2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 28

REVIEW Lecture 27:

- Finite Element (FE) Methods $\tilde{u}(x) = \sum_{i=1}^{n} a_i \, \phi_i(x) \quad \Rightarrow \quad L\left(\tilde{u}(x)\right) - f(x) = R(x) \neq 0$
  - Method of Weighted Residuals: Galerkin, Subdomain and Collocation
  - General Approach to FEs, Set-up and Examples $\int_V R(\mathbf{x}) \, w_i(\mathbf{x}) \, d\mathbf{x} = 0, \quad i = 1, 2, \ldots, n$
  - Computational Galerkin Methods for PDE: general case
    - Variations of MWR: summary
    - Isoparametric finite elements and basis functions on local coordinates (1D, 2D, triangular)
- Turbulent Flows and their Numerical Modeling
  - Properties of Turbulent Flows
    - Stirring and Mixing
    - Energy Cascade and Scales
    - Turbulent Wavenumber Spectrum and Scales
  - Numerical Methods for Turbulent Flows: Classification
  - Direct Numerical Simulations (DNS) for Turbulent Flows
  - Reynolds-averaged Navier-Stokes (RANS)

Project Presentations: Schedule
15 minutes each, including questions

Notes:

i) "Project" Office Hours As needed

ii) Need Draft Titles by Monday Dec 12

iii) Reports latest at noon on Tue Dec 20

- 4 to 25 pages of text, single space, 12ft
- 1 to 10 pages of figures

References and Reading Assignments

- Chapter 9 on “Turbulent Flows” of “J. H. Ferziger and M. Peric, *Computational Methods for Fluid Dynamics*. Springer, NY, 3rd edition, 2002”
- Chapter 3 on “Turbulence and its Modelling” of H. Versteeg, W. Malalasekra, An Introduction to Computational Fluid Dynamics: The Finite Volume Method. Prentice Hall, Second Edition.
- Chapter 4 of “I. M. Cohen and P. K. Kundu. *Fluid Mechanics*. Academic Press, Fourth Edition, 2008”
- Chapter 3 on “Turbulence Models” of T. Cebeci, J. P. Shao, F. Kafyeke and E. Laurendeau, *Computational Fluid Dynamics for Engineers*. Springer, 2005.

Energy Cascade and Scales

British meteorologist Richarson's famous quote:

"Big whorls have little whorls,
Which feed on their velocity,
And little whorls have lesser whorls,
And so on to viscosity".

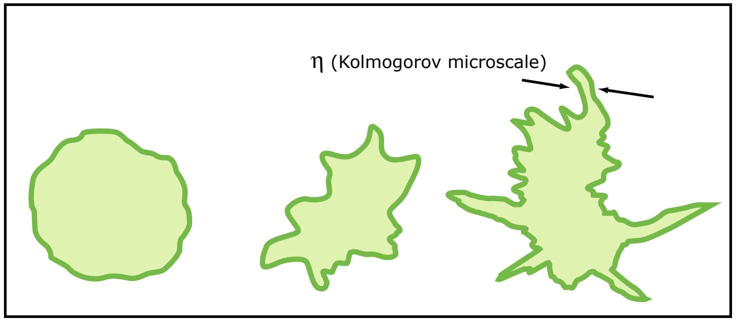


Image by MIT OpenCourseWare.

Dimensional Analyses and Scales (Tennekes and Lumley, 1972, 1976)

- Largest eddy scales: L, T, U = Distance/Time over which fluctuations are correlated and U = large eddy velocity (usually all three are very close to these of mean flow)
- Viscous scales: η, τ, u_v = viscous length (Kolmogorov scale), time and velocity scales

Hypothesis: rate of turbulent energy production ≈ rate of viscous dissipation

- ❖ Length-scale ratio: $\boxed{L / \eta \sim O(\mathrm{Re}_L^{3/4}) \qquad \mathrm{Re}_L = UL/\nu}$
- ❖ Time-scale ratio: $\boxed{T / \tau \sim O(\mathrm{Re}_L^{1/2})}$
- ❖ Velocity-scale ratio: $\boxed{U / u_v \sim O(\mathrm{Re}_L^{1/4})}$

Re_L= largest eddy Re $\sim \mathrm{Re}_{\mathrm{mean}}$ to $0.01\ \mathrm{Re}_{\mathrm{mean}}$

Turbulent Wavenumber Spectrum and Scales

- Turbulent Kinetic Energy Spectrum $S(K)$: $\overline{u'^2} = \int_0^\infty S(K)\, dK$
- In the inertial sub-range, Kolmogorov argued by dimensional analysis that

$$S = S(K, \varepsilon) = A\, \varepsilon^{2/3} K^{-5/3} \qquad \ell^{-1} \ll K \ll \eta^{-1}$$

$A \sim 1.5$ found to be universal for turbulent flows

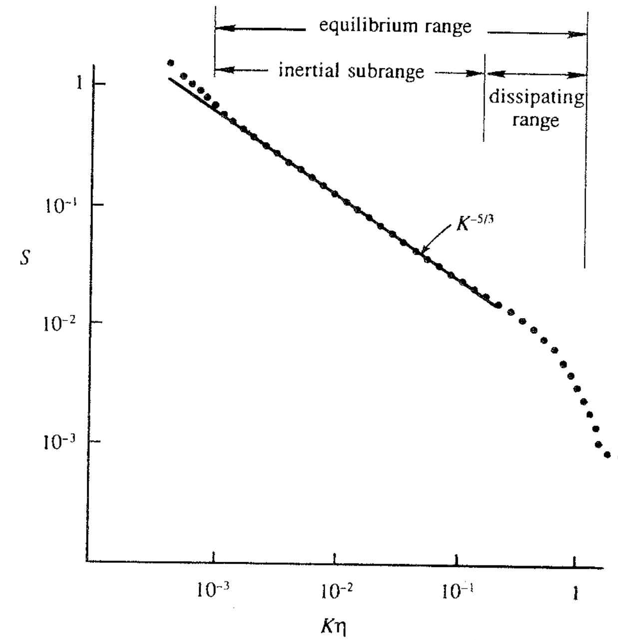


Figure 13.12 A typical wavenumber spectrum observed in the ocean, plotted on a log–log scale. The unit of S is arbitrary, and the dots represent hypothetical data.

