REVIEW of Lecture 5

- Continuum Hypothesis and conservation laws
- Macroscopic Properties

Material covered in class: Differential forms of conservation laws

- Material Derivative (substantial/total derivative)
- Conservation of Mass
  - Differential Approach
  - Integral (volume) Approach
    - Use of Gauss Theorem
  - Incompressibility
- Reynolds Transport Theorem
- Conservation of Momentum (Cauchy’s Momentum equations)
- The Navier-Stokes equations
  - Constitutive equations: Newtonian fluid
  - Navier-stokes, compressible and incompressible
Fluid flow modeling: the Navier-Stokes equations and their approximations – Cont’d

Today’s Lecture

• References:

• For today’s lecture, any of the chapters above suffice
  – Note each provide a somewhat different perspective
Integral Conservation Law for a scalar \( \phi \)

\[
\left\{ \frac{d}{dt} \int_{C_M} \rho \phi dV \right\} = \frac{d}{dt} \int_{C_{V_{\text{fixed}}}} \rho \phi dV + \int_{C_S} \rho \phi (\vec{v} \cdot \vec{n}) dA = \int_{C_S} \vec{q}_\phi \cdot \vec{n} dA + \sum \int_{C_{V_{\text{fixed}}}} s_\phi dV
\]

Advective fluxes ("convective" fluxes)  
Other transports (diffusion, etc)  
Sum of sources and sinks terms (reactions, etc)

Applying the Gauss Theorem, for any arbitrary CV gives:

\[
\frac{\partial \rho \phi}{\partial t} + \nabla \cdot (\rho \phi \vec{v}) = -\nabla \cdot \vec{q}_\phi + s_\phi
\]

For a common diffusive flux model (Fick’s law, Fourier’s law):

\[
\vec{q}_\phi = -k \nabla \phi
\]
Strong-Conservative form of the Navier-Stokes Equations ($\phi \Rightarrow \mathbf{v}$)

Cons. of Momentum:

\[
\frac{d}{dt} \int_{CV} \rho \mathbf{v} dV + \int_{CS} \rho \mathbf{v} (\mathbf{v}, \mathbf{n}) dA = \int_{CS} -p \mathbf{n} dA + \int_{CS} \mathbf{\tau} \mathbf{n} dA + \int_{CV} \rho \mathbf{g} dV = \sum \bar{F} \\
= \int_{CV} (-\nabla p + \nabla \mathbf{\tau} + \rho \mathbf{g}) dV
\]

Applying the Gauss Theorem gives:

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

For any arbitrary CV gives:

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

With Newtonian fluid + incompressible + constant $\mu$:

Momentum:

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Mass:

\[
\nabla \cdot \mathbf{v} = 0
\]

Equations are said to be in “strong conservative form” if all terms have the form of the divergence of a vector or a tensor. For the $i$th Cartesian component, in the general Newtonian fluid case:

With Newtonian fluid only:

\[
\frac{\partial \rho v_i}{\partial t} + \nabla \cdot (\rho v_i v) = \nabla \cdot \left( -p \mathbf{e}_i + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \mathbf{e}_j - \frac{2}{3} \mu \frac{\partial u_j}{\partial x_j} \mathbf{e}_i + \rho g_i \mathbf{e}_i \right)
\]
Navier-Stokes Equations: For an Incompressible Fluid with constant viscosity

Fluid Velocity Field

\[ V = iu + jv + kw \]

Conservation of Mass

\[ \text{div} V = \nabla \cdot V = 0 \]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

Navier-Stokes Equation

\[ \frac{D V}{D t} = \frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\frac{1}{\rho} \nabla P + \nu \nabla^2 V \]

Hydrostatic Pressure:

\[ -\rho g z \text{ for } z \text{ positive upward} \]

Dynamic Pressure

\[ P = P_{\text{actual}} - \rho g z \]

Kinematic viscosity \( \nu \)

Density \( \rho \)
Incompressible Fluid Pressure Equation

Navier-Stokes Equation

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V}
\]

Conservation of Mass

\[\nabla \cdot \mathbf{V} = 0\]

Divergence of Navier-Stokes Equation

\[\text{div}(\mathbf{V} \cdot \nabla \mathbf{V}) = \frac{1}{\rho} \nabla^2 P\]

