to finish our proof we must show that \( \mu^* \) is a countably additive measure

**Lemma.** \( \mu^* \) is countably additive on \( \mathcal{M}_F \)

**Proof.** Want to show that \( B_i \in \mathcal{M}_F, \ B_i \cap B_j = \emptyset, \ i \neq j \sum_{i=1}^{\infty} B_i = B \in \mathcal{M}_F, \) then \( \mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B_i) \)

We can easily prove inequality one way.

\[
B_N = \bigcup_{i=1}^{N} B_i \implies B_N \subset B
\]

and we get the following strings of inequalities

\[
\mu^*(B) \geq \mu^*(B_N) = \sum_{i=1}^{N} \mu^*(B_i) \implies \mu^*(B) \geq \sum_{i=1}^{N} \mu^*(B_i)
\]

now we need to show countably sub-additivity, that is \( \mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) \). Given \( \epsilon > 0, \ \forall i, \exists \{A_{i,j}\}_{j=1}^{\infty} \subset \mathcal{R} \) a cover of \( A_i \) such that

\[
\mu^*(A_i) + \frac{\epsilon}{2i} \geq \sum_{j=1}^{\infty} \mu^*(A_{i,j})
\]

the collection \( \{A_{i,j}\}_{i,j=1}^{\infty} \) covers \( \bigcup_{i=1}^{\infty} A_i \), then

\[
\mu^* \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i,j=1}^{\infty} \mu^*(A_{i,j}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu^*(A_{i,j}) \leq \sum_{i=1}^{\infty} \left[ \mu^*(A_i) + \frac{\epsilon}{2i} \right] = \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i)
\]

this is true \( \forall \epsilon \), so \( \mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i) \), and so \( \mu^* \) is countably additive on \( \mathcal{M}_F \). \( \Box \)

**Lemma.** \( \mathcal{M}_F = \{ A \in \mathcal{M} | \mu^*(A) < \infty \} \)

**Proof.** We would like to prove that if \( \mu^*(A) < \infty \) and \( A \in \mathcal{M} \) then \( A \in \mathcal{M}_F \). Since \( A \in \mathcal{M} \) we know that \( A = \bigcup_{j=1}^{\infty} B_j, B_j \in \mathcal{M}_F \). We can replace the \( B_j \)'s by a disjoint sequence in \( \mathcal{M}_F \), but with the same measure.

\[
A = \bigcup_{j=1}^{\infty} B_j', \quad B_N' = B_N \setminus \bigcup_{i=1}^{N-1} B_i \in \mathcal{M}_F, \ B_i \text{ disjoint}
\]

Then

\[
\bigcup_{j=1}^{N} B_j' \subset A \implies \mu^* \left( \bigcup_{j=1}^{N} B_j' \right) \leq \mu^*(A) < \infty, \quad \forall N
\]

and we have that and so

\[
\mu^* \left( \bigcup_{j=1}^{N} B_j' \right) = \sum_{i=1}^{N} \mu^*(B_j') \leq \mu^*(A)
\]
let \( C_N = \bigcup_{j=1}^{N} B'_j \) then \( C_N \in \mathcal{M}_F \) and so
\[
\mu^*(A \ominus C_N) = \mu^* \left( \bigcup_{j>N} B'_j \right) \leq \sum_{j>N} \mu^*(B'_j) < \epsilon
\]
for any \( \epsilon \) for \( N \) large enough. And so \( \mu^*(A \ominus B_N) \to 0 \), \( A \in \mathcal{M}_F \) since \( \mathcal{M}_F \) is closed under \( \mu^*(X \ominus X) \).

\[\Box\]

**Definition.** A collection of subsets \( \mathcal{N} \subset 2^X \) is a \( \sigma \)-ring if

1. \( \mathcal{N} \) is a ring
2. If \( A_i \in \mathcal{N} \) then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{N} \). [closure under countable unions]

**Theorem.** \( \mathcal{M} \) is a \( \sigma \)-ring.

**Proof.** Suppose \( A_i \in \mathcal{M} \) we need to show that \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{M} \). Well,
\[
A_i = \bigcup_{i=1}^{\infty} B_{ij}, B_{ij} \in \mathcal{M}_F \implies \bigcup_{i=1}^{\infty} A_i = \bigcup_{i,j=1}^{\infty} B_{ij}
\]
countable union of countable unions is still countable, so this is indeed in \( \mathcal{M} \).

We also need to prove that its a ring. We have already proved the condition for unions. So we have to prove that \( A \ominus B \in \mathcal{M} \) if \( A, B \in \mathcal{M} \), it suffices to show that \( A \setminus B \in \mathcal{M} \), since we already know closure under unions. Now,
\[
A \setminus B = \left( \bigcup_{i=1}^{\infty} A_i \right) \cap \left( \bigcup_{i=1}^{\infty} B_i \right)^c = \bigcup_{i=1}^{\infty} A_i \cap \bigcap_{i=1}^{\infty} B_i^c = \bigcup_{i=1}^{\infty} (A_i \setminus B), \quad B = \bigcap_{i=1}^{\infty} B_i^c
\]
\( A_i \setminus B \) is in \( \mathcal{M}_F \) (Melrose says think about it) \[\Box\]

**Theorem.** If \( A_i \in \mathcal{M} \) and \( A_i \cap A_j = \emptyset, i \neq j \) then
\[
\mu^* \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu^*(A_i)
\]
i.e. \( \mu^* \) is a measure on \( \mathcal{M} \)

**Proof.** Let \( A = \bigcup_{i=1}^{\infty} A_i \) there are two cases

- \( \mu^*(A) = \infty \). Then
  \[
  \mu^*(A) = \sum_{i=1}^{\infty} \mu^*(A_i)
  \]
  so \( \sum \mu^*(A_i) = \infty \)
\[
\mu^*(A) \neq \infty \text{ then } A \in \mathcal{M}_F \text{ and } \bigcup_{i=1}^N A_i \subset A \text{ and so }
\]
\[
\mu^*\left(\bigcup_{i=1}^N A_i\right) \leq \mu^*(A) \implies \bigcup_{i=1}^N A_i \in \mathcal{M}_F
\]

So we have extended \(\mu\) from \(\mathcal{R}\) to \(\mathcal{M}_F\) to \(\mathcal{M}\).