- Turbulent energy dissipation

$$\varepsilon \sim \frac{\text{Turb. energy}}{\text{Turb. time scale}} = u'^2 \times \frac{u'}{L} = \frac{u'^3}{L}$$

- Komolgorov microscale:
  - Size of eddies depend on turb. dissipation ε and viscosity ν
  - Dimensional Analysis: $\eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

$$\ell = L \quad \text{(large eddies scale)}$$

$$u = U + u'$$

Numerical Methods for Turbulent Flows

Primary approach (used to be) is experimental

Numerical Methods classified into methods based on:

1) Correlations: useful mostly for 1D problems, e.g.: $f = f(\mathrm{Re}, \varepsilon)$
$Nu = \varphi(\mathrm{Re}, \mathrm{Pr}, Ra)$

- Moody chart or friction factor relations for turbulent pipe flows, Nusselt number for heat transfer as a function of Re and Pr, etc.

2) Integral equations:

- Integrate PDEs (NS eqns.) in one or more spatial coordinates
- Solve using ODE schemes (time-marching)

3) Averaged equations

- Averaged over time or over an (hypothetical) ensemble of realizations
- Often decompositions into mean and fluctuations: $u = \overline{u} + u' ; \quad \phi = \overline{\phi} + \phi'$
- Require closure models and lead to a set of PDEs, the Reynolds-averaged Navier-Stokes (RANS) eqns. (“*One-point closure*” methods)

Numerical Methods for Turbulent Flows

Numerical Methods classification, Cont'd:

4) Large-Eddy Simulations (LES)

- Solves for the largest scales of motions of the flow
- Only approximates or parameterizes the small scale motions
- Compromise between RANS and DNS

5) Direct Numerical Simulations (DNS)

- Solves for all scales of motions of the turbulent flow (full Navier-Stokes)

- The methods 1) to 5) make less and less approximations, but computational time increases from 1) to 5).
- Conservation PDEs are solved as for laminar flows: major challenge is the much wider range of scales (of motions, heat transfer, etc)

Direct Numerical Simulations (DNS) for Turbulent Flows

- Most accurate approach
  - Solve NS with no averaging or approximation other than numerical discretizations whose errors can be estimated/controlled
- Simplest conceptually, all is resolved:
  - Size of domain must be at least a few times the distance L over which fluctuations are correlated (L= largest eddy scale)
  - Resolution must capture all kinetic energy dissipation, i.e. grid size must be smaller than viscous scale, the Kolmogorov scale, η
  - For homogenous isotropic turbulence, uniform grid is fine, hence number of grid points (DOFs) in each direction is (Tennekes and Lumley, 1976): $L / \eta \sim O(\mathrm{Re}_L^{3/4}) \qquad \mathrm{Re}_L = UL/\nu$
  - In 3D, total cost (if time-step scales as grid size): $\sim O(\mathrm{Re}_L^3)$
- CPU and RAM limit size of problem: 1 Peta-FLOP ~ O(hours) for $\mathrm{Re}_L \sim O(100{,}000)$

Direct Numerical Simulations (DNS) for Turbulent Flows: **Numerics**

- DNS likely gives more information that many engineers need (closer to experimental data)
- But, it can be used for turbulence studies, e.g. coherent structures dynamics and other fundamental research
  - Allow to construct better RANS models or even correlation models
- Numerical Methods for DNS
  - All NS solvers we have seen useful
  - Small time-steps required for bounded errors:
    - Explicit methods are fine in simple geometries (stability satisfied due to small time-step needed for accuracy)
    - Implicit methods near boundaries or complex geometries (larger derivatives in viscous terms normal to the walls can lead to numerical instabilities $\Rightarrow$ treated implicitly)

Direct Numerical Simulations (DNS) for Turbulent Flows: **Numerics, Cont'd**

- Time-marching methods commonly used
  - Explicit 2nd to 4th order accurate (Runge-Kutta, Adams-Bashforth, Leapfrog): R-K's often more accurate for same cost
    - For same order of accuracy, R-K's allow larger time-steps for same accuracy
  - Crank-Nicolson often used for implicit schemes
- Must be conservative, including kinetic energy
- Spatial discretization schemes should have low dissipation
  - Upwind schemes often too diffusive: error larger than molecular diffusion!
  - High-order finite difference
  - Spectral methods (use Fourier series to estimate derivatives)
    - Mainly useful for simple geometries (FFT)
    - Use spectral elements instead (Patera, Karnadiakis, etc)
  - (Discontinous)-Galerkin Methods (FE schemes)

Direct Numerical Simulations (DNS) for Turbulent Flows: **Numerics, Cont'd**

- Challenges:
  - Storage for states at all intermediate time steps (⇒ R-K's of low storage)
  - Total discretization error and turbulence spectrum
    - Total error: both order of discretization and values of derivatives (spectrum)

      ⇒ Measure of total error: integrate over whole turbulent spectrum
  - Difficult to measure accuracy due to (unstable) nature of turbulent flow
    - Due to predictability limit of turbulence
    - Hence, statistical properties of two solutions are often compared
      - Simplest measure: turbulent spectrum
  - Generating initial conditions: as much art as science
    - Initial conditions remembered over significant "eddy-turnover" time
    - Data assimilation, smoothing schemes to obtain ICs
  - Generating boundary conditions
    - Periodic for simple problems, Radiating/Sponge conditions for realistic cases

Example: Spatial Decay of Turbulence Created by an Oscillating Boundary

Briggs et al (1996)

- Grid oscillations creates turbulence near the wall
- Decays in intensity away from the wall by “turbulent diffusion”: stirring+mixing
- Use spectral method, periodic, 3rd order R-K
- Results used to test turbulence “closure” models
    - Don’t work well because not derived for that “type” of turbulence

