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# On the performance of affine policies for two-stage adaptive optimization: a geometric perspective

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**Abstract** We consider two-stage adjustable robust linear optimization problems with uncertain right hand side  $\mathbf{b}$  belonging to a convex and compact uncertainty set  $\mathcal{U}$ . We provide an a priori approximation bound on the ratio of the *optimal affine (in  $\mathbf{b}$ ) solution* to the optimal adjustable solution that depends on two fundamental geometric properties of  $\mathcal{U}$ : (a) the “*symmetry*” and (b) the “*simplex dilation factor*” of the uncertainty set  $\mathcal{U}$  and provides deeper insight on the power of affine policies for this class of problems. The bound improves upon a priori bounds obtained for robust and affine policies proposed in the literature. We also find that the proposed a priori bound is quite close to a posteriori bounds computed in specific instances of an inventory control problem, illustrating that the proposed bound is informative.

**Keywords** Robust optimization, Adjustable optimization, Affine policies

## 1 Introduction

We consider the following two-stage adaptive robust linear optimization problem,  $\Pi_{Adapt}(\mathcal{U})$  with an uncertain right hand side.

$$\begin{aligned} z_{Adapt}(\mathcal{U}) &= \min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} + \max_{\mathbf{b} \in \mathcal{U}} \min_{\mathbf{y}(\mathbf{b})} \mathbf{d}^T \mathbf{y}(\mathbf{b}) \\ \mathbf{Ax} + \mathbf{By}(\mathbf{b}) &\geq \mathbf{b}, \forall \mathbf{b} \in \mathcal{U} \\ \mathbf{x}, \mathbf{y}(\mathbf{b}) &\geq \mathbf{0}, \end{aligned} \tag{1}$$

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where  $\mathbf{A} \in \mathbb{R}^{m \times n_1}$ ,  $\mathbf{B} \in \mathbb{R}^{m \times n_2}$ ,  $\mathbf{c} \in \mathbb{R}_+^{n_1}$ ,  $\mathbf{d} \in \mathbb{R}_+^{n_2}$ ,  $\mathcal{U} \subseteq \mathbb{R}_+^m$  is a convex and compact uncertainty set of possible values of the right hand side of the constraints. For any  $\mathbf{b} \in \mathcal{U}$ ,  $\mathbf{y}(\mathbf{b})$  denotes the value of the second-stage variables in the scenario when the right hand side is  $\mathbf{b}$ . Problem (1) models a wide variety of adaptive optimization problems such as the set covering, security-commitment for power generation (Dhamdhere et al., 2005; Bertsimas et al., 2013) and inventory control problems (see our reformulation in Section 5).

In this paper, we study for Problem (1) the power of affine policies, i.e., solutions  $\mathbf{y}(\mathbf{b})$  that are affine functions of  $\mathbf{b}$ . Given that finding the best such policy is tractable, affine policies are widely used to solve multi-stage dynamic optimization problems. While it has been observed empirically that these policies perform well in practice, to the best of our knowledge, there are no theoretical performance bounds for affine policies in general.

**Hardness.** The problem  $\Pi_{Adapt}(\mathcal{U})$  is intractable in general. Feige et al. (2007) prove that a special case of Problem (1) can not be approximated within a factor better than  $O(\log m)$  in polynomial time unless  $\text{NP} \subseteq \text{TIME}\left(2^{O(\sqrt{n})}\right)$ , where  $n$  refers to the problem input size. However, an exact or an approximate solution to Problem (1) can be computed efficiently in certain special cases.

