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On the frequencies of inhomogeneous soil strata: Dobry's paradox

Eduardo Kausel¹

Abstract

This brief article elaborates on some clearly unlikely predictions made some four decades ago by R. Dobry within the context of his doctoral dissertation, which concern the resonant frequencies of soil strata whose stiffness starts at zero at ground level and then increases continuously as some power of the depth. Although the problem was eventually traced to subtle changes in the boundary conditions which take place when the soil parameter exceeds a threshold value, and the discrepancy was ultimately fully resolved, Dobry initially countered to the technical objections of the writer by presenting an extremely elegant counter-proof based on dimensional analysis whose strength emanated from the fact that it was free from the issue of the boundary conditions, and thus forced the writer to seek alternative explanations for the contradictory results. It is this fruitful exchange of ideas which provides the motivation for this brief technical note expounding on an apparent paradox, which may prove useful to the soil dynamics community for its potential didactical value, not to mention as an outright tool.

The Problem

In section 2.4 of his rather massive doctoral dissertation at MIT^{1,2} concerned with soil properties and one-dimensional wave amplification in soil deposits, R. Dobry considered the amplification of shear waves in a soil stratum of finite thickness h on rigid rock, where the mass density ρ of the stratum is uniform but the shear wave velocity is zero at the ground surface and increases steadily with depth in the following form:

$$V_s = Az^{p/2} = V_h \left(\frac{z}{h} \right)^{p/2}, \quad Ah^{p/2} = V_h \quad (1)$$

where A, p are constants, V_h is the shear wave velocity at the base of the stratum of depth h , and z is the depth measured down from the free surface. Although Dobry suggested that typical values for p lie in the range $0.33 < p < 1$ for most normally consolidated soils at small strains, his formulation was quite general and included other values of p . The special case $p = 1$, in which the shear modulus increases linearly with depth, is often referred to as a Gibson soil.

After solving in closed form the differential equation with a boundary condition of zero strain at the surface, Dobry concluded —correctly— that the fundamental period of the inhomogeneous stratum is proportional to the depth raised to some power, namely (p. 98)

$$T_1 \sim h^{(1-\frac{1}{2}p)} \quad (2)$$

¹ Professor of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139

From here, he made the following inferences about the fundamental period of the stratum T_1 namely that it

- increases as h increases for $p < 2$
- is independent of h for $p = 2$
- decreases as h increases for $p > 2$

He then went on to state that

It is usually accepted that, in the analysis of soil deposits, [addition of] deeper and stiffer layers will always tend to increase the natural period. However, these results show that a very rapid increase of velocity with depth may decrease the period, as more layers are considered!

Both this paragraph as well as the last two bullet items above did not seem likely to the writer, and this motivated the contact and exchange with Dobry (1990's). Indeed, there are fundamental lemmas in dynamics which indicate that any addition of layers *must* necessarily increase the period, *whatever* the soil properties. A simple way to visualize this principle is to consider the same stratum augmented with an arbitrarily thick layer of infinite stiffness, which of course at first changes nothing, except for adding a set of infinite frequencies associated with the rigid layer. However, as the stiffness of that additional layer is steadily softened to its final values, then some or all frequencies must necessarily drop, including those that were infinitely large, and certainly no frequency can increase².

In a nutshell, the problem with the formula was ultimately traced to the boundary conditions at the surface, which on close inspection ceased to be valid when $p \geq 2$. Still, the mathematical details of this rather subtle condition are now irrelevant, for they currently lack in either significance or interest. Instead, the purpose of this note is to report on the paradoxical contradiction that arose after Dobry — who was worried enough by the writer's objections to seek an independent verification— came up with an elegant alternative derivation of the natural frequencies of the soil deposit based on dimensional analysis. This new proof was not only interesting in its own right, but at first seemed to corroborate the earlier observations in Dobry's dissertation as well as contradict the fundamental lemmas referred to above. Thus, although we demonstrated the falsehood of the boundary conditions when $p \geq 2$, this was contradicted by Dobry's dimensional proof which made no use of such false boundary conditions in the first place, and so that new proof was a-priori known to be correct. This gave rise to the intriguing paradox reported herein as well its final resolution by the writer.

Dobry's dimensional analysis formula

To counteract our initial objections, Dobry (1990's) provided handwritten notes to the writer containing the following gem of a proof, based on dimensional analysis rooted in Buckingham's PI Theorem^{4,5}. The

² A more careful analysis would need to consider a soil where the reference shear modulus is normalized at some pre-determined depth, to avoid varying simultaneously two parameters affecting the period, namely the total depth of the stratum and the variation of shear modulus with depth. Although this would complicate somewhat the equations being considered, it would not materially change the conclusions herein.

problem at hand involves the following dimensional parameters in which the symbols L, M, T denote length, mass, and time, respectively:

Parameter	Dimensional Formula
h = thickness	L
ρ = mass density	ML^{-3}
V_h = sh. w. veloc. at base	LT^{-1}
p = exponent	0
T_1 = fund. period	T
ν = Poisson's ratio	0

