diffusion"
  - Characterized by "Coherent Structures"
    - CS are often spinning, i.e. eddies
    - Turbulence: wide range of eddies' size, in general, wide range of scales



Turbulent flow in a BL: Large eddy has the size of the BL thickness

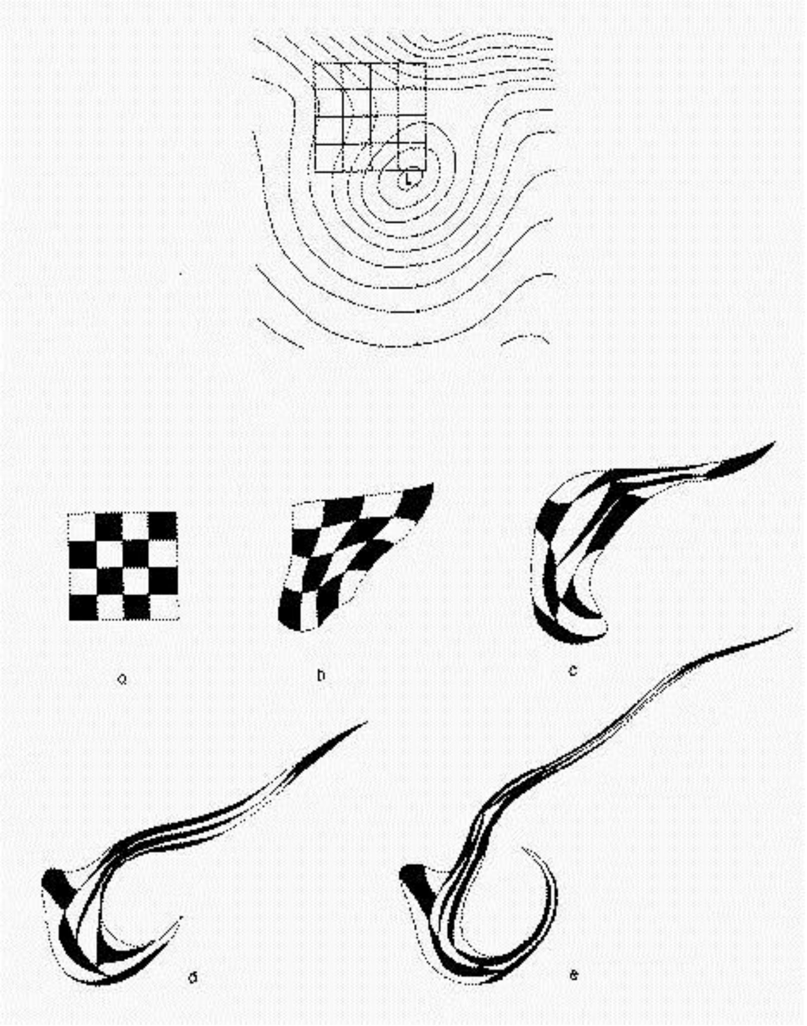
Image by MIT OpenCourseWare.

Stirring and Mixing

Welander's "scrapbook".

Welander P. Studies on the general development of motion in a two-dimensional ideal fluid. *Tellus, 7:141–156, 1955.*

- His numerical solution illustrates differential advection by a simple velocity field.
- A checkerboard pattern is deformed by a numerical quasigeostrophic barotropic flow which models atmospheric flow at the 500mb level. The initial streamline pattern is shown at the top. Shown below are deformed check board patterns at 6, 12, 24 and 36 hours, respectively.
- Notice that each square of the checkerboard maintains constant area as it deforms (conservation of volume).



Source: Fig. 2 from Welander, P. "Studies on the General Development of Motion in a Two-Dimensional, Ideal Fluid." *Tellus* 7, no. 2 (1955).

Energy Cascade and Scales

British meteorologist Richarson's famous quote:

"Big whorls have little whorls,
Which feed on their velocity,
And little whorls have lesser whorls,
And so on to viscosity".

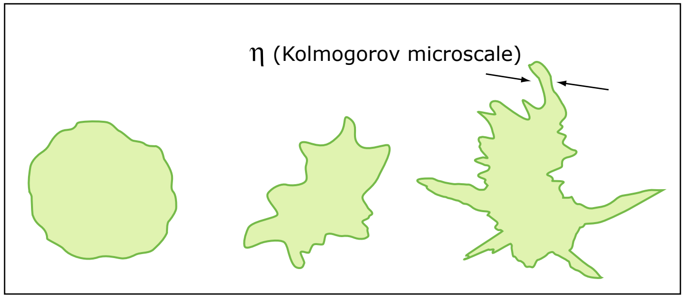


Image by MIT OpenCourseWare.

