Proposition. \( \mu_L \) is countably additive.

Proof. Given \( A_i \in \mathcal{R} \), \( A_i \cap A_j = \emptyset \), we know that \( \bigcup_{i=1}^{\infty} A_i = A \in \mathcal{R}_L \) we must show that \( \mu_L(A) \leq \sum_{i=1}^{\infty} \mu_L(A_i) \), and then we are done, since we already know by the previous theorem that \( \mu_L(A) \geq \sum_{i=1}^{\infty} \mu_L(A_i) \).

Lemma. If \( A \in \mathcal{R}_L \), \( \mu(A) > 0 \)

1. \( \exists F \in \mathcal{R}_L \) such that \( F \) is closed and \( F \subset A \) and \( \mu_L(F) \geq \mu_L(A) - \epsilon \)

2. If \( A \in \mathcal{R}_L \) then \( \exists G \in \mathcal{R}_L \), \( G \) open and \( G \supset A \) and \( \mu(G) \leq \mu(A) + \epsilon \).

Proof. It suffices to show this for multi-intervals. Let

\[ I = (a_1, b_1) \times \cdots \times (a_n, b_n) \]

(assume \( b_i > a_i \)), I have denoted the interval here as open, but it may be open closed or semi-open. Define \( F \) as follows

\[ F = [a_1 + \delta, b_1 - \delta] \times \cdots \times [a_n + \delta, b_n - \delta] \subset I \]

then

\[ \mu(F) = \prod_{j=1}^{n} (b_j - a_j - 2\delta) = \prod_{j=1}^{n} (b_j - a_j) - \delta F(\delta) \leq \mu_L(A). \]

if we choose \( \delta \) small enough (\( \delta F(\delta) \) is a polynomial in \( \delta \) and vanishes as \( \delta \to 0 \).

We can define \( G \) similarly by

\[ G = (a_1 - \delta, b_1 + \delta) \times \cdots \times (a_n - \delta, b_n + \delta) \supset I \]
We continue with our proof.

Given \( \delta > 0 \) apply the above lemma to get \( F \subset A, \mu_L(F) \geq \mu(A) - \delta \), given \( \epsilon > 0 \) apply lemma to each \( A_i \) to get \( G_i \) open such that \( A_i \subset G_i \) and \( \mu(G_i) \leq \mu(A_i) + \epsilon/2^i \). Now \( F \subset A = \bigcup_{i=1}^{\infty} A_i \) and so \( F \) is closed and bounded and thus compact, by Heine-Borel. So there is a finite subcover. This implies that

\[
F \subset \bigcup_{i=1}^{N} A_i
\]

for some \( N \) (since we have a finite subcover, we might as well just take the first \( N \)). But we know that we have finite additivity of \( \mu \), so

\[
\mu(F) \leq \sum_{i=1}^{N} \mu(G_i)
\]

and

\[
\mu(A) - \delta \leq \mu(F) \leq \sum_{i=1}^{N} \mu(G_i) \leq \sum_{i=1}^{N} \left( \mu(A_i) + \frac{\epsilon}{2^i} \right) \leq \sum_{i=1}^{\infty} \mu(A_i) + \epsilon
\]

then we have

\[
\mu(A) - \delta \leq \sum_{i=1}^{\infty} \mu(A_i) + \epsilon
\]

so \( \mu_L(A) \leq \sum_{i=1}^{\infty} \mu_L(A_i) \)

We try to enlarge \( \mathcal{R} \). First we try to measure every subset of \( X \). Suppose \( B \subset X \). Try to cover \( B \) by a countable collection of elements of \( \mathcal{R} \).

Is \( B \subset \bigcup_{i=1}^{\infty} A_i, A_i \in \mathcal{R} \)

Definition. **Outer Measure**: \( \mu^*: 2^X \to [0, \infty] \) has two conditions

1. If it is not possible to cover the set then we define \( \mu^*(B) = \infty \)

2. If it is possible we say

\[
\mu^* = \inf_{\text{covers}} \sum_{i=1}^{\infty} \mu(A_i) \in [0, \infty]
\]

Lemma. If \( A \in \mathcal{R} \) then \( \mu^*(A) = \mu(A) \)

Proof. \( A \) is a cover of itself, so automatically, \( \mu^*(A) \leq \mu(A) \). Suppose \( \{A_i\}_{i=1}^{\infty} \) is a cover of \( A_i \) and \( A_i \in \mathcal{R} \). We would like that

\[
\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)
\]

Because then, in particular \( \mu(A) \leq \inf \sum \mu(A_i) = \mu^*(A) \) and we would be done.
Set $A'_i = A_i \cap A \in \mathcal{R}$, then $A = \bigcup_{i=1}^{\infty} A'_i$. So the $A'_i$ are in $\mathcal{R}$, but they are not disjoint. So to make them disjoint we do the following

\begin{align*}
A''_1 &= A'_1 \\
A''_2 &= A'_2 \setminus A'_1 \\
& \vdots \\
A''_N &= A'_N \setminus \left( \bigcup_{i=1}^{N-1} A'_i \right) \subset \mathcal{R}
\end{align*}

Then $A'_N \subset \bigcup_{i=1}^{N} A''_i$ and this construction means that $A'_h \cap A'_l = \emptyset$, $h \neq l$ and now

\[
A = \bigcup_{i=1}^{\infty} A''_i, A''_h \cap A''_l = \emptyset \Rightarrow \mu(A) = \sum_{i=1}^{\infty} \mu(A''_i) \leq \sum_{i=1}^{\infty} \mu(A'_i)
\]

the last part follows since $A''_i \subset A'_i$ and since $A'_i \subset A$, then

\[
\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j)
\]

and we are done. \hfill \Box

**Caratheodory’s Idea** What subsets of $X$ should be measurable.

**Definition. Measureable Set** Call $\mathcal{M}$ the set of measurable sets. Then

\[
(\dagger) \quad A \in \mathcal{M} \Leftrightarrow \mu^*(C) = \mu^*(C \cap A) + \mu^*(C \cap A^c), \forall C \in 2^X
\]

**Definition. Set of Measure 0** Consider $\mu$ countably additive, $B \subset X$ has measure 0 if $\mu^*(B) = 0$, i.e. $\forall \epsilon > 0 \ \exists A_i \subset \mathcal{R}$ such that $B \subset \bigcup_{i=1}^{\infty} A_i$ such that

\[
\sum_{i=1}^{\infty} \mu(A_i) < \epsilon
\]

**Theorem. A set of measure 0 has the Caratheodory property.**

**Proof.** Assume $\mu^*(A) = 0$. Then $\mu^*(C \cap A) = 0\ \forall C$, because a cover of $A$ is a cover of $C \cap A$. So then we try to show $\mu^*(C) = \mu^*(C \cap A^c)$.

It is clear that $\mu^*(C) \geq \mu^*(C \cap A^c)$. And by subadditivity $\mu^*(C \cap A) + \mu^*(C \cap A^c) \geq \mu^*(C)$, so $(\dagger)$ holds. \hfill \Box