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Laurent Phenomenon Sequences

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Abstract In this paper, we undertake a systematic study of sequences generated by recurrences $x_{m+n}x_m = P(x_{m+1}, \ldots, x_{m+n-1})$ which exhibit the Laurent phenomenon. Some of the most famous among these are the Somos and the Gale-Robinson sequences. Our approach is based on finding period 1 seeds of Laurent phenomenon algebras of Lam-Pylyavskyy. We completely classify polynomials P that generate period 1 seeds in the cases of $n = 2, 3$ and of mutual binomial seeds. We also find several other interesting families of polynomials P whose generated sequences exhibit the Laurent phenomenon. Our classification for binomial seeds is a direct generalization of a result by Fordy and Marsh, that employs a new combinatorial gadget we call a double quiver.

1 Introduction

The goal of this paper is to understand the Laurent phenomenon ([1]) appearing from (discrete) nonlinear recurrence relations, i.e., recurrences of the form

$$x_m x_{m+n} = P(x_{m+1}, x_{m+2}, \ldots, x_{m+n-1}), \qquad m = 0, 1, 2, \ldots \qquad (1.1)$$

where P is a polynomial.

Definition 1.1 *A Laurent phenomenon sequence is a sequence $(x_m)_{m\geq 0}$ of Laurent polynomials in the initial data, that is $x_m \in \mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ for all $m \geq 0$, generated by a nonlinear recurrence of order n of the form (1.1).*

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The prototypical example of such a sequence is the Somos-n sequence, generated by the recurrence:

$$x_m x_{m+n} = \sum_{1 \le i \le \lfloor \frac{n}{2} \rfloor} x_{m+i} x_{m+n-i}.$$

Another key example is the Gale-Robinson sequence, generated by:

$$x_{m+n} x_m = \alpha x_{m+r} x_{m+n-r} + \beta x_{m+p} x_{m+n-p} + \gamma x_{m+q} x_{m+n-q}, \qquad (1.2)$$

for some $p, q, r > 0$ with $p + q + r = n$.

It is clear that the terms of the sequences generated from these recurrences can be written as rational functions of the first n terms. Remarkably, in a Somos-n sequence, for $1 \le n \le 7$, or any Gale-Robinson sequence, each term can, in fact, be written as a Laurent polynomial in the first n terms. This Laurent phenomenon for Gale-Robinson sequences was first proven in [1], where S. Fomin and A. Zelevinsky initated a study of Somos-type recurrences related to cluster algebras.

In [2], these types of sequences are studied as exchange relations in cluster mutation-periodic quivers. If mutating at a vertex v in a quiver Q results in a permutation of the vertices of Q, then the exchange polynomial P associated with v yields a Laurent phenomenon sequence. It is found in [2] that 3-term Gale-Robinson sequences, which are of the form (1.2) with $\gamma = 0$, are exactly the polynomials P we can obtain in this way. We also refer to [3], where an enumerative interpretation of the Gale-Robinson sequences, as the numbers of the perfect matches of graphs, was provided.

However, there are limitations to what the exchange polynomials can be in a cluster algebra; they need to be binomials which come from a quiver. Gale-Robinson sequences and other natural examples show that the Laurent phenomenon can arise when P has other forms. In [4], Lam and Pylyavskyy introduced a generalization of cluster algebras that removes these constraints, called Laurent phenomenon algebras, or LP algebras. They showed that LP algebras exhibit the Laurent phenomenon, which will imply that a period 1 LP algebra yields a Laurent phenomenon sequence.

In this paper, we study LP algebras arising from period 1 seeds to find more far-reaching results than [2]. We prove classification results for period 1 seeds when $n = 2, 3$. For $n = 2$, we find, by comparing our seeds with the classification of Laurent phenomenon sequences given by Musiker [5], which is attributed to Speyer, that a polynomial generates a period 1 LP seed if and only if it generates a Laurent phenomenon sequence. We also classify mutual binomial period 1 seeds. As exchange polynomials in a cluster algebra are all binomials, this result will generalize the classification theorem in [2] by taking advantage of the lessened constraints of LP algebras. In fact, our classification is described using a generalization of quivers that we introduce, called double quivers, which operates within the machinery of LP algebras.

We also give large families of polynomials that generate period 1 seeds. For many of these, to the best of our knowledge, the Laurent phenomenon had not been proven. Similar to [2, §9], we also investigate conserved quantities, k-invariants and (multi)linearizations of these families, which yields insights into their integrability.

The remainder of the paper is organized as follows. In Section 2, we introduce the relevant notions of LP algebras. We give an algorithm that, given a polynomial P, finds the unique candidate for a period 1 LP algebra, and would show that (1.1) has the Laurent phenomenon. We also link to our implementation of the algorithm in Sage. In Section 3, we summarize our results. In Sections 4 and 5, we prove our results about mutual binomial seeds and small n, respectively. In Section 6, we give the period 1 seeds corresponding to our families of examples. Finally, in Section 7 we investigate the conserved quantities and integrability of some of these recurrences.

2 Laurent Phenomenon Algebras

Before we state our main results, we introduce period 1 Laurent phenomenon algebra seeds and their important properties.

2.1 Seeds and Mutations

In this subsection, we define Laurent phenomenon (LP) algebras and related notions from [4].

Let $\mathcal{F}$ be the field of rational functions in n independent variables over $\mathbb{Q}$. A *seed* t is a pair $(\mathbf{x}, \mathbf{P})$ where:

- $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$ is a transcendence basis for $\mathcal{F}$ over $\mathbb{Q}$.

- $\mathbf{P} = \{P_0, \ldots, P_{n-1}\}$ is a collection of polynomials in $\mathcal{P} = \mathbb{Z}[x_0, \ldots, x_{n-1}]$ satisfying:

  (LP1) $P_i \in \mathcal{P}$ is irreducible and is not divisible by any x_j.

  (LP2) P_i does not depend on x_i.

Equivalently, if we denote by $\mathcal{P}_i$ the polynomials in $\mathcal{P}$ that satisfy (LP1) and (LP2), then we say that a seed $(\mathbf{x}, \mathbf{P})$ consists of a collection of pairs (x_i, P_i), $0 \leq i \leq n-1$, such that $P_i \in \mathcal{P}_i$ for all i.

Borrowing terminology from the theory of cluster algebras, the set $\{x_0, \ldots, x_{n-1}\}$ is called a *cluster*, each of $x_0, \ldots, x_{n-1}$ is called a *cluster variable*, and the polynomials $P_0, \ldots, P_{n-1}$ are the associated *exchange polynomials*.

We now define mutation in LP algebras. For $k \in \{0, \dots, n-1\}$, we say that a seed $(\mathbf{x}', \mathbf{P}')$ is obtained from $(\mathbf{x}, \mathbf{P})$ by *mutation at* k, which we denote $\mu_k(\mathbf{x}, \mathbf{P}) = (\mathbf{x}', \mathbf{P}')$, if $t' = (\mathbf{x}', \mathbf{P}')$ comes from $t = (\mathbf{x}, \mathbf{P})$ via the following sequence of steps:

1. Let $\mathcal{L}(t) = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ be the Laurent polynomial ring in the cluster variables. Define the *exchange Laurent polynomials* $\{\widehat{P}_0, \dots, \widehat{P}_{n-1}\} \subset \mathcal{L}(t)$ to be the unique set of Laurent polynomials satisfying:

   - For each $j \in \{0, \dots, n-1\}$ there are $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{n-1} \in \mathbb{Z}_{\leq 0}$ such that $\widehat{P}_j = x_1^{a_1} \dots x_{j-1}^{a_{j-1}} x_{j+1}^{a_{j+1}} \dots x_{n-1}^{a_{n-1}} P_j$

   - For each $i, j \in \{0, \dots, n-1\}$ with $i \neq j$, if we let $\mathcal{L}_j(t) = \mathbb{Z}[x_0^{\pm 1}, \dots, x_{j-1}^{\pm 1}, x^{\pm 1}, x_{j+1}^{\pm 1}, \dots x_{n-1}^{\pm 1}]$, then we have $\widehat{P}_i|_{x_j \leftarrow P_j/x} \in \mathcal{L}_j(t)$ and $\widehat{P}_i|_{x_j \leftarrow P_j/x}$ is not divisible by P_j in $\mathcal{L}_j(t)$.

2. The new cluster $\mathbf{x}' = \{x_0', \dots, x_{n-1}'\}$ is given by:

$$
x_i' = \begin{cases} x_i & \text{if } i \neq k, \\ \widehat{P}_k/x_k & \text{if } i = k. \end{cases}
$$

3. Define the polynomial

$$
G_j = P_j \bigg|_{x_k \leftarrow \frac{\widehat{P}_k|_{x_j \leftarrow 0}}{x_k'}}.
$$

4. Define H_j to be the result of removing all common factors with $\widehat{P}_k|_{x_j \leftarrow 0}$ from G_j (in the unique factorization domain $\mathbb{Z}[x_0, \dots, x_{k-1}, x_{k+1}, \dots, x_{j-1}, x_{j+1}, \dots x_{n-1}]$).

5. Define the new exchange polynomial $P_j' = M_j H_j$, where M_j is the unique Laurent monomial in $x_0', \dots, x_{n-1}'$ for which $M_j H_j$ is not divisible by any Laurent monomial.

6. The new seed is given by $\mu_k(\mathbf{x}, \mathbf{P}) = (\mathbf{x}', \mathbf{P}') = (\{x_0', \dots, x_{n-1}'\}, \{P_0', \dots, P_{n-1}'\})$.

Remark 2.1 *If P_j does not depend on x_k, then $G_j = H_j = P_j$ and $M_j = 1$, implying that $P_j' = P_j$.*

All the necessary existence and uniqueness conditions to show that the above mutation gives a unique valid seed can be found in [4, §2]. We will often abuse notation and write:

$$
\mu_0(\{x_0, \dots, x_{n-1}\}, \{P_0, \dots, P_{n-1}\}) = (\{x_1, \dots, x_n\}, \{P_1', \dots, P_n'\}),
$$

where $x_n = x_0'$ and $P_n' = P_0'$, since $x_i' = x_i$ for all $0 < i \leq n-1$. More generally, we will write:

$$
\mu_i(\{x_i, \dots, x_{n+i-1}\}, \{P_i, \dots, P_{n+i-1}\}) = (\{x_{i+1}, \dots, x_{n+i}\}, \{P_{i+1}', \dots, P_{n+i}'\}),
$$

where $x_{n+i} = x_i'$ and $P_{n+i}' = P_i'$. For any seed t, the *Laurent phenomenon algebra* $\mathcal{A}(t)$ is the commutative subring of $\mathbb{Q}(x_0, \ldots, x_{n-1})$ generated by the cluster variables of the seeds that can be obtained after a finite sequence of mutations from t. We call t the *initial seed* of $\mathcal{A}(t)$. The importance of this definition has much to do with the next theorem.

Theorem 2.2 *[4, Theorem 5.1] Let $\mathcal{A}$ be a Laurent phenomenon algebra and $t = (\mathbf{x}, \mathbf{P})$ a seed of $\mathcal{A}$. If $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$, then any cluster variable of $\mathcal{A}$ belongs to the Laurent polynomial ring $\mathcal{L}(t) = \mathbb{Z}[x_0^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$.*

2.2 Period 1 Seeds

In this paper, we are primarily interested in period 1 seeds that will be defined shortly.

Definition 2.3 *For any polynomial $P \in \mathbb{Q}[x_{-1}, x_0, x_1, x_2, \ldots]$, the upshift of P is the polynomial*

$$Q = P|_{x_i \leftarrow x_{i+1}} \; \forall i \in \mathbb{Q}[x_0, x_1, x_2, \ldots],$$

while the downshift of P is

$$R = P|_{x_i \leftarrow x_{i-1}} \; \forall i \in \mathbb{Q}[x_{-2}, x_{-1}, x_0, \ldots].$$

For our purposes, we use the downshifts of polynomials P which do not depend on x_{-1}, so that their downshifts are in the smaller ring $\mathbb{Q}[x_{-1}, x_0, x_1, \ldots]$.

Example 2.4 *The upshift and downshift of $x_2 x_3^2 + x_7 - 3x_9$ are $x_3 x_4^2 + x_8 - 3x_{10}$ and $x_1 x_2^2 + x_6 - 3x_8$, respectively.*

Definition 2.5 *Let $t = (\{x_0, \ldots, x_{n-1}\}, \{P_0, \ldots, P_{n-1}\})$ be a seed and $\mu_0(t) = (\{x_1, \ldots, x_n\}, \{P_1', \ldots, P_n'\})$ be its mutation at x_0 ($x_n = x_0'$ and $P_n' = P_0'$). Then t is a period 1 seed if P_i is the downshift of P_{i+1}' for all $0 \leq i < n-1$, and P_{n-1} is the downshift of P_0' (or equivalently of P_0).*

Period 1 seeds are interesting in light of Theorem 2.2, as they provide the machinery to prove that some recurrence sequences satisfy the Laurent phenomenon:

Corollary 2.6 *Let $P \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ be any irreducible polynomial, not divisible by any x_j. If there exists a period 1 seed $t = (\{x_0, \ldots, x_{n-1}\}, \{P_0, \ldots, P_{n-1}\})$ with $P_0 = P$, then the sequence $(x_i)_{i \geq 0}$ of rational functions of $x_0, \ldots, x_{n-1}$, defined by*

$$x_{m+n} = \frac{P(x_{m+1}, \ldots, x_{m+n-1})}{x_m} \text{ for all } m \geq 0,$$

is a Laurent phenomenon sequence.

Proof The sequence $(x_i)_{i\geq 0}$ consists of the cluster variables we get by applying the mutations $\mu_0, \mu_1, \ldots$ in order to t.

Example 2.7 *For $n = 3$, the polynomial $P = x_1 x_2 + 1$ is in the period 1 seed $\{(x_0, x_1 x_2 + 1), (x_1, x_0 + x_2), (x_2, x_0 x_1 + 1)\}$ and generates the sequence*

$$x_0, x_1, x_2, \frac{x_1 x_2 + 1}{x_0}, \frac{x_1 x_2^2 + x_0 + x_2}{x_0 x_1}, \frac{(x_1 x_2 + 1)(x_1 x_2^2 + x_0 + x_2) + x_0^2 x_1}{x_0^2 x_1},$$
$$\frac{(x_1 x_2^2 + x_0 + x_2)^2 + x_0^2 x_1 (x_0 + x_2)}{x_0^2 x_1^2 x_2}, \ldots$$

This example is a special case considered in [2] (equation (30)).

Corollary 2.8 *If $P \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ satisfies the conditions in Corollary 2.6, then the sequence $(x_i)_{i \geq 0}$ defined by*

$$x_{m+n} = \frac{P(x_{m+1}, \ldots, x_{m+n-1})}{x_m} \text{ for all } m \geq 0,$$

with initial values $x_i = 1$, for all $0 \leq i < n$, consists entirely of integers. If the coefficients of P are positive, then the sequence consists entirely of positive integers.

Definition 2.9 *For a period 1 seed $t = (\{x_0, \ldots, x_{n-1}\}, \{P_0, \ldots, P_{n-1}\})$, we will say that t is the period 1 seed generated by P_0, or that P_0 generates t. We call P_0 a period 1 polynomial, or say that P_0 is 1 periodic.*

At this point, it is worth noting that the converse of Corollary 2.6 is not true. For instance, consider $n = 3$ and $P(x_1, x_2) = x_1 + x_2 + 1$. After reading the next section, the reader should be able to easily confirm that P does not generate a period 1 seed. However, the sequence generated by P is periodic and satisfies the Laurent phenomenon:

$$x_1, x_2, x_3, \frac{x_2 + x_3 + 1}{x_1}, \frac{x_1 x_3 + x_1 + x_2 + x_3 + 1}{x_0 x_1}, \frac{x_1 x_2 + x_2^2 + x_1 x_3 + x_2 x_3 + x_1 + 2x_2 + x_3 + 1}{x_1 x_2 x_3},$$
$$\frac{x_1 x_3 + x_1 + x_2 + x_3 + 1}{x_0 x_1}, \frac{x_2 + x_3 + 1}{x_1}, x_1, x_2, x_3, \ldots$$

2.3 Generation of period 1 seeds

In this section, we propose a method for obtaining period 1 seeds generated by particular polynomials, using a variant of the method in [1].

Given a polynomial $P \in \mathcal{P}_0 = \mathbb{Z}[x_1, \ldots, x_{n-1}]$, we define a map $\tau = \tau_P : \mathcal{P} \to \mathcal{P}$, which takes polynomials in $\mathcal{P}_i$ to polynomials in $\mathcal{P}_{i-1}$, for all $i > 0$. If $Q \in \mathcal{P}_i$, then $\tau_P(Q) \in \mathcal{P}_{i-1}$ is computed according to the following algorithm:

1. Let $G = G(x_1, \ldots, \widehat{x}_i, \ldots, x_i) = Q\Big|_{x_0 \leftarrow \frac{P|_{x_i=0}}{x_{n+1}}} \in \mathcal{P}[x_{n+1}^{\pm 1}]$.

2. If d is the factor of G shared with $P\big|_{x_i=0}$ i.e., $d = \gcd(G, (P|_{x_i=0})^k)$ in $\mathbb{Z}[x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1}]$ for some sufficiently large $k \in \mathbb{N}$, then let $H = G/d$.

3. Finally, define $\tau_P(Q)$ to be the downshift of MH, where $M \in \mathcal{L}(x_1, \ldots, \widehat{x}_i, \ldots, x_n)$ is such that $MH \in \mathbb{Z}[x_1, \ldots, \widehat{x}_i, \ldots, x_n]$ and MH is not divisible by any x_j.

