Detection Networks[1]

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Abstract

Each member of a team of decision agents receives a conditionally independent observation about some underlying discrete hypothesis. Subject to causality constraints, the agents seek to optimize a team cost functional by making discrete decisions which are conveyed to other agents on capacity constrained channels. This paper derives optimal decision rules for a class of problems of this type and discusses their properties.

I.Introduction

Purpose: Team theory has long sought to unify the joint estimation/ communication/ control problems that arise when several agents that receive different information attempt to cooperate [2]. These problems have proven to be rather difficult in many cases [11], although a suprising number of them can be reduced to an equivalent static framework [3,4] through reformulation.

Building on [8], this paper describes the solution of a number of specific team problems with non-partially nested (non-NP) information structures. The problem is made tractable by assuming that the underlying natural randomness takes the form of a fixed, discrete hypothesis and that the decision agents receive conditionally independent observations. This permits explicit computation of decision rules, even when communication (preassigned flow pattern) is allowed between agents. The decision rules themselves display an interesting structure which aids interpretation of the solution (e.g. in terms of an optimal quantization of the local information). The insight gained from this work will serve as a stepping-stone for the development of a general solution procedure for distributed hypothesis testing with communication constraints.

Related Work: As the logical sequel to [8], Sections 2 and 3 present results for two special classes of structures: tandem and hierarchical. The

problems discussed are endowed with digital communication links between

the decision-makers (DMs). A predetermined causal precedence ordering

is imposed as each DM needs the decisions of his predecessors before

it can generate its own decision. We retain however, the assumption that

the actions of the DMs do not in any way affect any system dynamics.

However, unlike problems with predetermined communication variables
(e.g. one-step delay information sharing [7], [5] and [10]), the communicated

variables are limited to m discrete values with their content to be

determined. Messages are conveyed error free, i.e. we do not consider

noisy channels [1], although generalizations to this case can readily

be drawn. While the m discrete value channels provide encoding capability,

we are not interested in coding per se [1] but rather seek the overall

optimal communication and decision strategy. Note that this allows explicit

signalling [9] from a DM to "downstream" DMs through the communication

links.

Problem Statement: Find $\gamma_i^* : Z_i \to U_i$, $i \in \{0,1,\ldots,N\}$, to minimize

$J(\gamma_0, \gamma_1, \ldots\ldots \gamma_N) = E\{J(u_0, H)\}$ where u_0 is the output of the detection

network; $\{u_1, u_2, \ldots, u_N\}$ are internal communication variables; and Z_i is

the information available to DMi as specified by its access to: (1) a

local observation y_i and (2) other decisions specified by the topology

of the communication network (Figure 1). Finally H is the underlying

discrete state of nature.

Overview of results: Solutions are available for binary problems where

$H \in \{H^0, H^1\}$ and $U_i \in \{0,1\}$, and a number of network topologies. Extensions to
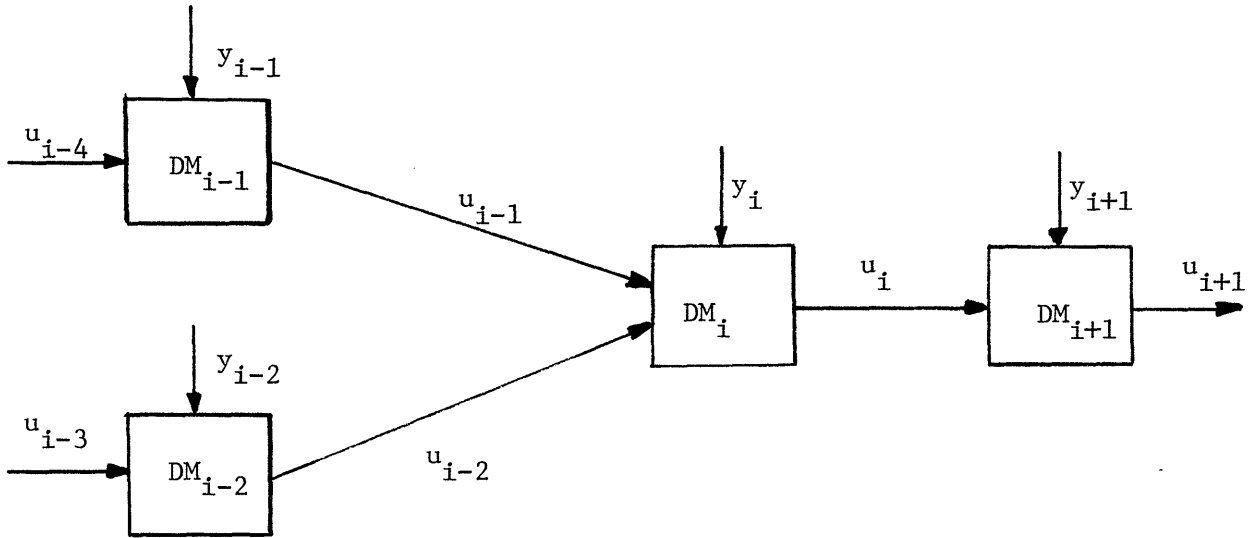
$$Z_i = \left\{ u_{i-1}, u_{i-2}, y_i \right\}$$

