Proof. \( A, B \in \mathcal{M}_F \), by assumption there exists approximating sequences \( A_i, B_i \in \mathcal{R} \) such that \( \mu^*(A_i \triangle A) \to 0 \) and \( \mu^*(B_i \triangle B) \to 0 \).

We compute \( \mu^*((A \cup B) \triangle (A_i \cup B_i)) = (A \cup B) \setminus (A_i \cup B_i) \cup (A_i \cup B_i) \setminus (A \cap B) \). We just compute one side of this union, because the argument would be symmetric, and we already know that the union will be in the ring,

\[
(A \cup B) \setminus (A_i \cup B_i) = (A \cup B) \cap (A_i \cup B_i)^c = (A \cup B) \cap (A_i^c \cap B_i^c)
\]

\[
= (A \cap A_i^c \cap B_i^c) \cup (B \cap A_i^c \cap B_i^c) \subset (A \setminus A_i) \cup (B \setminus B_i)
\]

and also \((A_i \cup B_i) \setminus (A \cup B) \subset (A_i \setminus A) \cup (B_i \setminus B)\). So the entire set contains \((A \triangle A_i) \cup (B \triangle B_i)\).

So then

\[
\mu^*(A \cup B, A_i \cup B_i) \leq \mu^*(A \triangle A_i, B \triangle B_i) \to 0
\]

so \( A \cup B \in \mathcal{M}_F \). Also like to show that \( A \triangle B \in \mathcal{M}_F \). Since \( (A \triangle B) = (A \setminus B) \cup (B \setminus A) \).

It suffices to show that \( A \setminus B \in \mathcal{M}_F \), since we know that the union is in \( \mathcal{M}_F \) already. We again compute half of \( \mu^*((A \setminus B) \triangle (A_i \setminus B_i)) \), that is \((A \setminus B) \setminus (A_i \setminus B_i)\)

\[
(A \setminus B) \setminus (A_i \setminus B_i) = (A \cap B^c) \cap (A_i \cap B_i^c) = (A \cap B^c) \cap (A_i^c \cup B_i)
\]

\[
= ((A \cap B^c \cap A_i^c) \cup (A \cap B^c \cap B_i)) \subseteq (A \setminus A_i) \cup (B_i \setminus B) \subseteq A \triangle A_i \cup B \triangle B_i
\]

if we work out the other half we get the same thing, so

\[
((A \setminus B) \setminus (B \setminus B_i)) \subseteq (A \triangle A_i) \cup (B \triangle B_i)
\]

and so

\[
\mu^*(A \setminus B) \setminus (B \setminus B_i) \leq \mu^*(A \triangle A_i) + \mu^*(B \triangle B_i) \to 0
\]

\[\square\]

In order that \( \mu^* \) is a measure, it must be finite, so we want to show that \( \mu^* < \infty \) for all \( A \in \mathcal{M}_F \).

**Lemma.** \( \mu^* < \infty \)

**Proof.**

\[
|\mu^*(A) - \mu^*(A_i)| \leq \mu^*(A \triangle A_i) = d(A, A_i) \to 0, \quad \mu^*(A_i) = \mu(A_i)
\]

So \( \mu^*(A) \leq \mu(A_i) + 1 \) for some \( i \)

\[\square\]

**Lemma.** \( \mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B) \)

**Proof.** We know this for \( A_i, B_i \in \mathcal{R} \) that is

\[
\mu^*(A_i \cup B_i) \to \mu^*(A \cup B), \quad \mu^*(A_i \cap B_i) \to \mu^*(A \cap B)
\]

and we know that

\[
|\mu^*(A \cup B) - \mu^*(A_i \cup B_i)| \leq \mu^*((A \cup B) \triangle (A_i \cup B_i))
\]

So now we have a finitely additive measure.

\[\square\]
to finish our proof we must show that $\mu^*$ is a countably additive measure

**Lemma.** $\mu^*$ is countably additive on $\mathcal{M}_F$

**Proof.** Want to show that $B_i \in \mathcal{M}_F$, $B_i \cap B_j = \emptyset$, $i \neq j \bigcup_{i=1}^{\infty} B_i = B \in \mathcal{M}_F$, then $\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B_i)$

