MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING
CAMBRIDGE, MASSACHUSETTS 02139
2.29 NUMERICAL FLUID MECHANICS — FALL 2011

QUIZ 2

The goals of this quiz 2 are to: (i) ask some general higher-level questions to ensure you understand broad concepts and are able to discuss such concepts and issues with others familiar with numerical fluid mechanics; (ii) show that you understand and can apply concepts, methods and schemes that you learned; and (iii), show that you are able to read numerical codes and recognize what they accomplish. Partial credit will be given to partial answers.

Shorter Concept Questions

Problem 1 (25 points)

Please provide a brief answer to the following questions (a few words to a few sentences is enough, with or without a few equations depending on the question).

a) Provide two limitations of the von Neumann stability analysis.

b) Explain how the discretization error and order of accuracy can be estimated numerically, without knowing the true solution. If you find that the estimated order of accuracy is not as expected, what are three possible causes?

c) You evaluate the stability of your advection scheme and you find that the Courant-Friedrichs-Lewy condition does not give you the same condition as a von Neumann stability analysis. Which of the two conditions are you going to utilize, if any? Why?

d) Briefly state three advantages that Multistep methods have over Runge-Kutta methods and three advantages that Runge-Kutta methods have over Multistep methods.

e) The effective wave speeds of numerical representations of advection schemes are usually smaller than the real wave speed and usually dependent on the wavenumber. Briefly explain what this implies for numerical simulations.

f) A friend tells you that for your FV scheme, you should always initialize your solution by computing the integral of your initial conditions over each finite volume. Otherwise, she says, your nodal values will not represent the volume average over the cell. Another friend tells you that you can initialize with the initial conditions evaluated at the nodal centers and that you should not bother doing integrals since it will be within the truncation errors. Are both of your friends right or wrong, or is only one of them right? Briefly explain why.

g) To simulate a turbulent pipe flow systems with bends and corners, your friend recommends a non-orthogonal shape-following finite-volume grid, with a staggered pressure and velocity arrangements, specifically: pressure at nodal points and Cartesian-coordinate velocities at the boundaries of these pressure finite-volumes. Is this a good idea or not? Briefly explain why.

Idealized Computational Problems

Problem 2: *Local Stability of a Numerical Scheme* (30 points)

Consider the following one-dimensional temperature diffusion,

$$\frac{\partial T}{\partial t} = \nu \frac{\partial^2 T}{\partial x^2}$$

and its discretization,

$$\frac{T_j^{n+1}-T_j^n}{\Delta t}=\nu\left(\theta\frac{T_{j+1}^{n+1}-2T_j^{n+1}+T_{j-1}^{n+1}}{\Delta x^2}+(1-\theta)\frac{T_{j+1}^{n}-2T_j^{n}+T_{j-1}^{n}}{\Delta x^2}\right)\text{ with }0\le\theta\le 1$$

and define $r=\nu\dfrac{\Delta t}{\Delta x^2}$.

a) Determine the leading term of the local truncation error in terms of r,θ and temperature derivatives at time *n* and point *j*. What are the orders of accuracy of the discretized equation in time and in space?

b) Using your result in a) provide an expression for $\theta^*(r)$ that maximizes these orders of accuracy in time and space for a given *r*. What are these maximum orders of accuracy?

c) Using a von Neumann stability analysis, determine the stability criterion for the discretized equation.

d) Based on your results in c), discuss 4 cases: $\theta=0$, $\theta<\frac{1}{2}$, $\theta\ge\frac{1}{2}$ and $\theta=1$, expressing in each case the stability conditions on *r*. Relate your results to the forward Euler (classic explicit), backward Euler (classic implicit) and Crank-Nicolson schemes.

e) How does the cost of the scheme obtained in b) with $\theta^*(r)$ compare to that of the Crank-Nicolson scheme?

f) Using your von Neumann analysis in c-d), provide a condition which guarantees no oscillation of the errors and a condition which guarantees constant signs for discrete temperatures (and their errors). Very briefly discuss the results.

g) If you were to utilize the above discretization to determine the steady state temperature, which range of values for θ would you choose? Why? Can you utilize any values for *r*?

Problem 3: *Finite Volumes on Triangles.* (25 points)

Consider the conservative form of a partial differential equation governing the two-dimensional (2D) in space advection-diffusion-reaction of a variable ϕ per unit volume (a surface in 2D):

$$\frac{\partial\phi}{\partial t}+\nabla.\,(\phi\vec{v})=\nabla.\,(\nu\,\nabla\phi)+s_\phi$$

A finite-volume discretization is employed to solve this equation numerically, using equilateral triangles over the given domain. A representative triangle is illustrated on Figure 1: triangles have sides of length *L* and are fixed in time and space. The finite-volume discretization is node-centered for ϕ. Velocities are staggered and defined normal and outward to the edges of each triangle.

a) Utilizing a mid-point rule for the surface (volume) integrals, provide the spatial discretization of the time-rate of change term and of the source term for the center triangle P.

