

Circular Symmetry in Topological Quantum Field Theory and the Topology of the Index Bundle

by

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M.S. in Mathematics, University of Bucharest, 1991

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Abstract

We extend the Mathai-Quillen construction for the Euler form to the equivariant case and use it to generalize the Atiyah-Jeffrey interpretation of Donaldson-Witten theory (DWT) in an equivariant situation. This allows us to interpret topological non-abelian Seiberg-Witten theory (TNSW) and its massive version (MTNSW) geometrically.

We state the infinite-dimensional analogue of the localization theorem for an S^1 -action and explore the consequences of the assumption that our infinite-dimensional version is true, for example that the correlation functions of MTNSW can be expressed as integrals over the moduli spaces for DWT and (topological) abelian Seiberg-Witten theory (TASW).

Under further assumptions we conclude that the integrals over the DWT moduli space are integrals of certain Segre classes of the index bundle, which we can express in terms of the usual μ -classes. For the TASW moduli space integrals, if we restrict to four-manifolds of simple type, we get a linear combination of Seiberg-Witten invariants.

Since the semiclassical approximation is supposed to be exact for a topological quantum field theory, we are led to a conjecture relating a ratio of determinants to the total Segre class of the index bundle.

Thesis Supervisor: I.M. Singer

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To my father and to the memory of my mother.

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Chapter 1

Introduction

In [33], Witten introduced the first example of a topological quantum field theory (TQFT). By twisting $N = 2$ supersymmetric Yang-Mills theory, he found a Lagrangian and quantum field theory in which Donaldson invariants are described as correlation functions of some natural observables. This quantum field theory is frequently called Donaldson-Witten theory (DWT).

The Donaldson-Witten Lagrangian can be obtained in different ways. On the physics side, Baulieu and Singer [4] started with a purely topological action, the Pontrjagin form, and derived the Donaldson-Witten Lagrangian and the observables via the standard gauge-fixing BRST formalism. On the mathematical side, Atiyah and Jeffrey [1] interpreted the Lagrangian as the Euler form of an infinite-dimensional bundle (the bundle of self-dual two-forms over the space of connections modulo gauge transformations, on a four-dimensional base manifold). This interpretation relies on a finite-dimensional construction of Mathai and Quillen [19] which, given an oriented vector bundle endowed with a section, produces a differential-form representative of the Euler form which decays rapidly away from the zero-locus of the section. Atiyah and Jeffrey formally apply the Mathai-Quillen method to the above-mentioned infinite-dimensional bundle, choosing the (Fredholm) section which associates to any connection the self-dual part of its curvature. Expectation values of the observables are obtained in this way: since the 'functional-integral measure' (i.e. the exponential of the Lagrangian) is supposed to be an Euler form, the integrals of products of observables against it are expressible, by 'Poincaré duality', as integrals on the zero-set of the section, which in this case is Donaldson's moduli space \mathcal{M}_D of anti-self-dual connections (instantons). As proved previously in [8], the integrals over \mathcal{M}_D admit a rigorous definition as intersection numbers on the Uhlenbeck compactification $\overline{\mathcal{M}}_D$, leading to the Donaldson invariants. For DWT, this implies that the correlation functions in [4] and [33] are path-integral representations of diffeomorphism invariants of the base manifold.

Of course all the statements in the previous paragraph have to be read with extreme caution,

largely because the passage from finite dimensions to infinite dimensions presents many problems. First, the group of gauge transformations doesn't act freely on reducible connections—so that the quotient space is not really smooth; secondly, it is not clear how the Mathai-Quillen formalism (originally devised for a compact, finite-dimensional manifold) applies to a non-compact, infinite-dimensional situation. When it does, how does it take into account the compactification of \mathcal{M}_D required in the definition of Donaldson invariants? A striking example of the limitations of the above model is this: formally, DWT predicts that the correlation functions are *independent of the metric* on the base X , whereas Donaldson invariants are metric independent only if $b_2^+(X) \geq 2$. The discrepancy arises because DWT assumes no boundary effects (see the first section of [33] and [22]).

The original aim of DWT was to study Donaldson's four-manifold invariants using ideas from quantum field theory. The case of Kähler surfaces with $b_2^+ \geq 3$ was understood by Witten [37], by using cluster decomposition in the large scaling limit. Finally, Seiberg and Witten [29, 30] succeeded in the general case by using renormalization group techniques, far deeper than formal path-integral arguments. Their work applies more generally to the physical counterpart of DWT, $N = 2$ supersymmetric Yang-Mills; its consequence for the topological theory was the introduction of a new set of topological invariants which turned out to be much simpler than Donaldson invariants. The mathematical implications of their work, as well as the conjectured relationship between Donaldson invariants and those of Seiberg-Witten can be found in [36]. It is worth pointing out that only recently has the case of $b_2^+ = 1$ been explained from the physics viewpoint; see Moore and Witten [22].

Mathematicians have had high hopes that the geometric/topological interpretation of DWT in [1] would, via mathematically traditional methods like localization and equivariant cohomology, explain the relationship between Donaldson and Seiberg-Witten invariants. The origin of these hopes lies in a mathematical idea of Pidstrigach and Tyurin: in [24] they construct a cobordism between Donaldson and Seiberg-Witten moduli spaces via the space of solutions of a non-abelian version of the Seiberg-Witten equations. Getting explicit information out of this cobordism is however very hard mathematically, so one can ask whether there exists instead a quantum field theoretic reformulation of the Pidstrigach-Tyurin idea which leads to concrete, albeit non-rigorous, results.

There does exist a TQFT, *non-abelian Seiberg-Witten theory* (TNSW), which leads, in the Atiyah-Jeffrey picture, to the non-abelian Seiberg-Witten equations. The field content of TNSW suggests that the quantum field theoretic analog of the Pidstrigach-Tyurin cobordism is to regard TNSW as an interpolation between DWT and topological abelian Seiberg-Witten theory (TASW).

TNSW is equally important because of its relationship to $N = 4$ supersymmetric Yang-Mills theory. It turns out [15] that TNSW is a topologically twisted version of $N = 4$ supersymmetric Yang-Mills. In addition to TNSW, $N = 4$ super-Yang-Mills admits two other topological twists; one of them, considered by Vafa and Witten in [32] provided the first tests of the S-duality conjecture. It is therefore interesting to compare TNSW to the Vafa-Witten twist and study whether it gives

any information on S-duality.

Besides its topological invariance, TNSW admits an additional S^1 -symmetry and leads to an S^1 -equivariant version of the theory via an equivariant extension of the Mathai-Quillen construction. We will call S^1 -equivariant TNSW massive topological non-abelian Seiberg-Witten theory (MTNSW) because of the physical meaning of the additional terms in the Lagrangian.

The key properties of MTNSW result from the interplay between its topological character and the S^1 -symmetry. The path-integrals defining the partition function and correlation functions of MTNSW are formal, infinite-dimensional integrals of equivariantly closed differential forms. In finite dimensions, such integrals localize to the fixed-point set of the S^1 -action, and one can explore whether a similar result is true in the infinite dimensional setting of TQFT's. At least the localization *statement* can be generalized to infinite dimensions, and sometimes a formal path-integral justification can be made. This justification follows one of the proofs of the finite-dimensional localization theorem, which relies only on semiclassical approximation and Gaussian integrals.

In the case of MTNSW, the fixed points of the S^1 -action correspond precisely to Donaldson-Witten and abelian Seiberg-Witten configurations. Assuming that the localization property holds for MTNSW, it follows that its correlation functions can be expressed as combinations of correlation functions in DWT and TASW, i.e Donaldson and Seiberg-Witten invariants.

We now summarize the contents of this thesis. In Chapter 2 we review the Mathai-Quillen construction of a representative of the Euler form by a rapidly decaying form (Section 2.2). We then extend the construction to equivariant vector bundles with equivariant section to obtain an equivariant Euler form (Section 2.3). In Sections 2.4 and 2.5 we generalize the Atiyah-Jeffrey strategy to the equivariant case, which we will need later in order to interpret some quantum field theories.

We begin Chapter 3 with a review of DWT as interpreted in [1]. In Section 3.2 we extend this interpretation to the case of abelian Seiberg-Witten (TASW) theory, the quantum field theory which gives the Seiberg-Witten invariants. We present a similar interpretation of the twisted $N = 4$ supersymmetric Yang-Mills theory (Vafa-Witten [32]) in Section 3.3. For special manifolds, the partition function of the twisted theory reduces to integrals over the moduli space of instantons \mathcal{M}_D . In 3.4 we state two mathematical conjectures which are a consequence of this fact and the S-duality conjecture for $N = 4$ supersymmetric Yang-Mills.

One application of the equivariant extensions in Chapter 2 is the geometric interpretation of topological non-abelian Seiberg-Witten theory, which we begin to study in Section 3.5. The additional symmetry that we exploit is very simple: TNSW contains a complex matter field with its scalar S^1 -action. We study two versions of the theory, according to whether we use the original Mathai-Quillen construction or its equivariant extension. Plain TNSW (which uses the standard Mathai-Quillen form) turns out to be a topologically twisted version of $N = 4$ supersymmetric Yang-Mills theory in four dimensions (albeit a different twist than the Vafa-Witten one, as can be seen

by comparing with Section 3.3). As for S^1 -equivariant TNSW (which involves the S^1 -equivariant Mathai-Quillen form) we call it massive topological non-abelian Seiberg-Witten theory (MTNSW) because the difference between its Lagrangian and the Lagrangian of TNSW (i.e. the extra terms which make the Lagrangian S^1 -equivariant) turns out to be a supersymmetric mass term [12, 16].

The reason for studying the additional symmetry and the massive version of TNSW lies in the possibility of localizing the path-integral. In finite dimensions, if S^1 acts on a compact oriented manifold, integrals of equivariantly closed differential forms can be expressed in terms of integrals over the fixed-point set of the action. We investigate to what extent a similar result holds in infinite dimensions in Chapter 4.

We review the finite-dimensional case for an S^1 -action in Section 4.1. We formally apply the localization theorem from Section 4.1 to the case of MTNSW in Section 4.2. We find that, under the assumption that the localization theorem remains true in this infinite-dimensional setting, the correlation functions of MTNSW are expressible as integrals over the moduli space of instantons \mathcal{M}_D and the moduli space of abelian Seiberg-Witten monopoles.

Now we want to evaluate these integrals by relating the integrands to the standard cohomology classes (the μ -classes) used in gauge theory. We will discuss two different approaches to this question, one which uses the topological content of the Mathai-Quillen construction, the other based on the semiclassical approximation.

We present the topological approach in Sections 4.3 and 4.4. According to the Mathai-Quillen interpretation, the partition function of MTNSW is the integral of an equivariantly closed differential (actually the S^1 -equivariant Euler class of an infinite-dimensional bundle). In finite dimensions, the abelian localization theorem states that such an integral reduces to an integral over the fixed-point set. The integrand over the fixed points is in general the original integrand divided by the S^1 -equivariant Euler class of the normal bundle; in our case this is a quotient of equivariant Euler classes.

Unfortunately we are in infinite dimensions and the two Euler classes in question are only formal objects, they are not well-defined mathematically. The key point is that *there exists nevertheless a well-defined candidate for their quotient*. For the component of the fixed point set corresponding to Donaldson-Witten configurations, the quotient we are investigating is the product of the Euler class of Donaldson-Witten theory and the equivariant Euler class of the bundle of sections of negative spinors, divided by the equivariant Euler class of the bundle of sections of positive spinors. The presence of the Donaldson-Witten Euler class in the integral over Donaldson-Witten configuration space shows (by Poincaré duality) that the integral reduces to an integral over \mathcal{M}_D .

As for the quotient of the (formal) Euler classes of the bundles of spinors, we again resort to the comparison with a finite-dimensional analog. If the two bundles involved were finite-dimensional, the quotient of equivariant Euler classes would equal the equivariant Euler class of the difference bundle.

In infinite dimensions, the natural interpretation of the difference bundle is the index bundle. With this interpretation, we show in Section 4.3 that a natural candidate for the quotient of Euler classes is the total Segre class of the index bundle.

We restate the new conjecture: the quotient of two infinite-dimensional Euler classes (which are formal objects) equals a characteristic class of the index bundle (a well-defined mathematical object). Assuming the conjecture to be true, we conclude that the contribution from Donaldson-Witten configurations to the partition function of MTNSW consists of integrals over Donaldson moduli spaces of certain Segre classes of the index bundle.

In Section 4.4 we carry out the similar analysis for abelian Seiberg-Witten configurations (the other component of the fixed-point set). The quotient of Euler classes appearing in the localization formula is a product of two Segre classes times the Euler class appearing in abelian Seiberg-Witten theory.

At this point it is useful to restrict to four-manifolds which satisfy the Seiberg-Witten simple type condition. This condition requires that the only non-zero Seiberg-Witten invariants arise from moduli spaces of virtual dimension zero. It is in fact conjectured that all simply-connected four-manifolds with $b_2^+ \geq 2$ satisfy it. All manifolds with $b_2^+ \geq 2$ for which the Seiberg-Witten invariants are known are of simple type (these include Kähler surfaces, blow-ups and rational blow-downs)—so the simple type condition is a rather mild restriction.

The simple type condition simplifies the contribution from abelian Seiberg-Witten configurations to the partition function of MTNSW: only the degree zero part in the Segre classes is now relevant, and we show that it is in fact a constant. We conclude that, under the simple type hypothesis, the contribution from abelian Seiberg-Witten configurations is simply a linear combination of Seiberg-Witten invariants.

Let us summarize the implications of these results for the partition function of MTNSW. We are making two assumptions: the first is that the abelian localization theorem holds in our infinite-dimensional framework; the second is that the quotient of formal Euler classes equals the Segre class of the index bundle. We also restrict to simple type four-manifolds. Assuming that all these conditions are fulfilled, the results of Sections 4.3 and 4.4 provide a concrete expression for the partition function of MTNSW as a sum of Seiberg-Witten invariants and characteristic numbers of Donaldson moduli spaces (integrals of the Segre classes of the index bundle).

We study the integrals over \mathcal{M}_D of the Segre classes of the index bundle in Chapter 5. This chapter is completely rigorous and relatively independent of the part preceding it. The first step, which we carry out in Sections 5.1 and 5.2, is to relate the Segre classes to the μ -classes used in gauge theory to describe the cohomology of the space of connections modulo gauge transformations (over a simply connected four-manifold). In Section 5.1, we apply the families index theorem in the spirit of [2] to find differential-form representatives for the Chern character forms of the index

bundle. We achieve the similar goal for the total Segre class in Section 5.2, by using Section 5.1 and a simple exercise in characteristic classes. The result provides explicit formulae for the total Segre class in terms of the μ -classes.

The second step is to use these formulae to calculate the integrals of the Segre classes on Donaldson moduli spaces. The simplest way to organize the calculation is to restrict to four-manifolds of simple type, and use the Kronheimer-Mrowka structure theorem for Donaldson invariants. The result is an expression of the integrals of the Segre classes as suitable coefficients in a generating function, involving only elementary functions and Seiberg-Witten invariants. (Section 5.3.)

In Chapter 6 we return to the study of localization. In Chapter 4, we had used a formal extension of the abelian localization theorem to reduce the partition function of MTNSW to contributions from DWT and TASW. Besides the topological argument that we used in Sections 4.3 and 4.4, another approach is possible, based on the semiclassical approximation. Instead of identifying the various terms in the Lagrangian as formal characteristic classes, we can evaluate the path-integral expression of the partition function by the rules of quantum field theory.

Since we are dealing with a topological theory, the semiclassical approximation is exact. As a consequence, we show that the path-integral localizes to the moduli spaces of instantons and abelian Seiberg-Witten pairs. As usual, the quadratic integrals in the normal directions to these moduli spaces lead to quotients of determinants of elliptic operators, which we identify in Sections 6.1 and 6.2.

In the case of abelian Seiberg-Witten solutions, if we assume again that the base manifold has simple type (so that the abelian Seiberg-Witten moduli space is of virtual dimension zero) then the operators in the numerator and denominator differ only by a factor of i , and so the quotient of determinants reduces to ± 1 . We check easily that the result of the semiclassical computation agrees in this case with the formal geometric result of section 4.4.

The integral in the normal directions to Donaldson moduli space, however, is more delicate. In this case, the quotient of determinants turns out to be non-trivial (to be precise, it is a differential form on Donaldson moduli space). By comparing with Section 4.3, we see that the quotient of determinants accounts for the same terms in the path-integral as the quotient of formal equivariant Euler classes occurring in that section.

This provides an alternative way of regularizing the quotient of formal equivariant Euler classes (recall that we had previously interpreted the quotient of Euler classes as the total Segre class of the index bundle). Contrasting this new regularization with our geometric argument in Section 4.3 leads to a conjecture relating two well-defined mathematical objects: the total Segre class of the index bundle and a certain quotient of determinants. The fact that both arise as possibilities of defining a quotient of formal Euler classes rigorously suggests that they might be equal. Notice that we are now faced with a purely mathematical conjecture, which we state independently of any path-integrals

in Section 7.1. We explain how our conjecture generalizes in fact similar results relating the first Chern class of the determinant line bundle to determinants of elliptic operators, see for instance [26] and [6].

The rest of Chapter 7 investigates the status of the conjecture in some particular cases. We explain why this conjecture should be understood as an equality at the level of cohomology classes rather than differential form representatives, a point which is particularly important in the infinite-dimensional setting of our problem. In fact, a locality argument shows that, if we work on a *fixed* Riemannian base X , the conjecture cannot hold at the level of differential forms, basically because the quotient of determinants involves some non-local quantities (Section 7.1).

However, if we rescale the metric on X , say by a constant factor μ , two interesting things happen in the adiabatic limit (i.e. as $\mu \rightarrow \infty$): the quotient of determinants admits an asymptotic expansion in powers of μ , and the coefficients of the expansion are *local* expressions (see Section 7.3). This means that the conjecture might hold at the level of differential forms *in the adiabatic limit*. In fact, we study this form of the conjecture in the case of line bundles over a Riemann surface in Section 7.4.

We extend the reasoning of Section 7.4 to any vector bundle over a two-dimensional base in Section 7.5. Although we currently don't have a complete proof, we present some of the necessary steps and a heuristic argument which suggests that the conjecture does hold in this case. Sections 7.4 and 7.5 also explain the major simplification which occurs in two dimensions: the asymptotic expansion of the ratio of determinants has a finite limit as $\mu \rightarrow \infty$.

This is no longer true in four dimensions (the case which is relevant to topological non-abelian Seiberg-Witten theory): indeed, an $O(\mu)$ term is present and one should reformulate the conjecture in order to get a sensible result (for instance, we should find a consistent way of removing the divergent part of the asymptotic expansion).

We conclude by an Appendix in which we describe our approach to the two problems mentioned earlier, the S-duality conjecture and the relationship between Donaldson and Seiberg-Witten invariants. The techniques explored in this thesis are unfortunately not sufficient for the solution of either of them.

According to Chapter 4, the partition function of MTNSW localizes to integrals over Donaldson and Seiberg-Witten moduli spaces. These are completely well-defined in the Seiberg-Witten case, since the moduli space is compact. In the Donaldson-Witten case we have to integrate over Donaldson moduli space, which is non-compact, hence we have to explain what the integrals mean. We follow the approach used in Donaldson-Witten theory, which works at least when $b_2^+ > 1$: we interpret the relevant integrals as integrals over the Uhlenbeck compactification of Donaldson moduli space.

With this assumption, we use the explicit results of Chapter 5 to obtain an expression for the

partition functions of TNSW and MTNSW in the case of a $K3$ surface. The resulting formula leads to conclusions which are in disagreement with both the S-duality conjecture and the relationship between the Donaldson and Seiberg-Witten invariants for this particular case. Therefore the integrals over Donaldson moduli space in topological non-abelian Seiberg-Witten theory have to be analyzed further.

Chapter 2

Equivariant Cohomology and the Mathai-Quillen Construction

In this chapter we review the Mathai-Quillen construction of differential form representatives of the Euler class of an oriented vector bundle [19] (Section 2.2). In section 2.3 we extend the Mathai-Quillen technique to the case of equivariant vector bundles in order to construct differential form representatives of the equivariant Euler class.

We work in the following framework: let G be a compact Lie group, M a compact oriented manifold, and P a principal G -bundle over M . Let V be an oriented vector space of dimension $2v$ with inner product and let $\rho_V : G \rightarrow SO(V)$ be a representation of G . Let E be the vector bundle $P \times_{\rho_V} V$.

In Sections 2.4-2.6 we use the Mathai-Quillen forms to generate integral formulae for certain characteristic numbers of the vector bundle. If the rank of E equals the dimension of the base manifold M , we obtain an integral formula for the Euler number (Section 2.4). In the case of a vector bundle whose rank is less than that of the base manifold, integrating the exterior product of the Euler form with suitable DeRham classes on the base manifold leads to integral formulae for intersection numbers on the zero set of a generic section (Section 2.5).

A priori, such formulae involve integration over the base manifold; we translate the integrals over M into equivalent integrals over the total space P of the principal bundle and the Lie algebra \mathfrak{g} of G . In Section 2.6 we discuss similar integral formulae (and their formulations as integrals over $P \times \mathfrak{g}$) for the case of equivariant bundles.

The importance of the formulae expressing characteristic numbers as integrals over $P \times \mathfrak{g}$ stems from their applications to topological quantum field theory, which will be discussed in Chapter 3.

2.1 Equivariant cohomology and the Chern-Weil theory

This section reviews the definition of (the Cartan model of) the equivariant cohomology of a manifold N endowed with the action of a connected compact Lie group G with Lie algebra \mathfrak{g} .

Given such an action, one is mainly interested in the orbit space N/G : for instance, this could be the configuration space of a physical system with symmetry group G . If the G -action is free then N/G is also a manifold, so if the main interest lies in its cohomology one can use the usual machinery of differential forms and DeRham cohomology. However, if the action has fixed points, the quotient space is usually no longer a manifold and $H^*(N/G)$ becomes harder to study by differential geometric means. This problem can be circumvented by the use of equivariant cohomology. It is known that, even in the topological category, equivariant cohomology is a substitute for the cohomology of the quotient space: in the case of a free action, the two coincide, but in general the equivariant cohomology contains more information. What is important for us is that in the smooth category there is a nice realization of equivariant cohomology as the cohomology of a complex of differential forms.

We will adopt this realization (usually called the Cartan model) as the definition of equivariant cohomology in the smooth category. Let N be the manifold on which the Lie group acts and consider the graded algebra of *equivariant differential forms*

$$\Omega_G(N) = [S(\mathfrak{g}^*) \otimes \Omega(N)]^G \otimes \mathbb{C}$$

whose elements are regarded as $\Omega(N)$ -valued invariant polynomials on \mathfrak{g} . The grading is the usual one on $\Omega(N)$ and twice the usual grading on $S(\mathfrak{g}^*)$. The superscript G denotes the subalgebra of G -invariant elements, where the action of G on differential forms is obtained from its action as diffeomorphisms of N and the action on $S(\mathfrak{g}^*)$ comes from the symmetric powers of the coadjoint action. The *equivariant DeRham differential* d_G on $\Omega_G(N)$ is defined by

$$d_G \alpha(X) = d[\alpha(X)] - \iota_{\rho(X)}[\alpha(X)], \quad \alpha \in \Omega_G(N),$$

where d is the usual DeRham differential, ι denotes interior product, and $\rho : \mathfrak{g} \rightarrow TN$ is the infinitesimal action of \mathfrak{g} , so that $\rho(X)$ is a vector field on N . We have

Lemma 2.1. $d_G^2 = 0$.

Proof. Since $d_G \alpha(X) = [(d - \iota_{\rho(X)})\alpha](X)$,

$$\begin{aligned} d_G^2 \alpha(X) &= [(d - \iota_{\rho(X)})^2 \alpha](X) \\ &= [(-d\iota_{\rho(X)} - \iota_{\rho(X)}d)\alpha](X) \\ &= -\mathcal{L}_{\rho(X)}\alpha(X) = 0 \end{aligned}$$

because α is G -invariant. We have used the Cartan formula for the Lie derivative of differential forms $\mathcal{L}_Y = d\iota_Y + \iota_Y d$.

Actually d_G is a graded differential of degree +1: if $\alpha \in S^i(\mathfrak{g}^*) \otimes \Omega^j(N)$ then $d\alpha \in S^i(\mathfrak{g}^*) \otimes \Omega^{j+1}(N)$ and $\iota_{\rho(\cdot)}\alpha \in S^{i+1}(\mathfrak{g}^*) \otimes \Omega^{j-1}(N)$. The first statement is obvious as is the fact that the form degree of $\iota_{\rho(\cdot)}\alpha$ is $j - 1$; the easiest way to see that $\iota_{\rho(\cdot)}\alpha$ is a polynomial of degree $i + 1$ is by looking at its scaling degree under $X \rightarrow tX$: indeed,

$$\iota_{\rho(tX)}\alpha(tX) = t\iota_{\rho(X)}\alpha(tX) = t^{i+1}\iota_{\rho(X)}\alpha(X).$$

Let us look at the particular case when $G = S^1$. The coadjoint action is trivial, so $\Omega_{S^1}(N) = \mathbb{C}[m] \otimes \Omega(N)^{S^1}$, where m is a generator of $(s^1)^*$ and

$$\Omega(N)^{S^1} = \{\omega \in \Omega(N) \mid \mathcal{L}_X\omega = 0\},$$

where X is the vector field on N corresponding to the generator of $\text{Lie}(S^1)$ dual to m . If $\omega \in \Omega_{S^1}(N)$ and $k \in \mathbb{N}$ then

$$d_{S^1}(m^k\omega) = m^k d\omega - m^{k+1}\iota_X\omega.$$

Definition 2.1. The G -equivariant cohomology of N (with complex coefficients), $H_G^*(N)$, is the cohomology of the above complex:

$$H_G^*(N) := H^*(\Omega_G(N), d_G).$$

We will now review the basic results of Chern-Weil theory. Assume that the G -action on N is free, so that the projection map $\pi : N \rightarrow N/G$ is a principal G -fibration. Choose any G -invariant connection form on N , $\theta \in \Omega^1(N) \otimes \mathfrak{g}$. Let $\Omega = d\omega + \frac{1}{2}[\theta, \theta] \in \Omega^2(N) \otimes \mathfrak{g}$ be its curvature.

Definition 2.2. The *Chern-Weil map*

$$CW_\theta : S(\mathfrak{g}^*) \rightarrow \Omega(N)$$

is the algebra homomorphism given on $X^* \in \mathfrak{g}^*$ by $CW_\theta(X^*) := X^*(\Omega) \in \Omega(N)$.

In other words, given any polynomial P on the Lie algebra \mathfrak{g} , CW_θ applies it to the Lie algebra part of the curvature Ω , the result being a (scalar) differential form on N .

There are several important subalgebras of $\Omega(N)$: the *horizontal forms* are those satisfying $\iota_{\rho(X)}\omega = 0$ for any $X \in \mathfrak{g}$ and the *invariant forms* are characterized by $\mathcal{L}_{\rho(X)}\omega = 0$ for any $X \in \mathfrak{g}$. The *basic subalgebra* $\Omega_{\text{bas}}(N)$ consists of differential forms that are both horizontal and invariant. It is easy to see that a form $\omega \in \Omega(N)$ is basic if and only if it is the pullback of a differential form on N/G . In fact, the pullback map π^* gives an isomorphism $\Omega(N/G) \rightarrow \Omega_{\text{bas}}(N)$.

Notice that, since the curvature Ω is a horizontal form, the image of the Chern-Weil map lies in the subalgebra of horizontal forms. If we restrict CW_θ to the G -invariant polynomials on \mathfrak{g} then the image is contained in the subalgebra of basic forms. Therefore, after identifying basic forms on N

with forms on N/G , the Chern-Weil map determines an algebra homomorphism (also denoted by CW_θ)

$$CW_\theta : S(\mathfrak{g}^*)^G \longrightarrow \Omega(N/G).$$

The key properties of the Chern-Weil map are

Proposition 2.1. *For any $P \in S(\mathfrak{g}^*)^G$, $CW_\theta(P)$ is a closed form on N/G .*

and

Proposition 2.2. *The cohomology class of $CW_\theta(P)$ is independent of the choice of a G -invariant connection θ .*

These properties lead to the differential-geometric construction of characteristic classes of the principal G -bundle N : namely, any G -invariant polynomial P as above determines a cohomology class in $H^*(N/G)$. For instance, if $G = \text{SU}(2)$ then the second Chern form equals

$$CW\left(\frac{1}{8\pi^2} \text{Tr } \Omega^2\right).$$

This example illustrates an integrality issue which is quite general: in order to obtain integral cohomology classes, one should normalize the elements of $S(\mathfrak{g}^*)$ appropriately.

The Chern-Weil map admits an extension to the algebra of equivariant differential forms, whose image lies in the invariant subalgebra, i.e. there exists a map

$$CW_\theta : \Omega_G(N) \longrightarrow \Omega(N)^G.$$

The connection θ determines a horizontal projection operator $\text{Hor} : \Omega(N) \longrightarrow \Omega_{\text{hor}}(N)$ so that the composition $\text{Hor} \circ CW_\theta$ can be regarded as a map

$$\text{Hor} \circ CW_\theta : \Omega_G(N) \longrightarrow \Omega_{\text{bas}}(N) \simeq \Omega(N/G).$$

Propositions 2.1 and 2.2 have some close cousins in this extended situation; namely,

Proposition 2.3. *The homomorphism $\text{Hor} \circ CW_\theta$ is a chain map with respect to the differentials d_G on $\Omega_G(N)$ and d on $\Omega(N/G)$; therefore $\text{Hor} \circ CW_\theta$ descends to a map from $H_G^*(N)$ to $H^*(N/G)$.*

Proposition 2.4. *The map*

$$\text{Hor} \circ CW_\theta : H_G^*(N) \longrightarrow H^*(N/G)$$

is independent of the connection θ .

The relationship between the Chern-Weil construction and equivariant cohomology is explained by

Theorem 2.1. *If the G -action is free then N/G is a manifold and we have an isomorphism*

$$H_G^*(N) \xrightarrow{\text{Hor} \circ CW} H^*(N/G),$$

The inverse of $\text{Hor} \circ CW$ is the pullback map π^ .*

2.2 The Mathai-Quillen construction of an Euler form

In this section we review the construction of differential form representatives of the Thom class of an oriented vector bundle E , following Mathai and Quillen [19]. Our goal is in fact to obtain explicit representatives for the Euler class: such representatives are obtained immediately from the Thom forms by pulling them back via suitable sections of E .

Recall the significance of the Thom class: given an oriented vector bundle E of rank r over a compact oriented manifold M , there is an isomorphism

$$H^*(M) \rightarrow H_{cv}^{*+r}(E),$$

where the subscript cv denotes compact support in the vertical directions. Explicitly, given a cohomology class β in $H^*(M)$, its image is obtained by pulling β back to E and taking its wedge product with the Thom class, $Th(E)$, (which is an element of $H_{cv}^r(E)$). The inverse of the Thom isomorphism is given by integrating differential forms on E with compact vertical supports along the fibres. It is also a well-known result that, given a section $s : M \rightarrow E$, the pullback $e(E) = s^*Th(E) \in H^*(M)$ is the Euler class of the bundle E . Usually the section s is taken to be the zero section, but we will need other choices when extending this construction to an infinite-dimensional case.

There is also an equivariant extension of the above construction: assume E is a G -equivariant bundle over M , endowed with a G -orientation (i.e. an orientation preserved by the action of G as linear maps between suitable fibres of E). The G -equivariant Thom isomorphism

$$H_G^*(M) \rightarrow H_{G,cv}^{*+r}(E)$$

is also obtained by pulling back and wedging with the G -equivariant Thom class $Th_G(E) \in \Omega_G(E)$. Similarly, $e_G(E) = s^*Th_G(E)$ equals the equivariant Euler class of E for any equivariant section $s : M \rightarrow E$.

The Mathai-Quillen construction uses a slight variation of the Thom isomorphism. Differential forms with compact support in each fibre are replaced by forms which are rapidly decreasing in the fibre directions. It is easy to check that the groups H_{cv}^* and H_{rd}^* (i.e. the cohomology of the complex of rapidly decaying forms) are isomorphic. The differential form representatives of the Thom class provided by the Mathai-Quillen construction are rapidly decaying, but are not compactly supported.

As stated in the introduction to this chapter, let G be a compact Lie group, M a compact oriented manifold, and P a principal G bundle over M . Let V be an oriented vector space of dimension 2ν with inner product and let $\rho_V : G \rightarrow SO(V)$ be a representation of G . Let $E = P \times_{\rho_V} V$. We later consider the equivariant situation, i.e. P will be an H -equivariant principal bundle, with H another Lie group.

We start by constructing a family of *universal Thom forms* $U_t \in \Omega_G(V)$, where $t \in \mathbb{R}$. To express it most conveniently for our purposes, we first recall the definition of the Berezin integral.

Definition 2.3. Let A be a commutative superalgebra and let $A \otimes \Lambda(V)$ be the graded tensor product of A with the exterior algebra of a vector space V . Choose a volume element vol on V , i.e. a non-zero element of $\Lambda^{\text{top}}(V)$. The *Berezin integral* is the map

$$A \otimes \Lambda(V) \xrightarrow{\mathcal{B}} A$$

which assigns to every element $a \in A \otimes \Lambda(V)$ the coefficient $\mathcal{B}(a)$ of its component $a^{\text{top}} \in A \otimes \Lambda^{\text{top}}(V) \simeq A \otimes \mathbb{R} \cdot \text{vol}$; $\mathcal{B}(a)$.

Note that if χ_1, \dots, χ_{2v} is an oriented orthonormal basis of V , so that $\Lambda(V) \simeq \Lambda[\chi_1, \dots, \chi_{2v}]$, Berezin integration amounts to taking the coefficient of $\chi_1 \chi_2 \cdots \chi_{2v}$.

Let now A be the commutative superalgebra $\Omega(V) \otimes S(\mathfrak{g}^*)$.

Definition 2.4. For each $t \in \mathbb{R}_+$, the *universal Thom form* $U_t \in S(\mathfrak{g}^*) \otimes \Omega(V)$ is given by

$$U_t := \left(\frac{1}{2\pi t} \right)^v \mathcal{B}_\chi \left(\exp \frac{t}{2} \left(-\|x\|^2 - 2\sqrt{-1} \sum_i dx_i \cdot \chi_i + \sum_{i,j} \chi_i \cdot \phi_V \chi_j \right) \right), \quad (2.1)$$

where x_i are coordinates on V (dual to the χ_i 's) and $\phi_V \in S^1(\mathfrak{g}^*) \otimes \mathfrak{so}(V)$ is obtained through the representation ρ from the *universal Weil element* $\phi \in S^1(\mathfrak{g}^*) \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g} \simeq \text{End}(\mathfrak{g})$ corresponding to the identity endomorphism. The dot in $\chi_i \cdot \phi_V \chi_j$ denotes exterior product in $\Lambda(V)$, whereas $\sum_i dx_i \cdot \chi_i$ is a shorthand for $\sum_i dx_i \otimes 1 \otimes \chi_i \in S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V)$.

In the sequel we will use a more compact notation; for instance, U_t will be written as

$$U_t = \left(\frac{1}{2\pi t} \right)^v e^{-\frac{t}{2}\|x\|^2} \mathcal{B}_\chi \exp \frac{t}{2} (-2\sqrt{-1} dx \chi + \chi \phi \chi),$$

where x , dx , and χ are to be thought of as vectors and ϕ as a matrix (so that, for instance, $dx \chi = \sum_i dx_i \cdot \chi_i$ etc.). We will also suppress the representation ρ from the notation: the operators $\iota_{\rho(X)}$ and $\mathcal{L}_{\rho(X)}$ acting on differential forms will be denoted by ι_X and \mathcal{L}_X .

The factor $e^{-\frac{t}{2}\|x\|^2}$ guarantees the rapid decay of U_t as differential forms on V . As for the other terms in the exponent,

$$\begin{aligned} \mathcal{B}_\chi \left(\exp (-\sqrt{-1} dx \chi) \right) &= \frac{(-1)^{2v} (\sqrt{-1})^{2v}}{(2v)!} \mathcal{B}_\chi ((2v)! dx_1 \chi_1 dx_2 \chi_2 \cdots dx_{2v} \chi_{2v}) \\ &= \frac{(-1)^v}{(2v)!} \mathcal{B}_\chi \left((-1)^{2v^2 - v} (2v)! dx_1 \cdots dx_{2v} \chi_1 \cdots \chi_{2v} \right) \\ &= dx_1 \cdots dx_{2v}, \end{aligned}$$

which shows that $\mathcal{B}_\chi \left(\exp (\sqrt{-1} dx \chi) \right)$ is just a fancy way of writing the volume form $dx_1 \cdots dx_{2v}$. Being a top degree form, this is obviously closed. As will be explained below, we are in fact more interested in constructing *equivariantly* closed forms – the role of the term $t \chi \phi \chi$ is precisely to provide an equivariant extension of the volume form.

The properties of the U_t 's are summarized in the following

Proposition 2.5. (i) $U_t \in \Omega_G(V)$ i.e. U is G -invariant;

(ii) $d_G U_t = 0$;

(iii) $\int_V U_t = 1$ (remember that only the top part of U_t as a differential form on V is integrated).

These properties express the fact that, for any real t , $U_t \in H_{G,rd}^{2v}(V)$ is a G -equivariant Thom form for the vector space V (viewed as a G -equivariant bundle over a point). Notice that the proposition also implies that the differential forms $U_t(s)$ for various sections s and real numbers t represent the same cohomology class in $H_{G,rd}^{2v}(V)$, namely the class corresponding to $1 \in H_G^0(\text{point})$.

Before proving the proposition we warm up with a special case, namely $v = 1$ and $G = SO(2)$ acting by rotations on \mathbb{R}^2 . Let X be the element

$$\begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} \in so(2).$$

The universal Thom form is a $\Omega(V)$ -valued polynomial on $so(2)$ whose value at X is, according to ((2.4)),

$$\begin{aligned} U_t(X) &= \frac{1}{2\pi t} B_X \left(\exp \frac{t}{2} \left(-(x_1^2 + x_2^2) - 2\sqrt{-1} (dx_1 \chi_1 + dx_2 \chi_2) + (x_1 \chi_2 - x_2 \chi_1) \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \right) \right) \\ &= \frac{t}{4\pi} \exp \left(-\frac{t}{2}(x_1^2 + x_2^2) \right) B_X \left((-1) (dx_1 \chi_1 + dx_2 \chi_2)^2 - \frac{1}{t} m \chi_1 \chi_2 \right) \\ &= \frac{t}{4\pi} \exp \left(-\frac{t}{2}(x_1^2 + x_2^2) \right) (2dx_1 dx_2 - \frac{m}{t}). \end{aligned}$$

Hence

$$U_t = \frac{t}{4\pi} \exp \left(-\frac{t}{2}(x_1^2 + x_2^2) \right) (2dx_1 dx_2 - \frac{m}{t})$$

as an element of $\Omega(\mathbb{R}^2)^{SO(2)} \otimes \mathbb{C}[m]$.

Let us now check the statements of Proposition 2.5 in this special case. $SO(2)$ -invariance is equivalent to $\mathcal{L}_X U_t = 0$, where X is the vector field $(-mx_2, mx_1)$ on \mathbb{R}^2 . Since the group is abelian, $\mathcal{L}_X(m) = 0$ and both the exponential factor $\exp((-t/2)(x_1^2 + x_2^2))$ and the volume form $dx_1 dx_2$ are $SO(2)$ -invariant. Property (iii) follows from the normalization of the Gaussian integral. Showing that U_t is equivariantly closed is an easy computation. We have

$$\begin{aligned} d_{SO(2)} x_i(X) &= dx_i - \iota_X(x_i) = dx_i \\ d_{SO(2)}(dx_1)(X) &= d(dx_1) - \iota_X(dx_1) = mx_2 \\ d_{SO(2)}(dx_2)(X) &= d(dx_2) - \iota_X(dx_2) = -mx_1 \\ d_{SO(2)}(m) &= 0, \end{aligned}$$

so that

$$\begin{aligned} d_{SO(2)} U_t(X) &= \frac{t}{4\pi} \exp \left(-\frac{t}{2}(x_1^2 + x_2^2) \right) \\ &\quad \times (-t(x_1 dx_1 + x_2 dx_2)(2dx_1 dx_2 - tm) - 2m(x_1 dx_1 + x_2 dx_2)) = 0. \end{aligned}$$

Proof of Proposition 2.5. In order to establish the G -invariance of U_t we will extend the action of \mathfrak{g} on $S(\mathfrak{g}^*) \otimes \Omega(V)$ to the algebra $S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V)$. For $X \in \mathfrak{g}$, the operator

$$\mathcal{L}_X : S(\mathfrak{g}^*) \otimes \Omega(V) \longrightarrow S(\mathfrak{g}^*) \otimes \Omega(V)$$

is the (ungraded) derivation given by

$$\mathcal{L}_X = ad_X^* \otimes \text{id} + \text{id} \otimes \mathcal{L}_X.$$

The operator ad_X^* is the coadjoint action of X ; if $X^* \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$ then $ad_X^*(X^*)(Y) = X^*([X, Y])$. The \mathcal{L}_X in $\text{id} \otimes \mathcal{L}_X$ stands for the Lie derivative on $\Omega(V)$. The exterior powers of the representation $\rho : \mathfrak{g} \rightarrow so(V)$ determine, for any $X \in \mathfrak{g}$, an operator $\lambda_X : \Lambda(V) \rightarrow \Lambda(V)$. The desired extension of \mathcal{L}_X is

$$\tilde{\mathcal{L}}_X : S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V) \longrightarrow S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V)$$

$$\tilde{\mathcal{L}}_X = \mathcal{L}_X \otimes \text{id} + \text{id} \otimes \lambda_X.$$

Lemma 2.2. (i) $B_X \lambda_X = 0$; (ii) $\mathcal{L}_X B_X = B_X \tilde{\mathcal{L}}_X$.

Proof. (i) Since λ_X preserves the degree in $\Lambda(V)$, it is enough to show that $B_X \lambda_X(\chi_1 \chi_2 \cdots \chi_{2\nu}) = 0$. We have

$$\lambda_X(\chi_1 \chi_2 \cdots \chi_{2\nu}) = \sum_{i=1}^{2\nu} \chi_1 \cdots \chi_{i-1} (X \chi_i) \chi_{i+1} \cdots \chi_{2\nu}.$$

Since X is a skew-symmetric transformation of V we have $(X \chi_i) \cdot \chi_i = 0$; so

$$X \chi_i \in \text{Span}(\chi_1, \dots, \chi_{i-1}, \chi_{i+1}, \dots, \chi_{2\nu}).$$

This shows that $\lambda_X(\chi_1 \chi_2 \cdots \chi_{2\nu}) = 0$.

(ii) If $a \in S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V)$ then a can be written as $a = a_{\text{top}} \chi_1 \chi_2 \cdots \chi_{2\nu} + \text{lower order in } \chi$; hence

$$\mathcal{L}_X B_X a = \mathcal{L}_X a_{\text{top}}.$$

But

$$\begin{aligned} B_X \tilde{\mathcal{L}}_X a &= B_X (\mathcal{L}_X a_{\text{top}} \chi_1 \chi_2 \cdots \chi_{2\nu} + a_{\text{top}} \lambda_X(\chi_1 \chi_2 \cdots \chi_{2\nu}) + \text{lower order in } \chi) \\ &= \mathcal{L}_X a_{\text{top}}, \end{aligned}$$

because of (i).

We can now prove the G -invariance of U_t , i.e. $\mathcal{L}_X U_t = 0$ for any $X \in \mathfrak{g}$. From the previous lemma,

$$\begin{aligned} (2\pi t)^\nu \mathcal{L}_X U_t &= B_X \left(\tilde{\mathcal{L}}_X \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \right) \right) \\ &= B_X \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \tilde{\mathcal{L}}_X \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \right). \end{aligned}$$

Now $\tilde{\mathcal{L}}_X(tx^2/2) = \tilde{\mathcal{L}}_X(dx \cdot \chi) = 0$ because X is a skew-symmetric transformation. This can be seen as follows:

$$\tilde{\mathcal{L}}_X(dx \cdot \chi) = \mathcal{L}_X(dx) \cdot \chi + dx \cdot X\chi = X(dx) \cdot \chi + dx \cdot X\chi = 0.$$

As for $\tilde{\mathcal{L}}_X(\chi\phi\chi)$, recall that $\chi\phi\chi$ is actually a linear map on \mathfrak{g} with values in $\Lambda(V)$: if $Y \in \mathfrak{g}$ then

$$\chi\phi\chi(Y) = \chi \cdot Y\chi.$$

(As explained before, the symbol χ denotes a vector whose components are the basis elements $\chi_1, \dots, \chi_{2\nu}$. The \cdot is just the dot product of such vectors.) Therefore

$$\begin{aligned} \left[\tilde{\mathcal{L}}_X(\chi\phi\chi) \right] (Y) &= [(X\chi)\phi\chi](Y) + \chi\phi([X, Y])\chi + [\chi\phi(X\chi)](Y) \\ &= (X\chi) \cdot (Y\chi) + \chi \cdot (XY - YX)\chi + \chi \cdot Y(X\chi) \\ &= (X\chi) \cdot (Y\chi) + \chi \cdot (XY\chi) = 0, \end{aligned}$$

because X is skew-symmetric. Hence

$$\tilde{\mathcal{L}}_X(-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) = 0,$$

and $\mathcal{L}_X U_i = 0$, as claimed.

In order to prove part (ii) we introduce some interior product operators as follows. We first define

$$\iota_{\chi_i} : \Lambda(V) \longrightarrow \Lambda(V)$$

as the graded derivation satisfying $\iota_{\chi_i}(\chi_j) = \delta_{ij}$.

Lemma 2.3. $B_\chi \iota_{\chi_i} = 0$ for any i .

Proof. Since ι_{χ_i} lowers the degree by 1, there is no top part left.