Figure 1: Topology of Communication Network

M-ary hypotheses and m_i symbol channels are straightforward and discussed in Sections 2 and 3.

II. Tandem Team Configuration

Assumptions: The tandem network topology is illustrated in Figure 2. H^o and H^1 arise with known a priori probabilities

$$P(H^o) = P^o \qquad\qquad P(H^1) = P^1 \qquad\qquad (1)$$

The joint conditional observation distribution is

$$P(y_0, y_1/H^j) \qquad\qquad \text{for } j\epsilon\{0,1\} \qquad\qquad (2)$$

where the subscripts denote the DM. (The y_i's may be random vectors generated by "preprocessing" of the original measurements). The global cost function is defined by $J(u_o, H)$ where

$$J : \{0,1\} \times \{H^o, H^1\} \rightarrow \mathbb{R} \qquad\qquad (3)$$

The objective of the team is to minimize

$$E\left\{J(u^o, H)\right\} \qquad\qquad (4)$$

and we will interpret

$$u_o = \begin{cases} 0 & H^o \text{ is declared} \\ \\ 1 & H^1 \text{ is declared} \end{cases} \qquad\qquad (5)$$

The following assumptions simplify in the derivation of the

Phenomenon   H

y_1

y_0

DM1

u_1

DM0

u_0
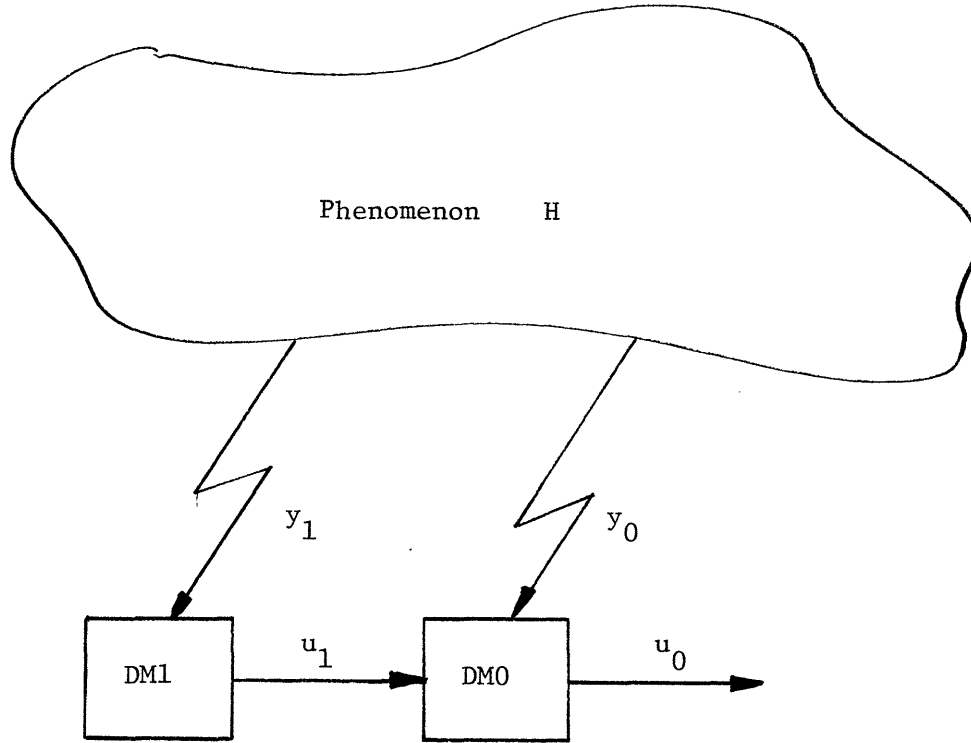
Figure 2: Two DM Tandem Topology

decision rules:

Assumption 1: The observations y_o and y_1 are statistically independent, i.e.

$$p(y_1/y_o,H) = p(y_1/H)$$
$$p(y_o/y_1,H) = p(y_o/H)$$

(6)

(See [6] for a case of dependent measurements without communication).