If  $\mathcal{U}$  is a polytope with a small (polynomial) number of extreme points, Problem (1) can be solved efficiently. Instead of considering the constraint  $\mathbf{Ax} + \mathbf{By}(\mathbf{b}) \geq \mathbf{b}$  for all  $\mathbf{b} \in \mathcal{U}$ , it suffices to consider the constraint only for all the extreme points of  $\mathcal{U}$ . Thus, the resulting expanded formulation of Problem (1) has only a small (polynomial) number of variables and constraints which can be solved efficiently. Dhamdhere et al. (2005) consider Problem (1) with  $\mathbf{A} = \mathbf{B}$ . In this case the problem can model the set covering, Steiner tree and facility location problems. In this case, and when the set  $\mathcal{U}$  has a small number of extreme points, Dhamdhere et al. (2005) give approximation algorithms with similar performance bounds as the deterministic versions. Feige et al. (2007) extend to a special case of the uncertainty set with an exponential number of extreme points and give a polylogarithmic approximation for the set covering problem in this setting. Khandekar et al. (2008) consider a similar uncertainty set with an exponential number of extreme points as Feige et al. (2007) and give constant factor approximations for several network design problems such as the Steiner tree and uncapacitated facility location problems. In these papers the algorithm and the analysis is very specific to the specific problem addressed and does not provide insights for a tractable solution for the general two-stage Problem (1).

**Approximability.** Given the intractability of  $\Pi_{Adapt}(\mathcal{U})$ , it is natural to consider simplifications of  $\Pi_{Adapt}(\mathcal{U})$ . A natural approximation is to consider solutions that are non-adaptive – that is  $\mathbf{y}(\mathbf{b}) = \mathbf{y}$ . In this case, Problem  $\Pi_{Adapt}(\mathcal{U})$  reduces to the robust optimization problem

$$\begin{aligned} z_{Rob}(\mathcal{U}) &= \min \mathbf{c}^T \mathbf{x} + \mathbf{d}^T \mathbf{y} \\ \mathbf{Ax} + \mathbf{By} &\geq \mathbf{b}, \forall \mathbf{b} \in \mathcal{U} \\ \mathbf{x}, \mathbf{y} &\geq \mathbf{0}. \end{aligned} \tag{2}$$

Problem (2) is solvable efficiently. Bertsimas et al. (2011), generalizing earlier work in Bertsimas and Goyal (2010), provide a bound for the performance of robust

solutions which only depends on the geometry of the uncertainty set. In particular, for a general convex, compact and full-dimensional uncertainty set  $\mathcal{U} \subseteq \mathbb{R}_+^m$ , they provide the following bound for two-stage adaptive optimization problems:

$$z_{Adapt}(\mathcal{U}) \leq z_{Rob}(\mathcal{U}) \leq \left(1 + \frac{\rho}{s}\right) \cdot z_{Adapt}(\mathcal{U}),$$

where  $\rho$  and  $s$  are the *translation factor* and the *symmetry* of the uncertainty set  $\mathcal{U}$  which described in Section 2.

**Affine Policies.** Under this class of policies,  $\mathbf{y}(\mathbf{b}) = \mathbf{P}\mathbf{b} + \mathbf{q}$ , that is the second stage decisions depend linearly on  $\mathbf{b}$ :

$$\begin{aligned} z_{Aff}(\mathcal{U}) &= \min \mathbf{c}^T \mathbf{x} + \max_{\mathbf{b} \in \mathcal{U}} \min_{\mathbf{y}(\mathbf{b})} \mathbf{d}^T \mathbf{y}(\mathbf{b}) \\ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y}(\mathbf{b}) &\geq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U} \\ \mathbf{x}, \mathbf{y}(\mathbf{b}) &\geq \mathbf{0} \\ \mathbf{y}(\mathbf{b}) &= \mathbf{P}\mathbf{b} + \mathbf{q}. \end{aligned} \tag{3}$$

Affine solutions were introduced in the context of stochastic optimization in Rockafellar and Wets (1978) and then later in robust optimization in Ben-Tal et al. (2004) and also extended to linear systems theory (Ben-Tal et al., 2005, 2006). Affine policies have also been considered extensively in control theory of dynamic systems (see Bemporad et al. (2003); Goulart et al. (2006); Lofberg (2003); Bertsimas and Brown (2007); Skaf and Boyd (2010); and Ben-Tal and Nemirovski (2008) and the references therein). In all these papers, the authors show how to reformulate the multi-stage linear optimization problem such that an optimal affine policy can be computed efficiently by solving convex optimization problems such as linear, quadratic, conic and semi-definite. Goulart et al. (2006) first consider the problem of theoretically analyzing the properties of such policies and show that, under suitable conditions, the affine policy has certain desirable properties such as stability and robust invariance. There are two complementary approaches for studying the performance of affine policies: **(a)** a priori methods which evaluate the worst-case approximation ratio of affine policies over a class of instances and **(b)** a posteriori methods that estimate the approximation error for each problem instance individually. The bound we propose in this paper is an a priori bound. There have been various studies in the literature for developing a posteriori bounds to evaluate the performance of affine policies for particular instances (Van Parys et al., 2012a, 2013, 2012b; Hadjiyiannis et al., 2011a; Goulart et al., 2006). Kuhn et al. (2011) consider one-stage and multi-stage stochastic optimization problems, give efficient algorithms to compute the primal and dual approximations using affine policies, and determine a posteriori performance bounds for affine policies on an instance by instance basis in contrast to the a priori performance bounds we derive in this paper.

