2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 5

REVIEW Lecture 4

- Roots of nonlinear equations: "Open" Methods
  - Fixed-point Iteration (General method or Picard Iteration), with examples
    - Iteration rule: $x_{n+1} = g(x_n)$ or $x_{n+1} = x_n - h(x_n) f(x_n)$
    - Error estimates, Convergence Criteria: $|g'(x)| \leq k < 1, \quad x \in I$
  - Order of Convergence p: $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = C$ (for Fixed-Point, usually linear, $p \sim 1$)
  - Newton-Raphson
    - Examples and Issues $x_{n+1} = x_n - \frac{1}{f'(x_n)} f(x_n)$
    - Quadratic Convergence ($p=2$)
  - Secant Method $f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$
    - Examples
    - Convergence ($p=1.62$) and efficiency
  - Extension of Newton-Raphson to systems of nonlinear eqns. (slower conver.)

Secant Method: Order of convergence

Absolute Error $\epsilon_n = x_n - x^e$

$$\epsilon_{n+1} = x_{n+1} - x^e = \frac{f(x^e+\epsilon_n)(x^e+\epsilon_{n-1}) - f(x^e+\epsilon_{n-1})(x^e+\epsilon_n)}{f(x^e+\epsilon_n) - f(x^e+\epsilon_{n-1})} - x^e$$

Using Taylor Series, up to 2nd order

Absolute Error $\epsilon_{n+1} \simeq \frac{1}{2}\epsilon_{n-1}\epsilon_n \frac{f''(x^e)}{f'(x^e)}$

Relative Error $\frac{\epsilon_{n+1}}{|x^e|} \simeq \frac{\epsilon_{n-1}}{|x_e|}\frac{\epsilon_n}{|x_e|}\frac{f''(x^e)}{2f'(x^e)}x^e$

Convergence Order/Exponent

By definition:

$$\epsilon_n = A(x^e)\epsilon_{n-1}^m \Rightarrow \epsilon_{n-1} = \left(\frac{1}{A}\epsilon_n\right)^{1/m} = B(x^e)\epsilon_n^{1/m}$$

Then:

$$\epsilon_{n+1} = C(x^e)\epsilon_n\epsilon_{n-1} = D(x^e)\epsilon_n\epsilon_n^{1/m} = D(x^e)\epsilon_n^{1+1/m}$$

$$\Rightarrow \quad 1 + \frac{1}{m} = m \Leftrightarrow m = \frac{1}{2}(1+\sqrt{5}) \simeq 1.62$$

Error improvement for each function call

Secant Method $\epsilon_{n+1}^* \simeq \epsilon_n^{1.62}$

Newton-Raphson $\epsilon_{n+1}^* = \epsilon_n^{2}$

Fluid flow modeling: the Navier-Stokes equations and their approximations
Today's Lecture

- References :
  - Chapter 1 of "J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, New York, third edition, 2002."
  - Chapter 4 of "I. M. Cohen and P. K. Kundu. *Fluid Mechanics*. Academic Press, Fourth Edition, 2008"
  - Chapter 4 in "F. M. White, *Fluid Mechanics*. McGraw-Hill Companies Inc., Sixth Edition"
- For today's lecture, any of the chapters above suffice
  - Note each provide a somewhat different prospective

Conservation Laws

- Conservation laws can be derived either using a
  - Control Mass approach (CM)
    - Considers a fixed mass (useful for solids) and its extensive properties (mass, momentum and energy)
  - Control Volume approach (CV)
    - CV is a certain spatial region of the flow, possibly moving with fluid parcels/system
    - Its surfaces are control surfaces (CS)
  - Each approach leads to a class of numerical methods
- For an extensive property, the conservation law "relates the rate of change of the property in the CM to externally determined effects on this property"
- To derive local differential equations, assumption of continuum is made
  - Knudsen number (mean free path over length-scale, $\lambda/L < 0.01$)
    - => Sufficiently "well behaved" continuous functions
    - Non-continuum flows: space shuttle in reentry, low-pressure processing
  - Note CFD is also used for Newton's law applied to each constituent molecules (simple, but computational cost often growths as N^2 or more)