Assumption 2:

(a) $J(1,H^o) > J(0,H^o)$ (7)

    as per the interpretation of J

(b) $p(u_o=0/u_1=0,y_o) \geq p(u_o=0/u_1=1,y_o)$ (8)

    this disambiguates two possible symmetric solutions

Derivation: We begin with

Lemma 1:

$$\sum_{u_o} J(u_o,H^o)[p(u_o/u_1=0,y_1) - p(u_o/u_1=1,y_o)] < 0$$

(9)

Proof: Expanding expression (9)

$$J(0,H^o)p(u_o=0/u_1=0,y_0) + J(1,H^o)p(u_o=1/u_1=0,y_o)$$
$$-J(0,H^o)p(u_o=0/u_1=1,y_o) - J(1,H^o)p(u_o=1/u_1=1,y_o)$$

(10)

But
$$p(u_o=1/u_1=0,y_o) = 1 - p(u_o=0/u_1=0,y_o)$$
$$p(u_o=1/u_1=1,y_o) = 1 - p(u_o=0/u_1=1,y_o)$$

(11)

Substituting (11) into (10) and simplifying

$$[J(0,H^o) - J(1,H^o)] \left\{ p(u_o=0/u_1=0,y_o) - p(u_o=0/u_1=1,y_o) \right\}$$ (12)

and Assumption 2 gives expression (12) is less than 0.   Q.E.D.

This lemma is required for proving:

<u>Theorem 1</u>: The decision rule for DM0 (Figure 2) is given by

$$U_0 = \gamma_0 (y_0, u_1)$$
(13)

and $\gamma_0(\cdot)$ is the following likelihood ratio test:

$$\Lambda^0_{01} \triangleq \frac{P(y_0/H^0)}{P(y_0/H^1)} \underset{u_0=1}{\overset{u_0=0}{\gtrless}} \frac{P(H^1)P(u_1/H^1)[J(0,H^1)-J(1,H^1)]}{P(H^0)P(u_1/H^0)[J(1,H^0)-J(0,H^0)]}$$
(14)

$$\triangleq \begin{cases} t_0^0 & if \ u_1=0 \\ t_0' & if \ u_1=1 \end{cases}$$
(15)

The decision rule for DM1 is:

$$U_1 = \gamma_1(y_1)$$
(16)

and $\gamma_1(\cdot)$ is the following likelihood ratio test:

$$\Lambda^1_{01} \triangleq \frac{P(y_1/H^0)}{P(y_1/H^1)} \underset{u_1=1}{\overset{u_1=0}{\gtrless}} \frac{\sum_{u_0} P(H^1)J(u_0,H^1)[P(u_0/u_1=0,H^1)-P(u_0/u_1=1,H^1)]}{\sum_{u_0} P(H^0)J(u_0,H^0)[P(u_0/u_1=1,H^0)-P(u_0/u_1=0,H^0)]}$$
(17)

$$\triangleq t_1$$
(18)

<u>Proof</u>: This proof parallels the one presented in [8].

The objective

$$\min E \left\{ J(u_o, H) \right\} \tag{19}$$

is expanded explicitly as:

$$\sum_{\substack{H, u_o \\ u_1}} \int_{y_0, y_1} P(u_0, u_1, y_0, y_1, H) J(u_0, H) dy_0 dy_1$$

$$= \sum_{\substack{H, u_o \\ u_1}} \int_{y_0, y_1} P(H) p(u_0/u_1, y_0, y_1, H) p(u_1, y_0, y_1/H) J(u_0, H) dy_0 dy_1 \tag{20}$$

Invoking appropriate independence assumptions yields

$$\sum_{\substack{H, u_o \\ u_1}} \int_{y_0, y_1} P(H) p(u_0/u_1, y_0) p(u_1, y_1/H) p(y_0/H) J(u_0, H) dy_0 dy_1 \tag{21}$$

Explicitly summing over u_1 and ignoring constant term, the expression to

be minimized reduces to

$$\sum_{u_1} \int_{y_0} P(u_0=0/u_1, y_0) \sum_{H} \int_{y_1} P(H) p(y_1, u_1/H) p(y_0/H) \left[J(0,H) - J(1,H) \right] dy_0 dy_1 \tag{22}$$

This expression is minimized by setting

$$P(u_0=0/u_1, y_0) = \begin{cases} 0 & \text{if } \sum_{H} P(H) p(u_1/H) p(y_0/H) \left[J(0,H) - J(1,H) \right] > 0 \\ 1 & \text{otherwise} \end{cases} \tag{23}$$

Expanding over H and invoking Assumption 2(a) yields the decision rule

for DMO given in expression (14).