There are six parameters in this table, and because the number of primary dimensions (or units) is three, Buckingham's Theorem indicates that there are $6 - 3 = 3$ independent dimensionless products Π_1, Π_2, Π_3 involved. The first two of these products are simply

$$\Pi_1 = p, \quad \Pi_2 = \nu$$

both of which are *eo ipso* dimensionless. Hence, only one dimensional product remains, which is

$$\Pi_3 = h^\alpha \rho^\beta V_h^\gamma T_1^\delta$$

where $\alpha, \beta, \gamma, \delta$ are unknown constants. It follows that

$$(L)^\alpha (ML^{-3})^\beta (LT^{-1})^\gamma (T)^\delta = (L)^{\alpha-3\beta+\gamma} (M)^\beta (T)^{-\gamma+\delta}$$

so

$$\begin{aligned} \alpha - 3\beta + \gamma &= 0 & \alpha &= -\gamma \\ \beta &= 0 & \Rightarrow \beta &= 0 \\ -\gamma + \delta &= 0 & \delta &= \gamma \end{aligned}$$

Replacing these into the definition of Π_3 , we obtain

$$\Pi_3 = h^{-\gamma} \rho^0 V_h^\gamma T_1^\gamma = \left(\frac{T_1 V_h}{h} \right)^\gamma = \left(\frac{T_1 A h^{\frac{1}{2}p}}{h} \right)^\gamma = \left(\frac{T_1 A}{h^{1-\frac{1}{2}p}} \right)^\gamma$$

but if Π_3 is dimensionless, then so is also $\Pi_3^{1/\gamma}$, in which case Π_3 can legitimately be redefined for the purpose of the PI Theorem as the expression above without the exponent γ . Returning then to the main expression of Buckingham's PI Theorem, it states that there exists a function $g(\)$ such that

$$g(\Pi_1, \Pi_2, \Pi_3) = g\left(p, \nu, \frac{T_1 A}{h^{1-\frac{1}{2}p}}\right) = 0$$

or equivalently,

$$\frac{T_1 A}{h^{1-\frac{1}{2}p}} = f(p, \nu)$$

where $f(\)$ is another function. Since p, ν are constants which do not change when h changes, then for the purposes of this analysis $f(p, \nu) = B$ is a constant, in which case

$$T_1 = \frac{B}{A} h^{1-\frac{1}{2}p} = C h^{1-\frac{1}{2}p} \quad (3)$$

where $C = B / A$ is also a constant independent of h that in principle depends on both p and ν . In reality, because only shear wave propagation is involved, it can be shown that $C = C(p)$ is only a function of p which does not depend on Poisson's ratio ν . The determination of the function $C(p)$ cannot be done by using dimensional analysis alone, so eq. 3 is the final result of the dimensional analysis for this problem. Observe that this result not only corroborates eq. 2, but that it is *not* subjected to any errors in boundary conditions for any value of p , for no such conditions were used in the derivation of eq. 3 in the first place! However, eq. 3 too contradicts the fundamental dynamic principles alluded to earlier, in which case it cannot possibly be true, or at least not always. If so, then where is the hidden error in the dimensional analysis?

As it turns out, there is nothing wrong with this application of Buckingham's Theorem, and in fact, eq. 3 correctly predicts that $T_1 \sim h^{1-\frac{1}{2}p}$ when $p < 2$. However, such a dimensional analysis can only provide the *form* of the solution *if it exists*, but without guaranteeing that such a solution actually *exists*. In fact, an examination of formula 3 in the context of the fundamental mechanical principle already alluded to indicates that when $p \geq 2$, the results are absurd, and therefore, that *no solution can exist in that case*. The inescapable conclusion is that a layer of thickness h and $p \geq 2$ cannot possibly have a period T_1 , and therefore, that such a problem must be mathematically (and physically) ill-posed. Just as importantly, observe that this fact has been established a priori without ever solving a differential equation!

And why is it that for $p \geq 2$ no solution exists? The reason relates to the pathological nature of this problem when $p \geq 2$ *and* the shear modulus vanishes at the surface. Indeed, the problem is eminently suspect even when p is within appropriate bounds, because the stress boundary condition $\tau_0 = G_0 \gamma_0$ at the surface translates into the identity $0 = 0 \times \gamma_0$, in which case the strain there need not be zero as is normally assumed. On the other hand, if the shear modulus decreases too fast as the surface is approached from underneath (case of $p \geq 2$), then the top infinitesimally thin layer is not rigid enough to resist its inertia and simply slips away during vibration, i.e. $\gamma_0 \rightarrow \infty$. If so, the problem becomes physically ill-posed, and no fundamental mode with period T_1 exists.

Conclusion

To the best of the writer's knowledge, Dobry's novel strategy of using Buckingham's PI Theorem to infer the fundamental period of an inhomogeneous soil stratum is new and has not been reported before within the soil dynamics community. An additional important lesson to be learned in Soil Dynamics —a lesson which is well known in other areas of science and engineering— is that dimensional analysis can be of invaluable help in establishing the form of some physical phenomena, but it cannot guarantee that such formulas will be physically meaningful. For that purpose, other methods and principles must be invoked, and when taken together, they may provide useful answers without having to solve a complicated differential equation.

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