Macroscopic Properties

- Continuum hypothesis allows to define macroscopic fluid properties
- Density (ρ): mass of material per unit volume [kg/m^3]
  - If the density is independent of pressure, the fluid is said incompressible
  - A measure of the flow compressibility is the Mach number:
    - $\mathrm{Ma} = \frac{v}{a}$ where $a^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$ (If Ma<0.3, variations of ρ can be assumed to be negligible)
  - Typical values:
    - Water: a = 1,400 m/s; Air: a = 300 m/s
- Viscosity (μ): measure of the resistance of the fluid to deformation under stress [$Pa.s$]
  - A solid sustains external shear stresses: intermolecular forces balance the stress
  - A fluid does not: the deformation increases with time
    - If the deformation increase is linear with the stress, the fluid is said Newtonian
  - Typical values of dynamic viscosity:
    - Air: $\mu = 1.8 \times 10^{-5}\ kg/ms$; Water: $\mu = 10^{-3}\ kg/ms$; SAE Oil (car): $\mu = 240\ 10^{-3}\ kg/ms$
- The ratio of the inertial (nonlinear) forces to the viscous force is measured by the Reynolds number:

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

Observed Influence of the Reynolds Number

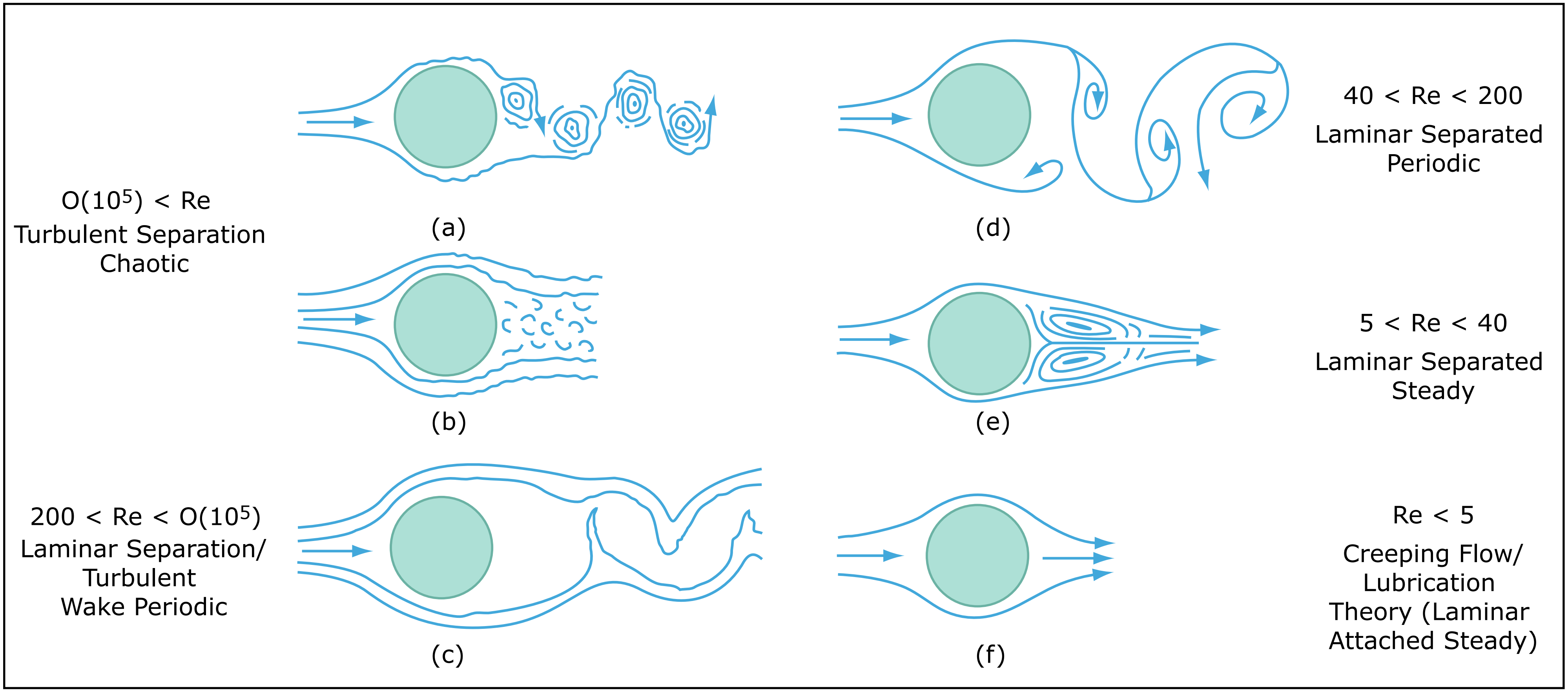


Image by MIT OpenCourseWare.

Conservation of Mass and Momentum for a CM

- Conservation of Mass
  - Mass is neither created nor destroyed in the flows of our engineering interests:

$$\frac{dM_{CM}}{dt} = 0$$

- Conservation of Momentum (Newton's second law)
  - Rate of change of momentum can be modified by the action of forces

$$\frac{d(M\,\mathbf{v})_{CM}}{dt} = \sum \vec{F}$$