We can easily prove inequality one way.

$$B_N = \bigcup_{i=1}^{N} B_i \implies B_N \subseteq B$$

and we get the following strings of inequalities

$$\mu^*(B) \geq \mu^*(B_N) = \sum_{i=1}^{N} \mu^*(B_i) \implies \mu^*(B) \geq \sum_{i=1}^{N} \mu^*(B_i)$$

now we need to show countably sub-additivity, that is $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$. Given $\epsilon > 0$, $\forall i$, $\exists \{A_{i,j}\}_{j=1}^{\infty} \subset \mathcal{R}$ a cover of $A_i$ such that

$$\mu^*(A_i) + \frac{\epsilon}{2i} \geq \sum_{j=1}^{\infty} \mu^*(A_{i,j})$$

the collection $\{A_{i,j}\}_{i,j=1}^{\infty}$ covers $\bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu^*(A_{i,j}) \leq \sum_{i=1}^{\infty} \left[\mu^*(A_i) + \frac{\epsilon}{2i}\right] = \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i)$$

this is true $\forall \epsilon$, so $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, and so $\mu^*$ is countably additive on $\mathcal{M}_F$  

**Lemma.** $\mathcal{M}_F = \{ A \in \mathcal{M} | \mu^*(A) < \infty \}$

**Proof.** We would like to prove that if $\mu^*(A) < \infty$ and $A \in \mathcal{M}$ then $A \in \mathcal{M}_F$. Since $A \in \mathcal{M}$ we know that $A = \bigcup_{j=1}^{\infty} B_j, B_j \in \mathcal{M}_F$. We can replace the $B_j$’s by a disjoint sequence in $\mathcal{M}_F$, but with the same measure.

$$A = \bigcup_{j=1}^{\infty} B'_j, \quad B'_N = B_N \setminus \bigcup_{i=1}^{N-1} B_i \in \mathcal{M}_F, B_i \text{ disjoint}$$

Then

$$\bigcup_{j=1}^{N} B'_j \subset A \implies \mu^*\left(\bigcup_{j=1}^{N} B'_j\right) \leq \mu^*(A) < \infty, \quad \forall N$$

and we have that and so

$$\mu^*\left(\bigcup_{j=1}^{N} B'_j\right) = \sum_{i=1}^{N} \mu^*(B'_j) \leq \mu^*(A)$$
let \( C_N = \bigcup_{j=1}^{N} B_j \) then \( C_N \in \mathcal{M}_F \) and so

\[
\mu^*(A \ominus C_N) = \mu^*\left( \bigcup_{j>N} B_j \right) \leq \sum_{j>N} \mu^*(B_j) < \epsilon
\]

for any \( \epsilon \) for \( N \) large enough. And so \( \mu^*(A \ominus B_N) \to 0 \), \( A \in \mathcal{M}_F \) since \( \mathcal{M}_F \) is closed under \( \mu^*(X \ominus X) \).

\[\square\]

**Definition.** A collection of subsets \( \mathcal{N} \subset 2^X \) is a \( \sigma \)-ring if

1. \( \mathcal{N} \) is a ring
2. If \( A_i \in \mathcal{N} \) then \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{N} \) [closure under countable unions]

**Theorem.** \( \mathcal{M} \) is a \( \sigma \)-ring.

**Proof.** Suppose \( A_i \in \mathcal{M} \) we need to show that \( \bigcup_{i=1}^{\infty} A_i \in \mathcal{M} \). Well,

\[
A_i = \bigcup_{i=1}^{\infty} B_{ij}, B_{ij} \in \mathcal{M}_F \implies \bigcup A_i = \bigcup B_{ij}
\]

countable union of countable unions is still countable, so this is indeed in \( \mathcal{M} \).