Bertsimas and Goyal (2012) show that the best affine policy can be  $\Omega(m^{1/2-\delta})$  times the optimal cost of a fully-adaptable two-stage solution for  $\Pi_{Adapt}(\mathcal{U})$  for any  $\delta > 0$ . Furthermore, for the case  $\mathbf{A} \in \mathbb{R}_+^{n_1 \times m}$ , they show that the worst-case cost of an optimal affine policy for  $\Pi_{Adapt}(\mathcal{U})$  is  $O(\sqrt{m})$  times the worst-case cost of an optimal fully-adaptable two-stage solution. However, unlike the bounds in Bertsimas et al. (2011), these bounds do not depend on  $\mathcal{U}$ .

Our main contribution in this paper is a new approximation bound on the power of affine policies relative to optimal adaptive policies that depends on fundamental geometric properties of  $\mathcal{U}$  and offers deeper insight on the power of affine policies. Specifically, we show that

$$z_{Adapt}(\mathcal{U}) \leq z_{Aff}(\mathcal{U}) \leq \frac{\lambda(1 + \frac{\rho}{s})}{\lambda + \frac{\rho}{s}} \cdot z_{Adapt}(\mathcal{U}),$$

where  $\lambda$ ,  $\rho$  and  $s$  are the simplex dilation factor, the translation factor and the symmetry of the uncertainty set  $\mathcal{U}$  described in Section 2. We also show that for many interesting uncertainty sets this new bound improves the bounds discussed in Bertsimas and Goyal (2012). The bound demonstrates theoretically that the improvement on the performance of affine policies compared to robust policies can be significant. We further provide computational evidence in the context of an inventory control problem that shows that the geometric a priori bound we obtain in this paper is quite close to the a posteriori bounds that we get for random instances of the inventory control problem. This suggests that our geometric a priori bound is indeed informative.

The structure of the paper is as follows. In the next section, we introduce the geometric properties of convex sets we will utilize and in Section 3, we state and prove the main approximation bound. In Section 4, we compute the bounds for specific examples of  $\mathcal{U}$  and compare it with earlier bounds illustrated in Bertsimas et al. (2011). Finally, in Section 5, we demonstrate the effectiveness of the proposed bound in the context of an inventory control problem.

## 2 Geometric Properties of Convex Sets

In this section, we review several geometric properties of convex sets  $\mathcal{U}$  that we use in the paper.

### 2.1 The Symmetry of a Convex Set

Given a nonempty compact convex set  $\mathcal{U} \subset \mathbb{R}^m$  and a point  $\mathbf{b} \in \mathcal{U}$ , we define the symmetry of  $\mathbf{b}$  with respect to  $\mathcal{U}$  as follows:

$$\mathbf{sym}(\mathbf{b}, \mathcal{U}) := \max\{\alpha \geq 0 : \mathbf{b} + \alpha(\mathbf{b} - \mathbf{b}') \in \mathcal{U}, \forall \mathbf{b}' \in \mathcal{U}\}. \quad (4)$$

In order to develop geometric intuition on  $\mathbf{sym}(\mathbf{b}, \mathcal{U})$ , we first show the following result. For this discussion, we assume that  $\mathcal{U}$  is full-dimensional.