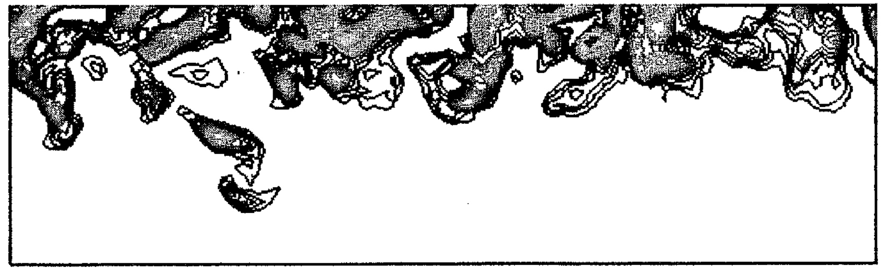
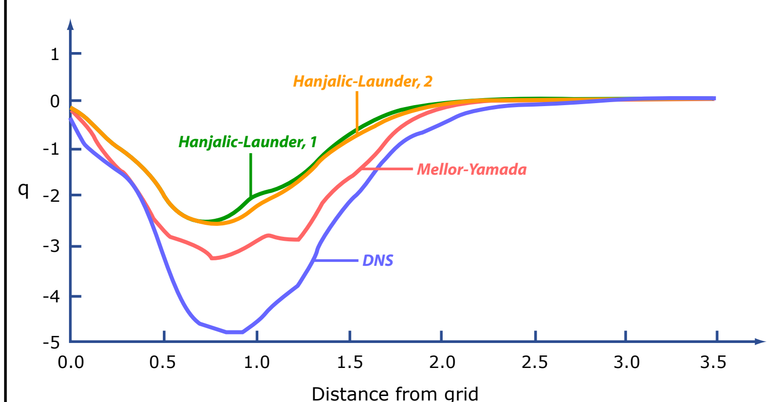


Fig. 9.1. Contours of the kinetic energy on a plane in the flow created by an oscillating grid in a quiescent fluid; the grid is located at the top of the figure. Energetic packets of fluid transfer energy away from the grid region. From Briggs et al. (1996)

 Source: Ferziger, J., and M. Peric. *Computational Methods for Fluid Dynamics*. 3rd ed. Springer, 2001.



The flux profile of q, turbulent kinetic energy, compared with the predictions of common turbulence models.

Note: DNS have at times found out when laboratory set-up was not proper

Reynolds-averaged Navier-Stokes (RANS)

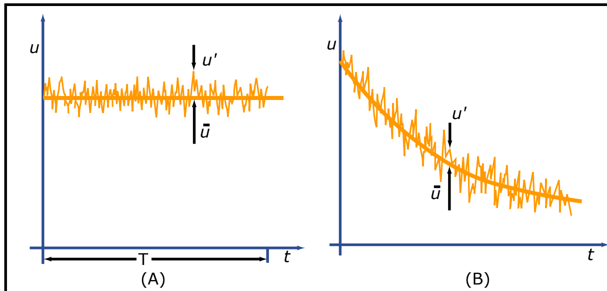
- Many science and engineering applications focus on averages
- RANS models: based on ideas of Osborne Reynolds
  - All “unsteadiness” regarded as part of turbulence and averaged out
  - By averaging, nonlinear terms in NS eqns. lead to new product terms that must be modeled
- Separation into mean and fluctuations $(\tau \ll T_0 \ll T)$
  - Moving time-average: $u = \bar{u} + u'$; $\boxed{\phi(x_i,t) = \bar{\phi}(x_i,t) + \phi'(x_i,t)}$

$$\text{where } \boxed{\bar{\phi}(x_i,t) = \frac{1}{T_0}\int_{t-T_0/2}^{t+T_0/2} \phi(x_i,t)dt \ ; \quad \text{i.e.} \quad \overline{\phi'}(x_i,t) = 0}$$

  - Ensemble average:

$$\boxed{<\phi(x_i,t)> = \frac{1}{N}\sum_{r=1}^{N} \phi^r(x_i,t)}$$

  - Reynolds-averaging: any of these



Graphs showing (A) the time average for a statistically steady flow and (B) the ensemble average for an unsteady flow.

Image by MIT OpenCourseWare.

Reynolds-averaged Navier-Stokes (RANS)

- Variance, r.m.s. and higher-moments:

$$\overline{\phi'^{2}}(x_i,t) = \frac{1}{T_0}\int_{t-T_0/2}^{t+T_0/2} \phi'^{2}(x_i,t)\,dt \qquad \overline{u_i\phi} = \overline{(\overline{u_i}+u')(\overline{\phi}+\phi')} = \overline{u_i}\;\overline{\phi} + \overline{u_i'\,\phi'}$$

$$\phi_{\mathrm{rms}} = \sqrt{\overline{\phi'^{2}}}$$

$$\overline{\phi'^{p}}(x_i,t) = \frac{1}{T_0}\int_{t-T_0/2}^{t+T_0/2} \phi'^{p}(x_i,t)\,dt \quad \text{with } p = 3,4, \quad etc$$

- Correlations:
  - In time: $R(t_1,t_2;x) = \overline{u'(x,t_1)\,u'(x,t_2)}$ for a stationary process: $R(\tau;x) = \overline{u'(x,t)\,u'(x,t+\tau)}$
  - In space:

    $R(x_1,x_2;t) = \overline{u'(x_1,t)\,u'(x_2,t)}$ for a homogeneous process: $R(\ell;t) = \overline{u'(x,t)\,u'(x+\ell,t)}$

- Turbulent kinetic energy: $k = \frac{1}{2}\overline{u'^{2} + v'^{2} + w'^{2}}$
  - Note: some arbitrariness in the decomposition $u = \overline{u} + u'$; $\phi = \overline{\phi} + \phi'$

    and in the definition that "fluctuations = turbulence"

Reynolds-averaged Navier-Stokes (RANS)

- Continuity and Momentum Equations, incompressible:

$$\frac{\partial \rho u_i}{\partial x_i} = 0$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial (\rho u_i u_j)}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

  – Applying either the ensemble or time-averages to these equations leads to the RANS eqns.
  – In both cases, averaging any linear term in conservation equation gives the identical term, but for the average quantity

- Average the equations, inserting the decomposition: $u_i = \overline{u_i} + u'$
  – the time and space derivatives commute with the averaging operator

$$\overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial \overline{u_i}}{\partial x_i}$$

$$\overline{\frac{\partial u_i}{\partial t}} = \frac{\partial \overline{u_i}}{\partial t}$$

Reynolds-averaged Navier-Stokes (RANS)

- Averaged continuity and momentum equations:

$$\frac{\partial \rho \overline{u_i}}{\partial x_i} = 0$$

$$\frac{\partial \rho \overline{u_i}}{\partial t} + \frac{\partial (\rho \overline{u_i}\, \overline{u_j} + \rho \overline{u_i' u_j'})}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j}$$

where $\overline{\tau_{ij}} = \mu \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right)$

(incompressible, no body forces)

- For a scalar conservation equation

$$\frac{\partial \rho \overline{\phi}}{\partial t} + \frac{\partial (\rho \overline{\phi}\, \overline{u_j} + \rho \overline{\phi' u_j'})}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\Gamma \frac{\partial \overline{\phi}}{\partial x_j} \right)$$