Definition 2.5. Let

$$\mathcal{I} : S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V) \longrightarrow S(\mathfrak{g}^*) \otimes \Omega(V) \otimes \Lambda(V)$$

be the operator $\mathcal{I} = \text{id} \otimes \text{id} \otimes \iota_{\chi}$, where

$$\iota_{\chi} : \Lambda(\cdot) \longrightarrow C^\infty(V) \otimes \Lambda(V), \quad \iota_{\chi} = \sum_i \chi_i \iota_{\chi_i}.$$

The previous lemma shows that $B_\chi \mathcal{I} = 0$. This enables us to prove that $d_G U_i = 0$:

$$\begin{aligned} (2\pi t)^\nu d_G U_i &= d_G B_\chi \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) \right) \\ &= B_\chi d_G \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) \right) \\ &= B_\chi \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) d_G \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) \right) \\ &= B_\chi \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi\phi\chi) (-tx dx + t\sqrt{-1}(\phi\chi)\chi) \right). \end{aligned}$$

For the last equality we have used the properties of d_G

$$d_G x = dx \quad d_G(dx) = -\phi x \quad d_G \phi = 0.$$

The key remark is that

$$\begin{aligned} \mathcal{I}(-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) &= -2\sqrt{-1} x dx + x \cdot \phi \chi - \chi \cdot \phi x \\ &= -2\sqrt{-1} x dx - 2(\phi x) \chi, \end{aligned}$$

since ϕ is skew-symmetric. This shows that

$$-tx dx + t\sqrt{-1}(\phi x)\chi = \frac{t}{2\sqrt{-1}} \mathcal{I} \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi)$$

and so

$$\begin{aligned} (2\pi t)^v d_G U_t &= \frac{t}{2\sqrt{-1}} B_\chi \left(\exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \mathcal{I} \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \right) \\ &= \frac{t}{2\sqrt{-1}} B_\chi \mathcal{I} \exp \frac{t}{2} (-x^2 - 2\sqrt{-1} dx \chi + \chi \phi \chi) \\ &= 0. \end{aligned}$$

Part (iii) is easy: indeed, only the term $-\sqrt{-1} dx \chi$ in Definition 2.4 contributes to the top degree part of U_t as a differential form on V , and we have already seen that

$$B_\chi(-\sqrt{-1} dx \chi) = dx_1 \cdots dx_{2v}.$$

Together with the chosen normalization, this proves (iii).

The elements U_t can be used to construct Thom forms for E . Recall that $\Omega(E) = \Omega(P \times_G V) \simeq \Omega_{bas}(P \times V)$, and the homomorphism

$$\Omega_G(V) \xrightarrow{CW} \Omega(P \times V)^G \xrightarrow{Hor} \Omega(P \times V)_{bas} \simeq \Omega(E)$$

is a chain map with respect to the differentials d_G and d on $\Omega_G(V)$ and $\Omega(E)$, respectively. Therefore $Th_t = Hor \circ CW(U_t)$ is a Thom form for the vector bundle E ; that is

Proposition 2.6. (i) Th_t is a differential form of degree $2v$ on E ;

(ii) $dTh_t = 0$;

(iii) $\int_{fibres} Th_t = 1$.

Proof. The image of ϕ by the Chern-Weil homomorphism is the curvature Ω , a 2-form. After substituting Ω for ϕ in (2.4), each term in the exponent has the same degree as a differential form as its degree in χ . The Berezin integral picks the term of degree $2v$ in χ and so Th_t also has degree $2v$ as a differential form. (ii) follows from Proposition 2.3 and (iii) from Proposition 2.5 part (iii).

Differential form representatives $e_t(E, s)$ for the Euler class are obtained by pulling back $Th_t(E)$ via a section $s : M \rightarrow E$ (which can be viewed alternatively as an equivariant map $s : P \rightarrow V$). In fact, the situation is summarized by the following commutative diagram:

$$\begin{array}{ccc} U_t \in \Omega_G(V) & \xrightarrow{Hor \circ CW} & \Omega(E) \ni Th_t(E) \\ \downarrow s^* & & \downarrow s^* \\ W_t(s) \in \Omega_G(P) & \xrightarrow{Hor \circ CW} & \Omega(M) \ni e_t(E, s). \end{array} \quad (2.2)$$

Therefore the image $W_t(s)$ of $e_t(E, s)$ in the equivariant cohomology $H_G^*(P)$ is expressed as the pullback of U_t through s :

$$W_t(s) = B_\chi \left(\left(\frac{1}{2\pi t} \right)^v \exp \frac{t}{2} \left(-\|s\|^2 - 2\sqrt{-1} ds \chi + \chi \phi \chi \right) \right). \quad (2.3)$$

As an application, the above construction can be used to give an analytic proof of the Poincaré-Hopf theorem. If $\dim V = \dim M$ then the Mathai-Quillen construction provides an integral formula for the Euler number $e_\#(E) = \int_M e_t(E)$; note that any of the forms e_t can be used. For $t \rightarrow \infty$, (2.3) shows that the integral localizes to the zeroes of the section s ; if s is transversal to the zero section, $e_\#(E)$ can thus be reduced to a sum of contributions from the finitely many zeroes of s . These contributions turn out, after a more careful analysis, to be just ± 1 's, corresponding to a certain orientation.

2.3 The case of equivariant vector bundles

We now extend the Mathai-Quillen construction to the case of equivariant vector bundles. Let P be an H -equivariant principal G -bundle over M , where G and H are compact Lie groups and M is compact oriented manifold. The H -equivariance means that H acts on both P and M and its action on P commutes with that of G . Let V be (as before) an oriented vector space of dimension $2v$ endowed with an inner product and representations $\rho : G \rightarrow SO(V)$ and $\lambda : H \rightarrow SO(V)$. The H -action on P induces an action of H on the associated bundle $E = P \times_G V$, so that E is naturally an H -equivariant bundle over M . Also assume that H is connected, which implies that its action on E preserves the given orientation (so that the equivariant Thom isomorphism holds without further restrictions).

One change needed in the construction carried out in the previous section is the replacement of the Chern-Weil homomorphism by its equivariant analogue. Choose an H -invariant connection on the principal bundle P . Denote by $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ the corresponding connection form and by $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ the curvature form.

Definition 2.6. (i) The *moment* \mathcal{J} of the connection θ is the element of $\mathfrak{h}^* \otimes C^\infty(P) \otimes \mathfrak{g}$ given by $\mathcal{J}(Y) = \iota_Y \theta$ for any $y \in \mathfrak{h}$. The symbol Y in the right-hand side of the equality denotes the vector

field on P determined by the element $Y \in \mathfrak{h}$ through the infinitesimal action of \mathfrak{h} on P .

(ii) The H -equivariant curvature Ω_H of the connection θ , is the element of $S(\mathfrak{g}^*) \otimes \Omega(P) \otimes \mathfrak{g}$ given by $\Omega_H := \Omega - \mathcal{J}$.

We now summarize the properties of the equivariant curvature.

Proposition 2.7. (i) For any $X \in \mathfrak{g}$,

$$\mathcal{L}_X(\Omega_H) = -ad_X \circ \Omega_H;$$

(ii) Since θ is an H -invariant connection form, i.e. $\mathcal{L}_Y\theta = 0$ for any $Y \in \mathfrak{h}$, the 1-form θ can be regarded as an element of $\Omega_H^1(P) \otimes \mathfrak{g}$. In this context,

$$\Omega_H = d_H\theta + \frac{1}{2}[\theta, \theta]$$

(the equivariant Maurer-Cartan equation).

(iii) If D denotes the covariant derivative operator associated with the connection θ , and we define the equivariant covariant derivative D_H on $\Omega_H(P)$ by

$$D_H\omega(Y) := D\omega(Y) - \iota_Y[\omega(Y)] \quad \text{for } Y \in \mathfrak{h},$$

then

$$D_H\Omega_H = 0$$

(the equivariant Bianchi identity).

Proof. (i) It is well-known that $\mathcal{L}_X\Omega = -ad_X \circ \Omega$. It remains to check that \mathcal{J} has the same property, namely $\mathcal{L}_X\mathcal{J} = -ad_X \mathcal{J}$ for any $X \in \mathfrak{g}$. Indeed, if $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ then

$$\mathcal{L}_X(\iota_Y\theta) = \iota_{\mathcal{L}_X Y}\theta + \iota_Y(-ad_X\theta) = -ad_X(\iota_Y\theta).$$

($\mathcal{L}_X Y = 0$ because the actions of G and H commute.) Therefore

$$\mathcal{L}_X(\mathcal{J}(Y)) = ad_X(\mathcal{J}(Y))$$

for any $Y \in \mathfrak{h}$, as stated.

(ii) $d_H\theta(Y) = d\theta - \iota_Y\theta = d\theta - \mathcal{J}(Y)$ or, as elements of $\Omega_H^2(P) \otimes \mathfrak{g}$,

$$d_H\theta + \frac{1}{2}[\theta, \theta] = d\theta - \mathcal{J} + \frac{1}{2}[\theta, \theta] = \Omega_H.$$

(iii) We know that

$$\begin{aligned} D_H\Omega_H(Y) &= D(\Omega - \mathcal{J}(Y)) - \iota_Y(\Omega - \mathcal{J}(Y)) \\ &= -D(\mathcal{J}(Y)) - \iota_Y\Omega \end{aligned}$$

because $\mathcal{J}(Y)$ is a zero-form so $\iota_Y \mathcal{J}(Y) = 0$, and Ω satisfies the usual Bianchi identity $D\Omega = 0$.

But

$$\begin{aligned} d(\mathcal{J}(Y)) &= d\iota_Y\theta = \mathcal{L}_X\theta - \iota_Y d\theta = -\iota_Y \left(\Omega - v\frac{1}{2}[\theta, \theta] \right) \\ &= -\iota_Y\Omega + [\iota_Y\theta, \theta]. \end{aligned}$$

The first term is obviously horizontal whereas the second is vertical because $\iota_Y\theta$ is a zero-form and θ is a vertical 1-form, hence

$$D(\mathcal{J}(Y)) = d[(\mathcal{J}(Y))]_{hor} = -\iota_Y\Omega,$$

and so $D_H\Omega_H(Y) = 0$.

Definition 2.7. The H -equivariant Chern-Weil map

$$CW_{H,\theta} : S(\mathfrak{g}^*) \longrightarrow \Omega_H(P)$$

is the algebra homomorphism given on $X^* \in \mathfrak{g}^*$ by $CW_{H,\theta}(X^*) := X^*(\Omega_H) \in \Omega_H(P)$. This map extends to a map from the algebra of $G \times H$ -equivariant forms to the algebra of G -invariant, H -equivariant forms on P ,

$$CW_{H,\theta} : \Omega_{G \times H}(P) \longrightarrow \Omega_H(P)^G.$$

By composing this map with the horizontal projection operator determined by the connection θ we get the H -equivariant Chern-Weil homomorphism

$$Hor \circ CW_{H,\theta} : \Omega_{G \times H}(P) \longrightarrow \Omega_{H,bas}(P) \simeq \Omega_H(M).$$

The following propositions extend the properties of the usual Chern-Weil homomorphism.

Proposition 2.8. *The homomorphism $Hor \circ CW_\theta$ is a chain map with respect to the differentials $d_{G \times H}$ on $\Omega_G(P)$ and d_H on $\Omega_H(M)$, therefore $Hor \circ CW_\theta$ descends to a map from $H_{G \times H}^*(P)$ to $H_H^*(M)$.*

Proposition 2.9. *The map*

$$Hor \circ CW_{H,\theta} : H_{G \times H}^*(P) \longrightarrow H_H^*(M)$$

is independent of the connection θ .

Proposition 2.10. *The map $Hor \circ CW_H$ is an isomorphism between $H_{G \times H}^*(P)$ and $H_H^*(M)$.*

We can now extend the Mathai-Quillen construction to equivariant vector bundles. Start with the universal Thom forms $U_t \in \Omega_{G \times H}(V)$ constructed in Section 2. Let

$$Th_{H,\theta}(E) := Hor \circ CW_H(U_t) \in \Omega_H(E).$$

The above map is the composition

$$\Omega_{G \times H}(V) \xrightarrow{CW_H} \Omega_H(P \times V)^G \xrightarrow{Hor} \Omega_H(P \times V)_{bas} \simeq \Omega_H(E).$$

We claim that $Th_{H,t}(E)$ is an H -equivariant Thom form for the vector bundle E . The proof of the following assertion is identical to that of Proposition 2.6.

Proposition 2.11. (i) *The degree of $Th_{H,t}$ as an H -equivariant form on E is $2v$;*

(ii) $d_H Th_{H,t} = 0$;

(iii) $\int_{fibre} Th_{H,t} = 1$.

The analogue of diagram (2.2) in the equivariant context is

$$\begin{array}{ccc} U_t \in \Omega_{G \times H}(V) & \xrightarrow{Hor \circ CW_H} & \Omega_H(E) \ni Th_{H,t}(E) \\ \downarrow s^* & & \downarrow s^* \\ W_{H,t}(s) \in \Omega_{G \times H}(P) & \xrightarrow{Hor \circ CW_H} & \Omega_H(M) \ni e_{H,t}(E, s), \end{array} \quad (2.4)$$

where $s : M \rightarrow E$ is an H -equivariant section. Explicitly,

$$W_{H,t}(s) = B_\chi \left(\left(\frac{1}{2\pi t} \right)^v \exp \frac{t}{2} \left(-\|s\|^2 - 2\sqrt{-1} ds \chi + \chi(\phi_G + \phi_H)\chi \right) \right). \quad (2.5)$$

The elements ϕ_G and ϕ_H are the universal Weil elements for the actions of G and H , respectively.

Notice that $W_{H,t}(s)$ itself is an equivariant Euler form: namely, it is a representative for the $G \times H$ -equivariant Euler class of the bundle $P \times V \rightarrow P$ (which, although topologically trivial, is not trivial as a $G \times H$ -equivariant bundle). Note however a subtlety about this statement which will become very important in Chapter 4. Assume for simplicity that H is abelian (in our applications it will be S^1); when regarding the bundle $P \times V$ as an H -equivariant bundle, the action of H on V is really the *opposite* of the chosen representation $\lambda : H \rightarrow \text{SO}(V)$. This is due to the definition of the associated bundle—recall that $P \times_G V$ is the quotient of $P \times V$ by the G -action given by $g(p, v) = (pg, g^{-1}v)$. The same fact can be seen by looking at the differential form $W_t(s)$ introduced in diagram (2.2). $W_t(s)$ represents the G -equivariant Euler class of the trivial bundle $P \times V \rightarrow P$, but it can be also interpreted as the result of applying (2.5) to the case of the trivial structure group G and changing the name of H to G .

2.4 Integration over P versus integration over M

In the application to the proof of the Poincaré-Hopf theorem, we use the expression of the Euler number as an integral over M . For later purposes, we will have to express the Euler number of the vector bundle E when $\text{rk } E = \dim M$ as an integral over the total space of the bundle P . This section shows how to compare integrals over P and M .

The easiest way of relating integrals over the manifold M and the total space of the bundle P is to introduce a vertical volume element v (i.e. a volume element along the fibres of $\pi : P \rightarrow M$), normalized so that its integral over each fibre is equal to one. For $\beta \in \Omega^{\text{top}}(M)$ we would then have

$$\int_P \pi^* \beta \wedge v = \int_M \beta.$$

We now turn to the G -equivariant extension of this property.

Definition 2.8. The *equivariant vertical volume element* γ_G is the element of $\Omega(P) \otimes S(\mathfrak{g})$ constructed as follows: let $\lambda_1, \dots, \lambda_{\dim \mathfrak{g}}$ be an orthonormal basis of \mathfrak{g} with respect to the Killing form, normalized so that $\text{vol}(G) = 1$. Choose a G -invariant connection θ on P with curvature Ω . Then

$$\gamma_G := e^{\sum_{\alpha} \Omega_{\alpha} \otimes \lambda_{\alpha}} \mathcal{B}_{\eta} \left(e^{\sum_{\alpha} \theta_{\alpha} \otimes \eta_{\alpha}} \right) \stackrel{\text{notation}}{=} e^{\Omega \otimes \lambda} \mathcal{B}_{\eta} \left(e^{\theta \otimes \eta} \right), \quad (2.6)$$

where \mathcal{B}_{η} denotes Berezin integration with respect to the Lie algebra variable η .

We explain more carefully the notation in the above formula: the differential forms θ and Ω take values in the Lie algebra \mathfrak{g} . Their components in the chosen basis of \mathfrak{g} are denoted by θ_{α} and Ω_{α} , respectively. The exponential $e^{\sum_{\alpha} \Omega_{\alpha} \otimes \lambda_{\alpha}}$ is to be computed in the algebra $\Omega(P) \otimes S(\mathfrak{g})$, whereas the Berezin integrand is in $\Omega(P) \otimes \Lambda(\mathfrak{g})$. Actually $\eta_1, \dots, \eta_{\dim \mathfrak{g}}$ are just a different notation for the elements of the same basis of \mathfrak{g} chosen before. Note that the Berezin integral in the above formula is just a fancy way of writing a vertical volume element.

The role of the first exponential factor is explained by the next proposition.

Proposition 2.12. (i) *The linear functional on $\Omega_G(P)$ defined by*

$$\alpha \mapsto \int_P \langle \alpha \wedge \gamma_G \rangle,$$

where $\langle \alpha \wedge \gamma_G \rangle$ denotes the contraction of the polynomial parts of α and γ_G , vanishes on d_G -exact forms, hence it descends to $H_G^*(P)$.

(ii) *Let α^u represent a cohomology class in $H_G^*(P)$ and α^d be its image in $H^*(M)$ by the isomorphism $\text{Hor} \circ \text{CW}$. Then*

$$\int_P \langle \alpha^u \wedge \gamma_G \rangle = \int_M \alpha^d. \quad (2.7)$$

Recall that $\alpha^u \in \Omega(P) \otimes S(\mathfrak{g}^*)$ and $\gamma_G \in \Omega(P) \otimes S(\mathfrak{g})$. The $\langle \cdot, \cdot \rangle$ symbol denotes the natural pairing between $S(\mathfrak{g}^*)$ and $S(\mathfrak{g})$. To better understand the proposition, note that when α^u is just the pullback of a top form on M , the only part of γ_G which matters is the usual vertical volume element (look at the degrees of the forms involved). However, the proposition wouldn't be true for all $\alpha^u \in H_G^*(P)$ if we used the usual vertical volume form instead of γ_G . It is in this sense that γ_G provides the suitable equivariant generalization.

The proposition is proved by Austin and Braam in [3]. The authors further explore the significance of γ_G from the (co-)homological viewpoint. They first define equivariant *homology* and show

that it can be computed by a complex of differential forms very similar to the Cartan model. It then turns out that γ_G is a cycle in this complex, and hence represents an equivariant homology class. The functional defined in the proposition is interpreted as a pairing between homology and cohomology.

The geometric content of the proposition (homology and cohomology set aside) is very easy to explain: the pairing of α and γ_G amounts to substituting the curvature in the polynomial part of α ; the result is then multiplied by the vertical volume form. This actually proves (ii) if α is the pull-back of a form on M , since then 2.7 reduces to the usual property of the vertical volume element. According to Proposition 2.1 it is therefore enough to prove part (i). We will do this in the proof of Proposition 2.13.

With the physical applications in mind we reformulate (2.12) in such a way that the pairing of the polynomial parts of equivariant forms is also expressed as an integral. This is achieved by means of the Fourier transform (normalized so that $\hat{1} = \delta$). If P and Q are polynomial functions on \mathfrak{g} then

$$\begin{aligned} \langle P(x), Q(x) \rangle_{\text{polyn}} &= \langle \widehat{P(\xi)}, Q(-ix) \rangle_{\text{distributions}} \\ &\stackrel{\text{formally}}{=} \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \int_{\xi \in \mathfrak{g}} \int_{x \in \mathfrak{g}} e^{i\langle x, \xi \rangle} P(\xi) Q(-ix). \end{aligned} \quad (2.8)$$

A more accurate way of writing the last equality is to insert an exponential convergence factor (i.e. a rapidly decaying test function):

$$\langle P(x), Q(x) \rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \int_{x \in \mathfrak{g}} \int_{\xi \in \mathfrak{g}} e^{i\langle x, \xi \rangle - \epsilon \langle \xi, \xi \rangle} P(\xi) Q(-ix). \quad (2.9)$$

Notice that in (2.12) the pairing was between an element in $S(\mathfrak{g}^*)$ and one in $S(\mathfrak{g})$, but this can be identified with the inner product in $S(\mathfrak{g}^*)$ via our choice of a nondegenerate bilinear form on \mathfrak{g} .

We can now obtain a formula expressing the Euler number as an integral over P by using the Mathai-Quillen construction in conjunction with (2.12) and (2.8):

$$e_{\#}(E) = \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \int_P e^{-\frac{1}{2} \|\theta\|^2} \mathcal{B}_{\eta}(e^{\theta \otimes \eta}) \int_{\lambda \in \mathfrak{g}} \int_{\phi \in \mathfrak{g}} e^{i\langle \phi, \lambda \rangle} e^{-i\Omega \otimes \lambda} \mathcal{B}_{\chi} \left(e^{-it d\theta \chi + \frac{1}{2} \chi \phi \chi} \right). \quad (2.10)$$

The variable ξ from (2.8) is in our previous notations ϕ whereas x corresponds to λ . As remarked before, to make this totally rigorous we should include the convergence factor as in (2.9) and then take the limit as $\epsilon \rightarrow 0$. Note here an additional strength of Proposition 2.12: instead of using the actual Euler form, we can use the simpler form $W_t(s)$ defined by the equation (2.3), since we know that the Euler form is obtained by applying $Hor \circ CW$ to it; this makes it unnecessary to apply the Chern-Weil homomorphism explicitly. We now summarize the discussion above.

Theorem 2.2. *Let E be a vector bundle as discussed above. Its Euler number can then be computed*

by the formula

$$e_{\#}(E) = \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \int_P e^{-\frac{1}{2}\|s\|^2} B_{\eta}(e^{\theta \otimes \eta}) \int \int_{\lambda, \phi \in \mathfrak{g}} e^{i(\phi, \lambda) - \epsilon(\phi, \phi)} e^{-i\Omega \otimes \lambda} B_{\chi} \left(e^{-it ds\chi + \frac{1}{2}\chi\phi\chi} \right). \quad (2.11)$$

This formula can be made even more explicit: the choice of a G -invariant Riemannian metric on P determines a particular connection (with connection form θ and curvature Ω) whose horizontal distribution is given by the orthogonal complements to the tangent spaces to the orbits of the G -action. If $C : \mathfrak{g} \rightarrow TP$ denotes the infinitesimal action of \mathfrak{g} and $C^* \in \Omega^1(P) \otimes \mathfrak{g}$ is its adjoint with respect to the inner products on \mathfrak{g} and TP , then

$$\theta = (C^*C)^{-1}C^* \quad \Omega_{Hor} = H\sigma(d\theta + \frac{1}{2}[\theta, \theta]) = (C^*C)^{-1}dC^*. \quad (2.12)$$

In the integral formula for the Euler number we can use Ω_{hor} instead of Ω since γ_G is a top degree vertical form. If we also make the change of variable $\lambda \rightarrow (C^*C)\lambda$ we can incorporate the Jacobian factor in the fermionic integral over η to write $\det(C^*C) B_{\eta}(e^{\theta \otimes \eta}) = B_{\eta}(e^{\langle \cdot, C\eta \rangle})$. The exponent $\langle \cdot, C\eta \rangle$ is a one-form on P whose value at the vector field ψ is the inner product of ψ with $C\eta$ (which is also a vector field on P). Borrowing the notation used in the physics literature, we could write this form as $\langle \psi, C\eta \rangle$; but the simplest notation comes from the fact that $\langle \psi, C\eta \rangle = \langle C^*\psi, \eta \rangle$, so $\langle \cdot, C\eta \rangle = \langle C^*, \eta \rangle$. We obtain

$$e_{\#}(E) = \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \int_P e^{-\frac{1}{2}\|s\|^2} B_{\eta}(e^{\langle C^*, \eta \rangle}) \int \int_{\lambda, \phi \in \mathfrak{g}} e^{i\langle \phi, C^*C\lambda \rangle} e^{-i dC^* \otimes \lambda} B_{\chi} \left(e^{-it ds\chi + \frac{1}{2}\chi\phi\chi} \right) \quad (2.13)$$

The last formula is particularly nice for the applications to quantum field theory since it involves no non-local operators (for instance inverses of differential operators, which appear in the a priori expression of the connection 1-form). Apparently the expression (2.13) doesn't involve any connection on P , relying instead on the G -invariant Riemannian metric. But it is not hard to see that the connection defined in (2.12) is actually involved. Namely, if

$$\Gamma_G := B_{\eta} \left(e^{\langle C^*, \eta \rangle} \right) e^{i\langle \phi, C^*C\lambda \rangle - i dC^* \otimes \lambda} \quad (2.14)$$

then the λ integration yields a δ -function in ϕ centered at $\phi = (C^*C)^{-1}dC^*$, i.e. ϕ is to be replaced by the curvature, as in the comment following Proposition 2.12.

There is one more cosmetic treatment which can be applied to (2.13) to make it look closer to standard formulae in quantum field theory. Namely Γ_G can be replaced by

$$\Gamma_G := B_{\eta} \left(e^{\langle C^*, \eta \rangle} \right) e^{it\langle \phi, C^*C\lambda \rangle - it dC^* \otimes \lambda}, \quad (2.15)$$

and the formula for the Euler number by

$$e_{\#}(E) = \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \int_P \int \int_{\lambda, \phi \in \mathfrak{g}} B_{\eta} B_{\chi} \exp \frac{t}{2} \left(-\|s\|^2 + \langle C^*, \eta \rangle + i\langle \phi, C^*C\lambda \rangle - i dC^* \otimes \lambda - 2i ds\chi + \chi\phi\chi \right), \quad (2.16)$$

in which the appearance of the 'coupling constant' t is exactly as in formula 3.1 of [33]. To see the equivalence of (2.13) and (2.16) it is enough to perform the change of variables $\lambda \rightarrow t\lambda/2$, $\eta \rightarrow t\eta/2$ and notice that the resulting determinants cancel out since the λ integral is bosonic and the η one is fermionic.

Finally, in order to eliminate the overall constant $(2\pi t)^{-\dim V/2}$ we can introduce an auxiliary bosonic variable $H \in V$ and use the fact that

$$\int_{H \in V} \exp\left(-it\langle s, H \rangle - \frac{t}{2}\|H\|^2\right) = (2\pi t)^{-\dim V/2} \exp\left(-\frac{t}{2}\|s\|^2\right),$$

which enables us to rewrite (2.16) as

$$e_{\#}(E) = \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \int_P \int_V \int_{\lambda, \phi \in \mathfrak{g}} B_{\eta} B_{\chi} \exp \frac{t}{2} (-2i\langle s, H \rangle - \|H\|^2 + \langle C^*, \eta \rangle + i\langle \phi, C^* C \lambda \rangle - i dC^* \otimes \lambda - 2i ds\chi + \chi\phi\chi) \quad (2.17)$$

Later on, when carrying out localization arguments similar to the remarks concluding Section 2.2, the form (2.13) will be easier to use, but we will also keep in mind its equivalence to (2.16) and (2.17).

Of course (2.13) should be understood in the distributional sense; if we want the integrand to only include smooth functions and differential forms we should write the analog of (2.11).

Theorem 2.3. *Let $G \rightarrow P \rightarrow M$ be a principal bundle, where G is a compact Lie group and M is an oriented compact manifold. Choose a G -invariant Riemannian metric on P and a normalized metric on \mathfrak{g} . Denote by $C : \mathfrak{g} \rightarrow TP$ the infinitesimal action of \mathfrak{g} and by C^* its adjoint. Consider the vector bundle $E = P \times_G V$ associated to a special orthogonal representation of G on the vector space V of even dimension $\dim V = \dim M$. The Euler number of E is then given by*

$$e_{\#}(E) = \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \lim_{t \rightarrow 0} \int_P e^{-\frac{t}{2}\|s\|^2} B_{\eta} \left(e^{\langle C^*, \eta \rangle} \int_{\lambda, \phi \in \mathfrak{g}} e^{i\langle \phi, C^* C \lambda \rangle - \epsilon \langle \phi, \phi \rangle} e^{-i dC^* \otimes \lambda} B_{\chi} \left(e^{-it ds\chi + \frac{1}{2}\chi\phi\chi} \right) \right). \quad (2.18)$$

2.5 Intersection numbers

We have dealt so far with the case $\dim V = \dim M$. In this case, for a generic choice of the section s (i.e. transversal to the zero section), the Euler number of the bundle equals the signed number of points in the zero set $Z(s)$, the sign being determined by a certain orientation. The previous sections provided integral formulas for computing this number. The Mathai-Quillen formalism is also useful in the case $\dim V \leq \dim M$; the Euler class can be multiplied by closed forms on M and the result integrated over M . For a section s that is transversal to the zero section, so that $Z(s)$ is an oriented submanifold of M , and $\mu_1, \dots, \mu_k \in H^*(M)$ such that $\deg \mu_1 + \dots + \deg \mu_k = \dim M - \dim V$,

Poincaré duality yields

$$\langle \mu_1 \mu_2 \cdots \mu_k \rangle := \int_{Z(s)} \mu_1 \wedge \cdots \wedge \mu_k = \int_M e(E) \wedge \mu_1 \wedge \cdots \wedge \mu_k. \quad (2.19)$$

Therefore $\langle \mu_1 \mu_2 \cdots \mu_k \rangle$, which is by definition the intersection number of the cohomology classes μ_1, \dots, μ_k restricted to $Z(s)$, can be expressed as an integral over M .

Let us assume we are in the 'good' situation when G acts freely on P and E is obtained as an associated bundle over $M = P/G$. Denote by $\hat{\mu}_1, \dots, \hat{\mu}_k$ the images in $H_G^*(P)$ of the given cohomology classes via the isomorphism $H^*(M) \simeq H_G^*(P)$. By using the Mathai-Quillen construction and Proposition 2.12 we get

Theorem 2.4.

$$\begin{aligned} \langle \mu_1 \mu_2 \cdots \mu_k \rangle &= \frac{1}{(2\pi)^{\dim \mathfrak{g}}} \frac{1}{(2\pi t)^{\dim V/2}} \int_P \hat{\mu}_1 \wedge \cdots \wedge \hat{\mu}_k \wedge e^{-\frac{1}{2}\|\cdot\|^2} \mathcal{B}_\eta \left(e^{(C^*, \eta)} \right) \\ &\quad \int \int_{\lambda, \phi \in \mathfrak{g}} e^{i(\phi, C^* C \lambda)} e^{-idC^* \otimes \lambda} \mathcal{B}_\chi \left(e^{-itd\phi \chi + \frac{1}{2}\chi \phi \chi} \right). \end{aligned} \quad (2.20)$$

2.6 Integration over P versus integration over M : an S^1 -equivariant extension

For later purposes we will need an extension of Proposition 2.12 to the case when P is endowed with an action of $G \times S^1$ such that the action of G is free. As stated in Proposition 2.10, there is an isomorphism between $H_{G \times S^1}^*(P)$ and $H_{S^1}^*(M)$.

Definition 2.9. The G, S^1 -equivariant vertical volume element $\gamma_{G, S^1} \in \Omega(P) \otimes S(\mathfrak{g}) \otimes \mathbb{C}[[m]]$ is

$$\gamma_{G, S^1} := e^{(\Omega - \mathcal{J})} \otimes \lambda_{\mathcal{B}_\eta} \left(e^{\theta \otimes \eta} \right).$$

Recall that θ denotes an S^1 -invariant G -connection on P and $\Omega_{S^1} = \Omega - \mathcal{J}$ is its S^1 -equivariant curvature, where $\mathcal{J} = m\theta(X)$ (as usual, X is the vector field corresponding to a generator of s^1). The rest of the notation is explained before Proposition 2.12.

Proposition 2.13. (i) The linear functional on $\Omega_{G \times S^1}(P)$ (with values in $\mathbb{C}[[m]]$) defined by

$$\alpha \mapsto \int_P \langle \alpha \wedge \gamma_{G, S^1} \rangle,$$

where $\langle \alpha \wedge \gamma_{G, S^1} \rangle$ denotes the contraction of the parts of α and γ_{G, S^1} which are polynomials on \mathfrak{g} , vanishes on $d_{G \times S^1}$ -exact forms, hence it descends to $H_{G \times S^1}^*(P)$.

(ii) Let α^u represent a cohomology class in $H_{G \times S^1}^*(P)$ and α^d be its image in $H_{S^1}^*(M)$ by the isomorphism $\text{Hor} \circ CW_H$. Then

$$\int_P \langle \alpha^u \wedge \gamma_{G, S^1} \rangle = \int_M \alpha^d. \quad (2.21)$$

Proof. (i) Note that although γ_{S^1} is a power series in m , after pairing it with α we are left with a polynomial in m . This is due to the pairing of the polynomials on \mathfrak{g} —if α has degree a as a polynomial on \mathfrak{g} then only the powers $\leq a$ of \mathcal{J} in the expansion of $\exp(\Omega - \mathcal{J})$ contribute. Let $Y_1, \dots, Y_{\dim \mathfrak{g}}$ be the basis of \mathfrak{g}^* dual to $\lambda_1, \dots, \lambda_{\dim \mathfrak{g}}$. Let X be, as usual, the vector field corresponding to a generator of s^1 . It is enough to show that

$$\int_P \langle d_{G \times S^1} \beta \wedge \gamma_{G, S^1} \rangle = 0$$

for a ‘monomial’ $\beta = \beta_0 m^k \prod Y_i^{k_i}$ with $\beta_0 \in \Omega(P)$. We have

$$d_{G \times S^1} \beta = \left(d\beta_0 - \sum (\iota_{\lambda_i} \beta_0) Y_i - (\iota_X \beta_0) m \right) m^k \prod Y_i^{k_i}.$$

We need some results about $\gamma_{G \times S^1}$ as well; to do explicit computations we write

$$\gamma_{G \times S^1} = \theta_1 \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp \left(\sum (\Omega_i - m \iota_X \theta_i) \otimes \lambda_i \right).$$

We know that

$$\begin{aligned} d\theta_i &= \Omega_i - \frac{1}{2} [\theta, \theta]_i \\ d\Omega_i &= -[\theta, \Omega]_i \\ d\iota_X \theta_i &= -\iota_X \Omega_i + [\iota_X \theta, \theta]_i, \end{aligned}$$

the last formula being derived in the proof of Proposition 2.7. This leads to

$$\begin{aligned} d\gamma_{G \times S^1} &= \sum (-1)^{i-1} \theta_1 \wedge \dots \wedge d\theta_i \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp(\Omega_H \otimes \lambda) \\ &\quad + (-1)^{\dim \mathfrak{g}} \theta_1 \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \left(\sum (d\Omega_i - m d\iota_X \theta_i) \otimes \lambda_i \right) \wedge \exp(\Omega_H \otimes \lambda). \end{aligned}$$

Hence

$$\begin{aligned} d\gamma_{G \times S^1} &= \sum (-1)^{i-1} \theta_1 \wedge \dots \wedge \Omega_i \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp(\Omega_H \otimes \lambda) \\ &\quad + (-1)^{\dim \mathfrak{g}} \theta_1 \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge m \left(\sum \iota_X \Omega_i \otimes \lambda_i \right) \wedge \exp(\Omega_H \otimes \lambda) \end{aligned}$$

because any product of at least $\dim \mathfrak{g} + 1$ forms θ_i vanishes. We also have

$$\iota_{\lambda_i} \gamma_{G, S^1} = \sum (-1)^{i-1} \theta_1 \wedge \dots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp(\Omega_H \otimes \lambda)$$

because the Ω_i ’s are horizontal and $\iota_X \theta_i$ are zero-forms; similarly,

$$\begin{aligned} \iota_X \gamma_{G, S^1} &= \sum (-1)^{i-1} \theta_1 \wedge \dots \wedge \iota_X \theta_i \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp(\Omega_H \otimes \lambda) \\ &\quad + (-1)^{\dim \mathfrak{g}} \theta_1 \wedge \dots \wedge \theta_{\dim \mathfrak{g}} \wedge \left(\sum (\iota_X \Omega_i) \otimes \lambda_i \right) \wedge \exp(\Omega_H \otimes \lambda). \end{aligned}$$

It is now easy to check the above assertion by using Stokes’ theorem and the fact that

$$\int \iota_{\lambda_i} (\cdot) = \int \iota_X (\cdot) = 0.$$

We have

$$\begin{aligned} & \int_P \langle (d\beta_0 - (\iota_X \beta_0)m)m^k \prod Y_i^{k_i} \wedge \gamma_{G,S^1} \rangle \\ &= \int_P \beta_0 m^k \wedge \sum (-1)^{i-1} \theta_1 \wedge \cdots \wedge (\Omega_i - \iota_X \theta_i) \wedge \cdots \wedge \theta_{\dim \mathfrak{g}} \wedge \prod (\Omega_j - \iota_X \theta_j)^{k_j} \end{aligned}$$

whereas

$$\begin{aligned} & \int_P \langle \sum Y_i \iota_{\lambda_i} \beta_0 m^k \prod Y_i^{k_i} \wedge \gamma_{G,S^1} \rangle \\ &= \int_P \langle \sum Y_i \beta_0 m^k \prod Y_i^{k_i} \wedge \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_{\dim \mathfrak{g}} \wedge \exp(\Omega_H \otimes \lambda) \rangle \\ &= \int_P \beta_0 m^k \wedge \sum (-1)^{i-1} \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_{\dim \mathfrak{g}} \wedge (\Omega_i - \iota_X \theta_i) \prod (\Omega_j - \iota_X \theta_j)^{k_j}, \end{aligned}$$

which completes the proof of part (i).

(ii) This can be easily deduced from part (ii) of Proposition 2.12. Namely, if $\alpha \in H_{S^1}^{\dim M+2k}(M)$ then $\alpha = m^k \alpha^{top} + \text{lower degree forms}$, with $\alpha^{top} \in H^{top}(M)$ and so

$$\int_M \alpha = m^k \int_M \alpha^{top}.$$

Similar facts are true for the pull-back $\pi^* \alpha \in H_{G \times S^1}^*(P)$, namely $\pi^* \alpha^{top} \in H_G^*(P)$ and

$$\int_P \langle \pi^* \alpha \wedge \gamma_{G,S^1} \rangle = \int_P \langle \pi^* \alpha^{top} \wedge \gamma_G \rangle.$$

The explanation is that $\pi^* \alpha$ and $\pi^* \alpha^{top}$ are constant as polynomials on \mathfrak{g} and so they only pair to the zeroth order terms in the expansions of γ_{G,S^1} and γ_G , which coincide.

Remark. It is natural to ask what happens if m is regarded as a real number rather than an element of $(s^1)^*$. One can then define an operator $d_{G,m}$ on $\Omega_G(P)^{S^1}$ by

$$d_{G,m}(\omega) := d\omega - m \iota_X \omega$$

whose square is 0 and the corresponding cohomology group $H_{G,m}(P)$. The element $\gamma_{G,m}$ is then defined similarly to γ_{G,S^1} —except that now m is a real number, so that $\gamma_{G,m} \in \Omega(P) \otimes S(\mathfrak{g})$. The proposition proved above has an obvious analogue (with the same proof)—the values of the functional are now polynomial functions in m and for $m = 0$ we are back to (2.12). This will be used later in our quantum-field theoretic application.

Chapter 3

The Mathai-Quillen Construction in Topological Quantum Field Theory

The Mathai-Quillen construction can be used to give a (formal) geometric interpretation of the partition functions of certain topological quantum field theories. The general idea is that the partition function of these theories is in fact a path-integral representation of the equivariant Euler number of a certain infinite-dimensional bundle. More generally, the path-integrals describing correlation functions can be understood as representing intersection numbers of equivariant cohomology classes. Historically, these ideas were first used in the context of Donaldson-Witten theory, introduced in [33] as an example of a quantum field theory whose correlation functions are topological invariants of the base 4-manifold (the Donaldson invariants). The topological structure of the Donaldson-Witten Lagrangian leading to the above properties of the partition function and correlation functions were discovered by Baulieu and Singer [4] and Atiyah and Jeffrey [1].

In this chapter, we will present the analog of the Atiyah-Jeffrey viewpoint for a number of four-dimensional topological gauge theories. We will start with the cases of Donaldson-Witten theory and topological Seiberg-Witten theory—merely by rewriting of the Atiyah-Jeffrey argument in the light of the results of Chapter 2. Our strategy will be to exhibit a Riemannian manifold P with an isometric G -action, a vector space V on which G acts linearly, and a G -equivariant map $s : P \rightarrow V$ so that the integral formula (2.10) applied to the G -equivariant bundle $P \times V \rightarrow P$ endowed with the section s coincides with the partition function of the quantum field theory.

A slightly more interesting case is that of the topologically twisted version of $N = 4$ supersymmetric Yang-Mills in four dimensions studied by Vafa and Witten in [32]. The corresponding

Lagrangian can be understood in terms of a variant of the Mathai-Quillen construction which makes use of the particular structure of the Vafa-Witten theory. We describe geometrically this version of the Mathai-Quillen construction and we apply it formally to the Vafa-Witten theory, in order to recover the result of [32], which asserts that the partition function is (under certain assumptions on the base four-manifold), a generating function for the Euler characteristics of the moduli spaces of instantons. Even under more general assumptions, the partition function has a topological significance, although it is not so easy to describe it in general.

There is a remarkable conjecture, called the S -duality conjecture, about the physical counterpart of Vafa-Witten theory, untwisted $N = 4$ supersymmetric Yang-Mills. The conjecture predicts that the partition function has specific modular properties under an action of $SL(2, \mathbb{Z})$. Vafa and Witten prove that in some cases this property remains true for the topologically twisted version of the theory. The latter is therefore important because the topological meaning of its partition function makes it possible to check the S -duality conjecture by a purely mathematical argument.

The main point of this chapter is the study of a *different* topological twist of $N = 4$ supersymmetric Yang-Mills, which we call topological non-abelian Seiberg-Witten theory. The choice of this name is motivated by the fact that the minimum energy classical configurations of the theory turn out to be exactly the solutions of a non-abelian version of the Seiberg-Witten equations. The key property of topological non-abelian Seiberg-Witten theory is the presence of an additional S^1 -symmetry, which is explored by means of the equivariant extension of the Mathai-Quillen construction introduced in the previous chapter.

The S^1 -symmetry of topological non-abelian Seiberg-Witten theory produces significant simplifications in the topological structure of its partition function, therefore making it a good candidate for the test of the S -duality conjecture. We will in fact make the point that topological non-abelian Seiberg-Witten theory is in general a better candidate than the Vafa-Witten theory because of the current knowledge of the topological quantities which are involved—we will discuss all this in a separate chapter.

We work in the following context: Let X^4 be a simply-connected oriented compact smooth four-manifold and E a rank 2 complex vector bundle over X with structure group $SU(2)$; we will sometimes write E_k instead of E if $c_2(E) = k$. We will denote by \mathcal{A} the space of unitary connections on E , \mathcal{G} the group of unitary gauge transformations of E , and $\mathfrak{su}(E)$ the bundle of traceless skew-hermitian endomorphisms of E , so that $\Gamma(\mathfrak{su}(E)) = \text{Lie}(\mathcal{G})$. We assume that a Riemannian metric on X has been chosen and that the spaces of sections of the various bundles involved are endowed with the appropriate L^2 -metrics. We will also be interested in the Spin^c -structures on X . Let L be a hermitian line bundle over the simply-connected oriented Riemannian four-manifold X , with $c_1(L) = c$. Denote by W^\pm the corresponding Spin^c -bundles (i.e. $\Lambda^2 W^\pm \simeq L$). They can be constructed as follows: the oriented frame bundle of X^4 is an $SO(4)$ -bundle; it can be shown that

its structure group can always be reduced to $\text{Spin}^c(4)$ (recall that $\text{Spin}^c(4) = \text{Spin}(4) \times_{\pm 1} \text{U}(1)$ projects onto $\text{SO}(4)$). Associated to any such reduction there is a $\text{U}(1)$ -bundle obtained through the homomorphism $\text{Spin}(4) \times_{\pm 1} \text{U}(1) \xrightarrow{\det^2} \text{U}(1)$ acting on the second factor. It turns out that there is a unique reduction of the frame bundle to $\text{Spin}^c(4)$ such that the first Chern class of the associated $\text{U}(1)$ -bundle is c . The bundles W^\pm are associated to the \pm spin representations on \mathbb{C}^2 . It is a standard result that, for a simply-connected manifold X , its Spin^c -structures are in bijection with classes $c \in H^2(X, \mathbb{Z})$ (see, for instance, [18]).

3.1 The Donaldson-Witten theory

The so-called Donaldson-Witten theory was introduced in [33] as an example of a quantum field theory whose correlation functions are topological invariants of the base 4-manifold (the Donaldson invariants). It turned out that the Lagrangian of the theory has a geometric significance which was explained by Baulieu and Singer [4] and Atiyah and Jeffrey [1].

The latter paper related the partition function of the Donaldson-Witten theory to the Mathai-Quillen formalism. We will now present their argument in the light of the results of Chapter 1. The scheme developed there, in particular formula (2.13) expressing the G -equivariant Euler number $e_{\#}^G(P, V, s)$ as a functional integral, will be applied to the following data:

- $P := \mathcal{A}_E$
- $V := \Omega_+^2(\mathfrak{su}(E))$
- $G := \mathcal{G}_E$
- $s : P \rightarrow V, \quad s(A) := F_A^+$ (the self-dual part of the curvature of A).

Recall that the infinitesimal action $C : \text{Lie}(\mathcal{G}) \rightarrow T_A \mathcal{A}$ is given by $\text{Lie}(\mathcal{G}) \simeq \Omega^0(\mathfrak{su}(E)) \ni \lambda \rightarrow -D_A \lambda \in \Omega^1(\mathfrak{su}(E))$, where D_A is the covariant derivative operator defined by the connection A . If D_A^* denotes its formal adjoint with respect to the L^2 -inner products then the adjoint operator $C^* : T_A \mathcal{A} \rightarrow \text{Lie}(\mathcal{G})$ satisfies $C^* \psi = -D_A^* \psi$ for any $\psi \in \Omega^1(\mathfrak{su}(E)) \simeq T_A \mathcal{A}$.