**Lemma 1** *Let  $\mathcal{L}$  be the set of lines in  $\mathbb{R}^m$  passing through  $\mathbf{b}$ . For any line  $\ell \in \mathcal{L}$ , let  $\mathbf{b}'_\ell$  and  $\mathbf{b}''_\ell$  be the points of intersection of line  $\ell$  with the boundary of  $\mathcal{U}$ , denoted  $\delta(\mathcal{U})$  (these exist as  $\mathcal{U}$  is full-dimensional and compact). Then,*

$$\begin{aligned} a) \quad \mathbf{sym}(\mathbf{b}, \mathcal{U}) &\leq \min \left\{ \frac{\|\mathbf{b} - \mathbf{b}''_\ell\|_2}{\|\mathbf{b} - \mathbf{b}'_\ell\|_2}, \frac{\|\mathbf{b} - \mathbf{b}'_\ell\|_2}{\|\mathbf{b} - \mathbf{b}''_\ell\|_2} \right\}, \\ b) \quad \mathbf{sym}(\mathbf{b}, \mathcal{U}) &= \min_{\ell \in \mathcal{L}} \left\{ \frac{\|\mathbf{b} - \mathbf{b}''_\ell\|_2}{\|\mathbf{b} - \mathbf{b}'_\ell\|_2}, \frac{\|\mathbf{b} - \mathbf{b}'_\ell\|_2}{\|\mathbf{b} - \mathbf{b}''_\ell\|_2} \right\}. \end{aligned}$$

The symmetry of set  $\mathcal{U}$  is defined as

$$\mathbf{sym}(\mathcal{U}) := \max\{\mathbf{sym}(\mathbf{b}, \mathcal{U}) \mid \mathbf{b} \in \mathcal{U}\}. \quad (5)$$

An optimizer  $\mathbf{b}_0$  of (5) is called a point of symmetry of  $\mathcal{U}$ . This definition of symmetry can be traced back to Minkowski (1911) and is the first and the most widely used symmetry measure. We refer to Belloni and Freund (2008) for a broad investigation of the properties of the symmetry measure defined in (4) and (5). Note that the above definition generalizes the notion of perfect symmetry considered by Bertsimas and Goyal (2010). In Bertsimas and Goyal (2010), the authors define that a set  $\mathcal{U}$  is symmetric if there exists  $\mathbf{b}_0 \in \mathcal{U}$  such that, for any  $\mathbf{z} \in \mathbb{R}^m$ ,  $(\mathbf{b}_0 + \mathbf{z}) \in \mathcal{U} \Leftrightarrow (\mathbf{b}_0 - \mathbf{z}) \in \mathcal{U}$ . Equivalently,  $\mathbf{b} \in \mathcal{U} \Leftrightarrow (2\mathbf{b}_0 - \mathbf{b}) \in \mathcal{U}$ . According to the definition in (5),  $\mathbf{sym}(\mathcal{U}) = 1$  for such a set.

**Lemma 2 (Belloni and Freund (2008))** *For any nonempty convex compact set  $\mathcal{U} \subseteq \mathbb{R}^m$ , the symmetry of  $\mathcal{U}$  satisfies*

$$\frac{1}{m} \leq \mathbf{sym}(\mathcal{U}) \leq 1.$$

The symmetry of a convex set is at most 1, which is achieved for a perfectly symmetric set; and at least  $1/m$ , which is achieved by a standard simplex defined as  $\Delta := \{\mathbf{x} \in \mathbb{R}_+^m \mid \sum_{i=1}^m x_i \leq 1\}$ . The lower bound follows from the Löwner-John Theorem (see Belloni and Freund (2008)).

## 2.2 The Translation Factor of a Convex Set

For a convex compact set  $\mathcal{U} \subset \mathbb{R}_+^m$ , we define the *translation factor* of  $\mathbf{b} \in \mathcal{U}$  with respect to  $\mathcal{U}$ , as follows:

$$\rho(\mathbf{b}, \mathcal{U}) := \min\{\alpha \in \mathbb{R}_+ \mid \mathcal{U} - (1 - \alpha) \cdot \mathbf{b} \subset \mathbb{R}_+^m\}.$$

In other words,  $\mathcal{U}' := \mathcal{U} - (1 - \rho)\mathbf{b}$  is the maximum possible translation of  $\mathcal{U}$  in the direction  $-\mathbf{b}$  such that  $\mathcal{U}' \subset \mathbb{R}_+^m$ . Figure 1 depicts the geometry of the translation factor  $\rho$ .