For $\gamma_1(\cdot)$, write (21) as

$$\sum_{\substack{H, u_o \\ u_1}} \int_{y_0, y_1} P(H) p(u_0/u_1, y_0) p(u_1/y_1) p(y_1/H) p(y_0/H) J(u_0, H) dy_0 dy_1 \tag{24}$$

Explicitly summing over u_1, substituting $p(u_1=1/y_1) = 1 - p(u_1=0/y_1)$

and ignoring the constant term yields

$$\int_{y_1} P(u_1=0/y_1) \sum_{H, u_o} \int_{y_0} P(H) p(y_0/H) p(y_1/H) J(u_0, H) \left[p(u_0/u_1=0, y_0) - p(u_0/u_1=1, y_0) \right] dy_0 dy_1 \tag{25}$$

The expression is minimized for

$$P(u_1=0/y_1) = \begin{cases} 0 & \text{if } \sum_{H,u_0} \int_{y_0} P(H)P(y_0|H)P(y_1|H)\overline{J}(u_0,H)\left[\begin{array}{c} P(u_0|u_1=0,y_0) \\ -P(u_0|u_1=1,y_0) \end{array} \right]dy_0 \geq 0 \\ \\ 1 & \text{otherwise} \end{cases} \quad (26)$$

Expanding over H and invoking Lemma 1 yields the decision rule for

DM1 given in expression (17).   Q.E.D.

<u>Summary</u>: The person-by-person optimal decision structure is based on

three thresholds that must be calculated together:

$$\begin{aligned} t_1 &= f_1(\gamma_0(.)) \\ t_0{}^0 &= f_0(\gamma_1(.))/_{u_1=0} \\ t_1{}^1 &= f_0(\gamma_1(.))/_{u_1=1} \end{aligned} \quad (27)$$

where $f_1(.)$ and $f_0(.)$ are defined in expressions (14) and (17). The

simultaneous solution of these equations yields the three thresholds.

<u>Extensions</u>:

<u>Theorem 2</u>: For the tertiary hypothesis case, i.e. $H \in \{H^0, H^1, H^2\}$ and

tertiary symbol channel capacity, i.e. $u_i \in \{0,1,2\}$, $i \in \{0,1\}$, the

decision rules for DM0 (Figure 2) is given by

$$u_0 = \gamma_0(y_0, u_1) \quad (28)$$

and $\gamma_0(\cdot)$ is the following set of likelihood ratio tests:

$$\Lambda_{ij}^0 \triangleq \frac{P(y_0|H^i)}{P(y_0|H^j)} \underset{\substack{u_0=j \text{ or } k}}{\overset{\substack{u_0=i \text{ or } k}}{\gtrless}} \frac{P(H^j)P(u_1|H^j)[J(j,H^j)-J(i,H^j)]}{P(H^i)P(u_1|H^i)[J(i,H^i)-J(j,H^i)]} + \frac{P(H^k)P(u_1|H^k)\Lambda_{kj}[J(j,H^k)-J(i,H^k)]}{P(H^i)P(u_1|H^i)[J(i,H^i)-J(j,H^i)]} \quad (29)$$

for $(i,j,k) \in \{(0,1,2),(0,2,1),(1,2,0)\}$

The decision rule for DM1 is

$$U_1 = \gamma_1(y_1) \tag{30}$$

and $\gamma_1(\cdot)$ is given by the following set of likelihood ratio test:

$$\Lambda_{ij}^{1} \triangleq \frac{P(y_1/H^i)}{P(y_1/H^j)} \underset{\substack{u_1=j\,or\,k \\ <}}{\overset{\substack{u_1=i\,or\,k \\ >}}{}} \frac{\sum_{u_0} P(H^j)J(u_0,H^j)\left[\begin{smallmatrix}P(u_0/u_1=j,H^j)\\-P(u_0/u_1=i,H^j)\end{smallmatrix}\right]}{\sum_{u_0} P(H^i)J(u_0,H^i)\left[\begin{smallmatrix}P(u_0/u_1=i,H^i)\\-P(u_0/u_1=j,H^i)\end{smallmatrix}\right]} + \frac{\sum_{u_0} P(H^k)\Lambda_{kj}J(u_0,H^k)\left[\begin{smallmatrix}P(u_0/u_1=j,H^k)\\-P(u_0/u_1=i,H^k)\end{smallmatrix}\right]}{\sum_{u_0} P(H^i)J(u_0,H^i)\left[\begin{smallmatrix}P(u_0/u_1=i,H^i)\\-P(u_0/u_1=j,H^i)\end{smallmatrix}\right]} \tag{31}$$

