Proof. $A, B \in \mathcal{M}_F$, by assumption there exists approximating sequences $A_i, B_i \in \mathcal{R}$ such that $\mu^*(A_i \ominus A) \to 0$ and $\mu^*(B_i \ominus B) \to 0$.

We compute $\mu^*((A \cup B) \ominus (A_i \cup B_i)) = (A \cup B) \setminus (A_i \cup B_i) \cup (A_i \cup B_i) \setminus (A \cap B)$. We just compute one side of this union,because the argument would be symmetric, and we already know that the union will be in the ring.

$$\begin{aligned}(A \cup B) \setminus (A_i \cup B_i) &= (A \cup B) \cap (A_i \cup B_i)^c = (A \cup B) \cap (A_i^c \cap B_i^c) \\ &= (A \cap A_i^c \cap B_i^c) \cup (B \cap A_i^c \cap B_i^c) \subset (A \setminus A_i) \cup (B \setminus B_i)\end{aligned}$$

and also $(A_i \cup B_i) \setminus (A \cup B) \subset (A_i \setminus A) \cup (B_i \setminus B)$. So the entire set contains $(A \ominus A_i) \cup (B \ominus B_i)$. So then

$$\mu^*(A \cup B, A_i \cup B_i) \leq \mu^*(A \ominus A_i, B \ominus B_i) \to 0$$

so $A \cup B \in \mathcal{M}_F$. Also like to show that $A \ominus B \in \mathcal{M}_F$. Since $(A \ominus B) = (A \setminus B) \cup (B \setminus A)$. It suffices to show that $A \setminus B \in \mathcal{M}_F$, since we know that the union is in $\mathcal{M}_F$ already. We again compute half of $\mu^*((A \setminus B) \ominus (A_i \setminus B_i))$, that is $((A \setminus B) \setminus (A_i \setminus B_i))$

$$\begin{aligned}(A \setminus B) \setminus (A_i \setminus B_i) &= (A \cap B^c) \cap (A_i \cap B_i^c)^c = (A \cap B^c) \cap (A_i^c \cup B_i) \\ &= ((A \cap B^c \cap A_i^c) \cup (A \cap B^c \cap B_i)) \subseteq (A \setminus A_i) \cup (B_i \setminus B) \subseteq A \ominus A_i \cup B \ominus B_i\end{aligned}$$

if we work out the other have we get the same thing, so

$$((A \setminus B) \setminus (B \setminus B_i)) \subseteq (A \ominus A_i) \cup (B \ominus B_i)$$

and so

$$\mu^*((A \setminus B) \setminus (B \setminus B_i)) \subseteq \mu^*(A \ominus A_i) + \mu^*(B \ominus B_i) \to 0$$

□

In order that μ^* is a measure, it must be finite, so we want to show that $\mu^* < \infty$ for all $A \in \mathcal{M}_F$.

Lemma. $\mu^* < \infty$

Proof.

$$|\mu^*(A) - \mu^*(A_i)| \leq \mu^*(A \ominus A_i) = d(A, A_i) \to 0, \qquad \mu^*(A_i) = \mu(A_i)$$

So $\mu^*(A) \leq \mu(A_i) + 1$ for some i □

Lemma. $\mu^*(A \cup B) + \mu^*(A \cap B) = \mu^*(A) + \mu^*(B)$

Proof. We know this for $A_i, B_i \in \mathcal{R}$ that is

$$\mu^*(A_i \cup B_i) \to \mu^*(A \cup B), \qquad \mu^*(A_i \cap B_i) \to \mu^*(A \cap B)$$

and we know that

$$|\mu^*(A \cup B) - \mu^*(A_i \cup B_i)| \leq \mu^*((A \cup B) \ominus (A_i \cup B_i))$$

So now we have a finitely additive measure. □

to finish our proof we must show that μ^* is a countably additive measure

Lemma. *μ^* is countably additive on $\mathcal{M}_F$*

Proof. Want to show that $B_i \in \mathcal{M}_F$, $B_i \cap B_j = \emptyset$, $i \neq j$ $\bigcup_{i=1}^{\infty} B_i = B \in \mathcal{M}_F$, then $\mu^*(B) = \sum_{i=1}^{\infty} \mu^*(B_i)$

We can easily prove inequality one way.

$$B_N = \bigcup_{i=1}^{N} B_i \Longrightarrow B_N \subset B$$

and we get the following strings of inequalities

$$\mu^*(B) \geq \mu^*(B_N) = \sum_{i=1}^{N} \mu^*(B_i) \Longrightarrow \mu^*(B) \geq \sum_{i=1}^{N} \mu^*(B_i)$$

now we need to show countably sub-additiviy, that is $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$. Given $\epsilon > 0$, $\forall i$, $\exists \{A_{i,j}\}_{j=1}^{\infty} \subset \mathcal{R}$ a cover of A_i such that

$$\mu^*(A_i) + \frac{\epsilon}{2^i} \geq \sum_{j=1}^{\infty} \mu^*(A_{i,j})$$

the collection $\{A_{i,j}\}_{i,j=1}^{\infty}$ covers $\bigcup_{i=1}^{\infty} A_i$, then

