2.29 Numerical Fluid Mechanics
Fall 2011 – Lecture 6

REVIEW of Lecture 5

- Continuum Hypothesis and conservation laws
- Macroscopic Properties

Material covered in class: Differential forms of conservation laws

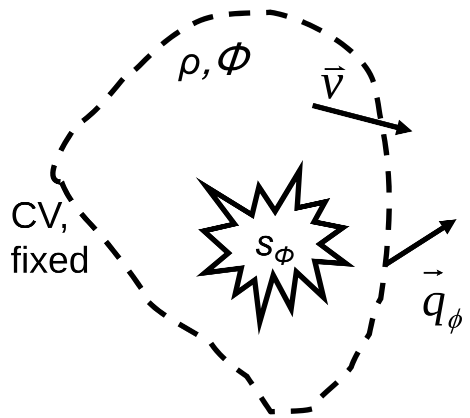
- Material Derivative (substantial/total derivative)
- Conservation of Mass
  - Differential Approach
  - Integral (volume) Approach
    - Use of Gauss Theorem
  - Incompressibility
- Reynolds Transport Theorem
- Conservation of Momentum (Cauchy's Momentum equations)
- The Navier-Stokes equations
  - Constitutive equations: Newtonian fluid
  - Navier-stokes, compressible and incompressible

Fluid flow modeling: the Navier-Stokes equations and their approximations – Cont'd
Today's Lecture

- References :
  - Chapter 1 of "J. H. Ferziger and M. Peric, Computational Methods for Fluid Dynamics. Springer, New York, third edition, 2002."
  - Chapter 4 of "I. M. Cohen and P. K. Kundu. *Fluid Mechanics*. Academic Press, Fourth Edition, 2008"
  - Chapter 4 in "F. M. White, *Fluid Mechanics*. McGraw-Hill Companies Inc., Sixth Edition"
- For today's lecture, any of the chapters above suffice
  - Note each provide a somewhat different perspective

Integral Conservation Law for a scalar ϕ

$$\left\{ \frac{d}{dt}\int_{CM} \rho\phi dV = \right\} \boxed{ \frac{d}{dt}\int_{CV_{\text{fixed}}} \rho\phi dV + \underbrace{\int_{CS} \rho\phi\,(\vec{v}.\vec{n})dA}_{\substack{\text{Advective fluxes} \\ \text{("convective" fluxes)}}} = \underbrace{-\int_{CS} \vec{q}_{\phi}.\vec{n}\; dA}_{\text{Other transports (diffusion, etc)}} + \underbrace{\sum \int_{CV_{\text{fixed}}} s_{\phi}\; dV}_{\substack{\text{Sum of sources and} \\ \text{sinks terms (reactions, etc)}}} }$$



Applying the Gauss Theorem, for any arbitrary CV gives:

$$\frac{\partial \rho\phi}{\partial t} + \nabla.(\rho\phi\vec{v}) = -\nabla.\,\vec{q}_{\phi} + s_{\phi}$$

For a common diffusive flux model (Fick's law, Fourier's law):

$$\vec{q}_{\phi} = -k\nabla\phi$$

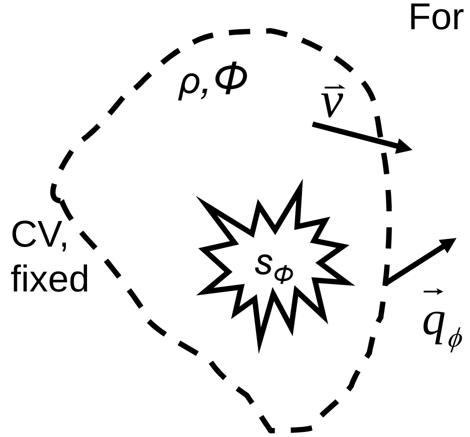
Conservative form of the PDE

$$\frac{\partial \rho\phi}{\partial t} + \nabla.\,(\rho\phi\vec{v}) = \nabla.\,(k\nabla\phi) + s_{\phi}$$

Strong-Conservative form of the Navier-Stokes Equations ($\phi \Rightarrow \mathbf{v}$)

Cons. of Momentum: $$\frac{d}{dt}\int_{CV} \rho\vec{v}dV + \int_{CS} \rho\vec{v}(\vec{v}.\vec{n})dA = \underbrace{\int_{CS} -p\,\vec{n}dA + \int_{CS} \vec{\vec{\tau}}.\vec{n}dA + \int_{CV} \rho\vec{g}dV}_{=\sum\vec{F}}$$