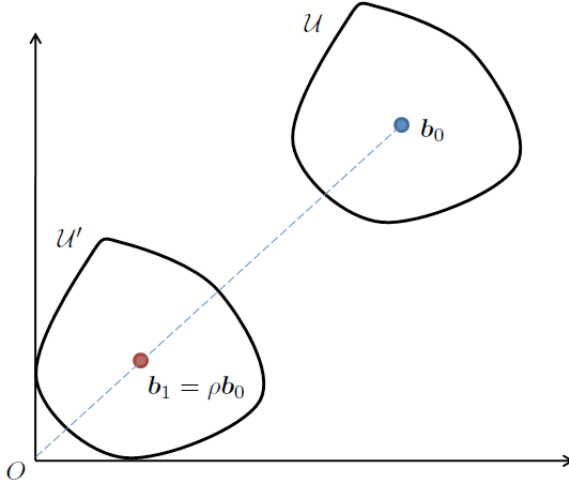
**Lemma 3 (Bertsimas et al. (2011))** *Let  $\mathcal{U} \subset \mathbb{R}_+^m$  be a convex and compact set such that  $\mathbf{b}_0$  is the point of symmetry of  $\mathcal{U}$ . Let  $s := \mathbf{sym}(\mathcal{U}) = \mathbf{sym}(\mathbf{b}_0, \mathcal{U})$  and  $\rho := \rho(\mathcal{U}) = \rho(\mathbf{b}_0, \mathcal{U})$ . Then,*

$$\left(1 + \frac{\rho}{s}\right) \cdot \mathbf{b}_0 \geq \mathbf{b}, \quad \forall \mathbf{b} \in \mathcal{U}.$$

## 2.3 The Simplex Dilation Factor of a Convex Set

In this section, we introduce a new geometric property of a convex set that is critical in bounding the performance of affine policies. For any convex set  $\mathcal{U}$  we associate a constant  $\lambda(\mathcal{U})$  in the following way.

In dimension  $m$ , a simplex  $S$  is a convex hull of  $m + 1$  affinely independent points. Bertsimas and Goyal (2012) show that the affine policy is optimal for two

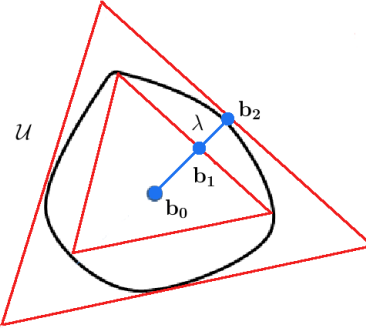


**Fig. 1** Geometry of the translation factor.

stage adaptive optimization problems when the uncertainty set is a simplex. For any convex set  $\mathcal{U} \in \mathbb{R}^m$ , we define the simplex dilation factor  $\lambda := \lambda(\mathcal{U})$  to be the smallest number such that there is a simplex  $S$  containing the symmetry point  $\mathbf{b}_0$  and

$$S \subset \mathcal{U} \subset \lambda \cdot (S - \mathbf{b}_0) + \mathbf{b}_0. \quad (6)$$

Figure 2 depicts the definition of  $\lambda$  and its calculation. We next provide some examples of calculation of  $\lambda$ .



**Fig. 2** The simplex dilation factor  $\lambda$  is  $\frac{\|\mathbf{b}_0 - \mathbf{b}_1\|_2}{\|\mathbf{b}_0 - \mathbf{b}_2\|_2}$ .

**Lemma 4** For a unit ball  $\mathcal{U} := \{\mathbf{b} : \|(\mathbf{b} - \bar{\mathbf{b}})\|_2 \leq 1\} \in \mathbb{R}^m$ ,  $\lambda(\mathcal{U}) = m$ .