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i,j=1}^{\infty} \mu^*(A_{i,j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu^*(A_{i,j}) \leq \sum_{i=1}^{\infty} \left[\mu^*(A_i) + \frac{\epsilon}{2^i}\right] = \epsilon + \sum_{i=1}^{\infty} \mu^*(A_i)$$

this is true $\forall \epsilon$, so $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$, and so μ^* is countably additive on $\mathcal{M}_F$ $\square$

Lemma. $\mathcal{M}_F = \{A \in \mathcal{M} | \mu^*(A) < \infty\}$

Proof. We would like to prove that if $\mu^*(A) < \infty$ and $A \in \mathcal{M}$ then $A \in \mathcal{M}_F$. Since $A \in \mathcal{M}$ we know that $A = \bigcup_{j=1}^{\infty} B_j, B_j \in \mathcal{M}_F$. We can replace the B_j's by a disjoint sequence in $\mathcal{M}_F$, but with the same measure.

$$A = \bigcup_{j=1}^{\infty} B'_j, \qquad B'_N = B_N \setminus \bigcup_{i=1}^{N-1} B_i \in \mathcal{M}_F, B_i \text{ disjoint}$$

Then

$$\bigcup_{j=1}^{N} B'_j \subset A \Longrightarrow \mu^*\left(\bigcup_{j=1}^{N} B'_j\right) \leq \mu^*(A) < \infty, \qquad \forall N$$

and we have that and so

$$\mu^*\left(\bigcup_{j=1}^{N} B'_j\right) = \sum_{i=1}^{N} \mu^*(B'_j) \leq \mu^*(A)$$

let $C_N = \bigcup_{j=1}^{N} B'_j$ then $C_N \in \mathcal{M}_F$ and so

$$\mu^*(A \ominus C_N) = \mu^*\left(\bigcup_{j>N} B'_j\right) \leq \sum_{j>N} \mu^*(B'_j) < \epsilon$$

for any ϵ for N large enough. And so $\mu^*(A \ominus B_N) \to 0$, $A \in \mathcal{M}_F$ since $\mathcal{M}_F$ is closed under $\mu^*(X \ominus X)$. □

Definition. A collection of subsets $\mathcal{N} \subset 2^X$ is a σ-ring if

1. $\mathcal{N}$ is a ring
2. If $A_i \in \mathcal{N}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{N}$.[closure under countable unions]

Theorem. *$\mathcal{M}$ is a σ-ring.*

Proof. Suppose $A_i \in \mathcal{M}$ we need to show that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$. Well,

$$A_i = \bigcup_{i=1}^{\infty} B_{ij}, B_{ij} \in \mathcal{M}_F \Longrightarrow \bigcup_{i=1}^{\infty} A_i = \bigcup_{i,j=1}^{\infty} B_{ij}$$

countable union of countable unions is still countable, so this is indeed in $\mathcal{M}$.

We also need to prove that its a ring. We have already proved the condition for unions. So we have to prove that $A \ominus B \in \mathcal{M}$ if $A, B \in \mathcal{M}$, it suffices to show that $A \setminus B \in \mathcal{M}$, since we already know closure under unions. Now,

$$A \setminus B = \left(\bigcup_{i=1}^{\infty} A_i\right) \cap \left(\bigcup_{i=1}^{\infty} B_i\right)^c = \bigcup_{i=1}^{\infty} A_i \cap \bigcap_{i=1}^{\infty} B_i^c = \bigcup_{i=1}^{\infty} (A_i \setminus B), \qquad B = \bigcap_{i=1}^{\infty} B_i^c$$

$A_i \setminus B$ is in $\mathcal{M}_F$ (Melrose says think about it) □

Theorem. *If $A_i \in \mathcal{M}$ and $A_i \cap A_j = \emptyset, i \neq j$ then*

$$\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu^*(A_i)$$

i.e. μ^* is a measure on $\mathcal{M}$

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$ there are two cases

- $\mu^*(A) = \infty$. Then

$$\mu^*(A) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$$

  so $\sum \mu^*(A_i) = \infty$

- $\mu^*(A) \neq \infty$ then $A \in \mathcal{M}_F$ and $\bigcup_{i=1}^N A_i \subset A$ and so

$$\mu^*\left(\bigcup_{i=1}^N A_i\right) \leq \mu^*(A) \Longrightarrow \bigcup_{i=1}^N A_i \in \mathcal{M}_F$$

□

So we have extended μ from $\mathcal{R}$ to $\mathcal{M}_F$ to $\mathcal{M}$.