We also need to work out the expression of the 2-form dC^* . Let $\psi_1, \psi_2 \in \Omega^1(\mathfrak{su}(E))$ be two tangent vectors. Since \mathcal{A}_E is an affine space modeled on $\Omega^1(\mathfrak{su}(E))$ we can regard ψ_1, ψ_2 as defining constant vector fields on \mathcal{A} . We then have

$$dC^*(\psi_1, \psi_2) = \psi_1(C^*(\psi_2)) - \psi_2(C^*(\psi_1)) - C^*([\psi_1, \psi_2]) = \psi_1(C^*(\psi_2)) - \psi_2(C^*(\psi_1))$$

since $[\psi_1, \psi_2] = 0$. The notation $C^*(\psi_2)$ stands for the function $A \mapsto C_A^*(\psi_2)$. Since

$$D_{A+t\psi_1} \lambda = D_A \lambda + t[\psi_1, \lambda] = D_A \lambda + t b_{\psi_1} \lambda,$$

where $b_{\psi_1} : \Omega^0(\mathfrak{su}(E)) \rightarrow \Omega^1(\mathfrak{su}(E))$, we have

$$C_{A+t\psi_1}^*(\psi_2) = -D_{A+t\psi_1}^* \psi_2 - t(b_{\psi_1})^* \psi_2.$$

We claim that $(b_{\psi_1})^* \psi_2 = - * [\psi_1, * \psi_2]$. Indeed

$$\langle [\psi_1, \lambda], \psi_2 \rangle = \int_X \text{Tr} ([\psi_1, \lambda] * \psi_2) = - \int_X \text{Tr} ([\lambda, \psi_1] * \psi_2) = - \int_X \text{Tr} (\lambda [\psi_1, * \psi_2]) = \langle \lambda, - * [\psi_1, * \psi_2] \rangle.$$

We have used the invariance of Tr and the property that $** = 1$ on 4-forms. Hence

$$C_{A+t\psi_1}^*(\psi_2) = -D_A^* \psi_2 + t * [\psi_1, * \psi_2]$$

and so, by taking the derivative at $t = 0$, $\psi_1(C^*(\psi_2)) = *[\psi_1, * \psi_2]$. Notice that this can be rewritten by using the contraction operator $\Omega_1 \otimes \Omega_1 \rightarrow \Omega_0$ induced by the Riemannian metric on X . Namely, if $[,]_0$ denotes the composition of the contraction operator on 1-forms with the Lie algebra bracket on their values then $*[\psi_1, * \psi_2] = [\psi_1, \psi_2]_0$. We finally get

$$dC^*(\psi_1, \psi_2) = 2[\psi_1, \psi_2]_0.$$

In the sequel we will write the formula above as

$$dC^* = [\psi, \psi]_0$$

with the understanding that ψ is used as a notation for both tangent and cotangent vectors on A (identified through the Riemannian metric).

We will also use a standard notation from physics papers which expresses integrals of differential forms on a manifold as supersymmetric integrals. Namely, let P be a real oriented vector space endowed with an inner product and x_i be a basis of P^* . We regard the x_i as coordinates on P and $\psi_i = dx_i$ as constant differential forms on P . The inner product determines a volume element vol_P , a constant top form on P which can be used to integrate functions with appropriate decay properties at infinity. The volume form vol_P can also be used to define Berezin integration in the algebra $\Lambda[\psi_1, \dots, \psi_{\dim P}]$. If ω is a (suitably decaying) differential form on P written in terms of x_i and dx_i we then have

$$\int_P \omega(x_i, dx_i) = \int_P \mathcal{B}_\psi \omega(x_i, \psi_i).$$

On the right-hand side, the result of the Berezin integration is a function whose integral on P is defined as above, by using the volume element.

The differential of the section s is given by $ds(\psi) = (D_A \psi)^+$ and the action of $\text{Lie}(\mathcal{G})$ on V is determined by the adjoint action on $su(E)$, so $\chi \phi \chi = \chi[\phi, \chi]$. The conclusion of the above discussion is the following functional-integral formula:

$$\begin{aligned} e_{\#}^G(P, V, s) &\sim \text{const} \int_A e^{-\frac{1}{2} \|F_A^+\|^2} \mathcal{B}_\eta \left(e^{-\langle D_A \psi, \eta \rangle} \right) \int \int_{\lambda, \phi \in \mathfrak{g}} e^{i \langle \phi, D_A^* D_A \lambda \rangle} \\ &\quad \cdot e^{-i \langle \psi, \psi \rangle_0 \lambda} \mathcal{B}_\chi \left(e^{-i \langle (D_A \psi)^+, \lambda \rangle + \frac{1}{2} \chi[\phi, \chi]} \right) \\ &\sim \text{const} \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\eta \mathcal{D}\lambda \mathcal{D}\phi \mathcal{D}\chi e^{-\int_X \text{Tr} \mathcal{L}_{DW}} \end{aligned} \quad (3.1)$$

where

$$\mathcal{L}_{DW} = \frac{t}{2} F_A^{+2} + \psi \wedge *D_A \eta - i * \phi D_A^* D_A \lambda + i * [\psi, \psi]_0 \lambda + it D_A \psi \wedge \chi - \frac{t}{2} \chi \wedge [\phi, \chi]. \quad (3.2)$$

We use the \sim sign in (3.1) to denote a formal equality, i.e. the fact that the path-integral on the right-hand side represents the Euler number on the left-hand side. The notation 'const' in the formula denotes the (infinite) constant

$$\frac{1}{(2\pi)^{\dim \Omega^0(\mathfrak{su}(E))/2}} \frac{1}{(2\pi t)^{\dim \Omega_+^2(\mathfrak{su}(E))/2}}. \quad (3.3)$$

As explained in Section 2.4 (equation (2.17)), we could get rid of this infinite constant at the expense of introducing the auxiliary variable H . We assume that formula (3.1) is correct for any $t > 0$, as would be the case if the objects involved were finite-dimensional. Since the argument leading to (3.1) is a formal one, it is important to relate it to the physical interpretation of its right-hand side.

Lemma 3.1. *The expression \mathcal{L}_{DW} defined in (3.2) is the Lagrangian of a quantum field theory called topologically twisted $N = 2$ supersymmetric Yang-Mills theory. Accordingly, the right-hand side of (3.1) is the partition function \mathcal{Z}_{DW} of the theory.*

The statement is a consequence of the discussion in Sections 2 and 3 of [33]. A more detailed exposition can be found in Chapter 15 of [7]. Another viewpoint, found in [4], explains how \mathcal{L}_{DW} can be obtained by gauge-fixing the Lagrangian

$$\mathcal{L}(A) := \int_X \text{Tr} (F_A \wedge F_A)$$

which is obviously topological since the result of the integral is just a multiple of the second Chern class of the connection A .

We have seen in Chapter 1 that the formula for the Euler number is meaningful only if $\text{rk } E = \dim M$; this implies, for a generic section s , that $\dim \mathcal{Z}(s) = \dim M - \text{rk } E = 0$. In the case $\dim \mathcal{Z}(s) > 0$ the Mathai-Quillen construction yields an integral formula for intersection numbers on $\mathcal{Z}(s)$. To understand the analogues of these cases in our infinite-dimensional problem we first cite some results of Donaldson theory. The first fact refers to the space of instantons, i.e. connections whose self-dual part of the curvature vanishes, modulo gauge equivalence.

Proposition 3.1. *If $b_2 > 0$ then, for a generic metric on X , there are no reducible anti-self-dual connections and the space \mathcal{M}_k of instantons on the bundle E_k is a finite-dimensional manifold of dimension $8k - 3(1 + b_2^+)$ (recall that $b_1(X) = 0$).*

Donaldson invariants are obtained as intersection numbers on a compactification of \mathcal{M}_k . Instead of giving a rigorous definition we are going to content ourselves with a heuristic one which is the starting point for a quantum field theoretic description. Let us first define the so-called μ -classes.

Definition 3.1. (i) Given $\Sigma \in H_2(X, \mathbf{Z})$ represented by the embedded oriented surface Σ , we define $\hat{\mu}(\Sigma) = \hat{\mu}^{2,0}(\Sigma) + \hat{\mu}^{0,1}(\Sigma) \in \Omega^2(\mathcal{A}) \oplus \Omega^0(\mathcal{A}) \otimes S^1(\text{Lie}(\mathcal{G})^*)$ by

$$\begin{aligned}\hat{\mu}^{2,0}(\Sigma)(\psi_1, \psi_2) &:= \frac{1}{8\pi^2} \int_{\Sigma} \text{Tr}(\psi_1 \wedge \psi_2) && \text{for } \psi_1, \psi_2 \in T\mathcal{A} \\ \hat{\mu}^{0,1}(\Sigma)(A, \phi) &:= \frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(\phi F_A) && \text{for } \phi \in \text{Lie}(\mathcal{G}) \text{ and } A \in \mathcal{A}.\end{aligned}$$

(ii) $\hat{\mu}(\Pi) \in S^2(\text{Lie}(\mathcal{G})^*)$ is determined by

$$\hat{\mu}(\Pi) := \frac{1}{8\pi^2} \int_{\Sigma} \text{Tr}(\phi^2) \quad \text{for } \phi \in \text{Lie}(\mathcal{G}).$$

Proposition 3.2. (i) The differential forms $\hat{\mu}(\Sigma)$ and $\hat{\mu}(\Pi)$ are \mathcal{G} -invariant, hence they define equivariant differential forms in $\Omega_{\mathcal{G}}^2(\mathcal{A})$ and $\Omega_{\mathcal{G}}^4(\mathcal{A})$, respectively.

(ii) The forms above are equivariantly closed, i.e. $d_{\mathcal{G}}\hat{\mu}(\Sigma) = 0$ and $d_{\mathcal{G}}\hat{\mu}(\Pi) = 0$, therefore they determine equivariant cohomology classes in $H_{\mathcal{G}}(\mathcal{A})$.

Proof. Part (i) is obvious, since the action of the group of gauge transformations is given pointwise by the adjoint action of $SU(2)$ which preserves Tr . As for part (ii), it is clear for $\hat{\mu}(\Pi)$ which is just a quadratic polynomial on $\text{Lie}(\mathcal{G})$. For $\hat{\mu}(\Sigma)$ we have $d\hat{\mu}^{2,0}(\Sigma) = 0$ because $\hat{\mu}^{2,0}(\Sigma)$ is a constant differential form on an affine space, and

$$[\iota_{\phi}\hat{\mu}^{2,0}(\Sigma)](\psi) = \frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(-D_A\phi \wedge \psi)$$

for $\phi \in \text{Lie}(\mathcal{G})$ and $\psi \in T\mathcal{A}$ (recall that the vector field on \mathcal{A} corresponding to ϕ is $A \mapsto -D_A\phi$). Since $d_{\mathcal{G}} = d - \iota_{\phi}$ we have $d_{\mathcal{G}}\hat{\mu}^{2,0}(\Sigma) \in \Omega^1(\mathcal{A}) \otimes \text{Lie}(\mathcal{G})^*$ and

$$[d_{\mathcal{G}}\hat{\mu}^{2,0}(\Sigma)](\psi, \phi) = \frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(D_A\phi \wedge \psi).$$

We now turn to $d_{\mathcal{G}}\hat{\mu}^{0,1}(\Sigma)$; $\hat{\mu}^{0,1}(\Sigma)$ is killed by ι_{ϕ} since it is a 0-form and

$$[d_{\mathcal{G}}\hat{\mu}^{0,1}(\Sigma)](\psi, \phi) = [d\hat{\mu}^{0,1}(\Sigma)](\psi, \phi) = \frac{1}{4\pi^2} \int_{\Sigma} \text{Tr}(\phi \wedge D_A\psi)$$

because $(d/dt)_{t=0}F_{A+t\psi} = D_A\psi$. The fact that A is a unitary connection and the properties of the covariant derivative operator D_A on $\Omega^*(su(E))$ yield

$$\text{Tr}(D_A\phi \wedge \psi + \phi \wedge D_A\psi) = d\text{Tr}(\phi \wedge \psi)$$

and so, by Stokes' theorem,

$$[d_{\mathcal{G}}\hat{\mu}(\Sigma)](\psi, \phi) = \frac{1}{4\pi^2} \int_{\Sigma} d\text{Tr}(\phi \wedge \psi) = 0,$$

as claimed.

By restriction to the subset \mathcal{A}^* of irreducible connections the forms $\hat{\mu}(\Sigma)$ and $\hat{\mu}(\Pi)$ determine equivariant cohomology classes in $H_{\mathcal{G}}^*(\mathcal{A}^*)$. Since the action of \mathcal{G} on \mathcal{A}^* is free, we can apply the Chern-Weil homomorphism

$$H_{\mathcal{G}}^*(\mathcal{A}^*) \xrightarrow{CW} H^*(\mathcal{A}^*/\mathcal{G})$$

to $\hat{\mu}(\Sigma)$ and $\hat{\mu}(\Pi)$ to obtain $\mu(\Sigma) \in H^2(\mathcal{A}^*/\mathcal{G})$ and $\mu(\Pi) \in H^4(\mathcal{A}^*/\mathcal{G})$. The cohomology classes $\mu(\Sigma)$ and $\mu(\Pi)$ are actually related to the characteristic classes of the *universal bundle* \mathcal{U} over $X \times \mathcal{A}/\mathcal{G}$: in fact $\mu(\Pi)$ is the second Chern class of \mathcal{U} restricted to \mathcal{A}/\mathcal{G} whereas $\mu(\Sigma)$ is the slant product of $[\Sigma] \in H^2(X)$ and the second Chern class of \mathcal{U} . A complete treatment can be found in [4] and [9].

Definition 3.2. The k -th *Donaldson polynomial* is the functional

$$\mathbb{D}_k : \mathbb{R}[\Sigma_1, \dots, \Sigma_{b_2}, \Pi]_{(k)} \longrightarrow \mathbb{R},$$

(where the subscript (k) denotes the polynomials of total degree $d(k) = 8k - 3(1 + b_2^+)$ if $\Sigma_1, \dots, \Sigma_{b_2}$ are assigned the degree 2 and Π the degree 4) given by

$$\mathbb{D}_k (\Sigma_1^{\alpha_1} \Sigma_2^{\alpha_2} \dots \Sigma_{b_2}^{\alpha_{b_2}} \Pi^\beta) := \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1} \dots \mu(\Sigma_{b_2})^{\alpha_{b_2}} \mu(\Pi)^\beta. \quad (3.4)$$

To make this heuristic definition complete one has to compactify \mathcal{M}_k , choose suitable representatives of the μ -classes which extend to the compactification and then prove independence of the choices—including independence of the representatives σ_i in their cohomology classes. Appropriate forms of the above properties are proved in [9].

\mathbb{D}_k is a polynomial of degree $d(k)$ on $H^2(X, \mathbb{Z}) \oplus H^0(X, \mathbb{Z})$ (with the above grading). We can assemble the polynomials \mathbb{D}_k into a single object by defining the *total Donaldson polynomial* \mathbb{D}

$$\mathbb{D} (\Sigma_1^{\alpha_1} \Sigma_2^{\alpha_2} \dots \Sigma_{b_2}^{\alpha_{b_2}} \Pi^\beta) := \sum_{k \in \mathbb{Z}} \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1} \dots \mu(\Sigma_{b_2})^{\alpha_{b_2}} \mu(\Pi)^\beta. \quad (3.5)$$

Alternatively one can regard \mathbb{D} as a generating series; introduce formal variables q_1, \dots, q_{b_2}, p and rewrite the above definition as

$$\mathbb{D} ((q_1 \Sigma_1)^{\alpha_1} (q_2 \Sigma_2)^{\alpha_2} \dots (q_{b_2} \Sigma_{b_2})^{\alpha_{b_2}} \Pi^\beta) := \sum_{k \in \mathbb{Z}} q_1^{\alpha_1} \dots q_{b_2}^{\alpha_{b_2}} p^\beta \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1} \dots \mu(\Sigma_{b_2})^{\alpha_{b_2}} \mu(\Pi)^\beta. \quad (3.6)$$

This notation explains the meaning of the left-hand side $\mathbb{D}(\exp(\sum_a q_a \Sigma_a))$ of Witten's conjecture stated in the Introduction.

According to the above heuristic definition we can think intuitively about Donaldson invariants as intersection numbers on the moduli spaces \mathcal{M}_k and therefore represent them as functional integrals by using (2.4) and (3.1). By analogy with (3.1) we can write

$$\mathbb{D}(\mathcal{O}) \sim \text{const} \sum_{k \in \mathbb{Z}} \int_{\mathcal{A}_k} \int_{\text{Lie}(\mathcal{G}_k)} \Gamma_{\mathcal{G}_k} \wedge W(\mathcal{A}_k, V_k, \mathfrak{g}_k) \wedge \mathcal{O}, \quad (3.7)$$

where \mathcal{O} is a product of $\hat{\mu}$ -classes. More concretely, the result of this procedure looks as follows:

$$\begin{aligned} \mathbb{D} (\Sigma_1^{\alpha_1} \Sigma_2^{\alpha_2} \dots \Sigma_{b_2}^{\alpha_{b_2}} \Pi^\beta) &\sim \text{const} \sum_{k \in \mathbb{Z}} \int_{\mathcal{A}_k, \text{Lie}(\mathcal{G}_k)} \mathcal{D}(\text{fields}) \\ &\hat{\mu}(\Sigma_1)^{\alpha_1} \wedge \dots \wedge \hat{\mu}(\Sigma_{b_2})^{\alpha_{b_2}} \wedge \hat{\mu}(\Pi)^\beta e^{-\int_X \text{Tr} \mathcal{L}_{DW}} \end{aligned} \quad (3.8)$$

where \mathcal{L}_{DW} is defined in (3.2). We can get a nicer formula if we rewrite both sides as generating series

$$\mathbb{D}(\exp(q\Sigma + p\Pi)) \sim \text{const} \sum_{k \in \mathbb{Z}} \int_{\mathcal{A}_k, \text{Lie}(\mathfrak{g}_k)} \mathcal{D}(\text{fields}) \exp\left(-\int_X \text{Tr} \mathcal{L}_{DW} + \frac{p}{8\pi^2} \int_X \text{Tr} \phi^2 + \frac{q}{8\pi^2} \int_\Sigma \text{Tr}(\psi \wedge \psi + 2\phi F_A)\right) \quad (3.9)$$

3.2 Topological Seiberg-Witten theory

We now give a concise description of the Seiberg-Witten invariants and their path-integral realization, in analogy with the one for Donaldson invariants from the previous section.

Definition 3.3. The *Seiberg-Witten equations* for a unitary connection $A \in \mathcal{A}_L$ and a section $S \in \Gamma(W^+)$ are

$$\begin{cases} \not{D}_A S = 0 \\ F_A^+ + i(S \otimes \bar{S})_0 = 0, \end{cases} \quad (3.10)$$

where \not{D}_A represents the twisted Dirac operator $\not{D}_A : \Gamma(W^+) \rightarrow \Gamma(W^-)$ (since $\text{Spin}^c(4) = \text{Spin}(4) \times_{\pm 1} U(1)$, a connection on W^\pm is obtained by combining the Levi-Civita connection on the frame bundle with a connection A on L). If $S \in \Gamma(W^+)$ then the endomorphism $iS(S, \cdot)$ is skew-hermitian so, if we denote by $i(S \otimes \bar{S})_0$ its traceless part, we have $i(S \otimes \bar{S})_0 \in \Gamma(\text{su}(W^+))$. To understand the second equation, recall that Clifford multiplication yields an isomorphism $\Lambda_+^2(X) \simeq \text{su}(W^+)$.

The relevant results about the Seiberg-Witten equations can be summarized as follows. The space of equivalence classes of solutions (also called the *moduli space* of solutions) is a finite-dimensional oriented compact smooth manifold. Compactness is the main difference from the Donaldson moduli spaces (see [33])—it is precisely this property which makes Seiberg-Witten theory simpler. Strictly speaking, the other properties can only be obtained after using some standard maneuvers in gauge theory (i.e. by perturbing the equations and the Riemannian metric). Seiberg-Witten invariants are defined as a signed count of the points in the moduli space if this is zero-dimensional (or as certain characteristic numbers if its dimension is positive). The virtual dimension of the moduli space turns out to be $(c^2 - 2\chi - 3\sigma)/4$, where χ and σ are the Euler characteristic and signature of the manifold X , respectively (so, if $b_1 = 0$, $2\chi + 3\sigma = 4 + 5b_2^+ - b_2^-$).

For each $c \in H^2(X, \mathbb{Z})$ such that $c^2 = 2\chi + 3\sigma$ we denote by $\text{SW}(c)$ the Seiberg-Witten invariant defined by the Spin^c -structure corresponding to c . We obtain a functional-integral realization of $\text{SW}(c)$ by applying the Mathai-Quillen construction to:

- $P := \mathcal{A}_L \times \Gamma(W^+)$
- $V := \Omega_+^2 \oplus \Gamma(W^-)$

- $G := \text{Map}(X, U(1))$
- $s : P \rightarrow V, \quad s(A, S) := (F_A^+ + i(S \otimes \bar{S})_0, \not{D}_A S)$.

We emphasize that although $e^{i\theta} \in G$ acts as the usual multiplication by $e^{i\theta}$ in the fibres of W^+ , its action on \mathcal{A}_L is through the gauge transformation $e^{2i\theta}$ since $L \simeq \Lambda^2 W^+$. This choice of actions is necessary for the Seiberg-Witten equations to be gauge-invariant. The quantities involved in the second equation are obviously invariant under scalar gauge transformations. As for the first equation, if $g = e^{i\theta}$ acts as $e^{2i\theta}$ on L then

$$\begin{aligned}
\not{D}_{gA}(gS) &= \not{D}_{A-2i\theta}(e^{i\theta}S) \\
&= \not{D}_A(e^{i\theta}S) - \text{cl}(i d\theta) S \\
&= \text{cl} \circ D_{LC,A}(e^{i\theta}S) - \text{cl}(i d\theta) S \\
&= \text{cl} \circ (I d\theta e^{i\theta}S + e^{i\theta} D_{LC,A} S) - \text{cl}(i d\theta) S \\
&= e^{i\theta} \not{D}_A S.
\end{aligned}$$

This shows that the section s is indeed G -equivariant (recall that by scalar multiplication on $\Gamma(W^-)$). In the above computation the key point is the relationship between \not{D}_A and $\not{D}_{A-2i d\theta}$ (acting on $\Gamma(W^+)$). The claim is that they differ by the operator $i \text{cl}(d\theta)$: the factor of 2 disappears when using the connection A together with the Levi-Civita connection LC to produce a connection on W^+ . This can be seen if we look at the projection $\text{Spin}(4) \times_{\pm 1} U(1) \rightarrow \text{SO}(4) \times U(1)$, which squares the second factor.

Let us work out explicitly the various quantities in (2.13). The infinitesimal action $C : \mathfrak{g} \rightarrow T_{(A,S)}P \simeq \Omega^1(X) \oplus \Gamma(W^+)$ is given by

$$C(\lambda) = (-2D_A \lambda, \lambda S).$$

The operator D_A on $\Omega^*(X) \simeq \Omega^*(\text{End}(L))$ is just the DeRham derivative d and so, if $\psi \in \Omega^1(X)$ and $\sigma \in \Gamma(W^+)$,

$$C^*(\psi, \sigma) = -2d^* \psi + C_0^* \sigma$$

where $\langle C_0^* \sigma, \lambda \rangle = \langle \sigma, \lambda S \rangle$ for any $\lambda \in \mathfrak{g}$. The 1-form $\langle C^*, \eta \rangle$ evaluated on the vector (ψ, σ) at (A, S) thus equals

$$\int_{X^4} -2\psi \wedge *d\eta + \langle \sigma, \eta S \rangle,$$

where the last $\langle \cdot, \cdot \rangle$ denoted the pointwise inner product on W^+ . There is no Tr in the above formula since we are working on the line bundle L .

For $\phi, \lambda \in \mathfrak{g}$ we have, at $(A, S) \in P$,

$$i\langle \phi, C^* C \lambda \rangle = \int_{X^4} 4i * \phi d^* d\lambda + i * \langle \lambda S, \phi S \rangle.$$

In order to calculate $dC^* \otimes \lambda$, first notice that it is enough to evaluate it on vectors of the form $(0, \sigma) \in TP$ since $C^*(\psi, 0) = -2d^* \psi$ is a constant differential form on the affine space \mathcal{A}_L . Denoting,

for short, the vectors $(0, \sigma_1)$ and $(0, \sigma_2)$ by σ_1 and σ_2 , we have:

$$\begin{aligned} dC^*(\sigma_1, \sigma_2)_{(A,S)} \otimes \lambda &= \sigma_1 (\langle C^*(\sigma_2), \lambda \rangle) - \sigma_2 (\langle C^*(\sigma_1), \lambda \rangle) \\ &= \sigma_1 (\langle \sigma_2, \lambda S \rangle) - \sigma_2 (\langle \sigma_1, \lambda S \rangle) \\ &= \langle \sigma_2, \lambda \sigma_1 \rangle - \langle \sigma_1, \lambda \sigma_2 \rangle = -2\langle \sigma_1, \lambda \sigma_2 \rangle \end{aligned}$$

because λ is skew-hermitian. Notice that $[\sigma_1, \sigma_2] = 0$ because the two vector fields are constant. Hence

$$-idC^* \otimes \lambda = i \int_{X^4} \langle \sigma, \lambda \sigma \rangle.$$

We can now write a concrete expression for the differential form Γ_G , the Fourier transform of the equivariant volume element.

Proposition 3.3.

$$\Gamma^{(SW)} \sim \int \mathcal{D}\eta \mathcal{D}\lambda \exp \left(\int_{X^4} -2\psi \wedge *d\eta + \langle \sigma, \eta S \rangle + 4i * \phi d^* d\lambda + i * \langle \lambda S, \phi S \rangle + i \langle \sigma, \lambda \sigma \rangle \right).$$

Let's now turn to the universal Euler form $W(s)$. If we denote the elements of $V = \Omega_+^2 \oplus \Gamma(W^-)$ by (χ, T) then the terms $\chi\phi\chi$, $\chi\phi T$, and $T\phi\chi$ vanish since ϕ acts by the (trivial) adjoint action of $U(1)$. The term denoted in the general case by $\chi\phi\chi$ therefore reduces in this case to $\langle T, \phi T \rangle$, where $\phi = e^{i\theta}$ acts by scalar multiplication on $T \in \Gamma(W^-)$. We also have

$$\|s\|^2 = \int_{X^4} F_A^{+2} + 2iF_A^+(S \otimes \bar{S})_0 + \|i(S \otimes \bar{S})_0\|^2 + *\|\mathcal{D}_A S\|^2$$

and

$$ds_{(A,S)}(\psi, \sigma) = \left(d^+ \psi + i(S \otimes \bar{\sigma} + \sigma \otimes \bar{S})_0, \mathcal{D}_A \sigma + \frac{1}{2} \text{cl}(\psi) S \right).$$

Proposition 3.4. *The universal Euler form for the Seiberg-Witten theory is represented by the following functional integral:*

$$\begin{aligned} W^{(SW)} \sim \text{const} \int \mathcal{D}\chi \mathcal{D}T \exp t \left(\int_{X^4} -\frac{1}{2} F_A^{+2} - F_A^+ \wedge i(S \otimes \bar{S})_0 - \frac{1}{2} \|i(S \otimes \bar{S})_0\|^2 - *\|\mathcal{D}_A S\|^2 \right) \\ -i (d\psi - i(S \otimes \bar{\sigma} + \sigma \otimes \bar{S})_0) \chi - i \langle \mathcal{D}_A \sigma + \frac{1}{2} \text{cl}(\psi) S, T \rangle + \frac{1}{2} \langle T, \phi T \rangle, \end{aligned}$$

for any $t > 0$. The (infinite) constant in the above formula is

$$\frac{1}{(2\pi)^{\dim \Omega^0/2}} \frac{1}{(2\pi t)^{(\dim \Omega_+^2 + \Gamma(W^-))/2}}. \quad (3.11)$$

We conclude with the functional-integral representation of the Seiberg-Witten invariants.

Proposition 3.5. *With $\Gamma^{(SW)}$ and $W^{(SW)}$ as above,*

$$SW(c) = \text{const} \int_{\mathcal{A}_L \times \Gamma(W^+) \times \text{Map}(X, u(1))} DADSD\phi \Gamma^{(SW)} \wedge W^{(SW)}. \quad (3.12)$$

3.3 The geometric interpretation of the Vafa-Witten Lagrangian

In their paper [32], Vafa and Witten introduced a topologically twisted version of $N = 4$ supersymmetric Yang-Mills theory on an arbitrary compact smooth four-manifold. Their construction relies on a variant of the Mathai-Quillen construction, whose particular geometric structure we explain in this section.

Recall that X^4 denotes an oriented compact smooth four-manifold, E a rank 2 complex vector bundle with structure group $SU(2)$, \mathcal{A} the space of unitary connections on E , \mathcal{G} the group of gauge transformations, and $\mathfrak{su}(E)$ the bundle of traceless skew-hermitian endomorphisms of E .

The result of Vafa and Witten can be summarized as follows:

Proposition 3.6. *The partition function of (the Vafa-Witten version of) topologically twisted $N = 4$ Yang-Mills theory is an integral representation of an (infinite-dimensional) Euler number. The exponential of the Lagrangian is the Mathai-Quillen Euler form corresponding to the following data:*

- $P := \mathcal{A} \times \Omega_+^2(\mathfrak{su}(E)) \times \Omega^0(\mathfrak{su}(E)) \ni (A, B, C)$
- $V := \Omega_+^2(\mathfrak{su}(E)) \oplus \Omega^1(\mathfrak{su}(E))$
- $G := \mathcal{G}$
- $s : P \rightarrow V, \quad s(A, B, C) = (F_A^+ + \frac{1}{2}[C, B] + \frac{1}{4}[\widehat{B}, \widehat{B}], *D_A B + D_A C),$

where $[\widehat{B}, \widehat{B}]$ denotes a partial contraction (using a metric g^{kl} , this is $[\widehat{B}, \widehat{B}]_{ij} = [B_{ik}, B_{jl}]g^{kl}$).

The bundle and the section involved in the Mathai-Quillen formulation of the Vafa-Witten theory possess an additional structure that we now describe.

We will start with a finite-dimensional analogue. Let $E \rightarrow M$ be an oriented Euclidian vector bundle with a fixed section s and let the Mathai-Quillen forms $MQ_t(E, M, s)$ represent the Euler class. If $\text{rk } E = \dim M$ and the section is non-degenerate then

$$e_{\#}(E) = \lim_{t \rightarrow \infty} \int_M MQ_t(E, M, s) = \sum_{\text{zeroes of } s} \pm 1.$$

If the section s is only assumed to be Bott-nondegenerate, i.e. its differential has constant corank on each component of its zero set, then a simple argument shows that

$$e_{\#}(E) = \lim_{t \rightarrow \infty} \int_M MQ_t(E, M, s) = \sum_{Z_m \text{ component of } Z(s)} \pm e_{\#}(\text{coker } Ds|_{Z_m}),$$

where Ds is really the vertical differential $Ds : TM \rightarrow T_{\text{vert}} E \simeq \pi^* E$.

Let us now repeat the Mathai-Quillen construction for the bundle $\mathcal{E} \rightarrow \mathcal{M}$ with the section S obtained as follows:

- $\mathcal{M} := E$ (total space of the bundle E considered before)
- $\mathcal{E} := \pi^* E \oplus \pi^*(T^* M)$
- $S(v) := (S(v), \langle Ds|_{\pi(v)}, v \rangle)$, where $v \in E$, $S : E \rightarrow \pi^* E$ is a section extending $s : M \rightarrow E$, M being included in E as the zero section, and \langle , \rangle denotes the fibre inner product.

We decompose the zero locus of S , $Z(S)$, as $Z_1(S) \cup Z_2(S)$, where $Z_1(S) = Z(S) \cap M$ and $Z_2(S) = Z(S) \setminus Z_1(S)$. We have $Z_1(S) = Z(s)$ because $S|_M = s$. Notice that $\text{rk } \mathcal{E} = \dim M$ so that we can write (at least formally since M is not compact):

$$e_{\#}(E) = \text{contributions from } Z_1(S) + \text{contributions from } Z_2(S).$$

We can be more explicit about the contributions from $Z_1(S) = Z(s)$ if the section $s : M \rightarrow E$ is non-degenerate; it turns out that, on the connected component Z_m of $Z(s)$, $\text{coker}(DS)|_{Z_m} = T^*Z_m$, so $e_{\#}((\text{coker } DS)|_{Z_m}) = \chi(Z_m)$ (the Euler characteristic). Notice that the \pm sign is actually $+$ because $DS|_{Z(s)}$ always has positive determinant. The bottom line is that:

$$e_{\#}(E) = \sum_{Z_m \text{ component of } Z(s)} \chi(Z_m) + \text{contributions from } Z_2(S). \quad (3.13)$$

Assume now that $Z(S) \subset M$. Then

$$e_{\#}(E) = \chi(Z(s)). \quad (3.14)$$

Notice that the above assumption is very peculiar; we only expect it to be true under quite restrictive hypotheses. However, the assumption turns out to be true in a few cases relevant to the analysis in [32]—in fact, as we shall soon explain, this is one of the key points in [32].

In order to understand the finite-dimensional analog of Proposition 3.6. we need a slight generalization of the above discussion for equivariant vector bundles. If $E \rightarrow P$ is a G -equivariant vector bundle and G acts freely on P then we apply the Mathai-Quillen scheme to

- $\mathcal{P} = E \times \mathfrak{g}$
- $\mathcal{E} = \pi^* E \oplus \pi^* T^* P$
- $S(v, g) = (S, \langle Ds, v \rangle - \rho(g))$, where $\rho : \mathfrak{g} \rightarrow TP$ represents the infinitesimal action and $g \in \mathfrak{g}$. Similar arguments to the ones above show that if $Z(S) \subset P \times \{0\} \subset E \times \mathfrak{g}$ then

$$e_{\#}^G(\mathcal{E}, \mathcal{P}, S) = \chi^G(Z(s)) = \chi(Z(s)/G) \quad (3.15)$$

To see the implications for the Vafa-Witten theory we formally apply the last construction to:

- $P = \mathcal{A}$, so that $T^*P = \mathcal{A} \times \Omega^1(\mathfrak{su}(E))$
- $E = P \times V = \mathcal{A} \times \Omega_+^2(\mathfrak{su}(E))$
- $G = \mathcal{G}$
- $s = (A \rightarrow F_A^+)$.

We thus have

- $\mathcal{P} = \mathcal{A} \times \Omega_+^2(\mathfrak{su}(E)) \times \Omega^0(\mathfrak{su}(E))$
- $\mathcal{E} = \mathcal{P} \times (\Omega_+^2(\mathfrak{su}(E)) \oplus \Omega^1(\mathfrak{su}(E)))$
- $S(A, B, C) = (S(A, B, C), *D_A B + D_A C)$, where

$$S(A, B, C) := F_A^+ + \frac{1}{2}[C, B] + \frac{1}{4}[\widehat{B}, \widehat{B}].$$

Notice that S is indeed an extension of s .

If we pretend that the action of the group of gauge transformations on the space of anti-self-dual connections is free (i.e. we ignore the problems caused by reducible instantons) then the upshot of the discussion is that, under favorable circumstances, the partition function of the Vafa-Witten theory computes the Euler characteristic of the moduli space of instantons. 'Favorable circumstances' means the fulfillment of the condition $Z(S) \subset P \times \{0\} \subset P \times \mathfrak{g}$, i.e. all the zeroes of S are instantons. Vafa and Witten give one case in which the condition holds, namely Kähler surfaces with nonnegative scalar curvature. This leads to

Statement 3.1. *If X^4 is a Kähler surface with nonnegative scalar curvature then $Z(S) \subset \mathcal{A} \times \{0\}$, hence the partition function of the Vafa-Witten theory computes the Euler characteristic of the moduli space of instantons on the bundle E .*

3.4 The modularity of the partition function of $N = 4$ supersymmetric Yang-Mills theory

The motivation for the work of Vafa and Witten comes from the Montonen-Olive conjecture about the S-duality of $N = 4$ supersymmetric Yang-Mills in four dimensions. This conjecture cannot be checked by the standard techniques of perturbation theory because it really involves the behavior of the theory in the strong coupling regime, and this is why the study of a topological twist can be particularly powerful. For instance, the physical theory and its twist coincide on a hyperkähler manifold, (e.g. a four-torus or a K3 surface); on such a manifold, the computation of the partition function of the physical theory reduces, according to the discussion in the previous section, to topological quantities (Euler characteristics).

The viewpoint of Vafa and Witten is that the S-duality property is preserved by the twist, so they test the S-duality conjecture through the following topological conjecture:

Conjecture 1. *Let E_k be the $SU(2)$ -bundle over X with $c_2(E) = k \in \mathbb{Z}$ and Z_k be the partition function of the topologically twisted $N = 4$ super-Yang-Mills theory corresponding to the fixed vector bundle E_k . Define the partition function*

$$Z(\tau) := \sum_{k \in \mathbb{Z}} e^{2\pi i k \tau} Z_k,$$

where τ is a complex number. Then $Z(\tau)$ is a modular form of weight $-\chi(X)/2$ for the usual action of $SL(2, \mathbb{Z})$ on τ .

Using the Propositions from the previous section, this can be reformulated in a particular case as

Conjecture 2. *If X^4 is a Kähler manifold with nonnegative scalar curvature then*

$$\sum_{k \in \mathbb{Z}} e^{2\pi i k \tau} \chi(\mathcal{I}_k)$$

is a modular form of weight $-\chi(X)/2$, where \mathcal{I}_k denotes the moduli space of k -instantons.

The last conjecture is checked in [32] in the case when X^4 is a $K3$ surface or $\mathbb{C}P^2$ (in the latter case, the conjecture needs to be slightly reformulated) by using existing mathematical results.

There are a few problems with the applicability of the Vafa-Witten topological twist. First, there could be other solutions to the equations of motion besides instantons, and it is not clear in general what they represent topologically. Secondly, even if the appropriate ‘vanishing’ theorem holds, i.e. the only zeroes of the section S are instantons, the computation of the partition function requires the knowledge of the Euler characteristics of the spaces of instantons—presently, the only results available are for $\mathbb{C}P^2$ and $K3$ surfaces.

In the next section we switch to a different topological twist of $N = 4$ supersymmetric Yang-Mills theory, which offers solutions to both problems mentioned above. We will find that the partition function localizes to moduli spaces of instantons and solutions to the (abelian) Seiberg-Witten equations, which makes it possible to express the partition function in terms of Donaldson and Seiberg-Witten invariants.

Remark. It would be interesting to study the explicit relationship between the two different topological twists of $N = 4$ Yang-Mills—a potential source of new mathematical results, as suggested by the analogy with the case of mirror symmetry (which is a consequence of the equivalence of two topological twistings of a supersymmetric sigma-model).

3.5 Topological non-abelian Seiberg-Witten theory

In this section we study the partition function of topological non-abelian Seiberg-Witten theory (TNSW). This is a non-abelian extension of the Seiberg-Witten theory described in Section 3.2 obtained by replacing the line bundle L by a vector bundle E (actually we will be mostly interested in the case when the rank of E is two).

The interesting new feature is the presence of an additional S^1 -symmetry; we can use this symmetry to construct a perturbation of topological non-abelian Seiberg-Witten theory, essentially by replacing all \mathcal{G} -equivariant objects by their $\mathcal{G} \times S^1$ extensions. The main ingredients will be the S^1 -equivariant Mathai-Quillen construction from Section 2.3 and the S^1 -equivariant volume element from Section 2.6. In physical terms, the change in the Lagrangian amounts to the addition of a mass term, therefore the perturbed theory will be called massive topological non-abelian Seiberg-Witten theory (MTNSW).

The objects entering our construction will be as follows. Let W^\pm denote a Spin^c -structure on the four-manifold X , E an $\text{SU}(2)$ complex vector bundle on X endowed with a hermitian metric, and $\mathfrak{sl}(E)$ the bundle of traceless endomorphisms of E . We will denote by \mathcal{A}_E the space of unitary connections on E and by \mathcal{G}_E the group of unitary gauge transformations of E of unit determinant. Let

- $P := \mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$;
- $G := \mathcal{G}_E / \{\pm 1\}$;
- $V := V_1 \oplus V_2$, where $V_1 := \Omega_+^2(\mathfrak{su}(E))$ and $V_2 := \Gamma(W^- \otimes \mathfrak{sl}(E))$;
- $s : P \rightarrow V$, $s = (s_1, s_2)$ where $s_1(A, \mathbf{S}) := F_A^+ + i[\mathbf{S}, \bar{\mathbf{S}}]_0$ and $s_2(A, \mathbf{S}) := \mathcal{D}_A \mathbf{S}$.

In the definition of s , the bracket $[,]$ denotes the Lie algebra bracket in $\mathfrak{sl}(E)$, the bar in $\bar{\mathbf{S}}$ denotes complex conjugation when $\mathfrak{sl}(E)$ is regarded as the complexification of $\mathfrak{su}(E)$, and \mathcal{D}_A denotes the coupled Dirac operator. The subscript 0 in $[,]_0$ denotes the projection onto the traceless part as an endomorphism of the Spin^c -bundle: since $\mathbf{S} \in \Gamma(W^+ \otimes \mathfrak{sl}(E))$, the complex conjugate $\bar{\mathbf{S}}$ belongs to $\Gamma(W^- \otimes \mathfrak{sl}(E))$ and $i[\mathbf{S}, \bar{\mathbf{S}}]$ belongs to $\Gamma(\mathfrak{u}(W^+) \otimes \mathfrak{su}(E))$; the term $i[\mathbf{S}, \bar{\mathbf{S}}]_0$ denotes its component in $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$. To make sense of the definition of s_1 we use the isomorphism $\Omega_+^2 \simeq \mathfrak{su}(W^+)$.

The definition of G chosen above is motivated by the fact that the constant gauge transformations ± 1 act trivially on connections as well as sections of $\mathfrak{sl}(E)$. We will pretend that the action of \mathcal{G}_E on $\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$ is free, i.e. we will ignore the subset $\mathcal{A}_{red} \times 0 = \{(A, 0) \mid A \text{ reducible}\}$. Therefore we can regard $E_1 = P \times V_1$ and $E_2 = P \times V_2$ as \mathcal{G} -equivariant bundles over $(\mathcal{A}_E \times \Gamma(W^+ \otimes E)) \setminus \mathcal{A}_{red} \times 0$.

A straightforward application of the Mathai-Quillen formalism gives

Proposition 3.7. *The Euler number $e_{\#}^G(P, V, s)$ corresponding to the above data is given by*

$$e_{\#}^G(P, V, s) \sim \text{const} \int_P \int_{\lambda, \phi, \eta \in \text{Lie}(G)} \Gamma_G(P) \wedge W_G(E_1, s_1) \wedge W_G(E_2, s_2), \quad (3.16)$$

where $W_G(E_1, s_1)$ and $W_G(E_2, s_2)$ are the universal Euler forms as in (2.4) (note that we are suppressing the dependence on $t > 0$ in the notation), and $\Gamma_G(P)$ is the \mathcal{G} -equivariant vertical volume element. Specifically,

$$\begin{aligned} W_G(E_1, s_1) &= B_\chi \exp t \int_X \text{Tr} \left[-\frac{1}{2} (F_A^+ + i[\mathbf{S}, \bar{\mathbf{S}}]_0)^2 - i D_A \psi \wedge \chi \right. \\ &\quad \left. - i(i[\sigma, \bar{\mathbf{S}}]_0 + i[\mathbf{S}, \bar{\sigma}]_0) \wedge \chi + \frac{1}{2} \chi[\phi, \chi] \right] \end{aligned} \quad (3.17)$$

$$\begin{aligned} W_G(E_2, s_2) &= B_{\mathbf{T}} \exp t \int_X * \left[-\frac{1}{2} |\mathcal{D}_A \mathbf{S}|^2 - i(\text{cl}(\psi) \mathbf{S} + \mathcal{D}_A \sigma, \mathbf{T}) \right. \\ &\quad \left. + \frac{1}{2} \langle \mathbf{T}, [\phi, \mathbf{T}] \rangle \right]. \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} \Gamma_G(P) &= \int_\lambda B_\eta \exp \int_X * [-\text{Tr} D_A^* \psi \wedge \eta + \langle \sigma, [\eta, \mathbf{S}] \rangle + i \text{Tr} \phi D_A^* D_A \lambda \\ &\quad + i \langle [\lambda, \mathbf{S}], [\phi, \mathbf{S}] \rangle - i \text{Tr} [\psi, \psi]_0 \lambda + i \langle \sigma, [\lambda, \sigma] \rangle]. \end{aligned} \quad (3.19)$$

The value of the infinite constant preceding the functional integral is

$$(2\pi)^{-\dim \Omega^0(\mathfrak{su}(E))/2} (2\pi t)^{-\frac{1}{2}(\dim \Omega_+^2(\mathfrak{su}(E)) + \dim \Gamma(W^- \otimes \mathfrak{sl}(E)))}.$$

It is natural to ask whether the formal equality (3.16) admits a physical interpretation similar to that of (3.1). The relevant result is proved (in the physical sense of the word) in Section 2.2 (2) of [15]:

Proposition 3.8. *If X is a spin manifold and W^\pm are its \pm Spin-bundles then the expression*

$$\begin{aligned} \mathcal{L}_{NSW}(t) := & t \int_X \text{Tr} \left[\frac{1}{2}(F_A^+ + i[\mathbf{S}, \bar{\mathbf{S}}]_0)^2 + iD_A \psi \wedge \chi + i(i[\sigma, \bar{\mathbf{S}}]_0 + i[\mathbf{S}, \bar{\sigma}]_0) \wedge \chi - \frac{1}{2}\chi[\phi, \chi] \right] \\ & + t \int_X * \left[\frac{1}{2}|\not{D}_A \mathbf{S}|^2 + i\langle \text{cl}(\psi)\mathbf{S} + \not{D}_A \sigma, \mathbf{T} \rangle - \frac{1}{2}\langle \mathbf{T}, [\phi, \mathbf{T}] \rangle \right] \\ & + \int_X * [\text{Tr} D_A^* \psi \wedge \eta - \langle \sigma, [\eta, \mathbf{S}] \rangle - i\text{Tr} \phi D_A^* D_A \lambda \\ & - i\langle [\lambda, \mathbf{S}], [\phi, \mathbf{S}] \rangle + i\text{Tr} \{\psi, \psi\}_0 \lambda + i\langle \sigma, [\lambda, \sigma] \rangle] \end{aligned} \quad (3.20)$$

is the Lagrangian of a quantum field theory obtained by a suitable topological twisting of $N = 4$ supersymmetric Yang-Mills theory. Therefore, (3.16) can be regarded as a formula for the partition function

$$Z_{NSW} \sim \text{const} \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\mathbf{S} \mathcal{D}\sigma \mathcal{D}\eta \mathcal{D}\lambda \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\mathbf{T} \exp(-\mathcal{L}_{NSW}). \quad (3.21)$$

There is a corresponding statement for the correlation functions of the theory; the interesting observables are products of the $\hat{\mu}$ -classes defined in (3.1). For such an observable \mathcal{O} we have

Proposition 3.9.

$$\langle \mathcal{O} \rangle_{NSW} \sim \text{const} \int \mathcal{D}A \mathcal{D}\psi \mathcal{D}\mathbf{S} \mathcal{D}\sigma \mathcal{D}\eta \mathcal{D}\lambda \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\mathbf{T} \exp(-\mathcal{L}_{NSW}) \wedge \mathcal{O}. \quad (3.22)$$

It is important to point out that TNSW results from a *different* topological twisting of $N = 4$ super-Yang-Mills than the Vafa-Witten theory described before; the two twists coincide only in the case of a hyperkähler four-manifold.