**Proof** Without loss of generality, let  $\mathcal{U}$  be a unit ball in  $\mathbb{R}^m$  (centered at the origin) and  $S$  be a regular (the distance of every two vertices is the same) maximum volume simplex inside the unit ball  $\mathcal{U}$ . Let  $\mathbf{v}_1, \dots, \mathbf{v}_{m+1}$  be vertices of such a simplex  $S$ . Then, we have:

1.  $\|\mathbf{v}_i\|_2 = 1$  for  $i = 1, \dots, m+1$ .
2.  $\mathbf{v}_i \cdot \mathbf{v}_j = -\frac{1}{m}$  for  $i \neq j$ .

We next show that  $S \subset \mathcal{U} \subset m \cdot S$ . We first claim that for any point  $\mathbf{w}$  in the boundary of  $S$ , we have  $\|\mathbf{w}\|_2 \geq \frac{1}{m}$ . Since  $\mathbf{w}$  is on the boundary of  $S$ , it is on one of its facets. Without loss of generality, we can assume that  $\mathbf{w}$  is on the facet containing the vertices  $\mathbf{v}_1, \dots, \mathbf{v}_m$ . Therefore,

$$\mathbf{w} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_m, \quad \alpha_1 + \dots + \alpha_m = 1.$$

We then have

$$\begin{aligned} \|\mathbf{w}\|_2^2 &= \mathbf{w}^T \cdot \mathbf{w} = \alpha_1^2 \|\mathbf{v}_1\|_2^2 + \dots + \alpha_m^2 \|\mathbf{v}_m\|_2^2 + 2 \sum_{i \neq j} \alpha_i \alpha_j \mathbf{v}_i^T \cdot \mathbf{v}_j \\ &= \alpha_1^2 + \dots + \alpha_m^2 - \frac{2}{m} \sum_{i \neq j} \alpha_i \alpha_j \\ &\geq \frac{1}{m} - \frac{(m-1)}{m^2} \geq \frac{1}{m^2}, \end{aligned}$$

implying that  $\|\mathbf{w}\|_2 \geq \frac{1}{m}$  for any point  $\mathbf{w}$  in the boundary of  $S$ . Thus, a ball of radius  $\frac{1}{m}$  is contained in  $S$ , i.e.,  $\frac{\mathcal{U}}{m} \subset S$ . So, we have:

$$S \subset \mathcal{U} \subset m \cdot (S - \mathbf{0}) + \mathbf{0}.$$

Therefore,  $\lambda(\mathcal{U}) \leq m$ .

On the other hand, consider

$$\hat{\mathbf{w}} := \frac{1}{m} \mathbf{v}_1 + \dots + \frac{1}{m} \mathbf{v}_m.$$

We then have

$$\begin{aligned} \|\hat{\mathbf{w}}\|_2^2 &= \hat{\mathbf{w}}^T \cdot \hat{\mathbf{w}} = \frac{1}{m}^2 \|\mathbf{v}_1\|_2^2 + \dots + \frac{1}{m}^2 \|\mathbf{v}_m\|_2^2 + 2 \sum_{i \neq j} \frac{1}{m}^2 \mathbf{v}_i^T \cdot \mathbf{v}_j \\ &= \frac{1}{m^2} + \dots + \frac{1}{m^2} - \frac{2}{m} \sum_{i \neq j} \frac{1}{m^2} \\ &= \frac{1}{m} - \frac{(m-1)}{m^2} = \frac{1}{m^2}, \end{aligned}$$

This implies that  $\|\hat{\mathbf{w}}\|_2 = \frac{1}{m}$ , and so  $m$  is the smallest number  $\lambda(\mathcal{U})$  such that

$$S \subset \mathcal{U} \subset \lambda(\mathcal{U}) \cdot (S - \mathbf{0}) + \mathbf{0}.$$

Therefore,  $\lambda(\mathcal{U}) = m$ . ■

We proceed with calculating the simplex dilation factor for different sets. We omit the proofs as they are not particularly illuminating.