Conservation Laws (Principles/Relations) for a CV

Mass conservation

(summation form):

$$\frac{dM_{CV}}{dt} = \sum_{in} \dot{m}_{in} - \sum_{out} \dot{m}_{out}$$

(integral form):

$$\frac{d}{dt}\int_{CV} \rho dV + \int_{CS} \rho\left(\vec{v}_r.\vec{n}\right) dA = 0$$

(differential form):

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0$$

Momentum conservation

(integral form):

$$\frac{d}{dt}\int_{CV} \rho \vec{v} dV = \underbrace{\int_{CS} -P\vec{n} dA + \int_{CS} \vec{\tau} dA + \int_{CV} \rho \vec{g} dV}_{=\sum \vec{F}} - \int_{CS} \rho \vec{v}(\vec{v}_r.\vec{n}) dA$$

$$\Leftrightarrow \sum \vec{F} = \int_{CS} -P\vec{n} dA + \int_{CS} \vec{\tau} dA + \int_{CV} \rho \vec{g} dV = \frac{d}{dt}\int_{CV} \rho \vec{v} dV + \int_{CS} \rho \vec{v}(\vec{v}_r.\vec{n}) dA$$

Conservation Laws (Principles/Relations) for a CV

Energy conservation (First Law)
(integral form):

$$\frac{d}{dt}\int_{CV}\rho\left(u+\frac{v^2}{2}+gz\right)dV=\dot{Q}-\dot{W}_{shaft}-\int_{CS}\rho\left(h+\frac{v^2}{2}+gz\right)\left(\vec{v}_r.\vec{n}\right)dA$$

(summation form):

$$\frac{dE_{CV}}{dt}=\dot{Q}-\dot{W}_{shaft}+\sum_{in}\dot{m}_{in}\left(h+\frac{v^2}{2}+gz\right)_{in}-\sum_{out}\dot{m}_{out}\left(h+\frac{v^2}{2}+gz\right)_{out}$$

Second Law of Thermodynamics
(integral form):

$$\frac{d}{dt}\int_{CV}\rho s dV=\sum_i\left(\frac{\dot{Q}}{T}\right)_i+\dot{S}_{gen}-\int_{CS}\rho s\left(\vec{v}_r.\vec{n}\right)dA$$

(summation form):

$$\frac{dS_{CV}}{dt}=\sum_i\left(\frac{\dot{Q}}{T}\right)_i+\dot{S}_{gen}+\sum_{in}\left(\dot{m}s\right)_{in}-\sum_{out}\left(\dot{m}s\right)_{out}$$