Notice that we have suppressed the parameter t from the notation: we wrote Z_{NSW} and $\langle \mathcal{O} \rangle_{NSW}$ instead of $Z_{NSW}(t)$ and $\langle \mathcal{O} \rangle_{NSW}(t)$. This is based on the following

Statement 3.2. *The quantities $Z_{NSW}(t)$ and $\langle \mathcal{O} \rangle_{NSW}(t)$ are independent of $t > 0$.*

The justification of the statement is the same as the one used for Donaldson-Witten theory in Section 3.1. In finite dimensions, independence of t follows from the properties of the Mathai-Quillen construction. The argument for the infinite-dimensional case is given in [33], equation (3.10): the key point is to show that the variation of the path-integral with respect to the coupling constant t vanishes since the Lagrangian is a BRST commutator.

If we look back at the data P, G, V, s introduced on page 55, we notice the existence of an additional circular symmetry. Namely, if S^1 acts by scalar multiplication on the sections of W^\pm and trivially on all other spaces involved, the bundles and sections under consideration are S^1 -equivariant. We will now study the analogues of (3.21) and (3.22) obtained by replacing the G -equivariant Euler classes and vertical volume element by their $G \times S^1$ -extensions.

Let $t, r > 0$ and m be real. We introduce $\mathcal{Z}_{NSW}(t, r, m)$ and $\langle \mathcal{O} \rangle_{NSW}(t, r, m)$ through the following path-integrals:

Definition 3.4.

$$\begin{aligned} \mathcal{Z}_{NSW}(t, r, m) &\sim \text{const} \int_P \int_{\lambda, \phi, \eta \in \text{Lie}(\mathcal{G})} \Gamma_{\mathcal{G}, m}(P) \wedge \\ &W_{\mathcal{G}}(E_1, s_1) \wedge W_{\mathcal{G}, m}(E_2, s_2) \wedge \exp\left(\frac{r}{2} d_{\mathcal{G}, m} \omega\right); \end{aligned} \quad (3.23)$$

$$\begin{aligned} \langle \mathcal{O} \rangle_{NSW}(t, r, m) &\sim \text{const} \int_P \int_{\lambda, \phi, \eta \in \text{Lie}(\mathcal{G})} \Gamma_{\mathcal{G}, m}(P) \wedge \\ &W_{\mathcal{G}}(E_1, s_1) \wedge W_{\mathcal{G}, m}(E_2, s_2) \wedge \exp\left(\frac{r}{2} d_{\mathcal{G}, m} \omega\right) \wedge \mathcal{O}, \end{aligned} \quad (3.24)$$

where ω is the differential 1-form on P whose value on the vector

$$(\psi, \sigma) \in \Omega^1(\text{su}(E)) \oplus \Gamma(W^+ \otimes \text{sl}(E))$$

at a point $(A, \mathbf{S}) \in P$ is

$$\omega(\psi, \sigma) = \int_{X^4} * \langle \sigma, im\mathbf{S} \rangle \quad (3.25)$$

(where $\langle \cdot, \cdot \rangle$ is a euclidian inner product, for instance the real part of the standard hermitian one).

The constant preceding the path-integral is the same as the one appearing in (3.16), in particular it is independent of r and m . The differential form $W_{\mathcal{G}, m}(E_2, s_2)$ is the \mathcal{G}, m -equivariant Euler form defined in (2.5) (note that the \mathcal{G}, m -equivariant Euler form $W_{\mathcal{G}, m}(E_1, s_1)$ coincides with $W_{\mathcal{G}}(E_1, s_1)$ because the S^1 -action on this bundle is trivial). The operator $d_{\mathcal{G}, m}$ is the $\mathcal{G} \times S^1$ -equivariant DeRham derivative.

An easy computation leads to

Proposition 3.10.

$$d_{\mathcal{G}, m} \omega = - \int_{X^4} * [\langle \sigma, im\sigma \rangle + m^2 \|\mathbf{S}\|^2 + \langle [\phi, \mathbf{S}], im\mathbf{S} \rangle]. \quad (3.26)$$

The above definition is motivated by physical results obtained in [16] and Section 3 of [12] which can be summarized in the following

Proposition 3.11. *If X is a spin four-manifold and the Spin^c -bundles W^\pm are its \pm Spin-bundles then $\mathcal{Z}_{NSW}(t, t/m, m)$ and $\langle \mathcal{O} \rangle_{NSW}(t, t/m, m)$ are, respectively, the partition function and the correlation function corresponding to the observable \mathcal{O} in topologically twisted of $N = 4$ supersymmetric Yang-Mills theory after adding the mass term of the $N = 2$ matter hypermultiplet.*

Remark. ‘Topological twisting’ is used here in the sense of [33], Section 2.1. A generalized twisting procedure is used in the papers [12], [16], and [17] to find analogues of the previous two Statements in the case when the manifold X is no longer assumed to be spin (and so a Spin^c -structure is used instead).

Remark. Our path-integral definition involves the three parameters t , r , and m . In contrast, the above physical statement involves only two parameters, i.e. it only applies to the case $r = t/m$. We will keep all three parameters in the notation in order to clarify the geometric origin of the various terms in the Lagrangian.

We want to examine more closely the difference between the Lagrangians $\mathcal{L}_{NSW}(t)$ and $\mathcal{L}_{NSW}(t, r, m)$ appearing in (3.21) and Definition 3.4. Using the results of Chapter 2, we find by explicit computation that

Proposition 3.12.

$$\Gamma_{\mathcal{G},m}(P) = \int_{\lambda} B_{\eta} \exp \int_X * [-\text{Tr} D_A^* \psi \wedge \eta + \langle \sigma, [\eta, \mathbf{S}] \rangle + i \text{Tr} \phi D_A^* D_A \lambda + i \langle im\mathbf{S}, [\lambda, \mathbf{S}] \rangle + i \langle [\lambda, \mathbf{S}], [\phi, \mathbf{S}] \rangle - i \text{Tr} [\psi, \psi]_0 \lambda + i \langle \sigma, [\lambda, \sigma] \rangle]; \quad (3.27)$$

$$W_{\mathcal{G},m}(E_2, s_2) = B_{\mathbf{T}} \exp t \int_X * \left[-\frac{1}{2} |\not{D}_A \mathbf{S}|^2 - i \langle \text{cl}(\psi) \mathbf{S} + \not{D}_A \sigma, \mathbf{T} \rangle + \frac{1}{2} \langle \mathbf{T}, [\phi, \mathbf{T}] + im\mathbf{T} \rangle \right]. \quad (3.28)$$

By substituting the result of the proposition in (3.16) and Definition 3.4 we get

Proposition 3.13. *The Lagrangians $\mathcal{L}_{NSW}(t)$ and $\mathcal{L}_{NSW}(t, r, m)$ are related by*

$$\mathcal{L}_{NSW}(t, r, m) = \mathcal{L}_{NSW}(t) + \int_X * [i \langle im\mathbf{S}, [\lambda, \mathbf{S}] \rangle + \frac{t}{2} \langle \mathbf{T}, im\mathbf{T} \rangle - \frac{r}{2} (\langle \sigma, im\sigma \rangle + m^2 \|\mathbf{S}\|^2 + \langle [\phi, \mathbf{S}], im\mathbf{S} \rangle)]. \quad (3.29)$$

As an immediate consequence,

Proposition 3.14. *We have $\mathcal{L}_{NSW}(t, r, 0) = \mathcal{L}_{NSW}(t)$ and so*

$$\mathcal{Z}_{NSW}(t, r, 0) = \mathcal{Z}_{NSW}(t) \quad (3.30)$$

and

$$\langle \mathcal{O} \rangle_{NSW}(t, r, 0) = \langle \mathcal{O} \rangle_{NSW}(t) \quad (3.31)$$

for any $t, r > 0$.

We have explained before that $\mathcal{Z}_{NSW}(t)$ and $\langle \mathcal{O} \rangle_{NSW}(t)$ are actually independent of t . A similar property holds for the dependence of $\mathcal{Z}_{NSW}(t, r, m)$ and $\langle \mathcal{O} \rangle_{NSW}(t, r, m)$ on t and r :

Statement 3.3. *The partition function $\mathcal{Z}_{NSW}(t, r, m)$ and the correlation functions $\langle \mathcal{O} \rangle_{NSW}(t, r, m)$ are independent of t and r .*

This statement involves quantities which are formally defined through path-integrals. We will present the proof of the corresponding finite-dimensional result along with the physical argument for the infinite-dimensional case. In finite dimensions, independence of t follows from Proposition 2.13, part (i), since the universal Euler forms $W(t)$ for $t > 0$ are cohomologous. In infinite dimensions, this can be restated as an argument in BRST cohomology, the relevant fact being the vanishing of the vacuum expectation value of a Q -exact operator.

Independence of r in finite dimensions is also a consequence of Proposition 2.13 since the terms involving r in the integrand of Definition 3.4 are equivariantly exact. In infinite dimensions, independence of r follows from independence under a BRST-trivial perturbation (for more details, see [35], beginning of Section 3.2). Notice that the additional subtleties involved in the infinite-dimensional case (i.e. non-degeneracy of the kinetic terms) could generate a difference between the results obtained for $r = 0$ on one hand and any $r > 0$ on the other, and consequently the Statement doesn't make any claim about how these two situations compare.

Chapter 4

Localization of Topological Non-abelian Seiberg-Witten Theory

4.1 Localization of integrals of equivariantly closed forms

We have described in the previous section the partition function and correlation functions of massive topological non-abelian Seiberg-Witten theory as integrals of $\mathcal{G} \times S^1$ -equivariantly closed differential forms on a certain configuration space. In finite dimensions integrals of equivariantly closed forms can be reduced to integrals over the components of the fixed-point set of the action. In this chapter we adapt such localization results for use in our infinite-dimensional setting and apply them to massive topological non-abelian Seiberg-Witten theory.

We begin by reviewing one version of the localization theorem for integrals of S^1 -equivariantly closed differential forms. A proof can be found in [5].

Proposition 4.1. (the abelian localization theorem) *Let N be a compact oriented manifold endowed with an S^1 -action; let $\alpha \in \Omega_{S^1}(N)$ be equivariantly closed. Denote by m a generator of $s^1 \simeq (s^1)^*$ and by X the corresponding vector field on N . If N_0 denotes the zero set of X (this coincides with the fixed point set of the S^1 -action) then*

- 1) N_0 is a (not necessarily connected) submanifold of N whose normal bundle \mathcal{N} is orientable;
- 2)

$$\int_N \alpha = (2\pi)^{\text{rk } \mathcal{N}/2} \int_{N_0} \frac{\alpha|_{N_0}}{e_{S^1}(\mathcal{N})},$$

where $e_{S^1}(\mathcal{N})$ is the equivariant Euler form of \mathcal{N} .

We should make a few comments about this result. The integrand on the left-hand side of the formula is an element of $\mathbf{C}[m] \otimes \Omega(N)^{S^1}$, hence the result of the integration is a polynomial in m . To understand the right-hand side, recall that $e_{S^1}(\mathcal{N}) \in H_{S^1}^{\text{rk } \mathcal{N}/2}(N_0)$ can be written as

$$e_{S^1}(\mathcal{N}) = m^{\text{rk } \mathcal{N}/2} e^{(0)} + m^{\text{rk } \mathcal{N}/2-1} e^{(2)} + \dots + e^{(\text{rk } \mathcal{N})}$$

with $e^{(i)} \in H^i(N_0)$. The key fact is that $e^{(0)}$ is a non-zero constant on each component of N_0 (equal, modulo a power of 2π , to the product of the weights of the S^1 -action on \mathcal{N}). This implies that

$$e_{S^1}(\mathcal{N})^{-1} = \frac{1}{m^{\text{rk } \mathcal{N}/2} e^{(0)}} \left(1 + \sum_{k \geq 1} \left(\frac{1}{e^{(0)}} \left(m^{-1} e^{(2)} + \dots + m^{-\text{rk } \mathcal{N}/2} e^{(\text{rk } \mathcal{N})} \right) \right)^k \right)$$

is a well-defined element of $\Omega_{S^1}(N_0)_m := \mathbf{C}[m, m^{-1}] \otimes \Omega(N_0)^{S^1}$ —actually a homogeneous element of degree $-\text{rk } \mathcal{N}/2$ (recall that $\deg m = 2$).

Let us look more closely at the right-hand side of the localization formula: the integrand is homogeneous of degree $(\deg \alpha - \text{rk } \mathcal{N})/2$ and the integral over N_0 picks up those terms whose usual degree as a form is $\dim N_0$. The result is therefore a multiple of $m^{(\deg \alpha - \dim N)/2}$ (note that the power of m turns out to be independent of the dimension of the various connected components of N_0).

The significance of the localization theorem depends on whether $\deg \alpha - \dim N$ is negative or not: if $\deg \alpha - \dim N \geq 0$ then the theorem provides a formula for the integral of α in terms of integrals over the fixed-point set, whereas if $\deg \alpha - \dim N < 0$ the integral of α over N is zero so the localization result can be thought of as a relationship between some integrals on the components of the fixed-point set. To understand this statement, note that although the only component of α relevant to the integral over Y is $\alpha^{(\dim N)} m^{(\deg \alpha - \text{rk } \mathcal{N})/2}$, other components of α are involved in the integrals over the fixed-point set, so each of the terms on the right-hand side can be non-zero.

The main idea of the proof of the localization theorem is the following. Consider an S^1 -invariant metric g on N and the differential form $\omega \in \Omega^1(P)$ defined by $\omega(v) := g(v, X)$ for $v \in TP$. It is easy to see that ω is G -invariant and so $\omega \in \Omega_{S^1}^1(P)$. Since α is equivariantly closed we have, for any $r > 0$,

$$\int_N \alpha = \int_N \alpha e^{r d_{S^1} \omega} = \int_N \alpha e^{-r \|X\|^2} e^{r d\omega}.$$

This shows that, as $r \rightarrow \infty$, the above integral is localized near the points where $X = 0$, i.e. at the fixed point set of the action (note that the factor $e^{r d\omega}$ has only polynomial growth in r since $d\omega$ is nilpotent). The theorem follows by analyzing the integrand near the fixed point set.

Remark. The last paragraph explains the geometric meaning of the 1-form ω appearing in Definition 3.4. If S^1 acts on $\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$ by scalar multiplication on the spinors, the vector field X calculated at a point (A, \mathbf{S}) has the value $(0, im\mathbf{S})$ so ω is its dual with respect to the metric.

We will adapt the localization argument to a $G = \mathcal{G}_E$ -equivariant situation so that it can be applied (formally) to the action of $\mathcal{G}_E \times S^1$ on $\mathcal{A}_E \times \Gamma(W^+ \otimes E)$.

Theorem 4.1. *Assume G and S^1 act on the manifold P so that the two actions commute and the G -action is free. Let α be a $G \times S^1$ -equivariantly closed differential form and $\gamma_{G,m}$ be the equivariant vertical volume element defined on page 37, namely*

$$\gamma_{G,m} = B_\eta (e^{\theta \otimes \eta}) e^{(\Omega - m\theta(X)) \otimes \phi},$$

where θ is the S^1 -invariant G -connection determined by a $G \times S^1$ -invariant Riemannian metric on P and Ω is its curvature. The integral

$$\int_P \langle \alpha \wedge \gamma_{G,m} \rangle$$

can be written as a sum of integrals on the components of the union of all the S^1 -invariant G -orbits.

Notice that the union of the S^1 -invariant G -orbits is indeed a submanifold of P : it is the inverse image through the projection $P \rightarrow P/G$ of the set of S^1 -fixed points on P/G .

As stated, the result is rather trivial: it can be obtained immediately from the usual S^1 -localization theorem if we use the isomorphism from Proposition 2.10 and the results preceding it as well as Proposition 2.13. It is however interesting to analyze the beginning of a direct proof that would mimic the proof of the abelian localization theorem. Consider again the 1-form $\omega \in \Omega_{G,m}(P)$ defined on the vector field v by $\omega(v) := \langle v, mX \rangle$. We have

$$d_{G,m}\omega = d\omega - \|mX\|^2 - \langle C\phi, mX \rangle.$$

Recall that C denotes the infinitesimal action of $\text{Lie}(G)$ and so the term $-\langle C\phi, mX \rangle$ is a linear functional on $\text{Lie}(G)$. We have explained before that the pairing with the equivariant vertical volume element amounts to the substitution $\phi \rightarrow \Omega - m\theta(X)$ so

$$\begin{aligned} \int_P \langle \alpha \wedge \gamma_{G,m} \rangle &= \int_P \langle \alpha \wedge e^{r d_{\sigma \times S^1} \omega} \wedge \gamma_{G,m} \rangle \\ &= \int_P B_\eta (e^{\theta \otimes \eta}) \exp r (-\|mX\|^2 + \langle C(C^*C)^{-1}C^*(mX), mX \rangle - \langle C\Omega, mX \rangle + d\Omega). \end{aligned}$$

Let us look at the 0-form part of the exponent:

$$-\|mX\|^2 + \langle C(C^*C)^{-1}C^*(mX), mX \rangle = -\|mX\|^2 + \langle mX_{vert}, mX \rangle = -m^2 \langle X_{hor}, X \rangle = -m^2 \|X_{hor}\|^2,$$

where X_{vert} and X_{hor} are respectively the vertical and horizontal parts of the vector field X with respect to the G -connection. The above function is always non-positive, and equals zero on the subset of P on which X_{hor} is zero, i.e. the vector field X is vertical. This condition is equivalent to requiring that the G -orbit through the given point of P be S^1 -invariant, and so we conclude that the integral localizes to the union of all such orbits (see the discussion after Proposition 4.1).

4.2 Localization in the framework of quantum field theory

Topological massive non-abelian Seiberg-Witten theory admits a substantial simplification due to its geometric structure described in Definition 3.4. If we assume that Theorem 4.1 extends to infinite

dimensions, applying it to the $\mathcal{G}_E \times S^1$ -equivariant objects appearing in the MTNSW path-integral implies that correlation functions can be expressed as integrals on the S^1 -invariant orbits of the group of gauge transformations. Such orbits are described by

Theorem 4.2. *The union \mathcal{F} of the S^1 -invariant gauge orbits in $\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$ (where $\theta \in S^1$ acts as $\theta(A, \mathbf{S}) := (A, e^{i\theta}\mathbf{S})$) consists of two types of points:*

- 1) $\mathbf{S} = 0$ and arbitrary A .
- 2) Reducible connections A preserving a decomposition $E = L \oplus L^{-1}$ and sections $\mathbf{S} \in \Gamma(W^+ \otimes L^2)$.

Proof. Any point with $\mathbf{S} = 0$ is obviously invariant under S^1 . If (A, \mathbf{S}) belongs to an S^1 -invariant gauge orbit then there exist gauge transformations g_θ so that

$$(A, e^{i\theta}\mathbf{S}) = (g_\theta A, g_\theta^{-1}\mathbf{S});$$

g_θ cannot be equal to the constant gauge transformations ± 1 for $\theta \neq 0$, which implies that $\text{Stab}_{\mathcal{G}} A$ is non-trivial and so A is a reducible connection. If A preserves a decomposition $E = L \oplus L^{-1}$ then

$$\text{Stab}_{\mathcal{G}} A \simeq \left\{ \begin{pmatrix} e^{i\theta'} & 0 \\ 0 & e^{-i\theta'} \end{pmatrix} \mid \theta' \in \mathbb{R} \right\}$$

and so, if we write

$$\mathbf{S} = \begin{pmatrix} S_0 & S_1 \\ S_2 & -S_0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{sl}(E) \simeq \mathbb{C} \oplus L^2 \oplus L^{-2}$, the condition

$$e^{i\theta} \begin{pmatrix} S_0 & S_1 \\ S_2 & -S_0 \end{pmatrix} = \left[\begin{pmatrix} e^{i\theta'} & 0 \\ 0 & e^{-i\theta'} \end{pmatrix}, \begin{pmatrix} S_0 & S_1 \\ S_2 & -S_0 \end{pmatrix} \right]$$

forces $S_0 = 0$ and either $S_1 = 0$ or $S_2 = 0$. If, for instance, $S_2 = 0$, then we have $\mathbf{S} \in \Gamma(W^+ \otimes L^2)$, as claimed.

In the next two sections we will investigate the fixed-point contributions to MTNSW by using the abelian localization theorem (under the assumption that it extends to our infinite-dimensional framework). In the remainder of this section we present an alternative approach, in which S^1 -localization combines with a semiclassical approximation.

As explained in the remarks following Theorem 4.1, the localization of $\mathcal{Z}_{NSW}(t, r, m)$ and $\langle \mathcal{O} \rangle_{NSW}(t, r, m)$ to the union of the \mathcal{G} -invariant gauge orbits can be carried out by considering the $r \rightarrow \infty$ limit of the path-integrals. Actually Statement 3.3 allows us to localize further by taking the $t \rightarrow \infty$ limit as well: the properties of the Mathai-Quillen universal Euler forms show that, in the large t limit, the path-integrals involved in Definition 3.4 localize to the common zero set \mathcal{M}_{NSW} of the sections s_1 and s_2 , and therefore we conclude that the path-integrals eventually localize to $\mathcal{M}_{NSW} \cap \mathcal{F}$.

The zero set \mathcal{M}_{NSW} consists of pairs (A, \mathbf{S}) satisfying (the following version of) the non-abelian Seiberg-Witten equations:

$$\begin{cases} \mathcal{D}_A \mathbf{S} = 0 \\ F_A^+ + i[\mathbf{S}, \bar{\mathbf{S}}]_0 = 0. \end{cases} \quad (4.1)$$

There are two types of solutions belonging to \mathcal{F} . First, there are solutions with $\mathbf{S} = 0$, which implies $F_A^+ = 0$, so these are pairs consisting of an instanton and the zero spinor. Secondly, solutions with $\mathbf{S} \neq 0$ turn out to be solutions of an *abelian* Seiberg-Witten equation. To understand the last statement, suppose that A is a reducible connection preserving the decomposition $E = L \oplus L^{-1}$ and that $\mathbf{S} = S_1$ is an element of $\Gamma(W^+ \otimes L^2)$. The decomposition $E = L \oplus L^{-1}$ enables the identification $W^+ \otimes \mathfrak{sl}(E) \simeq W^+ \otimes (\mathbb{C} \oplus L^2 \oplus L^{-2})$; with respect to this identification,

$$i[\mathbf{S}, \bar{\mathbf{S}}] = i \begin{bmatrix} S_1 \otimes \bar{S}_1 & 0 \\ 0 & -S_1 \otimes \bar{S}_1 \end{bmatrix}$$

and so

$$i[\mathbf{S}, \bar{\mathbf{S}}]_0 = \begin{bmatrix} i(S_1 \otimes \bar{S}_1)_0 & 0 \\ 0 & -i(S_1 \otimes \bar{S}_1)_0 \end{bmatrix}.$$

Since $S \in \Gamma(W^+ \otimes L^2)$, we can regard it as a spinor for a new Spin^c -structure, $W_{new}^\pm = W^\pm \otimes L^2$, with

$$c_1(L_{new}) = c_1(W_{new}^\pm) = c + 4c_1(L)$$

The connection $A \in \mathcal{A}_L$ gives rise to a connection $A_{new} \in \mathcal{A}_{\det(W^\pm) \otimes L^4}$, so that

$$F_{A_{new}} = 4F_A + 2F_{A_0}.$$

The equation $F_A^+ + i[\mathbf{S} \otimes \bar{\mathbf{S}}]_0 = 0$ becomes

$$F_{A_{new}}^+ - 2F_{A_0}^+ + 4i(S \otimes \bar{S})_0 = 0,$$

which is a perturbed Seiberg-Witten equation for the Spin^c -structure $c + 4c_1(L)$. Notice that in the Spin case, i.e. when $\det(W^\pm)$ are trivial, the above equation reduces to the original (unperturbed) Seiberg-Witten equation.

In conclusion, the correlation functions of topological massive non-abelian Seiberg-Witten theory can be computed in terms of integrals over the moduli spaces of instantons and abelian Seiberg-Witten monopoles.

4.3 Contributions from Donaldson-Witten configurations: a formal geometric approach

In this section we study the Donaldson-Witten contributions to the partition function and correlation functions of MTNSW by using the abelian localization theorem. Let us focus on $\langle \mathcal{O} \rangle_{NSW}(m)$, the

partition function $\mathcal{Z}_{NSW}(m)$ being the particular case $\mathcal{O} = 1$. We have suppressed the parameters t and r from the notation because the partition function and correlation functions of MTNSW are independent of them, as shown by Statement 3.3. The integrals introduced in Definition 3.4 are formally pairings between a $\mathcal{G} \times S^1$ -cohomology class and an equivariant volume element; assuming that the action of $G = \mathcal{G}$ is free (that is, ignoring the pairs consisting of a reducible connection and the zero spinor), we find, according to Proposition 2.13,

$$\begin{aligned} \langle \mathcal{O} \rangle_{NSW}(m) &= \int_{\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))} e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} d_{\mathcal{G}_E, S^1} \omega \wedge \gamma_{\mathcal{G}, S^1} \\ &= \int_{\mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^+ \otimes \mathfrak{sl}(E))} e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} CW_{S^1} d_{\mathcal{G}_E, S^1} \omega, \end{aligned} \quad (4.2)$$

where $CW_{S^1} d_{\mathcal{G}_E, S^1} \omega$ is obtained from $d_{\mathcal{G}, m} \omega$ by applying the equivariant version of the Chern-Weil homomorphism. Recall that this amounts to $\phi \rightarrow \Omega_{Hor} - \mathcal{J}$, where Ω_{Hor} is the universal curvature and \mathcal{J} is its S^1 -equivariant moment. Assuming that the abelian localization theorem applies to this infinite-dimensional context, $\langle \mathcal{O} \rangle_{NSW}(m)$ can be computed in terms of the contributions from the fixed point set of the S^1 -action. Recall the result of Proposition 4.2 which implies that there are two types of fixed points, corresponding to configurations with vanishing spinor and reducible configurations.

By a formal application of the abelian localization theorem we obtain

Statement 4.1. *The contribution $\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m)$ from the set $(\mathcal{A}_E/\mathcal{G}_E) \times 0$ to the correlation function $\langle \mathcal{O} \rangle_{NSW}(m)$ equals*

$$\int_{\mathcal{A}_E/\mathcal{G}_E} \frac{e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O}}{e_{S^1}(\mathcal{N}_{\mathcal{A}_E/\mathcal{G}_E}, 0)}. \quad (4.3)$$

In the above equation, $\mathcal{N}_{\mathcal{A}_E/\mathcal{G}_E}$ denotes the normal bundle of $\mathcal{A}_E/\mathcal{G}_E$ in $\mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^+ \otimes \mathfrak{sl}(E))$. Note that the differential form $d_{\mathcal{G}_E, S^1} \omega$ does not occur anymore in (4.3); this is due to the fact that the pullback of its Chern-Weil image to the fixed-point set vanishes, as shown by the following two lemmas (we include the appropriate facts for reducible configurations, which we will use in Section 4.4).

Lemma 4.1. *The S^1 -equivariant moment \mathcal{J} has the following properties:*

- (i) $\mathcal{J}|_{\mathcal{A}_E \times 0} = 0$;
- (ii) $\mathcal{J}|_{\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))} = \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix}$.

The proof of (i) is straightforward: since the vector field X generated by the S^1 -action vanishes on $\mathcal{A}_E \times 0$ we have $\mathcal{J} = (C^*C)^{-1}C^*(mX) \equiv 0$ on that subset. For the second part, note that $mX(A, S) = (0, imS)$ satisfies

$$mX|_{\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))} = C \left(\begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \right), \quad (4.4)$$

where the matrix notation for the element of $\text{Lie}(\mathcal{G})$ comes from the splitting $E = l \oplus l^{-1}$. Hence

$$\mathcal{J}|_{\mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})} = (C^*C)^{-1}C^*C \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} = \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix}.$$

We can now prove

Lemma 4.2. (i) $\iota_{\mathcal{A}_E \times 0}^* CW d_{\mathcal{G}, m\omega} = 0$;

(ii) $\iota_{\mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})}^* CW d_{\mathcal{G}, m\omega} = 0$.

Proof. The form $CW d_{\mathcal{G}, m\omega}$ is an inhomogeneous differential form, it consists of a 0-form and a 2-form. That its 0-form part vanishes on the fixed point set was explained in the outline of the localization procedure; as for its 2-form part, it equals $d\omega - \langle C\Omega_{Hor}, mX \rangle$, which is obviously zero on $\mathcal{A}_E \times 0$ (see Proposition 3.10). On $\mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})$ we know that

$$\langle C\Omega_{Hor}, X \rangle = \langle C^*C\Omega_{Hor}, \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \rangle$$

and the pullback of the expression $C^*C\Omega_{Hor}$ to the set \mathcal{R}_l of reducible configurations has already been computed in the section on the abelian Seiberg-Witten theory. The result was that

$$\langle C^*C\Omega_{Hor}, \lambda \rangle = - \int_{X^4} *(\sigma, \lambda\sigma)$$

where $\sigma \in \Gamma(W^+ \otimes l^{\otimes 2})$. Using this fact in the expression of $\langle C\Omega_{Hor}, X \rangle$ and comparing with Proposition 3.10 complete the proof of the lemma.

Let us now identify the various parts of the integrand in (4.3). We refer to Section 3.5, page 55 for the notation. Since $E_1 = \mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(\Omega_+^2)$, the S^1 -action is trivial on this bundle and therefore the equivariant Euler class $e_{S^1}(E_1, s_1)|_{\mathcal{A}_E/\mathcal{G}_E}$ coincides with the usual Euler class $e(E_1, s_1)|_{\mathcal{A}_E/\mathcal{G}_E}$. The latter is precisely the Euler class of Donaldson-Witten theory. Recall from page 55 that $s_2(A, S) = \mathcal{P}_A S$, which vanishes on $\mathcal{A}_E/\mathcal{G}_E \times 0$. This implies that the quotient

$$C := \frac{e_{S^1}(E_2, s_2)}{e_{S^1}(\mathcal{N}_{\mathcal{A}_E/\mathcal{G}_E}, 0)}$$

reduces to

$$C = \frac{e_{S^1}(\mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^- \otimes \mathfrak{sl}(E)), 0)}{e_{S^1}(\mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^+ \otimes \mathfrak{sl}(E)), 0)}. \quad (4.5)$$

With these remarks, and by using the properties of the Donaldson-Witten theory, (4.3) transforms into

$$\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m) = \int_{\mathcal{A}_E/\mathcal{G}_E} e(E_1, s_1) \wedge C \wedge \mathcal{O} = \int_{\mathcal{M}_D} C \wedge \mathcal{O}, \quad (4.6)$$

where \mathcal{M}_D denotes the moduli space of instantons.

The major problem is to understand the quotient C . Our goal is in fact to express C as a cohomology class on $\mathcal{A}_E/\mathcal{G}_E$, so that we can relate it to the observables of Donaldson-Witten theory. This would enable us to turn (4.6) into a correlation function of DWT, i.e. a Donaldson invariant.

To achieve this goal, we again take our cue from a finite-dimensional analog of the problem. Let Y be a smooth manifold and F be a complex vector bundles over Y . This bundle can be regarded as an S^1 -equivariant bundle with respect to the scalar action on the fibres and the trivial action on the base. Denote by $c_{\text{tot}}(F)(T)$ the generating function for the total Chern class of F , i.e.

$$c_{\text{tot}}(F)(T) := c_0(F) + Tc_1(F) + \cdots + T^{\text{rk } \mathcal{E}/2} c_{\text{rk } \mathcal{E}/2}(F)$$

We then have

Proposition 4.2. *The S^1 -equivariant Euler class of F is related to the total Chern class by*

$$e_{S^1}(F) = \left(\frac{m}{2\pi}\right)^{\text{rk } F} c_{\text{tot}}(F) \left(\frac{2\pi}{m}\right). \quad (4.7)$$

Proof. According to the splitting principle, it is enough to prove the proposition for a line bundle. An expression for the equivariant Euler class can be obtained from formula (2.5) for the universal Euler class $W_{H,t}(s)$ by applying the H -equivariant Chern-Weil homomorphism. For a complex line bundle L and $s = 0$, (2.5) yields

$$\begin{aligned} e_{S^1}(L) &= \frac{1}{2\pi t} \mathcal{B}_\chi \left(\frac{t}{2} \chi(\Omega - \mathcal{J} + \phi_{S^1}) \chi \right) \\ &= \frac{1}{2\pi i} (\Omega - \mathcal{J} + im), \end{aligned}$$

where Ω is the curvature of L , \mathcal{J} is the S^1 -equivariant moment (see Section 2.3), and ϕ_{S^1} is the universal Weil element. In our case, $\phi_{S^1} = m$ is the generator of the Lie algebra of S^1 , and $\mathcal{J} = 0$ because the S^1 -action is trivial on the base. Hence

$$e_{S^1}(L) = \frac{1}{2\pi i} (\Omega + im) = c_1(L) + \frac{m}{2\pi} = \frac{m}{2\pi} c_{\text{tot}}(L) \left(\frac{2\pi}{m}\right),$$

as claimed.

Assume now that we have two complex bundles F^\pm and a bundle homomorphism $u : F^+ \rightarrow F^-$. Let C denote the (a priori formal) quotient of equivariant Euler classes $e_{S^1}(F^-)/e_{S^1}(F^+)$. Using the previous proposition we see that $e_{S^1}(F^+)$ is an invertible element of the ring of formal power series with coefficients in $H^*(Y)$, and so C does make sense as an element of this ring. Moreover

Proposition 4.3. *The power series C defined above is the generating series for the total Chern class of the difference element $F^- - F^+ \in K(Y)$, i.e.*

$$\frac{e_{S^1}(F^-)}{e_{S^1}(F^+)} = \left(\frac{m}{2\pi}\right)^{(\text{rk } F^- - \text{rk } F^+)} c_{\text{tot}}(F^- - F^+) \left(\frac{2\pi}{m}\right). \quad (4.8)$$

Proof. This follows immediately from the previous proposition and the fact that $c_{\text{tot}}(F^- - F^+) = c_{\text{tot}}(F^-)/c_{\text{tot}}(F^+)$.

There is one other way of interpreting the proposition, which will be the key to our infinite-dimensional application. Although the kernel and the cokernel of the bundle homomorphism u are not necessarily bundles, their formal difference $\text{Ker } u - \text{Coker } u$, which we will denote by $\text{Ind } u$, can be defined as the difference element $F^+ - F^-$ in K-theory. This follows from the exact sequence

$$0 \longrightarrow \text{Ker } u \longrightarrow F^+ \longrightarrow F^- \longrightarrow \text{Coker } u \longrightarrow 0.$$

With this definition, the proposition can be reformulated as

Proposition 4.4.

$$\frac{e_{S^1}(F^-)}{e_{S^1}(F^+)} = \left(\frac{m}{2\pi}\right)^{(\text{rk } F^- - \text{rk } F^+)} c_{\text{tot}}(-\text{Ind } u) \left(\frac{2\pi}{m}\right). \quad (4.9)$$

Therefore the quotient of equivariant Euler classes equals the (generating series for) the total Segre class of the ‘index bundle’ (i.e. the difference element in this finite-dimensional case). Recall that the total Segre class is the inverse of the total Chern class, so the total Segre class of an element in K-theory equals the total Chern class of minus the element.

We can now return to the quotient of infinite-dimensional of Euler classes from (4.5). The bundles F^\pm are both associated to the principal bundle $\mathcal{A}_E \rightarrow \mathcal{A}_E/\mathcal{G}_E$, with fibres $\Gamma(W^\pm \otimes \mathfrak{sl}(E))$ respectively. The family of coupled Dirac operators $(\mathcal{D}_A)_{A \in \mathcal{A}_E}$ provides a bundle homomorphism between F^+ and F^- . Of course the equivariant Euler classes $e_{S^1}(F^\pm)$ are only formal objects. However their quotient has a well-defined mathematical meaning if we assume that Proposition 4.4 extends to infinite dimensions:

Statement 4.2. *Assuming that Proposition 4.4 holds in our infinite-dimensional framework, the quotient of equivariant Euler classes introduced in (4.5) is given by*

$$C = \left(\frac{m}{2\pi}\right)^{-\text{Ind}(\mathcal{D} \otimes \mathfrak{sl}(E))} c_{\text{tot}}(-\text{Ind } \mathcal{D}) \left(\frac{2\pi}{m}\right). \quad (4.10)$$

Alternatively, we can interpret Statement 4.2 as a definition: although C is not a priori well-defined, it admits an indirect mathematical definition through (4.2).

We conclude with the implications of Statement 4.2 for the Donaldson-Witten contributions to $\langle \mathcal{O} \rangle_{NSW}(m)$. We have seen in (4.6) that these contributions can be written as integrals over Donaldson moduli spaces; in fact, as is usual in quantum field theory, one is supposed to sum over all instanton numbers, so (4.6) becomes

$$\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m) = \sum_{k \geq 0} \int_{\mathcal{M}_k} \mathcal{O} \wedge C_k, \quad (4.11)$$

where \mathcal{M}_k denotes the moduli space of instantons with instanton number k . The notation C_k stands for the component of C of the appropriate degree: if the degree of \mathcal{O} as a differential form is α and

$q_k := (\dim \mathcal{M}_k - \alpha)/2$ then \mathcal{O} has to be integrated against a differential form of degree $2q_k$ in order to get a non-zero result. Hence

Statement 4.3.

$$\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m) = \sum_{\{k | \dim \mathcal{M}_k \geq \alpha\}} \left(\frac{m}{2\pi} \right)^{-\text{Ind}(\not{D} \otimes \text{sl}(E)) - q_k} \int_{\mathcal{M}_k} \mathcal{O} \wedge s_{q_k}(\text{Ind } \not{D}), \quad (4.12)$$

where s_q denotes the q -th Segre class.

The right-hand side of (4.12) involves only finite-dimensional integrals; however, in order to make (4.12) rigorous we still have to compactify the integration spaces. The same problem appears in the definition of Donaldson invariants, and the solution used in Donaldson theory is to replace \mathcal{M}_k by its Uhlenbeck compactification. We adopt the same approach: we will interpret the integrals on the right-hand side of (4.12) as integrals over the compactification $\bar{\mathcal{M}}_k$:

Statement 4.4.

$$\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m) = \sum_{\{k | \dim \mathcal{M}_k \geq \alpha\}} \left(\frac{m}{2\pi} \right)^{-\text{Ind}(\not{D} \otimes \text{sl}(E)) - q_k} \int_{\bar{\mathcal{M}}_k} \mathcal{O} \wedge s_{q_k}(\text{Ind } \not{D}) \quad (4.13)$$

There are two possible interpretations of Statement 4.4: it can either be regarded as a *calculation* of the correlation functions of MTNSW through a formal geometric argument or it can be considered a definition of the correlation functions. Either way, more work is needed in order to get explicit results out of (4.13). We will carry out the necessary steps in Chapter 5, where we relate the characteristic classes of the index bundle to the μ -classes and we use the Kronheimer-Mrowka structure theorem for Donaldson invariants to compute $\langle \mathcal{O} \rangle_{NSW}^{\text{pure gauge}}(m)$ concretely.

4.4 Contributions from reducible configurations

We now turn to the contribution of reducible configurations to the correlation functions and partition function of MTNSW.

We begin by setting up the notation. Inside the configuration space $P := \mathcal{A}_E \times \Gamma(W^+ \otimes \text{sl}(E))$ (recall that we actually delete from P the configurations consisting of a reducible connection and the zero spinor in order to make the action of the group of gauge transformations free) we consider the set \mathcal{R} of configurations (A, S) with A reducible and S as in Section 4.3, i.e. $S \in \Gamma(W^+ \otimes l^2)$. We have

$$\begin{aligned} \mathcal{R} &:= \bigcup_{\{E=l \oplus l^{-1}\}} \mathcal{A}_l \times \Gamma(W^+ \otimes l^2) \\ &= \bigcup \mathcal{R}_x, \text{ where } \mathcal{R}_x = \bigcup_{c_1(l)=x} \mathcal{A}_l \times (\Gamma(W^+ \otimes l^2) \setminus 0). \\ &\quad \left\{ \begin{array}{l} x \in H^2(X, \mathbb{Z}) \\ x^2 = -k \end{array} \right\} \end{aligned}$$

The union on the first line is over all line subbundles l of E such that $E = l \oplus l^{-1}$; on the second line we have partitioned the set of such line subbundles into topological types.

Let us denote by $\mathcal{N}_{\mathcal{R}_x}$ the normal bundle of \mathcal{R}_x in the configuration space $\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$. At a point $(A, S) \in \mathcal{A}_l \times \Gamma(W^+ \otimes l^2)$ the fibre of $\mathcal{N}_{\mathcal{R}_x}$ can be identified with

$$\{a \in \Omega^1(l^2) \mid D_A^* a = 0\} \oplus \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2})).$$

(Equivalently, the vector space $\{a \in \Omega^1(l^2) \mid D_A^* a = 0\}$ can be regarded as the quotient $\Omega^1(l^2)/D_A \Omega^0(l^2)$.) To see this, we use the fact that the normal space of \mathcal{A}_l in \mathcal{A}_E is isomorphic to $\Omega^1(l^2)$ (as a consequence of the isomorphism $\mathfrak{su}(E) \simeq \mathbb{R} \oplus l^2$). Since any two subbundles l_1 and l_2 of E with the same c_1 can be mapped into one another by a gauge transformation, the normal space of the set of all reducible connections in \mathcal{A}_E is the subspace of $\Omega^1(l^2)$ orthogonal to the gauge orbits, as claimed above.

The submanifolds \mathcal{R}_x are \mathcal{G}_E -invariant; if we choose one line bundle l_x for each x , we see that \mathcal{R}_x contains as a submanifold the space

$$\mathcal{R}_{l_x} := \mathcal{A}_{l_x} \times (\Gamma(W^+ \otimes l_x^2) \setminus 0),$$

which is invariant under the action of the subgroup $\mathcal{G}_{l_x} \simeq \text{Map}(X, U(1))$ of \mathcal{G}_E preserving the decomposition $E = l \oplus l^{-1}$. Note that the inclusion map descends to a diffeomorphism $\iota : \mathcal{R}_{l_x}/\mathcal{G}_{l_x} \xrightarrow{\sim} \mathcal{R}_x/\mathcal{G}_E$.

As we have already mentioned, if we ignore the elements of P consisting of a reducible connection and the zero spinor, the correlation functions of are given by (4.2)

$$\begin{aligned} \langle \mathcal{O} \rangle_{NSW}(m) &= \int_{\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))} e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} d_{\mathcal{G}_E, S^1} \omega \wedge \gamma_{\mathcal{G}, S^1} \\ &= \int_{\mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^+ \otimes \mathfrak{sl}(E))} e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} CW_{S^1} d_{\mathcal{G}_E, S^1} \omega. \end{aligned} \quad (4.14)$$

In Section 4.3 we applied abelian localization formally to (4.14) and discussed the contribution from Donaldson-Witten configurations. We now carry out the similar argument for reducible configurations.

Statement 4.5. *The contribution $\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m)$ from the set $\mathcal{R}/\mathcal{G}_E$ of reducible configurations to the correlation function $\langle \mathcal{O} \rangle_{NSW}(m)$ is given by*

$$\int_{\mathcal{R}/\mathcal{G}_E} \frac{e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} CW_{S^1} d_{\mathcal{G}_E, S^1} \omega}{e_{S^1}(\mathcal{N}_{\mathcal{R}}, 0)}, \quad (4.15)$$

where $\mathcal{N}_{\mathcal{R}}$ denotes the normal bundle of \mathcal{R} in P/\mathcal{G}_E .

The integral (4.15) is in fact a sum of integrals over the $\mathcal{R}_x/\mathcal{G}_E$ for various classes x ; it is useful to express the latter as integrals over $\mathcal{R}_{l_x}/\mathcal{G}_{l_x}$ by using the diffeomorphism ι described above. We

get

$$\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m) = \int_{\mathcal{R}_{I_x}/\mathcal{G}_{I_x}} \iota^* \left(\frac{e_{S^1}(E_1, s_1) \wedge e_{S^1}(E_2, s_2) \wedge \mathcal{O} \wedge \exp \frac{r}{2} CW_{S^1} d_{\mathcal{G}_E, S^1} \omega}{e_{S^1}(\mathcal{N}_{\mathcal{R}}, 0)} \right). \quad (4.16)$$

The structure group of the bundles E_1 , E_2 , and $\mathcal{N}_{\mathcal{R}}$ is \mathcal{G}_E (since they are associated to the principal fibration $P \rightarrow \mathcal{A}_E \times_{\mathcal{G}_E} \Gamma(W^+ \otimes \mathfrak{sl}(E))$). When pulled back to $\mathcal{R}_{I_x}/\mathcal{G}$, their structure group reduces to \mathcal{G}_I and their reductions decompose as follows (we drop the subscript x from the notation):

$$\begin{aligned} \iota^* E_1 &= \mathcal{R}_I \times_{\mathcal{G}_I} \Omega_+^2(\mathfrak{su}(E)) = \mathcal{R}_I \times_{\mathcal{G}_I} (\Omega_+^2 \oplus \Omega_+^2 \otimes l^2) =: E_1^I \oplus E_1^n; \\ \iota^* E_2 &= \mathcal{R}_I \times_{\mathcal{G}_I} \Gamma(W^- \otimes \mathfrak{sl}(E)) = \mathcal{R}_I \times_{\mathcal{G}_I} (\Gamma(W^- \otimes l^2) \oplus \Gamma(W^- \otimes (\mathbb{C} \oplus l^{-2}))) =: E_2^I \oplus E_0^n \oplus E_2^n; \\ \iota^* \mathcal{N}_{\mathcal{R}} &= \mathcal{N}_{\mathcal{R}_I} = \{a \in \Omega^1(l^2) \mid D_A^* a = 0\} \oplus \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2})) =: N_1 \oplus N_0 \oplus N_2. \end{aligned}$$

Substituting back in (4.16) and using Lemma 4.2, part (ii), we get

$$\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m) = \int_{\mathcal{R}_I/\mathcal{G}_I} \frac{e_{S^1}(E_1^I) \wedge e_{S^1}(E_1^n) \wedge e_{S^1}(E_2^I) \wedge e_{S^1}(E_0^n) \wedge e_{S^1}(E_2^n) \wedge \mathcal{O}}{e_{S^1}(N_1) \wedge e_{S^1}(N_0) \wedge e_{S^1}(N_2)}. \quad (4.17)$$

Equation (4.17) contains a (formal) quotient of equivariant Euler classes similar to the one we encountered in the previous section. We will use the method of Section 4.3 and apply Statement 4.2 to (4.17) in order to transform it into a finite-dimensional integral. To this end, we need to obtain more information about the topology of the integration space $\mathcal{C} := \mathcal{R}_I/\mathcal{G}_I$, the bundles E_1^I , E_1^n , E_2^I , E_0^n , E_2^n , N_1 , N_0 , and N_2 , and about the S^1 -action on them.

