18.785: Analytic Number Theory, MIT, spring 2007 (K.S. Kedlaya)
Error bounds in the prime number theorem in arithmetic progressions

In this unit, we summarize how to derive a form of the prime number theorem in arithmetic progressions with an appropriate uniformity in the modulus; many proofs are missing, and will not be included in this course. We will revisit this uniformity later in the Bombieri-Vinogradov theorem.

1 Uniformity in the explicit formula

Let χ be a primitive Dirichlet character of level N.

Lemma 1. *The number of critical zeroes of $L(s,\chi)$ with imaginary part in $[T, T+1]$ is $O(\log(NT))$, where the implied constant is absolute (i.e., it does not depend on N).*

Put

$$\psi(x,\chi) = \sum_{n \leq x} \chi(n)\Lambda(n).$$

Then as long as x is an integer and $T \leq x$, the explicit formula for $\psi(x,\chi)$ has the form

$$\psi(x,\chi) = -\sum_{\rho : |\operatorname{Im}(\rho)| < T} \frac{x^\rho}{\rho} - b(\chi) + O(xT^{-1}\log^2(Nx)),$$

where $b(\chi)$ is defined by

$$\frac{L'(s,\chi)}{L(s,\chi)} = \begin{cases} s^{-1} + b + O(s) & \chi(-1) = 1 \\ b + O(s) & \chi(-1) = -1. \end{cases}$$

The proof is as for von Mangoldt's formula; I will not redo it here. The point is that everything is uniform in N *except* the constant $b(\chi)$.

So to get uniform estimates, one must control $b(\chi)$, which we can do by expressing it in terms of zeroes of $L(s,\chi)$. For this we go back to the product expansion:

$$\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log(N/\pi) - \frac{1}{2}\frac{\Gamma'(s/2+a/2)}{\Gamma(s/2+a/2)} + B(\chi) + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where $a = 0$ if χ is even and $a = 1$ if χ is odd. The constant B here is not the same as b (it includes the contribution from the exponential part of the Hadamard product expansion), but no matter; we can eliminate it by comparing a given s with $s = 2$. Hence

$$\frac{L'(s,\chi)}{L(s,\chi)} = O(1) - \frac{1}{2}\frac{\Gamma'(s/2+a/2)}{\Gamma(s/2+a/2)} + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{2-\rho}\right).$$

If $a = 1$, everything is regular near $s = 0$; if $a = 0$, the two log derivatives both have a simple pole at $s = 0$, and the residues match. We can thus equate the constant terms of the expansions around $s = 0$, to obtain

$$b(\chi) = O(1) - \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{2-\rho} \right).$$

As happened with ζ, we can bound the contribution of the zeroes with $|\operatorname{Im}(\rho)| \geq 1$ by $O(\log N)$. The same goes for the term $1/(2-\rho)$ when $|\operatorname{Im}(\rho)| \leq 1$.

This means that we have

$$\psi(x,\chi) = -\sum_{\rho : |\operatorname{Im}(\rho)| < T} \frac{x^\rho}{\rho} + \sum_{\rho : |\operatorname{Im}(\rho)| < 1} \frac{1}{\rho} + O(xT^{-1}\log^2(Nx)).$$

2 Controlling the exceptional zeroes

The question that now remains is: how big is the contribution to $\psi(x,\chi)$ from the sum of $1/\rho$ over critical zeroes $L(s,\chi)$ in the range $|\operatorname{Im}(\rho)| < 1$?

To answer this, we need a uniform zero-free region near the real axis. Here's what happens when you try to produce this.

Theorem 2. *There is a constant $c > 0$ such that there is no zero of $L(s,\chi)$ with*

$$|\operatorname{Im}(\rho)| < 1, \qquad \operatorname{Re}(\rho) > 1 - \frac{c}{\log N}$$

except *perhaps if χ is real, in which case there may be* one *such zero (counting multiplicity), necessarily real.*

The Riemann Hypothesis for $L(s,\chi)$ implies that no such oddball zero exists, but to prove unconditional results we must allow for it. In particular, it helps to have a label for the hypothetical zero: we call it an *exceptional zero*, or *Siegel zero*, of $L(s,\chi)$. (Note that the criterion for being a Siegel zero depends on the choice of the cutoff parameter c.)