1.3 Extending Lebesgue Measure

Lebesgue measures $(X, \mathcal{R}, \mu)$ is countably additive and we extended it to $(X, \mathcal{M}, \mu)$ where $\mathcal{M}$ is a σ-ring. Apply these constructions to $X = \mathbb{R}^n$, $\mathcal{R} = \mathcal{R}_{\text{Leb}}$ (disjoint union of a finite number of rectangles). So we can generate $(\mathbb{R}^n, \mathcal{M}, \mu)$. $\mathcal{M}$ is closed under countable unions. $E \subset \mathcal{M}$ then $E = \bigcup_{i=1}^\infty E_i, E_i \in \mathcal{M}_F$. And $F \in \mathcal{M}_F$, there exist $A_i \in \mathcal{R}_{\text{Leb}}$ such that $\mu^*(E \ominus A) \to 0$.

Theorem. *Every open set in $\mathbb{R}^n$ is in $\mathcal{M}$ (i.e. is Lebesgue measurable)*

Proof. Consider elements of $\mathbb{R}^n$ with rational endpoints, $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Consider $U \subset \mathbb{R}^n$, open. And consider all multisets of the type above in U. Then

$$U = \bigcup_{R \subset U} R, \qquad R \text{ the rational sets}$$

So U is open and measurable. $\mathbb{R}^n$ is itself measurable, so closed sets are in $\mathcal{M}$, since for E open $\mathcal{R}^n \setminus E \in \mathcal{M}$ is open and so $E^c \in \mathcal{M}$. □

Recall that $\mathcal{M}$ is a σ-ring, closed under countable unions and intersections. $\mathcal{F}_1, \mathcal{F}_2$ are σ-ring subsets of $\mathbb{R}^n$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ is a σ-ring.

Definition. The Borel sets of $\mathbb{R}^n$ are the elements of the smallest σ-ring containing all open (and closed) sets of $\mathbb{R}^n$.

There *is* a smallest σ-ring because we can take intersections of all such σ-rings, and get one contained in all of them.

Theorem. *If $A \in \mathcal{M}_{Leb}$, then there exists $B \subset \mathcal{B}$ (the borel ring) such that $B \subset A$ and $\mu^*(A \setminus B) = 0$*

Proof. Do this in three steps:

1. If $A \in \mathcal{M}_{\text{Leb}}$, then there exists a borel set $G \subset \mathcal{B}$, such that $G \supset A$ and $\mu^*(G \setminus A) < \epsilon$ (this is not quite what we want, since G contains A)

   NB $\mathcal{R}_{\text{Leb}} \subset \mathcal{B}$, so if $A \in \mathcal{M}_F$, then $\exists A_i \in \mathcal{R}_{\text{Leb}}$ (A_i's are multi-intervals), $A \subset \bigcup_{i=1}^\infty A_i$ implies that $\mu(A) + \epsilon > \sum^\infty \mu(A_i)$, so $B = \bigcup_{i=1}^\infty A_i \in \mathcal{B}$, then $\mu(A \setminus B) < \epsilon$ (so if the measure is finite the definition automatically gives the theorem).

2 Measurable Functions

We're going to deal with measure spaces $(X, \mathcal{F}, \mu)$, $\mu : \mathcal{F} \to [0, \infty]$ is countably additive. Now we are interested in functions on X:

$$f : X \to [-\infty, \infty] = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

(using the extended real numbers allows us to take about infs and sups)

Define $\tilde{\mathcal{B}}$, it is a σ-ring of subsets of $[-\infty, \infty]$. That is

$$\tilde{\mathcal{B}} = \{A, A \cup \{\infty\}, A \cup \{-\infty\}, A \cup \{-\infty, \infty\} | A \subset B\}$$

this is the extended Borel σ-ring. And for X our "good sets" (measurable) are $\mathcal{F}$.

Definition. $f : X \to [-\infty, \infty]$ is **measurable** iff $f^{-1}(-\infty, a) \in \mathcal{F}$, that is

$$\{x | f(x) < a\} \in \mathcal{F}, \qquad \forall a \in \mathcal{M}$$

Proposition. *The following conditions on $f : X \to [-\infty, \infty]$ are equivalent*

1. $\{x | f(x) > a\} \in \mathcal{F}, \ \forall a$
2. $\{x | f(x) < a\} \in \mathcal{F}, \ \forall a$
3. $\{x | f(x) \leq a\} \in \mathcal{F}, \ \forall a$
4. $\{x | f(x) \geq a\} \in \mathcal{F}, \ \forall a$

Proof. Compare (1) and (3), (2) and (4), they are complements and so one in the σ-ring implies the other is in the σ-ring as well.