Angular momentum conservation
(integral form):

$$\sum\vec{T}=\frac{d}{dt}\int_{CV}\rho(\vec{r}\times\vec{v})dV+\int_{CS}\rho(\vec{r}\times\vec{v})(\vec{v}_r.\vec{n})dA$$

Bernoulli Equation (unsteady)

$$\int_1^2\frac{\partial v}{\partial t}ds+\frac{P_2-P_1}{\rho}+\frac{{v_2}^2-{v_1}^2}{2}+g\left(z_2-z_1\right)=0$$

Vector Operators

Cartesian Coordinates (x, y, z)

$$\nabla \psi = \frac{\partial \psi}{\partial x}\hat{i} + \frac{\partial \psi}{\partial y}\hat{j} + \frac{\partial \psi}{\partial z}\hat{k}$$

$$\nabla \bullet \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Cylindrical Coordinates (r, φ, z)

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \phi}\hat{\phi} + \frac{\partial \psi}{\partial z}\hat{z}$$

$$\nabla \bullet \vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \psi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Spherical Coordinates (r, θ, φ)

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{\phi}$$

$$\nabla \bullet \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial \psi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

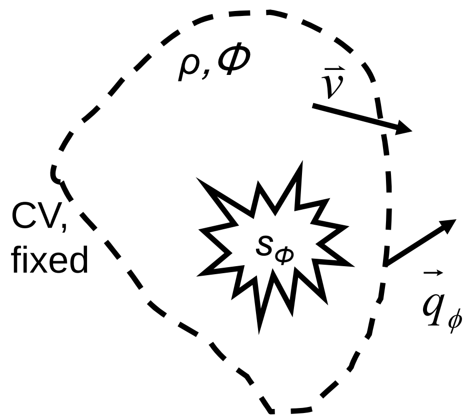
Material Covered in class

Differential forms of conservation laws (Mass, RTT, Mom. & N-S)

- Material Derivative (substantial/total derivative)
- Conservation of Mass
  - Differential Approach
  - Integral (volume) Approach
    - Use of Gauss Theorem
  - Incompressibility
- Reynolds Transport Theorem
- Conservation of Momentum (Cauchy's Momentum equations)
- The Navier-Stokes equations
  - Constitutive equations: Newtonian fluid
  - Navier-stokes, compressible and incompressible

Integral Conservation Law for a scalar ϕ

$$\left\{\frac{d}{dt}\int_{CM}\rho\phi dV = \right\} \frac{d}{dt}\int_{CV_{\text{fixed}}}\rho\phi dV + \underbrace{\int_{CS}\rho\phi\,(\vec{v}.\vec{n})dA}_{\substack{\text{Advective fluxes} \\ \text{("convective" fluxes)}}} = \underbrace{-\int_{CS}\vec{q}_{\phi}.\vec{n}\; dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum\int_{CV_{\text{fixed}}} s_{\phi}\; dV}_{\substack{\text{Sum of sources and} \\ \text{sinks terms (reactions, etc)}}}$$



Applying the Gauss Theorem, for any arbitrary CV gives:

$$\frac{\partial\rho\phi}{\partial t} + \nabla.(\rho\phi\vec{v}) = -\nabla.\vec{q}_{\phi} + s_{\phi}$$

For a common diffusive flux model (Fick's law, Fourier's law):

$$\vec{q}_{\phi} = -k\nabla\phi$$

Conservative form of the PDE

$$\frac{\partial\rho\phi}{\partial t} + \nabla.\,(\rho\phi\vec{v}) = \nabla.\,(k\nabla\phi) + s_{\phi}$$

Additional Handouts: Derivation of Reynolds Transport Theorem

Handouts extracted from pp. 91-93 in Whitaker, S. *Elementary Heat Transfer Analysis*. Pergamon Press, 1976. ISBN: 9780080189598

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