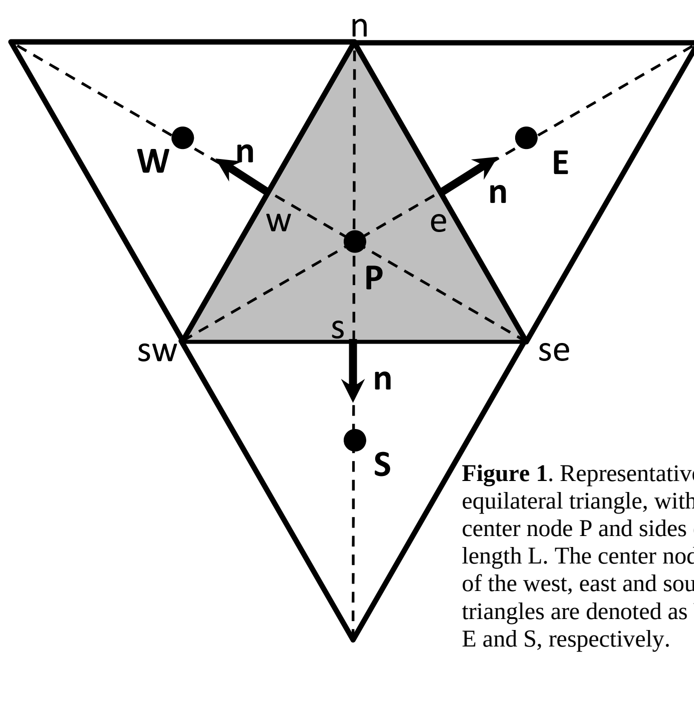

Figure 1. Representative equilateral triangle, with center node P and sides of length L. The center nodes of the west, east and south triangles are denoted as W, E and S, respectively.

b) Discretize the convective fluxes for the center triangle P, using a mid-point rule for the edge integral and linear interpolation to estimate the center edge values in terms of nodal values.
c) Discretize the diffusive fluxes for the center triangle P, using a mid-point rule for the edge integral and linear interpolation of fluxes (centered difference) to estimate the center edge fluxes in terms of nodal values.
d) For the given discretizations, do you need to resolve flux discontinuities in b) and/or c) to ensure conservative flux discretizations? If yes, please do so. If not, explain why.
e) Combining a)-d), write the resulting finite-volume discretization for center triangle P.
f) Should the triangle not be equilateral, what would be the order of accuracy of the spatial discretization obtained in e)? What is the main reason for this result?

Problem 4: *Code for solving a PDE* (20 points)

The MATLAB code for solving a PDE is given below:

```
% MATLAB Code for solving a PDE.

clear all,clc,clf,
close all;

% Initialize constants
Nx=21; % Number of grid points
x=1;   % Length of domain
dx=x/(Nx-1); % Grid spacing
Nt=601; % Total number of time steps
t=1;    % Final time
dt=t/(Nt-1); % Time step
c=0.5;
C = c*dt/(2*dx);

% Initial conditions
for i=1:Nx
    if (dx*(i-1) >0.5)
        u(i,1)=1;
    else
        u(i,1)=0;
    end
end

% Fill in Matrix H:
H = zeros(size(u,1),size(u,1));

for ii=3:size(u,1)
    H(ii,ii) = 1+3*C;
    H(ii,ii-1) = -4*C;
    H(ii,ii-2) = C;
end
H(1,1) = 1+3*C;
H(1,end) = -4*C;
H(1,end-1) = C;
H(2,1) = -4*C;
H(2,2) = 1+3*C;
H(2,end) = C;

% Fill in Matrix L;
```

```
L=diag(ones(1,Nx));
    for i=2:Nx-1
        L(i,i-1)=-C;
        L(i,i+1)= C;
    end
 L(1,2) = C;L(1,end) = -C;
 L(end,1) = C;L(end,end-1) = -C;
 L = inv(L);

% Time integration
for k = 2:Nt
    uold = L*u(:,k-1);
    unew = zeros(Nx,1);
    iter=0;
    while(norm(unew-uold)>0.0001)
       if (iter~=0) uold = unew; end
       unew = L*u(:,k-1) + (eye(Nx)-L*H)*uold;
       iter=iter+1;
    end
    u(:,k)=unew;
end
u_dc = u;
```

Answer the following questions:

a) Which equation is being solved by the given MATLAB code? What are the initial conditions? What type of boundary condition is implemented?
b) Which types of discretization in space and time do the L and H matrices correspond to?
c) Which scheme is being implemented in the code?
d) Give two reasons why one would want to use this scheme.