Prove (4) implies (1)

$$\bigcup_{n=1}^{\infty}\left\{x \in X | f(x) \geq a + \frac{1}{n}\right\} = \{x \in X | f(x) > a\}$$

and

$$\bigcap_{n=1}^{\infty}\left\{x \in X | f(x) > a - \frac{1}{n}\right\} = \{x \in X | f(x) \geq a\}$$

prove these makes good use of the σ-field properties (closed under complements, intersection and union). □

Theorem. $f : X \to [-\infty, \infty]$ *iff* $f^{-1}(B) \in \mathcal{F}$, $\forall B \in \tilde{\mathcal{B}}$.*(kind of like continuity, note that this makes continuous functions automatically measurable)*

Proof. (1) above implies that $f^{-1}((a, \infty]) \in \mathcal{F}$, $\forall a$. Note that $(a, b) = [-\infty, b) \cap (a, \infty]$, so we can get any Borel set as some union and intersection of $(a, \infty], [-\infty, a), [a, \infty)$.

Consider

$$\mathcal{C} = \{C \subset [-\infty, \infty] \text{ such that } f^{-1}(C) \in \mathcal{F}\}$$

Note that $\mathcal{C}$ is a σ-field, since

$$f^{-1}\left(\bigcup_{i=1}^{\infty} C_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(C_i), \qquad f^{-1}\left(\bigcap_{i=1}^{\infty} C_i\right) = \bigcap_{i=1}^{\infty} f^{-1}(C_i)$$

and $f^{-1}([-\infty, \infty])$ is in $\mathcal{C}$. Since f is measurable, then $\mathcal{C}$ contains all open sets, and so $\mathcal{C} \supset \tilde{\mathcal{B}}$, because $\mathcal{C}$ contains the open sets. Then $\{x | f(x) > a\} \in \mathcal{F}$ implies that $f^{-1}(B) \in \mathcal{F}$ for all $B \in \tilde{\mathcal{B}}$. The converse is easy. □

Theorem. *If* $f : \mathbb{R}^n \to \mathbb{R}$ *is continuous then it is measurable (Lebesgue)*

Proof. $f^{-1}(B) \in \mathcal{F}$. We know that $f^{-1}(A) \subset \mathbb{R}^n$ then $A \subset \mathbb{R}$ is open, so $f^{-1}(A)$ is Lebesgue measurable, because all open sets are. $\mathcal{C} = \{B \in \mathbb{R} | f^{-1}(B) \in \mathcal{M}\}$ is a σ-field containing the open sets. □

Theorem. *Suppose* f *and* g *are measurable functions then* $\max(f, g)$, $\min(f, g)$ *are measurable.*

Proof. $F^{-1}([\infty, a)) = \{x \in X | F(x) < a\} = \{x \in x | f(x) < a,\ g(x) < a\} = \{x \in X | f(x) < a\} \cap \{x \in X | g(x) < a\} \in \mathcal{F}$. □

Notice that if f is measurable then f_+ is measurable because $f_+ = \max(f, 0)$

Theorem. *Suppose* f_i *is a sequence of measurable functions,* $f_i : E \to [-\infty, \infty]$ *and let* $f = \sup_i f_i$, $f(x) = \sup_{i=1}^{\infty} f_i(x)$ *then* f *is measurable. (same thing for* $\inf$*)*

Proof. $\{x|f(x) > a\} = \bigcap_{j=1}^{\infty}\{x|f_j(x) > a\} \in \mathcal{F}$. $\square$

Theorem. *If f_i is a a sequence of measurable functions on $(X, \mathcal{F})$, then* $\limsup$ *and* $\liminf f_i$ *are measurable.*

Proof. $f(x) = \limsup_i f_i(x) = \lim_j \sup_{j \geq i} f_i(x)$. This is a decreasing set of numbers, convergent if bounded below. Then this is $\inf_i \sup_{j \geq i} f_j(x)$. Apply the above theorem twice. $\square$

Theorem. *If f_i is a sequence of measurable functions such that $f_i(x) \to f(x)$ (pointwise convergent), then f is measurable.*

Proof. Since $f_i(x)$ converges, $\limsup f_i = \liminf f_i = f$, so its measurable $\square$

This very different from normal analysis, where we normally do not know much about pointwise convergent functions.

Note that for some of the above properties, its the σ-algebra that is allowing us to do what we are, by taking unions and intersections of an infinite number of sets.

Theorem. *If $f : X \to \mathbb{R}$ is a measurable (note: X need not have a topology) and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.*

Proof. Consider

$$\{x \in X|g \circ f > a\} = f^{-1}\{t \in \mathbb{R}|g(t) > a\}$$

since $\{t \in \mathbb{R}|g(t) > a\}$ is open, then f^{-1} of it is open, since f is measurable and so the inverse contains Borel sets. $\square$