### 1.3 Extending Lebesgue Measure

Lebesgue measures \((X, \mathcal{R}, \mu)\) is countably additive and we extended it to \((X, \mathcal{M}, \mu)\) where \(\mathcal{M}\) is a \(\sigma\)-ring. Apply these constructions to \(X = \mathbb{R}^n\), \(\mathcal{R} = \mathcal{R}_{\text{Leb}}\) (disjoint union of a finite number of rectangles). So we can generate \((\mathbb{R}^n, \mathcal{M}, \mu)\). \(\mathcal{M}\) is closed under countable unions. \(E \subset \mathcal{M}\) then \(E = \bigcup_{i=1}^\infty E_i\), \(E_i \in \mathcal{M}_F\). And \(F \in \mathcal{M}_F\), there exist \(A_i \in \mathcal{R}_{\text{Leb}}\) such that \(\mu^*(E \ominus A) \to 0\).

**Theorem.** Every open set in \(\mathbb{R}^n\) is in \(\mathcal{M}\) (i.e. is Lebesgue measurable)

**Proof.** Consider elements of \(\mathbb{R}^n\) with rational endpoints, \((a_1, b_1) \times \cdots \times (a_n, b_n)\). Consider \(U \subset \mathbb{R}^n\), open. And consider all multisets of the type above in \(U\). Then

\[
U = \bigcup_{R \subset U} R, \quad R \text{ the rational sets}
\]

So \(U\) is open and measurable. \(\mathbb{R}^n\) is itself measurable, so closed sets are in \(\mathcal{M}\), since for \(E\) open \(\mathbb{R}^n \setminus E \subset \mathcal{M}\) is open and so \(E^c \subset \mathcal{M}\). \(\square\)

Recall that \(\mathcal{M}\) is a \(\sigma\)-ring, closed under countable unions and intersections. \(\mathcal{F}_1, \mathcal{F}_2\) are \(\sigma\)-ring subsets of \(\mathbb{R}^n\), then \(\mathcal{F}_1 \cap \mathcal{F}_2\) is a \(\sigma\)-ring.

**Definition.** The Borel sets of \(\mathbb{R}^n\) are the elements of the smallest \(\sigma\)-ring containing all open (and closed) sets of \(\mathbb{R}^n\).

There is a smallest \(\sigma\)-ring because we can take intersections of all such \(\sigma\)-rings, and get one contained in all of them.

**Theorem.** If \(A \in \mathcal{M}_{\text{Leb}}\), then there exists \(B \subset \mathcal{B}\) (the borel ring) such that \(B \subset A\) and \(\mu^*(A \setminus B) = 0\)

**Proof.** Do this in three steps:

1. If \(A \in \mathcal{M}_{\text{Leb}}\), then there exists a borel set \(G \subset \mathcal{B}\), such that \(G \supset A\) and \(\mu^*(G \setminus A) < \epsilon\) (this is not quite what we want, since \(G\) contains \(A\))

   **NB** \(\mathcal{R}_{\text{Leb}} \subset \mathcal{B}\), so if \(A \in \mathcal{M}_F\), then \(\exists A_i \in \mathcal{R}_{\text{Leb}}\) (\(A_i\)'s are multi-intervals), \(A \subset \bigcup_{i=1}^\infty A_i\) implies that \(\mu(A) + \epsilon > \sum_{i=1}^\infty \mu(A_i)\), so \(B = \bigcup_{i=1}^\infty A_i \in \mathcal{B}\), then \(\mu(A \setminus B) < \epsilon\) (so if the measure is finite the definition automatically gives the theorem).
In general $A = \bigcup_{i=1}^{\infty} E_i$, $E_i \in \mathcal{M}_F$. Apply the previous step to each $E_i$ find $B_i \in \mathcal{B}$, $B_i \supset E_i$ such that $\mu^*(E_i \setminus B_i) < \epsilon/2^i$. and then $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$, $\mu^*(A \setminus B) = \sum_{i=1}^{\infty} \mu^*(E_i \setminus B_i) < \epsilon$.

2. If $A \in \mathcal{M}$, $\exists B \in \mathcal{B}$, $B \subset A$ with $\mu(A \setminus B) < \epsilon$. Apply (1) to $\mathbb{R}^n \setminus A$ then we get $E \subset B$ such that $E \supset \mathbb{R}^n \setminus A$, $\mu^*(E \setminus (\mathbb{R}^n \setminus A)) < \epsilon$, but if we set $B = \mathbb{R}^n \setminus E$ then $B \subset A$ and $\mu^*(A \setminus B) < \epsilon$.

3. Finally, use (2) to construct $F_N \in \mathcal{B}$, $F_N \subset A$, such that $\mu^*(A \setminus F_N) < 1/N$, then $B = \bigcup_{N=1}^{\infty} F_N \in \mathcal{B}$, $B \subset A$ and $\mu^*(A \setminus B) \leq \mu^*(A \setminus F_N) < 1/N$, so $\mu^*(A \setminus B) = 0$. We are done

so the Lebesgue sets are "trapped" between Borel sets. \qed

MEASURE THEORY IS DONE.