1.1 Applications to Probability

\( \mathcal{B} \) is the bernoulli space of outcome of coin toss. Where as usual \( H \mapsto 1, \ T \mapsto 0, \) and \( \mathcal{B} \setminus \mathcal{B}_{\text{deg}} \to (0, 1] \). Let \( E \) be the set of outcomes in which an event occurs. Then the probability is \( \mu_L(E) \)

Define \( R_k \), the Radamacher function

\[
R_k(\omega) = 2a_k - 1, \quad \omega = .a_1a_2\ldots
\]

(sort of a heads you win, tails you lose function) The function looks like a square wave thingy.

We would like to measure how many heads in first \( N \) tosses. Let \( S_N \) be the number of heads in the first \( N \) tosses, then

\[
S_N(\omega) = a_1 + \cdots + a_n, \quad S_N : (0, 1] \to \mathbb{N}
\]

And we see that

\[
S_N = \frac{1}{2} \left( \sum_{i=1}^{N} R_i(\omega) + N \right)
\]

so

\[
S_N(\omega) - \frac{N}{2} = \frac{1}{2} \sum_{i=1}^{N} R_i(\omega)
\]

Let \( E \) be the number of heads in the first \( N \) tosses from \( N/2 \). The "distance" from \( N/2 \) that is.

\[
E = \left\{ \omega \in \mathcal{B} : \left| S_N(\omega) - \frac{N}{2} \right| > \epsilon \right\}
\]

the question is what is \( \lim_{N \to \infty} \mu_L(E_N) \)

This is the same as saying that

\[
E_N = \left\{ \omega \in \mathcal{B} : \left| \frac{S_N}{N} - \frac{1}{2} \right|^2 > \epsilon^2 \right\} = \left\{ \omega : \frac{1}{4N^2} \left( \sum_{i=1}^{N} R_i \right)^2 > \epsilon^2 \right\}
\]

Before we proceed,

**Theorem. Special Chebyshev** \( f \) is piecewise constant, non-negative on \([0, 1]\) then

\[
\mu\{f(\omega) > \alpha\} \leq \frac{1}{\alpha} \int_{[0,1]} f
\]

**Proof.** If \( f \) is piecewise constant then we have intervals such that

\[
\bigcup_i [x_i, x_{i+1}] = [0, 1], \quad (x_i, x_{i+1}) \text{ disjoint}
\]
so \( f(x) = c_i, x \in (x_i, x_{i+1}) \), \( c_i \geq 0 \) and then

\[
\int_{[0,1]} f = \sum_i c_i(x_{i+1} - x_i) \geq \sum_{c_i \geq \alpha} c_i(x_{i+1} - x_i) \geq \alpha \sum_{c_i > \alpha} (x_{i+1} - x_i) = \alpha \mu \{ f(\omega) > \alpha \}
\]

and so

\[
\mu \{ f(\omega) > \alpha \} \leq \frac{1}{\alpha} \int_{[0,1]} f
\]

\[ \square \]

**Theorem. Weak Law of Large Numbers**

\[
\lim_{N \to \infty} \mu_L(E_N) = 0
\]

**Proof.** \( E_N \) is equivalent to

\[
\{ \omega; f_N(\omega) > 4\varepsilon^2 N^2 \}, \quad f_N(\omega) = \left( \sum_{i=1}^{N} R_i(\omega) \right)^2
\]

then we apply Chebyshev with \( \alpha = 4\varepsilon^2 N^2, f_n = (\sum R_i)^2 \) then

\[
\int_{[0,1]} f = \int_{[0,1]} \left( \sum_{i=1}^{N} R_i^2 + \sum_{i=1}^{N} R_i R_j \right) = N
\]

So

\[
\mu(E_N) \leq \frac{N}{4\varepsilon^2 N^2} = \frac{1}{4\varepsilon^2 N^2}
\]

this tends to 0 as \( N \to \infty \) \[ \square \]

### 1.2 Continue Extending Measures

One of our methods was to define a measurable set, \( A \), as one in which

\[
\mu^*(C) = \mu^*(C \cap A) + \mu^*(C \cap A^c), \quad \forall C \in 2^X
\]

A second approach is to define a set \( \mathcal{M}_F \) as follows

\[
\mathcal{M}_F = \{ B \in 2^x | \exists \text{ a sequence } A_i \in \mathcal{R}, \text{ such that } \mu^*(B \ominus A_i) = 0 \}
\]

then we define \( \mathcal{M} \) as the set of countable unions of sets of \( \mathcal{M}_F \)

On \( 2^X \) consider the function

\[
d_\mu(A, B) = \mu^*(B \ominus A) \in [0, \infty]
\]
Idea This is almost a metric on the power set. We see its, symmetric and non-negative, but $d(A, B) = 0$ does not mean that $A = B$, but thats alright. However, we have to check that it has the triangle inequality:

$$
\mu^*(A \ominus B) \leq \mu^*(A \ominus C) + \mu^*(B \ominus C)
$$

this follows from the fact that

$$(A \ominus B) \subset (A \ominus C) \cup (B \ominus C)$$

(check above relation for self)