Remark 2.10 *If Q does not depend on x_0, then $H = G = Q$, $M = 1$ and so $\tau_P(Q)$ is simply the downshift of Q.*

It is not immediately clear that τ maps polynomials from $\mathcal{P}_i$ to $\mathcal{P}_{i-1}$ for all i. The two propositions below show that this is the case.

Proposition 2.11 *If $Q \in \mathcal{P}_i$, then $R = \tau_P(Q)$ does not depend on x_{i-1}.*

Proof If P_i does not depend on x_1, the statement follows because $P_{i-1} = \tau(P_i)$ is the downshift of P_i. If P_i depends on x_1, then in the computation of $\tau(P_i)$, we define G as the Laurent polynomial resulting from replacing x_0 in Q by an expression wherein we made the substitution $x_i = 0$. In particular, G does not depend on x_i. Then, H does not contain x_i and neither does M, by definition. Hence, $P_{i-1} = \tau(P_i)$, which is the downshift of MH, does not contain x_{i-1}.

Proposition 2.12 *If $Q \in \mathcal{P}_i$, then $R = \tau_P(Q)$ is irreducible in $\mathcal{P}$ and is not divisible by any of the x_j.*

Proof From the definition of τ, it is clear that $R = \tau(Q)$ is not divisible by any x_j. It then suffices to show R is irreducible. This is clear if Q does not depend on x_0, so assume Q depends on x_0. Write

$$Q = \sum_k f_k x_0^k,$$

where $f_k \in \mathbb{Z}[x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1}]$ for all k. Then,

$$G = Q\Big|_{x_0 \leftarrow \frac{P|_{x_i=0}}{x_n}} = \sum_k f_k \left(\frac{P|_{x_i=0}}{x_n} \right)^k,$$

and H is G divided by all common factors it shares with $P_0|_{x_i \leftarrow 0}$. Finally, R is the downshift of MH for some Laurent monomial M. As M is a unit in $\mathcal{P}$, it will suffice to show H is irreducible. Let d be a nonunit factor of H. From the definition of H, d is not a factor of $P|_{x_i=0}$.

If d is independent of x_n then $d \mid f_k$ for all k, which implies $d \mid Q$, contradicting the irreducibility of Q.

If d depends on x_n, write $d = d(x_n)$, so,

$$d\left(\frac{P_0|_{x_i \leftarrow 0}}{x_n} \right) \text{ divides } G\left(\frac{P_0|_{x_i \leftarrow 0}}{x_n} \right) = Q(x_n).$$

This again contradicts that Q is irreducible.

Given an irreducible polynomial $P = P(x_1, \ldots, x_{n-1}) \in \mathcal{P}$, we generate a seed $(\mathbf{x}, \mathbf{P})$ by letting $P_0 = P$, P_{n-1} be the downshift of P, and recursively defining $P_i = \tau_P(P_{i+1})$ for $i = n-1, n-2, \ldots, 1$. From Propositions 2.11 and 2.12, it is clear that $(\mathbf{x}, \mathbf{P})$ is a valid seed. For example, for $n = 3$, the polynomial $P = x_1 x_2 + x_3^2$ generates the seed $\{(x_0, x_1 x_2 + x_3^2), (x_1, x_2^3 + x_0 x_3^2), (x_2, x_0^2 + x_1 x_3), (x_3, x_0 x_1 + x_2^2)\}$. The following proposition gives a sufficient condition for asserting that $(\mathbf{x}, \mathbf{P})$ is a period 1 seed.

Proposition 2.13 *Let $\widehat{P}_0$ be the exchange Laurent polynomial of P_0 for the generated seed $(\mathbf{x}, \mathbf{P})$. If $P_0 = \tau_P(P_1)$ and $\widehat{P}_0 = P_0 = P(x_1, \ldots, x_{n-1})$, then $(\mathbf{x}, \mathbf{P})$ is a period 1 seed. In particular, P generates a Laurent phenomenon sequence.*

Proof We remarked above that $(\mathbf{x}, \mathbf{P})$ is a valid seed. It is also clear that P_{n-1} is the downshift of P_0. Finally, observe that if $\widehat{P}_0 = P_0$, then the definitions of τ_P and μ_0 coincide. Therefore, the seed $(\mathbf{x}, \mathbf{P})$ is a period 1 seed, as desired.

If $t = (\mathbf{x}, \mathbf{P})$ is a seed generated by $P \in \mathcal{P}_0$ and is such that $P_0 = \tau_P(P_1)$, we say t has *pseudoperiod* 1. Proposition 2.13 can then be rephrased as saying that if $\widehat{P}_0 = P_0$, then t has period 1. The following conjecture, in conjunction with Proposition 2.13, would show that period and pseudoperiod are equivalent definitions in this context.

Conjecture 2.14 *Let $\widehat{P}_0$ be the exchange Laurent polynomial of P_0 for the generated seed $(\mathbf{x}, \mathbf{P})$. Then $\widehat{P}_0 = P_0$.*

In the rest of this paper, we classify certain families of polynomials that generate pseudoperiod 1 seeds. In addition, we find many examples of pseudoperiod 1 seeds. In all cases, we can show that the seeds are, indeed, period 1 seeds, using the Lemma below and Proposition 2.13.

For simplicity of terminology, in cases where the Lemma below is satisfied, we will simply say period 1 instead of pseudoperiod 1.

Lemma 2.15 $\widehat{P}_0 = P_0$ *if either of the two conditions holds*

1. *P_j depends on x_0 whenever P_0 depends on x_j.*

2. *All polynomials P_j, $0 \le j \le n-1$, have the same number d of terms.*

Proof From the construction of the P_j, it is clear that $P_0\big|_{x_j \leftarrow P_j/x} \in \mathbb{Z}[x_0^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x^{\pm 1}, x_{j+1}^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$. It then suffices to show that $P_0\big|_{x_j \leftarrow P_j/x}$ is not divisible by P_j, or equivalently that $P_0\big|_{x_j = 0}$ is not divisible by P_j.

(1) If P_0 depends on x_j, then P_j depends on x_0 by assumption. From Proposition 2.11, P_0 does not depend on x_0 and therefore neither does $P_0\big|_{x_j=0}$. Then P_j cannot divide $P_0\big|_{x_j=0}$.

If P_0 does not depend on x_j, then $P_0\big|_{x_j=0} = P_0$. As both P_0 and P_j are irreducible and not divisible by any x_k, we only need that $P_j \neq P_0$ for $j > 0$. Let m, M be the minimum and maximum indices i such that P_0 depends on x_i. We claim that P_j either does not depend on x_M or it depends on some x_k with $k < m$; this immediately implies $P_j \neq P_0$ for $j > 0$. If there is no intermediate polynomial $P_{j'}$ with $j' > j$ that depends on x_0, then P_s is the downshift of P_{s+1} for all $s \geq j$. Since the maximum index upon which $P_{n-1} = P(x_0, \ldots, x_{n-2})$ depends is $M - 1$, then the maximum index upon which P_j depends is also smaller than M; in particular P_j does not depend on x_M. If there is some intermediate polynomial $P_{j'}$ with $j' > j$ that depends on x_0, let j_0 be the smallest such index $j_0 > j$ (so P_s is the downshift of P_{s+1} for all $j \leq s < j_0$). Recall that the polynomial P_{j_0-1} comes from

$$P_{j_0}\bigg|_{x_0 \leftarrow \frac{P|_{x_{j_0}=0}}{x_n}}.$$

Hence, P_{j_0-1} depends on x_{m-1} unless $P_{j_0}\big|_{x_0=0}$ is divisible by $P = P_0$. Since P_{j_0} depends on x_0, P_0 depends on x_{j_0}. From Proposition 2.11, P_{j_0} does not depend on x_{j_0}; therefore P_0 cannot divide $P_{j_0}|_{x_0=0}$. Therefore P_{j_0-1} depends on x_{m-1}. The polynomial P_j, which is the result of $j_0 - j - 1$ downshifts from P_{j_0-1}, then depends on $x_{m'}$ for some $m' < m$.

(2) If P_0 depends on x_j, then $P_0\big|_{x_j=0}$ has at most $d - 1$ terms. Thus P_j, which has d terms, cannot divide it.

If P_0 does not depend on x_j, then $P_0|_{x_j \leftarrow P_j/x} = P_0$. As both P_0 and P_j are irreducible and not divisible by any x_k, we only need to show $P_j \neq P_0$ for $j > 0$. The argument is the same as in part (1) except for the reason why P_0 does not divide $P_{j_0}|_{x_0=0}$. In this case, it is because $P_{j_0}|_{x_0=0}$ has at most $d - 1$ terms (as j_0 was defined as an index for which P_{j_0} depends on x_0) and P_0 has d terms.

Remark 2.16 *One can see that if P generates some period 1 seed, then such a seed must be the one described in Proposition 2.13. If we begin with $P \in \mathcal{P}$ and follow the process mentioned above (recursively obtain the intermediate polynomials P_j, $0 < j < n - 1$), we may have that one of the conditions in Proposition 2.13 is not satisfied; in that case, P is not a period 1 polynomial.*

We next use [4, Proposition 2.10], which says that if $(\mathbf{x}', \mathbf{F}') = \mu_i(\mathbf{x}, \mathbf{F})$, then $(\mathbf{x}, \mathbf{F}) = \mu_i(\mathbf{x}', \mathbf{F}')$, to devise an analogue of τ that instead takes polynomials from $\mathcal{P}_{i-1}$ to polynomials in $\mathcal{P}_i$.

We define the mapping $\kappa = \kappa_P$, that is the inverse of τ, as follows.

Given a polynomial $P \in \mathcal{P}_0 = \mathbb{Z}[x_1, \ldots, x_{n-1}]$, let $P' = P(x_0, x_1, \ldots, x_{n-2}) \in \mathcal{P}_{n-1}$ and $\kappa : \mathcal{P} \to \mathcal{P}$ a map which takes polynomials from $\mathcal{P}_i$ to polynomials in $\mathcal{P}_{i+1}$ for all $i \geq 0$. If $Q \in \mathcal{P}_i$, then $\kappa_P(Q) \in \mathcal{P}_{i+1}$ is computed according to the following rules:

1. Let $G' = G'(x_{-1}, x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1}) = Q\big|_{x_{n-1} \leftarrow \frac{P'|_{x_i=0}}{x_{-1}}} \in \mathcal{P}[x_{-1}^{\pm 1}]$.

2. If d' is the factor of G' shared with $P'\big|_{x_i=0}$, i.e., $d' = \gcd(G', (P'|_{x_i=0})^k)$ in $\mathbb{Z}[x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1}]$ for some sufficiently large $k \in \mathbb{N}$, then let $H' = G'/d'$.

3. Finally, let $R = \kappa_P(Q)$ be the upshift of $M'H'$, where $M' \in \mathcal{L}(x_{-1}, x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1})$ is such that $M'H' \in \mathbb{Z}[x_{-1}, x_1, \ldots, \widehat{x}_i, \ldots, x_{n-1}]$ and is not divisible by any x_j.

Remark 2.17 *If Q does not depend on x_{n-1}, then $H' = G' = Q$, $M' = 1$ and so $\kappa_P(Q)$ is simply the upshift of Q.*

The proof that κ is a well-defined map comes from the analogous statements of Propositions 2.11 and 2.12 to κ. Given an irreducible polynomial P, choose k such that $0 < k < n - 1$. We generate a seed $(\mathbf{x}, \mathbf{P})$ by letting $P_0 = P$, $P_{n-1} = P'$ be the downshift of P and recursively defining $P_i = \tau_P(P_{i+1})$ for all $k < i < n$ and $P_i = \kappa_P(P_{i-1})$ for all $0 < i < k$. A refinement of Proposition 2.13 is then

Proposition 2.18 *Let $\widehat{P}_0$ be the exchange Laurent polynomial of P_0 for the generated seed $(\mathbf{x}, \mathbf{P})$. If $P_k = \kappa(P_{k-1})$, or equivalently $P_{k-1} = \tau(P_k)$, and $\widehat{P}_0 = P_0$, then $(\mathbf{x}, \mathbf{P})$ is a period 1 seed.*

Remark 2.19 *We have implemented the above algorithm (with $k = \lfloor n/2 \rfloor$) in Sage ([6]) at* `https://bit.ly/LPsequences`. *This can be used to test whether a given polynomial P is period 1.*

3 Statements of results and conjectures

In this section, we present our main results. Their proofs will be presented in the remaining sections. In the first subsection, we give our classification theorems, while in the second subsection, we give a proposition asserting that several large families of polynomials are 1 periodic.

3.1 Classification theorems

We first classify all period 1 polynomials when $n = 2, 3$.

Theorem 3.1 *For $n = 2$, the only period 1 polynomials P are*

1. *Irreducible polynomials that are monic and palindromic, i.e., that satisfy $x^{\deg(P)} \cdot P(\frac{1}{x}) = P(x)$.*

2. *Irreducible polynomials of even degree that are monic and antipalindromic, i.e., that satisfy $x^{\deg(P)} \cdot P(-\frac{1}{x}) = -P(x)$.*

3. *Monic irreducible polynomials of degree 2.*

Remark 3.2 *In [5], G. Musiker found necessary and sufficient conditions for polynomials $P \in \mathbb{Q}[x]$ with $P(0) \neq 0$ to yield Laurent phenomenon sequences. The polynomials in Theorem 3.1 are special cases of Proposition 1 in [5]; more specifically, the polynomials in our theorem are the polynomials from the list of Musiker that have are monic, irreducible and have integer coefficients.*

Remark 3.3 *The Laurent phenomenon, for the case of palindromic polynomials P has recently been considered in the noncommutative setting (see [7]).*

Theorem 3.4 *For $n = 3$, the only period 1 polynomials P are*

1. *$P = x_1 x_2 + a x_1 + a x_2$, for any $a \in \mathbb{Z}, a \neq 0$,*

2. *$P = x_1 x_2 + a x_1 - a x_2$, for any $a \in \mathbb{Z}, a \neq 0$,*

3. *$P = x_1 - x_2 - 1$,*

4. *$P = -x_1 + x_2 - 1$,*

5. *$P = x_1 x_2 + a x_1 + a x_2 + b$, for any $a, b \in \mathbb{Z}$, not both of which are 0,*

6. *$P = x_1^2 + x_2^2 + a x_1 x_2 + b x_1 + b x_2 + c$, for any $a, b, c \in \mathbb{Z}$,*

7. *$P = -x_1^2 - x_2^2 + a x_1 x_2 + b$, for any $a, b \in \mathbb{Z}$,*

8. *$P = \pm x_1 x_2 + a$, for any $a \in \mathbb{Z}, a \neq 0$,*

9. *$P = 1 + x_1^m x_2^n + \sum_{\substack{0 < i < m \\ 0 < j < n}} C_{i,j}(x_1^i x_2^j + x_1^{m-i} x_2^{n-j})$, for any $C_{i,j} \in \mathbb{Z}$, $m, n \in \mathbb{N}_{>0}$,*

10. *$P = -1 + (-1)^{m+1} x_1^m x_2^n + \sum_{\substack{0 < i < m \\ 0 < j < n}} C_{i,j}(x_1^i x_2^j + (-1)^{m+j+i} x_1^{m-i} x_2^{n-j})$ for any $C_{i,j} \in \mathbb{Z}$, $m, n \in \mathbb{N}_{>0}$, $m \equiv n \mod 2$.*

Remark 3.5 *The arbitrary coefficients and exponents in Theorem 3.4 must be such that P is irreducible and not divisible by any x_j.*

The proofs of Theorems 3.1 and 3.4 are given in Section 5 below.

Our final classification theorem comes from our own definition of *Double Quivers* (see Section 4, especially Definition 4.1). The family of polynomials we found includes those that are classified by the main theorem in [2].

Theorem 3.6 *The binomial P generates a period 1 seed which corresponds to a double quiver if and only if it is of the form*

$$P = \prod_{1 \leq i \leq n} x_i^{a_i} + \prod_{1 \leq i \leq n} x_i^{b_i},$$

where $a_i, b_i \in \mathbb{Z}_{\geq 0}$ are such that $a_i = 0 \iff a_{n-i} = 0$ and $b_j = 0 \iff b_{n-j} = 0$.

The proof of Theorem 3.6, as well as the notion of a double quiver, is given in Section 4 below.

Finally, the families of polynomials we have found, and that we present in the next subsection, give rise to the following conjectures:

Conjecture 3.7 *If P is a multilinear polynomial with positive coefficients that generates a period 1 seed, then $P(x_1, x_2, \ldots, x_n) = P(x_n, x_{n-1}, \ldots, x_1)$.*

Conjecture 3.8 *If n is odd, no degree 1 polynomial with positive coefficients generates a period 1 seed. If n is even, the only degree 1 polynomial P with positive coefficients that generates a period 1 seed is $x_{n/2} + 1$.*

Conjecture 3.9 *The only symmetric polynomials P with positive coefficients that generate period 1 seeds are either of the form*

$$P = \sum_{i=1}^{n-1} x_i^2 + M(x_1, \ldots, x_{n-1}),$$

where M is any multilinear symmetric polynomial, or of the form

$$P = \sum_{1 \leq i < j \leq n-1} x_i x_j + A \sum_{i=1}^{n-1} x_i + B,$$

for odd n.

3.2 Families of period 1 polynomials

Theorem 3.10 *The following families of polynomials P are 1 periodic.*

1. *Gale-Robinson polynomial.*

$$P = A x_p x_{n-p} + B x_q x_{n-q} + C x_r x_{n-r},$$

   for any $p < q < r$ and $p + q + r = n$.

