1 Dirichlet series

The Riemann zeta function ζ is a special example of a type of series we will be considering often in this course. A *Dirichlet series* is a formal series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ with $a_n \in \mathbb{C}$. You should think of these as a number-theoretic analogue of formal power series; indeed, our first order of business is to understand when such a series converges absolutely.

Lemma 1. *There is an extended real number $L \in \mathbb{R} \cup \{\pm\infty\}$ with the following property: the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for $\mathrm{Re}(s) > L$, but not for $\mathrm{Re}(s) < L$. Moreover, for any $\epsilon > 0$, the convergence is uniform on $\mathrm{Re}(s) \geq L+\epsilon$, so the series represents a holomorphic function on all of $\mathrm{Re}(s) > L$.*

Proof. Exercise. □

The quantity L is called the *abscissa of absolute convergence* of the Dirichlet series; it is an analogue of the radius of convergence of a power series. (In fact, if you fix a prime p, and only allow a_n to be nonzero when n is a power of p, then you get an ordinary power series in p^{-s}. So in some sense, Dirichlet series are a strict generalization of ordinary power series.)

Recall that an ordinary power series in a complex variable must have a singularity at the boundary of its radius of convergence. For Dirichlet series with *nonnegative real* coefficients, we have the following analogous fact.

Theorem 2 (Landau)**.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with* nonnegative real *coefficients. Suppose $L \in \mathbb{R}$ is the abscissa of absolute convergence for $f(s)$. Then f cannot be extended to a holomorphic function on a neighborhood of $s = L$.*

Proof. Suppose on the contrary that f extends to a holomorphic function on the disc $|s-L| < \epsilon$. Pick a real number $c \in (L, L+\epsilon/2)$, and write

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} a_n n^{-c} n^{c-s} \\ &= \sum_{n=1}^{\infty} a_n n^{-c} \exp((c-s)\log n) \\ &= \sum_{n=1}^{\infty} \sum_{i=0}^{\infty} \frac{a_n n^{-c} (\log n)^i}{i!} (c-s)^i. \end{aligned}$$

Since all coefficients in this double series are nonnegative, everything must converge absolutely in the disc $|s-c| < \epsilon/2$. In particular, when viewed as a power series in $c-s$, this must give the Taylor series for f around $s = c$. Since f is holomorphic in the disc $|s-c| < \epsilon/2$, the Taylor series converges there; in particular, it converges for some real number $L' < L$.

But now we can run the argument backwards to deduce that the original Dirichlet series converges absolutely for $s = L'$, which implies that the abscissa of absolute convergence is at most L'. This contradicts the definition of L. □

2 Euler products

Remember that among Dirichlet series, the Riemann zeta function had the unusual property that one could factor the Dirichlet series as a product over primes:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}.$$

In fact, a number of natural Dirichlet series admit such factorizations; they are the ones corresponding to multiplicative functions.

We define an *arithmetic function* to simply be a function $f : \mathbb{N} \to \mathbb{C}$. Besides the obvious operations of addition and multiplication, another useful operation on arithmetic functions is the *(Dirichlet) convolution* $f * g$, defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Just as one can think of formal power series as the generating functions for ordinary sequences, we may think of a formal Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ as the "arithmetic generating function" for the multiplicative function $n \mapsto a_n$. In this way of thinking, convolution of multiplicative functions corresponds to ordinary multiplication of Dirichlet series:

$$\sum_{n=1}^{\infty} (f * g)(n) n^{-s} = \left(\sum_{n=1}^{\infty} f(n) n^{-s} \right) \left(\sum_{n=1}^{\infty} g(n) n^{-s} \right).$$

In particular, convolution is a commutative and associative operation, under which the arithmetic functions taking the value 1 at $n = 1$ form a group. The arithmetic functions taking all integer values (with the value 1 at $n = 1$) form a subgroup (see exercises).

We say f is a *multiplicative function* if $f(1) = 1$, and $f(mn) = f(m)f(n)$ whenever $m, n \in \mathbb{N}$ are coprime. Note that an arithmetic function f is multiplicative if and only if its Dirichlet series factors as a product (called an *Euler product*):

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p \left(\sum_{i=0}^{\infty} f(p^i) p^{-is} \right).$$

In particular, the property of being multiplicative is clearly stable under convolution, and under taking the convolution inverse.

We say f is *completely multiplicative* if $f(1) = 1$, and $f(mn) = f(m)f(n)$ for any $m, n \in \mathbb{N}$. Note that an arithmetic function f is multiplicative if and only if its Dirichlet series factors in a very special way:

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p}(1 - f(p)p^{-s})^{-1}.$$

In particular, the property of being completely multiplicative is *not* stable under convolution.

3 Examples of multiplicative functions

Here are some examples of multiplicative functions, some of which you may already be familiar with. All assertions in this section are left as exercises.

