5.1 The Riesz Representation Theorem

Back into pre-hilbert space.

**Lemma.** In a pre-Hilbert (no completeness) space the parallelogram law holds:

\[ \forall x, y \in H \quad 2(\|x\|^2 + \|y\|^2) = \|x - y\|^2 + \|x + y\|^2 \]

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Theorem. Conversely, any norm space in which this law holds is a pre-Hilbert space. (i.e. \( \exists \) an inner product such that \( \|x\|^2 = \langle x, x \rangle \).)

Proof. Define

\[ 4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2 \]

we also have to prove linearity, but thats not hard. \( \square \)

Proposition. If \( C \subset H \) is a closed convex subset of a Hilbert Space then \( \exists x^* \in C \) such that

\[ \|x^*\| = \inf_{x \in C} \|x\| \]

Proof. We first define convex. \( C \) convex implies that for \( x, y \in C \) then \( \frac{x+y}{2} \in C \) (a rather weak notion of it, but it works in our case)

Now continue with the proof. By definition \( \exists x_n \in C \) such that \( \|x_n\|^2 \to \inf_{x \in C} \|x\|^2 = 1^2. \)

We claim that \( \{x_n\} \) is automatically Cauchy. Given \( \epsilon > 0 \), \( \exists N \) such that

\[ n > N \Rightarrow \|x_n\|^2 = \inf_{x \in C} \|x\|^2 + \delta \]

Now, note that \( \|x_n + x_m\|^2 = 4\|\frac{x_n + x_m}{2}\|^2 \). Then

\[ \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \leq 4I^2 + 4\delta - 4I^2 \leq 4\delta \]

So if \( n, m \geq N \), then \( \|x_n - x_m\| \leq 4\delta \) so \( \{x_n\} \) is Cauchy. \( \square \)

Theorem. Riesz Representation Theorem If \( H \) is a hilbert space and \( g : H \to \mathbb{C} \) (a functional) is linear and continuous then \( \exists y \in H \) such that

\[ g(x) = \langle x, y \rangle, \quad \forall x \in H \]

First, what does continuous mean? \( g : H \to \mathbb{C} \) continuous iff \( g^{-1}(O) \subset H \) is open for \( O \subset \mathbb{C} \) open, which implies that \( g^{-1}(B(0,1)) \subset H \) is open. i.e. \( \{x \in H | |g(x)| < 1\} \subset H \) is open. (All of this is because \( g \) is linear, i.e. you only need continuity around the origin) or similarly \( \{x \in H : |g(x)| \leq 1\} \) is closed.

\( \{x \in H : |g(x)| < 1\} \) is open implies \( \exists \epsilon > 0 \) such that \( \{x \in H : \|x\| < \epsilon\} \Rightarrow \|g(x)\| < 1 \).

\( g \) is linear, so \( x \in H : \|x\| \leq 1 \Rightarrow \|\frac{1}{2}x\| < \epsilon, \) so \( |g(\frac{1}{2}x)| < 1 < 2, \) so

\[ g(x) = \left| \frac{2}{\epsilon} g \left( \frac{\epsilon}{2} x \right) \right| \leq \frac{4}{\epsilon} = c \]

So \( \exists \) a constant \( c \) such that

\[ \|x\| \leq 1 \Rightarrow |g(x)| \leq c \]

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that is, we have boundedness, $\forall x \in H, |g(x)| \leq C\|x\|$. Thus, by the boundedness around the origin,

$$
\left| g\left(\frac{x}{\|x\|}\right) \right| \leq C \Rightarrow \left| \frac{1}{\|x\|} g(x) \right| = \frac{1}{\|x\|} g(x), \quad \text{so } |g(x)| \leq C\|x\|
$$

so by continuity of $g$, $\exists C$ such that $|g(x)| \leq C\|x\|$, $\forall x \in H$.

But it works conversely. To get $\Leftarrow$ direction: If $x_n \to x$ then $x_n - x \to 0$, i.e. given $\epsilon > 0 \exists N$ such that $\|x_n - x\| < \epsilon$ if $n \geq N$, then

$$
|g(x_n) - g(x)| = |g(x_n - x)| \leq C\|x_n - x\| \to 0
$$

So for linear functions, boundedness is the same as continuity,

$$
|g(x)| \leq C\|x\|
$$

**Proof.** If $y \in H$ by Cauchy-Schwarz implies

$$
|\langle x, y \rangle| \leq \|x\|\|y\| = C\|x\|
$$

so this jibes with $|g(x)| \leq C\|x\|$.

Conversely, given $g : H \to \mathbb{C}$ linear and continuous find $y$ such that $g(x) = \langle x, y \rangle$. $g = 0$ implies that $y = 0$. $g$ non-zero

$$
g(x') = 1, \quad x' = \frac{x}{g(x)}
$$

Set

$$
C_g = \{ x \in H : g(x) = 1 \}
$$

This is closed (inverse image of a single point) and convex, since

$$
g\left(\frac{x + y}{2}\right) = \frac{1}{2} g(x) + \frac{1}{2} g(y) = 1
$$

This is closed, convex. By the proposition above $\exists! y^* \in C_g$ such that $\|y^*\| = \inf_{x \in C_g} \|x\|$. Then

$$
C_g = \{ y^* + y : y \in H, g(y) = 0 \}
$$

$y^* \perp N, N = \{ x \in H : g(x) = 0 \}$, the above can be re-written as $C_g = \{ y^* + y : y \in N \}$. So then $y \in N \Rightarrow \langle y, y^* \rangle = 0$, because

$$
\|y^* + ty\|^2 = \|y^*\|^2 + t\langle y, y^* \rangle + t\langle y, y^* \rangle + t^2\|y\|^2
$$

We claim

$$
x \in H \Rightarrow x = sy^* + y, \quad y \in N, s \in \mathbb{C}
$$

**Proof:**

$$
g(x) = 0 \Rightarrow x \in N$$

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so then
\[ g(x) = s \Rightarrow g \left( \frac{x}{s} \right) = 1 \Rightarrow \frac{x}{s} \in C_g \Rightarrow x = sy^* + y \]
then
\[ g(x) = g(sy^* + y) = sg(y^*) + g(y) = s \]

Then \( x \in H \) implies \( x = g(x)y^* + y, \quad g(y) = 0 \) then
\[ \langle x, y^* \rangle = g(x)\langle y^*, y^* \rangle \Rightarrow g(x) = \left\langle x, \frac{y^*}{\|y^*\|^2} \right\rangle \]

Example. If \( g : L^2(X, \mu) \to \mathbb{C} \) is continuous and linear, then \( \exists G \in L^2 \) such that
\[ g(f) = \int fGd\mu \]