for $(i,j,k) \in \{(0,1,2),(0,2,1),(1,2,0)\}$

Proof: Consider expression (20)

$$\sum_{\substack{H,u_0 \\ u_1}} \int_{y_0,y_1} P(H)p(u_0/u_1,y_0)p(u_1,y_1/H)p(y_0/H)J(u_0,H)\,dy_0\,dy_1$$

$$= \sum_{u_0,u_1} \int_{y_0} P(u_0/u_1,y_0) \sum_H P(H)p(u_1/H)p(y_0/H)J(u_0,H)\,dy_0 \tag{32}$$

Hence the decision rule is deterministic and is given by (29).

To show (31) rewrite (32) as

$$\int_{y_1} \sum_{u_1} P(u_1/y_1) \sum_{H,u_0} P(H)p(u_0/u_1,H)p(y_1/H)J(u_0,H)\,dy_1 \tag{33}$$

and parallel the proof of Theorem 1.   Q.E.D.

Theorem 3: For the Three Tandem DM case (Figure 3) and binary symbol

capacity channels, the decision rule for DM0 is biven by

$$U_0 = \gamma_0(y_0,u_1) \tag{34}$$

and $\gamma_0(\cdot)$ is the following likelihood ratio test:

$$\Lambda_{01}^{0} \triangleq \frac{P(y_0/H^0)}{P(y_0/H^1)} \underset{\substack{u_0=1 \\ <}}{\overset{\substack{u_0=0 \\ >}}{}} \frac{P(H^1)P(u_1/H^1)[J(0,H^1)-J(1,H^1)]}{P(H^0)P(u_1/H^0)[J(1,H^0)-J(0,H^0)]} \tag{35}$$

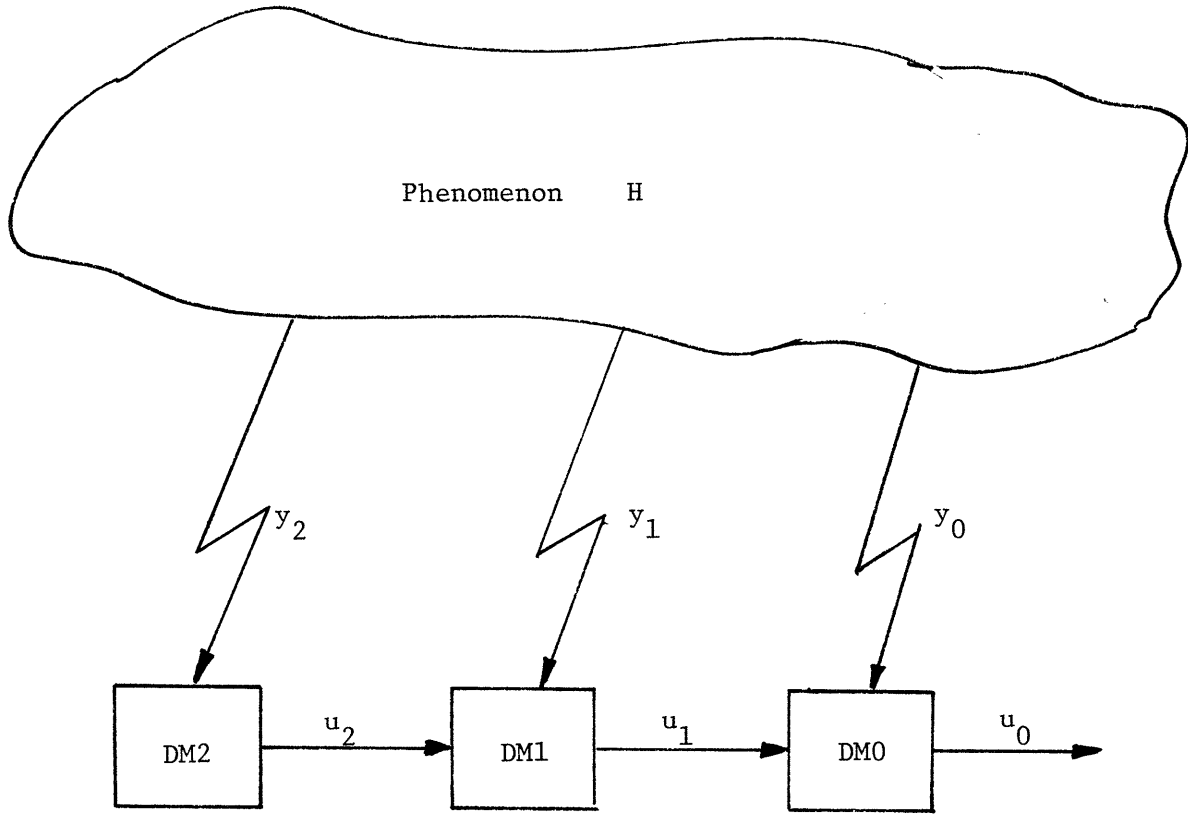$$\triangleq \begin{cases} t_0^0 & if \ u_1=0 \\ t_0^1 & if \ u_1=1 \end{cases} \tag{36}$$