2. *Symmetric with second powers polynomial.*

$$P = S + A_1 E_1 + \ldots A_{n-1} E_{n-1} + A,$$

   for any coefficients $A, A_1, \ldots, A_{n-1} \in \mathbb{Z}$, where $E_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n-1} x_{i_1} \ldots x_{i_k}$ for all $1 \leq k \leq n$ and $S = \sum_{i=1}^{n-1} x_i^2$.

   For example, $P = x_1^2 + x_2^2 + 2x_1 x_2 + 5$ when $n = 3$.

3. *Sink-type binomial.*

$$P = x_1^{a_1} x_2^{a_2} \ldots x_{n-1}^{a_{n-1}} + 1,$$

   where $a_i = 0 \iff a_{n-i} = 0$ for all i. For example, $P = x_1^2 x_3^3 x_5 + 1$ when $n = 6$.

4. *Extreme polynomial.*

$$P = x_1 x_{n-1} + A \cdot \sum_{i=1}^{n-1} x_i + B,$$

   for any coefficients $A, B \in \mathbb{Z}$. For example, $P = x_1 x_3 + 3(x_1 + x_2 + x_3) + 2$ when $n = 4$.

5. *Singleton polynomial. If $n \in \mathbb{N}$ is even, let P is a single variable monic irreducible polynomial that is palindromic $(x^{\deg(P)} \cdot P(1/x) = P(x))$, or antipalindromic $(x^{\deg(P)} \cdot P(1/x) = -P(x))$, or $P = x_{n/2}^2 + A x_{n/2} + B$ for any $A, B \in \mathbb{Z}$. For example, $P = x_2^2 + 2x_2 - 7$ when $n = 3$.*

6. *Chain polynomial. If $n \in \mathbb{N}, n > 2$ is odd,*

$$P = \sum_{i=1}^{n-2} x_i x_{i+1} + A \cdot \sum_{i=1}^{n-1} x_i + B,$$

   for any coefficients $A, B \in \mathbb{Z}$. For example, $P = x_1 x_2 + x_2 x_3 + x_3 x_4 + 2(x_1 + x_2 + x_3 + x_4) + 3$ when $n = 5$.

7. *Multilinear symmetric polynomial. If $n \in \mathbb{N}, n > 2$,*

$$P = E_2 + A \cdot E_1 + B,$$

   for any coefficients $A, B \in \mathbb{Z}$, where the E_i are the elementary symmetric polynomials. For example, $P = x_1 x_2 + x_2 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 - 3(x_1 + x_2 + x_3 + x_4) + 1$ when $n = 5$.

8. *r-Jumping polynomial. If $r, n \in \mathbb{N}$ are such that $n \geq 2r + 1$ and $n \equiv 1 \pmod{r}$,*

$$P = \sum_{i=0}^{\frac{n-1}{r}-1} x_{ri+1} \cdot x_{ri+r} + A,$$

   for any $A \in \mathbb{Z}$. For example, $P = x_1 x_3 + x_4 x_6$ when $n = 7$.

9. *r-Hopping polynomial. If $r, n \in \mathbb{N}$ be such that $n \geq 2r + 2$ and $n \equiv 1 \pmod{r}$,*

$$P = \sum_{i=0}^{\frac{n-1}{r}-1} x_{ri+1} \cdot x_{ri+r} + A \cdot \sum_{i=0}^{\frac{n-1}{r}-2} x_{ri+r} \cdot x_{ri+r+1} + B,$$

for any $A, B \in \mathbb{Z}$. For example, $P = x_1 x_3 - 2 x_3 x_4 + x_4 x_6 + 3$ when $n = 7$.

Note: *The r-Jumping polynomials are special cases of the r-Hopping polynomials (when $A = 0$). We distinguish them because we found a conserved quantity for sequences generated by r-Jumping polynomials, but not by r-Hopping polynomials (see Section 7).*

10. *Flip-symmetric binomial.* If $L, R \subset [n-1]$ are disjoint subsets such that $i \in L \iff n - i \in L$ and $i \in R \iff n - i \in R$, and if $a : L \cup R \to \mathbb{N}$ is any map into the positive integers, then,

$$P = \prod_{i \in L} x_i^{a(i)} + \prod_{i \in R} x_i^{a(i)}.$$

For example, $P = x_1^3 x_7^2 + x_4^3 x_2 x_6$ when $n = 8$.

Note: *The Somos-4 and Somos-5 polynomials ($x_1 x_3 + x_2^2$ and $x_1 x_4 + x_2 x_3$) are particular cases of flip-symmetric binomials. The family (2) of sink-type polynomials are also particular cases of flip-symmetric polynomials (when $R = \emptyset$).*

11. *Balanced polynomial.* If $L, R \subset [n-1]$ are disjoint subsets such that $i \in L \iff n - i \in L$ and $i \in R \iff n - i \in R$, and $a : L \cup R \to \mathbb{N}$ is any map into the positive integers. Then for any $m > 1$, write $M_1 = \prod_{i \in L} x_i^{a(i)}$, $M_2 = \prod_{i \in R} x_i^{b(i)}$ and,

$$P = M_1^m + M_2^m + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} A_i \cdot \left(M_1^i M_2^{m-i} + M_1^{m-i} M_2^i \right),$$

for arbitrary coefficients $A_i \in \mathbb{Z}$, $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$. For example, $P = x_3^{12} x_6^4 + x_2^4 x_7^8 + 2(x_2 x_3^9 x_6^3 x_7^2 + x_2^3 x_3^3 x_6 x_7^6) + 3 x_2^2 x_3^6 x_6^2 x_7^4$ when $n = 9$.

12. *Vector sum polynomial.* For $a_1, \ldots, a_{n-1} \in \mathbb{N}$ and a finite set B of vectors $(b_1, \ldots, b_{n-1}) \in \mathbb{N}^{n-1}$ such that $0 < b_i < a_i$ for all i, then,

$$P = 1 + x_1^{a_1} \ldots x_{n-1}^{a_{n-1}} + \sum_{b \in B} \left(C_b \cdot x_1^{b_1} \ldots x_{n-1}^{b_{n-1}} + C_b \cdot x_1^{a_1 - b_1} \ldots x_{n-1}^{a_{n-1} - b_{n-1}} \right),$$

for arbitrary coefficients $C_b \in \mathbb{N}$. For example, $P = 1 + x_1^3 x_2^2 x_3^4 x_4^2 + 2 x_1 x_2 x_3^2 x_4 + 2 x_1^2 x_2 x_3^2 x_4$ when $n = 5$.

13. *Little Pi polynomial.* For $k, n \in \mathbb{N}$ such that $n > 2k$ and $n \neq 3k$, then

$$P = A x_k + A x_{n-k} + x_{2k} x_{n-2k},$$

for any $A \in \mathbb{Z}$. For example, $P = 2 x_2 + 2 x_5 + x_4 x_3$ when $n = 7$, $k = 2$.

14. *Pi polynomial. For $k, n, a_1, b_1, a_2, b_2 \in \mathbb{N}$ such that $n > 2k$, $n \neq 3k$ and $a_1 + b_1 = a_2 + b_2$, then*

$$P = Ax_k^{a_1} x_{b_1}^b + Bx_k^{a_2} x_{n-k}^{b_2} + x_{2k}x_{n-2k}$$

for any $A, B \in \mathbb{Z}$. For example, $P = -2x_2^1 x_6^4 + 3x_2^2 x_6^3 + x_4^2$ when $n = 8$, $k = 2$.

Remark 3.11 *In each case of the theorem above, we omitted saying that the coefficients and exponents are such that P is irreducible and not divisible by any x_j. The following important corollary will also follow easily from the proof of Theorem 3.10 and Lemma 2.15.*

Remark 3.12 *Recently, Hone and Ward found independently the Laurent phenomenon for extreme polynomials (family (3) in Theorem 3.10). They do a thorough study of this family of polynomials in [8].*

The proof of Theorem 3.10 occupies most of Section 6 below. The following Corollary is immediate from the previous theorem and the machinery developed in Section 2.

Corollary 3.13 *All polynomials P from Theorem 3.10 generate Laurent phenomenon sequences.*

Conjecture 3.14 *Let $k, n, a_1, b_1, a_2, b_2 \in \mathbb{N}$ be such that $n > 2k$, $n \neq 3k$ and $a_1 + b_1 = a_2 + b_2$. Consider the polynomial*

$$P = (Ax_k^{a_1} x_{n-k}^{b_1} + Bx_k^{a_2} x_{n-k}^{b_2}) \cdot M + x_{2k}x_{n-2k},$$

for any $A, B \in \mathbb{Z}$ and monomial $M = \displaystyle\prod_{\substack{i=1 \\ i \neq 2k, n-2k}}^{n-1} x_i^{c_i}$, where $c_i = 0 \iff c_{n-i} = 0$ for all i.

Then P is a period 1 polynomial and generates a Laurent phenomenon sequence.

We also will prove in Section 6 the following lemmas that can be applied to known period 1 polynomials to yield new ones:

Lemma 3.15 (Expansion Lemma) *If $F = F(x_1, x_2, \ldots, x_{n-1})$ generates a period 1 seed, then for any $k \in \mathbb{N}$, so does the polynomial $G = G(x_1, x_2, \ldots, x_{nk-1}) = F(x_k, x_{2k}, \ldots, x_{(n-1)k})$. We call G the k-expansion of F.*

Lemma 3.16 (Reflection Lemma) *If $F = F(x_1, x_2, \ldots, x_{n-1})$ generates a period 1 seed, then so does $G = G(x_1, x_2, \ldots, x_{n-1}) = F(x_{n-1}, x_{n-2}, \ldots, x_1)$.*

Remark 3.17 *Observe that the reflection lemma, applied to the families of polynomials in Theorem 3.10, always gives another member of the same family.*

4 Polynomials arising from double quivers

4.1 Binomial Seeds and Double Quivers

In this section we find all period 1 binomials with a mild mutuality condition. To do this, we first introduce a new representation for binomial seeds, which we call a double quiver. The main constraint of a normal quiver that our double quiver removes is that binomial seeds represented by a quiver have to be mutual, i.e., if x_i appears in P_j, then x_j appears in P_i with the same degree.

Definition 4.1 *A* double quiver Q *is a finite set of vertices with directed half-edges between vertices. Between each pair of vertices i and j, there can be edges between them attached at i, as well as edges between them attached at j. We allow multiple half-edges at each vertex, but not 2-cycles, i.e., there cannot be edges from i to j as well as edges from j to i all attached at i. We also do not allow self-loops.*

The B-matrix $B = (b_{i,j})_{n \times n}$ *of a double quiver with n vertices is defined as follows. The magnitude $|b_{i,j}|$ is the number of half-edges between vertex i and vertex j that are attached at i. If the edges are outgoing from vertex i, then $b_{i,j} > 0$; if the edges are incoming to i, then $b_{i,j} < 0$. Conversely, each $n \times n$ integer matrix with 0's in its diagonal corresponds to a double quiver. For convenience, we will index the rows and columns of B from 0 to $n - 1$. The (LP algebra) seed corresponding to a B-matrix B is $(\mathbf{x}, \mathbf{P})$, where $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$ and the intermediate polynomials are, for all i:*

$$P_i = \prod_{j:b_{i,j}>0} x_j^{b_{i,j}} + \prod_{j:b_{i,j}<0} x_j^{-b_{i,j}}.$$

Example 4.2 *Figure 1 shows a double quiver with 3 vertices. There is a half-edge from x_1 to x_0 attached at x_0, a half-edge from x_1 to x_0 attached at x_1, two half-edges from x_0 to x_2 attached at x_0, a half-edge from x_0 to x_2 attached at x_2, three half-edges from x_2 to x_1 attached at x_1 and no half-edges from x_1 to x_2 attached at x_2.*

Definition 4.3 *A vertex i of a double quiver is* mutable *if, whenever there are half-edges between i and j attached at j, then there are also half-edges between i and j attached at i. In terms of the B-matrix, vertex i is mutable if for all other vertices j, $b_{j,i} \neq 0$ implies $b_{i,j} \neq 0$.*

Example 4.4 *In the double quiver of figure 1, x_0 and x_1 are mutable, but x_2 is not mutable since there are half-edges from x_2 to x_1 attached at x_1, but no half-edges between x_1 and x_2 attached at x_2.*

$$\{x_0 : x_1 + x_2^2\}$$
$$\{x_1 : x_0 + x_2^3\}$$
$$\{x_2 : x_0 + 1\}$$

$$B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -1 & 0 & 0 \end{bmatrix}$$
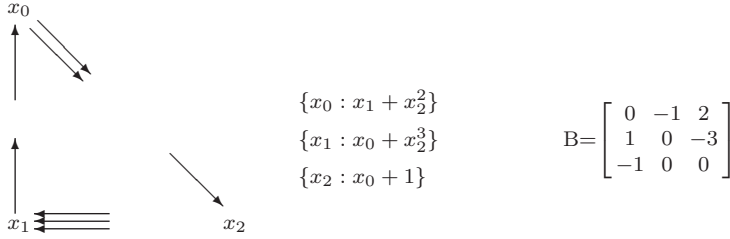
Fig. 1 Example of a Double Quiver

Definition 4.5 *We define* mutation *at a mutable vertex k of a double quiver Q with vertices $\{0, 1, \ldots, n-1\}$ to be the application of the map τ_k that takes Q to a new double quiver $\tau_k(Q)$ via the following steps:*

1. *Add a half-edge $i \to j$ attached at i, for each pair of half-edges $i \to k$ attached at i and $k \to j$ attached at k. Also add a half-edge $j \to i$ attached at i, for each pair of half-edges $j \to k$ attached at k and $k \to i$ attached at i.*

2. *Reverse the direction of half-edges between vertex k and node i, for all $i \neq k$.*

3. *Successively pick 2-cycles and remove both half-edges until no 2-cycles remain.*

The mutation of a double quiver corresponds to the mutation of the corresponding LP algebra seed. Let $(\mathbf{x}, \mathbf{P})$ be the LP algebra seed associated to the double quiver Q and $(\mathbf{x}', \mathbf{P}')$ the LP algebra seed associated to Q', the double quiver resulting from mutating Q at k. Then the intermediate polynomials P_j' are the intermediate polynomials of the seed $\mu_k(\mathbf{x}, \mathbf{P})$, where μ_k is seed mutation as defined in Section 2. We will be able to find all period 1 binomials P that satisfy some mild conditions regarding their corresponding double quiver Q. Observe that a period 1 seed whose exchange polynomials are all binomials has a corresponding period 1 double quiver Q. However, it will be easier to work with period 1 B-matrices; next, we give the corresponding definition of mutation for B-matrices.

Denote by $\mathbf{1}_S$ the indicator variable of S. Mutation at vertex k corresponds to a mutation of the B-matrix of the double quiver that maps it to $\tau_k(B) = \tilde{B} = (\tilde{b}_{i,j})_{n \times n}$, such that

$$\tilde{b}_{i,j} = \begin{cases} -b_{i,j} & i = k \ or \ j = k \\ b_{i,j} + b_{i,k} \cdot |b_{k,j}| \cdot \mathbf{1}_{\{b_{k,i} b_{k,j} < 0\}} & otherwise \end{cases} \tag{4.1}$$

Example 4.6 *If we mutate the double quiver in Figure (1) at x_0, we obtain the following double quiver*
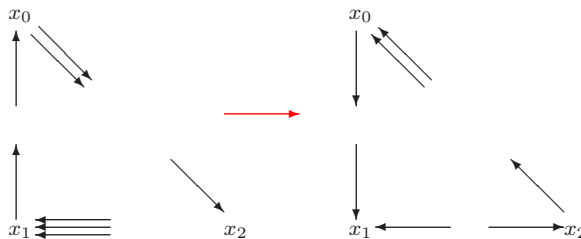
Fig. 2 Double Quiver Mutation at x_0

Remark 4.7 *Double quivers are generalizations of (normal) quivers in the following sense:*

1. *A quiver Q can be regarded as an example of a double quiver. Split each edge $i \to j$ into two half-edges. Then attach one of them to i and the other to j. The mutation rules for double quivers and for quivers agree with each other.*

2. *The cluster algebra $\mathcal{A}$ defined by any skew-symmetrizable matrix B can be realized as a double quiver. In fact, B is associated to a double quiver Q and to a seed t that gives rise to a LP algebra $\mathcal{A}(t)$ that is identical to $\mathcal{A}$ and the mutation rules agree. Furthermore, if v is a vertex in the double quiver $\tilde{Q}$, that is the result of mutating Q at v, then v is mutable in $\tilde{Q}$.*

3. *Fomin and Zelevinsky defined cluster algebras in their fundamental paper [9] by sign-skew-symmetric matrices. In this definition, it was required that any sequence of mutations yields another sign-skew-symmetric matrix. Our double quivers can be regarded as a direct generalization of cluster algebras defined by sign-skew-symmetric matrices. For one thing, we do not require the matrix B to be sign-skew-symmetric. For another, we have fewer restrictions on the mutation rules; we define mutability at a vertex, so that double quivers where some mutation sequences are invalid but others are not can still be considered.*

4.2 1 Periodicity

In this section, we examine more precisely the notion of a period 1 double quiver. We also prove a weaker version of Theorem 3.6.

Let Q be a double quiver and B be the matrix (not necessarily skew-symmetric) determined by Q. We say that Q *has period 1* if mutating at 0 and relabeling the vertices $(0, 1, 2, \ldots n-1) \to (n-1, 0, 1, \ldots n-2)$ gives back the original double quiver Q. In particular, if Q has period 1, then its vertex 0 is mutable, meaning in terms of B-matrices that $b_{k,0} \neq 0 \implies b_{0,k} \neq 0$.