Applying the Gauss Theorem gives: $$= \int_{CV} \left(-\nabla p + \nabla.\vec{\vec{\tau}} + \rho\vec{g}\right)dV$$



For any arbitrary CV gives: $$\frac{\partial\rho\vec{v}}{\partial t} + \nabla.(\rho\vec{v}\;\vec{v}) = -\nabla p + \nabla.\vec{\vec{\tau}} + \rho\vec{g}$$

With Newtonian fluid + incompressible + constant μ:

Momentum: $$\boxed{\frac{\partial\rho\vec{v}}{\partial t} + \nabla.(\rho\vec{v}\;\vec{v}) = -\nabla p + \mu\nabla^2\vec{v} + \rho\vec{g}}$$

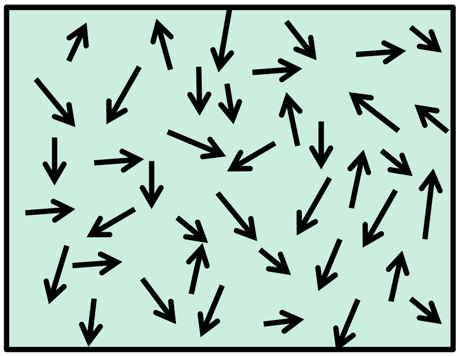
Mass: $$\boxed{\nabla.\vec{v} = 0}$$

Equations are said to be in "strong conservative form" if all terms have the form of the divergence of a vector or a tensor. For the i^{th} Cartesian component, in the general Newtonian fluid case:

With Newtonian fluid only: $$\boxed{\frac{\partial\rho v_i}{\partial t} + \nabla.(\rho v_i\;\vec{v}) = \nabla.\left(-p\;\vec{e}_i + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\vec{e}_j - \frac{2}{3}\mu\frac{\partial u_j}{\partial x_j}\vec{e}_i + \rho g_i x_i \vec{e}_i\right)}$$

Navier-Stokes Equations: For an Incompressible Fluid with constant viscosity

V(x,y,z)



Fluid Velocity Field

$$\mathbf{V} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

Hydrostatic Pressure:
$-\rho g z$ for z positive upward

Conservation of Mass

$$\mathrm{div}\mathbf{V} = \nabla \cdot \mathbf{V} = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Dynamic Pressure $P = P_{actual} - \rho g z$

Navier-Stokes Equation

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

Density ρ

Kinematic viscosity ν

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Incompressible Fluid
Pressure Equation

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

Conservation of Mass

$$\nabla \cdot \mathbf{V} = 0$$

Divergence of Navier-Stokes Equation

$$\mathrm{div}(\mathbf{V} \cdot \nabla\ \mathbf{V}) = -\frac{1}{\rho}\nabla^2 P$$

Dynamic Pressure Poisson Equation

$$\Rightarrow \quad \nabla^2 P = -\rho \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + 2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + 2\frac{\partial w}{\partial x}\frac{\partial u}{\partial z} + 2\frac{\partial w}{\partial y}\frac{\partial v}{\partial z} \right\}$$

More general than Bernoulli – Valid for unsteady and rotational flow

Incompressible Fluid
Vorticity Equation

Vorticity

$$\tilde{\omega} \equiv \mathrm{curl}\mathbf{V} \equiv \nabla \times \mathbf{V}$$

Navier-Stokes Equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

curl of Navier-Stokes Equation

$$\frac{D\tilde{\omega}}{Dt} = -(\tilde{\omega} \cdot \nabla)\mathbf{V} + \nu \nabla^2 \tilde{\omega}$$

Inviscid Fluid Mechanics
Euler's Equation

Navier-Stokes Equation: incompressible, constant viscosity

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{V}$$

If also inviscid fluid

$$\nu = 0$$

$\Rightarrow$ Euler's Equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial P}{\partial z}
\end{aligned}$$

Inviscid Fluid Mechanics
Bernoulli Theorems

Theorem 1

Irrotational Flow, incompressible

$$\nabla \times \mathbf{V} = \mathbf{0}$$

Flow Potential

$$\mathbf{V} = \nabla\phi$$

Define

$$H = \frac{1}{2}|\mathbf{V}|^2 + \frac{P}{\rho}$$

$$\frac{\partial\phi}{\partial t} + H = 0$$

Introduce P_T = Thermodynamic pressure

$$P_T = P - \rho g z$$

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}|\mathbf{V}|^2 + \frac{P_T}{\rho} + gz = 0}$$