Consider the subgroup \mathcal{G}_I^0 of \mathcal{G}_I consisting of gauge transformations equal to the identity at a given point of X . The quotient is a copy of $U(1)$:

$$1 \longrightarrow \mathcal{G}_I^0 \longrightarrow \mathcal{G}_I \longrightarrow U(1) \longrightarrow 1.$$

The integration space $\mathcal{C} = \mathcal{A}_I \times_{\mathcal{G}_I} (\Gamma(W^+ \otimes l^2) - 0)$ can be alternatively thought of as $(\mathcal{A}_I/\mathcal{G}_I^0) \times \mathbb{P}$, where $\mathbb{P} := (\Gamma(W^+ \otimes l^2) - 0)/U(1)$ is diffeomorphic to $\mathbb{R}_+^* \times \mathbb{C}\mathbb{P}^\infty$. Let us denote by $\mathcal{O}(1)$ the Serre bundle over \mathbb{P} and define the following bundles over $\mathcal{A}_I/\mathcal{G}_I^0$:

$$\begin{aligned} \mathcal{E}_1^I &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Omega_+^2 & \mathcal{E}_1^n &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} (\Omega_+^2 \otimes l^2) \\ \mathcal{E}_2^I &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Gamma(W^- \otimes l^2) & \mathcal{E}_0^n &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Gamma(W^-) & \mathcal{E}_2^n &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Gamma(W^- \otimes l^{-2}) \\ \mathcal{N}_0 &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Gamma(W^+) & \mathcal{N}_1 &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Omega^1(l^2)/D_A \Omega^0(l^2) & \mathcal{N}_2 &:= \mathcal{A}_I \times_{\mathcal{G}_I^0} \Gamma(W^+ \otimes l^{-2}). \end{aligned}$$

If π denotes the projection $\mathcal{C} \rightarrow \mathcal{A}_I/\mathcal{G}_I^0$ then

Proposition 4.5. *The following isomorphisms hold:*

$$\begin{aligned} E_1^I &\simeq \pi^* \mathcal{E}_1^I & E_1^n &\simeq \pi^* \mathcal{E}_1^n \otimes \mathcal{O}(1) \\ E_2^I &\simeq \pi^* \mathcal{E}_2^I & E_0^n &\simeq \pi^* \mathcal{E}_0^n & E_2^n &\simeq \pi^* \mathcal{E}_2^n \otimes \mathcal{O}(-1) \\ N_0 &\simeq \pi^* \mathcal{N}_0 & N_1 &\simeq \pi^* \mathcal{N}_1 \otimes \mathcal{O}(1) & N_2 &\simeq \pi^* \mathcal{N}_2 \otimes \mathcal{O}(-1). \end{aligned}$$

The proof is immediate, the only information needed being the residual action of $U(1)$ on the fibre of each of the bundles involved.

As for the S^1 -action (this is the original action on W^\pm , the one producing the *equivariant* Euler classes in (4.17)), recall that it is trivial on \mathcal{C} ; for the various bundles involved in (4.17) we have

Proposition 4.6. *S^1 acts by scalar multiplication on the fibres. The S^1 -action is trivial on E_1^l and E_2^l , has weight 1 on E_0^n and N_0 , weight -1 on E_1^n and N_1 , and weight 2 on E_2^n and N_2 .*

Proof. We will just present the proof for two of the bundles, say E_1^l and N_2 , since all cases are proved similarly. The bundle E_1^l is given by the projection

$$\mathcal{A}_L \times_{\mathcal{G}_l} (\Gamma(W^+ \otimes l^2) \oplus \Gamma(W^- \otimes l^2)) \longrightarrow \mathcal{A}_L \times_{\mathcal{G}_l} (\Gamma(W^+ \otimes l^2)).$$

The action of S^1 is given by $e^{i\theta}(a, s, t) = (a, e^{i\theta}s, e^{i\theta}t)$ and apparently this is non-trivial on the fibre $\Gamma(W^- \otimes l^2)$. However, we know that the action is trivial on the base. To make this explicit, we need to act with the gauge transformation $e^{-i\theta/2}$ on $(a, e^{i\theta}s, e^{i\theta}t)$, which shows that the action on the fibre is in fact trivial.

For the bundle N_2 , given by the projection

$$\mathcal{A}_L \times_{\mathcal{G}_l} (\Gamma(W^+ \otimes l^2) \oplus \Gamma(W^+ \otimes l^{-2})) \longrightarrow \mathcal{A}_L \times_{\mathcal{G}_l} (\Gamma(W^+ \otimes l^2)),$$

the action of S^1 is given by $e^{i\theta}(a, s, t) = (a, e^{i\theta}s, e^{i\theta}t) = (a, e^{i\theta}s, e^{i\theta}t)e^{-i\theta/2} = (a, s, e^{2i\theta}t)$. The element $e^{-i\theta/2}$ acting on the right represents a gauge transformation, so its effect on $t \in \Gamma(W^+ \otimes l^{-2})$ is scalar multiplication by $e^{+i\theta}$.

Returning now to (4.17), we see that the equivariant Euler classes $e_{S^1}(E_1^l)$ and $e_{S^1}(E_2^l)$ coincide with the usual Euler classes $e(E_1^l)$ and $e(E_2^l)$ (since the S^1 -action is trivial on these bundles). Furthermore, the product $e(E_1^l)e(E_2^l)$ equals the Euler class for abelian Seiberg-Witten theory (compare with Section 3.2) and so, if we assume that Poincaré duality holds in this infinite-dimensional case, (4.17) yields

$$\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m) = \int_{\mathcal{M}_{SW}(c+4c_1(l))} \frac{e_{S^1}(E_1^n)}{e_{S^1}(N_1)} \wedge \frac{e_{S^1}(E_0^n)}{e_{S^1}(N_0)} \wedge \frac{e_{S^1}(E_2^n)}{e_{S^1}(N_2)} \wedge \mathcal{O}, \quad (4.18)$$

where $\mathcal{M}_{SW}(c+4c_1(l))$ denotes the abelian Seiberg-Witten moduli space corresponding to the (new) Spin^c -structure $c+4c_1(l)$ (see the discussion in Section 4.2).

We can now apply Statement 4.2 to (4.18); some care is required since, strictly speaking, (4.10) only applies for a weight one action. For a weight k action, m has to be changed to km and so, using proposition 4.6 we get

$$\frac{e_{S^1}(E_1^n)}{e_{S^1}(N_1)} = \left(-\frac{m}{2\pi}\right)^{-\text{Ind}(D_A^+ + i(S \otimes \bar{S})_0 + D_A^*)} c_{\text{tot}}((-\text{Ind}(D_A^+ + i(S \otimes \bar{S})_0 + D_A^*))) \left(-\frac{2\pi}{m}\right) \quad (4.19)$$

$$\frac{e_{S^1}(E_0^n)}{e_{S^1}(N_0)} = \left(\frac{m}{2\pi}\right)^{-\text{Ind}\mathcal{D}} c_{\text{tot}}(-\text{Ind}\mathcal{D}) \left(\frac{2\pi}{m}\right) \quad (4.20)$$

$$\frac{e_{S^1}(E_2^n)}{e_{S^1}(N_2)} = \left(\frac{2m}{2\pi}\right)^{-\text{Ind}\mathcal{D} \otimes l^{-2}} c_{\text{tot}}(-\text{Ind}\mathcal{D} \otimes l^{-2}) \left(\frac{2\pi}{2m}\right). \quad (4.21)$$

These equalities, substituted into (4.18), provide an expression for $\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m)$ as an integral of a cohomology class over the finite-dimensional manifold \mathcal{M}_{SW} .

Remark. It is known that, for a generic metric on X , \mathcal{M}_{SW} is a *compact* orientable smooth manifold (see [36]). Unlike Statement 4.3, (4.18) equates the contributions from reducible configurations to the correlation functions of MTNSW to rigorously defined objects (cohomology pairings). There is no need in this case to compactify the moduli space, as we had to do in (4.12).

The right-hand side of (4.18) involves a cohomology pairing on \mathcal{M}_{SW} . We now describe the cohomology of \mathcal{M}_{SW} . Recall that $\mathcal{M}_{SW} \subset \mathcal{C}$, where $\mathcal{C} = \mathcal{A}_l \times_{\mathcal{G}_l} \Gamma(W^+ \otimes l^2) \simeq \mathcal{A}_l / \mathcal{G}_l^0 \times \mathbb{P}$. For a simply-connected four-manifold X , the space $\mathcal{A}_l / \mathcal{G}_l^0$ is contractible, and so \mathcal{C} is homotopy-equivalent to $\mathbb{C}\mathbb{P}^\infty$. Hence $H^*(\mathcal{C}) \simeq \mathbb{C}[\gamma]$, where $\gamma := c_1 \mathcal{O}(1)$ has degree two.

At this point it is useful to recall the definition of the Seiberg-Witten invariants: given a Spin^c -structure c with corresponding Spin^c -bundles W^\pm and $\det(W^\pm) \simeq l$, the Seiberg-Witten invariant $SW(c)$ is given by

$$SW(c) := \int_{\mathcal{M}_{SW}(c)} \gamma^{\dim \mathcal{M}_{SW}(c)/2} \quad (4.22)$$

(if the dimension of the moduli space is odd, the invariant is defined to be zero).

Note that that integrand on the right-hand side of (4.18) must be a multiple, say β of $\gamma^{\dim \mathcal{M}_{SW}(c)/2}$ (since it is the pull-back of a cohomology class on \mathcal{C}) and, moreover, that the integral is a multiple of a Seiberg-Witten invariant:

$$\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m) = \sum_x \beta(c + 4x) \cdot SW(c + 4x). \quad (4.23)$$

The sum in the above formula is over all $x \in H^2(X, \mathbb{Z})$ such that $x^2 = -k = -c_2(E)$. The values of the constants $\beta(c + 4x)$ (which are determined by the cohomology class of the integrand of (4.18)) can be determined in principle by applying the families index theorem to the various index bundles involved.

We will discuss the families index theorem in the next chapter; in the remainder of this section, we restrict ourselves to a case in which we can obtain explicit results in a much easier way. Namely, we restrict to simply-connected four-manifolds X which are of *Seiberg-Witten simple type*. By definition, the simple type condition requires that the only non-zero Seiberg-Witten invariants arise from Seiberg-Witten moduli spaces of virtual dimension zero. Since $\text{v.dim } \mathcal{M}_{SW} = (c^2 - 2\chi - 3\sigma)/4$ (as explained in Section 3.2), the simple type condition amounts to $SW(c) = 0$ if $c^2 \neq 2\chi + 3\sigma$. In other words, comparing with (4.22), this is equivalent to requiring that the cycle \mathcal{M}_{SW} be homologically trivial inside \mathcal{C} if $c^2 \neq 2\chi + 3\sigma$. We therefore conclude that the simple type condition guarantees that the contribution to $\langle \mathcal{O} \rangle_{NSW}(m)$ vanishes for all reductions $E = l \oplus l^{-1}$ with $c_1(l) = x$ and $(c + 4x)^2 \neq 2\chi + 3\sigma$.

It is in fact conjectured that all simply-connected four-manifolds with $b_2^+ > 1$ are of simple type. The conjecture is verified in all cases in which the Seiberg-Witten invariants are explicitly

known, including K3 surfaces, elliptic surfaces, and minimal surfaces of general type. Simple type is also stable under various topological operations such as blow-ups, connected sums, and rational blow-downs—which shows that the class of manifolds having this property is very rich and so, for our purposes, it is therefore a rather mild assumption.

Let us now work $\langle \mathcal{O} \rangle_{NSW}(m)$ under the simple type assumption. The contributions to $\langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m)$ from moduli spaces of virtual dimension zero, are considerably simpler, since only the degree zero components in the integrand matter. We will denote by $\mathcal{O}_0(c)$ the restriction of \mathcal{O} to $\mathcal{M}_{SW}(c)$. Concretely, if \mathcal{O} is a S^1 -equivariant differential form of degree $2d$, $\mathcal{O}_0(c)$ is a multiple of m^d .

The degree zero part of the total Chern forms in (4.19)–(4.21) is equal to one, so we only have to compute the values of the indices in the exponents; by applying the index theorem we obtain

$$\begin{aligned} \text{Ind}(D_A^+ + i(S \otimes \bar{S})_0 + D_A^*) &= 4k - \frac{1}{2}(\chi + \sigma) \\ \text{Ind } \mathcal{D} &= \frac{1}{8}(c \cdot c - \sigma) \\ \text{Ind}(\mathcal{D} \otimes l^{-2}) &= \frac{1}{8}(c \cdot c - \sigma) - c \cdot x - 2k, \end{aligned}$$

where $k = c_2(E)$ is the instanton number and $c_1(l) = x$ (so that $x \cdot x = -k$). Substituting into (4.18) yields

Statement 4.6. *For a simply-connected four-manifold of simple type, the contributions from reducibles to the correlation functions of MTNSW are given by*

$$\begin{aligned} \langle \mathcal{O} \rangle_{NSW}^{\text{reducibles}}(m) &= (-1)^{4k - (\chi + \sigma)/2} 2^{4k + (2\sigma - c \cdot c)/4} \\ &\quad \left(\frac{m}{2\pi}\right)^{-3(c \cdot c - 2\chi - 3\sigma)/8} \sum_x \mathcal{O}_0(c + 4x) SW(c + 4x), \end{aligned} \quad (4.24)$$

where the sum is over all $x \in H^2(X, \mathbb{Z})$ such that $x \cdot x = -k$ and $(c + 4x) \cdot (c + 4x) = 2\chi + 3\sigma$ (so that $\text{v.dim } \mathcal{M}_{SW}(c + 4x) = 0$).

The results of Sections 4.3 and 4.4 (Statements 4.4 and 4.6) provide an expression of the partition function and correlation functions of MTNSW in terms of well-defined finite-dimensional integrals. Statement 4.6 is more precise, since we have succeeded in this section in computing the relevant finite-dimensional integrals in terms of Seiberg-Witten invariants. In the next chapter, we turn to the similar problem for (4.13); we will first express the integrals over $\bar{\mathcal{M}}_k$ in terms of Donaldson invariants, and then, using the Kronheimer-Mrowka structure theorem for simple type manifolds, we will eventually calculate (4.13) in terms of Seiberg-Witten invariants.

Chapter 5

Characteristic Classes of the Index Bundle and Instanton Contributions

We have seen in the previous chapter that, in order to calculate the partition function of MTNSW explicitly, we need to express the Chern (or Segre) classes of the index bundle in terms of so-called μ -classes, i.e. the set of generators of the cohomology ring of the space of connections modulo gauge transformations, and then evaluate the corresponding Donaldson invariants. We will achieve this goal by using techniques similar in spirit to those of [2]. Throughout the chapter we work on a simply-connected four-manifold X ; in Section 5.3, we also assume that X is of simple type.

5.1 A review of the index bundle and its Chern character

Let X^4 be a simply-connected compact oriented Riemannian four-manifold and E be an $SU(2)$ complex vector bundle over X^4 endowed with a fixed hermitian metric. Fix c a Spin^c -structure on X with corresponding Spin^c -bundles W^\pm such that $\Lambda^2 W^\pm \simeq L$ and $c_1(L) = c$.

Recall that, since X is simply-connected, the space of all Spin^c -structures on X is a torsor for $H^2(X, \mathbb{Z})$ or, in more down-to-earth terms, given one Spin^c -structure any other one can be obtained by tensoring the Spin^c -bundles with a line bundle l :

$$W_{new}^\pm = W^\pm \otimes l.$$

Since $\Lambda^2(W_{new}^\pm) \simeq \Lambda^2 W^\pm \otimes l^2$, we see that $c_{new} = c + 2c_1(l)$; the general statement is that the set of Spin^c -structures on X coincides with the classes in $H^2(X, \mathbb{Z})$ which are congruent to $w_2(X)$ modulo 2.

Let us also fix a unitary connection A_0 on L and, for a unitary connection A on E , let us consider the coupled Dirac operator

$$\mathcal{D}_A : C^\infty(W^+ \otimes \text{sl}(E)) \longrightarrow C^\infty(W^- \otimes \text{sl}(E)).$$

The above definition makes use, of course, of the Levi-Civita connection on the spinor bundles. For varying A we get a family of elliptic operators over the space \mathcal{A} of unitary connections on E . The corresponding index bundle turns out to be equivariant with respect to the action of the group \mathcal{G}^\wedge of unitary gauge transformations of unit determinant. If we restrict to the subspace \mathcal{A}^* of irreducible connections, on which the group of gauge transformations acts freely, we therefore obtain an element $\text{Ind } \mathcal{D} \in K(\mathcal{A}^*/\mathcal{G})$. The above framework is described in detail in [2].

It is also explained in [2] how to compute a differential form representative for the Chern character of $\text{Ind } \mathcal{D}$ by using the families index theorem. Over $X \times \mathcal{A}^*/\mathcal{G}$ there exists a universal $\text{SO}(3) \simeq \text{PU}(2)$ -bundle endowed with a universal connection; let us denote by \mathcal{U} its complexification (this is an $\text{sl}(3)$ -bundle whose restriction to $X \times \{A\}$ is isomorphic to $\text{sl}(E)$ and the restriction of the universal connection is precisely the connection induced by A). The families index theorem gives

$$\text{ch}(\text{Ind } \mathcal{D}) = \int_X \widehat{A}_c(TX) \text{ch}(\mathcal{U}) \in H^*(\mathcal{A}^*/\mathcal{G}),$$

where \widehat{A}_c is the Spin^c - \widehat{A} -class given in this case by

$$\widehat{A}_c(TX) = (1 + \frac{1}{2}c + \frac{1}{8}c \wedge c)(1 - \frac{1}{24}p_1(X)) = 1 + \frac{1}{2}c + \frac{1}{8}(c \wedge c - \frac{1}{3}p_1).$$

The Chern character of the universal bundle can be obtained from the formulae for the universal curvature given in [2]. Notice that, since \mathcal{U} is an $\text{sl}(3)$ -bundle, $c_1(\mathcal{U}) = 0$ and $c_n(\mathcal{U}) = 0$ for $n \geq 3$ and so

$$\text{ch}(\mathcal{U}) = 1 + 2 \sum_{n \geq 0} \frac{[-c_2(\mathcal{U})]^n}{(2n)!}.$$

Since $c_1(\mathcal{U}) = 0$ we have

$$c_2(\mathcal{U}) = \frac{1}{8\pi^2} \text{Tr } \mathcal{F} \wedge \mathcal{F},$$

\mathcal{F} being the curvature of the universal connection. The 4-form $\mathcal{F}^2 := \text{Tr } \mathcal{F} \wedge \mathcal{F}$ on $X \times \mathcal{A}^*/\mathcal{G}$ has a decomposition

$$\mathcal{F}^2 = \mathcal{F}_{4,0}^2 + \mathcal{F}_{3,1}^2 + \mathcal{F}_{2,2}^2 + \mathcal{F}_{1,3}^2 + \mathcal{F}_{0,4}^2$$

in which the subscripts denote the degree as a differential form on X and $\mathcal{A}^*/\mathcal{G}$ respectively. Recall that

Proposition 5.1. *With the same notations as in [2],*

$$\mathcal{F}_{4,0}^2 = \text{Tr}(F_A \wedge F_A); \quad (5.1)$$

$$\mathcal{F}_{3,1}^2 = \text{Tr}(2\psi \wedge F_A); \quad (5.2)$$

$$\mathcal{F}_{2,2}^2 = \text{Tr}(2\phi \wedge F_A + \psi \wedge \psi); \quad (5.3)$$

$$\mathcal{F}_{1,3}^2 = \text{Tr}(2\phi \wedge \psi); \quad (5.4)$$

$$\mathcal{F}_{0,4}^2 = \text{Tr}(\phi^2). \quad (5.5)$$

Trace in the above expressions means trace in the adjoint representation since we are computing the curvature of the associated bundle \mathcal{U} .

Since X is simply-connected, $H^1(X) = 0$ and $H^3(X) = 0$ and so

$$H^4(X \times \mathcal{A}^*/G) \simeq H^4(X) \otimes H^0(\mathcal{A}^*/G) \oplus H^2(X) \otimes H^2(\mathcal{A}^*/G) \oplus H^0(X) \otimes H^4(\mathcal{A}^*/G).$$

It is therefore easy to compute the cohomology class $\xi := c_2(\mathcal{U})$ by integrating the differential form $c_2(\mathcal{U})$ over cycles in X (of dimension 4, 2, 0, respectively). If $\xi = \xi_{4,0} + \xi_{2,2} + \xi_{0,4}$ with respect to the above decomposition, Σ is a 2-cycle in X , and Π is a 0-cycle then

$$\begin{aligned} \xi_{4,0}(X) &= \frac{1}{8\pi^2} \int_X \mathcal{F}_{4,0}^2 = c_2(\text{sl}(E)) = 4k \\ \xi_{2,2}(\Sigma) &= \frac{1}{8\pi^2} \int_\Sigma \mathcal{F}_{2,2}^2 = 4\hat{\mu}(\Sigma) \\ \xi_{0,4}(\Pi) &= \frac{1}{8\pi^2} \mathcal{F}_{0,4}^2|_\Pi = 4\hat{\mu}(\Pi). \end{aligned}$$

The factors of 4 arise because Tr denotes the trace in the adjoint representation, which is four times the trace in the fundamental representation of $\text{SU}(2)$ (used in the definition of the μ -classes). If $\text{vol} \in H^4(X, \mathbf{Z})$ is the positive generator, $\Sigma_1, \dots, \Sigma_{b_2}$ is a basis of $H_2(X, \mathbf{Z})$ with dual basis $\Sigma_1^*, \dots, \Sigma_{b_2}^*$ in $H^2(X, \mathbf{Z})$, and $\Pi \in H^0(X, \mathbf{Z})$ the class of a point then we find that

$$\xi = 4 \left(k \text{vol} \otimes 1 + \sum_{i=1}^{b_2} \Sigma_i^* \otimes \hat{\mu}(\Sigma_i) + \hat{\mu}(\Pi) \right). \quad (5.6)$$

Let us denote by ch'_n the characteristic class $n! ch_n$. The last equation implies the following formulae for the components of the Chern character:

$$\begin{aligned} ch'_{0,4n}(\mathcal{U}) &= (-4)^n \hat{\mu}(\Pi)^n; \\ ch'_{2,4n-2}(\mathcal{U}) &= (-4)^n n \left(\sum_{i=1}^{b_2} \Sigma_i^* \otimes \hat{\mu}(\Sigma_i) \right) \hat{\mu}(\Pi)^{n-1} \quad (n \geq 1); \\ ch'_{4,4n-4}(\mathcal{U}) &= (-4)^n n k \text{vol} \hat{\mu}(\Pi)^{n-1} + (-4)^n \frac{n(n-1)}{2} \sum_{i,j} \Sigma_i^* \wedge \Sigma_j^* \hat{\mu}(\Pi)^{n-2} \quad (n \geq 2). \end{aligned}$$

The application of the families index theorem yields

$$ch \text{Ind } \mathcal{D} = \int_X \left[1 + \frac{1}{2}c + \frac{1}{8}(c \wedge c - \frac{1}{3}p_1) \right] \wedge \left[1 + 2 \sum_{n \geq 0} \frac{(-\xi)^n}{(2n)!} \right]. \quad (5.7)$$

Expanding the right-hand side according to the previous formulae gives

Proposition 5.2.

$$ch_0(\text{Ind } \mathcal{D}) = \frac{3}{8}(c \cdot c - 2\chi - 3\sigma) - 4k; \quad (5.8)$$

$$ch'_{2n}(\text{Ind } \mathcal{D}) = 2(-4)^n \hat{\mu}(\Pi)^{n-1} \left[\left(\frac{1}{8}(c \cdot c - \sigma) - 4(n+1)k \right) \hat{\mu}(\Pi) - 4 \frac{n(n+1)}{2} \sum_{i,j} \Sigma_i^* \cdot \Sigma_j^* \hat{\mu}(\Sigma_i) \hat{\mu}(\Sigma_j) \right]; \quad (5.9)$$

$$ch'_{2n+1}(\text{Ind } \mathcal{D}) = \frac{1}{2}(-4)^{n+1} \hat{\mu}(\Pi)^n \sum_i (\Sigma_i \cdot c) \hat{\mu}(\Sigma_i). \quad (5.10)$$

5.2 Chern and Segre classes of the index bundle

The above proposition contains explicit expressions for the components of the Chern character in terms of $\hat{\mu}$ -classes. To do the same for the Chern and classes, we now have to use the ‘Newton formula’ (see [20]):

$$ch'_n - c_1 ch'_{n-1} + \cdots + (-1)^{n-1} c_{n-1} ch'_1 + (-1)^n n c_n = 0, \quad (5.11)$$

where c_n are the Chern classes. Finally, the Segre classes s_n are related to the Chern classes by

$$s_n + s_{n-1} c_1 + \cdots + s_1 c_{n-1} + c_n = 0. \quad (5.12)$$

Let us first reformulate Proposition 5.2 slightly so that the problem reduces to pure algebra. Consider the following cohomology classes:

$$\begin{aligned} X &:= -4\hat{\mu}(\Pi) \in H_{\mathcal{G}}^4(\mathcal{A}); \\ Y &:= 4 \sum_{i,j} \Sigma_i^* \cdot \Sigma_j^* \hat{\mu}(\Sigma_i) \hat{\mu}(\Sigma_j) \in H_{\mathcal{G}}^4(\mathcal{A}); \\ Z &:= 2 \sum_i (\Sigma_i \cdot c) \hat{\mu}(\Sigma_i) \in H_{\mathcal{G}}^2(\mathcal{A}). \end{aligned}$$

The results of Proposition 5.2 can be rewritten in this new notation as

$$ch'_{2n}(\text{Ind } \mathcal{D}) = [a + b(n+1)]X^n + 4n(n+1)X^{n-1}Y; \quad (5.13)$$

$$ch'_{2n+1}(\text{Ind } \mathcal{D}) = -ZX^n, \quad (5.14)$$

with $a = (c \cdot c - \sigma)/4$ and $b = -8k$. Therefore we can work in the subring generated by X , Y , and Z in $H_{\mathcal{G}}^*(\mathcal{A})$ or, equivalently, in the polynomial ring $\mathbb{Q}[X, Y, Z]$.

The easiest approach is through the use of the following generating series (in a formal variable T)

$$\begin{aligned} c(T) &:= 1 + c_1 T + c_2 T^2 + \cdots; \\ ch(T) &:= ch_1 T + ch_2 T^2 + \cdots; \\ s(T) &:= 1 + s_1 T + s_2 T^2 + \cdots. \end{aligned}$$

The recurrence relations (5.11) and (5.12) become

$$\begin{aligned} c(T)ch(-T) &= \sum_{n \geq 1} -nc_n T^n = -T \frac{d}{dT} c(T) \\ c(T)s(T) &= 1. \end{aligned}$$

The key point is that the first one leads to the differential equation

$$\frac{c'}{c}(T) = -\frac{ch(-T)}{T} \quad (5.15)$$

and so

$$\ln c(T) = -\int \frac{ch(-T)}{T} + C, \quad (5.16)$$

where C is independent of T (in our case C will be a function of X, Y, Z). Notice that the generating function for the Segre classes can be computed immediately since $\ln s(T) = -\ln c(T)$. As a general remark, also notice that (5.15) is valid for any K-theory element over any manifold, so it can be used in general to relate the corresponding characteristic classes. As an immediate implication of (5.16), we have

Theorem 5.1. *For any element in K-theory, the total Segre class is related to the Chern character by*

$$s(T) = \text{e.r.p} \left(\sum_{k \geq 1} \frac{(-1)^k}{k} ch_k T^k \right) \quad (5.17)$$

In our problem, the formulae for the Chern character given above imply that

$$\begin{aligned} ch(T) &= \sum_{n \geq 1} ch_{2n}(\text{Ind } \mathcal{D}) T^{2n} + \sum_{n \geq 0} ch_{2n+1}(\text{Ind } \mathcal{D}) T^{2n+1} \\ &= a \sum_{n \geq 1} (XT^2)^n + b \sum_{n \geq 1} (n+1)(XT^2)^n + 4 \sum_{n \geq 1} n(n+1)X^{n-1}Y + \sum_{n \geq 0} -ZX^n T^{2n+1} \\ &= \frac{aXT^2}{1-XT^2} + b \left[\frac{1}{(1-XT^2)^2} - 1 \right] + \frac{4YT^2}{(1-XT^2)^3} - \frac{ZT}{1-XT^2}. \end{aligned}$$

We therefore have to solve (5.15) for this particular $ch(T)$. Notice that the recurrence relations (5.11) and (5.12) admit unique solutions, and hence (5.15) must have a unique solution which is a power series in T —this remark will be important when integrating $ch(-T)/T$.

Direct integration shows that

$$\ln s(T) = -\frac{a+b}{2} \ln(1-XT^2) + \frac{b}{2(1-XT^2)} + \frac{Y}{X} \frac{1}{(1-XT^2)^2} + \frac{Z}{2\sqrt{X}} \ln \frac{1-\sqrt{X}T}{1+\sqrt{X}T} + C. \quad (5.18)$$

The constant C is determined by the requirement that the right-hand side of (5.18) have a well-defined power series exponential: for instance, $\exp(1/(1-XT^2))$ is not well-defined since the exponent is not divisible by T , but $\exp(-1 + 1/(1-XT^2))$ is. We conclude that

$$C = -\frac{b}{2} - \frac{Y}{X}$$

and so

$$s(T) = (1 - XT^2)^{-\frac{a+b}{2}} \exp\left(\frac{bXT^2}{2(1 - XT^2)}\right) \exp\left(\frac{2YT^2 - XYT^4}{(1 - XT^2)^2}\right) \left(\frac{1 - \sqrt{XT}}{1 + \sqrt{XT}}\right)^{\frac{Z}{2\sqrt{X}}}. \quad (5.19)$$

Equation (5.19) can be read as a formula for the total Segre class if we substitute $T = 1$ and we regard the resulting expression as an inhomogeneous cohomology class:

$$s_{tot} = (1 - X)^{-\frac{a+b}{2}} \exp\left(\frac{bX}{2(1 - X)}\right) \exp\left(\frac{2Y - XY}{(1 - X)^2}\right) \left(\frac{1 - \sqrt{X}}{1 + \sqrt{X}}\right)^{\frac{Z}{2\sqrt{X}}}. \quad (5.20)$$

5.3 Contributions from instantons revisited

Our task now is to use the results of Sections 5.1 and 5.2 in order to compute the right-hand side of (4.13) explicitly. In (5.20) we have expressed the Segre classes of the index bundle in terms of the cohomology classes X, Y, Z , which are in turn defined in terms of μ -classes. In principle, this reduces the integrals in (4.13) to integrals involving only products of μ -classes, i.e. Donaldson invariants. This section is joint work with Alexandru Zaharescu.

To get concrete results, we have to compute integrals of the form

$$\int_{\mathcal{M}_k} X^m Y^n Z^r,$$

where \mathcal{M}_k is the Donaldson moduli space corresponding to $c_2(E) = k$ and $2m + 2n + r = 4k - (3/2)(1 + b_2^+)$. As usual, b_2^+ is the number of positive eigenvalues of the intersection form of X .

We will achieve this goal under the assumption that the manifold X^4 is of *Donaldson simple type*. This condition (defined below) is not the same as Seiberg-Witten simple type (the condition introduced in Section 4.3), but it is very closely related to it. All simply-connected four-manifolds with $b_2^+ \geq 2$ for which the Donaldson invariants are known satisfy the Donaldson simple type condition; moreover, Donaldson simple type is stable under all the usual topological constructions (blow-up, rational blow-down)—similarly to Seiberg-Witten simple type. Under the Donaldson simple type assumption, Kronheimer and Mrowka [14] proved a structure theorem for Donaldson invariants, which will be very useful for our computations.

We now review the definition of Donaldson simple type and the Kronheimer-Mrowka result. Recall that $\Sigma_1, \dots, \Sigma_{b_2}$ denotes a basis of $H_2(X^4, \mathbf{Z})$ and $\Sigma_1^*, \dots, \Sigma_{b_2}^*$ denotes the dual basis in $H^2(X^4, \mathbf{Z})$. The class of a point in $H^0(X^4, \mathbf{Z})$ is denoted by Π . We shall denote the Donaldson polynomials by the notation $\langle \dots \rangle$, where $\langle \Pi^\alpha \text{Sigma}_1^{\beta_1} \dots \Sigma_{b_2}^{\beta_{b_2}} \rangle$ denotes the integral of the corresponding product of μ -classes over $\tilde{\mathcal{M}}_k$, as in Donaldson theory.

Donaldson simple type is the following statement about the generating function of Donaldson invariants:

$$\left(\frac{d^2}{dp^2} - 4\right) \langle \exp(p\Pi + q\Sigma) \rangle = 0.$$

Theorem 5.2. (Kronheimer-Mrowka) *Let X^4 be a simply-connected manifold of simple type. Then there exist finitely many classes $\lambda_1, \dots, \lambda_B \in H^2(X^4, \mathbf{Z})$ (called basic classes) and non-zero integers SW_1, \dots, SW_B such that*

$$\begin{aligned} \langle \exp(p\Pi + q\Sigma) \rangle &= \\ &= \left\{ \sum_{i=1}^B SW_i \exp \left(2p + q(\Sigma \cdot \lambda_i) + \frac{1}{2}q^2(\Sigma \cdot \Sigma) \right) \right\}_{r \pmod 4}, \end{aligned} \quad (5.21)$$

where $r = -(3/2)(1 + b_2^+)$ and (5.21) is to be interpreted as an equality between formal power series. The notation $\{ \}_{r \pmod 4}$ means that, in the right-hand side, only the terms $p^\alpha q^\beta$ of degree $2\alpha + \beta \equiv r \pmod 4$ are to be considered.

Actually, in (5.21) $q\Sigma$ is just a short-hand for $\sum_j q_j \Sigma_j$ and q^β really means $\prod_j q_j^{\beta_j}$. Let us explain the significance of the condition $2\alpha + \beta \equiv r \pmod 4$: the left-hand side vanishes if the condition is not satisfied because of the mismatch between the degree of the integrand and the dimensions of the various \mathcal{M}_k 's. Alternatively we could have suppressed this condition and written the right-hand side as the combination of four exponentials.

This type of equality between power series will appear repeatedly in the sequel and we will use the notation $\{ \}_m$ or $\{ \}_{m \pmod n}$ to denote the terms of degree m and $(m \pmod n)$ respectively. Similarly, for a formal power series in the variable T the notation $\{ \}_{T^m}$ will stand for the coefficient of T^m .

The link between the two simple type conditions is provided by the conjecture that in fact the cohomology classes $\lambda_1, \dots, \lambda_B$ correspond exactly to the Spin^c -structures for which the Seiberg-Witten invariants are non-zero, and SW_i are equal to the corresponding Seiberg-Witten invariants. From now on, we will restrict to simply-connected four-manifolds which satisfy *both* simple type conditions, i.e. Donaldson and Seiberg-Witten, and we will use simple type to mean simple type in both definitions.

We begin our computations by evaluating

$$\langle \underbrace{\mu \cdots \mu}_{4l+r} \rangle,$$

with r as above, from (5.21). For this purpose we can substitute $p = 0$ in (5.21) to get

$$\langle \exp(q\Sigma) \rangle = SW \exp \left\{ \left(\frac{1}{2}q^2(\Sigma^* \cdot \Sigma^*) + q(\Sigma^* \cdot \lambda) \right) \right\}_{4l+r},$$

where the various summation notations have been omitted. For instance, we have written the right-hand side as if there was only one basic class—something we may do until just before the final answers since the right-hand side is additive.

We have

$$\begin{aligned}
& \left\{ \exp \left(\frac{q^2(\Sigma \cdot \Sigma)}{2} + q(\Sigma \cdot \lambda) \right) \right\}_{4l+r} \\
&= (4l+r)! \left\{ \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \left[\frac{q^2(\Sigma \cdot \Sigma)}{2} \right]^{n_1} \sum_{n_2=0}^{\infty} \frac{1}{n_2!} [q(\Sigma \cdot \lambda)]^{n_2} \right\}_{4l=r} \\
&= (4l+r)! \sum_{0 \leq n \leq 2l + [\frac{r}{2}]} \frac{(\Sigma \cdot \Sigma)^n (\Sigma \cdot \lambda)^{4l+r-2n}}{2^n n! (4l+r-2n)!}. \tag{5.22}
\end{aligned}$$

Notice that the Kronheimer-Mrowka formula is an invariant statement, i.e. it is obviously independent of the chosen basis for $H_2(X^4, \mathbb{Z})$. For later purposes, it would be very convenient if we could choose this basis so that it diagonalizes the intersection form

$$H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \xrightarrow{\mathcal{E}} H^4(X, \mathbb{Z}) \simeq \mathbb{Z}. \tag{5.23}$$

This can't be done over \mathbb{Z} , but, since intersection kills the torsion anyway, we can work instead in a basis w_1, \dots, w_{b_2} of $H_2(X^4, \mathbb{R})$ in which the intersection matrix has the form $\mathcal{E} = \text{diag}(\delta_1, \dots, \delta_{b_2})$ with $\delta_j \in \{1, -1\}$ (actually b_2^+ of the δ 's are equal to 1 and $b_2^- = b_2 - b_2^+$ equal to -1).

With respect to the new basis and its dual $w_1^*, \dots, w_{b_2}^*$, we can write $q\Sigma = \sum q_j w_j^*$ and $\lambda = \sum \lambda_j w_j^*$. (As explained before, we keep writing the formulae as if there existed only one basic class). With these notations,

$$\begin{aligned}
\langle \underbrace{\Sigma \cdots \Sigma}_{4l+r} \rangle &= SW \sum_{0 \leq n \leq 2l + [\frac{r}{2}]} \frac{(4l+r)!}{n! 2^n (4l+r-2n)!} \cdot \left(\sum_j q_j^2 \delta_j \right)^n \cdot \left(\sum_j q_j \lambda_j \delta_j \right)^{4l+r-2n} \\
&= SW \sum_{0 \leq n \leq 2l + [\frac{r}{2}]} \frac{(4l+r)!}{n! 2^n (4l+r-2n)!} \cdot \left(\sum_{\beta_1 + \dots + \beta_{b_2} = n} \frac{n!}{\beta_1! \cdots \beta_{b_2}!} \right. \\
&\quad \left. \prod_j q_j^{2\beta_j} \delta_j^{\beta_j} \right) \cdot \left(\sum_{\gamma_1 + \dots + \gamma_{b_2} = 4l+r-2n} \frac{(4l+r-2n)!}{\gamma_1! \cdots \gamma_{b_2}!} \prod_j (q_j \lambda_j \delta_j)^{\gamma_j} \right) \\
&= SW \sum_{0 \leq n \leq 2l + [\frac{r}{2}]} \frac{(4l+r)!}{2^n} \sum_{\substack{\beta_j, \gamma_j \\ \sum \beta_j = n \\ \sum \gamma_j = 4l+r-2n}} \prod_{j=1}^{b_2} \frac{1}{\beta_j! \gamma_j!} q_j^{2\beta_j + \gamma_j} \lambda_j^{\gamma_j} \delta_j^{\gamma_j + \beta_j} \\
&= (4l+r)! SW \sum_{\substack{0 \leq \beta_j, \gamma_j \\ 2 \sum \beta_j + \sum \gamma_j = 4l+r}} \prod_{j=1}^{b_2} \frac{1}{2^{\beta_j} \beta_j! \gamma_j!} q_j^{2\beta_j + \gamma_j} \lambda_j^{\gamma_j} \delta_j^{\gamma_j + \beta_j}. \tag{5.24}
\end{aligned}$$

On the other hand, since $\Sigma = \sum q_j w_j$, we have

$$\langle \underbrace{\Sigma \cdots \Sigma}_{4l+r} \rangle = \sum_{\sum n_j = 4l+r} \frac{(4l+r)!}{n_1! \cdots n_{b_2}!} \prod_j q_j^{n_j} \langle \prod_j w_j^{n_j} \rangle, \tag{5.25}$$

which leads to explicit formulae for $\langle \prod_j w_j^{n_j} \rangle$ if we use (5.24). The result is that if $\sum n_j = 4l + r$ then

$$\begin{aligned}
\langle \prod_j w_j^{n_j} \rangle &= \prod_j n_j! \left\{ \langle \underbrace{\Sigma \cdots \Sigma}_{4l+r} \rangle \right\}_{\prod_j q_j^{n_j}} \\
&= \prod_j n_j! SW \sum_{\substack{0 \leq \beta_j, \gamma_j \\ 2\beta_j + \gamma_j = n_j}} \prod_{j=1}^{b_2} \frac{1}{2^{\beta_j} \beta_j! \gamma_j!} \lambda_j^{\gamma_j} \delta_j^{\gamma_j + \beta_j} \\
&= \prod_j n_j! SW \prod_j \sum_{\substack{0 \leq \beta_j, \gamma_j \\ 2\beta_j + \gamma_j = n_j}} \frac{\lambda_j^{\gamma_j} \delta_j^{\gamma_j + \beta_j}}{2^{\beta_j} \beta_j! \gamma_j!} \\
&= SW \prod_j \sum_{k=0}^{\lfloor \frac{n_j}{2} \rfloor} \lambda_j^{n_j - 2k} \delta_j^{n_j - k} \frac{n_j!}{2^k k! (n_j - 2k)!}. \tag{5.26}
\end{aligned}$$

The notation on the first line stands for the coefficient of the $\prod_j q_j^{n_j}$ term in the expansion of the right-hand side as a power series. On the second line we have permuted the summation and the product, whereas on the last line k is a new notation for β_j .

Recall that our present goal is the computation of $\langle X^m Y^n Z^r \rangle$ for any m, n, r . With the information we have so far we can already compute $\langle Y^n \rangle$, in the case when $r = -(3/2)(1 + b_2^+)$ is even and $n = 2l + r/2$. (If n doesn't satisfy this condition, $\langle Y^n \rangle = 0$ by definition).

The original expression for Y , given in Section 3, was

$$Y := 4 \sum_{i,j} \Sigma_i^* \cdot \Sigma_j^* \hat{\mu}(\Sigma_i) \hat{\mu}(\Sigma_j).$$

An easy computation shows that, in terms of the new basis $\{w_j\}$,

$$Y = 4 \sum_{j=1}^{b_2} \delta_j \theta_j^2 \hat{\mu}(w_j)^2,$$

where θ_j are the eigenvalues of the intersection matrix \mathcal{E} . We infer that

$$\begin{aligned}
Y^n &= 4^n \left(4 \sum_{j=1}^{b_2} \delta_j \theta_j^2 \hat{\mu}(w_j)^2 \right)^n \\
&= \sum_{\sum m_j = n} \frac{n!}{m_1! \cdots m_{b_2}!} \prod_j \delta_j^{m_j} \theta_j^{2m_j} \hat{\mu}(w_j)^{2m_j}
\end{aligned}$$

and this implies, in conjunction with (5.26) in which n_j is replaced by $2m_j$, that

$$\begin{aligned}
\langle Y^n \rangle &= 4^n \sum_{\sum m_j = n} \frac{n!}{m_1! \cdots m_{b_2}!} \prod_j \delta_j^{m_j} \theta_j^{2m_j} \\
&\quad \cdot SW \prod_j \sum_{k=0}^{m_j} \lambda_j^{2m_j - 2k} \delta_j^{2m_j - k} \frac{2m_j!}{2^k k! (2m_j - 2k)!}. \tag{5.27}
\end{aligned}$$

The equality (5.27) can be reorganized if we introduce the notations

$$\begin{aligned} T_j &:= \delta_j \theta_j^2 \lambda_j^2 \\ S_j &:= 2\delta_j \lambda_j^2; \end{aligned} \quad (5.28)$$

with these notations,

$$\begin{aligned} \langle Y^n \rangle &= 4^n SW n! \sum_{\sum m_j = n} \prod_j T_j^{m_j} \\ &\cdot \prod_j \sum_{k=0}^{m_j} S_j^{-k} \frac{(2m_j)!}{2^k k! (2m_j - 2k)!}. \end{aligned} \quad (5.29)$$

This is best understood in the language of generating functions; namely, let $G(S, T)$ be the formal power series in variables S and T given by

$$G(S, T) = \sum_{m \geq 0} T^m \left(\sum_{k=0}^m S^{-k} \frac{(2m)!}{k! m! (2m - 2k)!} \right). \quad (5.30)$$

It is then easy to see that

$$\langle Y^n \rangle = 4^n SW n! \sum_{\sum m_j = n} \prod_j \{G(S_j, T_j T)\}_{T^{m_j}}, \quad (5.31)$$

(recall our convention that the braces followed by a monomial subscript mean the coefficient of that monomial in the power series). The last formula simplifies even more after permuting the sum and the product, leaving us with

$$\langle Y^n \rangle = 4^n SW n! \left\{ \prod_{j=1}^{b_2} G(S_j, T_j T) \right\}_{T^n}. \quad (5.32)$$

In order to complete the computation of $\langle Y^n \rangle$ we have to have a closer look at the power series $G(S, T)$.