Mutating at vertex 0 yields the new B-matrix $\tilde{B}$ given by:

$$\tilde{b}_{i,j} = \begin{cases} -b_{i,j} & i = 0 \text{ or } j = 0 \\ b_{i,j} + b_{i,0} \cdot |b_{0,j}| \cdot \mathbf{1}_{\{b_{0,i}b_{0,j}<0\}} & \text{otherwise} \end{cases}$$

The B-matrix of the mutated quiver $\tau(Q)$ is

$$\tau(B) = \begin{pmatrix} 0 & -b_{0,1} & -b_{0,2} & \cdots & -b_{0,n-1} \\ -b_{1,0} & 0 & b_{1,2}+\epsilon_{1,2} & \cdots & b_{1,n-1}+\epsilon_{1,n-1} \\ -b_{2,0} & b_{2,1}+\epsilon_{2,1} & 0 & \cdots & b_{2,n-1}+\epsilon_{2,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -b_{n-1,0} & b_{n-1,1}+\epsilon_{n-1,1} & b_{n-1,2}+\epsilon_{n-1,2} & \cdots & 0 \end{pmatrix}$$

where $\epsilon_{i,j} = b_{i,j} + b_{i,0} \cdot |b_{0,j}| \cdot \mathbf{1}_{\{b_{0,i}b_{0,j}<0\}}$. The double quiver Q has period 1 if $\tau(B)$ and $\mu B \mu^{-1}$ represent the same binomial seed, where μ is the permutation matrix such that $\mu B \mu^{-1}$ corresponds to the seed after the relabeling $(0, 1, 2, \ldots n - 1) \to (n - 1, 0, 1, \ldots n - 2)$,

$$\mu B \mu^{-1} = \begin{pmatrix} 0 & b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-2} \\ b_{0,n-1} & 0 & b_{0,1} & \cdots & b_{0,n-2} \\ b_{1,n-1} & b_{2,1} & 0 & \cdots & b_{2,n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-2,n-1} & b_{n-2,0} & b_{n-2,1} & \cdots & 0 \end{pmatrix}$$

Therefore Q is a period 1 double quiver if

$$\tau(B) = \mu B \mu^{-1}. \tag{4.2}$$

Equivalently, Q is a period 1 double quiver if

$$\begin{aligned} b_{n-1,i} &= -b_{0,i+1}, & 0 \le i \le n-2, & \tag{4.3} \\ b_{i,n-1} &= -b_{i+1,0}, & 0 \le i \le n-2, \text{ and} & \tag{4.4} \\ b_{i,j} &= b_{i+1,j+1} + \epsilon_{i+1,j+1}, & 0 \le i,j \le n-2. & \tag{4.5} \end{aligned}$$

Solving these equations leads to the following equations

$$\begin{aligned} -b_{i+1,0} = b_{i,n-1} &= b_{i-1,n-2} - \epsilon_{i,n-1} \\ &= b_{i-2,n-3} - \epsilon_{i-1,n-2} - \epsilon_{i,n-1} \\ &\quad\vdots \\ &= b_{0,n-i-1} - \epsilon_{1,n-i} - \epsilon_{2,n-i+1} - \ldots - \epsilon_{i,n-1}. \tag{4.6} \\ -b_{0,i+1} = b_{n-1,i} &= b_{n-2,i-1} - \epsilon_{n-1,i} \\ &= b_{n-3,i-2} - \epsilon_{n-2,i-1} - \epsilon_{n-1,i} \\ &\quad\vdots \\ &= b_{n-i-1,0} - \epsilon_{n-i,1} - \epsilon_{n-i+1,2} - \ldots - \epsilon_{n-1,i}. \tag{4.7} \end{aligned}$$

Using the same terminology as [2] for quivers, vertex i of the double quiver Q is said to be a *sink* if all the half-edges incident to i are directed inwards. A double quiver is said to be a period 1 *sink-type* double quiver if vertex 0 is sink, and the double quiver has period 1. From above, we can obtain the following theorem classifying all period 1 sink-type double quivers:

Theorem 4.8 *Let B be the matrix of a sink-type double quiver Q. Then Q is a period 1 double quiver if and only if the following conditions hold:*

1. *$b_{0,i}$ and $b_{0,n-i}$ are either both negative, or both zero, for $i = 1, 2 \ldots n - 1$.*

2. *$b_{i,0} = -b_{0,n-i}$, for $i = 1, 2, \ldots n - 1$.*

3. *$b_{i,j} = b_{0,j-i}$ if $0 < i < j \leq n - 1$ and $b_{i,j} = -b_{0,n-i+j}$ if $0 < j < i \leq n - 1$.*

Proof Since Q is of sink type, then $b_{0,i} \leq 0$ for all i. Therefore $\epsilon_{i,j} = 0$ for all $0 < i, j \leq n - 1$.

If all three conditions above are satisfied, then (4.3), (4.4) and (4.5) are trivially satisfied.

Conversely, let us assume Q has period 1, so (4.3), (4.4) and (4.5) are satisfied. From (4.5), we have

$$b_{i,j} = b_{0,j-i}, \ \ if \ 0 < i < j \leq n - 1,$$
$$b_{i,j} = b_{i-j,0}, \ \ if \ 0 < j < i \leq n - 1.$$

Combining with (4.3) and (4.4), we have,

$$b_{i,n-1} = b_{0,n-i-1} = -b_{i+1,0},$$
$$b_{n-1,i} = -b_{0,i+1} = b_{n-i-1,0}.$$

It follows that $b_{0,i+1} = 0 \implies b_{n-i-1,0} = 0$. Since 0 is a mutable vertex, we have $b_{0,n-i-1} = 0$ for all i.

The seed $t = (\mathbf{x}, \mathbf{p})$ corresponding to a sink-type double quiver Q is such that p_0 is of the form $\prod_j x_j^{a_j} + 1$. If Q has period 1, then $a_i = 0 \iff a_{n-i} = 0$ follows from (1) in the theorem above. Conversely, any polynomial of this form generates a period 1 seed as item (2) of Theorem 3.10 shows. Thus Theorem 4.8 can be restated as:

Theorem 4.9 *The only period 1 binomials such that the quiver corresponding to $(\mathbf{x}, \mathbf{P})$ is of sink-type and has period 1 are those of the form $P = x_1^{a_1} x_2^{a_2} \cdots x_{n-1}^{a_{n-1}} + 1$, where $a_i = 0 \iff a_{n-i} = 0$, for all i.*

4.3 Mutual Double Quiver

In general, given a binomial seed (all the exchange polynomials are binomials), the corresponding double quiver is not unique. For example, if we reverse all

the half-edges attached at a certain vertex, the new double quiver represents the same seed. However, there is a canonical choice, which coincides with usual quivers (if we regard a quiver as a double quiver, as it was done in (1) of Remark 4.7). From the mutability of vertex 0, if there are half-edges between 0 and i attached to i, then there are half-edges between 0 and i attached to 0. We make all these half-edges point in the same direction by reversing all the half-edges attached at i, if necessary. In terms of the B-matrix, the resulting canonical quiver is such that $b_{0,i}$ and $b_{i,0}$ are of opposite sign. A double quiver with this condition is said to be *mutual* at vertex 0. Such double quiver is said to be the *canonical double quiver* associated to the seed. In this subsection, we prove Theorem 4.10 regarding period 1 mutual (at 0) double quivers. By translating this into the language of period 1 polynomials, this is equivalent to Theorem 3.6 in Section 3.

Theorem 4.10 *Let B be the matrix associated to a canonical mutual double quiver Q. Then Q has period 1 if and only if the following conditions hold:*

1. *$b_{i,0}$ and $b_{0,i}$ are of opposite signs, or both zero, for $0 < i \leq n - 1$.*

2. *$b_{i,0} = -b_{0,n-i}$, for $i = 1, 2, \ldots n - 1$.*

3. *$b_{i,j} = -\sum_{k=0}^{i} \epsilon_{i-k,j-k} + b_{0,j-i}$ if $0 < i < j \leq n - 1$.*

4. *$b_{i,j} = -\sum_{k=0}^{j} \epsilon_{i-k,j-k} - b_{0,n-i+j}$ if $0 < j < i \leq n - 1$.*

Proof We assume Q is a canonical mutual double quiver. If all four conditions above are satisfied, then (4.3), (4.4) and (4.5) are trivially satisfied.

Conversely, let us assume Q has period 1, so (4.3), (4.4) and (4.5) are satisfied. We first prove by induction

$$b_{0,i} = -b_{n-i,0} \text{ and } b_{i,0} = -b_{0,n-i}. \tag{4.8}$$

Setting $i = 0$ in (4.6) and (4.7) gives the base cases $b_{0,1} = -b_{n-1,0}$ and $b_{1,0} = -b_{0,n-1}$. Now, assume that (4.8) holds for $i = 0, 1, 2 \ldots k$; we prove it for $i = k + 1$. Note that,

$$-b_{k+1,0} = b_{0,n-k-1} - \sum_{j=1}^{k} \epsilon_{j,n-k+j-1}$$

$$= b_{0,n-k-1} - \sum_{j=1}^{k} b_{j,0} \cdot |b_{0,n-k+j-1}| \cdot \mathbf{1}_{\{b_{0,j}b_{0,n-k+j-1}<0\}}$$

$$= b_{0,n-k-1} - \frac{1}{2} \cdot \Big(\sum_{j=1}^{k} b_{j,0} \cdot |b_{0,n-k+j-1}| \cdot \mathbf{1}_{\{b_{0,j}b_{0,n-k+j-1}<0\}} \tag{4.9}$$

$$+ \sum_{j=1}^{k} b_{k-j+1,0} \cdot |b_{0,n-j}| \cdot \mathbf{1}_{\{b_{0,k-j+1}b_{0,n-j}<0\}} \Big)$$

From the inductive hypothesis, $b_{j,0} = b_{0,n-j}$ and $b_{k-j+1,0} = b_{0,n-k+j-1}$, so,

$$b_{j,0} \cdot |b_{0,n-k+j-1}| \cdot \mathbf{1}_{\{b_{0,j}b_{0,n-k+j-1}<0\}} + b_{k-j+1,0} \cdot |b_{0,n-j}| \cdot \mathbf{1}_{\{b_{0,k-j+1}b_{0,n-j}<0\}}$$
$$= b_{0,n-j} \cdot |b_{0,n-k+j-1}| \cdot \mathbf{1}_{\{b_{0,j}b_{k-j+1,0}<0\}} + b_{0,n-k+j-1} \cdot |b_{0,n-j}| \cdot \mathbf{1}_{\{b_{0,k-j+1}b_{j,0}<0\}} \quad (4.10)$$

From the mutuality assumption, $\mathbf{1}_{\{b_{0,k-j+1}b_{j,0}<0\}} = \mathbf{1}_{\{b_{0,j}b_{k-j+1,0}<0\}}$. If both of them are 0, then (4.10) is zero. If both of them are 1, then $b_{0,n-k+j-1}$ and $b_{0,n-j}$ are of opposite sign, therefore (4.10) is zero. Substituting back into (4.9), we have $-b_{k+1,0} = b_{0,n-k-1}$. From (4.7), by a similar argument, we obtain $-b_{0,k+1} = b_{n-k-1,0}$.

Finally, conditions (3) and (4) follow from equations (4.8) and (4.3).

Remark 4.11 *Theorem 3.6 generalizes the main theorem of [2]. For example, the polynomial $P = x_1^a x_2^b + 1$, with $a \neq b$, produces Laurent phenomenon sequences; it is predicted by Theorem 3.6, but not by [2, Theorem 6.6].*

Remark 4.12 *A double quiver with a skew-symmetrizable matrix B-matrix is mutual at each vertex. Hence, restricting B to be skew-symmetric, Theorem 4.10 provides a classification of period 1 cluster algebras over the coefficient ring $\mathbb{Z}$.*

5 Classification of period 1 Seeds for Small n

In this section we prove the classifications stated in Section 3 of all period 1 seeds when $n = 2$ and $n = 3$.

5.1 Proof of Theorem 3.1

Let $P = P(x_1) = \sum_{i=0}^{d} a_i x_1^i$ be an (irreducible) polynomial of degree $d > 0$ that generates a period 1 seed. Since P is not divisible by x_1, we know $a_0 \neq 0$ and $P|_{x_1=0} = a_0 \neq 0$. From the definition of τ, P generates a period 1 seed if and only if

$$P(x_1) = a_0^{-1} x_1^d \cdot \sum_{i=0}^{d} a_i \left(\frac{a_0}{x_1}\right)^i = \sum_{i=0}^{d} (a_i a_0^{i-1}) x_1^{d-i}. \quad (5.1)$$

For each $0 \leq i \leq d$, by equating the coefficient of x_1^i on both sides of (5.1), we see that

$$a_i = a_{d-i} a_0^{d-i-1} \text{ for all } i. \quad (5.2)$$

In particular, when $i = d$, we obtain $a_d = 1$, so P has to be monic.

If $d > 2$, we have that $a_i = a_{d-i}a_0^{d-i-1} = (a_i a_0^{i-1})a_0^{d-i-1} = a_i a_0^{d-2}$ for all i. Setting $i = d$ gives that $a_0 = \pm 1$. If $a_0 = 1$ then (5.2) implies $a_i = a_{d-i}$ for all i, or equivalently, P is palindromic, and all such polynomials satisfy (5.1). If $a_0 = -1$, then (5.2) implies $a_i = (-1)^{d-i-1}a_{d-i}$ for all i. When d is odd, these relations when $i = i_0, d - i_0$ imply $a_{i_0} = 0$ for all $1 \le i_0 \le d - 1$. But then $P = x_1^d - 1$ is not irreducible. When d is even, we find that $a_i = -a_{d-i}$ for all i, or equivalently, P is antipalindromic.

If $d = 2$, (5.2) is trivially satisfied for any a_1, a_2.

If $d = 1$, (5.2) with $i = 0$ gives $a_0 = a_1 = 1$.

5.2 Proof of Theorem 3.4

5.2.1 Bounding Degrees

Assume the following is a period 1 seed

$$\{x_0, P(x_1, x_2)\},$$
$$\{x_1, Q(x_0, x_2)\},$$
$$\{x_2, P(x_0, x_1)\},$$

where P is a two-variable irreducible polynomial not divisible by x_1 or x_2. By Proposition 2.12, $Q(x_0, x_2)$ is also irreducible and not divisible by x_0 or x_2. Then $P(x_0, 0)$ and $P(0, x_1)$ are not 0. It is not hard to see that there do not exist period 1 polynomials P that do not depend on x_0 or x_1. Thus assume that P depends on both of these variables. Let m be the degree of x_0 in $P(x_0, x_1)$; we can write

$$P(x_0, x_1) = \sum_{k=0}^{m} f_k(x_1) x_0^k, \tag{5.3}$$

where the f_k are single variable polynomials for $k = 0, 1, \ldots m$. Let $\tilde{Q}$ be the intermediate polynomial G at step (2) of applying τ to Q. Then

$$\tilde{Q}(x_0, x_2) = P\left(\frac{P(x_0, 0)}{x_2}, x_0\right) = \sum_{k=0}^{m} f_k(x_0) \cdot \frac{P^k(x_0, 0)}{x_2^k}.$$

Let $d(x_0)$ be the maximal factor of $\tilde{Q}(x_0, x_2)$, which is in the form $x_0^{s_1} \cdot p(x_0)$, where $f(x_0)$ is a factor of $P(x_0, 0)^K$ for some K. From the rules for computing τ, we have

$$Q(x_0, x_2) = \sum_{k=0}^{m} \frac{f_k(x_0)P^k(x_0, 0)}{d(x_0)} x_2^{m-k}. \tag{5.4}$$

In view of (5.4), the coefficient of x_2^m in Q is

$$\frac{f_0(x_0)}{d(x_0)} = \frac{P(0, x_0)}{d(x_0)},$$

which is a non-vanishing polynomial. Therefore $d(x_0)$ divides $f_0(x_0) = P(x_0, 0)$, and x_2 is of degree m in $Q(x_0, x_2)$.

We can similarly obtain $P(x_1, x_2)$ from $Q(x_0, x_2)$. Let n be the degree of x_0 in $Q(x_0, x_2)$; we can write

$$P(x_1, x_2) = \sum_{k=0}^{n} \frac{g_k(x_1) P^k(0, x_1)}{t(x_1)} x_2^{n-k}. \qquad (5.5)$$

The coefficient of x_2^n in P is

$$\frac{g_0(x_1)}{t(x_1)} = \frac{P(x_1, 0)}{t(x_1)},$$

which is a nonzero polynomial. Therefore $t(x_1)$ divides $g_0(x_1) = Q(0, x_1)$ and the degree of x_2 in $P(x_1, x_2)$ is n.

If we let $x_2 = 0$ in (5.4), then

$$Q(x_0, 0) = \frac{f_m(x_0) P^m(x_0, 0)}{d(x_0)} \qquad (5.6)$$

and similarly

$$P(x_0, 0) = \frac{g_n(x_0) P^n(0, x_0)}{t(x_0)}. \qquad (5.7)$$

Comparing the degree of both sides of (5.6) and (5.7) and recalling the divisibility relations $d(x_0) \mid P(0, x_0)$, $t(x_0) \mid Q(0, x_0)$, we arrive at the inequalities

$$\deg f_m(x_0) + m \deg P(x_0, 0) = \deg Q(x_0, 0) + \deg d(x_0) \le \deg Q(x_0, 0) + \deg P(0, x_0) \qquad (5.8)$$

$$\deg g_n(x_0) + n \deg P(0, x_0) = \deg P(x_0, 0) + \deg t(x_0) \le \deg P(x_0, 0) + \deg Q(0, x_0) \qquad (5.9)$$

Summing (5.8) and (5.9), and noticing that $\deg Q(0, x_0) \le m$, $\deg Q(x_0, 0) \le n$, we obtain the following inequality:

$$2 \ge \deg g_n(x_0) + \deg f_m(x_0) + (m-1)\big(\deg P(x_0, 0) - 1\big) + (n-1)\big(\deg P(0, x_0) - 1\big). \qquad (5.10)$$

From this inequality, the classification of period 1 polynomials is decomposed into the following five cases:

1. $\deg P(x_0, 0) = 2$ and $\deg P(0, x_0) = 1$

2. $\deg P(x_0, 0) = 1$ and $\deg P(0, x_0) = 2$

3. $\deg P(x_0, 0) = \deg P(0, x_0) = 1$

4. $\deg P(x_0, 0) = \deg P(0, x_0) = 2$

5. Either $\deg P(x_0, 0) = 0$ or $\deg P(0, x_0) = 0$

Lemma 5.1 *If $P(0, 0) \neq 0$, then the bound (5.9) can be refined to*

$$\deg g_n(x_0) + n \deg P(0, x_0) = \deg P(x_0, 0) + \deg t(x_0) \leq 2 \deg P(x_0, 0). \tag{5.11}$$

Proof Since $d(x_0) \mid P(0, x_0)$ and $x_0 \nmid P(0, x_0)$, we see that $x_0 \nmid d(x_0)$.