Theorem 2

Steady, Incompressible, inviscid, no shaft work, no heat transfer

Navier-Stokes Equation

$$\cancel{\frac{\partial\mathbf{V}}{\partial t}} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla P$$

$$\mathbf{V} \times \tilde{\omega} = \nabla H$$

Along stream lines and vortex lines

$$\boxed{\begin{aligned} H &= \frac{1}{2}|\mathbf{V}|^2 + \frac{P}{\rho} \\ &= \frac{1}{2}|\mathbf{V}|^2 + \frac{P_T}{\rho} + gz = \text{const} \end{aligned}}$$

Potential Flows

Integral Equations

Irrotational Flow

$$\nabla \times \mathbf{V} = 0$$

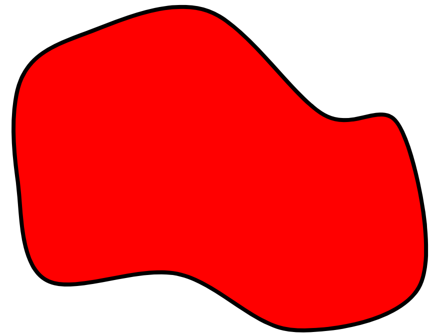
Flow Potential

$$\mathbf{V} = \nabla \phi$$

Conservation of Mass

$$\begin{aligned} \nabla \cdot \mathbf{V} &= 0 \\ &\Rightarrow \\ \nabla \cdot (\nabla \phi) &= 0 \end{aligned}$$

$$\boxed{\nabla^2 \phi = 0}$$



"Mostly" Potential Flows:

Only rotation occurs at boundaries due to viscous terms

In 2D:

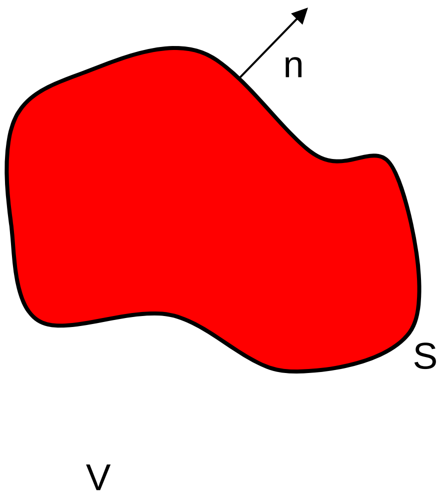
$$\text{Velocity potential}: \quad u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

$$\text{Stream function}: \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

- Since Laplace equation is linear, it can be solved by superposition of flows, called panel methods
- What distinguishes one flow from another are the boundary conditions and the geometry: there are no intrinsic parameters in the Laplace equation

Potential Flow

Boundary Integral Equations



Green's Theorem

$$\int_S \left[G(\mathbf{x},\mathbf{x}_0)\frac{\partial \phi(\mathbf{x}_0)}{\partial n} - \phi(\mathbf{x}_0)\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} \right] dS_0$$

$$= \int_V \left[\phi(\mathbf{x}_0)\nabla^2 G(\mathbf{x},\mathbf{x}_0) - G(\mathbf{x},\mathbf{x}_0)\nabla^2\phi(\mathbf{x}_0) \right] dV_0$$

Green's Function

$$G(\mathbf{x},\mathbf{x}_0) = \frac{1}{r} = \frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \psi(\mathbf{x})$$

Homogeneous Solution

$$\nabla^2\psi = 0$$

$$\nabla^2 G((\mathbf{x},\mathbf{x}_0) = -\delta(\mathbf{x}-\mathbf{x}_0)$$

Boundary Integral Equation

$$\phi(\mathbf{x}) = \int_S \left[G(\mathbf{x},\mathbf{x}_0)\frac{\partial \phi(\mathbf{x}_0)}{\partial n} - \phi(\mathbf{x}_0)\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n} \right] dS_0 - \int_V \left[G(\mathbf{x},\mathbf{x}_0)\nabla^2\phi(\mathbf{x}_0) \right] dV_0$$

Discretized Integral Equation

$$\sum_{j=0}^{N-1} A_{ij} w_j = B_i$$

Linear System of Equations

$$\overline{\overline{\mathbf{A}}}\mathbf{u} = \mathbf{b}$$

Panel Methods

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