Figure 3: Three DM Tandem Topology.

The decision rule for DM1 is

$$u_1 = \gamma_1(y_1, u_2) \tag{37}$$

and $\gamma_1(\cdot)$ is the following likelihood ratio test:

$$\Lambda_{o1}' \triangleq \frac{P(y_1/H^o)}{P(y_1/H')} \underset{u_1=1}{\overset{u_1=0}{\gtrless}} \frac{\sum_{u_o} P(H')P(u_2/H')\mathcal{J}(u_o,H')[P(u_o/u_1=0,H') - P(u_o/u_1=1,H')]}{\sum_{u_o} P(H^o)P(u_2/H^o)\mathcal{J}(u_o,H^o)[P(u_o/u_1=1,H^o) - P(u_o/u_1=0,H^o)]} \triangleq \begin{cases} t_1^o & \text{if } u_2: \\ t_1' & \text{if } u_2: \end{cases} \tag{38}$$

The decision rule for DM2 is

$$u_2 = \gamma_2(y_2) \tag{39}$$

and $\gamma_2(\cdot)$ is the following likelihood ratio test:

$$\Lambda_{o1}^2 = \frac{P(y_2/H^o)}{P(y_2/H')} \underset{u_2=1}{\overset{u_2=0}{\gtrless}} \frac{\sum_{u_o} P(H')\mathcal{J}(u_o,H')[P(u_o/u_2=0,H') - P(u_o/u_2=1,H')]}{\sum_{u_o} P(H^o)\mathcal{J}(u_o,H^o)[P(u_o/u_2=1,H^o) - P(u_o/u_2=0,H^o)]} \triangleq t_2 . \tag{40}$$

Proof: Invoking similar steps as in Theorem 1. Q.E.D.

Extensions to M-ary hypothesis, m_i symbol channels and N-tandem DMs are straightforward.

III. Tree Team Configuration

Assumption: Consider same assumptions as in Section II. First consider the simplest tree topology (Data Fusion) illustrated in Figure 4.

Theorem 4: The decision rule for DM0 is given by

$$\Lambda_{o1}^o \triangleq \frac{P(y_o/H^o)}{P(y_o/H')} \underset{u_o=1}{\overset{u_o=0}{\gtrless}} \frac{P(H')P(u_1/H')P(u_2/H')[\mathcal{J}(0,H') - \mathcal{J}(1,H')]}{P(H^o)P(u_1/H^o)P(u_2/H^o)[\mathcal{J}(1,H^o) - \mathcal{J}(0,H^o)]} \tag{41}$$

$$\triangleq \begin{cases} t_0^{00} & \text{if } u_1=0, \ u_2=0 \\ t_0^{01} & \text{if } u_1=0, \ u_2=1 \\ t_0^{10} & \text{if } u_1=1, \ u_2=0 \\ t_0^{11} & \text{if } u_1=1, \ u_2=1 \end{cases} \tag{42}$$