- The unit function ε: $\varepsilon(1) = 1$ and $\varepsilon(n) = 0$ for $n > 1$. This is the identity under $*$.
- The constant function 1: $1(n) = 1$.
- The *Möbius function* μ: if n is squarefree with d distinct prime factors, then $\mu(n) = (-1)^d$, otherwise $\mu(n) = 0$. This is the inverse of 1 under $*$.
- The identity function id: $\mathrm{id}(n) = n$.
- The k-th power function id^k: $\mathrm{id}^k(n) = n^k$.
- The *Euler totient function* ϕ: $\phi(n)$ counts the number of integers in $\{1, \dots, n\}$ coprime to n. Note that $1 * \phi = \mathrm{id}$, so $\mathrm{id} * \mu = \phi$.
- The divisor function d (or τ): $d(n)$ counts the number of integers in $\{1, \dots, n\}$ dividing n. Note that $1 * 1 = d$.
- The divisor sum function σ: $\sigma(n)$ is the sum of the divisors of n. Note that $1 * \mathrm{id} = d * \phi = \sigma$.
- The divisor power sum functions σ_k: $\sigma_k(n) = \sum_{d|n} d^k$. Note that $\sigma_0 = d$ and $\sigma_1 = \sigma$. Also note that $1 * \mathrm{id}^k = \sigma_k$.

Of these, only $\varepsilon, 1, \mathrm{id}, \mathrm{id}^k$ are completely multiplicative. We will deal with some more completely multiplicative functions, the Dirichlet characters, in a subsequent unit.

Note that all of the Dirichlet series corresponding to the aforementioned functions can be written explicitly in terms of the Riemann zeta function; see exercises. An important non-multiplicative function with the same property is the *von Mangoldt function* $\Lambda = \mu * \log$; see exercises.

Exercises

1. Prove Lemma 1. Then exhibit examples to show that a Dirichlet series with some abscissa of absolute convergence $L \in \mathbb{R}$ may or may not converge absolutely on $\mathrm{Re}(s) = L$.

2. Give a counterexample to Theorem 2 in case the series need not have nonnegative real coefficients. (Optional, and I don't know the answer: must a Dirichlet series have a singularity *somewhere* on the abscissa of absolute convergence?)

3. Let $f : \mathbb{N} \to \mathbb{Z}$ be an arithmetic function with $f(1) = 1$. Prove that the convolution inverse of f also has values in $\mathbb{Z}$; deduce that the set of such f forms a group under convolution. (Likewise with $\mathbb{Z}$ replaced by any subring of $\mathbb{C}$, e.g., the integers in an algebraic number field.)

4. Prove the assertions involving $*$ in Section 3. Then use them to write the Dirichlet series for all of the functions introduced there in terms of the Riemann zeta function.

5. Here is a non-obvious example of a multiplicative function. Let $r_2(n)$ be the number of pairs (a, b) of integers such that $a^2 + b^2 = n$. Prove that $r_2(n)/4$ is multiplicative, using facts you know from elementary number theory.

6. We defined the *von Mangoldt function* as the arithmetic function $\Lambda = \mu * \log$. Prove that
$$\Lambda(n) = \begin{cases} \log(p) & n = p^i, i \geq 1 \\ 0 & \text{otherwise} \end{cases}$$
and that the Dirichlet series for Λ is $-\zeta'/\zeta$.

7. For t a fixed positive real number, verify that the function
$$Z(s) = \zeta^2(s)\zeta(s + it)\zeta(s - it)$$
is represented by a Dirichlet series with nonnegative coefficients which does not converge everywhere. (Hint: check $s = 0$.)

8. Assuming that $\zeta(s) - s/(s - 1)$ extends to an entire function (we'll prove this in a subsequent unit), use the previous exercise to give a second proof that $\zeta(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$.

9. (Dirichlet's hyperbola method) Suppose f, g, h are arithmetic functions with $f = g * h$, and write
$$G(x) = \sum_{n \leq x} g(n), \qquad H(x) = \sum_{n \leq x} h(n).$$

Prove that (generalizing a previous exercise)

$$\sum_{n \le x} f(n) = \left(\sum_{d \le y} g(d) H(x/d) \right) + \left(\sum_{d \le x/y} h(d) G(x/d) \right) - G(y) H(x/y).$$

10. Prove that the abscissa of absolute convergence L of a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ satisfies the inequality

$$L \le \limsup_{n \to \infty} \left(1 + \frac{\log |a_n|}{\log n} \right)$$

(where $\log 0 = -\infty$), with equality if the $|a_n|$ are bounded away from 0. Then exhibit an example where the inequality is strict. (Thanks to Sawyer for pointing this out.) Optional (I don't know the answer): is there a formula that computes the abscissa of absolute convergence in general? Dani proposed

$$\limsup_{n \to \infty} \frac{\log \sum_{m \le n} |a_m|}{\log n}$$

but Sawyer found a counterexample to this too.