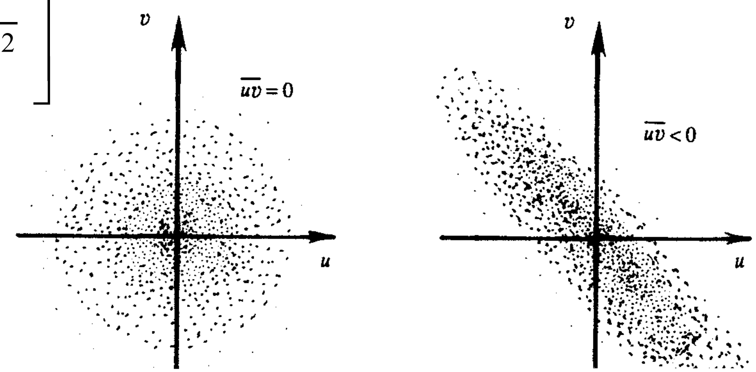
  – e.g. for $\overline{\phi} = c_p \overline{T} \Rightarrow$ mean internal energy
  – Terms that are products of fluctuations remain:
    - Reynolds stresses: $-\rho \overline{u_i' u_j'}$
    - Turbulent scalar flux: $-\rho \overline{u_i' \phi'}$
  – Equations are thus not closed (more unknown variables than equations)
    - Closure requires specifying $\rho \overline{u_i' u_j'}$ and $\rho \overline{u_i' \phi'}$ in terms of the mean quantities and/or their derivatives (any Taylor series decomposition of mean quantities)

Reynolds Stresses

- Total stress acting on mean flow: $\tau_{ij} = \overline{\tau}_{ij} - \rho\,\overline{u_i' u_j'} = \overline{\tau}_{ij} + \tau_{ij}^{\mathrm{Re}}$

$$\overline{\tau}_{ij} = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) \quad \text{and} \quad \tau^{\mathrm{Re}} = -\rho\begin{bmatrix} \overline{u'^2} & \overline{u'\,v'} & \overline{u'\,w'} \\ & \overline{v'^2} & \overline{v'\,w'} \\ & & \overline{w'^2} \end{bmatrix}$$

- If turbulent fluctuations are isotropic:
  - Off diagonal elements of τ^{Re} cancel
  - Diagonal elements equal: $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$
- Average of product of fluctuations not zero
  - Consider mean shear flow: $\dfrac{\partial \overline{u}}{\partial y} > 0$
  - If parcel is going up ($v' > 0$), it slows down neighbors, hence $u' < 0$ (opposite for $v' < 0$)
  - Hence: $\overline{u'\,v'} < 0 \quad \text{for} \quad \dfrac{\partial \overline{u}}{\partial y} > 0$ (acts as turb. "diffus.")
- Other meanings of Reynolds stress:
  - Rate of mean momentum transfer by turb. fluctations
  - Average flux of j-momentum along i-direction



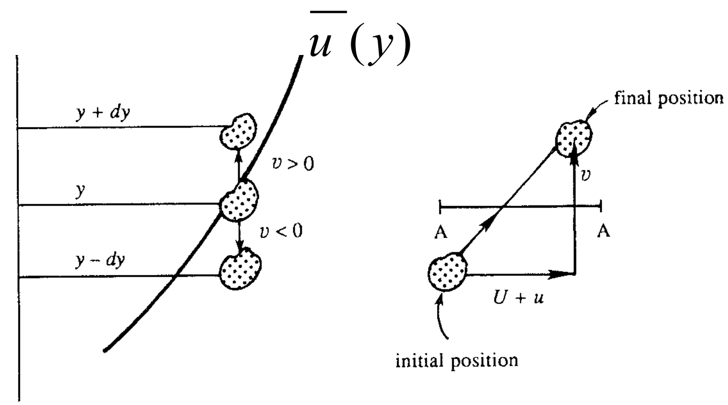


Figure 13.7 Movement of a particle in a turbulent shear flow.

Simplest Turbulence Closure Model

- Eddy Viscosity and Eddy Diffusivity Models
  - Effect of turbulence is to increase stirring/mixing on the mean-fields, hence increase effective viscosity or effective diffusivity
  - Hence, "Eddy-viscosity" Model and "Eddy-diffusivity" Model

$$\boxed{\tau_{ij}^{\mathrm{Re}} = -\rho \overline{u_i' u_j'} \cong \mu_t \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) - \frac{2}{3}\rho k \,\delta_{ij}} \qquad \boxed{-\rho \overline{u_i' \phi_j'} \cong \Gamma_t \frac{\partial \overline{\phi}}{\partial x_j}}$$

- Last term in Reynolds stress is required to ensure correct results for the sum of normal stresses:

$$\tau_{ii}^{\mathrm{Re}} = -\rho \overline{u_i' u_i'} = \mu_t 2 \frac{\partial \overline{u_i}}{\partial x_i} - \frac{2}{3}\rho k\, 3 = 0 - 2\rho \frac{1}{2}\overline{u'^2 + v'^2 + w'^2} = -\rho\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\equiv -2\rho k$$

- The use of scalar μ_t, $\Gamma_t \Rightarrow$ assumption of isotropic turbulence, which is often inaccurate
- Since turbulent transports (momentum or scalars, e.g. internal energy) are due to "average stirring" or "eddy mixing", we expect similar values for μ_t and Γ_t. This is the so-called Reynolds analogy: $\sigma_t = \dfrac{\mu_t}{\Gamma_t} \cong 1$

Turbulence Closures: **Mixing-Length Models**

– Mixing length models attempt to vary unknown μ_t as a function of position

– Main parameters available: turbulent kinetic energy k $[\mathrm{m}^2/\mathrm{s}^2]$ or velocity u^*, large eddy length scale L

=> Dimensional analysis: $\boxed{\begin{aligned} k &= \left(u^*\right)^2/2 \\ \mu_t &= C_\mu \, \rho \, u^* \, L \end{aligned}}$ $\quad C_\mu$ dimensionless constant

– Observations and assumptions:

- Most k is contained in largest eddies of mixing-length L
- Largest eddies interact most with mean flow

$\Rightarrow u^* = f(\overline{u}_i, \frac{\partial \overline{u}_i}{\partial x_j}, \frac{\partial^2 \overline{u}_i}{\partial x_j^2}, \cdots)$

$$\Rightarrow \text{in} \sim \text{2D, mostly} \quad \tau_{xy}^{\mathrm{Re}} = -\rho\, \overline{u'v'} \quad \Rightarrow \quad \boxed{u^* \cong f\left(\frac{\partial \overline{u}}{\partial y}\right) = c\, L \left|\frac{\partial \overline{u}}{\partial y}\right|}$$

– Hence, $\boxed{\mu_t \cong \rho\, L^2 \left|\frac{\partial \overline{u}}{\partial y}\right|}$. This is Prandtl's "mixing length model.