We also need to prove that its a ring. We have already proved the condition for unions. So we have to prove that \( A \ominus B \in \mathcal{M} \) if \( A, B \in \mathcal{M} \), it suffices to show that \( A \setminus B \in \mathcal{M} \), since we already know closure under unions. Now,

\[
A_i \setminus B = \left( \bigcup_{i=1}^{\infty} A_i \right) \cap \left( \bigcup_{i=1}^{\infty} B_i \right)^c = \bigcup_{i=1}^{\infty} A_i \cap \bigcap_{i=1}^{\infty} B_i^c = \bigcup_{i=1}^{\infty} (A_i \setminus B), \quad B = \bigcap_{i=1}^{\infty} B_i^c
\]

\( A_i \setminus B \) is in \( \mathcal{M}_F \) (Melrose says think about it) \[\square\]

**Theorem.** If \( A_i \in \mathcal{M} \) and \( A_i \cap A_j = \emptyset, i \neq j \) then

\[
\mu^* \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu^*(A_i)
\]

\( i.e. \) \( \mu^* \) is a measure on \( \mathcal{M} \)

**Proof.** Let \( A = \bigcup_{i=1}^{\infty} A_i \) there are two cases

- \( \mu^*(A) = \infty \). Then

\[
\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)
\]

so \( \sum \mu^*(A_i) = \infty \)
• \( \mu^*(A) \neq \infty \) then \( A \in \mathcal{M}_F \) and \( \bigcup_{i=1}^{N} A_i \subseteq A \) and so
\[
\mu^* \left( \bigcup_{i=1}^{N} A_i \right) \leq \mu^*(A) \implies \bigcup_{i=1}^{N} A_i \in \mathcal{M}_F
\]

So we have extended \( \mu \) from \( \mathcal{R} \) to \( \mathcal{M}_F \) to \( \mathcal{M} \).

1.3 Extending Lebesgue Measure

Lebesgue measures \((X, \mathcal{R}, \mu)\) is countably additive and we extended it to \((X, \mathcal{M}, \mu)\) where \( \mathcal{M} \) is a \( \sigma\)-ring. Apply these constructions to \( X = \mathbb{R}^n \), \( \mathcal{R} = \mathcal{R}_{\text{Leb}} \) (disjoint union of a finite number of rectangles). So we can generate \((\mathbb{R}^n, \mathcal{M}, \mu)\). \( \mathcal{M} \) is closed under countable unions. \( E \subseteq \mathcal{M} \) then \( E = \bigcup_{i=1}^{\infty} E_i, E_i \in \mathcal{M}_F \). And if \( F \in \mathcal{M}_F \), there exist \( A_i \in \mathcal{R}_{\text{Leb}} \) such that \( \mu^*(E \ominus A) \to 0 \).

**Theorem.** Every open set in \( \mathbb{R}^n \) is in \( \mathcal{M} \) (i.e. is Lebesgue measurable)

**Proof.** Consider elements of \( \mathbb{R}^n \) with rational endpoints, \((a_1, b_1) \times \cdots \times (a_n, b_n)\). Consider \( U \subseteq \mathbb{R}^n \), open. And consider all multisets of the type above in \( U \). Then
\[
U = \bigcup_{R \in U} R, \quad R \text{ the rational sets}
\]

So \( U \) is open and measurable. \( \mathbb{R}^n \) is itself measurable, so closed sets are in \( \mathcal{M} \), since for \( E \) open \( \mathbb{R}^n \setminus E \subseteq \mathcal{M} \) is open and so \( E^c \subseteq \mathcal{M} \).

Recall that \( \mathcal{M} \) is a \( \sigma\)-ring, closed under countable unions and intersections. \( \mathcal{F}_1, \mathcal{F}_2 \) are \( \sigma\)-ring subsets of \( \mathbb{R}^n \), then \( \mathcal{F}_1 \cap \mathcal{F}_2 \) is a \( \sigma\)-ring.

**Definition.** The Borel sets of \( \mathbb{R}^n \) are the elements of the smallest \( \sigma\)-ring containing all open (and closed) sets of \( \mathbb{R}^n \).

There is a smallest \( \sigma\)-ring because we can take intersections of all such \( \sigma\)-rings, and get one contained in all of them.