Dimensional Analyses and Scales (Tennekes and Lumley, 1972, 1976)

- Largest eddy scales: L, T, U = Distance/Time over which fluctuations are correlated and U = large eddy velocity
- Viscous scales: η, τ, u' = viscous length (Kolmogorov scale), time and velocity scales

Hypothesis: rate of turbulent energy production ≈ rate of viscous dissipation

- Length-scale ratio: $L/\eta \sim O(\mathrm{Re}_L^{3/4})$ $\quad \mathrm{Re}_L = u' L/\nu$
- Time-scale ratio: $T/\tau \sim O(\mathrm{Re}_L^{1/2})$
- Velocity-scale ratio: $U/u' \sim O(\mathrm{Re}_L^{1/4})$

Turbulent Wavenumber Spectrum and Scales

- Turbulent Kinetic Energy Spectrum $S(K)$: $\overline{u'^2} = \int_0^\infty S(K)\,dK$
- In the inertial sub-range, Kolmogorov argued by dimensional analysis that

$$S = S(K,\varepsilon) = A\,\varepsilon^{2/3} K^{-5/3} \qquad \ell^{-1} \ll K \ll \eta^{-1}$$

$A \sim 1.5$ found to be universal for turbulent flows

- Turbulent energy dissipation

$$\varepsilon \sim \frac{\text{Turb. energy}}{\text{Turb. time scale}} = u'^2 \times \frac{u'}{L} = \frac{u'^3}{L}$$

- Komolgorov microscale:
  - Size of eddies depend on turb. dissipation ε and viscosity ν
  - Dimensional Analysis: $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

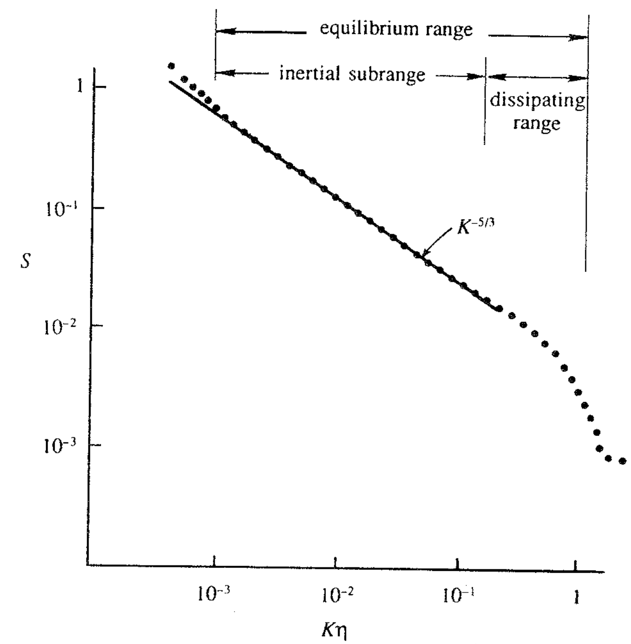


Figure 13.12 A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale. The unit of S is arbitrary, and the dots represent hypothetical data.

Numerical Methods for Turbulent Flows

Primary approach (used to be) is experimental

Numerical Methods classified into methods based on:

1) Correlations: useful mostly for 1D problems, e.g.: $f = f(\text{Re}, \varepsilon)$ $\quad Nu = \varphi(\text{Re}, \text{Pr}, Ra)$

- Moody chart or friction factor relations for turbulent pipe flows, Nusselt number for heat transfer as a function of Re and Pr, etc.

2) Integral equations:

- Integrate PDEs (NS eqns.) in one or more spatial coordinates
- Solve using ODE schemes (time-marching)

3) Averaged equations

- Averaged over time or over an (hypothetical) ensemble of realizations
- Often decompositions into mean and fluctuations: $u = \bar{u} + u' ; \quad \phi = \bar{\phi} + \phi'$
- Leads to a set of PDEs, the Reynolds-averaged Navier-Stokes (RANS) equations ("*One-point closure*" methods)

Numerical Methods classification, Cont'd:

4) Large-Eddy Simulations (LES)

- Solves for the largest scales of motions of the flow
- Only Approximates or parameterizes the small scale motions
- Compromise between RANS and DNS

5) Direct Numerical Simulations (DNS)

- Solves for all scales of motions of the turbulent flow (full Navier-Stokes)

- The methods 1-to-5 make less and less approximations, but computational time increases
- Conservation PDEs are solved as for laminar flows: major challenge is the much wider range of scales (of motions, heat transfer, etc)

2.29 Numerical Fluid Mechanics
Fall 2011