- For a plate flow, Prandtl assumed: $L \approx \kappa y \quad \Rightarrow \quad u^* \cong \kappa y \frac{\partial \overline{u}}{\partial y} \quad \Rightarrow \quad \frac{\overline{u}}{u^*} = \frac{1}{\kappa} \ln y + \text{const.}$

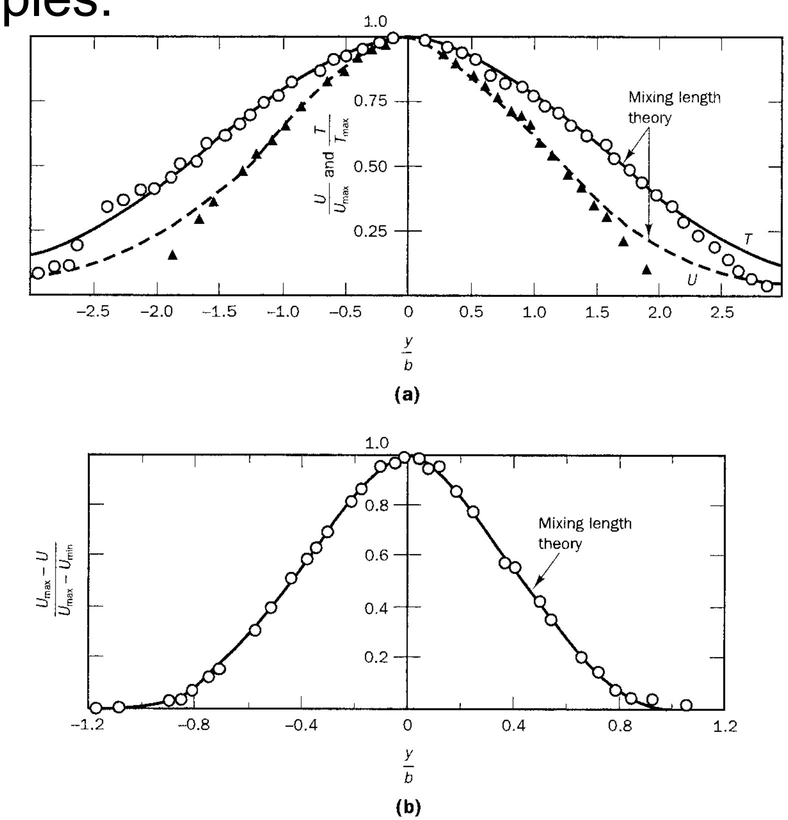
– Mixing-length turbulent Reynolds stress: $\tau_{xy}^{\mathrm{Re}} = -\rho\, \overline{u'v'} \cong \rho\, L^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \frac{\partial \overline{u}}{\partial y}$

– Mixing length model can also be used for scalars:

$$-\rho\, \overline{u'\phi'} \cong \rho\, \sigma_t\, L^2 \left|\frac{\partial \overline{u}}{\partial y}\right| \frac{\partial \overline{\phi}}{\partial y}$$

Mixing Length Models: What is L (ℓ_m)?

- In simple 2D flows, mixing-length models agree well with data
- In these flows, mixing length L proportional to physical size (D, etc)
- Here are some examples:



 Source: Schlichting, H. *Boundary Layer Theory.* 7th ed. The McGraw-Hill Companies, 1979.

- But, in general turbulence, more than one space and time scale!

Turbulence Closures: k - ε Models

- Mixing-length = "zero-equation" Model
- One might find a PDE to compute $\tau_{ij}^{\mathrm{Re}} = -\rho\overline{u_i' u_j'}$ and $-\rho\overline{u_i'\phi'}$ as a function of k and other turbulent quantities
  - Turbulence model requires at least a length scale and a velocity scale, hence two PDEs?
- Kinetic energy equations (incompressible flows)
  - Define $\text{Total KE} = \frac{1}{2}\left(\overline{u_i}\right)^2 + \frac{1}{2}\overline{\left(u_i'\right)^2}$, $\mathrm{K} = \frac{1}{2}\left(\overline{u_i}\right)^2$ and $\overline{\tau_{ij}} = \mu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) = 2\mu\,\overline{e_{ij}}$
  - Mean KE: Take mean mom. eqn., multiply by $\overline{u_i}$ to obtain:

$$\frac{\partial \rho K}{\partial t} + \frac{\partial(\rho K\,\overline{u_j})}{\partial x_j} = \frac{\partial\left(-\overline{p}\;\overline{u_j}\right)}{\partial x_j} + \frac{\partial\left(2\mu\,\overline{e_{ij}}\;\overline{u_i} - \rho\,\overline{u_i' u_j'}\;\overline{u_i}\right)}{\partial x_j} - 2\mu\,\overline{e_{ij}}\;\overline{e_{ij}} + \rho\,\overline{u_i' u_j'}\,\frac{\partial \overline{u_i}}{\partial x_j}$$

Rate of change of K + Advection of K = Transport of K by pressure + Transport of K by mean viscous stresses + Transport of K by Reynolds stresses - Rate of viscous dissipation of K - Rate of decay of K due turbulence production

Turbulence Closures: k - ε Models, Cont'd

- Turbulent kinetic energy equation
  - Obtain momentum eqn. for the turbulent velocity u_i'
  - Define the fluctuating strain rate: $e_{ij}' \equiv \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$
  - Multiply by u_i' (sum) and average to obtain the eqn. for $k = \frac{1}{2}\overline{u_i' u_i'}$

$$\frac{\partial \rho k}{\partial t} + \frac{\partial(\rho k\, \overline{u}_j)}{\partial x_j} = \frac{\partial\left(-\overline{p' u_j'}\right)}{\partial x_j} + \frac{\partial\left(2\mu\overline{e_{ij}' u_i'} - \rho\frac{1}{2}\overline{u_i' u_j' . u_i'}\right)}{\partial x_j} - 2\mu\overline{e_{ij}' . e_{ij}'} - \rho\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}$$