Dynamic Pressure Poisson Equation

\[
\Rightarrow \nabla^2 P = -\rho \left\{ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + 2 \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial w}{\partial x} \frac{\partial u}{\partial z} + 2 \frac{\partial w}{\partial y} \frac{\partial v}{\partial z} \right\}
\]

More general than Bernoulli – Valid for unsteady and rotational flow
Incompressive Fluid
Vorticity Equation

\[ \tilde{\omega} \equiv \text{curl} \mathbf{V} \equiv \nabla \times \mathbf{V} \]

Navier-Stokes Equation

\[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V} \]

\text{curl of Navier-Stokes Equation}

\[ \frac{D \tilde{\omega}}{Dt} = - (\tilde{\omega} \cdot \nabla) \mathbf{V} + \nu \nabla^2 \tilde{\omega} \]
Inviscid Fluid Mechanics

Euler’s Equation

Navier-Stokes Equation: incompressible, constant viscosity

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{V}
\]

If also inviscid fluid

\[ \nu = 0 \]

\[ \Rightarrow \text{Euler's Equations} \]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial x}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial y}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z}
\]
**Inviscid Fluid Mechanics**

**Bernoulli Theorems**

**Theorem 1**

Irrotational Flow, incompressible

\[ \nabla \times \mathbf{V} = 0 \]

Flow Potential

\[ \mathbf{V} = \nabla \phi \]

Define

\[ H = \frac{1}{2} |\mathbf{V}|^2 + \frac{P}{\rho} \]

\[ \frac{\partial \phi}{\partial t} + H = 0 \]

Introduce \( P_T \) = Thermodynamic pressure

\[ P_T = P - \rho g z \]

\[ \frac{\partial \phi}{\partial t} + \frac{1}{2} |\mathbf{V}|^2 + \frac{P_T}{\rho} + g z = 0 \]

**Theorem 2**

Steady, Incompressible, inviscid, no shaft work, no heat transfer

Navier-Stokes Equation

\[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla P \]

\[ \mathbf{V} \times \mathbf{\omega} = \nabla H \]

Along stream lines and vortex lines

\[ H = \frac{1}{2} |\mathbf{V}|^2 + \frac{P}{\rho} \]

\[ = \frac{1}{2} |\mathbf{V}|^2 + \frac{P_T}{\rho} + g z = \text{const} \]
Potential Flows
Integral Equations

Irrotational Flow

\[ \nabla \times \mathbf{V} = 0 \]

Flow Potential

\[ \mathbf{V} = \nabla \phi \]

Conservation of Mass

\[ \nabla \cdot \mathbf{V} = 0 \]

\[ \Rightarrow \]

\[ \nabla \cdot (\nabla \phi) = 0 \]

\[ \nabla^2 \phi = 0 \]

“Mostly” Potential Flows:
Only rotation occurs at boundaries due to viscous terms

In 2D:

Velocity potential: \[ u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y} \]

Stream function: \[ u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \]

• Since Laplace equation is linear, it can be solved by superposition of flows, called panel methods

• What distinguishes one flow from another are the boundary conditions and the geometry: there are no intrinsic parameters in the Laplace equation
Potential Flow
Boundary Integral Equations

Green’s Theorem

\[ \int_S \left[ G(x, x_0) \frac{\partial \phi(x_0)}{\partial n} - \phi(x_0) \frac{\partial G(x, x_0)}{\partial n} \right] dS_0 = \int_V \left[ \phi(x_0) \nabla^2 G(x, x_0) - G(x, x_0) \nabla^2 \phi(x_0) \right] dV_0 \]

Green’s Function

\[ G(x, x_0) = \frac{1}{r} = \frac{1}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} + \psi(x) \]

Homogeneous Solution

\[ \nabla^2 \psi = 0 \]

\[ \nabla^2 G((x, x_0) = -\delta(x - x_0) \]

Boundary Integral Equation

\[ \phi(x) = \int_S \left[ G(x, x_0) \frac{\partial \phi(x_0)}{\partial n} - \phi(x_0) \frac{\partial G(x, x_0)}{\partial n} \right] dS_0 - \int_V \left[ G(x, x_0) \nabla^2 \phi(x_0) \right] dV_0 \]

Discretized Integral Equation

\[ \sum_{j=0}^{N-1} A_{ij} w_j = B_i \]

Linear System of Equations

\[ \tilde{A}u = b \]

Panel Methods