We consider now

$$\begin{aligned} \tilde{G}(S, T) &:= \sum_{m \geq 0} \sum_{k=0}^m T^m S^k \frac{(2m)!}{(2k)! m! (m - k)!} \\ &= \sum_{k \geq 0} \frac{S^k}{(2k)!} \sum_{m \geq k} T^m \frac{(2m)!}{m! (m - k)!} \\ &= \sum_{k \geq 0} \frac{S^k}{(2k)!} H_k(T), \end{aligned} \quad (5.33)$$

where

$$H_k(T) = \sum_{m \geq k} T^m \frac{(2m)!}{m! (m - k)!}. \quad (5.34)$$

Finally let us introduce the formal power series

$$F(T) := \sum_{m \geq 0} T^m \frac{(2m)!}{m! m!} \quad (5.35)$$

and make the crucial observation that the k -th derivative of F is

$$F^{(k)}(T) = \sum_{m \geq k} \frac{(2m)!}{m!m!} T^{m-k} \frac{m!}{(m-k)!}$$

and so

$$H_k(T) = T^k F^{(k)}(T). \quad (5.36)$$

This shows that

$$\tilde{G}(S, T) = \sum_{k \geq 0} \frac{S^k T^k}{(2k)!} F^{(k)}(T). \quad (5.37)$$

The point of this approach is that $F(T)$ is in fact the MacLaurin expansion of an elementary function, namely

Theorem 5.3. For $|T| < \frac{1}{4}$,

$$\sum_{m \geq 0} T^m \frac{(2m)!}{m!m!} = (1 - 4T)^{-1/2}. \quad (5.38)$$

We won't give the proof of the theorem since a posteriori it only involves elementary calculus—the hard part is to find the elementary function on the right-hand side!

Once $F(T)$ is known in closed form, we can work out $\tilde{G}(S, T)$ explicitly. Since

$$\begin{aligned} F^{(k)}(T) &= (-4)^k (1 - 4T)^{-\frac{1}{2}-k} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{2^k} \\ &= 2^k (2k-1)!! (1 - 4T)^{-\frac{1}{2}-k}, \end{aligned}$$

we get

$$\begin{aligned} \tilde{G}(S, T) &= \sum_{k \geq 0} \frac{2^k S^k T^k}{(2k)!!} (1 - 4T)^{-\frac{1}{2}-k} \\ &= \sum_{k \geq 0} \frac{S^k T^k}{k!} (1 - 4T)^{-\frac{1}{2}-k} \\ &= (1 - 4T)^{-\frac{1}{2}} \exp\left(\frac{ST}{1 - 4T}\right). \end{aligned} \quad (5.39)$$

We obtain for $G(S, T)$:

$$G(S, T) = \tilde{G}\left(S, \frac{T}{S}\right) = \left(1 - 4\frac{T}{S}\right)^{-\frac{1}{2}} \exp\left(\frac{T}{1 - 4\frac{T}{S}}\right). \quad (5.40)$$

This implies that

$$\begin{aligned} \prod_{j=1}^{b_2} G(S_j, T_j T) &= \prod_{j=1}^{b_2} (1 - 2\theta_j^2 T)^{-\frac{1}{2}} \exp\left(\frac{\delta_j \theta_j^2 \lambda_j^2 T}{1 - 2\theta_j^2 T}\right) \\ &= \det(\text{Id} - 2\mathcal{E}^2 T)^{-\frac{1}{2}} \exp\left(\sum_j \frac{\delta_j \theta_j^2 \lambda_j^2 T}{1 - 2\theta_j^2 T}\right). \end{aligned} \quad (5.41)$$

If we substitute (5.41) in (5.32) we finally obtain

$$\begin{aligned}
\langle Y^n \rangle &= 4^n n! \left\{ \det(\text{Id} - 2\mathcal{E}^2 T)^{-\frac{1}{2}} \right. \\
&\quad \left. \cdot \sum_{i=1}^B SW_i \exp \left(\sum_j \frac{\delta_j \theta_j^2 \lambda_{ij}^2 T}{1 - 2\theta_j^2 T} \right) \right\}_{T^n} \\
&= n! \{K(T)\}_{T^n},
\end{aligned} \tag{5.42}$$

where $\lambda_{ij} = (\lambda_i \cdot w_j)$ and

$$K(T) = \det(\text{Id} - 2\mathcal{E}^2 T)^{-\frac{1}{2}} \cdot \sum_{i=1}^B SW_i \exp \left(\sum_j \frac{\delta_j \theta_j^2 \lambda_{ij}^2 T}{1 - 2\theta_j^2 T} \right). \tag{5.43}$$

The next goal is to carry out a similar argument in order to compute $\langle X^m Y^n \rangle$ for $m + n \equiv -(\tau/2) \pmod{2}$ (the result being automatically zero if $m + n \not\equiv -(\tau/2) \pmod{2}$). Due to the simple type property we have

$$\langle X^{m+2s} Y^n \rangle = 2^{6s} \langle X^m Y^n \rangle \tag{5.44}$$

(recall that X is minus four times the μ class of a point; two extra insertions of $\mu(\text{point})$ introduce a factor of 4, whereas two extra insertions of X introduce a factor of 64). This reduces the computation of $\langle X^m Y^n \rangle$ to that of either $\langle Y^n \rangle$ or $\langle XY^n \rangle$.

We have done the work for $\langle Y^n \rangle$. The argument for $\langle XY^n \rangle$ is identical, except that now we must start from

$$\langle X \underbrace{\Sigma \cdots \Sigma}_{4l+r-2} \rangle.$$

We claim that

$$\langle XY^n \rangle = (-8)4^n n! \{K(T)\}_{T^n}, \tag{5.45}$$

with $K(T)$ as in (5.43). The proof is entirely analogous to the one for $\langle Y^n \rangle$ so we won't present it here.

The nicest way of formulating the results we have obtained so far is through the use of generating series. Assuming that τ is even (otherwise all the $\langle X^m Y^n \rangle$ vanish), we have

Theorem 5.4. *The following two formal power series coincide in all degrees $T^m U^n$ for which $m + n \equiv (\tau/2) \pmod{2}$:*

$$\left\{ \left\langle \frac{\exp(YT)}{1 - XU} \right\rangle \right\}_{(\tau/2) \pmod{2}} = \left\{ \frac{K(4T)}{1 + 8U} \right\}_{(\tau/2) \pmod{2}}, \tag{5.46}$$

with $K(T)$ as in (5.43). Equivalently, since the left-hand side vanishes anyway in degrees not $\equiv (\tau/2) \pmod{2}$, this can be rewritten as

$$\left\langle \frac{\exp(YT)}{1 - XU} \right\rangle = \frac{1}{2} \left(\frac{K(4T)}{1 + 8U} + (-1)^{(\tau/2)} \frac{K(-4T)}{1 - 8U} \right). \tag{5.47}$$

The proof of the theorem is immediate, since (5.44) and (5.45) can be restated as

$$\langle X^m Y^n \rangle = (-8)^m 4^n n! \{K(T)\}_{T^n}. \quad (5.48)$$

We now address the (slightly) more complicated case of $\langle X^m Y^n Z^p \rangle$ for $u > 0$. Recall that we are now working in the basis $\{w_j\}$ for H_2 . If the components of the Spin^c-class $c \in H^2$ in the dual basis are c_j , i.e. $c = \sum c_j w_j^*$, then

$$Z = \sum_{j=1}^{b_2} (w_j^* \cdot c) \hat{\mu}(w_j) = \sum_{j=1}^{b_2} \delta_j c_j \hat{\mu}(w_j).$$

If we denote by $c_j^* := \delta_j c_j$ then the above formula reads $Z = \sum c_j^* \hat{\mu}(w_j)$.

As before, we will discuss in detail the argument leading to an expression for $\langle Y^n Z^p \rangle$ and explain briefly the changes needed to include X^m for $m > 0$. We begin with

$$Z^p = \sum_{p_1 + \dots + p_{b_2} = p} \prod_j c_j^{*p_j} \hat{\mu}(w_j)^{p_j}$$

which implies that

$$Y^n Z^p = \sum_{p_1 + \dots + p_{b_2} = p} \prod_j c_j^{*p_j} Y^n \hat{\mu}(w_j)^{p_j}$$

and so

$$\langle Y^n Z^p \rangle = \sum_{p_1 + \dots + p_{b_2} = p} \prod_j c_j^{*p_j} \langle Y^n \hat{\mu}(w_j)^{p_j} \rangle. \quad (5.49)$$

Let us analyze separately the invariant appearing in the right-hand side of (5.49). We will assume that $2n + \sum p_j = 4k + r$ for some integer k (the usual dimension compatibility).

$$\begin{aligned} & \langle Y^n \hat{\mu}(w_j)^{p_j} \rangle \\ &= 4^n \langle \sum_{\sum m_j = n} \frac{n!}{m_1! \dots m_{b_2}!} \prod_j \delta_j^{m_j} \theta_j^{2m_j} \hat{\mu}(w_j)^{2m_j + p_j} \rangle \end{aligned} \quad (5.50)$$

$$\begin{aligned} &= 4^n \sum_{\sum m_j = n} \frac{n!}{m_1! \dots m_{b_2}!} \prod_j \delta_j^{m_j} \theta_j^{2m_j} \\ & \cdot SW \prod_j \sum_{k=0}^{m_j + [\frac{p_j}{2}]} \lambda_j^{2m_j + p_j - 2k} \delta_j^{2m_j + p_j - k} \frac{(2m_j + p_j)!}{2^k k! (2m_j + p_j - 2k)!}. \end{aligned} \quad (5.51)$$

The formula (5.51) was obtained by applying (5.26) for $n_j = 2m_j + p_j$. It can be rearranged if we use the notations T_j and S_j introduced in (5.28):

$$\begin{aligned} \langle Y^n \hat{\mu}(w_j)^{p_j} \rangle &= 4^n n! SW \prod_{j=1}^{b_2} (\delta_j \lambda_j)^{p_j} \\ & \cdot \sum_{\sum m_j = n} \prod_{j=1}^{b_2} T_j^{m_j} \sum_{k=0}^{m_j + [\frac{p_j}{2}]} S_j^{-k} \frac{2m_j + p_j}{k! m_j! (2m_j + p_j)!} \\ &= 4^n n! SW \prod_{j=1}^{b_2} (\delta_j \lambda_j)^{p_j} \{ \prod_{j=1}^{b_2} G(S_j, T_j, p_j) \}_{T^n}, \end{aligned} \quad (5.52)$$

where the formal power series in S, T (and depending on the parameter p) $G(S, T, p)$ is defined by

$$G(S, T, p) = \sum_{m \geq 0} T^m \sum_{k=0}^{m, +[\frac{p}{2}]} S^{-k} \frac{(2m+p)!}{k!m!(2m+p-2k)!}. \quad (5.53)$$

Notice that for $p = 0$ the series $G(S, T, p)$ reduces to the (now familiar) series $G(S, T)$. We will therefore try to express $G(S, T, p)$ in closed form similarly to the ideas used for $G(S, T)$. We change the summation index in (5.53) by substituting $m + [p/2] - k$ with k and we consider

$$\tilde{G}(S, T, p) := G(S, ST, p)S^{[\frac{p}{2}]} \quad (5.54)$$

instead of G . If we denote by ϵ_p the binary rest of p , i.e. $\epsilon_p := p - 2[p/2]$, we have the following expression of $\tilde{G}(S, T, p)$:

$$\begin{aligned} \tilde{G}(S, T, p) &= \sum_{m \geq 0} T^m \sum_{k=0}^{m, +[\frac{p}{2}]} S^k \frac{(2m+p)!}{m! (m + [\frac{p}{2}] - k)! (2k + \epsilon_p)!} \\ &= \sum_{k \geq 0} \frac{S^k}{(2k + \epsilon_p)!} \sum_{\substack{m \geq k - [\frac{p}{2}] \\ m \geq 0}} T^m \frac{(2m+p)!}{m! (m + [\frac{p}{2}] - k)!} \\ &= \sum_{k \geq 0} \frac{S^k}{(2k + \epsilon_p)!} H_k(T, p). \end{aligned} \quad (5.55)$$

The line before (5.55) was obtained by interchanging the order of the two summations.

We will now study the formal power series

$$H_k(T, p) := \sum_{\substack{m \geq k - [\frac{p}{2}] \\ m \geq 0}} T^m \frac{(2m+p)!}{m! (m + [\frac{p}{2}] - k)!}. \quad (5.56)$$

Notice that $H_0(T, p = 0)$ coincides with $F(T)$ defined in (5.35), which admits the nice closed form expression given in Theorem 5.3. Our hope is that the same is true for $H_k(T, p)$, so we will try to relate the latter to $F(T)$. Let us introduce the more general power series $H(T, a, p)$ depending on the (non-negative integer) parameters a and p

$$H(T, a, p) := \sum_{\substack{m \geq 0 \\ m \geq -a}} T^m \frac{(2m+p)!}{m!(m+a)!}. \quad (5.57)$$

Since, for $a \geq 0$,

$$H(T, a, p) = \sum_{m \geq 0} T^m \frac{(2m)!}{m!m!} \frac{(2m+1)(2m+2) \cdots (2m+p)}{(m+1) \cdots (m+a)}$$

and, for $p \geq 2a$,

$$\frac{(2m+1)(2m+2)\cdots(2m+p)}{(m+1)\cdots(m+a)} = \underbrace{(2m+1)(2m+3)\cdots(2m+2a-1)}_{\text{odd integers}} \underbrace{(2m+2a+1)\cdots(2m+p)}_{\text{all integers}},$$

we see that, for $p \geq 2a$, $H(T, a, p)$ is an iterated derivative of $F(T)$. Specifically, the previous result implies that, for $p \geq 2a$ (and $a \geq 0$), we have

$$T^{2a}H(T, a, p) = (T^p H(T^2, a, 2a))^{(p-2a)}. \quad (5.58)$$

As for $H(T^2, a, 2a)$,

$$H(T, a, 2a) = \sum_{m \geq 0} T^m \frac{(2m)!}{m!m!} 2^{2a} (m + \frac{1}{2})(m + \frac{3}{2}) \cdots (m + a - \frac{1}{2})$$

yields, after multiplication by $T^{-1/2}$,

$$T^{-\frac{1}{2}}H(T, a, 2a) = \sum_{m \geq 0} T^{m-\frac{1}{2}} \frac{(2m)!}{m!m!} 2^{2a} (m + \frac{1}{2})(m + \frac{3}{2}) \cdots (m + a - \frac{1}{2}).$$

This fact, in conjunction with

$$2^{2a}T^{a-\frac{1}{2}}F(T) = \sum_{m \geq 0} \frac{(2m)!}{m!m!} 2^{2a}T^{m+a-\frac{1}{2}}$$

implies that

$$H(T, a, 2a) = 2^{2a}T^{\frac{1}{2}} \left(T^{a-\frac{1}{2}}F(T) \right)^{(a)}. \quad (5.59)$$

The last formula can be improved if we use the analytic properties of $F(T)$, i.e. the fact that it represents the MacLaurin expansion of $1/\sqrt{1-4T}$. Using the Leibniz formula we deduce that

$$\begin{aligned} \left(T^{a-\frac{1}{2}}F(T) \right)^{(a)} &= \sum_{k=0}^a \binom{a}{k} \left(T^{a-\frac{1}{2}} \right)^{(a-k)} F(T)^{(k)} \\ &= \sum_{k=0}^a \binom{a}{k} 4^k \left(\frac{1}{2} \frac{3}{2} \cdots (a - \frac{1}{2}) \right) T^{k-\frac{1}{2}} (1-4T)^{-\frac{1}{2}-k} \\ &= \frac{1}{2} \frac{3}{2} \cdots (a - \frac{1}{2}) T^{-\frac{1}{2}} (1-4T)^{-\frac{1}{2}} \sum_{k=0}^a \binom{a}{k} \left(\frac{4T}{1-4T} \right)^k \\ &= \frac{1}{2} \frac{3}{2} \cdots (a - \frac{1}{2}) T^{-\frac{1}{2}} (1-4T)^{-\frac{1}{2}-a}. \end{aligned}$$

Substituting the result back in (5.59) we find that

$$H(T, a, 2a) = 2^a(2a-1)!!(1-4T)^{-\frac{1}{2}-a}$$

But $H(T, a, 2a) = H_0(T, 2p)$ as can be seen from (5.56) and so we have the following result for $H_0(T, 2p)$:

$$H_0(T, 2p) = 2^p(2p-1)!!(1-4T)^{-\frac{1}{2}-p}. \quad (5.60)$$

To get the corresponding formula for $H_0(T, 2p-1)$ let us remark a recurrence formula which follows from (5.56):

$$\begin{aligned} H_0(T, 2p) &= \sum_{m \geq 0} T^m \frac{(2m+2p)!}{m!(m+p)!} \\ &= \sum_{m \geq 0} T^m \frac{(2m+2p-1)!}{m! (m + \lfloor \frac{2p-1}{2} \rfloor)!} \frac{2m+2p}{m+p} \\ &= 2H_0(T, 2p-1). \end{aligned}$$

It follows that

$$H_0(T, 2p-1) = 2^{p-1} (2p-1)!! (1-4T)^{-\frac{1}{2}-p}; \quad (5.61)$$

the last formula and (5.60) can be compressed into

$$H_0(T, p) = 2^{\lfloor \frac{p}{2} \rfloor} (p-1 + \epsilon_p)!! (1-4T)^{-\frac{1}{2} - \lfloor \frac{p+1}{2} \rfloor}. \quad (5.62)$$

We now move on to the more complicated series $H_k(T, p)$ using the same strategy as before: we will try to express H_k as an iterated derivative of H_0 . Recall that

$$\begin{aligned} H_k(T, p) &= \sum_{\substack{m \geq k - \lfloor \frac{p}{2} \rfloor \\ m \geq 0}} T^m \frac{(2m+p)!}{m! (m + \lfloor \frac{p}{2} \rfloor - k)!} \\ &= \sum_{\substack{m \geq k - \lfloor \frac{p}{2} \rfloor \\ m \geq 0}} T^m \frac{(2m+p)!}{m! (m + \lfloor \frac{p}{2} \rfloor)!} \frac{(m + \lfloor \frac{p}{2} \rfloor)!}{(m + \lfloor \frac{p}{2} \rfloor - k)!}. \end{aligned}$$

The first line is just (5.56) and the second is an attempt at describing H_k as an iterated derivative, in the spirit of the derivation of (5.59). We easily see, after multiplication by $T^{\lfloor p/2 \rfloor - k}$, that

$$\begin{aligned} T^{\lfloor \frac{p}{2} \rfloor - k} H_k(T, p) &= \left[\sum_{\substack{m \geq k - \lfloor \frac{p}{2} \rfloor \\ m \geq 0}} T^m \frac{(2m+p)!}{m! (m + \lfloor \frac{p}{2} \rfloor)!} \right]^{(k)} \\ &= \left[\sum_{m \geq 0} T^m \frac{(2m+p)!}{m! (m + \lfloor \frac{p}{2} \rfloor)!} \right]^{(k)}, \end{aligned} \quad (5.63)$$

where (5.63) is obtained by using the familiar fact that the k -th derivative of $T^{k'}$ vanishes if $k' < k$ (therefore, when changing the range of the summation index m to include the values $1, \dots, k - \lfloor p/2 \rfloor$ in case the latter is positive, all the new terms in the sum are equal to zero). Relation (5.63) is

crucial because the range of the summation would have otherwise made it quite difficult to handle H_k .

If we also notice that the quantity whose derivative is taken on the right-hand side of (5.63) is just $T^{[p/2]}H_0(T, p)$ (compare with (5.56)) then we can conclude that

$$H_k(T, p) = T^{k-[k/2]} \left[T^{[k/2]} H_0(T, p) \right]^{(k)}. \quad (5.64)$$

The last result enables us to go back to $\tilde{G}(S, T, p)$ (see (5.55)). If we use the form of H_k obtained above in (5.55), we find

$$\tilde{G}(S, T, p) = T^{-[p/2]} \sum_{k \geq 0} \frac{S^k T^k}{(2k + \epsilon_p)!} \left[T^{[k/2]} H_0(T, p) \right]^{(k)}. \quad (5.65)$$

To understand the general pattern of (5.65) we first discuss a few special cases, namely $p = 1$ and $p = 2$ ($p = 0$ is already known, since we have obtained $G(S, T)$ in closed form. For $p = 1$, (5.62) gives $H_0(T, 1) = (1 - 4T)^{-3/2}$; using this in (5.65) we infer that

$$\begin{aligned} \tilde{G}(S, T, 1) &= \sum_{k \geq 0} \frac{S^k T^k}{(2k + 1)!} \left[(1 - 4T)^{-3/2} \right]^{(k)} \\ &= \sum_{k \geq 0} \frac{S^k T^k}{(2k + 1)!} (-4)^k (1 - 4T)^{-3/2 - k} \left(-\frac{3}{2}\right) \left(-\frac{5}{2}\right) \cdots \left(-\frac{3}{2} - k + 1\right) \\ &= (1 - 4T)^{-3/2} \sum_{k \geq 0} \frac{2ST}{(2k + 1)!} (1 - 4T)^{-k} (2k + 1)!! \\ &= (1 - 4T)^{-3/2} \sum_{k \geq 0} \frac{1}{k!} \left(\frac{ST}{1 - 4T} \right)^k \\ &= (1 - 4T)^{-3/2} \exp \left(\frac{ST}{1 - 4T} \right). \end{aligned}$$

We can argue similarly for $p = 2$. Since $H_0(T, 2) = 2H_0(T, 1) = 2(1 - 4T)^{-3/2}$, the formula for $\tilde{G}(S, T, 2)$ reads

$$\begin{aligned} \tilde{G}(S, T, 2) &= 2 \sum_{k \geq 0} \frac{S^k T^{k-1}}{(2k)!} \left[T(1 - 4T)^{-3/2} \right]^{(k)} \\ &= \frac{1}{2} \sum_{k \geq 0} \frac{S^k T^{k-1}}{(2k)!} \left[(1 - 4T)^{-3/2} - (1 - 4T)^{-1/2} \right]^{(k)} \\ &= \frac{1}{2} \sum_{k \geq 0} \frac{S^k T^{k-1}}{(2k)!} \left[2^k (1 - 4T)^{-3/2 - k} (2k + 1)!! \right] - \frac{1}{2} \frac{\tilde{G}(S, T, 0)}{T} \\ &= \frac{(1 - 4T)^{-3/2}}{2T} \sum_{k \geq 0} \frac{2k + 1}{k!} \left(\frac{ST}{1 - 4T} \right)^k - \frac{1}{2T} \tilde{G}(S, T, 0) \\ &= \frac{(1 - 4T)^{-3/2}}{2T} \exp \left(\frac{ST}{1 - 4T} \right) + \frac{2ST}{1 - 4T} \frac{(1 - 4T)^{-3/2}}{2T} \exp \left(\frac{ST}{1 - 4T} \right) \\ &\quad - \frac{1}{2T} (1 - 4T)^{-1/2} \exp \left(\frac{ST}{1 - 4T} \right) \\ &= \frac{4T - 8T^2 + S}{(1 - 4T)^{3/2}} \exp \left(\frac{ST}{1 - 4T} \right). \end{aligned}$$

A general pattern emerges: for any p , the power series $\tilde{G}(S, T, p)$ is the MacLaurin expansion of a rational function in S, T times $\exp(ST/1 - 4T)$. To convince ourselves of this fact, we only have to use (5.62) and (5.64) in the formula (5.55). The latter is a little bit subtle, due to the difference between the cases p odd and p even, so, in order to fix the ideas, we will deal with the even case first, assuming $p = 2l$. The application of the Leibniz formula yields

$$\begin{aligned} 2^l(2l-1)!!\tilde{G}(S, T, 2l) &= \frac{(-1)^l}{4^l}(1-4T)^{-\frac{1}{2}}T^{-l}\sum_{s=0}^l\binom{l}{s}\frac{(-1)^s(1-4T)^{-s}}{(2s-1)!!} \\ &\cdot\sum_{k\geq 0}\frac{1}{k!}\left(\frac{ST}{1-4T}\right)^k(2k+1)\cdots(2s+2k-1), \end{aligned} \quad (5.66)$$

which shows that $\tilde{G}(S, T, 2l)$ has indeed the stated form (a similar argument holds for odd p).

However, this is not entirely satisfactory since we would want to have $\tilde{G}(S, T, p)$ in closed form—the miracle is that this goal can be achieved despite the complications suggested by (5.66)! To see this, we undertake a more detailed analysis of the power series appearing on line (5.66). Let

$$\begin{aligned} L_s(V) &:= \sum_{k\geq 0}\frac{1}{k!}V^k(2k+1)\cdots(2s+2k-1) \\ &= 2^sV^{\frac{1}{2}}\cdot\sum_{k\geq 0}\frac{1}{k!}V^{k-\frac{1}{2}}\left(k+\frac{1}{2}\right)\cdots\left(k+s-\frac{1}{2}\right). \end{aligned}$$

From the identity

$$V^{s-\frac{1}{2}}\exp V = \sum_{k\geq 0}\frac{V^{k+s-\frac{1}{2}}}{k!},$$

we obtain, by differentiating s times,

$$\left(V^{s-\frac{1}{2}}\exp V\right)^{(s)} = \sum_{k\geq 0}\left(k+\frac{1}{2}\right)\cdots\left(k+s-\frac{1}{2}\right)\frac{V^{k-\frac{1}{2}}}{k!},$$

which proves that

$$L_s(V) = 2^sV^{\frac{1}{2}}\left(V^{s-\frac{1}{2}}\exp V\right)^{(s)}.$$

It is actually more useful to put this identity in a different form by applying the Leibniz rule and carrying out the differentiations, to the effect that

$$L_s(V) = 2^s\exp V\sum_{t=0}^s\binom{s}{t}\left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\cdots\left(t+\frac{1}{2}\right)V^t. \quad (5.67)$$

Going back to (5.66), we get

$$\begin{aligned}
& 2^l(2l-1)!!\tilde{G}(S, T, 2l) \Big/ \frac{(-1)^l}{4^l}(1-4T)^{-\frac{1}{2}}T^{-l} \exp\left(\frac{ST}{1-4T}\right) \\
&= \frac{(-1)^l}{4^l}(1-4T)^{-\frac{1}{2}}T^{-l} \sum_{s=0}^l \binom{l}{s} \frac{(-1)^s(1-4T)^{-s}}{(2s-1)!!} 2^s \sum_{t=0}^s \binom{s}{t} \left(s-\frac{1}{2}\right)\left(s-\frac{3}{2}\right)\cdots\left(t+\frac{1}{2}\right)V^t \\
&= \sum_{t=0}^l V^t \sum_{s=t}^l \binom{l}{s} \binom{t}{s} \frac{(-1)^s(1-4T)^{-s}}{(2s-1)!!} 2^t(2s-1)(2s-3)\cdots(2t+1) \\
&= \sum_{t=0}^l \frac{(2V)^t}{(2t-1)!!} \frac{l!}{(l-t)!t!} \frac{1}{(4T-1)^t} \sum_{s=t}^l \binom{l-t}{s-t} (4T-1)^{-s+t} \\
&= \sum_{t=0}^l \frac{(2V)^t}{(2t-1)!!} \frac{l!}{(l-t)!t!} \frac{1}{(4T-1)^t} \left(1+\frac{1}{4T-1}\right)^{l-t} \\
&= \sum_{t=0}^l \frac{1}{(2t-1)!!} \binom{l}{t} \left(\frac{2V}{4T-1}\right)^t \left(\frac{4T}{4T-1}\right)^{l-t} \\
&= K_l \left(\frac{V}{2T}\right) \left(\frac{4T}{4T-1}\right)^l,
\end{aligned}$$

where K_l is defined by

$$K_l(V) := \sum_{t=0}^l \frac{1}{(2t-1)!!} \binom{l}{t} V^t. \quad (5.68)$$

Therefore we obtain for \tilde{G}

$$\begin{aligned}
2^l(2l-1)!!\tilde{G}(S, T, 2l) &= \frac{(-1)^l}{4^l}(1-4T)^{-\frac{1}{2}}T^{-l} \left(\frac{4T}{4T-1}\right)^l \exp\left(\frac{ST}{1-4T}\right) K_l\left(\frac{S}{2(1-4T)}\right) \\
&= (1-4T)^{-\frac{1}{2}-l} \exp\left(\frac{ST}{1-4T}\right) K_l\left(\frac{S}{2(1-4T)}\right).
\end{aligned} \quad (5.69)$$

The next task we are faced with is to write K_l in closed form. The following remark will be very useful: if $c_{2m}(X)$ denotes the coefficient of R^{2m} in $\exp(RX^2 + R)$ then, as an immediate consequence of the binomial formula,

$$c_{2m}(X) = \{\exp(RX^2 + R)\}_{R^{2m}} = \sum_{k=0}^m \frac{X^k}{k!(2m-2k)!}. \quad (5.70)$$

Notice that K_l can be rewritten as follows:

$$K_l(V) = \sum_{t=0}^l \frac{1}{(2t-1)!!} \binom{l}{t} V^t = \sum_{t=0}^l \frac{l!}{(2t)!(l-t)!} (2V)^t$$

and so we can conclude, after a slight change of notations, that

$$K_l(V) = (2V)^l l! \left\{ \exp\left(\frac{R^2}{2V} + R\right) \right\}_{R^{2l}}. \quad (5.71)$$

We can summarize the results obtained so far (recall that $V = S/2(1-4T)$) as follows (see (5.69)):

Theorem 5.5.

$$(2l)! \tilde{G}(S, T, 2l) = \left\{ (1 - 4T)^{-\frac{1}{2} - 2l} S^l \exp\left(\frac{ST}{1 - 4T}\right) \exp\left(R + \frac{R^2(1 - 4T)}{S}\right) \right\}_{R^{2l}}. \quad (5.72)$$

Finally, the closed form expression we have obtained for $\tilde{G}(S, T, 2l)$ enables us to complete the computation of $G(S, T, 2l)$. The definition (5.54) of \tilde{G} implies that

$$\begin{aligned} G(S, T, 2l) &= \tilde{G}\left(S, \frac{T}{S}, 2l\right) S^{-l} \\ &= \frac{1}{(2l)!} \left\{ \left(1 - 4\frac{T}{S}\right)^{-\frac{1}{2} - 2l} \exp\left(\frac{T}{1 - 4\frac{T}{S}}\right) \exp\left(R + \frac{R^2(1 - 4\frac{T}{S})}{S}\right) \right\}_{R^{2l}}. \end{aligned}$$

We have dealt so far with the case p even ($p = 2l$). The arguments for odd p are entirely similar (one has to go through exactly the same computational steps) and we will only write down the result.

Theorem 5.6. *For any non-negative integer p ,*

$$G(S, T, p) = \frac{1}{p!} \left\{ \left(1 - 4\frac{T}{S}\right)^{-\frac{1}{2} - p} \exp\left(\frac{T}{1 - 4\frac{T}{S}}\right) \exp\left(R + \frac{R^2(1 - 4\frac{T}{S})}{S}\right) \right\}_{R^p}. \quad (5.73)$$

We are now in a position to return to the computation of the invariants $\langle Y^n Z^p \rangle$. Combining (5.49) and (5.52) we have

$$\langle Y^n Z^p \rangle = 4^n n! SW \sum_{p_1 + \dots + p_{b_2} = p} \prod_{j=1}^{b_2} c_j^{p_j} \prod_{j=1}^{b_2} (\delta_j \lambda_j)^{p_j} \left\{ \prod_{j=1}^{b_2} G(S_j, T_j T, p_j) \right\}_{T^n}.$$

In the last expression we now want to substitute the formula (5.73) for \tilde{G} . Notice that the factors $V_j := c_j^* \delta_j \lambda_j$ can be absorbed in the exponentials after rescaling R ; explicitly, if we introduce formal variables R_1, \dots, R_{b_2} then

$$\begin{aligned} \prod_{j=1}^{b_2} c_j^{p_j} \prod_{j=1}^{b_2} (\delta_j \lambda_j)^{p_j} \prod_{j=1}^{b_2} G(S_j, T_j T, p_j) &= \\ \prod_{j=1}^{b_2} \left(1 - \frac{4T_j T}{S_j}\right)^{-\frac{1}{2}} \exp\left(\frac{T_j T}{1 - \frac{4T_j T}{S_j}}\right) \exp\left(\frac{V_j}{1 - \frac{4T_j T}{S_j}} R_j + \frac{V_j^2 / S_j}{1 - \frac{4T_j T}{S_j}} R_j^2\right). \end{aligned}$$

Therefore we conclude that

$$\langle Y^n Z^p \rangle = 4^n n! p! SW \cdot \left\{ \prod_{j=1}^{b_2} \left(1 - \frac{4T_j T}{S_j}\right)^{-\frac{1}{2}} \exp\left(\frac{T_j T}{1 - \frac{4T_j T}{S_j}}\right) \exp\left(\frac{V_j}{1 - \frac{4T_j T}{S_j}} R_j + \frac{V_j^2 / S_j}{1 - \frac{4T_j T}{S_j}} R_j^2\right) \right\}_{R^p}. \quad (5.74)$$

The last identity can be put in a nicer generating series form. Recall that n and p are supposed to satisfy the condition $2n + p \equiv r \pmod{4}$ and that we actually have to sum over basic classes, i.e. introduce the numbers SW_i for $i = 1, \dots, B$. Also recall that

$$\begin{aligned} T_{ij} &= \delta_j \theta_j^2 \lambda_{ij}^2 \\ S_{ij} &= 2\delta_j \lambda_j^2 \\ T_{ij}/S_{ij} &= \theta_j^2/2 \\ V_j &= c_j^* \delta_j \lambda_j. \end{aligned}$$

With all these preparations, the result is that

$$\left\{ \left\langle \exp\left(\frac{1}{4}YT + ZR\right) \right\rangle_{2 \deg T + \deg R \equiv r \pmod{4}} \right\} = \sum_{i=1}^B SW_i. \quad (5.75)$$

$$\left\{ \prod_{j=1}^{b_2} \left(1 - \frac{\theta_j^2}{2}T\right)^{-\frac{1}{2}} \exp\left(\frac{T_{ij}T}{1 - \frac{\theta_j^2}{2}T}\right) \exp\left(\frac{V_j}{1 - \frac{\theta_j^2}{2}T}R_j + \frac{V_j^2/S_{ij}}{1 - \frac{\theta_j^2}{2}T}R_j^2\right) \right\}_{2 \deg T + \deg R \equiv r \pmod{4}}.$$

Finally we have to discuss the insertion of X^m ; this is very easy to do (see the comments preceding Theorem 4.3). The final result is summarized below.

Theorem 5.7. *There is an equality between the corresponding terms in the following two power series in all degrees satisfying*

$$2 \deg T + 2 \deg U + \deg R \equiv r \pmod{4} :$$

$$\left\{ \left\langle \frac{\exp(\frac{1}{4}YT + ZR)}{(1 - XU)} \right\rangle_{2 \deg T + 2 \deg U + \deg R \equiv r \pmod{4}} \right\} = \sum_{i=1}^B SW_i. \quad (5.76)$$

$$\left\{ \frac{1}{1 + 8U} \prod_{j=1}^{b_2} \left(1 - \frac{\theta_j^2}{2}T\right)^{-\frac{1}{2}} \exp\left(\frac{T_{ij}T}{1 - \frac{\theta_j^2}{2}T}\right) \exp\left(\frac{V_j}{1 - \frac{\theta_j^2}{2}T}R_j + \frac{V_j^2/S_j}{1 - \frac{\theta_j^2}{2}T}R_j^2\right) \right\}_{2 \deg T + 2 \deg U + \deg R \equiv r \pmod{4}}.$$

The series on the left-hand side vanishes in all degrees which don't fulfill this condition and so we could rewrite (5.76) as an equality between the full series on the left-hand side and a combination of four series of the type appearing on the right-hand side.

Chapter 6

Semiclassical MTNSW

In chapter 4 we studied the localization properties of massive topological non-abelian Seiberg-Witten theory by using a formal infinite-dimensional generalization of the abelian localization theorem. A different approach is possible: the path integrals defining the partition function and correlation functions of MTNSW (see Definition 3.4) are independent of the parameters t and r . Therefore, one can compute them (exactly) in the $t, r \rightarrow \infty$ limit, i.e. in the semiclassical approximation. As explained in Section 4.2, in the semiclassical approximation the path-integrals localize to the fixed points of the S^1 -action on the moduli space \mathcal{M}_{NSW} , i.e. the moduli spaces of Donaldson instantons and abelian Seiberg-Witten pairs. Some quadratic integrals have to be performed in the normal directions, followed by integrals over these moduli spaces. We will find in this chapter the (quotients of) determinants which result from the normal integrals and use them to express the contributions from instantons and abelian Seiberg-Witten pairs to the partition function of MTNSW as integrals over the moduli spaces.

6.1 Contributions from instantons

In this section we study the instanton contributions to the partition function $Z_{NSW}(t, r, m)$ by applying a quadratic approximation argument to the $t, r \rightarrow \infty$ limit of their path-integral expressions. According to Statement 3.11, from a physical viewpoint it is natural to restrict to the case $r = t/m$ and so we will restrict to considering the double limit in this sense. Let $\widehat{\mathcal{M}}_D$ denote the space of anti-self-dual connections on E and \mathcal{M}_D the corresponding moduli space of instantons, i.e. the quotient of $\widehat{\mathcal{M}}_D$ by the group of unitary gauge transformations of unit determinant.

In the path-integral expression (3.24) for the correlation functions, the part of the integrand coming from $\Gamma_{\mathcal{G}, m}(P)$ is independent of t and r , therefore, to leading order in t and r , all we have to do regarding these terms is to restrict them to the space of instantons $\widehat{\mathcal{M}}_D$. The parameter t appears in the universal Euler forms $W_{\mathcal{G}}(E_1, s_1)$ and $W_{\mathcal{G}, m}(E_2, s_2)$ and the parameter r in the 'localizing

form' $d_{G,m}\omega$, therefore we have to find the quadratic part in their expansion normal to $\widehat{\mathcal{M}}_D$. We will use the following coordinates for the normal directions: since the localization is at $\mathbf{S} = 0$, we can just use \mathbf{S} and σ as normal coordinates for the spinor fields; as for the connections, recall that the normal space to the set of anti-self dual connections at a connection A is

$$\mathcal{N}_A \widehat{\mathcal{M}}_D(A) = \{a \in \Omega^1(\text{su}(E)) \mid D_A^* a = 0\}$$

so we can work with a (bosonic) and ψ (fermionic) so that $D_A^* a = D_A^* \psi = 0$.

Proposition 6.1. *IN the 'physical' case $r = t/\tau n$, the quadratic expansions in the normal directions to the space of instantons at a fixed point (A, \mathbf{S}) are*

$$\text{Quadr } W_G(E_1, s_1) = B_\chi \exp t \int_X \text{Tr} \left[-\frac{1}{2} |D_A a|^2 - i D_A a \wedge \chi \right]; \quad (6.1)$$

$$\text{Quadr } W_{G,m}(E_2, s_2) = B_T \exp t \int_X * \left[-\frac{1}{2} |\mathcal{D}_A \mathbf{S}|^2 - i \langle \mathcal{D}_A \sigma + \text{cl}(\psi) \mathbf{S}, \mathbf{T} \rangle + \frac{1}{2} \langle \mathbf{T}, (\text{ad } \phi + im) \mathbf{T} \rangle \right];$$

$$\text{Quadr } \exp \frac{t}{2} d_{G,m} \omega = \exp \frac{r}{2} \int_{X^4} * [-\langle \sigma, i\sigma \rangle - m \|\mathbf{S}\|^2 - \langle (\phi, \mathbf{S}), i\mathbf{S} \rangle]. \quad (6.3)$$

Remark. The notation $(\text{ad } \phi) \mathbf{S}$ replaces $[\phi, \mathbf{S}]$ and similarly for $(\text{ad } \phi) \mathbf{T}$. To understand correctly the meaning of the proposition, there is something we have to explain about the field ϕ . In the above formulae the field ϕ is *fixed*. Its value is the restriction of the field ϕ to the space of instantons—its components in the normal directions give lower order contributions in t, r since they only occur in cubic terms. This value will be used later on when doing the integral over ϕ . In the language of equivariant differential forms, we have seen that the ϕ integral in (3.24) amounts to substituting ϕ by the equivariant curvature and so, in this language, we have to restrict the curvature to $\widehat{\mathcal{M}}_D$. The same comment applies to ψ which has to be regarded as a differential 1-form on $\widehat{\mathcal{M}}_D$.

We infer that the quadratic integral $I_{\widehat{\mathcal{M}}_D}^{\text{normal}}$ which has to be performed in the normal directions is

$$I_{\widehat{\mathcal{M}}_D}^{\text{normal}} = I_{\mathbf{S}, \mathbf{T}, \sigma} \int D a D \psi D \chi \exp t \int_X \text{Tr} \left[-\frac{1}{2} |D_A a|^2 - i D_A \psi \wedge \chi \right], \quad (6.4)$$

where

$$I_{\mathbf{S}, \mathbf{T}, \sigma} = \int D \mathbf{S} D \mathbf{T} D \sigma \exp \left[\frac{t}{2} \int_X * [-|\mathcal{D}_A \mathbf{S}|^2 - 2i \langle \mathcal{D}_A \sigma + \text{cl}(\psi) \mathbf{S}, \mathbf{T} \rangle + \langle \mathbf{T}, (\text{ad } \phi + im) \mathbf{T} \rangle \right. \\ \left. + \frac{t}{2} \int_{X^4} * [-\langle \sigma, i\sigma \rangle - m \|\mathbf{S}\|^2 - \langle (\text{ad } \phi) \mathbf{S}, i\mathbf{S} \rangle] \right].$$

As explained before, in the above integrals A, \mathbf{S} , and ϕ are fixed. Notice that if $I_{\mathbf{S}, \mathbf{T}, \sigma}$ were not present, the integral $I_{\widehat{\mathcal{M}}_D}^{\text{normal}}$ would coincide with the quadratic normal integral obtained in Donaldson-Witten theory when localizing the correlation functions to the space of instantons. In fact it is well known (or it can be easily seen from (6.4)) that the integral over a, ψ , and χ equals ± 1 , and the sign is determined by the orientation of Donaldson moduli space. Therefore

$$I_{\widehat{\mathcal{M}}_D}^{\text{normal}} = \pm I_{\mathbf{S}, \mathbf{T}, \sigma}.$$

Let us now analyze a simplified version $I'_{\mathbf{S},\mathbf{T},\sigma}$ of $I_{\mathbf{S},\mathbf{T},\sigma}$. Notice that with the exception of the term $-it\langle \text{cl}(\psi)\mathbf{S}, \mathbf{T} \rangle$ in the integrand of $I_{\mathbf{S},\mathbf{T},\sigma}$ the bosonic and fermionic parts are decoupled from each other so if we consider

$$I'_{\mathbf{S},\mathbf{T},\sigma} = \int \mathcal{D}\mathbf{S}\mathcal{D}\mathbf{T}\mathcal{D}\sigma \exp \left[\frac{t}{2} \int_X * [-|\mathcal{D}_A \mathbf{S}|^2 - 2i\langle \mathcal{D}_A \sigma, \mathbf{T} \rangle + \langle \mathbf{T}, (\text{ad } \phi + im)\mathbf{T} \rangle + \frac{t}{2} \int_{X^4} * [-\langle \sigma, i\sigma \rangle - m\|\mathbf{S}\|^2 - \langle (\text{ad } \phi)\mathbf{S}, i\mathbf{S} \rangle] \right]$$

then $I'_{\mathbf{S},\mathbf{T},\sigma}$ can be calculated by doing the bosonic and fermionic integrals separately.

We begin with the bosonic integral

$$\int \mathcal{D}\mathbf{S} \exp \frac{t}{2} \int_X * [-\langle \mathbf{S}, (\mathcal{D}_A^2 + m - i \text{ad } \phi)\mathbf{S} \rangle] = \frac{(2\pi/t)^{\frac{1}{2} \dim \Gamma(W^+ \otimes \mathfrak{sl}(E))}}{\det(\mathcal{D}_A^2 + m - i \text{ad } \phi)|_{\Gamma(W^+ \otimes \mathfrak{sl}(E))}}.$$

We next compute the fermionic integral

$$\begin{aligned} & \int \mathcal{D}\sigma \mathcal{D}\mathbf{T} \exp \int_X * \left[-\frac{1}{2} \langle \sigma, i\sigma \rangle - it\langle \mathcal{D}_A \sigma, \mathbf{T} \rangle + \frac{t}{2} \langle \mathbf{T}, (\text{ad } \phi + im)\mathbf{T} \rangle \right] \\ &= (-t)^{\frac{1}{2} \dim \Gamma(W^+ \otimes \mathfrak{sl}(E))} \int \mathcal{D}\mathbf{T} \exp \int_X * \left[\frac{t}{2} \langle \mathcal{D}_A \mathbf{T}, i\mathcal{D}_A \mathbf{T} \rangle + \frac{t}{2} \langle \mathbf{T}, (\text{ad } \phi + im)\mathbf{T} \rangle \right] \\ &= \pm t^{\frac{1}{2} \dim \Gamma(W^+ \otimes \mathfrak{sl}(E))} t^{\frac{1}{2} \dim \Gamma(W^- \otimes \mathfrak{sl}(E))} \det(\mathcal{D}_A^2 + m - i \text{ad } \phi)|_{\Gamma(W^- \otimes \mathfrak{sl}(E))}. \end{aligned}$$

In the previous paragraphs, \dim denotes the *real* dimension; as for the determinants, there are no square roots involved since we are dealing with Gaussian integrals over *complex* vector spaces. Recall that the definition (3.23) of the partition function $\mathcal{Z}_{NSW}(t, r, m)$ includes the infinite constant

$$(2\pi)^{-\dim \Omega^0(\mathfrak{su}(E))/2} (2\pi t)^{-\frac{1}{2}(\dim \Omega_+^2(\mathfrak{su}(E)) + \dim \Gamma(W^- \otimes \mathfrak{sl}(E)))}.$$

It is natural to think of this constant as the product of the constant appearing in Donaldson-Witten theory (see (3.3)) and

$$(2\pi t)^{-\frac{1}{2} \dim \dim \Gamma(W^- \otimes \mathfrak{sl}(E))},$$

which we can incorporate in the normal integral.

After including this constant into the normal integral, we get

$$I'_{\mathbf{S},\mathbf{T},\sigma} = (2\pi)^{\text{Ind}(\mathcal{D} \otimes \mathfrak{sl}(E))} \frac{\det(\mathcal{D}_A^2 + m - i \text{ad } \phi)|_{\Gamma(W^- \otimes \mathfrak{sl}(E))}}{\det(\mathcal{D}_A^2 + m - i \text{ad } \phi)|_{\Gamma(W^+ \otimes \mathfrak{sl}(E))}}, \quad (6.5)$$

where $\zeta(0)$ is the value at 0 of the ζ -function of the elliptic operator $\mathcal{D}_A^2 + m - i \text{ad } \phi$ on minus spinors. Notice that there are no zero-modes to worry about in the above formula, due to the presence of the scalar term m —a nice non-degeneracy property of the functional integral.