We moreover claim that $x_1 \nmid t(x_1)$. Assume otherwise that $x_1 \mid t(x_1)$. From (5.5), $t(x_1)$ divides $g_k(x_1) P^k(0, x_1)$ for all $k = 0, 1, 2 \ldots n$ and $x_1 \nmid P(0, x_1)$. Therefore $x_1 \mid g_k(x_1)$ for all $k = 0, 1, \ldots n$. This implies that $x_1 \mid Q(x_0, x_1)$, which contradicts the irreducibility of $Q(x_0, x_1)$.

Since $P(x_0, 0) = \sum_{k=0}^{m} f_k(0) x_0^k$, then $f_k(0) = 0$ if $k > \deg p(x_0, 0)$. Hence

$$t(x_1) \mid Q(0, x_1) = \sum_{k=0}^{m} \frac{f_k(0) P^k(0, 0)}{d(0)} x_1^{m-k}$$

$$= \left(\sum_{k=0}^{\deg P(x_0, 0)} \frac{f_k(0) P^k(0, 0)}{d(0)} x_1^{\deg P(x_0, 0) - k} \right) \cdot x_1^{m - \deg P(x_0, 0)}.$$

Therefore $\deg t(x_1) \leq \deg P(x_0, 0)$ and (5.9) leads to the desired inequality

$$\deg g_n(x_0) + n \deg P(0, x_0) = \deg P(x_0, 0) + \deg t(x_0) \leq 2 \deg P(x_0, 0).$$

Lemma 5.2 *If $f_m(x_1)$ has a nonzero constant term, then $m = \deg P(x_0, 0)$.*

Proof In view of (5.5), $P(x_0, x_1)$ contains the term $f_m(x_1) x_0^m$. Since $f_m(x_1)$ contains a nonzero constant term, then $f_m(0) \neq 0$. Thus, $P(x_0, 0)$ contains the term $f_m(0) x_0^m$. This implies $\deg P(x_0, 0) \geq m$ and since m is the degree of x_0 in $P(x_0, x_1)$, we have $\deg P(x_0, 0) \leq m$. Therefore $m = \deg P(x_0, 0)$.

5.2.2 Analysis of Cases

Case 1: $\deg P(x_0, 0) = 2$ and $\deg P(0, x_0) = 1$.

From (5.10), we either have $m = 2$ or $m = 3$.

If $m = 3$, we obtain $\deg f_3(x_1) = 0$ from (5.10). However, Lemma 5.2 says that $P(x_0, 0) = 3$, thus implying $f_3(x_1) \neq 0$, a contradiction.

If $m = 2$, (5.8) tells us that $\deg f_2(x_0) + 4 = degQ(x_0, 0) + \deg d(x_0) \leq n + 1$, thus $n \geq 3$. From (5.3), we have

$$P(x_0, x_1) = f_2(x_1)x_0^2 + f_1(x_1)x_0 + f_0(x_1).$$

Since the degree of x_1 in $P(x_0, x_1)$ is $n \geq 3$, $\deg f_2(x_1) \leq 1$ and $\deg f_0(x_1) = 2$, we must have $\deg f_1(x_1) = n$. In view of (5.5),

$$Q(x_0, x_2) = \frac{f_0(x_0)}{d(x_0)}x_2^2 + \frac{f_1(x_0)P(x_0, 0)}{d(x_0)}x_2 + \frac{f_2(x_0)P^2(x_0, 0)}{d(x_0)}.$$

As the degree of x_0 in $Q(x_0, x_2)$ is n, we have

$$n \geq \deg_{x_0}\left(\frac{f_1(x_0)P(x_0, 0)}{d(x_0)}\right)$$

Then

$$n \geq \deg f_1(x_0) + \deg P(x_0, 0) - \deg d(x_0) \geq n + 2 - 1 = n + 1.$$

Therefore there are no period 1 polynomials in this case.

Case 2: $\deg P(x_0, 0) = 1$ and $\deg P(0, x_0) = 2$.

From (5.10), we either have $n = 2$ or $n = 3$.

If $n = 3$, we obtain $\deg f_m(x_1) = 0$ from (5.10). From Lemma (5.2), we have $m = \deg P(x_0, 0) = 1$. Hence in (5.3),

$$P(x_0, x_1) = f_1(x_1)x_0 + f_0(x_1).$$

However, $\deg f_1(x_1) = 0$ and $\deg f_0(x_1) = 2$, so the degree of x_1 in $P(x_0, x_1)$ is $2 \neq n$, a contradiction.

If $n = 2$, inequalities (5.8) and (5.9) yield

$$\deg f_m(x_0) + m \leq 4,$$
$$\deg g_n(x_0) + 3 \leq m.$$

Thus $m \geq 3 > 1 = \deg P(x_0, 0)$. From Lemma (5.2), we have $\deg f_m(x_1) \geq 1$. These inequalities yield $m = 3$, $\deg f_3(x_1) = 1$ and $\deg g_n(x_0) = 0$. Moreover, from (5.9), $\deg t(x_0) = 3$. Since these values do not satisfy (5.11), Lemma (5.1) tells us that $P(0, 0) = 0$, i.e., $P(x_0, x_1)$ does not have a constant term. Taking $m = 3$ in (5.4) yields

$$Q(x_0, x_2) = \frac{f_0(x_0)}{d(x_0)}x_2^3 + \frac{f_1(x_0)P(x_0, 0)}{d(x_0)}x_2^2 + \frac{f_2(x_0)P^2(x_0, 0)}{d(x_0)}x_2^2 + \frac{f_3(x_0)P^3(x_0, 0)}{d(x_0)}.$$

From (5.8) and the values already found, we also have $\deg d(x_0) = 2$. Since $f_3(x_0)$ and $P(x_0, 0)$ are both linear polynomials without constant terms, $f_3(x_0)P^3(x_0, 0) = ax_0^4$ for some $a \in \mathbb{Z}$. Since d has degree 2 and divides $(x_0) \mid f_3(x_0)P^3(x_0, 0)$, then $d(x_0) = bx_0^2$ for some

$b \in \mathbb{Z}, b \neq 0$. From $d(x_0) \mid f_0(x_0)$ and $d(x_0) \mid f_1(x_0)P(x_0, 0)$, we have $x_0 \mid f_0(x_0)$ and $x_0 \mid f_1(x_0)$. From equation (5.3), we have

$$P(x_0, 0) = f_3(0)x_0^3 + f_2(0)x_0^2 + f_1(0)x_0 + f_0(0) = f_3(0)x_0^3 + f_2(0)x_0^2.$$

Since in this case, $\deg P(x_0, 0) = 1$, then $P(x_0, 0) = 0$. This contradicts the fact that $P(x_0, x_1)$ is not divisible by x_1.

Case 3: $\deg P(x_0, 0) = \deg P(0, x_0) = 1$.

If $P(x_0, x_1)$ contains no constant term, then we can write

$$P(x_1, x_2) = ax_1 + bx_2 + x_1 x_2 R(x_1, x_2),$$

where $a, b \in \mathbb{Z}$ are both nonzero and $R(x_1, x_2)$ is a polynomial of degree $m-1$ in x_1 and degree $n-1$ in x_2. We next obtain $\tau_P(P(x_0, x_1))$ by replacing x_0 with $\frac{ax_1}{x_3}$, downshifting and then multiplying by a monomial M. This monomial has to be such that the resulting Q is a Laurent polynomial, not divisible by any x_i and its coefficients have greatest common divisor 1. Assume $\tilde{M}$ is M, but with coefficient 1 and let $\tilde{Q}$ be the resulting polynomial. Thus $\tilde{Q} = cQ$ for some constant c. We can write $\tilde{Q}$ as

$$\tilde{Q}(x_0, x_2) = a^2 x_2^{m-1} + bx_2^m + ax_0 R(\frac{ax_0}{x_2}, x_0)x_2^{m-1}$$

From this polynomial, we analogously obtain $\tilde{P}$ by omitting a constant factor for the *adjusting monomial*

$$\tilde{P}(x_1, x_2) = a^2 x_2^{k+1} + bx_1 x_2^{k+1} + abx_1 R(\frac{ab}{x_2}, \frac{bx_1}{x_2})x_2^k,$$

where k is the least integer for which $abx_1 R(\frac{ab}{x_2}, \frac{bx_1}{x_2})x_2^k$ is a polynomial. In view of the above equation, $k+1 = \deg \tilde{P}(0, x_2) = P(0, x_2) = 1$. Therefore $k = 0$ and so $R(x_0, x_1)$ is a contant; write $R = R(x_0, x_1)$. Then $\tilde{P}(x_1, x_2) = abRx_1 + a^2 x_2 + bx_1 x_2$ must be equal to $\lambda P(x_1, x_2) = \lambda ax_1 + \lambda bx_2 + \lambda Rx_1 x_2$, where λ is a nonzero integer. After equating coefficients, we obtain the polynomials in items (1) and (2) of Theorem 3.4 whose generated seeds are

$$\{x_0, ax_1 + ax_2 + x_1 x_2\}, \{x_1, a + x_0 + x_2\}, \{x_2, ax_0 + ax_1 + x_0 x_1\}$$

and

$$\{x_0, ax_1 - ax_2 + x_1 x_2\}, \{x_1, a + x_0 - x_2\}, \{x_2, ax_0 - ax_1 + x_0 x_1\}.$$

If $P(x_0, x_1)$ has a nonzero constant term, we can write

$$P(x_0, x_1) = c + ax_0 + bx_1 + x_0 x_1 R(x_0, x_1).$$

From Lemma 5.1, we have $n \leq 2$. Substituting into (5.8),

$$\deg f_m(x_0) + m \leq \deg d(x_0) + 2.$$

If $a \neq b$, then $\gcd(P(x_0, 0), P(0, x_0)) = 1$, so $\deg d(x_0) = 0$. We can then write constant $d(x_0)$ as d. By the same argument as in Lemma 5.2, we have $m = 1$. Then (5.4) reads

$$Q(x_0, x_2) = \frac{f_1(x_0)P(x_0, 0)}{d} + \frac{f_0(x_0)}{d}x_2,$$

and since the degree of x_0 in $Q(x_0, x_2)$ is $n \leq 2$, we must have $\deg f_1(x_0) \leq 1$. Then we can write $P(x_0, x_1) = c + ax_0 + bx_1 + Rx_0x_1$ for some $R \in \mathbb{Z}$ and where $a \neq b$. By the same argument as above, we obtain the period 1 polynomials in items (3) and (4) of Theorem 3.4 whose generated seeds are

$$\{x_0, x_1 - x_2 - 1\}, \{x_1, -x_0x_2 + x_0 - x_2 - 1\}, \{x_2, x_0 - x_1 - 1\}$$

and

$$\{x_0, -x_1 + x_2 - 1\}, \{x_1, x_0x_2 + x_0 - x_2 + 1\}, \{x_2, -x_0 + x_1 - 1\}.$$

If $a = b$ then $m = 1$ or 2. If $m = 1$, from (5.5) and (5.4),

$$Q(x_0, x_2) = f_1(x_0) + x_2, \qquad P(x_1, x_2) = \frac{x_2^n}{t(x_1)}\left(f_1(\frac{f_0(x_1)}{x_2}) + x_1\right).$$

Since $f_1(x_0)$ has constant term a, $P(x_1, x_2)$ contains the term x_2^n. Therefore $n = 1 = \deg f_1(x_0)$. We can then write $P(x_0, x_1) = c + ax_0 + ax_1 + Rx_0x_1$. The same argument as above yields the period 1 polynomials in (5) of Theorem 3.4, whose associated seed is

$$\{x_0, x_1x_2 + ax_1 + ax_2 + c\},$$
$$\{x_1, x_0 + x_2 + a\},$$
$$\{x_2, x_0x_1 + ax_0 + ax_1 + c\}.$$

If $m = 2$, then $m \neq \deg P(x_0, 0)$. From Lemma 5.1, f_m does not contain constant term, so $\deg f_m \geq 1$. Since $\deg d \leq \deg P(0, x_0) = 1$, plug them into (5.8), we get $n = 2$ and $\deg f_m = 1$. Morover, from Lemma 5.2, $\deg t \leq 1$. Plug them into (5.9), we get $\deg g_n = 0$. From (5.3)

$$P(x_0, x_1) = f_2(x_1)x_0^2 + f_1(x_1)x_0 + f_0(x_1),$$

and the degree of x_1 is 2, it must be that $\deg f_1 = 2$. Mutating at x_0 to obtain $Q(x_0, x_2)$ gives (notice $P(x_0, 0) = P(0, x_0)$),

$$Q(x_0, x_2) = f_2(x_0)P(x_2, 0) + f_1(x_0)x_2 + x_2^2.$$

This contains the term $x_0^2x_2$, which contradicts that $\deg g_2 = 0$.

Case 4: $\deg P(x_0, 0) = \deg P(0, x_0) = 2$.

From (5.11), we see that $m = n = \deg P(x_0, 0) = \deg P(0, x_0) = 2$ and $\deg g_n(x_0) = \deg f_m(x_0) = 0$. Hence,

$$P(x_0, x_1) = f_2(x_1)x_0^2 + f_1(x_1)x_0 + f_0(x_1),$$
$$Q(x_0, x_2) = \frac{f_2(x_0)P^2(x_0, 0)}{d(x_0)} + \frac{f_1(x_0)P(x_0, 0)}{d(x_0)}x_2 + \frac{f_0(x_0)}{d(x_0)}x_2^2. \quad (5.12)$$

The first equation gives $\deg f_0(x_1) = \deg P(0, x_1) = 2$. From the second, by looking at the coefficient of x_2^2, we have $\deg d(x_0) \leq \deg f_0(x_0) = 2$. Moreover, remember we had $Q(x_0, x_2) = g_2(x_2)x_0^2 + g_1(x_2)x_0 + g_0(x_2)$. Since $\deg g_2(x_0) = 0$, Q does not contain terms divisible by $x_2 x_0^2$. By looking at the coefficient of x_2 in Q in equation (5.12), we have that $\deg f_1(x_0) + \deg P(x_0, 0) - \deg d(x_0) \leq 1$, from which $\deg f_1(x_0) \leq 1$. Since $\deg f_2(x_0) = 0$, from (5.12) we have that $d(x_0) \mid P^2(x_0, 0)$.

If $d(x_0) \nmid P(x_0, 0)$, then $d(x_0) = cr^2(x_0)$ for some monic linear factor $r(x_0)$ and constant $c \in \mathbb{Z}$. We will omit the constant c as it will factor later, so simply write $d(x_0) = r^2(x_0)$. In particular, we have $\deg d(x_0) = 2$, from which $\deg f_0(x_0) = 2$. Moreover, since $d(x_0) \mid f_1(x_0)P(x_0, 0)$, then $\deg f_1(x_0) = 1$. From the divisibility relations, we can write

$$P(x_0, 0) = t(x_0)r(x_0), \qquad f_0(x_0) = Ar^2(x_0),$$
$$f_1(x_0) = Br(x_0), \qquad f_2(x_0) = C,$$

for some polynomial t and constants $A, B, C \in \mathbb{Z}$. Expression (5.12) can then be simplified:

$$Q(x_0, x_2) = Ax_2^2 + Bt(x_0)x_2 + Ct^2(x_0)$$

We also have

$$Ar^2(x_2) = P(0, x_2) = Ax_2^2 + Bt(0)x_2 + Ct^2(0).$$

From both equations, we have

$$t^2(0)Q(x_0, x_2) = Ar^2\left(\frac{x_2 t(0)}{t(x_0)}\right)t^2(x_0).$$

Since $Q(x_0, x_2)$ is irreducible, then $t(0) = 0$. From above, we have $Ar^2(x_2) = Ax_2^2$ and so $d(x_0) = r^2(x_0) = x_0^2$. Moreover, $t(0) = 0$ implies $x_0 \mid t(x_0)$ and so $d(x_0) = x_0^2 \mid t(x_0)r(x_0) = P(x_0, 0)$. This is a contradiction with our initial assumption.

Now assume $d(x_0) \mid P(x_0, 0)$.