**Lemma 5** *The simplex dilation factors for the following sets  $\mathcal{U}$  are:*

1. For  $\mathcal{U}_1 := \{\mathbf{b} : \|\mathbf{b}\|_p \leq 1, \mathbf{b} \geq \mathbf{0}\}$ ,  $\lambda(\mathcal{U}_1) = m^{\frac{p-1}{p}}$ .
2. For  $\mathcal{U}_2 := \{\mathbf{b} : \|\mathbf{E}(\mathbf{b} - \bar{\mathbf{b}})\|_2 \leq 1\} \subset \mathbb{R}_+^m$  where  $\mathbf{E}$  is invertible,  $\lambda(\mathcal{U}_2) = m$ .
3. For  $\mathcal{U}_3 := \left\{ \mathbf{b} \in [0, 1]^m \mid \sum_{i=1}^m b_i = k \right\}$ , for  $k \in \{1, \dots, m\}$ ,  $\lambda(\mathcal{U}_3) \leq k$ .



Finally we provide bounds for  $\lambda(\mathcal{U})$ .

**Theorem 1 (Lagarias and Ziegler (1989))** *Let  $\mathcal{U}$  be a convex body in  $\mathbb{R}^m$  and let  $S$  be a simplex of maximal volume contained in  $\mathcal{U}$  and assume that the centroid of  $S$  is at the origin. Then  $S \subset \mathcal{U} \subset (m+2) \cdot S$ , i.e.,  $\lambda(\mathcal{U}) \leq m+2$ .*

### 3 On the Performance of Affine Policies

In this section, we state and prove a new bound on the suboptimality of affine policies. We first state a result from the literature that affine policies are optimal when the uncertainty set is a simplex.

**Theorem 2 (Bertsimas and Goyal (2012))** *Consider the problem  $\Pi_{Adapt}(\mathcal{U})$  such that  $\mathcal{U}$  is a simplex, i.e.,  $\mathcal{U} = \text{conv}\{\mathbf{b}^1, \dots, \mathbf{b}^{m+1}\}$ , where  $\mathbf{b}^j \in \mathbb{R}^m$  for all  $j = 1, \dots, m$  such that  $\mathbf{b}^1, \dots, \mathbf{b}^{m+1}$  are affinely independent. Then, there is an optimal two-stage solution  $\mathbf{x}, \mathbf{y}(\mathbf{b})$  for all  $\mathbf{b} \in \mathcal{U}$  such that for all  $\mathbf{b} \in \mathcal{U}$ ,  $\mathbf{y}(\mathbf{b}) = \mathbf{P}\mathbf{b} + \mathbf{q}$ , where  $\mathbf{P}$  is a real matrix of dimension  $n_2$  and  $m$ .*

We next present a lemma that prepares the ground for our main result.

**Lemma 6** *Consider a convex and compact uncertainty set  $\mathcal{U}$ , with a symmetry point  $\mathbf{b}_0$ , symmetry  $s$ , translation factor  $\rho$  and simplex dilation factor  $\lambda$ . For  $\alpha = \frac{\lambda(1 + \frac{\rho}{s})}{\lambda + \frac{\rho}{s}}$  we have*

$$\alpha \cdot \left( \frac{1}{\lambda}(\mathbf{b} - \mathbf{b}_0) + \mathbf{b}_0 \right) \geq \mathbf{b}.$$

**Proof** Let  $\alpha = \frac{\lambda(1 + \frac{\rho}{s})}{\lambda + \frac{\rho}{s}}$ , leading to

$$1 + \frac{\rho}{s} = \frac{\alpha - \frac{\alpha}{\lambda}}{1 - \frac{\alpha}{\lambda}}.$$

Applying Lemma 3, and substituting the above expression for  $1 + \frac{\rho}{s}$ , we obtain that for all  $\mathbf{b} \in \mathcal{U}$ ,

$$\left( \alpha - \frac{\alpha}{\lambda} \right) \mathbf{b}_0 \geq \left( 1 - \frac{\alpha}{\lambda} \right) \mathbf{b},$$

leading to

$$\alpha \left( \frac{1}{\lambda}(\mathbf{b} - \mathbf{b}_0) + \mathbf{b}_0 \right) \geq \mathbf{b}.$$

■

We are now ready to prove the main result of the paper.