**Theorem.** If \( A \in \mathcal{M}_{\text{Leb}} \), then there exists \( B \subseteq \mathcal{B} \) (the borel ring) such that \( B \subseteq A \) and \( \mu^*(A \setminus B) = 0 \)

**Proof.** Do this in three steps:

1. If \( A \in \mathcal{M}_{\text{Leb}} \), then there exists a borel set \( G \subseteq \mathcal{B} \), such that \( G \supset A \) and \( \mu^*(G \setminus A) < \epsilon \) (this is not quite what we want, since \( G \) contains \( A \))

NB \( \mathcal{R}_{\text{Leb}} \subseteq \mathcal{B} \), so if \( A \in \mathcal{M}_F \), then \( \exists A_i \in \mathcal{R}_{\text{Leb}} \) (\( A_i \)'s are multi-intervals), \( A \subseteq \bigcup_{i=1}^{\infty} A_i \) implies that \( \mu(A) + \epsilon > \sum_{i=1}^{\infty} \mu(A_i) \), so \( B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{B} \), then \( \mu(A \setminus B) < \epsilon \) (so if the measure is finite the definition automatically gives the theorem).
2 Measurable Functions

We’re going to deal with measure spaces $(X, \mathcal{F}, \mu)$, $\mu : \mathcal{F} \to [0, \infty]$ is countably additive. Now we are interested in functions on $X$:

$$f : X \to [-\infty, \infty] = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

(using the extended real numbers allows us to take about infs and sups)

Define $\mathcal{B}$, it is a $\sigma$-ring of subsets of $[-\infty, \infty]$. That is

$$\mathcal{B} = \{A, A \cup \{\infty\}, A \cup \{-\infty\}, A \cup \{-\infty, \infty\} | A \subset B\}$$

this is the extended Borel $\sigma$-ring. And for $X$ our ”good sets” (measurable) are $\mathcal{F}$.

**Definition.** $f : X \to [-\infty, \infty]$ is measurable iff $f^{-1}(-\infty, a) \in \mathcal{F}$, that is

$$\{x|f(x) < a\} \in \mathcal{F}, \quad \forall a \in \mathcal{M}$$

**Proposition.** The following conditions on $f : X \to [-\infty, \infty]$ are equivalent

1. $\{x|f(x) > a\} \in \mathcal{F}, \forall a$

2. $\{x|f(x) < a\} \in \mathcal{F}, \forall a$

3. $\{x|f(x) \leq a\} \in \mathcal{F}, \forall a$

4. $\{x|f(x) \geq a\} \in \mathcal{F}, \forall a$
\textbf{Proof}. Compare (1) and (3), (2) and (4), they are complements and so one in the \(\sigma\)-ring implies the other is in the \(\sigma\)-ring as well.

Prove (4) implies (1)
\[
\bigcup_{n=1}^{\infty} \left\{ x \in X | f(x) \geq a + \frac{1}{n} \right\} = \{ x \in X | f(x) > a \}
\]
and
\[
\bigcap_{n=1}^{\infty} \left\{ x \in X | f(x) > a - \frac{1}{n} \right\} = \{ x \in X | f(x) \geq a \}
\]
prove these makes good use of the \(\sigma\)-field properties (closed under complements, intersection and union).

\textbf{Theorem.} \(f : X \to [-\infty, \infty] \) iff \(f^{-1}(B) \in \mathcal{F}, \forall B \in \mathcal{B}\). (kind of like continuity, note that this makes continuous functions automatically measurable)

\textbf{Proof.} (1) above implies that \(f^{-1}((a, \infty]) \in \mathcal{F}, \forall a\). Note that \((a, b) = [-\infty, b) \cap (a, \infty]\), so we can get any Borel set as some union and intersection of \((a, \infty], [-\infty, a], [a, \infty)\).