If $P(x_0, x_1)$ has a nonzero constant term, then since $d(x_0) \mid P(x_0, 0)$ and $d(x_0) \mid P(0, x_0)$, we see $P(x_0, 0) = P(0, x_0)$. $P(x_0, x_1)$ must be

of the form $P(x_0, x_1) = ax_0^2 + ax_1^2 + bx_0x_1 + cx_0 + cx_1 + d$. In this
case, we obtain the period 1 polynomials in item (6) of Theorem 3.4,
whose generated seeds are

$$\{x_0, x_1^2 + x_2^2 + ax_1x_2 + bx_1 + bx_2 + c\},$$
$$\{x_1, x_0^2 + x_2^2 + ax_0x_2 + bx_0 + bx_2 + c\},$$
$$\{x_2, x_0^2 + x_1^2 + ax_0x_1 + bx_0 + bx_1 + c\},$$

and

$$\{x_0, -x_1^2 - x_2^2 + ax_1x_2 + c\}, \{x_1, x_0^2 + x_2^2 + ax_0x_2 - c\}, \{x_2, -x_0^2 - x_1^2 + ax_0x_1 + c\}.$$

If $P(x_0, x_1)$ does not have a constant term, then $d(x_0) = ax_0^2 + bx_0$ and
$P(x_0, x_1)$ is of form $P(x_0, x_1) = d_1(ax_0^2 + bx_0) + d_2(ax_1^2 + bx_1) + cx_0x_1$.
In this case, we obtain a special case of (6) and the general polynomial
in (7) of Theorem 3.4. Their generated seeds are

$$\{x_0, x_1^2 + x_2^2 + ax_1x_2 + bx_1 + bx_2\},$$
$$\{x_1, x_0^2 + x_2^2 + ax_0x_2 + bx_0 + bx_2\},$$
$$\{x_2, x_0^2 + x_1^2 + ax_0x_1 + bx_0 + bx_1\}$$

and

$$\{x_0, -x_1^2 - x_2^2 + ax_1x_2\}, \{x_1, x_0^2 + x_2^2 - ax_0x_2\}, \{x_2, -x_0^2 - x_1^2 + ax_0x_1\}.$$

Case 5: Either $\deg P(x_0, 0) = 0$ or $\deg P(0, x_0) = 0$.

From Lemma (5.1), $\deg P(x_0, 0) = 0 \implies \deg P(0, x_0) = 0$. Thus,
we only consider the case where $\deg P(0, x_0) = 0$, i.e., $P(0, x_0)$ is a
nonzero constant $a \in \mathbb{Z}$. Observe that $d(x_0) \mid P(0, x_0)$ implies that
$d(x_0)$ is a constant d. Equation (5.5) can then be simplified to

$$P(x_1, x_2) = \sum_{k=0}^{n} \frac{g_k(x_1)a^k}{t(x_1)} x_2^{n-k}, \tag{5.13}$$

from which $t(x_1) \mid g_k(x_1)$ for all k. Therefore $t(x_1) \mid Q(x_0, x_1)$. Since
$Q(x_0, x_1)$ is irreducible, $t(x_1)$ is a constant t. Equation (5.13) with
$x_2 = 0$ and $x_1 = 0$ yield

$$P(0, x_2) = a = \sum_{k=0}^{n} \frac{g_k(0)a^k}{t(x_1)} x_2^{n-k},$$
$$P(x_1, 0) = \frac{a^n g_n(x_1)}{t} = \frac{a g_n(x_1)}{g_n(0)}.$$

From the first one, we have $t = g_n(0)a^{n-1}$ and $x_1 \mid g_k(x_1)$ for $0 \leq k \leq n - 1$. Now mutating $P(x_1, x_2)$ at x_0 gives,

$$
\begin{aligned}
Q(x_0, x_2) &= \frac{x_2^m}{d} P\left(\frac{P(x_0, 0)}{x_2}, x_0\right) \\
&= \frac{x_2^m}{td} \sum_{k=0}^{n} g_k\left(\frac{ag_n(x_0)}{g_n(0)x_2}\right) a^k x_0^{n-k} \qquad (5.14) \\
&= \sum_{k=0}^{n} g_k(x_2) x_0^k \qquad (5.15) \\
&= \sum_{l=0}^{m} h_k(x_0) x_2^l.
\end{aligned}
$$

Next we compute h_m. Since $x_2 \mid g_k(x_2)$ for $k = 0, 1, 2 \ldots, n - 1$, in (5.14), only the term $\frac{x_2^m}{td} g_n\left(\frac{ag_n(x_0)}{g_n(0)x_2}\right) a^n$ contains term x_2^m. Indeed,

$$
h_m(x_0) x_2^m = \frac{g_n(0)a^n}{td} x_2^m.
$$

Therefore $h_m(x_0)$ is a constant, so only $g_0(x_2)$ contains the term x_2^m, say $g_0(x_2) = bx_2^m + \ldots$. Since $\deg g_k < m$ for $k = 1, 2, \ldots, n - 1$,

$$
x_2 \mid \frac{x_2^m}{td} g_k\left(\frac{ag_n(x_0)}{g_n(0)x_2}\right) a^k x_0^{n-k}, \quad k = 1, 2, \ldots n - 1.
$$

Setting $x_2 = 0$, the above expressions all vanish, so

$$
Q(x_0, 0) = h_0(x_0) = \frac{b}{td}\left(\frac{ag_n(x_0)}{g_n(0)}\right)^m x_0^n.
$$

Since $\deg Q(x_0, 0) \leq n$, $\deg g_n(x_0) = 0$ (or $m = 0$, then $P(x_0, x_1)$ only depends on x_0). (5.14) is simplified as

$$
\frac{x_2^m}{td} \sum_{k=0}^{n} g_k\left(\frac{a}{x_2}\right) a^k x_0^{n-k} = \sum_{k=0}^{n} g_{n-k}(x_2) x_0^{n-k},
$$

Comparing coefficients on both sides, we see,

$$
\frac{x_2^m}{td} g_k\left(\frac{a}{x_2}\right) a^k = g_{n-k}(x_2), \quad k = 0, 1, 2 \ldots n.
$$

Taking $k = 0$ in above equation, and noticing that $t = g_n(0)a^{n-1}$, we have $n = 1$ or $a = \pm 1$. We thus obtain the period 1 polynomials in items (8), (9) and (10) of Theorem 3.4. Their generated seeds are

$$
\{x_0, \pm x_1 x_2 + a\}, \{x_1, \pm x_0 + x_2\}, \{x_2, \pm x_0 x_1 + a\}
$$

and

$$\{x_0, 1 + x_1^m x_2^n + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}(x_1^i x_2^j + x_1^{m-i} x_2^{n-j})\},$$

$$\{x_1, x_0^n + x_2^m + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}(x_0^j x_2^{m-i} + x_0^{n-j} x_2^i)\},$$

$$\{x_2, 1 + x_0^m x_1^n + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}(x_0^i x_1^j + x_0^{m-i} x_1^{n-j})\}.$$

When $m \equiv n \mod 2$,

$$\{x_0, -1 + (-1)^{m+1} x_1^m x_2^n + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}(x_1^i x_2^j + (-1)^{m+j+i} x_1^{m-i} x_2^{n-j})\},$$

$$\{x_1, -x_0^n - x_2^m + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}((-1)^j x_0^j x_2^{m-i} + (-1)^i x_0^{n-j} x_2^i)\},$$

$$\{x_2, -1 + (-1)^{m+1} x_0^m x_1^n + \sum_{\substack{0<i<m \\ 0<j<n}} C_{i,j}(x_0^i x_1^j + (-1)^{m+j+i} x_0^{m-i} x_1^{n-j})\}.$$

6 Examples of period 1 polynomials and seeds

In this section, we prove that several families of polynomials $P = P(x_1, \ldots, x_{n-1})$ are 1 periodic. In our first subsection, we prove the Expansion and Reflection Lemmas (Lemmas 3.15 and 3.16), which can be applied to period 1 polynomials to generate more period 1 polynomials. In the second subsection, we prove Theorem 3.10.

The proof that P is a period 1 polynomial for each item in Theorem 3.10 will simply consist of writing down the intermediate polynomials P_i. In general, it is easy to verify that $P_{i-1} = \tau_P(P_i)$ for all i (and $P_{n-1} = \tau_P(P_0)$), showing the seed is a period 1 seed.

The importance of period 1 polynomials stems from Proposition 2.13 that says that if $\widehat{P}_0 = P_0$, then P generates a Laurent phenomenon sequence. Lemma 2.15 gives sufficient conditions for $\widehat{P}_0 = P_0$ to be satisfied. In most of the seeds given below, the reader can easily verify that the intermediate polynomials P_i that depend on x_0 are the ones for which $P = P_0$ depends on x_i, and so condition (1) of Theorem 2.15 is satisfied. The only exceptions will be the families in Subsections 6.2.10 and 6.2.11, but these families satisfy condition (2) of Lemma 2.15 instead. Hence, the truth of Corollary 3.13 will follow from the seeds for the polynomials in Theorem 3.10 that we give below.

6.1 Proofs of the Expansion and Reflection Lemmas

6.1.1 Proof of Lemma 3.15

Let $t = (\mathbf{x}, \mathbf{F})$ be the period 1 seed generated by F and let $k \in \mathbb{N}$ be any positive integer. We prove that $G(x_1, x_2, \ldots, x_{nk-1}) = F(x_k, x_{2k}, \ldots, x_{(n-1)k})$ generates a period 1 seed. Let $\mathbf{G} = (G_1, \ldots, G_{kn})$, where $G_{k(i-1)+j}(x_1, \ldots, \widehat{x}_{k(i-1)+j}, \ldots, x_{kn}) = F_i(x_j, \ldots, \widehat{x}_{k(i-1)+j}, \ldots, x_{k(n-1)+j})$ for all $1 \leq i \leq n$, $1 \leq j \leq k$. It is clear that $t' = (\mathbf{y}, \mathbf{G})$ is a seed and $G_{kn} = G$. It will then suffice to show that $t' = (\mathbf{y}, \mathbf{G})$ has period 1. Observe that $G_1(x_2, \ldots, x_{kn}) = F_1(x_{k+1}, \ldots, x_{k(n-1)+1})$ and $G_{kn}(x_1, \ldots, x_{kn-1}) = F_n(x_k, \ldots, x_{k(n-1)})$, so G_{kn} is the downshift of G_1. We need to verify $G_s = \tau_{x_1, G_1}(G_{s+1})$ for all $1 \leq s \leq nk-1$.

If $s \neq 0 \pmod k$, then $s = k(i-1) + j$ for some i and $1 \leq j < k$. In this case, observe that G_s is the downshift of G_{s+1} by definition of $\mathbf{G}$. Moreover, the polynomial G_{s+1} (and also) only depends on the variables $x_{j+1}, \ldots, \widehat{x}_{k(i-1)+j+1}, \ldots, x_{k(n-1)+j+1}$, and in particular, not x_1. Hence, $\tau(G_{s+1})$ is the downshift of G_{s+1}, which is G_s as remarked above.

If $s = 0 \pmod k$, then $s = k(i-1) + k$ for some positive integer i, and so $s + 1 = ki + 1$. In this case, $G_s = F_i(x_k, \ldots, \widehat{x}_s, \ldots, x_{k(n-1)+1})$ and $G_{s+1} = F_{i+1}(x_1, \ldots, \widehat{x}_{s+1}, \ldots, x_{k(n-1)+1})$. Since $(\mathbf{x}, \mathbf{F})$ is a period 1 seed, we have that $\tau_{x_1, F_1}(F_i) = F_{i-1}$. Hence, $\tau_{x_1, G_1}(G_{s+1}) = F_{i-1}(x_k, \ldots, \widehat{x}_s, \ldots, x_n) = G_s$.

6.1.2 Proof of Lemma 3.16

Let $(\mathbf{x}, \mathbf{F})$ be the period 1 seed whose intermediate polynomials are F_i, $0 \leq i \leq n - 1$. Define $G_i(x_0, \ldots, \widehat{x}_i, \ldots, x_{n-1}) = F_{n-i-1}(x_{n-1}, \ldots, \widehat{x}_i, \ldots, x_1)$ for all i, and $\mathbf{G} = (G_1, \ldots, G_n)$. We show that $(\mathbf{x}, \mathbf{G})$ is also a period 1 seed generated by G.

From the relation $\kappa_F(F_{n-i}) = F_{n-i+1}$, we have that replacing x_n with $\dfrac{F_{n-1}|_{x_{n-i}=0}}{x_{-1}}$ in F_{n-i} and upshifting yields

$$F_{n-i}\left(x_1, \ldots, \widehat{x}_{n-i+1}, \ldots, x_{n-1}, \frac{F_{n-1}(x_1, \ldots, x_{n-1})|_{x_{n-i+1}=0}}{x_0}\right). \qquad (6.1)$$

Then, F_{n-i+1} comes from dividing (6.1) by the largest power of $F_{n-1}(x_1, \ldots, x_{n-1})|_{x_{n-i+1}=0}$ that divides it, and adjusting by a monomial factor.

We show that $(\mathbf{x}, \mathbf{G})$ is a period 1 seed by verifying that $\tau_G(G_i) = G_{i-1}$ for all i. Similar to before, replacing x_0 with $\dfrac{G_0|_{x_i=0}}{x_n}$ in G_i and downshifting

yields

$$G_i\Big(\frac{G_0(x_0, x_1, \ldots, x_{n-2})|_{x_{i-1}=0}}{x_{n-1}}, x_0, x_1, \ldots, \widehat{x}_{i-1}, \ldots, x_{n-2}\Big)$$
$$= F_{n-i-1}(x_{n-2}, \ldots, \widehat{x}_{i-1}, \ldots, x_1, x_0, \frac{F_{n-1}(x_{n-2}, \ldots, x_0)|_{x_{i-1}=0}}{x_{n-1}}). \qquad (6.2)$$

Then, $\tau_G(G_i)$ comes from dividing (6.2) by the largest power of $G_0(x_0, \ldots, x_{n-2})|_{x_{i-1}=0}$ that divides it, and adjusting by a monomial factor.

Notice that replacing x_j by x_{n-j-1} for all j in (6.1) gives (6.2). Therefore,

$$\tau_G(G_i) = F_{n-i+1}(x_{n-1}, \ldots, \widehat{x}_{i-1}, \ldots, x_1) = G_{i-1}(x_0, \ldots, \widehat{x}_{i-1}, \ldots, x_{n-1}) = G_{i-1}.$$

6.2 Period 1 polynomials and their generated seeds

As remarked at the beginning of the section, we will show the seeds generated by the polynomials P in Theorem 3.10. In all cases, we obviously have $P_0 = P$ and P_{n-1} be the downshift of P, so it will suffice to show the intermediate polynomials P_i for $0 < i < n - 1$.

6.2.1 Gale-Robinson seed.

We begin with the *Gale-Robinson polynomial* of the form (1) in Theorem 3.10. There are many cases to consider when writing the seed for P. As the Laurent property for this polynomial is already well-known, we only write the seed in the case that $r < n/2$ (the other cases $q < n/2 < r$, $p < n/2 < r$, $n/2 < p$ and where there are some equalities among some of these quantities, are similar).

We first give the intermediate polynomials P_i for $i \in \{p, q, r, n-p, n-q, n-r\}$:

- $P_p = ABx_q x_{2p} x_{p+r} + ACx_r x_{2p} x_{p+q} + Cx_0 x_{p+r} x_{n+p-r} + Bx_0 x_{p+q} x_{n+p-q}.$

- $P_q = ABx_{q-p} x_p x_{2q} x_{q+r} + ABx_0 x_{2q-p} x_{p+q} x_{q+r} + ACx_0 x_{r+q-p} x_{p+q} x_{2q} + BCx_{q-p} x_r x_{p+q} x_{2q} + Cx_0 x_{q-p} x_{q+r} x_{n+q-r}.$

- $P_r = ABx_0 x_{r-p} x_{p+r-q} x_{q+r} x_{2r} + ACx_{r-p} x_p x_{r-q} x_{q+r} x_{2r} + ABx_0 x_{q+r-p} x_{r-q} x_{p+r} x_{2r} + BCx_{r-p} x_q x_{r-q} x_{p+r} x_{2r} + ACx_0 x_{r-q} x_{2r-p} x_{p+r} x_{q+r} + BCx_0 x_{r-p} x_{2r-q} x_{p+r} x_{q+r}.$

- $P_{n-r} = ABx_q x_{p+q-r} x_{2p} x_{n+q-r} + ACx_q x_{2p+q-r} x_p x_{n+q-r} + Cx_0 x_{p+q-r} x_{n+p-r} x_{n+q-r} + ABx_{p+q-r} x_{2q} x_p x_{n+p-r} + BCx_q x_{n+q-2r} x_p x_{n+p-r}.$

- $P_{n-q} = ABx_r x_p x_{n+p-2q} + ACx_r x_{2p} x_{p+r-q} + Bx_0 x_{p+r-q} x_{n+p-q} + Cx_p x_{r-q} x_{n+p-q}.$

- $P_{n-p} = Ax_0 x_{n-2p} + Bx_{q-p} x_r + Cx_{r-p} x_q.$

For the remaining polynomials P_j, pick the largest $i < j$ in the set $\{0, p, q, r, n-p, n-q, n-r\}$, and let P_j be P_i after upshifting $j - i$ times.

6.2.2 Symmetric with second powers seed.

If P is of the form (2) in Theorem 3.10, the intermediate polynomials are $P_i = P(x_0, \ldots, \widehat{x_i}, \ldots, x_{n-1})$ for all $0 < i < n - 1$.

Example 6.1 *When $n = 3$, these family of polynomials accounts for the family (6) in Theorem 3.4.*

6.2.3 Sink-type binomial seed.

If P is of the form (3) in Theorem 3.10, the intermediate polynomials are

$$P_i = \prod_{j=0}^{i-1} x_j^{a_{n-i-j}} + \prod_{j=1}^{n-i-1} x_{i+j}^{a_j} \text{ for all } 0 < i < n - 1.$$

Example 6.2 *When $n = 6$, the polynomial $P = x_1^2 x_3^3 x_5 + 1$ generates the period 1 seed*

$$\{x_0, x_1^2 x_3^3 x_5 + 1\}, \{x_1, x_2^2 x_4^3 + x_0\}, \{x_2, x_3^2 x_5^3 + x_1\}, \{x_3, x_0^3 x_2 + x_4^2\}, \{x_4, x_1^3 x_3 + x_5^2\}, \{x_5, x_0^2 x_2^3 x_4 + 1\}.$$

6.2.4 Extreme seed.

If P is of the form (4) in Theorem 3.10, the intermediate polynomials are $P_i = x_{i-1} + x_{i+1} + A$ for all $0 < i < n - 1$.