The decision rule for DMi ($i \in \{1,2\}$) is

$$u_i = \gamma_i(y_i) \qquad \text{for} \quad i \in \{1,2\}. \tag{43}$$

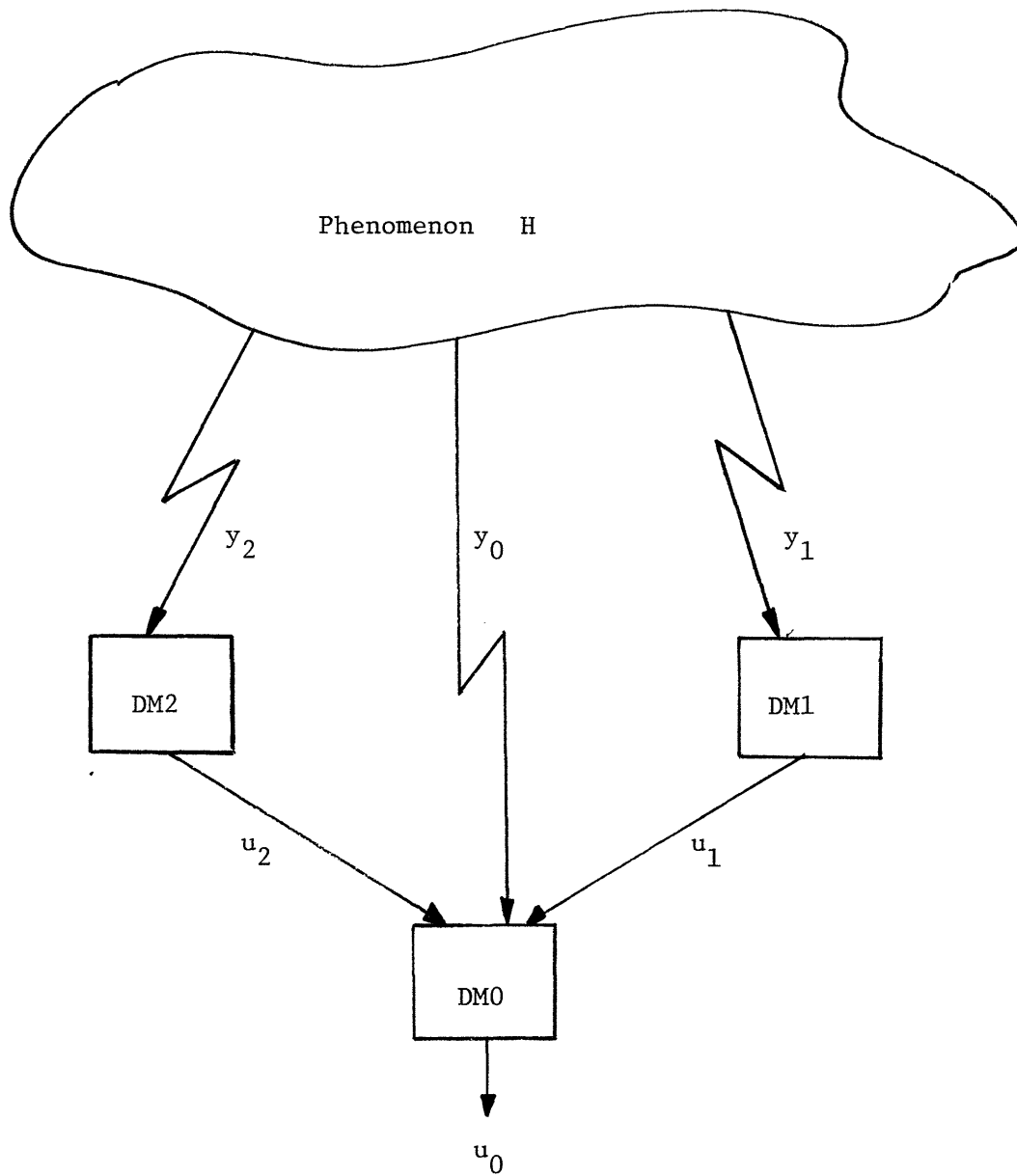Figure 4: Optimal Data Fusion Topology

and $\gamma_i(\cdot)$ is the following likelihood ratio test:

$$\Lambda_{o_I}^i \triangleq \frac{P(y_i/H^o)}{P(y_i/H')} \begin{array}{c} {\scriptstyle u_i=0} \\ > \\ < \\ {\scriptstyle u_i=1} \end{array} \frac{\sum_{u_o} P(H') \tilde{J}(u_0, H')[P(u_0/u_i=0, y_0) - P(u_0/u_i=1, y_0)]}{\sum_{u_o} P(H^o) \tilde{J}(u_0, H^o)[P(u_0/u_i=1, y_0) - P(u_0/u_i=0, y_0)]} \qquad (44)$$

$$\triangleq t_i \qquad (45)$$

Proof: This proof parallels the one in Theorem 1.   Q.E.D.

Extensions: Consider DM1 in the tree structure illustrate in Figure 5.