Consider
\[
\mathcal{C} = \{ C \subseteq [-\infty, \infty] \text{ such that } f^{-1}(C) \in \mathcal{F} \}
\]
Note that \(\mathcal{C}\) is a \(\sigma\)-field, since
\[
f^{-1} \left( \bigcup_{i=1}^{\infty} C_i \right) = \bigcup_{i=1}^{\infty} f^{-1}(C_i), \quad f^{-1} \left( \bigcap_{i=1}^{\infty} C_i \right) = \bigcap_{i=1}^{\infty} f^{-1}(C_i)
\]
and \(f^{-1}([-\infty, \infty])\) is in \(\mathcal{C}\). Since \(f\) is measurable, then \(\mathcal{C}\) contains all open sets, and so \(\mathcal{C} \supseteq \mathcal{B}\), because \(\mathcal{C}\) contains the open sets. Then \(\{ x | f(x) > a \} \in \mathcal{F}\) implies that \(f^{-1}(B) \in \mathcal{F}\) for all \(B \in \mathcal{B}\). The converse is easy.

\textbf{Theorem.} If \(f : \mathbb{R}^{n} \to \mathbb{R}\) is continuous then it is measurable (Lebesgue)

\textbf{Proof.} \(f^{-1}(B) \in \mathcal{F}\). We know that \(f^{-1}(A) \subseteq \mathbb{R}^{n}\) then \(A \subseteq \mathbb{R}\) is open, so \(f^{-1}(A)\) is Lebesgue measurable, because all open sets are. \(\mathcal{C} = \{ B \in \mathbb{R} | f^{-1}(B) \in \mathcal{M} \}\) is a \(\sigma\)-field containing the open sets.

\textbf{Theorem.} Suppose \(f\) and \(g\) are measurable functions then \(\max(f, g), \min(f, g)\) are measurable.

\textbf{Proof.} \(F^{-1}((\infty, a)) = \{ x \in X | F(x) < a \} = \{ x \in X | f(x) < a, g(x) < a \} = \{ x \in X | f(x) < a \} \cap \{ x \in X | g(x) < a \} \in \mathcal{F}. \)

Notice that if \(f\) is measurable then \(f_{+}\) is measurable because \(f_{+} = \max(f, 0)\)

\textbf{Theorem.} Suppose \(f_{i}\) is a sequence of measurable functions, \(f_{i} : E \to [-\infty, \infty]\) and let \(f = \sup_{i} f_{i}, f(x) = \sup_{i=1}^{\infty} f_{i}(x)\) then \(f\) is measurable. (same thing for \(\inf\))
Proof. \( \{ x | f(x) > a \} = \bigcap_{j=1}^{\infty} \{ x | f_j(x) > a \} \in \mathcal{F}. \)

**Theorem.** If \( f_i \) is a sequence of measurable functions on \((X, \mathcal{F})\), then \( \limsup \) and \( \liminf \) \( f_i \) are measurable.

**Proof.** \( f(x) = \limsup f_i(x) = \lim_j \sup_{j \geq i} f_i(x) \). This is a decreasing set of numbers, convergent if bounded below. Then this is \( \inf_i \sup_{j \geq i} f_j(x) \). Apply the above theorem twice.

**Theorem.** If \( f_i \) is a sequence of measurable functions such that \( f_i(x) \to f(x) \) (pointwise convergent), then \( f \) is measurable.

**Proof.** Since \( f_i(x) \) converges, \( \limsup f_i = \liminf f_i = f \), so its measurable

This very different from normal analysis, where we normally do not know much about pointwise convergent functions.

Note that for some of the above properties, its the \( \sigma \)-algebra that is allowing us to do what we are, by taking unions and intersections of an infinite number of sets.

**Theorem.** If \( f : X \to \mathbb{R} \) is a measurable (note: \( X \) need not have a topology) and \( g : \mathbb{R} \to \mathbb{R} \) is continuous, then \( g \circ f \) is measurable.

**Proof.** Consider
\[
\{ x \in X | g \circ f > a \} = f^{-1} \{ t \in \mathbb{R} | g(t) > a \}
\]
since \( \{ t \in \mathbb{R} | g(t) > a \} \) is open, then \( f^{-1} \) of it is open, since \( f \) is measurable and so the inverse contains Borel sets.