Due to the cross-term $-it\langle \text{cl}(\psi)\mathbf{S}, \mathbf{T} \rangle$ the computation of $I_{\mathbf{S},\mathbf{T},\sigma}$ is harder. The Berezin integral over σ can be carried out as before, but the coupling of fermions and bosons will give rise to a superdeterminant when integrating $(\mathbf{S}, \mathbf{T}) \in \Gamma^b(W^+ \otimes \mathfrak{sl}(E)) \oplus \Gamma^f(W^- \otimes \mathfrak{sl}(E))$. (The superscripts

b and f denote the even/odd parts of the above super vector space.) Denoting by Q^\pm the operator $\mathcal{D}_A^2 + m - i \operatorname{ad} \phi$ on positive/negative spinors we have

$$\begin{aligned} I_{S,T,\sigma} &= (2\pi)^{\operatorname{Ind}(\mathcal{P} \otimes \mathfrak{sl}(E))} \operatorname{sdet}^{-1} \begin{bmatrix} Q^+ & -i \operatorname{cl}(\psi) \\ i \operatorname{cl}(\psi) & Q^- \end{bmatrix} \\ &= (2\pi)^{\operatorname{Ind}(\mathcal{P} \otimes \mathfrak{sl}(E))} \frac{\det(Q^- - \operatorname{cl}(\psi)(Q^+)^{-1} \operatorname{cl}(\psi))}{\det Q^+}. \end{aligned} \quad (6.6)$$

The above quotient of determinants depends on ϕ and ψ (basically it can be thought of as a differential form on \mathcal{A}/G). The ‘physical’ situation corresponds to $t = r$ in which case we can factor t out both in the denominator and the numerator. Except for the overall factors $t^\zeta(0)$ the resulting quotient of determinants doesn’t contain t (so we don’t have to compute a limit anymore). The result is

$$I_{S,T,\sigma} = (2\pi)^{\operatorname{Ind}(\mathcal{P} \otimes \mathfrak{sl}(E))} \frac{\det(\mathcal{D}_A^2 + m - i \operatorname{ad} \phi - \operatorname{cl}(\psi)(Q^+)^{-1} \operatorname{cl}(\psi))_{|\Gamma(W^- \otimes \mathfrak{sl}(E))}}{\det(\mathcal{D}_A^2 + m - i \operatorname{ad} \phi)_{|\Gamma(W^+ \otimes \mathfrak{sl}(E))}}. \quad (6.7)$$

6.2 Contributions from abelian Seiberg-Witten solutions: zero-dimensional case

In this section we discuss the contributions from (abelian) Seiberg-Witten solutions to the partition function of MTNSW. Let us denote generically the space of such solutions by \mathcal{M}_{SW} (we omit the class of the Spin^c -structure in the notation).

We will concentrate on a special situation, namely the case when the (virtual) dimension of \mathcal{M}_{SW} is zero. This case suffices for the computation of the correlation functions of massive topological non-abelian Seiberg-Witten theory defined on four-manifolds of Seiberg-Witten simple type.

The assumption that the moduli spaces have zero virtual dimension considerably simplifies the quadratic integral in the normal directions. Recall from Section 6.1 that in the case of instantons, the integral $I_{S,T,\sigma}$ on page 101 was complicated by the ‘coupling’ of bosons and fermions (in fact $I_{S,T,\sigma}$ is a differential form on moduli space). On a zero-dimensional moduli space there are no positive-degree differential forms to take into account, so in the expression of $I_{S,T,\sigma}$ we can eliminate all terms which involve the differential forms ψ and ϕ . The result simplifies considerably—in fact we will show in this section that the integral in the normal directions to \mathcal{M}_{SW} reduces to a constant times a power of $m/2\pi$.

Let us repeat the notation introduced in Section 4.4. Inside the configuration space $P := \mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$ we consider the set \mathcal{R} of configurations (A, S) with A reducible and S as in Section

4.2. We have

$$\begin{aligned} \mathcal{R} &:= \bigcup_{\{E=l\oplus l^{-1}\}} \mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2}) \\ &= \bigcup \left\{ \begin{array}{l} \mathcal{R}_x, \text{ where } \mathcal{R}_x = \bigcup_{c_1(l)=x} \mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2}). \\ x \in H^2(X, \mathbb{Z}) \\ x^2 = -k \end{array} \right\} \end{aligned}$$

The union on the first line is over all line subbundles l of E such that $E = l \oplus l^{-1}$; on the second line we have partitioned the set of such line subbundles into topological types. As in Section 4.4, in the actual computation we ignore the pairs consisting of reducible connections and the zero spinor in order to be able to pretend that the action of \mathcal{G}_l is free.

Let us denote by $\mathcal{N}_{\mathcal{R}_x}$ the normal bundle of \mathcal{R}_x in the configuration space $\mathcal{A}_E \times \Gamma(W^+ \otimes \mathfrak{sl}(E))$. At a point $(A, S) \in \mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})$, the fibre of $\mathcal{N}_{\mathcal{R}_x}$ can be identified with

$$\{a \in \Omega^1(l^{\otimes 2}) \mid D_A^* a = 0\} \oplus \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2})).$$

To see this, use the fact that the normal space of \mathcal{A}_l in \mathcal{A}_E is isomorphic to $\Omega^1(l^{\otimes 2})$ (as a consequence of the isomorphism $\mathfrak{su}(E) \simeq \mathbb{R} \oplus l^{\otimes 2}$). Since any two subbundles l_1 and l_2 of E with the same c_1 can be mapped into one another by a gauge transformation, the normal space of the set of all reducible connections in \mathcal{A}_E is the subspace of $\Omega^1(l^{\otimes 2})$ orthogonal to the gauge orbits, as claimed above.

The quadratic approximation will involve an integral over $\mathcal{N}_{\mathcal{R}_x}$ and one over \mathcal{R}_x for each x . The submanifolds \mathcal{R}_x are \mathcal{G}_E -invariant; if we choose one line bundle l_x for each x we see that \mathcal{R}_x contains as a submanifold the space

$$\mathcal{R}_{l_x} := \mathcal{A}_{l_x} \times \Gamma(W^+ \otimes l_x^{\otimes 2}),$$

which is endowed with an action of the subgroup \mathcal{G}_l of \mathcal{G}_E . Note that there is a diffeomorphism $\mathcal{R}_x/\mathcal{G}_E \xrightarrow{\sim} \mathcal{R}_{l_x}/\mathcal{G}_l$. We claim that the relevant integral over \mathcal{R}_x can be rewritten as an integral over \mathcal{R}_{l_x} . Indeed, the integrands of interest to us are products of vertical volume elements times equivariant differential forms so, at least formally, if we ignore the problems caused by fixed points, we have

$$\int_{\mathcal{R}_x} \text{vol}_{\mathcal{G}_E} \wedge \pi^* \alpha = \int_{\mathcal{R}_x/\mathcal{G}_E} \alpha = \int_{\mathcal{R}_{l_x}/\mathcal{G}_l} \alpha = \int_{\mathcal{R}_{l_x}} \text{vol}_{\mathcal{G}_l} \wedge \pi^* \alpha. \quad (6.8)$$

This is a ‘partial gauge fixing’—except that no determinants will appear since our integrands are top differential forms and not functions (usually, physicists use a similar procedure for functions, so they need a measure to integrate them against; the determinants arise when comparing the measures on the two integration spaces). Alternatively, we can say that since we gauge-fix bosons as well as fermions, the resulting determinants cancel out.

We will denote points of $\mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})$ by (A_l, S_l) . The coordinates on the normal space of the set of reducible configurations are $(a, S_n) \in \Omega^1(l^{\otimes 2}) \oplus \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2}))$. The tangent vectors

to $\mathcal{A}_l \times \Gamma(W^+ \otimes l^{\otimes 2})$ are of the form (ψ_l, σ_l) , whereas the tangent vectors in the normal directions are written as $(\psi_n, \sigma_n) \in \Omega^1(l^{\otimes 2}) \ni \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2}))$. The Berezin integration variables χ and \mathbf{T} appearing in (3.17) and (3.18) will be decomposed as $\chi_l + \chi_n$ and $\mathbf{T}_l + \mathbf{T}_n$ respectively, with $\chi_l \in \Omega_+^2$, $\chi_n \in \Omega_+^2(l^{\otimes 2})$, $\mathbf{T}_l \in \Gamma(W^- \otimes l^{\otimes 2})$, and $\mathbf{T}_n \in \Gamma(W^- \otimes (\mathbb{C} \oplus l^{-2}))$. Furthermore, we will denote the components of $S_n, \sigma_n, \mathbf{T}_n \in \Gamma(W^+) \oplus \Gamma(W^+ \otimes l^{-2})$ by the superscripts 0 and -2 , respectively (so $S_n = S_n^0 + S_n^{-2}$ and so on).

We will need some results about the behavior of the localizing form $d_{\mathcal{G}, m\omega}$ in the neighborhood of the \mathcal{R}_{l_*} . We have proved in Lemma 4.2 that the pullback of $CW d_{\mathcal{G}, m\omega}$ to \mathcal{R}_{l_*} is equal to zero. There is an obvious difference between the pullback and the restriction of a 2-form to a submanifold: the two differ by the normal components of the restriction—and in the semiclassical approximation it is actually the restriction and not the pullback which has to be used in the normal quadratic integral. We therefore adapt the proof of Lemma 4.2 to calculate the *restriction* of the 2-form part of $d_{\mathcal{G}, m\omega}$ to \mathcal{R}_{l_*} ; we only have to use the restriction of Ω_{Hor} to \mathcal{R}_l instead of its pullback. Since

$$\langle C^* C \Omega_{Hor}, \lambda \rangle = \int_{\mathcal{X}^4} \text{Tr} ([\psi, \psi]_0 \lambda) - * \langle \sigma, \lambda \sigma \rangle,$$

we conclude that

Lemma 6.1. *The restriction of the 2-form part of $CW d_{\mathcal{G}, m\omega}$ to \mathcal{R}_{l_*} equals*

$$\int_{\mathcal{X}^4} -2 * \langle \sigma_n, im \sigma_n \rangle + \text{Tr} \left([\psi_n, \psi_n]_0 \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \right). \quad (6.9)$$

In the above expression, $(\psi_n, \sigma_n) \in \Omega^1(l^{\otimes 2}) \oplus \Gamma(W^+ \otimes (\mathbb{C} \oplus l^{-2}))$ denotes a normal vector to \mathcal{R}_l . The last term can be simplified if we express ψ_n in matrix form:

$$\begin{aligned} \text{Tr} \left([\psi_n, \psi_n]_0, \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \right) &= \text{Tr} \left(\psi_n \begin{bmatrix} 0 & -\bar{\psi}_n \\ \psi_n & 0 \end{bmatrix}, \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix} \right) \\ &= \text{Tr} \left(\begin{bmatrix} 0 & -\bar{\psi}_n \\ \psi_n & 0 \end{bmatrix} \begin{bmatrix} 0 & 2im\bar{\psi}_n \\ 2im\psi_n & 0 \end{bmatrix} \right) \\ &= -2im\psi_n\bar{\psi}_n \\ &=: -2\langle \psi_n, im\psi_n \rangle, \end{aligned}$$

where the ‘fermionic’ notation on the last line stands for a differential two-form whose arguments are denoted by ψ_n . The symbol Tr stands (as always throughout the argument) for minus one half the trace in the fundamental representation.

In order to complete the calculation of $\text{Quadr } CW d_{\mathcal{G}, m\omega}$ we also need to calculate the Hessian of the 0-form part of $CW d_{\mathcal{G}, m\omega}$ —this is probably the trickiest part of the computation. There are two terms in this 0-form: one is $-\iota_{m\chi}\omega = -m^2\|\mathbf{S}\|^2$, whose Hessian is clearly $-m^2\|S_n\|^2$; the other one is $\langle C\mathcal{J}, m\mathcal{X} \rangle$. We claim that (with the notations introduced before)

Lemma 6.2.

$$\text{Quadr } \langle C\mathcal{J}, X \rangle = -3m^2 \|S_n\|^2 - 4m^2 \|a\|^2. \quad (6.10)$$

Proof. Start from the definition of \mathcal{J} , which implies, for any $\lambda \in \text{Lie}(\mathcal{G})$, that

$$\langle C^*C\mathcal{J}, \lambda \rangle = \langle mX, C^*\lambda \rangle = \langle imS, \lambda S \rangle \quad (6.11)$$

at a point (A, S) in configuration space. Let us differentiate (6.11) in the direction $(0, \sigma)$ with $\sigma \in \Gamma(W^+ \otimes (C \oplus l^{-2}))$; we get (after switching to global inner products so that we don't have to write the integrals over the manifold X^4 anymore)

$$\langle C^*C\mathcal{J}'_\sigma, \lambda \rangle + \langle \mathcal{J}\sigma, \lambda S \rangle + \langle \mathcal{J}S, \lambda\sigma \rangle = \langle im\sigma, \lambda S \rangle + \langle imS, \lambda\sigma \rangle. \quad (6.12)$$

On the other hand we have

$$\frac{d}{d\sigma} \langle C\mathcal{J}, mX \rangle = \langle C\mathcal{J}'_\sigma, mX \rangle + \langle \mathcal{J}\sigma, mX \rangle + \langle C\mathcal{J}, im\sigma \rangle. \quad (6.13)$$

The key point is the fact that mX is a vertical vector at the points of \mathcal{R}_{l^*} , namely the image of $\lambda = \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix}$, and so the right-hand side of (6.13) can be computed from (6.12) by using this particular λ . One finds that $(d/d\sigma)\langle C\mathcal{J}, mX \rangle \equiv 0$ on \mathcal{R}_{l^*} , as expected. The second derivative is computed by differentiating (6.12) and (6.13) once more with respect to σ and using the same argument. The conclusion is that

$$\frac{d^2}{d\sigma^2} \langle C\mathcal{J}, mX \rangle = -6m^2 \|\sigma\|^2.$$

The next step is to compute $(d/da)\langle C\mathcal{J}, mX \rangle$ for $a \in \Omega^1(l^{\otimes 2})$. The reasoning is similar, but there is one important point that we would like to emphasize. If we start from (6.11) and we differentiate in the direction a we obtain:

$$\langle C^*C\mathcal{J}'_a, \lambda \rangle + \langle \{a, D_A\mathcal{J}\}_0, \lambda \rangle + \langle D_A^*[a, \mathcal{J}], \lambda \rangle = 0. \quad (6.14)$$

We know that $\mathcal{J} \equiv \begin{bmatrix} im & 0 \\ 0 & -im \end{bmatrix}$ on \mathcal{R}_{l^*} and so $D_A\mathcal{J} = 0$ and $[a, \mathcal{J}] = 2ima$; the last formula follows for instance if we write a in matrix form as

$$\begin{bmatrix} 0 & -\bar{a} \\ a & 0 \end{bmatrix}.$$

The normal directions correspond to those a satisfying $D_A^*a = 0$, hence $(d/da)\langle C\mathcal{J}, mX \rangle = 0$ in these directions.

Continuing along the same lines we conclude that on \mathcal{R}_{l^*} we have

$$(d^2/da d\sigma)\langle C\mathcal{J}, mX \rangle = 0 \quad (6.15)$$

$$(d^2/da^2)\langle C\mathcal{J}, mX \rangle = -8m^2 \|a\|^2. \quad (6.16)$$

This completes the proof of the Lemma (if we remember that the quadratic part of a Taylor expansion is preceded by a 1/2 factor). Hence

Proposition 6.2. *The quadratic expansion of $d_{\mathcal{G},m}\omega$ in the normal directions to \mathcal{R}_l is given by*

$$\text{Quadr } d_{\mathcal{G},m}\omega = -4m^2\|S_n\|^2 - 4m^2\|a\|^2 - 2\langle\sigma_n, im\sigma_n\rangle - 2\langle\psi_n, im\psi_n\rangle. \quad (6.17)$$

The next task is to find which terms in the universal Euler forms (3.17) and (3.28) are relevant in the quadratic integral normal to \mathcal{M}_{SW} . It is easy to see that

Lemma 6.3. *The quadratic parts of the universal Euler forms in the directions normal to the set of reducible configurations, evaluated at a solution (A_l, S_l) of the perturbed abelian Seiberg-Witten equations, are given by*

$$\begin{aligned} \text{Quadr } W_{\mathcal{G}}(E_1, s_1) &= -\frac{t}{2}\|D_{A_l}^+ a + i[S_n, \bar{S}_l]_0 + i[S_l, \bar{S}_n]_0\|^2 - it\langle D_{A_l}^+ \psi_n + i[\sigma_n, \bar{S}_l]_0 + i[\sigma_l, \bar{S}_n]_0, \chi_n \rangle \\ &\quad - \frac{t}{2}\langle \chi_n, im\chi_n \rangle \end{aligned} \quad (6.18)$$

$$\begin{aligned} \text{Quadr } W_{\mathcal{G},m}(E_2, s_2) &= -\frac{t}{2}\|\mathcal{P}_A S_n + \text{cl}(a)S_l\|^2 - it\langle \mathcal{P}_A \sigma_n + \text{cl}(\psi_n)S_n, \mathbf{T}_n \rangle + \frac{t}{2}\langle \mathbf{T}_n^0, im\mathbf{T}_n^0 \rangle \\ &\quad + t\langle \mathbf{T}_n^{-2}, im\mathbf{T}_n^{-2} \rangle. \end{aligned} \quad (6.19)$$

We can now work out the quadratic integral in the directions normal to the set of reducible configurations; let us denote this integral by $\langle 1 \rangle_{\mathcal{R}}^{\text{normal}}(m, r, t)$ (note that observables don't contribute to this because they don't include the coupling constants; therefore, in the semiclassical limit, observables only enter through their restriction to the moduli space). We have

$$\begin{aligned} \langle 1 \rangle_{\mathcal{R}}^{\text{normal}}(m, r, t) &= \int \mathcal{D}a \mathcal{D}\psi_n \mathcal{D}S_n \mathcal{D}\sigma_n \mathcal{D}\chi \mathcal{D}\mathbf{T} \exp \\ &\quad - \frac{t}{2}\|D_{A_l}^+ a + i[S_n, \bar{S}_l]_0 + i[S_l, \bar{S}_n]_0\|^2 - it\langle D_{A_l}^+ \psi_n + i[\sigma_n, \bar{S}_l]_0 + i[\sigma_l, \bar{S}_n]_0, \chi_n \rangle \\ &\quad - \frac{t}{2}\langle \chi_n, im\chi_n \rangle \\ &\quad - \frac{t}{2}\|\mathcal{P}_A S_n + \text{cl}(a)S_l\|^2 - it\langle \mathcal{P}_A \sigma_n + \text{cl}(\psi_n)S_n, \mathbf{T}_n \rangle + \frac{t}{2}\langle \mathbf{T}_n^0, im\mathbf{T}_n^0 \rangle + t\langle \mathbf{T}_n^{-2}, im\mathbf{T}_n^{-2} \rangle \\ &\quad - 2tm\|S_n\|^2 - 2tm\|a\|^2 - t\langle\sigma_n, i\sigma_n\rangle - t\langle\psi_n, i\psi_n\rangle. \end{aligned} \quad (6.20)$$

Despite appearance to the contrary, the integral (6.20) can be evaluated very easily. The key point lies in its particular 'supersymmetry'—the bosonic integrals involved exactly match the fermionic ones, and they are not coupled to each other. In down-to-earth terms, let us assume that the Berezin integrations over χ_n and \mathbf{T} have already been carried out (this part is easy, since the operators defining the quadratic parts are just multiplication by multiples of im) and if we denote by b the bosonic variables (a, S_n) and by f the fermionic ones (ψ_n, σ_n) , our integral has the form

$$\int \mathcal{D}b \mathcal{D}f \exp \left[-\frac{1}{2}\langle b, Qb \rangle + \frac{1}{2}\langle f, \frac{Q}{im}f \rangle \right], \quad (6.21)$$

where Q is a positive self-adjoint operator (to see that Q is indeed nondegenerate, it is enough to look at the last line of (6.20) which shows that Q is the sum of a non-negative operator and a positive multiple of the identity). The precise form of Q is not important, since the integral is supersymmetric: the determinants produced by the bosonic and fermionic integrations cancel out. Therefore the integral (6.21) simply reduces to constant times a power of $m/2\pi$. Explicitly, for the physical case ($t = r$) and after including the overall power of t given by the Mathai-Quillen construction,

Proposition 6.3.

$$\langle 1 \rangle_{\mathcal{R}}^{normal}(m, r, t) = (-1)^{\frac{1}{2}(\dim \Omega_+^2(l^{\otimes 2}) - \dim \Omega^1(l^{\otimes 2}) + \dim \Omega^0(l^{\otimes 2}))} 2^{-\text{Ind } \mathcal{D} \otimes l^{-2}} \left(\frac{m}{2\pi} \right)^{-\text{Ind}(\mathcal{D} \otimes (\mathbb{C} \oplus l^{-2})) + \frac{1}{2}(\dim \Omega_+^2(l^{\otimes 2}) - \dim \Omega^1(l^{\otimes 2}) + \dim \Omega^0(l^{\otimes 2}))}. \quad (6.22)$$

The alternating sum of infinite dimensions appearing in the exponent of (6.22) can be regularized as the index of the half DeRham complex, and so the exponent equals $-\text{Ind}(\mathcal{D} \otimes (\mathbb{C} \oplus l^{-2})) + 4x^2 + (1/2)(\chi + \sigma)$. It is easy to see that (6.22) coincides with the coefficient of the Seiberg-Witten invariant appearing in (4.24), as expected.

Let us analyze the restriction of the integral (3.24) with $\mathcal{O} = 1$ to the set of reducible configurations \mathcal{R}_{l_x} ; as explained before, we can rewrite this integral as an integral over \mathcal{R}_{l_x} after having chosen a particular line bundle l_x with $c_1 = x$. We claim that (modulo some infinite power of t which we ignore)

$$\langle 1 \rangle_{\mathcal{R}_{l_x}}(m, r, t) = \text{SW}(c + 4x).$$

This follows by looking at the restriction of the integrand of (3.24) to the submanifold \mathcal{R}_{l_x} ; the crucial fact is the vanishing of the localization form $CWd_{\mathcal{G}, m\omega}$ when pulled back to \mathcal{R}_{l_x} , proved in lemma (4.2). Due to this vanishing, the path-integral along \mathcal{R}_{l_x} only involves the pull-back of the universal Euler forms (3.17) and (3.28) to \mathcal{R}_{l_x} —therefore it reduces precisely to the path-integral defining abelian Seiberg-Witten theory (cf. Section 3.2).

We conclude that

Proposition 6.4. *The contribution $Z_{NSW}^l(m, r, t)$ to the partition function of the massive topological non-abelian Seiberg-Witten theory obtained from abelian Seiberg-Witten pairs compatible with a decomposition $E = l \oplus l^{-1}$ such that $c_1(l) = x$ and $(c + 4x) \cdot (c + 4x) = 2\chi + 3\sigma$ (so that $\text{v. dim } \mathcal{M}_{SW} = 0$ is*

$$Z_{NSW}^l(m, r, t) = (-1)^{4k - (\chi + \sigma)/2} 2^{4k + (2\sigma - c \cdot c)/4} \left(\frac{m}{2\pi} \right)^{-3(c \cdot c - 2\chi - 3\sigma)/8} \text{SW}(c + 4x), \quad (6.23)$$

in agreement with Statement 4.6.

Chapter 7

Determinants and the Topology of the Index Bundle

In this chapter we study a conjecture generated by the comparison between the formal geometric argument of Section 4.3 and the semiclassical approach from Section 6.1. In Section 4.3 we expressed the contribution from Donaldson-Witten configurations to the correlation functions of MTNSW as integrals of ratios of equivariant Euler classes. We identified such ratios with the generating series for the total Segre class of an index bundle. In Section 6.1 we expressed the quadratic integral $I_{\mathcal{M}_E}^{normal}$ as the quotient of determinants $I_{\mathbf{S},\mathbf{T},\sigma}$ appearing in (6.7). Notice that both $I_{\mathbf{S},\mathbf{T},\sigma}$ and the Segre class of the index bundle are rigorously defined objects. They arose as two different expressions for the contributions from Donaldson-Witten configurations to the partition function of MTNSW. It is therefore natural to conjecture that the total Segre class of the index bundle from (4.2) is represented by the differential form $I_{\mathbf{S},\mathbf{T},\sigma}$.

We will begin by stating a more general form of the above conjecture; the generalization of the statement is obvious, but its main interest lies in potential application to a wide variety of semiclassical computations in supersymmetric quantum field theories. Despite the fact that at this stage we don't have a complete proof of the conjecture, we can control a number of particular cases, in particular we have a relatively good understanding of the case of vector bundles over a Riemann surface, described in Sections 7.4 and 7.5. We build the necessary technical tools in Sections 7.2 and 7.3.

7.1 A conjecture on determinants and characteristic classes of the index bundle

A comparison between the results of Sections 4.3 and 6.1 leads to the following conjecture about the total Segre class of the index bundle of a family of coupled Dirac operators over a four-dimensional compact oriented Riemannian manifold. Choose a Spin^c -structure on the base manifold X^4 with corresponding spinor bundles W^\pm and an $\text{SU}(2)$ -bundle E . Let \mathcal{A}/\mathcal{G} be the space of (irreducible) unitary connections on E modulo gauge transformations, and denote by $\text{Ind}(\mathcal{D})$ the index of the family of Dirac operators $\mathcal{D}_A : \Gamma(W^+ \otimes \text{sl}(E)) \rightarrow \Gamma(W^- \otimes \text{sl}(E))$, regarded as an element of $K(\mathcal{A}/\mathcal{G})$. The conjecture can then be stated as follows.

Conjecture 3. *The cohomology class of the total Segre class*

$$s_{\text{tot}}(\text{Ind } \mathcal{D}) := \left(\frac{m}{2\pi}\right)^{-\text{Ind}(\mathcal{D})} \sum_{k \geq 0} \left(\frac{2\pi}{m}\right)^k s_k(\text{Ind } \mathcal{D}) \in H^*(\mathcal{A}/\mathcal{G})$$

admits the De Rham representative

$$(2\pi)^{\text{Ind}(\mathcal{D})} \frac{\det(\mathcal{D}_A^2 + m - i \text{ad } \phi - \text{cl}(\psi)(\mathcal{D}_A^2 + m - i \text{ad } \phi)^{-1} \text{cl}(\psi))|_{\Gamma(W^- \otimes \text{sl}(E))}}{\det(\mathcal{D}_A^2 + m - i \text{ad } \phi)|_{\Gamma(W^+ \otimes \text{sl}(E))}}, \quad (7.1)$$

where ϕ denotes the $(0,2)$ -part of the curvature of the universal bundle over \mathcal{A}/\mathcal{G} introduced in Section 5.1. In fact, in order to interpret (7.1) as a differential form on \mathcal{A}/\mathcal{G} , one has to consider its horizontal part as a form on \mathcal{A} with respect to the action of \mathcal{G} .

We can now give an obvious generalization of the conjecture. If X is a compact even-dimensional oriented Riemannian manifold, W^\pm and E are complex vector bundles over X endowed with hermitian metrics, and $Q : \Gamma(W^+) \rightarrow \Gamma(W^-)$ is an elliptic first order operator then we can consider the family of elliptic operators $Q_A : \Gamma(W^+ \otimes E) \rightarrow \Gamma(W^- \otimes E)$ for A a unitary connection on E with trivial stabilizer in the group of gauge transformations. The index of the family is an element in $K(\mathcal{A}_E/\mathcal{G}_E)$ (we restrict ourselves to the subspace of connections on which the group of gauge transformations acts freely), and we have then

Conjecture 4. *The total Segre class of the index bundle is represented in DeRham cohomology by the differential form*

$$\mathcal{D} := (2\pi)^{\text{Ind } Q} \frac{\det(QQ^* + m - i\phi - q(\psi)^*(Q^*Q + m - i\phi)^{-1}q(\psi))|_{\Gamma(W^- \otimes E)}}{\det(Q^*Q + m - i\phi)|_{\Gamma(W^+ \otimes E)}}, \quad (7.2)$$

where ϕ is the universal curvature in the $\mathcal{A}_E/\mathcal{G}_E$ -directions, $*$ denotes the hermitian adjoint, and q is the symbol of Q .

In the above, $q(\psi)$ is the operator-valued 1-form on \mathcal{A} defined as follows: if ψ denotes a tangent vector to \mathcal{A} , i.e. $\psi \in \Gamma(T^*X \otimes \text{End}(E))$, then $q(\psi)$ is the algebraic operator in $\Gamma(\text{Hom}(W^+ \otimes$

$E, W^- \otimes E$) obtained by applying q to ψ (recall that, by definition, q is pointwise a linear function on T^*X with values in $\text{Hom}(W^+, W^-)$).

A slightly different viewpoint on the conjecture is provided by the discussion in Sections 5.1 and 5.2, where we used the families index theorem to compute the total Segre class of the index bundle. In those sections, we were interested in the Segre class *as a cohomology class* since the purpose was then to relate it to the μ -classes relevant to Donaldson theory. In fact, as pointed out in [2], the families index theorem leads to *differential form representatives* for the Segre class if one uses the curvature of the universal connection on $X \times A/G$ in the expression of the Chern character of the universal bundle. Therefore the conjecture can be reformulated if we consider this particular representative on the Segre class:

Conjecture 5. *The total Segre form of the index bundle computed via the families index theorem using the universal curvature \mathcal{F} coincides with the differential form on A/G given by (7.2).*

In the remainder of this chapter we will investigate Conjectures 4 and 5 if the base manifold is either two-dimensional. In order to do this, we first need to review some facts about determinants of elliptic operators and their variation in families.

7.2 Determinants and their derivatives

We use the Ray-Singer definition of determinants [27] $\det(\Delta) := \exp(-\zeta'_\Delta(0))$, where Δ is a self-adjoint positive-definite Riemannian manifold.

The function ζ_Δ is defined by $\zeta_\Delta(s) := \text{Tr}(\Delta^{-s})$ for $\text{Re}(s) \gg 0$ and the properties of the Mellin transform

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\Delta}) dt,$$

show that, if $\dim X$ is even, this function is meromorphic in the s -plane, holomorphic for $\text{Re } s \gg 0$, and $s = 0$ is not a pole. The function ζ_Δ has at most simple poles at $1, 2, \dots, \dim X/2$. The residues at these poles as well as the value $\zeta_{\Delta}(0)$ are locally computable from the coefficients of the asymptotic expansion of the heat kernel of Δ .

Remark. The above discussion can be extended to self-adjoint differential operators which are merely non-negative or even bounded from below—in which case one has to be careful about the zero eigenvalue. The result is a definition of the determinant of the restriction of the operator to the orthocomplement of its kernel (and similarly for the dimension). Fortunately for us, the ‘mass terms’ introduced in Section 3.5 in connection with topological non-abelian Seiberg-Witten theory kill the zero eigenvalue of the operators appearing in (6.7), so we can ignore this subtlety for our present purposes.

We now turn to the study of the variation of the determinants in families; we will again restrict to a special case, namely we will consider affine families of the form

$$\Delta_u := \Delta + uR,$$

with R self-adjoint and either algebraic (i.e. a multiplication operator) or a pseudodifferential operator of negative order (for instance the inverse of Δ composed with some algebraic operators, as needed in (6.7)). The real number u is assumed to be small enough so that Δ_u be positive-definite, in order to avoid any delicate problems related to its kernel.

Our main goal in this section is to compute the derivatives

$$\frac{dk}{du^k} \zeta'_{\Delta_u}(0).$$

The Mellin transform tells us that, for $\operatorname{Re}(s) \gg 0$,

$$\zeta_{\Delta_u}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} (e^{-t\Delta_u}) dt, \quad (7.3)$$

and so, by taking derivatives with respect to s ,

$$\zeta'_{\Delta_u}(s) = -\frac{\Gamma'(s)}{\Gamma(s)} \zeta_{\Delta_u}(s) + \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \log t \operatorname{Tr} (e^{-t\Delta_u}) dt. \quad (7.4)$$

By differentiating with respect to u , and still for $\operatorname{Re}(s) \gg 0$, we obtain

$$\frac{d}{du} \zeta'_{\Delta_u}(s) = \frac{\Gamma'(s)}{\Gamma(s)} \frac{1}{\Gamma(s)} \int_0^\infty t^s \operatorname{Tr} (Re^{-t\Delta_u}) dt - \frac{1}{\Gamma(s)} \int_0^\infty t^s \log t \operatorname{Tr} (Re^{-t\Delta_u}) dt. \quad (7.5)$$

If we now denote by $\xi_{\Delta}(s)$ the function defined for $\operatorname{Re}(s) \gg 0$ by

$$\xi_{\Delta}(s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \log t \operatorname{Tr} (e^{-t\Delta_u}) dt \quad (7.6)$$

then (7.5) becomes

$$\frac{d}{du} \zeta'_{\Delta_u}(s) = s \frac{\Gamma'(s)}{\Gamma(s)} \zeta_{\Delta_u, R}(s+1) - s \xi_{\Delta_u, R}(s+1), \quad (7.7)$$

where $\zeta_{\Delta, T}(s)$ and $\xi_{\Delta, T}(s)$ are defined by

$$\begin{aligned} \zeta_{\Delta, T}(s) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} (Te^{-t\Delta}) dt \\ \xi_{\Delta, T}(s) &:= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \log t \operatorname{Tr} (Te^{-t\Delta}) dt \end{aligned} \quad (7.8)$$

for $\operatorname{Re}(s) \gg 0$.

By differentiating (7.5) once more with respect to u we get

$$\frac{d^2}{du^2} \zeta'_{\Delta_u}(s) = -\frac{\Gamma'(s)}{\Gamma(s)} \frac{1}{\Gamma(s)} \int_0^\infty t^{s+1} \text{Tr} (R^2 e^{-t\Delta_u}) dt + \frac{1}{\Gamma(s)} \int_0^\infty t^{s+1} \log t \text{Tr} (R^2 e^{-t\Delta_u}) dt, \quad (7.9)$$

where we have used the fact that the family Δ_u is affine, i.e. $d^2/du^2 \Delta_u = 0$ (otherwise there would have been extra terms). With the new notations, (7.9) can be rewritten as

$$\frac{d^2}{du^2} \zeta'_{\Delta_u}(s) = -\frac{\Gamma'(s)}{\Gamma(s)} s(s+1) \zeta_{\Delta_u, R^2}(s+2) + s(s+1) \xi_{\Delta_u, R^2}(s+2). \quad (7.10)$$

By iterating the above reasoning we can derive an expression for $d^k/du^k \zeta'_{\Delta_u}(s)$:

Proposition 7.1. *For any s with $\text{Re}(s) \gg 0$ and any integer $k \geq 1$,*

$$\frac{d^k}{du^k} \zeta'_{\Delta_u}(s) = (-1)^{k-1} s(s+1) \cdots (s+k-1) \left[\frac{\Gamma'(s)}{\Gamma(s)} \zeta_{\Delta_u, R^k}(s+k) - \xi_{\Delta_u, R^k}(s+k) \right] \quad (7.11)$$

We would now want to analytically extend the right-hand side of (7.11) to $s = 0$ in order to get a formula for the variation of $\zeta'_{\Delta_u}(0)$. It is clear from (7.11) that this requires knowledge about the behavior of the functions $\zeta_{\Delta, T}$ and $\xi_{\Delta, T}$ at positive integers. The necessary information can be obtained in analogy with the properties of the usual ζ -function. We will assume that T is a pseudodifferential operator of order $-2a$ (so $a = 0$ for an algebraic operator). For the function $\zeta_{\Delta, T}$ we have

Proposition 7.2. *The function $\zeta_{\Delta, T}$ defined a priori for $\text{Re}(s) \gg 0$ can be analytically continued to a meromorphic function on \mathbb{C} with at most simple poles at $1 - a, 2 - a, \dots, \dim X/2 - a$ and holomorphic everywhere else.*

A proof of the proposition can be derived similarly to that of Theorem 1.12.2 in [11]. We can easily deal with $\xi_{\Delta, T}$ if we remark that the definition (7.8) implies that, for $\text{Re}(s) \gg 0$,

$$\Gamma(s) \xi_{\Delta, T}(s) = \frac{d}{ds} [\Gamma(s) \zeta_{\Delta, T}(s)].$$

This proves that

Proposition 7.3. *The function $\xi_{\Delta, T}(s)$ extends meromorphically to \mathbb{C} with poles at most at integers less or equal to $\dim X/2 - a$. The poles at positive integers k are double poles with no residues and the coefficient of the singular part $1/(s-k)^2$ in the Laurent expansion of $\xi_{\Delta, T}$ around $s = k$ equals $-\text{Res}_k \zeta_{\Delta, T}$.*

The final part of the proposition follows immediately from the fact that the gamma functions has neither zeroes nor poles at positive integers. Actually the function $\xi_{\Delta, T}$ has simple poles at negative integers because of the poles of the gamma function at those points, but our interest lies only in the behavior of $\xi_{\Delta, T}$ at positive integers so we won't elaborate on this point.

We are now in a position to simplify (7.11). Recall the Laurent expansion about $s = 0$ for the logarithmic derivative of the gamma function

$$\frac{\Gamma'(s)}{\Gamma(s)} \sim -\frac{1}{s} - \gamma + \frac{\pi^2}{6}s + O(s^2),$$

where γ is Euler's constant. Applying this and the previous two propositions to (7.11) yields

$$\begin{aligned} \frac{d^k}{du^k} \zeta'_{\Delta_u}(s)|_{s=0} &\sim (-1)^{k-1} (k-1)! s \left[\left(-\frac{1}{s} - \gamma + O(s) \right) \left(\frac{\text{Res}_k \zeta_{\Delta_u, T^k}}{s} + \zeta_{\Delta_u, T^k}(k) + O(s) \right) \right. \\ &\quad \left. - \left(\frac{-\text{Res}_k \zeta_{\Delta_u, T^k}}{s^2} + O(1) \right) \right]. \end{aligned}$$

Notice that the singular part (i.e. $O(1/s)$) drops out so we are left with

Theorem 7.1. *Under the assumptions stated before, the k -th derivative of $\zeta'_{\Delta_u}(0)$ is*

$$\frac{d^k}{du^k} \zeta'_{\Delta_u}(0) = (-1)^k (k-1)! (\gamma \text{Res}_k \zeta_{\Delta_u, T^k} + \zeta_{\Delta_u, T^k}(k)), \quad (7.12)$$

where $\zeta_{\Delta_u, T^k}(k)$ is either the value of the ζ -function at k if k is not a pole or its finite part at k otherwise.

The theorem can be reformulated as a statement about a Taylor expansion:

Theorem 7.2. *For small enough $|u|$,*

$$\zeta'_{\Delta+uR}(0) = \zeta'_{\Delta}(0) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (\gamma \text{Res}_k \zeta_{\Delta, R^k} + \zeta_{\Delta, R^k}(k)), \quad (7.13)$$

at least if we assume that the series on the right-hand side has a positive radius of convergence.

This theorem already sheds some light on Conjecture 5. The two ingredients of the conjecture, namely the representative for the Segre class obtained from the families index theorem and the quotient of determinants are quite different in nature: the Segre form computed from the universal curvature on A/G is a *local* expression, i.e. all its components can be expressed as integrals over X of local quantities (this is easily seen from the discussion in Section 5.1). On the other hand, if we apply Theorem 7.2 to the quotient of determinants, for $\Delta = QQ^* + m^2$ and $\Delta = Q^*Q + m^2$ and appropriate R , we see that the quotient of determinants involves the values of the ζ -function at all positive integers, which are definitely *non-local* quantities.

This suggests that Conjecture 5 is not true, although it doesn't rule out Conjecture 4. Despite this, we don't want to dismiss Conjecture 5 completely, and actually in the next section we will show that it admits an improved version in which all non-local quantities are eliminated. Needless to say, an equality between two differential forms would be much more powerful, especially in this infinite-dimensional context, than an equality between cohomology classes. This is why we shift our viewpoint in the next section in order to formulate a more plausible variant of Conjecture 4 that would still predict an equality at the level of differential form representatives.

7.3 Zeta functions and adiabatic limits

We have so far, in our analysis of topological quantum field theories, worked on a *fixed* Riemannian base, i.e. we have tacitly assumed that a fixed Riemannian metric had been chosen on the four-manifold X (or on the even-dimensional manifold X in the previous section). However, the partition function and correlation functions of these theories are in general *diffeomorphism* invariants (this is strictly speaking true only under some topological restrictions—in the case of four-manifolds, we should restrict to those with $b_2^+ > 1$) and so we could try to vary the metric in order to simplify the evaluation of expressions such as (6.7) which appear in the semiclassical analysis.

The simplest possible procedure is to rescale the metric by a constant factor, i.e. $g \rightarrow \mu g$, and analyze the $\mu \rightarrow 0$ or $\mu \rightarrow \infty$ limits. It turns out that the latter, so-called *adiabatic limit*, leads to significant simplifications in the study of the quotients of determinants from our quadratic approximations.

We illustrate now this phenomenon by a simple case which captures the key properties of zeta-functions in the adiabatic limit. Our main interest lies in the determinants of operators of the form Laplacian $+m$ + perturbation (where we assume that m is positive and big enough to that the operator be positive definite); if we ignore the perturbation for the time being and we only take into account the scaling properties of the Laplacian, we see that, after rescaling the metric of the base, the operator under consideration has the form (power of μ)·Laplacian $+m$ (for instance, the power of μ for the Laplacian of functions is $-\dim X/2$). Our result is

Theorem 7.3. *Let $\Delta : \Gamma(F) \rightarrow \Gamma(F)$ be a non-negative self-adjoint second-order differential operator, so that the ζ -function ζ_Δ is meromorphic in \mathbb{C} , with poles at most at $1, 2, \dots, \dim X/2$. If we denote by ζ_μ the ζ -function of the operator $(\Delta/\mu) + m$ then*

$$\lim_{\mu \rightarrow \infty} \left[\zeta_\mu(s) - \frac{1}{m^s} \left(\zeta_\Delta(0) + \sum_{c=1}^{\dim X/2} \frac{(m\mu)^c \text{Res}_c \zeta_\Delta}{(s-1)\cdots(s-c)} \right) \right] = 0 \quad (7.14)$$

uniformly on compact subsets of $\mathbb{C} \setminus \{1, 2, \dots, \dim X/2\}$.

Proof. The Mellin transform shows that

$$\begin{aligned} \zeta_\mu(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tm} \text{Tr} \left(e^{-t \frac{\Delta}{\mu}} \right) dt \\ &= \frac{\mu^s}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tm\mu} \text{Tr} \left(e^{-t\Delta} \right) dt, \end{aligned}$$

after the change of variables $t \rightarrow \mu t$. For large μ , the integral from 0 to ∞ localizes to the neighborhood of $t = 0$ because of the term $e^{-tm\mu}$. Therefore we can use the short-time asymptotics of the heat kernel of Δ along the diagonal,

$$e^{-t\Delta}(x, x) \sim \sum_{c \geq -\dim X/2} t^c a_c(x).$$

Recall that the integrals of the coefficients a_c over X have a very specific meaning, namely

$$\int_X \text{Tr } a_{-c}(x) d\text{vol}(x) = \text{Res }_c \zeta_\Delta \quad \text{for } c = 1, 2, \dots, \dim X/2$$

and

$$\int_X \text{Tr } a_0(x) d\text{vol}(x) = \zeta_\Delta(0)$$

and that

$$\int_0^\infty t^{s-1+c} e^{-t m \mu} dt = \frac{\Gamma(s+c)}{(m\mu)^{s+c}}$$

if $s+c \neq 0$. We conclude that

$$\zeta_\mu(s) \sim \frac{1}{m^s} \left(\zeta_\Delta(0) + \sum_{c=1}^{\dim X/2} \frac{(m\mu)^c \text{Res }_c \zeta_\Delta}{(s-1)\cdots(s-c)} \right) + O\left(\frac{1}{\mu}\right)$$

for $s \in \mathbb{C} \setminus \{1, 2, \dots, \dim X/2\}$. The expansion is easily seen to be uniform for s varying in a compact set, which completes the proof of the theorem.

Remark. The proof shows that the theorem also holds for a generalized ζ -function of the form $\zeta_{(\Delta/\mu)+m,T}$ if T is an *algebraic* operator. We will discuss further generalizations of the theorem in the next section.

The theorem illustrates the dramatic simplification of the ζ -function which occurs in the adiabatic limit. Let's assume, for instance, that s is an integer bigger than $\dim X/2$, so that the ζ -function is regular at s . Although $\zeta_{\Delta+m}(s)$ is not locally expressible, the value $\zeta_{(\Delta/\mu)+m}(s)$ has an asymptotic expansion in powers of μ whose coefficients are local quantities.

This immediately suggests that Conjecture 5 might be saved at least in part if we used some scaling limit of the quotient of determinants (although it is not obvious at this point whether the relevant limit is the adiabatic one, because of the terms in the asymptotic expansion which contain positive powers of μ and so blow up as $\mu \rightarrow \infty$). Hence the next logical step is to experiment with some particular cases in order to find the right candidate for a scaling limit of the quotient of determinants.

Remark. The previous theorem relies essentially on the *mass gap*, i.e. the mass term present in the operator—an illustration of a quite general fact in quantum field theory, namely that theories with a mass gap simplify at large scales. On the other hand, the mass term has in this case a geometric meaning related to equivariant cohomology, which makes it plausible that this kind of result might be useful in other geometric applications as well.

7.4 Topology of the index bundle I: line bundles

In this section we study the adiabatic limit of the quotient of determinants (7.2) in the simplest possible case. Assume now that X is a Riemann surface, W^+ is the trivial bundle, $W^- = \Lambda^{0,1}(X)$,

and, in addition, assume that $E = L$ is a complex *line bundle*. Let the operator Q be the $\bar{\delta}$ -operator (to be specific, the $\bar{\delta}$ -operator on X coupled to a connection on L , which can be thought of as a $\bar{\delta}$ -operator on L). Fix a metric g on X , let μ be positive, and denote by \mathcal{D}_μ the quotient (7.2) in the metric μg . We don't rescale the metric on L , and we consider the natural rescaling of the metrics on W^\pm : the metric on $\Lambda^{0,1}(X)$ is conformally invariant, hence it doesn't scale, whereas the metric on $C^\infty(X)$ is rescaled by μ .

Lemma 7.1. *The universal curvature $\phi \in \Omega^2(\mathcal{A}_L/\mathcal{G}_L)$ vanishes if L is a line bundle.*

Proof. The proof of the lemma already appears in Section 3.1 on abelian Seiberg-Witten theory. Recall from (2.12) that, with the notations of Section 2.4, $\phi = \Omega_{Hor} = (C^*C)^{-1} dC^*$; but on page 49 we showed that $C^*(\psi) = -2d^*\psi$ is a constant differential form on the affine space \mathcal{A}_L and so $dC^* = 0$.