Rate of change of k + Advection of k = Transport of k by turbulent pressure + Transport of k by "turbulent" viscous stresses + Transport of k by Reynolds stresses - Rate of viscous dissipation of k + Rate of production of k

  - This equation is similar than that of K, but with prime quantities
  - Last term is now opposite in sign: is the rate of shear production of k : $\boxed{P_k = -\rho\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j}}$
  - Next to last term = rate of viscous dissipation of k : $\boxed{\varepsilon = 2\nu\overline{e_{ij}' . e_{ij}'}}$ $(= 2\mu\overline{e_{ij}'\, e_{ij}'}\ p.u.m)$
  - These two terms often of the same order (this is how Kolmogorov microscale is defined)
    - e.g. consider steady state turbulence over any volume
  - If Boussinesq fluid, the two KE eqns. contain buoyant loss/production terms

Turbulence Closures: k - ε Models, Cont'd

- Parameterizations for the standard k equation:
  - For incompressible flows, the viscous transport term is: $\boxed{\dfrac{\partial\left(2\mu\overline{e'_{ij}\,u'_i}\right)}{\partial x_j} = \dfrac{\partial}{\partial x_j}\left(\mu\dfrac{\partial k}{\partial x_j}\right)}$
  - The other two turbulent energy transport terms are thus modeled using:

$$\boxed{-\overline{p'u'_j} + (-\rho\frac{1}{2}\overline{u'_i u'_j . u'_i}) \approx \frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}} \qquad (\sigma_t = \frac{\mu_t}{\Gamma_t} \cong 1)$$

    - This is analogous to an "eddy-diffusion of a scalar" model, recall: $-\rho\overline{u'_i\phi'_j} \cong \Gamma_t\dfrac{\partial\overline{\phi}}{\partial x_j}$
    - In some models, eddy-diffusions are tensors
  - The production term: using again an eddy diffusion model for the Rey. Stresses

$$\boxed{P_k = -\rho\overline{u'_i u'_j}\frac{\partial\overline{u}_i}{\partial x_j} \approx \left(\mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right) - \frac{2}{3}\rho k\,\delta_{ij}\right)\frac{\partial\overline{u}_i}{\partial x_j} = \mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right)\frac{\partial\overline{u}_i}{\partial x_j}}$$

  - All together, we have all "unknown" terms for the k equation parameterized, as long as $\varepsilon = 2\nu\overline{e'_{ij}.e'_{ij}}$ the rate of viscous dissipation of k is known:

$$\boxed{\frac{\partial\rho k}{\partial t} + \frac{\partial(\rho k\,\overline{u}_j)}{\partial x_j} = \frac{\partial}{\partial x_k}\left(\frac{\mu_t}{\sigma_t}\frac{\partial k}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}\right) - \rho\,\varepsilon + \mu_t\left(\frac{\partial\overline{u}_i}{\partial x_j}+\frac{\partial\overline{u}_j}{\partial x_i}\right)\frac{\partial\overline{u}_i}{\partial x_j}}$$

Turbulence Closures: k - ε Models, Cont'd

- The standard k - ε model equations (Launder and Spalding, 1974)
  - They are several choices for $\varepsilon = 2\nu\,\overline{e'_{ij}\,.\,e'_{ij}}$ $\left([\varepsilon] = m^2/s^3\right)$ the standard popular one is based on the "equilibrium turbulent flows" hypothesis:
    - In "equilibrium turbulent flows", ε the rate of viscous dissipation of k is in balance with P_k the rate of production of k (i.e. the energy cascades):

$$P_k = -\rho\,\overline{u'_i u'_j}\,\frac{\partial \bar{u}_i}{\partial x_j} \approx \mu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\frac{\partial \bar{u}_i}{\partial x_j} = O\left(\mu_t \left(u^*\right)^2 / L^2\right) \approx -2\mu\,\overline{e'_{ij}\,.\,e'_{ij}} = \rho\,\varepsilon$$

    - Recall the scalings: $k = \left(u^*\right)^2/2 \qquad \mu_t = C_\mu\,\rho\,u^*\,L$
    - This gives the length scale and the turbulent viscosity scalings:

$$\boxed{L \approx \frac{k^{3/2}}{\varepsilon} \qquad \mu_t = C_\mu\,\rho\,\frac{k^2}{\varepsilon}}$$

  - As a result, one can obtain an equation for ε (with a lot of assumptions):

$$\boxed{\frac{\partial \rho\varepsilon}{\partial t} + \frac{\partial(\rho\,\varepsilon\,\bar{u}_j)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + C_{\varepsilon 1} P_k \frac{\varepsilon}{k} - C_{\varepsilon 2}\,\rho\,\frac{\varepsilon^2}{k^2}}$$

where $\mu_t = C_\mu\,\rho\,\dfrac{k^2}{\varepsilon}$ and $\sigma_\varepsilon, C_{\varepsilon 1}, C_{\varepsilon 2}$ are constants. The production and destruction terms of ε are assumed proportional to those of k (the ratios ε/k and ε^2/k are for dimensions)

Turbulence Closures: k - ε Models, Cont'd

- The standard k - ε model (RANS) equations are thus:

$$\boxed{\begin{aligned} &\frac{\partial \rho k}{\partial t}+\frac{\partial(\rho k\, \overline{u}_j)}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right)+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial k}{\partial x_j}\right)+2\mu_t\, \overline{e}_{ij}\,.\,\overline{e}_{ij}-\rho\,\varepsilon \\ &\frac{\partial \rho \varepsilon}{\partial t}+\frac{\partial(\rho\, \varepsilon\, \overline{u}_j)}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)+C_{\varepsilon 1}P_k\frac{\varepsilon}{k}-C_{\varepsilon 2}\,\rho\,\frac{\varepsilon^2}{k}\end{aligned}}$$

with $\mu_t = C_\mu\, \rho\, \dfrac{k^2}{\varepsilon}$

Rate of change of k or ε	+	Advection of k or ε	=	Transport of k or ε by "eddy-diffusion" stresses	+	Rate of production of k or ε	-	Rate of viscous dissipation of k

- The Reynolds stresses are obtained from: $\tau_{ij}^{\mathrm{Re}} = -\rho\, \overline{u_i' u_j'} \cong \mu_t\left(\dfrac{\partial \overline{u}_i}{\partial x_j}+\dfrac{\partial \overline{u}_j}{\partial x_i}\right)-\dfrac{2}{3}\rho k\,\delta_{ij}$
- The most commonly used values for the constants are:

$$C_\mu = 0.09, \quad C_{\varepsilon 1} = 1.44, \quad C_{\varepsilon 2} = 1.92, \quad \sigma_k = 1.0, \quad \sigma_\varepsilon = 1.3$$