Example 6.3 *When $n = 4$, the polynomial $P = x_1 x_3 + 3(x_1 + x_2 + x_3) + 2$ generates the period 1 seed*

$$\{x_0, x_1 x_3 + 3(x_1 + x_2 + x_3) + 2\}, \{x_1, x_0 + x_2 + 3\}, \{x_2, x_1 + x_3 + 3\}, \{x_3, x_0 x_2 + 3(x_0 + x_1 + x_2) + 2\}.$$

6.2.5 Singleton seed.

If P is a single variable polynomial of the form (5) in Theorem 3.10, the intermediate polynomials are $P_i = P\left(x_{i + \frac{n}{2} \pmod{n}}\right)$.

Example 6.4 *When $n = 4$, the polynomial $P = x_2^2 + 2x_2 - 7$ generates the period 1 seed*

$$\{x_0, x_2^2 + 2x_2 - 7\}, \{x_1, x_3^2 + 2x_3 - 7\}, \{x_2, x_0^2 + 2x_0 - 7\}, \{x_3, x_1^2 + 2x_1 - 7\}, \{x_4, x_3^2 + 2x_3 - 7\}.$$

Remark 6.5 *These polynomials correspond to $(\frac{n}{2})$-expansions of the period 1 polynomials found in Theorem 3.1.*

6.2.6 Chain seed.

If P is of the form (6) in Theorem 3.10, the intermediate polynomials are

$$P_{2j-1} = x_0 + x_{n-1} + A, \text{ for all } 0 < j \leq (n-1)/2$$
$$P_{2j} = F(x_0, \ldots, \widehat{x}_{2j}, \ldots, x_{n-1}), \text{ for all } 0 < j < (n-1)/2.$$

6.2.7 Multilinear symmetric seed.

If P is of the form (7) in Theorem 3.10, the intermediate polynomials are

$$P_{2j-1} = E_1(x_0, \ldots, \widehat{x}_{2j-1}, \ldots, x_{n-1}) + A, \text{ for all } 0 < j \leq (n-1)/2$$
$$P_{2j} = F(x_0, \ldots, \widehat{x}_{2j}, \ldots, x_{n-1}), \text{ for all } 0 < j < (n-1)/2.$$

6.2.8 r-Jumping seed.

Let $r, n, A \in \mathbb{N}$ be constants and P a polynomial in the setup of (8) of Theorem 3.10. For any $a \geq 0$ such that $a + r < n$, define

$$F_a = \sum_{k=0}^{\lfloor \frac{n-a}{r} \rfloor - 1} x_{a+rk} \cdot x_{a+rk+r-1} + A.$$

With this definition, notice that $P = F_1$. The intermediate polynomials are:

- $P_j = \left(\sum_{i=1}^{j+1} a_i^j \cdot F_{j+2-i}\right)\big|_{x_j=0}$ and $P_{n-j-1} = \left(b_0^j \cdot F_0 + \sum_{i=1}^{j} b_i^j \cdot F_{r-i}\right)\big|_{x_{n-j-1}=0}$

  for all $1 \leq j \leq r-2$, where $a_i^j = \prod_{k=0}^{j-i} x_k \cdot \prod_{k=0}^{i-2} x_{n-r+j-k}$ and $b_i^j = \prod_{k=0}^{j-i-1} x_{r-j+k} \cdot$

  $\prod_{k=1}^{i} x_{n-k}.$

- $P_j = \left(\sum_{i=1}^{r} a_i \cdot F_{r-i+1}\right)\big|_{x_j=0}$ for all $r-1 \leq j \leq n-r$, where $a_i = \prod_{k=0}^{r-1-i} x_k \cdot$

  $\prod_{k=0}^{i-2} x_{n-k-1}.$

Example 6.6 *When $n = 7$, the polynomial $P = x_1 x_3 + x_4 x_6$ generates the period 1 seed*

$$\{x_0, \quad x_1 x_3 + x_4 x_6\},$$
$$\{x_1, \quad x_0 x_2 x_4 + x_4 x_5 x_6\},$$
$$\{x_2, \quad x_0 x_1 x_3 x_5 + x_1 x_3 x_5 x_6 + x_4 x_5 x_6^2\},$$
$$\{x_3, \quad x_0^2 x_1 x_2 + x_0 x_2 x_4 x_6 + x_4 x_5 x_6^2\},$$
$$\{x_4, \quad x_0^2 x_1 x_2 + x_0 x_1 x_3 x_5 + x_1 x_3 x_5 x_6\},$$
$$\{x_5, \quad x_0 x_1 x_2 + x_2 x_4 x_6\},$$
$$\{x_6, \quad x_0 x_2 + x_3 x_5\}$$

Observe that this is an example of a binomial that generates a period 1 seed whose intermediate polynomials are not all binomials. Jumping polynomials are not classified by Theorem 3.6.

6.2.9 r-Hopping seed.

Let $r, n, A, B \in \mathbb{N}$ be constants and P a polynomial in the setup of (9) of Theorem 3.10. For any $0 \le a < n - r$, define

$$F_a = \sum_{k=0}^{\lfloor \frac{n-a}{r} \rfloor - 1} x_{a+rk} \cdot x_{a+rk+r-1} + A \sum_{k=0}^{\lfloor \frac{n-a}{r} \rfloor - 2} x_{a+rk+r-1} \cdot x_{a+rk+r} + B.$$

With this definition, notice that $P = F_1$. The intermediate polynomials are:

$-\ P_j = \big(\sum_{i=1}^{j+1} a_i^j \cdot F_{j+2-i}\big)\big|_{x_j=0}$ and $P_{n-j-1} = \big(b_0^j \cdot F_0 + \sum_{i=1}^{j} b_i^j \cdot F_{r-i}\big)\big|_{x_{n-j-1}=0}$

for all $1 \le j \le r-2$, where $a_i^j = \prod_{k=0}^{j-i} x_k \cdot \prod_{k=0}^{i-2} x_{n-r+j-k}$ and $b_i^j = \prod_{k=0}^{j-i-1} x_{r-j+k} \cdot$

$\prod_{k=1}^{i} x_{n-k}.$

$-\ P_j = \big(\sum_{i=1}^{r} a_i \cdot F_{r-i+1}\big)\big|_{x_j=0}$ for all $r - 1 \le j \le n - r$, where $a_i = \prod_{k=0}^{r-1-i} x_k \cdot$

$\prod_{k=0}^{i-2} x_{n-k-1}.$

Remark 6.7 *In the definitions of a_i^j, b_i^j and a_i in the jumping and hopping seeds, a product $\prod_{k=L}^{M} X_k$ is defined to be 1 if $M < L$.*

6.2.10 Flip-symmetric binomial seed.

These are the seeds discussed on Section 4. We give an explicit description here for consistency.

Let $L, R \subset [n-1]$ be disjoint subsets, $a : L \cup R \to \mathbb{N}$ a map and P a polynomial in the setup of (10) of Theorem 3.10.

Let $\mathbf{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathbb{Z}_{\geq 0}^n$ be the vector with nonnegative entries such that $a_i = a(i+1)$ if $i+1 \in L$, $a_j = -a(j+1)$ if $j+1 \in R$ and $a_k = 0$ for the remaining indices k. Define the vectors $\mathbf{b^{(1)}}, \ldots, \mathbf{b^{(n-1)}}$ recursively as follows. Let $\mathbf{b^{(i)}} = (b_0^{(i)}, \ldots, b_{n-1}^{(i)})$, $\mathbf{c^{(i)}} = (c_0^{(i)}, \ldots, c_{n-1}^{(i)})$ for all i and begin defining $\mathbf{b^{(1)}} = \mathbf{a}$. Then let $c_0^{(i)} = -b_0^{(i-1)}$ and $c_j^{(i)} = b_j^{(i-1)} + b_0^{(i-1)} \cdot |a_j| \cdot \mathbf{1}_{\{a_{n-i}a_j < 0\}}$ for all $0 \leq j \leq n-1$ and $i > 1$ (the indicator function $\mathbf{1}_{\{a_{n-i}a_j < 0\}}$ is 1 if $a_{n-i}a_j < 0$ and is 0 otherwise). Finally, let $\mathbf{b^{(i)}}$ be the vector that comes from permuting $\mathbf{c^{(i)}}$ with the permutation $(0, 1, 2, \ldots, n-1) \to (1, 2, \ldots, n-1, 0)$. We can now show the intermediate polynomials P_i for $0 < i < n-1$. Polynomial P_i is derived from vector $\mathbf{b^{(n-i)}}$ as follows: let L_i (resp. R_i) be the set of indices $0 \leq k \leq n-1$ such that $b_k^{(n-i)} > 0$ (resp. $b_k^{(n-i)} < 0$). Then $P_i =$

$$\prod_{k \in L_i} x_i^{b_k^{(n-i)}} + \prod_{k \in R_i} x_i^{-b_k^{(n-i)}}.$$

From Theorem 4.10 and the definition of B-matrix mutation in (4.1), the resulting seed $(\mathbf{x}, \mathbf{P})$ is a period 1 seed.

Example 6.8 *When $n = 8$, the polynomial $P = x_1^3 x_7^2 + x_4^3 x_2 x_6$ generates the period 1 seed*

$$\{x_0, x_1^3 x_7 + x_2 x_4^3 x_6\}, \{x_1, x_0^2 x_3 x_5^3 x_7 + x_2^5 x_4^6 x_6^2\}, \{x_2, x_0 x_3^5 x_5^6 + x_1^5 x_4 x_6^3\}, \{x_3, x_1 x_4^5 x_6^6 + x_2^5 x_5 x_7^3\},$$
$$\{x_4, x_0^3 x_2 x_5^5 + x_1^9 x_3^5 x_6\}, \{x_5, x_1^3 x_3 x_6^5 + x_2^9 x_4^5 x_7\}, \{x_6, x_0 x_2^3 x_4 x_7^3 + x_1^3 x_3^9 x_5^5\}, \{x_7, x_0^3 x_6 + x_1 x_3^3 x_5\}$$

We now demonstrate how to obtain $\mathbf{b^{(5)}}$ from $\mathbf{b^{(4)}}$. We have $\mathbf{a} = (0, 3, -1, 0, -3, 0, -1, 2)$ (corresponding to polynomial P) and $\mathbf{b^{(4)}} = (3, -9, 1, -5, 0, 5, -1, 0)$ (corresponding to polynomial P_4).

Notice that $a_5 = -3$ implies $a_5 a_j < 0$ if and only if $a_j > 0$. The only indices j for which $a_j > 0$ are 1 and 7. Then $c_1^{(0)} = -b_1^{(4)}$, $c_1^{(5)} = b_1^{(4)} + b_0^{(4)} \cdot |a_1| = -9 + (3)(3) = 0$, $c_7^{(5)} = b_7^{(4)} + b_0^{(4)} \cdot |a_7| = 0 + (3)(2) = 6$ and $c_j^{(5)} = b_j^{(4)}$ for the remaining indices j. Thus $\mathbf{c^{(5)}} = (-3, 0, 1, -5, 0, 5, -1, 6)$ and $\mathbf{b^{(5)}} = (0, 1, -5, 0, 5, -1, 6, -3)$. This corresponds to $P_3 = x_1 x_4^5 x_6^6 + x_2^5 x_5 x_7^3$.

6.2.11 Balanced seed.

Let $L, R \subset [n-1]$ be disjoint subsets, $a : L \cup R \to \mathbb{N}$ a map, M_1, M_2 monomials, $m > 1$ an integer and P a polynomial in the setup of (11) of Theorem 3.10.

Notice that $P' = M_1^m + M_2^m$ is a binomial with flip-symmetry seed, so it generates a period 1 seed. Let P'_j be the intermediate polynomials of this seed. From the analysis in 6.2.10, we see that each P'_i is a binomial of the form $M_1^{(i)} + M_2^{(i)}$, where each of the monomials $M_1^{(i)}, M_2^{(i)}$ is an m-th power of a monomial. Say that $(N_1^{(i)})^m = M_1^{(i)}$ and $(N_2^{(i)})^m = M_2^{(i)}$, then the intermediate polynomials are $P_i = (N_1^{(i)})^m + (N_2^{(i)})^m + \sum_{k=1}^{\lfloor m/2 \rfloor} (A_i \cdot (N_1^{(i)})^k (N_2^{(i)})^{m-k} + A_i \cdot (N_1^{(i)})^{m-k} (N_2^{(i)})^k)$.

6.2.12 Vector sum seed.

Let $a_1, \ldots, a_{n-1} \in \mathbb{N}$ be constants, B a finite set of vectors $\mathbb{N}^{n-1}$ and P a polynomial in the setup of (12) in Theorem 3.10. For ease of notation, let $B' = B \cup \{(0, \ldots, 0)\}$ and $C_{(0,\ldots,0)} = 1$. The intermediate polynomials are

$$P_i = \sum_{b \in B'} (C_b \cdot x_0^{b_{n-i}} \ldots x_{i-1}^{b_{n-1}} x_{i+1}^{a_1-b_1} \ldots x_{n-1}^{a_{n-1-i}-b_{n-1-i}})$$

$$+ \sum_{b \in B'} (C_b \cdot x_0^{a_{n-i}-b_{b_{n-i}}} \ldots x_{i-1}^{a_{n-1}-b_{n-1}} x_{i+1}^{b_1} \ldots x_{n-1}^{b_{n-1-i}}), \text{ for all } 0 < i < n-1.$$

Example 6.9 *The polynomial* $P = 1 + x_1^3 x_2^2 x_3^4 x_4^2 + 2x_1 x_2 x_3^2 x_4 + 2x_1^2 x_2 x_3^2 x_4$ *generates the period 1 seed*

$$\begin{aligned}
\{x_0, & \quad 1 + x_1^3 x_2^2 x_3^4 x_4^2 + 2x_1 x_2 x_3^2 x_4 + 2x_1^2 x_2 x_3^2 x_4\}, \\
\{x_1, & \quad x_0^2 + x_2^3 x_3^2 x_4^4 + 2x_0 x_2^2 x_3 x_4^2 + 2x_0 x_2 x_3 x_4^2\}, \\
\{x_2, & \quad x_3^3 x_4^2 + x_0^4 x_1^2 + 2x_0^2 x_1 x_3^2 x_4 + 2x_0^2 x_1 x_3 x_4\}, \\
\{x_3, & \quad x_4^3 + x_0^2 x_1^4 x_2^2 + 2x_0 x_1^2 x_2 x_4^2 + 2x_0 x_1^2 x_2 x_4\}, \\
\{x_4, & \quad 1 + x_0^3 x_1^2 x_2^4 x_3^2 + 2x_0 x_1 x_2^2 x_3 + 2x_0^2 x_1 x_2^2 x_3\}
\end{aligned}$$

6.2.13 Little Pi Seed.

Let $k, n \in \mathbb{N}$ be constants and P a polynomial in the setup of (13) of Theorem 3.10. We show the intermediate polynomials P_j for $j \in \{k, 2k, n-2k, n-k\}$. For the general P_i, if j is the largest integer with $j < i$ and $j \in J$, then P_i comes from $i - j$ upshifts to P_j.

Case 1: If $n > 4k$, so that $k < 2k < n - 2k < n - k$, then

- $P_k = A x_0 x_{2k} + A x_{2k} x_{n-2k} + x_0 x_{3k} x_{n-k} + A^2 x_{n-k}$

- $P_{2k} = x_0 x_{3k} + x_k x_{4k} + A^2$

- $P_{n-2k} = A x_k x_{n-3k} + A x_{n-3k} x_{n-k} + x_0 x_{n-4k} x_{n-k} + A^2 x_0$

$-\ P_{n-k} = Ax_0 + Ax_{n-2k} + x_k x_{n-3k}.$

Case 2: If $n = 4k$, so that $k < 2k = n - 2k = 2k < n - k = 3k$, then

$-\ P_k = Ax_0 x_{2k} + Ax_{2k}^2 + x_0 x_{3k}^2 + A^2 x_{3k}$

$-\ P_{2k} = Ax_k^2 + Ax_k x_{3k} + x_0^2 x_{3k} + A^2 x_0$

$-\ P_{3k} = Ax_0 + Ax_{2k} + x_k^2.$

Case 3: If $4k > n > 3k$, so that $k < n - 2k < 2k < n - k$, then

$-\ P_k = Ax_0 x_{2k} + Ax_{2k} x_{n-2k} + x_0 x_{3k} x_{n-k} + A^2 x_{n-k}$

$-\ P_{n-2k} = x_0 x_{n-3k} x_{n-k} + x_0 x_{2n-5k} x_{n-k} + x_{n-3k} x_k x_{2n-4k} + x_{n-3k} x_{n-k} x_{2n-4k} + Ax_0 x_{2n-4k}$

$-\ P_{2k} = Ax_k x_{5k-n} + Ax_k x_{3k} + x_0 x_k x_{3k} + A^2 x_{4k-n}$

$-\ P_{n-k} = Ax_0 + Ax_{n-2k} + x_k x_{n-3k}.$

Case 4: If $3k > n > 2k$, so that $n - 2k < k < n - k < 2k$, then

$-\ P_{n-2k} = x_{2n-4k} x_k + x_{2n-4k} x_{n-k} + x_0 x_{2k} + x_0 x_{2n-3k}$

$-\ P_k = x_0 x_{n-k} x_{4k-n} + x_0 x_{n-k} x_{2k} + x_0 x_{3k-n} x_{2k} + x_{n-2k} x_{3k-n} x_{2k} + Ax_{3k-n} x_{n-k}$

$-\ P_{n-k} = x_{n-2k} x_k + x_{2n-3k} x_{n-2k} + x_0 x_{2n-3k} + x_k x_{2n-4k}$

$-\ P_{2k} = Ax_{3k-n} + Ax_k + x_0 x_{4k-n}.$

6.2.14 Pi Seed.

Let $k, n, a, b \in \mathbb{N}$ be constants and P a polynomial in the setup of (14) in Theorem 3.10. We show the intermediate polynomials P_j for $j \in \{k, 2k, n - 2k, n - k\}$. We obtain the remaining intermediate polynomials P_i as before. Without loss of generality, assume $a_2 \geq a_1$ and $b_1 \geq b_2$.