Therefore \mathcal{D}_μ reduces in this case to

$$\mathcal{D}_\mu = \frac{\det(\frac{\bar{\delta}\bar{\delta}^*}{\mu} + m - \frac{1}{\mu}\psi^{0,1}(\frac{\bar{\delta}^*\bar{\delta}}{\mu} + m)^{-1} * \psi^{1,0})|_{\Gamma(\Lambda^{0,1} \otimes L)}}{\det(\frac{\bar{\delta}^*\bar{\delta}}{\mu} + m)|_{\Gamma(L)}}. \quad (7.15)$$

In (7.15) the dependence on μ has been factored out explicitly, i.e. all the operations which involve the metric have been written *with respect to the original metric g* (for instance the adjoint $\bar{\delta}^*$ and the Riemannian duality operator $*$: $\Omega^{1,1} \rightarrow \Omega^0$). Before going any further, we should explain what the action of the operator

$$\psi^{0,1} \left(\frac{\bar{\delta}^*\bar{\delta}}{\mu} + m \right)^{-1} * \psi^{1,0}$$

really is. The 1-forms $\psi^{1,0}$ and $\psi^{0,1}$ are to be thought of as tangent vectors to the space of connections on L , so that the above expression is an operator-valued 2-form on the space of connections. The operator acts on a section of $\Lambda^{0,1} \otimes L$ by first multiplying it with $\psi^{1,0}$; the result (which is a section of $\Lambda^{1,1} \otimes L$) is then transformed into a section of L by the Riemannian star operator. After applying the inverse of the differential operator $(\bar{\delta}^*\bar{\delta}/\mu) + m$, the result is taken back into $\Lambda^{0,1} \otimes L$ by multiplication with $\psi^{0,1}$.

We can use Theorem 7.2 to analyze the denominator of \mathcal{D}_μ . Let us define

$$\begin{aligned} \Delta_\mu^+ &:= \frac{\bar{\delta}^*\bar{\delta}}{\mu} + m \\ \Delta_\mu^- &:= \frac{\bar{\delta}\bar{\delta}^*}{\mu} + m \quad \text{and} \\ R_\mu &:= -\frac{1}{\mu}\psi^{0,1}(\Delta_\mu^+)^{-1} * \psi^{1,0}. \end{aligned}$$

Equation (7.13) yields

$$\log \det(\Delta_\mu^- + R_\mu) = \log \det \Delta_\mu^- - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\gamma \text{Res}_k \zeta_{\Delta_\mu^-, R_\mu^k} + \zeta_{\Delta_\mu^-, R_\mu^k}(k) \right).$$

We are being a little bit formal, since we should actually make sure that we are dealing with positive definite operators and that the appropriate convergence properties of the right-hand side hold. Notice that the operator R_μ^k is pseudodifferential of order $-2k$, so Proposition 7.2 implies that for $k \geq 1$ the ζ -function $\zeta_{\Delta_\mu^-, R_\mu^k}$ has only one pole (at 0), hence it is regular at k (recall that we are now working over a Riemann surface, $\dim X = 2$). It follows that

$$\log \det(\Delta_\mu^- + R_\mu) = \log \det \Delta_\mu^- - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \zeta_{\Delta_\mu^-, R_\mu^k}(k).$$

The first term on the right-hand side combines nicely with the logarithm of the denominator of \mathcal{D}_μ : we have

$$\frac{\det \Delta_\mu^-}{\det \Delta_\mu^+} = m^{-\text{Ind}(\bar{\delta})}$$

because, for any operator Q , the operators QQ^* and Q^*Q have the same spectrum except for the zero eigenvalue. We infer that

$$\log \mathcal{D}_\mu = -\text{Ind}(\bar{\delta}) \log m - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \zeta_{\Delta_\mu^-, R_\mu^k}(k). \quad (7.16)$$

At this point it is clear that a generalization of Theorem 7.3 that would apply to the function $\zeta_{\Delta_\mu^-, R_\mu^k}$ ($k \geq 1$) would express the $\mu \rightarrow \infty$ limit of (7.16) as a function of local quantities. By definition,

$$\zeta_{\Delta_\mu^-, R_\mu^k}(s) = \text{Tr} (R_\mu^k (\Delta_\mu^-)^{-s}). \quad (7.17)$$

The right-hand side would be much simpler if we could permute the operators involved in order to separate all the powers of Δ_μ^- . If we denote by Ψ the algebraic operator $\psi^{0,1} \circ * \circ \psi^{1,0}$ acting on sections of $\Lambda^{0,1} \otimes L$ then, modulo a number of commutations, the operator on the right hand side of (7.17) has the form

$$\frac{1}{(\mu)^k} \Psi^k (\Delta_\mu^-)^{-s-k}$$

whose trace equals

$$\frac{1}{(\mu)^k} \zeta_{\Delta_\mu^-, \Psi^k}(s+k).$$

This is very promising since the remark after Theorem 7.3 shows that the large μ limit of this generalized ζ -function satisfies the analogue of (7.14). Asymptotically in μ , we have

$$\zeta_{\Delta_\mu^-, \Psi^k}(s+k) \sim \frac{1}{m^{(s+k)}} \left(\zeta_{\Delta_\mu^-, \Psi^k}(0) + \frac{m\mu \text{Res}_1 \zeta_{\Delta_\mu^-, \Psi^k}}{s+k-1} \right) + O\left(\frac{1}{\mu}\right).$$

It is reasonable to assume that the commutation needed above doesn't affect the highest order term in the asymptotic expansion in μ . Assuming this to be true, we would get

$$\zeta_{\Delta_\mu^-, R_\mu^k}(s) \sim \frac{1}{(\mu)^k m^{(s+k)}} \left(\zeta_{\Delta_\mu^-, \Psi^k}(0) + \frac{m\mu \text{Res}_1 \zeta_{\Delta_\mu^-, \Psi^k}}{s+k-1} \right) + O\left(\frac{1}{\mu}\right), \quad (7.18)$$

as $\mu \rightarrow \infty$. Recall that in (7.16) we actually need the values of the ζ -function at $s = k \geq 1$, and (7.18) shows that the only contribution which survives as $\mu \rightarrow \infty$ is the second term $m^{-1} \text{Res}_1 \zeta_{\Delta^-, \psi}$ obtained for $k = 1$. We conclude that

$$\lim_{\mu \rightarrow \infty} \zeta_{\Delta^-, R_\mu^k}(k) = \begin{cases} m^{-1} \text{Res}_1 \zeta_{\Delta^-, \psi} & \text{for } k = 1 \\ 0 & \text{for } k = 2, 3, \dots \end{cases} \quad (7.19)$$

and finally, by using the asymptotics of the heat kernel, we get

$$\text{Res}_1 \zeta_{\Delta^-, \psi} = -\frac{1}{4\pi} \int_X \psi \wedge \psi. \quad (7.20)$$

Remark. Notice how the $(1/\mu)$ factor present in R compensates for the positive powers of μ that can a priori appear in the asymptotic expansion (7.14). This phenomenon will occur again later and is the key to the existence of the adiabatic limit of \mathcal{D}_μ determinants; (7.14) allows for terms that blow up for large μ and in fact in dimension 4, a power-counting analysis of the quotient of determinants (6.7) relevant to topological non-abelian Seiberg-Witten theory shows that its asymptotic expansion does contain positive powers of μ . Therefore the large scaling limit in the four-dimensional case should be interpreted further.

Returning to (7.16) and exponentiating we find that \mathcal{D}_μ has a limit as $\mu \rightarrow \infty$ and

$$\lim_{\mu \rightarrow \infty} \mathcal{D}_\mu = m^{-\text{Ind}(\bar{\delta})} \exp\left(-\frac{1}{4\pi m} \int_X \psi \wedge \psi\right). \quad (7.21)$$

It remains to compare this limit to the expression for the total Segre form of the index bundle provided by the families index theorem.

Let us recapitulate the discussion in Section 5.1 for the present situation. The curvature \mathcal{F} of the universal bundle \mathcal{L} over $X \times \mathcal{A}_L/\mathcal{G}_L$ is given by

Proposition 7.4. *With the notations of [2], and if A denotes a connection on L ,*

$$\mathcal{F}_{2,0} = F_A; \quad (7.22)$$

$$\mathcal{F}_{1,1} = \psi; \quad (7.23)$$

$$\mathcal{F}_{0,2} = 0. \quad (7.24)$$

Proof. Statement (7.24) is the same as Lemma 7.1. The other two statements follow as in [2]. The only part worth explaining is the notation in (7.23)—given a tangent vector at a point of X and a tangent vector to \mathcal{A} , i.e. a 1-form, ψ is just supposed to evaluate the 1-form on the vector in X .

Since \mathcal{L} is a line bundle, its Chern character is given by $ch(\mathcal{L}) = \exp c_1(\mathcal{L}) = \exp(\mathcal{F}/2\pi i)$. The characteristic form associated with the $\bar{\delta}$ -operator being $1 + c_1(X)/2$ we get

$$ch(\text{Ind } \bar{\delta}) = \int_X (1 + c_1(X)/2) \sum_{n=0}^{\infty} \frac{(\mathcal{F}_{2,0} + \mathcal{F}_{1,1})^n}{n!(2\pi i)^n},$$

which yields $ch_k(\text{Ind } \bar{\partial}) = 0$ for $k \geq 2$ and

$$ch_1(\text{Ind } \bar{\partial}) = \frac{1}{8\pi^2} \int_X \psi \wedge \psi. \quad (7.25)$$

We can now use Theorem 5.1 to derive the generating series for the total Segre class from the Chern character:

$$s_{\text{tot}}(\text{Ind } \bar{\partial})(T) = \exp(-ch_1 T) = \exp\left(-T \frac{1}{8\pi^2} \int_X \psi \wedge \psi\right), \quad (7.26)$$

which gives

$$s_{\text{tot}}(\text{Ind } \bar{\partial})\left(\frac{2\pi}{m}\right) = \exp\left(-\frac{1}{4\pi m} \int_X \psi \wedge \psi\right). \quad (7.27)$$

Comparing (7.21) and (7.26), we see the result of our naive computation for of the quotient of determinants \mathcal{D}_μ does indeed agree with the total Segre class of the index bundle. It remains, of course, to find a proof of (7.18) (if true).

7.5 Topology of the index bundle II: vector bundles

In the previous section we studied the quotient of determinants \mathcal{D} introduced in (7.2) over a two-dimensional base for a family of $\bar{\partial}$ operators coupled to a line bundle. The major simplification was the vanishing of ϕ (i.e. the universal curvature), so we will now try to understand what happens when ϕ is non-zero. This is always the case if the structure group of E is non-abelian, so we will consider here the case of an arbitrary vector bundle E over a Riemann surface. The bundles W^\pm will be as is the previous section, i.e. W^+ is trivial and $W^- := \Lambda^{0,1}(X)$.

As before, we rescale the metric on X by a constant factor μ . In order to study the ‘rescaled’ quotient of determinants \mathcal{D}_μ , we now have to know the scaling properties of the universal curvature ϕ (this was unnecessary in last section, since ϕ vanished then). The result is very easy.

Proposition 7.5. *The curvature ϕ of the bundle $A_E \rightarrow \mathcal{G}_E$ is invariant under rescaling the metric on the base manifold by a constant factor.*

Proof. We know from formula (2.12) that $\phi = (C^*C)^{-1}dC^*$, where $C : \Omega^0(\text{su}(E)) \rightarrow \Omega^1(\text{su}(E))$ maps λ into $-D_A\lambda$. The operator C doesn’t depend on the metric on X , but its adjoint C^* does: under a rescaling of the metric by μ , the operator C^* gets scaled by $1/\mu$ (because of the different scaling properties of 0-forms and 1-forms). Nevertheless, (2.12) is independent of μ because the factors of μ cancel each other, which proves the proposition.

Remark. This proof shows that the proposition is true over a base X of arbitrary dimension, although we should point out that in general C^* scales by a factor $\mu^{-\dim X/2}$.

In order to study \mathcal{D}_μ more easily, we split it as $\mathcal{D}_\mu = \mathcal{D}_\mu^1 \mathcal{D}_\mu^2$, where

$$\mathcal{D}_\mu^1 = \frac{\det(\bar{\partial}\bar{\partial}^* + m - i\phi)|_{\Gamma(\Lambda^{0,1} \otimes E)}}{\det(\bar{\partial}^*\bar{\partial} + m - i\phi)|_{\Gamma(E)}}$$

and

$$\mathcal{D}_\mu^2 = \frac{\det\left(\frac{\bar{\partial}\partial^*}{\mu} + m - i\phi - \frac{1}{\mu}q(\psi)^*\left(\frac{\bar{\partial}^*\partial}{\mu} + m - i\phi\right)^{-1}q(\psi)\right)_{\Gamma(\Lambda^{0,1} \otimes E)}}{\det\left(\frac{\bar{\partial}\partial^*}{\mu} + m - i\phi\right)_{\Gamma(\Lambda^{0,1} \otimes E)}}.$$

We now apply (7.13) to the logarithm of \mathcal{D}_μ^1 , for $\Delta = \Delta^\pm$ and $R = -i\phi$:

$$\begin{aligned} \log \mathcal{D}_\mu^1 &= -\zeta'_{\Delta_\mu^-(0)} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\gamma \text{Res}_k \zeta_{\Delta_\mu^-, (-i\phi)^k} + \zeta_{\Delta_\mu^-, (-i\phi)^k}(k) \right) \\ &\quad + \zeta'_{\Delta_\mu^+(0)} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\gamma \text{Res}_k \zeta_{\Delta_\mu^+, (-i\phi)^k} + \zeta_{\Delta_\mu^+, (-i\phi)^k}(k) \right). \end{aligned} \quad (7.28)$$

As explained in the previous section,

$$-\zeta'_{\Delta_\mu^-(0)} + \zeta'_{\Delta_\mu^+(0)} = \log \frac{\det \Delta_\mu^-}{\det \Delta_\mu^+} = -\text{Ind}(\bar{\partial} \otimes E) \log m.$$

Proposition 7.2 tells us that the above ζ -functions are regular at any $k \geq 2$, so the only residue to be taken into account is at $k = 1$. However, we claim that

Proposition 7.6.

$$\text{Res}_1 \zeta_{\Delta_\mu^-, (-i\phi)} = \text{Res}_1 \zeta_{\Delta_\mu^+, (-i\phi)}. \quad (7.29)$$

Proof. The above residues are locally computable from the heat kernel asymptotics. In two dimensions, the residue at 1 equals the integral of the trace of the first coefficient in the asymptotic expansion (see the proof of Theorem 7.3). To be specific, if the integral kernels of the operators $\phi \exp(-t\Delta_\mu^\pm)$ satisfy

$$\phi e^{-t\Delta_\mu^\pm}(x, x) \sim \sum_{c \geq -1} t^c a_c^\pm(x) \quad \text{for } t \rightarrow 0$$

then the residues at 1 are

$$\text{Res}_1 \zeta_{\Delta_\mu^\pm, (-i\phi)} = -i \int_X \text{Tr} a_{-1}^\pm(x) d\text{vol}(x).$$

But ϕ is an algebraic operator so the integral kernel of $\phi \exp(-t\Delta_\mu^\pm)$ is just ϕ times the integral kernel of $\exp(-t\Delta_\mu^\pm)$. Since the first coefficients in the asymptotic expansions of Δ_μ^\pm are (the same) multiple of the identity, we conclude that $a_{-1}^-(x) = a_{-1}^+(x)$ for any $x \in X$, which completes the proof of the proposition.

Therefore (7.28) takes the simpler form

$$\log \mathcal{D}_\mu^1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\zeta_{\Delta_\mu^+, (-i\phi)^k}(k) - \zeta_{\Delta_\mu^-, (-i\phi)^k}(k) \right). \quad (7.30)$$

As remarked earlier, this is non-local and very hard to compute for fixed μ , but Theorem 7.3 and the discussion following it show that one can compute its asymptotics as $\mu \rightarrow \infty$, and the coefficients

of the expansion are local. The easiest way to study the asymptotics of (7.30) is to use the analogues of (7.14) for $\zeta_{\Delta_{\mu}^{\pm}, (-i\phi)^k}$. The proof of Theorem 7.3 implies that

$$\lim_{\mu \rightarrow \infty} \left[\zeta_{\Delta_{\mu}^+, (-i\phi)^k}(s) - \zeta_{\Delta_{\mu}^-, (-i\phi)^k}(s) - \frac{1}{m^s} \int_X \text{Tr}(-i\phi)^k (a_0^+ - a_0^-) \right] = 0$$

for any $s \in \mathbb{C}$, where now a_0^{\pm} denote the second coefficient in the short-time asymptotics of the heat kernels for $\bar{\partial}^* \bar{\partial}$ and $\bar{\partial} \bar{\partial}^*$, respectively. Notice that the heat-kernel proof of the index theorem tells us that in fact $a_0^+ - a_0^-$ is the characteristic form appearing in the index formula for $\bar{\partial}_E : \Gamma(E) \rightarrow \Gamma(\Lambda^{0,1} \otimes E)$, i.e. $\frac{\text{rk } E}{2} c_1(X) + c_1(E)$. Hence

$$\lim_{\mu \rightarrow \infty} \left[\zeta_{\Delta_{\mu}^+, (-i\phi)^k}(k) - \zeta_{\Delta_{\mu}^-, (-i\phi)^k}(k) \right] = \int_X \text{Tr} \left[\left(-i \frac{\phi}{m} \right)^k \left(\frac{\text{rk } E}{2} c_1(X) + c_1(E) \right) \right]$$

and so (7.30) becomes

$$\lim_{\mu \rightarrow \infty} \log \mathcal{D}_{\mu}^1 = \sum_{k=1}^{\infty} \frac{1}{k} \int_X \text{Tr} \left[\left(i \frac{\phi}{m} \right)^k \left(\frac{\text{rk } E}{2} c_1(X) + c_1(E) \right) \right]. \quad (7.31)$$

As for \mathcal{D}_{μ}^2 , we also begin by applying Theorem 7.2 to its logarithm; for the determinant in the numerator we regard

$$R_0 := -i\phi - \frac{1}{\mu} q(\psi)^* \left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m - i\phi \right)^{-1} q(\psi)$$

as the perturbation to Δ_{μ}^- , whereas the denominator will be treated as we did for \mathcal{D}_{μ}^1 , i.e. $R = -i\phi$.

We introduce the following notation which will be very useful in the sequel: let A, B be two operators and k, l two non-negative integers. Then

$$\{A^k, B^s\} := \sum_{\substack{0 \leq r \leq k \\ 0 \leq s \leq l}} A^r B^s A^{k-r} B^{l-s}.$$

If we denote by

$$R_k := \sum_{l=1}^k \left\{ (-i\phi)^{k-l}, \left(-\frac{1}{\mu} q(\psi)^* \left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m - i\phi \right)^{-1} q(\psi) \right)^l \right\} \quad (7.32)$$

then (7.13) implies that

$$\log \mathcal{D}_{\mu}^2 = - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \zeta_{\Delta_{\mu}^-, R_k}(k). \quad (7.33)$$

The residues appearing in the right-hand side of (7.13) vanish in the case of (7.33) because for any $k \geq 1$ the operator R_k has *negative* order, which means that the ζ -function $\zeta_{\Delta_{\mu}^-, R_k}$ is regular at any positive integer (cf. Proposition 7.2).

We now want to study the asymptotics of (7.33) as $\mu \rightarrow \infty$ —although we should first reassert that at this stage we will content ourselves with a naive argument. There are two main points to emphasize: first, we will expand the inverse operator

$$\left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m - i\phi\right)^{-1}$$

in a power series

$$\left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m - i\phi\right)^{-1} = \left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m\right)^{-1} \sum_{n \geq 0} \left(i\phi \left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m\right)^{-1}\right)^n. \quad (7.34)$$

Secondly, as we have already done in the previous section, we assume that the leading term in the asymptotic expansion doesn't change if we permute the algebraic operator ϕ as needed.

It is easy to deduce from the Mellin transform that because of the various factors of $1/\mu$ only the term corresponding to $l = 1$ from (7.32) leads to a contribution to (7.33) which is $O(1)$ in μ , whereas the terms for $l > 1$ vary as negative powers of μ . We can therefore use R'_k instead of R_k in the large μ limit, where

$$R'_k = \left\{ (-i\phi)^{k-1}, \left(-\frac{1}{\mu} q(\psi)^* \left(\frac{\bar{\partial}^* \bar{\partial}}{\mu} + m - i\phi \right)^{-1} q(\psi) \right) \right\}. \quad (7.35)$$

Combining (7.33), (7.34), and (7.35), we obtain

$$\log \mathcal{D}_\mu^2 \sim - \sum_{k=1}^{\infty} \frac{(-1)^k}{k\mu} \text{Tr} \left\{ (-i\phi)^{k-1}, q(\psi)^* (\Delta_\mu^+)^{-1} \sum_{n=0}^{\infty} (i\phi (\Delta_\mu^+)^{-1})^n q(\psi) \right\} (\Delta_\mu^-)^{-k} + O\left(\frac{1}{\mu}\right). \quad (7.36)$$

Assuming that we can permute all the algebraic operators in (7.36) to the left of the inverse Laplacians without affecting the leading order term in the asymptotic expansion as $\mu \rightarrow \infty$, we get

$$\log \mathcal{D}_\mu^2 \sim - \sum_{k=1}^{\infty} \frac{(-1)^k}{k\mu} \text{Tr} \left\{ (-i\phi)^{k-1}, q(\psi)^* (i\phi)^n q(\psi) \right\} (\Delta_\mu^-)^{-k-n-1} + O\left(\frac{1}{\mu}\right). \quad (7.37)$$

Theorem 7.3 (see the Remark after the proof) shows that

$$\begin{aligned} \log \mathcal{D}_\mu^2 &\sim - \sum_{k=1}^{\infty} \frac{1}{k+1} \int_X \text{Tr} * \left\{ (i\phi)^k, q(\psi)^* \sum_{n=0}^{\infty} (i\phi)^n q(\psi) \right\} \frac{m^{-(n+k+1)}}{n+k+1} + O\left(\frac{1}{\mu}\right) \\ &= - \sum_{0 \leq k \leq l} \frac{m^{-l-1}}{(k+1)(l+1)} \int_X \text{Tr} * \left\{ (i\phi)^k, \frac{1}{m} q(\psi)^* (i\phi)^{l-k} q(\psi) \right\} + O\left(\frac{1}{\mu}\right) \\ &= - \sum_{0 \leq k \leq l} \frac{1}{(k+1)(l+1)} \int_X \text{Tr} \left\{ \left(i\frac{\phi}{m}\right)^k, \frac{1}{m} \psi \left(i\frac{\phi}{m}\right)^{l-k} \psi \right\} + O\left(\frac{1}{\mu}\right); \end{aligned} \quad (7.38)$$

recall that we are now studying the case $\dim X = 2$ so there is only one term ($c = 1$) in the sum in (7.14) and that in fact we are evaluating a ζ -function at $n + k + 1 = l + 1$.

We have now completed the computation of the adiabatic limit of the ratio of determinants. It remains to compare the result to the total Segre form of the index bundle by using the families index theorem, as we did at the end of the previous section for line bundles.

Theorem 5.1 implies that

$$\log s_{\text{tot}}(\text{Ind } \bar{\delta})(T) = \sum_{k \geq 1} (k-1)! ch_k(\text{Ind } \bar{\delta}) T^k. \quad (7.39)$$

Applying the families index theorem we find

$$ch(\text{Ind } \bar{\delta}) = \int_X (1 + c_1(X)/2) \sum_{n=0}^{\infty} \frac{(\mathcal{F}_{2,0} + \mathcal{F}_{1,1} + \mathcal{F}_{0,2})^n}{n!(2\pi i)^n},$$

where $\mathcal{F} := \mathcal{F}_{2,0} + \mathcal{F}_{1,1} + \mathcal{F}_{0,2}$ is the curvature of the universal bundle \mathcal{E} (see Section 5.1). Recall that $\mathcal{F}_{1,1} = \psi$ and $\mathcal{F}_{0,2} = \phi$, whereas $\mathcal{F}_{2,0}$ is the curvature of E . Hence the n -th Chern character form of the index bundle for $n \geq 1$ is given by

$$ch_n(\text{Ind } \bar{\delta}) = \int_X \frac{c_1(X)}{2} \frac{\text{Tr } \phi^n}{n!} + \frac{\text{Tr } \{\mathcal{F}_{2,0}, \phi^n\}}{(n+1)!} + \frac{\text{Tr } \{\psi^2, \phi^{n-1}\}}{(n+1)!} \quad (7.40)$$

and so

$$\begin{aligned} \log s_{\text{tot}}(\text{Ind } \bar{\delta}) \left(\frac{2\pi}{m} \right) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_X \text{Tr} \left(\frac{c_1(X)}{2} + \mathcal{F}_{2,0} \right) \left(\frac{\phi}{im} \right)^n \\ &\quad + \frac{(-1)^n}{n(n+1)} \int_X \text{Tr} \left\{ \frac{\psi^2}{m}, \left(\frac{\phi}{im} \right)^{n-1} \right\}. \end{aligned} \quad (7.41)$$

The first two terms in the right-hand side of (7.41) exactly match the adiabatic expression for $\log \mathcal{D}_\mu^1$ given in (7.31). We claim that the third coincides with the adiabatic formula (7.38) for $\log \mathcal{D}_\mu^2$. To see this, we use the following combinatorial exercise:

Lemma 7.2.

$$\sum_{0 \leq k \leq l} \frac{1}{(k+1)(l+1)} \text{Tr} \{ \phi^k, \psi \phi^{l-k} \psi \} = \sum_{0 \leq n \leq l} \frac{1}{(n+1)(n+2)} \text{Tr} \{ \psi^2, \phi^n \}. \quad (7.42)$$

Proof. The trace is invariant under cyclic permutations of the factors, so we can arrange that the first factor in each term of the expansions of $\{ \phi^k, \psi \phi^{l-k} \psi \}$ and $\{ \psi^2, \phi^n \}$ be ψ . After this rearrangement, we see that both sides of (7.42) equal

$$\sum_{0 \leq k \leq l} \frac{1}{l+1} = \sum_{0 \leq k \leq l} \text{Tr } \psi \phi^k \psi \phi^{l-k}.$$

This completes the check of our conjecture in the case of a family of $\bar{\delta}$ operators on a Riemann surface, modulo the assumptions about the possibility to commute certain operators without affecting the highest-order term in some asymptotic expansions.

Appendix A

Some Comments on Two Problems Related to TNSW

In the Introduction to this thesis we mentioned two problems (the relationship between Donaldson and Seiberg-Witten invariants and the topological S-duality conjecture) whose formulation involves TNSW. In fact, part of the original motivation for this project was the hope that one can use TNSW to provide path-integral solutions to these problems. Unfortunately, we have found that the localization ideas used in this thesis are not sufficient to resolve either problem. The purpose of this Appendix is to explain the limitations of our approach and suggest a few directions for future research.

A.1 The relationship between TNSW and MTNSW

In this section we analyze the relationship between the correlation functions and partition function of TNSW and of its massive version. This relationship is needed if we want to use the computations from Chapters 4 and 5 to verify particular cases of the S-duality conjecture. To explain this, recall that our formulae in Chapters 4 and 5 refer to MTNSW, the massive version of the theory, whereas, as will be summarized below, the S-duality conjecture makes certain predictions about the partition function of TNSW.

We will denote by Z_{NSW} the partition function of TNSW (see (3.21)) and by $Z_{NSW}(m)$ the partition function of the massive version of the theory (see (3.4) for the definition). Similarly, the vacuum expectation values in TNSW and MTNSW of the observable \mathcal{O} will be denoted by $\langle \mathcal{O} \rangle_{NSW}$ and $\langle \mathcal{O} \rangle_{NSW}(m)$, respectively. Recall that in Section 3.5 we suppressed the parameters t and r from the discussion since we proved that all the above quantities are independent of them.

Statement A.1. $\langle \mathcal{O} \rangle_{NSW} = 0$ if $\deg \mathcal{O} \neq \text{v.dim } \mathcal{M}_{NSW}$, where $\text{v.dim } \mathcal{M}_{NSW}$ denotes the virtual dimension of the moduli space \mathcal{M}_{NSW} .

The virtual dimension equals, by definition, the index of the Fredholm section $\#$ used in the infinite-dimensional Mathai-Quillen construction. Recall that the operators \mathcal{O} under consideration are products of differential forms (actually, products of the μ -classes defined in Section 3.1) and so, at least formally, the statement follows from the Mathai-Quillen formalism. In this context, ‘formally’ means that we ignore the potential problems created by the non-compactness of the moduli space \mathcal{M}_{NSW} . In other words, we pretend that \mathcal{M}_{NSW} is a smooth, oriented, compact manifold. Although the moduli space is orientable and smoothness can be achieved by choosing a generic metric of the base, compactness fails, and so a rigorous proof of the statement would require the Uhlenbeck compactification to be taken into account.

In the physics literature on BRST cohomology, the statement that the integral of a differential form of the wrong degree (i.e. not equal to the dimension of the integration space) is equal to zero is called the ‘cancellation of the ghost number anomaly’. To the best of the author’s knowledge, there are currently no further arguments to support such a cancellation except the above dimension count (sometimes this statement is given as a claim about the presence of a supersymmetry, which is mathematically equivalent to our formulation).

Statement A.1 does not extend to the correlation functions of massive TNSW. In the massive theory, the operators \mathcal{O} are S^1 -equivariant differential forms (in particular, \mathcal{O} has a different meaning in TNSW and in MTNSW). The operators \mathcal{O} arise from the equivariant cohomology of the space of connections (with respect to the the group of gauge transformations). To obtain the integrand over the moduli space, one applies the Chern-Weil homomorphism in the case of TNSW and its S^1 -equivariant generalization in the case of MTNSW. If $\deg_{S^1} \mathcal{O} > \text{v.dim } \mathcal{M}_{NSW}$, the (inhomogeneous) differential form \mathcal{O} (in MTNSW) can still include components whose differential form degree equals the dimension of the integration space, leading to a non-zero contribution to the correlation function. However, if the degree of \mathcal{O} is smaller than the dimension of the moduli space we have

Statement A.2. If $\deg_{S^1} \mathcal{O} < \text{v.dim } \mathcal{M}_{NSW}$ then $\langle \mathcal{O} \rangle_{NSW} (m) = 0$.

The formal justification is obvious: since the differential-form degree of all the components of \mathcal{O} is less than the dimension of the integration space, none of them contributes to the integral.

Finally, the case when the degree of \mathcal{O} equals the dimension of the moduli space is described by

Statement A.3. If $\deg_{S^1} \mathcal{O} = \text{v.dim } \mathcal{M}_{NSW}$ then $\langle \mathcal{O} \rangle_{NSW} = \langle \mathcal{O} \rangle_{NSW} (m)$ for any real m .

Formally this follows from the fact that the top component of $CW_{S^1}(\mathcal{O})$ equals $CW(\mathcal{O})$ and the lower-degree components of $CW_{S^1}(\mathcal{O})$ don’t matter in the integral over moduli space.

We should emphasize again that the previous statements are rigorous only in the cases when \mathcal{M}_{NSW} is compact. Otherwise, one should analyze the integrals over \mathcal{M}_{NSW} further.

Let us also formulate the particular form of Statements A.2 and A.3 for the partition functions of TNSW and MTNSW, i.e. when $\mathcal{O} = 1$:

Statement A.4. i) If $\text{v.dim } \mathcal{M}_{NSW} > 0$ then $\mathcal{Z}_{NSW} = \mathcal{Z}_{NSW}(m) = 0$.

ii) If $\text{v.dim } \mathcal{M}_{NSW} = 0$ then $\mathcal{Z}_{NSW} = \mathcal{Z}_{NSW}(m)$.

iii) If $\text{v.dim } \mathcal{M}_{NSW} < 0$ then $\mathcal{Z}_{NSW} = 0$ but $\mathcal{Z}_{NSW}(m)$ could be non-zero.

By applying the index theorem to the coupled Dirac operator and using Proposition 3.1 we get

Proposition A.1. *The virtual dimension of \mathcal{M}_{NSW} is given by*

$$\text{v.dim } \mathcal{M}_{NSW} = \frac{3}{4}(c \cdot c - 2\chi - 3\sigma), \quad (\text{A.1})$$

where c denotes the class of the Spin^c -structure, χ the Euler characteristic, and σ the signature of the four-manifold.

Proof.

$$\begin{aligned} \text{v.dim } \mathcal{M}_{NSW} &= 8k - 3(1 + b_2^+) + 2\text{Ind}_{\mathcal{C}}(\not{D} \otimes \text{sl}(E)) \\ &= 8k - 3(1 + b_2^+) + 2 \left(\frac{3}{8}(c \cdot c - \sigma) - 4k \right) \\ &= \frac{3}{4}(c \cdot c - 2\chi - 3\sigma), \end{aligned} \quad (\text{A.2})$$

where $k = c_2(E)$ and c is the class of the Spin^c -structure.

Remark. The remarkable feature of the above formula is that $\text{dim } \mathcal{M}_{NSW}$ is *independent* of the instanton number k .

We will restrict now to the particular case when $\text{v.dim } \mathcal{M}_{NSW} = 0$. This implies that $\langle \mathcal{O} \rangle_{NSW} = 0$ for any operator \mathcal{O} of positive degree, so the only quantity of interest is the partition function \mathcal{Z}_{NSW} . The previous index computation yields

Statement A.5. *If $c \cdot c = 2\chi + 3\sigma$, so that $\text{dim } \mathcal{M}_{NSW} = 0$, then*

$$\mathcal{Z}_{NSW} = \mathcal{Z}_{NSW}(m) \quad (\text{A.3})$$

for any real m .

As with the statements on Section 3.5, we present the analogous statement in finite dimensions and quote the physical argument for the infinite-dimensional case. As we have already explained for Statement A.3, in finite dimensions, if E is an oriented S^1 -equivariant vector bundle over the compact manifold M (on which S^1 acts) and $\text{rk}(E) = \text{dim}(M)$ then

$$\int_M e(E) = \int_M e_{S^1}(E)$$

because the difference $e_{S^1}(E) - e(E)$ has no top degree component. The analogy with the infinite-dimensional statement is now clear since \mathcal{Z}_{NSW} is a formal Euler number and $\mathcal{Z}_{NSW}(m)$ is the

corresponding S^1 -equivariant object. The appropriate physical argument is given (in a different context) at the beginning of Section 5 in [32]: although the addition of the mass term breaks part of the supersymmetry, there is enough supersymmetry left to preserve the topological character of the theory, in particular the partition function is unchanged.

Remark. One has to be careful about this supersymmetry argument. A rigorous mathematical definition of the partition function might very well lead to a ‘spontaneous supersymmetry breaking’, especially because of the non-compactness of \mathcal{M}_{NSW} (i.e. the operations needed to compactify the moduli space could destroy the supersymmetry).

The condition $c \cdot c = 2\chi + 3\sigma$ is easily fulfilled with a suitable choice of c , but instead of discussing it in general we will focus on an important special case, namely the case of spin four-manifolds and the ‘spin’ Spin^c -structure, i.e. $c = 0$. The above condition reduces to $2\chi + 3\sigma = 0$, which, in the case of a Kähler surface gives $K^2 = 2\chi + 3\sigma = 0$. This can happen either if the Kodaira dimension is zero or one and the surface is minimal, or for a suitable blow-up of a surface of general type. A rich class of examples consists of simply connected (spin) elliptic surfaces (i.e. with at most two singular fibres). The simplest such four-manifold is the $K3$ surface, for which $\chi = 24$ and $\sigma = -16$. In general, if the geometric genus of the elliptic surface is p_g (which has to be odd for the surface to be spin) then $b_2 = 12p_g + 10$, $b_2^+ = 2p_g + 1$, $b_2^- = 10p_g + 9$, and so $\chi = 12p_g + 12$ and $\sigma = -8p_g - 8$, and so $\dim \mathcal{M}_{NSW} = 0$ in these cases.

A.2 S-duality and the partition function of topological non-abelian Seiberg-Witten theory on a $K3$ surface

We now discuss the possibility of applying Chapters 4 and 5 to the S-duality conjecture. Our present results are not sufficient to obtain further tests of S-duality along the lines of [32]. In order to illustrate the limits of our existing computations, we specialize even more and restrict to the case of $K3$ surfaces. This case already illustrates the main points of the argument and the relationship with the physical results mentioned in Section 3.4. The goal is to compare our results to the S-duality predictions of Vafa and Witten [32]. Although topological non-abelian Seiberg-Witten theory is in general a *different* topological twisting of $N = 4$ Yang-Mills than the Vafa-Witten twist, it does coincide with the latter on a hyperKähler manifold, in particular on a $K3$ surface (see [16]). Therefore the partition function \mathcal{Z}_{NSW} should coincide in this case with the partition function of the Vafa-Witten twist, which is explicitly computed in [32].

The relevant result from [32] is formula (4.17) for the Vafa-Witten partition function:

$$\mathcal{Z}_{\text{Vafa-Witten}}^{K3}(q) = q^2 \left(\frac{1}{8}G(q^2) + \frac{1}{4}G(q^{1/2}) + \frac{1}{4}G(-q^{1/2}) \right), \quad (\text{A.4})$$

where

$$G(q) := \frac{1}{q \prod_{n=1}^{\infty} (1 - q^n)^{24}}$$

is the -24 -th power of the Dedekind eta function. In the above formula, $\mathcal{Z}_{\text{Vafa-Witten}}^{K3}(q)$ is obtained after summing over instanton numbers, as in Conjecture 1 from Section 3.4, i.e.

$$\mathcal{Z}_{\text{Vafa-Witten}}^{K3}(q) = \sum_{k=0}^{\infty} q^k \mathcal{Z}_k,$$

where \mathcal{Z}_k is the partition function corresponding to connections on a fixed bundle E with $c_2(E) = k$.

There is one important subtlety about \mathcal{Z}_0 : although the virtual dimension of \mathcal{M}_0 is negative, (which would naively suggest that $\mathcal{Z}_0 = 0$), the Vafa-Witten formula (A.4) predicts ‘somewhat mysteriously’ that $\mathcal{Z}_0 = 1/4$ (see formula (4.16) of [32])—the non-vanishing of $\mathcal{Z}_0 = 0$ being interpreted as a contribution from the trivial connection.

We therefore have to compute

$$\mathcal{Z}_{NSW}^{K3} := \sum_{k \in \mathbb{Z}} q^k \mathcal{Z}_{NSW,k}^{K3},$$

where $\mathcal{Z}_{NSW,k}$ denotes the partition function of topological non-abelian Seiberg-Witten theory computed from configurations with instanton number k . Note that, according to Statement A.5, $\mathcal{Z}_{NSW,k}^{K3}$ equals the corresponding partition function in massive TNSW (note that $\dim \mathcal{M}_{NSW,k} = 0$ for all integers k). As elaborated in Sections 4.3, 4.4, and 5.3, the latter partition function is a combination of Donaldson and Seiberg-Witten invariants—which are completely known for a $K3$ surface.

In fact, for a $K3$ surface, the only Seiberg-Witten basic class is $\lambda = 0$ with $SW(\lambda) = 1$ and Theorem 5.2 reads, in this particular case,

$$\langle \exp(p\Pi + q\Sigma) \rangle = \left\{ \exp \left(2p + \frac{1}{2}q^2(\Sigma \cdot \Sigma) \right) \right\}_{2 \pmod 4}. \quad (\text{A.5})$$

The only contribution of abelian Seiberg-Witten pairs to $\mathcal{Z}_{NSW,k}^{K3}(m)$ is therefore the one obtained from $k = 0$ and $x = 0$, where x is as in Statement 4.6. Actually (4.24) shows that

$$\text{contribution of reducible configurations to } \mathcal{Z}_{NSW,k}^{K3}(m) = \begin{cases} 1/4 & \text{if } k = 0 \\ 0 & \text{if } k \in \mathbb{Z} \setminus 0. \end{cases} \quad (\text{A.6})$$

To check the last statement, recall that $\chi = 24$, $\sigma = -16$, $x = 0$ (i.e. l is the trivial line bundle), and

$$\text{Ind}(\mathcal{D} \otimes l^{-2}) = \text{Ind}(\mathcal{D}) = (-\sigma/8) = 2.$$

Remark. The result of (A.6) is consistent with Statement A.5, which implies that $\mathcal{Z}_{NSW}(m)$ is independent of m under the assumption that $\dim \mathcal{M}_{NSW} = 0$ (for a four-manifold of simple type).

Therefore the contribution from reducible configurations to $\mathcal{Z}_{NSW,0}^{K3}(m)$ does agree with the prediction of Vafa and Witten for instanton number 0 and provides a geometric explanation for the ‘somewhat mysterious’ $1/4$.

The harder task is to compute the contributions to $\mathcal{Z}_{NSW,k}(m)$ from ‘pure gauge’ configurations, i.e. configurations with vanishing spinor. With the notations of Section 4.3, and according to (4.13),

$$\mathcal{Z}_{NSW,k}^{K3, \text{ pure gauge}}(m) = \int_{\tilde{\mathcal{M}}_k} s_{\dim \mathcal{M}_k/2}(\text{Ind } \mathcal{D} \otimes \text{sl}(E_k)). \quad (\text{A.7})$$

Notice that there is no power of m appearing as a prefactor on the right-hand side of (A.7), in agreement with the prediction that $\mathcal{Z}_{NSW}^{K3}(m)$ is independent of m .

We can therefore state the following

Statement A.6.

$$\mathcal{Z}_{NSW}^{K3, \text{ pure gauge}}(m) = \sum_{k=2}^{\infty} q^k \int_{\mathcal{M}_k} s_{4k-6}(\text{Ind } \mathcal{D} \otimes \text{sl}(E_k)). \quad (\text{A.8})$$

The lower limit of the summation is worth explaining: since $\dim \mathcal{M}_k = 8k - 12$, \mathcal{M}_k is generically empty for $k \leq 1$, so there are no contributions from k in this range. In addition, we have already included the contribution from the trivial connection in the contribution from reducible configurations.

We conclude this section by analyzing the implications of the Statement. The ingredients needed in (A.8) are the expression of the Segre class of the index bundle in terms of μ -classes from Section 5.2 and the computations of topological invariants from Section 5.3.

The Segre class of the index bundle was computed in (5.20); the $K3$ surface being spin, $Z = 0$; moreover $a = -\sigma/4 = 4$ and $b = -8k$. Hence

$$s_{\text{tot}} \text{Ind}(\mathcal{D} \otimes E_k) = (1 - X)^{4k-2} \exp\left(\frac{-8X}{2(1-X)}\right) \exp\left(\frac{2Y - XY}{(1-X)^2}\right), \quad (\text{A.9})$$

where E_k denotes the $SU(2)$ -bundle of instanton number k . Recall that X and Y are degree-four cohomology classes so s_{4k-6} is the part of degree $2k - 3$ in X, Y .

As for Theorem 5.4 which gives $\langle X^m Y^n \rangle$ for any m and n , we need to work out the function $K(T)$ in (5.46), where $K(T)$ was defined in (5.43). There is only one basic class for a $K3$ surface, namely $\lambda = 0$. Therefore in this case

$$K(T) = \det(\text{Id} - 2\mathcal{E}^2 T)^{-\frac{1}{2}}, \quad (\text{A.10})$$

where \mathcal{E} is the intersection matrix, which is well-known to be

$$\mathcal{E} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 2(-E_8),$$

with E_8 the suitable Cartan matrix.

We haven’t been able so far to use (A.9) and (A.10) to find a closed form expression for the right-hand side of Statement A.6. A comparison with the Vafa-Witten formula (A.4) can nevertheless be made for a few coefficients by computing them individually. The author has used Maple to check

the first ten coefficients, but unfortunately they don't agree with the corresponding coefficients in the Vafa-Witten expression.

The main reason for the disagreement seems to be the non-compactness of the moduli space \mathcal{M}_{NSW} . We have tried to circumvent this problem by using the abelian localization theorem first, in order to reduce to Donaldson moduli spaces, for which the Uhlenbeck compactification is well-understood and leads to Donaldson invariants. We have tacitly assumed the the ansatz of Donaldson-Witten theory, i.e. that the integrals over Donaldson moduli spaces can to be interpreted as integrals over the Uhlenbeck compactification, can be applied to massive non-abelian Seiberg-Witten theory as well.

A more careful analysis is probably needed. The manifold \mathcal{M}_{NSW} does admit a compactification, but the locus at infinity could contribute as well in the localization procedure of Section 4.2—therefore extra terms would have to be added to the partition function.

A.3 Donaldson versus Seiberg-Witten invariants

Similar problems arise when attempting to relate the Donaldson and Seiberg-Witten invariants by using MTNSW. As explained in the first section of the Appendix, in the case when $v.\dim \mathcal{M}_{NSW} > 0$, one can still try to evaluate the expectation value for an operator whose degree is less than $v.\dim \mathcal{M}_{NSW}$ by using the abelian localization theorem. Formally, the expectation value should be zero because of the dimension mismatch (see Statement A.2), which would imply that the fixed point contributions add up to zero. This would lead to an equality between the 'pure gauge' contributions and the contributions from reducible configurations, i.e. a relationship between some Donaldson and Seiberg-Witten invariants. This reasoning however leads to formulae which are mathematically incorrect. This can be checked by comparison with cases in which the Donaldson and Seiberg-Witten invariants are known, and we will illustrate this in the case of a $K3$ surface.

Consider an $SU(2)$ -bundle E with $c_2(E) = k$ over the $K3$ surface and the topological non-abelian Seiberg-Witten theory whose fields are connections on this bundle and sections of the positive Spin bundle. Seiberg-Witten contributions arise from decompositions $E = l \oplus l^{-1}$, with $c_1(l)^2 = -k$. It is known that the only non-zero Seiberg-Witten invariant of $K3$ corresponds to the trivial line bundle, so we would conclude that all Donaldson invariants corresponding to a positive instanton number vanish, in contradiction with known results (see, for instance, the Kronheimer-Mrowka structure theorem in Section 5.3). Even summing over instanton numbers doesn't fix the problem, which leads us to the same conclusion as the comments in the previous section: we need a more careful definition of the integral over \mathcal{M}_{NSW} that would take into account its non-compactness before applying the localization theorem.

A worthwhile topic for further research is to produce a rigorous definition of the integrals over

moduli spaces of topological gauge theories that would be consistent with various operations such as the localization procedure analyzed in this thesis. In other words, the definition of the integrals for various gauge groups and representations giving the 'matter part' should satisfy compatibility requirements that we will pursue in the future.

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