- Two new PDEs are relatively simple to implement (same form as NS)
  - But, time-scales for k - ε are much shorter than for the mean flow
- Other $k - \varepsilon$ models: Spalart-Allmaras $v - L$, Wilcox or Menter k - ω, anisotropic $k - \varepsilon$'s, etc

Turbulence Closures: k - ε Models, Cont'd

- Numerics for standard k - ε models
  - Since time-scales for k - ε are much shorter than for the mean flow, their equations are treated separately
    - Mean-flow NS outer iteration first performed using old k - ε
    - Strongly non-linear equations for k - ε are then integrated (outer-iteration) with smaller time-step and under-relaxation
  - Smaller space scales requires finer-grids near walls for $k - \varepsilon$ eqns
    - Otherwise, too low resolution leads to wiggles and negative k - ε
    - If grids are the same, need to use schemes that reduce oscillations
- Boundary conditions for k - ε models
  - Similar than for other scalar eqns., excepted at solid walls
    - Inlet: k, ε given (from data or from literature)
    - Outlet or symmetry axis: normal derivatives set to zero (or other OBCs)
    - Free stream: k, ε given or zero-derivatives
    - Solid walls: depends on Re

Turbulence Closures: k - ε Models, Cont'd

- Solid-walls boundary conditions for k - ε models, Cont'd
  - No-slip BC would be standard:
    - Hence, appropriate for to set $k = 0$ at the wall
    - But, dissipation not zero at the wall $\rightarrow$ use : $\varepsilon = \nu \left.\frac{\partial^2 k}{\partial n^2}\right|_{\text{wall}}$ or $\varepsilon = 2\nu \left.\left(\frac{\partial k^{1/2}}{\partial n}\right)^2\right|_{\text{wall}}$
  - At high-Reynolds numbers:
    - One can avoid the need to solve $k - \varepsilon$ right at the wall by using an analytical shape "wall function":
    - At high-Re, in logarithmic layer :

$$L \approx \kappa y \;\Rightarrow\; u^* \cong \kappa y \frac{\partial \bar{u}}{\partial y} \;\Rightarrow\; u^+ = \frac{\bar{u}}{u^*} = \frac{1}{\kappa} \ln y + \text{const.}$$

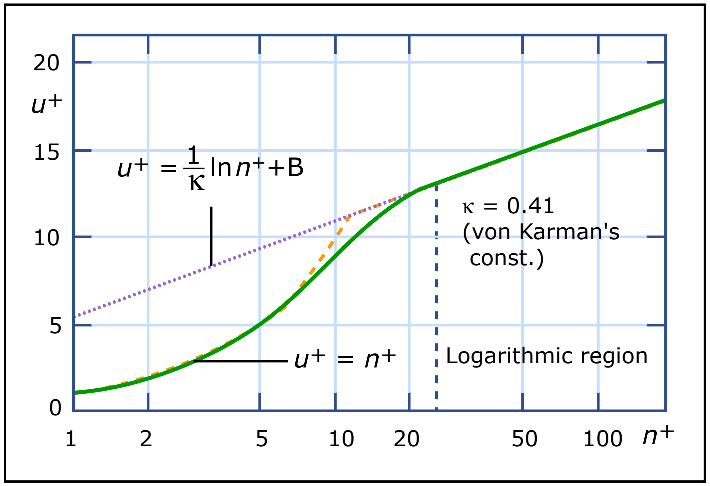


Image by MIT OpenCourseWare.

- If dissipation balances turbulence production, recall: $L \approx \frac{k^{3/2}}{\varepsilon}; \quad \mu_t = C_\mu \, \rho \frac{k^2}{\varepsilon}$
- Combining, one obtains: $\varepsilon \approx \frac{k^{3/2}}{L} = \frac{(u^*)^3}{\kappa y}$ and one can match: $\tau_{\text{wall}} = \rho (u^*)^2$ without resolving the viscous sub-layer
- For more details, including low-Re cases, see references

Turbulence Closures: k - ε Models, Cont'd

- Example: Flow around an engine Valve (Lilek et al, 1991)
  - $k - \varepsilon$ model, 2D axi-symmetric
  - Boundary-fitted, structured grid
  - 2nd order CDS, 3-grids refinement
  - BCs: wall functions at the walls
  - Physics: separation at valve throat
  - Comparisons with data not bad
  - Such CFD study can reduce number of experiments/tests required

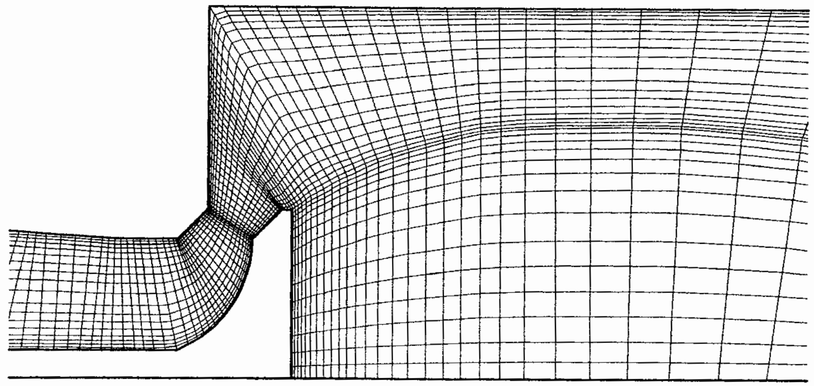


Fig. 9.12. Section of a grid (level two) used to calculate flow around a valve (from Lilek et al., 1991)

Please also see figures 9.13 and 9.14 from Figs 9.13 and 9.14 from Ferziger, J., and M. Peric. *Computational Methods for Fluid Dynamics*. 3rd ed. Springer, 2001.