Case 1: If $n > 4k$, so that $k < 2k < n - 2k < n - k$, then

$-\ P_k = Ax_{2k}^{a_2+b_2} x_{n-2k}^{b_1} + Bx_{2k}^{a_2+b_2} x_{n-2k}^{b_2} x_0^{b_1-b_2} + x_0^{b_1} x_{3k} x_{n-k}.$

$-\ P_{2k} = x_0 x_{3k}^{a_2+b_2} + x_k^{a_1+b_1} x_{4k}.$

$-\ P_{n-2k} = Ax_k^{a_1} x_{n-3k}^{a_1+b_1} x_{n-k}^{a_2-a_1} + Bx_k^{a_2} x_{n-3k}^{a_2+b_2} + x_0 x_{n-4k} x_{n-k}^{a_2}.$

$-\ P_{n-k} = Ax_0^{a_1} x_{n-2k}^{b_1} + Bx_0^{a_2} x_{n-2k}^{b_2} + x_k x_{n-3k}.$

Case 2: If $n = 4k$, so that $k < 2k = n - 2k < n - k = 3k$, then

$-\ P_k = Ax_{2k}^{a_2+b_1+b_2} + Bx_{2k}^{a_2+2b_2} x_0^{a_2-a_1} + x_0^{b_1} x_{3k}^2.$

$-\ P_{2k} = Ax_k^{2a_1+b_1} x_{3k}^{a_2-a_1} + Bx_k^{2a_2+b_2} + x_0^2 x_{3k}^{a_2}.$

$-\ P_{3k} = Ax_0^{a_1} x_{2k}^{b_1} + Bx_0^{a_2} x_{2k}^{b_2} + x_k^2.$

Case 3: If $4k > n > 3k$, so that $k < n - 2k < 2k < n - k$, then

- $P_k = Ax_{2k}^{a_2+b_2}x_{n-2k}^{b_1} + Bx_0^{a_2-a_1}x_{2k}^{a_2+b_2}x_{n-2k}^{b_2} + x_0^{b_1}x_{3k}x_{n-k}.$

- $P_{n-2k} = Ax_0x_{n-3k}^{a_1}x_{2n-5k}^{b_1}x_{n-k}^{a_2} + Bx_0x_{2n-5k}^{b_2}x_{n-3k}^{a_2}x_{n-k}^{a_2} + Ax_{n-3k}^{b_1}x_k^{a_1}x_{2n-4k}x_{n-k}^{a_2-a_1} + Bx_{n-3k}^{b_1}x_k^{a_2}x_{2n-4k}.$

- $P_{2k} = Ax_{3k}^{a_2-a_1}x_{5k-n}^{a_1}x_k^{a_1+b_1} + Bx_k^{a_2+b_2}x_{5k-n}^{a_2} + x_0x_{4k-n}x_{3k}^{a_2}.$

- $P_{n-k} = Ax_0^{a_1}x_{n-2k}^{b_1} + Bx_0^{a_2}x_{n-2k}^{b_2} + x_kx_{n-3k}.$

Case 4: If $3k > n > 2k$, so that $n - 2k < k < n - k < 2k$. Two cases will arise; for simplicity, let us only do the case $a_1 \le b_2$.

- $P_{n-2k} = Ax_{2n-4k}x_k^{a_1}x_{n-k}^{b_1-a_1} + Bx_{2n-4k}x_k^{a_2}x_{n-k}^{b_1-a_2} + Ax_0x_{2n-3k}^{b_1} + Bx_0x_{n-k}^{a_2-a_1}x_{2n-3k}^{b_2}.$

- $P_k = Ax_0^{b_1}x_{2k}^{a_2-a_1}x_{4k-n}^{a_1}x_{n-k} + Bx_0^{b_1}x_{n-k}x_{4k-n}^{a_2} + Ax_{3k-n}x_{n-2k}^{b_1}x_{2k}^{a_2} + Bx_0^{a_2-a_1}x_{n-2k}^{b_2}x_{2k}^{a_2}x_{3k-n}.$

- $P_{n-k} = Ax_0^{a_1}x_{n-2k}^{b_1-a_1}x_{2n-3k} + Bx_0^{a_2}x_{2n-3k}x_{n-2k}^{b_2-a_1} + Ax_kx_{2n-4k}^{b_1} + Bx_{n-2k}^{a_2-a_1}x_{2n-4k}^{b_2}x_k.$

- $P_{2k} = Ax_{3k-n}^{a_1}x_k^{b_1} + Bx_{3k-n}^{a_2}x_k^{b_2} + x_0x_{4k-n}.$

Example 6.10 *When $n = 8$, $k = 2$ (Case $n = 4k$), $a = 3$ and $b = 2$, the polynomial $P = -2x_2^3x_6^2 + 3x_2^2x_6^3 + x_4^2$ generates the period 1 seed*

$$\{x_0, -2x_2^3x_6^2 + 3x_2^2x_6^3 + x_4^2\}, \{x_1, -2x_3^3x_7^2 + 3x_3^2x_7^3 + x_5^2\}, \{x_2, -2x_0x_4^7 + 3x_4^8 + x_0^3x_6^2\},$$
$$\{x_3, -2x_1x_5^7 + 3x_5^8 + x_1^3x_7^2\}, \{x_4, -2x_2^8 + 3x_2^7x_6 + x_0^2x_6^3\}, \{x_5, -2x_3^8 + 3x_3^7x_7 + x_1^2x_7^3\},$$
$$\{x_6, -2x_0^3x_4^2 + 3x_0^2x_4^3 + x_2^2\}, \{x_7, -2x_1^3x_5^2 + 3x_1^2x_5^3 + x_3^2\}.$$

7 Conserved quantities and k-invariants

In this section, we investigate the integrability of the recurrences which generate some period 1 polynomials. Our general approach, for each recurrence, is to find a *conserved quantity*, which we will denote by J. A conserved quantity is a rational polynomial function depending on any n consecutive terms of the sequence, i.e., $J_{m,n} = J(x_m, x_{m+1}, \ldots, x_{m+n-1})$, which is independent of m; in other words $J_{m+1,n} = J_{m,n} = J$. Using this conserved quantity, we *multilinearize* the recurrence $x_{m+n} = P(x_{m+1}, \ldots, x_{m+n-1})/x_m$ by writing it in the equivalent form

$$x_{m+n} = L(x_m, x_{m+1}, \ldots, x_{m+n-1}),$$

where L is a multilinear polynomial with coefficients in $\mathbb{Q}[x_0, \ldots, x_{n-1}]$. If L is linear, we say that the recurrence has been *linearized*. Notice that when the coefficients of L are all Laurent polynomials in $x_0, \ldots, x_{n-1}$, this multilinearization provides an alternate proof of the Laurent phenomenon for these sequences. In some cases, we find a *k-invariant* instead of a conserved quantity.

This is a rational polynomial function J such that $J_{m,n} = J(x_m, x_{m+1}, \ldots, x_{m+n-1})$ depends on the residue of m modulo k; in other words $J_{m+k} = J_{m,n}$.

In addition to finding conserved quantities, we also perform the singularity confinement test which, although insufficient for proving integrability, is commonly used to identify integrable discrete systems. We refer to [10] for a discussion on the singularity confinement test.

In the setting of rational recurrences, integrability corresponds to the discrete version of Liouville integrability, as first described by Maeda [11]. To be clear, the results in this section should be regarded as the first step towards the investigation of the integrability of the recurrences in Theorem 3.10. Fully proving the integrability of a recurrence is technical and involves more detailed discussion for each sequence than we provide here. For a more detailed discussion on integrability of recurrences, including cluster algebras and Poisson geometry, we refer to [12, 13, 14, 15].

7.1 Special case of symmetric with second powers polynomial

We obtain a conserved quantity for the recurrence defined by

$$P = \sum_{i=1}^{n-1} x_i^2 + A \sum_{i=1}^{n-1} x_i + B.$$

The recurrence at indices $m + n$ and $m + n + 1$ are:

$$x_{m+n}x_m = \sum_{i=1}^{n-1} x_{m+i}^2 + A \sum_{i=1}^{n-1} x_{m+i} + B$$

$$x_{m+n+1}x_{m+1} = \sum_{i=1}^{n-1} x_{m+i+1}^2 + A \sum_{i=1}^{n-1} x_{m+i+1} + B$$

After subtracting the former from the latter and rearranging, we obtain

$$\frac{x_{m+1} + x_{m+n+1} + A}{\prod_{i=1}^{n-1} x_{m+i+1}} = \frac{x_m + x_{m+n} + A}{\prod_{i=1}^{n-1} x_{m+i}}.$$

Therefore $\dfrac{x_m + x_{m+n} + A}{\prod_{i=1}^{n-1} x_{m+i}}$ is a conserved quantity for our recurrence that we will write as $J_{m,n}$. By multiplying the numerator and denominator by x_m, and using $x_{m+n}x_m = \sum_{i=1}^{n-1} x_{m+i}^2 + A \sum_{i=1}^{n-1} x_{m+i} + B$ in the numerator, we see that $J_{m,n}$ can be written as

$$J_{m,n} = \frac{\sum_{i=0}^{n-1} x_{m+i}^2 + A \cdot \sum_{i=0}^{n-1} x_{m+i} + B}{\prod_{i=0}^{n-1} x_{m+i}}.$$

Then we have

$$J_{m+1,n} = J_{m,n} = J = \frac{\sum_{i=0}^{n-1} x_i^2 + A \cdot \sum_{i=0}^{n-1} x_i + B}{\prod_{i=0}^{n-1} x_i},$$

and the mutilinear recurrence

$$x_{m+n} = J \cdot \prod_{i=1}^{n-1} x_{m+i} - x_m - A, \qquad m = 0, 1, 2 \ldots \tag{7.1}$$

We show that the recurrence generated by P passes the singularity confinement test described in [16]. Assume we had a singularity at x_{m+n}, i.e., $x_{m+n} = \epsilon \to 0$. Then we have $\epsilon x_m = P(x_{m+1}, x_{m+2}, \ldots x_{m+n-1}) = O(\epsilon)$. From (7.1), $x_{m+n} = J \cdot \prod_{i=1}^{n-1} x_{m+i} - x_m - A$, we can show inductively that $x_{n+m+i} = -x_{n+i} - A + O(\epsilon)$ for $i = 1, \ldots, m-1$. It is therefore clear that $x_{n+2m} = O(1)$, that is, the singularity is confined. It is interesting to observe that, even though the sequence passes this singularity confinement test, it is not Diophantine integrable, as shown in [16].

Another interesting fact is that if we define a sequence $(y_m)_{m=0}^{\infty}$ such that $y_0 = y_1 = \ldots y_{n-1} = 1$ and $y_{m+n} = (\sum_{i=1}^{n-1} y_{m+i}^2 + A \sum_{i=1}^{n-1} y_{m+i} + B)/y_m$ for all $m \geq 0$, then any successive n-tuples $(y_m, y_{m+1}, \ldots, y_{m+n-1})$ are integer solutions of the the following quadratic Diophantine equation

$$\sum_{i=0}^{n-1} x_i^2 + A \cdot \sum_{i=0}^{n-1} x_i + B = (n(1+A) + B) \cdot \prod_{i=0}^{n-1} x_i,$$

which is closely related to the Markoff-Hurwitz equation

$$\sum_{i=0}^{n-1} x_i = a \prod_{i=0}^{n-1} x_i.$$

We refer to [17] ot the study of the solvobility of the Markoff-Hurwitz equation.

7.2 r-Jumping polynomial

A conserved quantity for the r-Jumping polynomial

$$P = \sum_{i=0}^{\frac{n-1}{r}-1} x_{ri+1} \cdot x_{ri+r} + A,$$

when $n \geq 2r + 1$ and $n \equiv 1 \pmod{r}$, is

$$J_{m,n} = \frac{x_{m+1} + x_{m+n+r}}{\prod_{i=r+1}^{n} x_{m+i}}.$$

We then have

$$J_{m+1,n} = J_{m,n} = J = \frac{x_1 + x_{n+r}}{\prod_{i=r+1}^{n} x_i},$$

as well as the multilinear recurrence

$$x_{m+n+r} = J \cdot \prod_{i=r+1}^{n} x_{m+i} - x_{m+1}, \qquad m = 0, 1, 2 \ldots$$

7.3 Special case of sink-type binomial

This is the first example of polynomial for which we find a k-invariant instead of a conserved quantity. For the polynomial $P = x_k x_{n-k} + 1$, where $0 < k < n$, there is a $(n-k)$-invariant which is

$$J_{m,n} = \frac{x_m + x_{m+2k}}{x_{m+k}}.$$

We then have

$$J_{m+n-k,n} = J_{m,n}.$$

The quantity $J_{m,n}$ will depend on the residue of m modulo $n - k$; more specifically:

$$J_{m,n} = J_i = \frac{x_i + x_{i+2k}}{x_{i+k}} \text{ if } m \equiv i \pmod{n-k} \text{ and } 0 \leq i < n - k;$$

moreover, we obtain the linear recurrence

$$x_{m+2k} = J_{(m \bmod n-k)} x_{m+k} - x_m, \qquad m = 0, 1, 2 \ldots$$

This recurrence is thoroughly discussed in [2], where it is shown that the sequence is given by its initial values and a recurrence $x_{m+n} = G(x_m, x_{m+1}, \ldots, x_{m+n-1})$ for a linear function G. Moreover, it is shown in [12] for $k = 1$, and in [13] for general k that the sequence is *completely integrable*.

7.4 Extreme polynomial

A $(n-1)$-invariant for

$$P = x_1 x_{n-1} + A \cdot \sum_{i=1}^{n-1} x_i + B$$

is

$$J_{m,n} = \frac{x_{m+2} + x_m + A}{x_{m+1}}.$$

We then have
$$J_{m+n-1,n} = J_{m,n}.$$

The quantity $J_{m,n}$ will depend on the residue of m modulo $n-1$; more specifically:

$$J_{m,n} = J_i = \frac{x_{i+2} + x_i + A}{x_{i+1}} \quad \text{if } m \equiv i \pmod{n-1} \text{ and } 0 \le i < n-1;$$

moreover, we obtain the linear recurrence

$$x_{m+2} = J_{(m \bmod n-1)} x_{m+1} - x_m - A, \qquad m = 0, 1, 2 \ldots$$

7.5 Chain polynomial

A 2-invariant for

$$P = \sum_{i=1}^{n-2} x_i x_{i+1} + A \cdot \sum_{i=1}^{n-1} x_i + B,$$

when n is odd, is

$$J_{m,n} = \frac{x_{m+n-1} + x_m + A}{\prod_{i=0}^{\frac{n-3}{2}} x_{m+2i+1}}.$$

We then have

$$J_{m+2,n} = J_{m,n} = J_0 = \frac{x_{n-1} + x_0 + A}{\prod_{i=0}^{\frac{n-3}{2}} x_{2i+1}}, \qquad 2 \mid m$$

$$J_{m+2,n} = J_{m,n} = J_1 = \frac{\sum_{i=1}^{n-2} x_i x_{i+1} + A \sum_{i=1}^{n-1} x_i + x_1 + A + B}{\prod_{i=0}^{\frac{n-3}{2}} x_{2i+2}}, \qquad 2 \nmid m$$

and the mutilinear recurrence

$$x_{m+n-1} = J_{\langle m \bmod 2 \rangle} \cdot \prod_{i=0}^{\frac{n-3}{2}} x_{m+2i+1} - x_m - A, \qquad m = 0, 1, 2 \ldots$$

7.6 Multilinear symmetric polynomial

A 2-invariant for

$$P = \sum_{1 \le i < j \le n-1} x_i x_j + A \sum_{i=1}^{n-1} x_i + B,$$

when n is odd, is

$$J_{m,n} = \frac{\sum_{i=0}^{n-1} x_{m+i} + A}{\prod_{i=0}^{\frac{n-3}{2}} x_{m+2i+1}}.$$

We then have

$$J_{m+2,n} = J_{m,n} = J_0 = \frac{\sum_{i=0}^{n-1} x_i + A}{\prod_{i=0}^{\frac{n-3}{2}} x_{2i+1}}, \qquad 2 \mid m$$

$$J_{m+2,n} = J_{m,n} = J_1 = \frac{\sum_{0 \le i < j \le n-1} x_i x_j + A \sum_{i=0}^{n-1} x_i + B}{\prod_{i=1}^{\frac{n-1}{2}} x_{2i}}, \qquad 2 \nmid m.$$

and the mutilinear recurrence

$$x_{m+n-1} = J_{\langle m \bmod 2 \rangle} \cdot \prod_{i=0}^{\frac{n-3}{2}} x_{m+2i+1} - \sum_{i=0}^{n-2} x_m - A, \qquad m = 0, 1, 2, 3 \ldots$$

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