Assume the pair of decision rules in DM4 and DM0 are fixed. Then a cost

function of u_1 and H at DM1, $\tilde{J}(u_1, H)$, can be obtained by taking the

expectation over y_0 and y_4, i.e.

$$\tilde{J}(u_1, H) = \underset{y_0, y_4}{E} \left\{ \tilde{J}(u_0, H) \mid \gamma_4(\cdot), \gamma_0(\cdot) \right\}$$

Given this cost, one can solve for decision rules γ_1, γ_2 and γ_3 using

Theorem 4. Inserting these decision rules into the the DMs, one can

compute $p(u_1/H)$. Then one can solve for the decision rules for DM0

and DM4 with u_1 (described by $p(u_1/H)$) by using the results from the

tandem case (Theorem 1). Note that u_1 acts as an additional measurement

to DM0. The result of this process is an iterative procedure for alternately

computing $\{\gamma_4, \gamma_0\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$, which exploits a decomposition

resulting from the independence of observations when conditioned on

H (Assumption 1). This suggests a procedure for extending the results for these
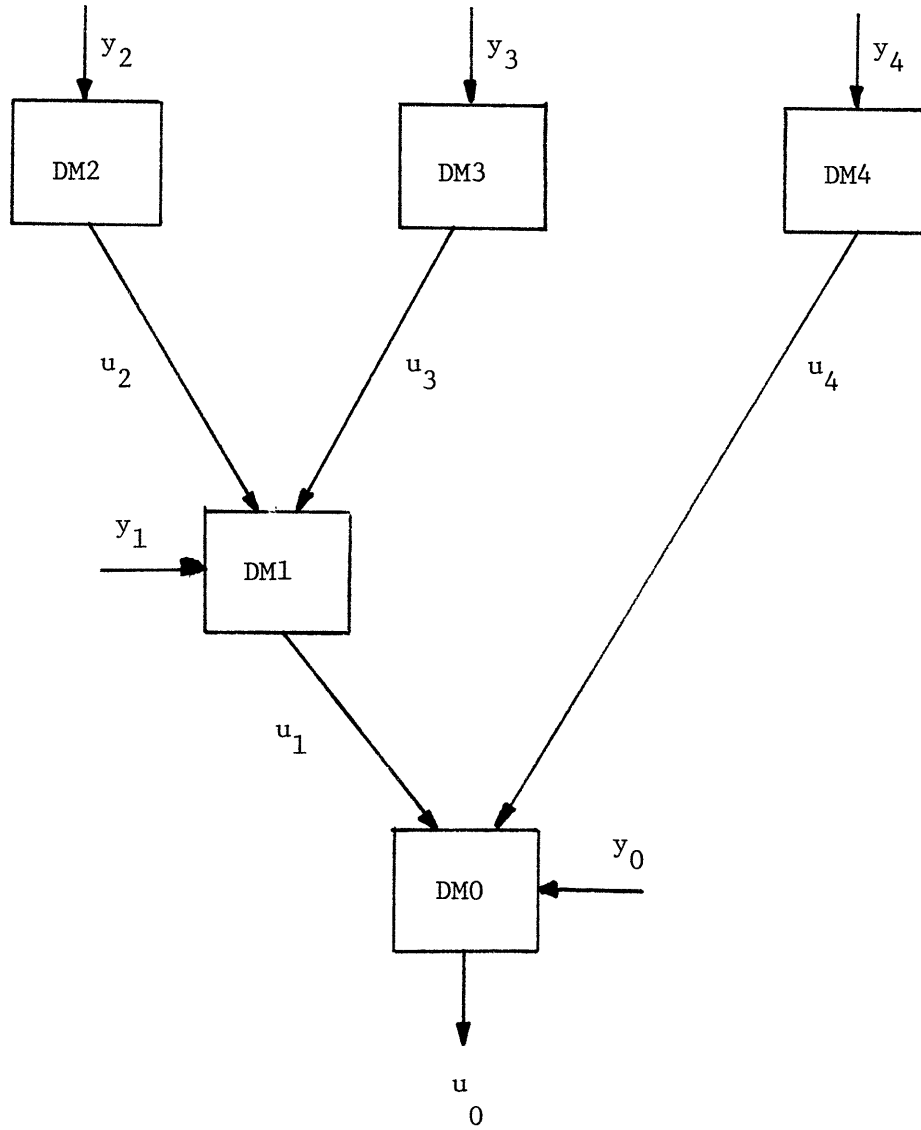
simple cases to larger networks.

Figure 5: Tree Hierarchical Topology

IV.Summary

Optimal decision rules for a static non-PN detection team have been derived. The decision rules are likelihood ratios in the actual data, with thresholds determined by incoming communicated messages. The number of thresholds at each DM is equal to the number of combinations of these discrete inputs. Moreover, it is apparent that, at least for some tree structured problems, a decomposition principle (along the lines of spatial dynamic programming) can be found for methodically computing decision rules. This is a topic of current research and will be reported in a subsequent paper.

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