Reynolds-Stress Equation Models (**RSMs**)

- Underlying assumption of "Eddy viscosity/diffusivity" models and of k - ε models is that of isotropic turbulence, which fails in many flows
  - Some have used anisotropic eddy-terms, but not common
- Instead, one can directly solve transport equations for the Reynolds stresses themselves: $-\rho \overline{u_i' u_j'}$ and $-\rho \overline{u_i' \phi'}$
  - These are among the most complex RANS used today. Their equations can be derived from NS
  - For momentum, the six transport equations, one for each Reynolds stresses, contain: diffusion, pressure-strain and dissipation/production terms which are unknown
    - In these "2nd order models", assumptions are made on these terms and resulting PDEs are solved, as well as an equation for ε
    - Extra 7 PDEs to be solved increase cost and mostly used for academic research (assumptions on unknown terms still being compared to data)

Reynolds-Stress Equation Models (RSMs), Cont'd

- Equations for $\tau_{ij}^{\mathrm{Re}} = -\rho\,\overline{u_i' u_j'}$

$$\boxed{\frac{\partial \tau_{ij}^{\mathrm{Re}}}{\partial t} + \frac{\partial (\tau_{ij}^{\mathrm{Re}}\,\overline{u}_j)}{\partial x_j} = -\overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - \left(\tau_{im}^{\mathrm{Re}} \frac{\partial \overline{u}_j}{\partial x_m} + \tau_{jm}^{\mathrm{Re}} \frac{\partial \overline{u}_i}{\partial x_m}\right) + \rho\,\varepsilon_{ij} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \tau_{ij}^{\mathrm{Re}}}{\partial x_j} + C_{ijm}\right)}$$

Rate of change of Reynolds Stress	+	Advection of Reynolds Stress	=	Pressure-strain redistribution of Reynolds stresses	+	Rate of production of Reynolds stress	+	Rate of tensor dissipation of Reynolds stress	-	Rates of viscous and turbulent diffusions (dissipations) of Reynolds stress

where

- The dissipation (as ε but now a tensor) is : $\varepsilon_{ij} = 2\nu\,\overline{e_{ik}' . e_{jk}}$
- The 3rd order turbulence diffusions are: $C_{ijm} = \rho\,\overline{u_i' u_j' . u_m'} + \overline{p' u_i'}\,\delta_{jk} + \overline{p' u_j'}\,\delta_{ik}$

– Simplest and most common 3rd order closures:

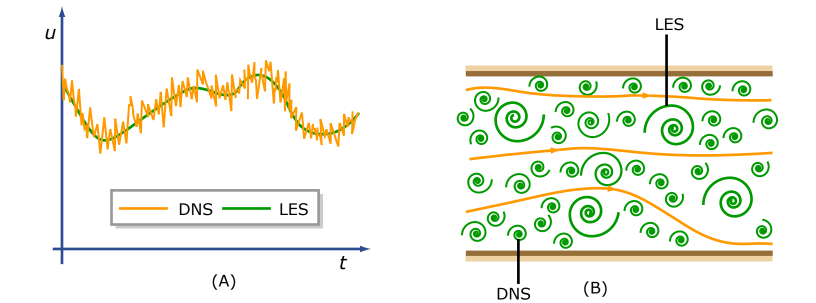
- Isotropic dissipation: $\varepsilon_{ij} = \frac{2}{3}\varepsilon\,\delta_{ij} = \frac{4}{3}\nu\,\overline{e_{ij}' . e_{ij}'}\,\delta_{ij}$ $\rightarrow$ the ε PDE must be solved
- Several models for pressure-strain used (attempt to make it more isotropic), see Launder et al)
- The 3rd order turbulence diffusions: usually modeled using an eddy-flux model, but nonlinear models also used
- Active research

Large Eddy Simulation (LES)

- Turbulent Flows contain large range of time/space scales

- However, larger-scale motions often much more energetic than small scale ones

- Smaller scales often provide less transport



(A) The time dependence of a component of velocity at one point; (B) Representation of turbulent motion.

Image by MIT OpenCourseWare.

→ simulation that treats larger eddies more accurately than smaller ones makes sense: = LES:

- Instead of time-averaging, LES uses spatial filtering to separate large and small eddies
- Models smaller eddies as a “universal behavior”
- 3D, time-dependent and expensive, but much less than DNS
- Preferred method at very high Re or very complex geometry

Large Eddy Simulation (LES), Cont'd

- Spatial Filtering of quantities
  - The larger-scale (the ones to be resolved) are essentially a local spatial average of the full field
  - For example, the filtered velocity is: $\boxed{\overline{u_i}(\mathbf{x},t) = \int_V G(\mathbf{x},\mathbf{x}';\Delta)\; u_i(\mathbf{x}',t)\; dV'}$

    where $G(\mathbf{x},\mathbf{x}';\Delta)$ is the filter kernel, a localization function of support/cutoff width Δ

    - Example of Filters: Gaussian, box, top-hat and spectral-cutoff (Fourier) filters
- When NS, incompressible flows, constant density is averaged, one obtains

$$\boxed{\begin{aligned} &\frac{\partial \rho \overline{u_i}}{\partial x_i} = 0 \\ &\frac{\partial \rho \overline{u_i}}{\partial t} + \frac{\partial (\rho \overline{u_i u_j})}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)\right] \end{aligned}}$$

  - Continuity is linear, thus filtering does not change its shape
  - Simplifications occur if filter does not depend on positions: $G(\mathbf{x},\mathbf{x}';\Delta) = G(\mathbf{x}-\mathbf{x}';\Delta)$

Large Eddy Simulation (LES), Cont'd

- LES sub-grid-scale stresses
  - It is important to note that $\rho\overline{u_i u_j} \neq \rho\overline{u_i}\ \overline{u_j}$
  - This quantity is hard to compute
  - One introduces the sub-grid-scale Reynolds Stresses, which is the difference between the two:

$$\tau_{ij}^{SG} = -\rho\left(\overline{u_i u_j} - \overline{u_i}\ \overline{u_j}\right)$$

    - It represents the large scale momentum flux caused by the action of the small or unresolved scales (SG is somewhat is a misnomer)
  - Example of models:
    - Smagorinsky: it is an eddy viscosity model

$$\tau_{ij}^{SG} - \frac{1}{3}\tau_{kk}^{SG}\,\delta_{ij} \cong \mu_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) = 2\mu_t\,\overline{e_{ij}}$$

    - Higher-order SGS models
    - More advanced models (mixed models, dynamic models, deconvolution models, etc)

Examples (see Durbin and Medic, 2009)

Please see figures 6.1, 6.2, 6.22, 6.23, 6.26, and 6.27 in Durbin, p. and G. *Medic. Fluid Dynamics with a Computational Perspective*. Vol. 10. Cambridge University Press, 2007. See the figures now via Google Books Preview

2.29 Numerical Fluid Mechanics
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