If we exclude any exceptional zero β and also its mirror image $1-\beta$, then the sum of $1/\rho$ over the remaining zeroes in the range $|\operatorname{Im}(\rho)| < 1$ is $O((\log N)^2)$, since there are $O(\log N)$ such zeroes and each term contributes $O(\log N)$ to the sum. We then have

$$\psi(x,\chi) = -\sum_{\rho : |\operatorname{Im}(\rho)| < T}^{\sim} \frac{x^\rho}{\rho} - \frac{x^\beta}{\beta} - \frac{x^{1-\beta}-1}{1-\beta} + O(xT^{-1}\log^2(Nx)),$$

where the tilde means don't count β and $1-\beta$ as zeroes. The term $(x^{1-\beta}-1)/(1-\beta)$ is $O(x^c \log x)$, but controlling the term x^β/β requires preventing the exceptional zero from getting too close to 1. Here's one way to do that.

Theorem 3 (Siegel)**.** *For any $\epsilon > 0$, there exists a constant $c = c(\epsilon)$ with the following property: for any real primitive Dirichlet character χ of level N, every real zero β of $L(s,\chi)$ satisfies*

$$\beta \leq 1 - cN^{-\epsilon}.$$

(The proof of this uses the previous theorem; the idea is to show that if you have an exceptional zero for one real character, it "repels" real zeroes for other characters.)

This is enough to get the following form of the prime number theorem in arithmetic progressions with error term, called the Siegel-Walfisz theorem. For N a positive integer and a an integer coprime to N, put

$$\pi(x,N,a) = \sum_{p \leq x, p \equiv a(N)} 1$$
$$\psi(x,N,a) = \sum_{n \leq x, n \equiv a(N)} \Lambda(n).$$

Theorem 4. *Fix $\epsilon > 0$ and $A > 0$. Then*

$$\pi(x,N,a) = \frac{\mathrm{li}(x)}{\phi(N)} + O(x \log^{-A} x)$$
$$\psi(x,N,a) = \frac{x}{\phi(N)} + O(x \log^{-A} x)$$

where the implied and explicit constants depend only on A and ϵ, not on N.

One can improve this error bound a bit unconditionally, but not much. On the other hand, under the *Generalized Riemann Hypothesis* (i.e., the critical zeroes of every $L(s,\chi)$ lie on the line $\mathrm{Re}(s) = 1/2$), you get errors of $O(x^{1/2+\epsilon})$.

With the error bound as is, the theorem only has content for N no bigger than a fixed power of $\log x$. You can prove much better results, say for N up to x^C for a fixed $c < 1/2$, if you are willing to accept an *average* statement about the error bounds. More on this when we discuss the Bombieri-Vinogradov theorem.

3 Why the exceptional zero?

One can see where the possibility of an exceptional zero arises by beginning to imitate for $L(s,\chi)$ the proof we gave of the zero-free region for ζ. We have

$$-\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-\mathrm{Re}(s)} \chi(n) e^{-i\,\mathrm{Im}(s)\log n},$$

and using the trigonometric inequality, we have for $\sigma > 1$

$$-3\frac{L'(\sigma,\chi_0)}{L(\sigma,\chi_0)} - 4\,\mathrm{Re}\,\frac{L'(\sigma+it,\chi)}{L(\sigma+it,\chi)} - \mathrm{Re}\,\frac{L'(\sigma+2it,\chi^2)}{L(\sigma+2it,\chi)} \geq 0.$$

Here χ_0 is the principal character of the same level as χ.

The argument to exclude zeroes close to the edge of the critical strip proceeds as before if $\operatorname{Im}(\rho)$ is bounded away from 0, say $|\operatorname{Im}(\rho)| > c/(\log N)$. For χ nonreal, you do better: χ^2 is nonprincipal and so $L(\sigma + 2it, \chi^2)$ stays bounded as $\sigma \to 1^+$. So you get an inequality of the form

$$\frac{4}{\sigma - \operatorname{Im}(\rho)} < \frac{3}{\sigma - 1} + O(\log N + \log(|\operatorname{Im}(\rho)| + 1)),$$

and that gives you a zero-free region all the way down to the real line.

Unfortunately, if χ is real, then $L(\sigma+2it, \chi^2)$ blows up at $\chi = 1$, and our present methods cannot exclude a single zero very close to 1: you only end up with an inequality of the form

$$\frac{4}{\sigma - \operatorname{Im}(\rho)} < \frac{3}{\sigma - 1} + \operatorname{Re}\left(\frac{1}{\sigma - 1 + 2i\operatorname{Im}(\rho)}\right) + O(\log N + \log(|\operatorname{Im}(\rho)| + 1)).$$

However, we can exclude two such zeroes ρ_1, ρ_2, by writing

$$-\frac{L'(s,\chi)}{L(s,\chi)} < -\frac{1}{\sigma - \rho_1} - \frac{1}{\sigma - \rho_2} + O(\log N + \log(|\operatorname{Im}(\rho)| + 1))$$

and so on.