**Theorem 3** *Consider a convex and compact uncertainty set  $\mathcal{U}$ , with a symmetry point  $\mathbf{b}_0$ , symmetry  $s$ , translation factor  $\rho$  and simplex dilation factor  $\lambda$ . Then,*

$$z_{Adapt}(\mathcal{U}) \leq z_{Aff}(\mathcal{U}) \leq \frac{\lambda(1 + \frac{\rho}{s})}{\lambda + \frac{\rho}{s}} \cdot z_{Adapt}(\mathcal{U}).$$

**Proof**

First we show that when the uncertainty set is the simplex  $S \subset \mathcal{U}$ , there is a two-stage affine optimal solution  $\mathbf{x}$  and  $\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}$ , where  $\mathbf{q} \geq \mathbf{0}$ .

Applying Theorem 2, we know that when the uncertainty set is the simplex  $S \subset \mathcal{U}$ , there is an optimal affine solution  $\mathbf{x}$  and  $\mathbf{P}(\mathbf{b}) + \bar{\mathbf{q}}$ . We define  $\mathbf{q} := \bar{\mathbf{q}} + \mathbf{P}(\mathbf{b}_0)$ . Therefore  $\mathbf{x}$  and  $\mathbf{y}(\mathbf{b}) = \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}$  is an optimal affine solution for the simplex  $S \subset \mathcal{U}$ . By the properties of  $\Pi_{Adapt}(\mathcal{U})$ ,  $\mathbf{y}(\mathbf{b}_0) = \mathbf{P}(\mathbf{b}_0 - \mathbf{b}_0) + \mathbf{q} = \mathbf{q} \geq \mathbf{0}$ . Therefore, there is a two-stage optimal affine solution  $\mathbf{x}$  and  $\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}$ , where  $\mathbf{q} \geq \mathbf{0}$ .

Starting with the solutions  $\mathbf{x}$  and  $\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}$  and letting  $\alpha = \frac{\lambda(1 + \frac{\rho}{s})}{\lambda + \frac{\rho}{s}}$ , we define the following affine solution for the uncertainty set  $\mathcal{U}$  as follows:

$$\hat{\mathbf{x}} := \alpha \mathbf{x},$$

and

$$\hat{\mathbf{y}}(\mathbf{b}) := \frac{\alpha}{\lambda} \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \alpha \mathbf{q}.$$

We first show that  $\hat{\mathbf{x}}$  and  $\hat{\mathbf{y}}(\mathbf{b})$  is a feasible affine solution for  $\Pi_{Adapt}(\mathcal{U})$ .

1.  $\mathbf{x} \geq \mathbf{0}$ , therefore we have  $\hat{\mathbf{x}} = \alpha \mathbf{x} \geq \mathbf{0}$ .
2. Now we show that

$$\hat{\mathbf{y}}(\mathbf{b}) = \frac{\alpha}{\lambda} \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \alpha \mathbf{q} \geq \mathbf{0}.$$

Considering the definition of the simplex dilation factor, we know that

$$\mathcal{U} \subset \lambda \cdot (S - \mathbf{b}_0) + \mathbf{b}_0.$$

Therefore, for any  $\mathbf{b} \in \mathcal{U}$ ,  $\hat{\mathbf{b}} = \frac{\mathbf{b} - \mathbf{b}_0}{\lambda} + \mathbf{b}_0 \in S$ .

We know that  $\mathbf{x}$  and  $\mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q}$  is a feasible solution for  $\Pi_{Adapt}(S)$ . So,

$$\begin{aligned} & \mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q} \geq \mathbf{0} \\ \Rightarrow & \frac{1}{\lambda} \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q} \geq \mathbf{0} \\ \Rightarrow & \alpha \left( \frac{1}{\lambda} \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q} \right) \geq \mathbf{0} \\ \Rightarrow & \hat{\mathbf{y}}(\mathbf{b}) = \frac{\alpha}{\lambda} \mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \alpha \mathbf{q} \geq \mathbf{0}. \end{aligned}$$

3. Now, we show that

$$\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{y}}(\mathbf{b}) \geq \mathbf{b}, \forall \mathbf{b} \in \mathcal{U}.$$

Again, considering the definition of the simplex dilation factor, we know that

$$\mathcal{U} \subset \lambda \cdot (S - \mathbf{b}_0) + \mathbf{b}_0.$$

Therefore, for any  $\mathbf{b} \in \mathcal{U}$ ,  $\hat{\mathbf{b}} = \frac{\mathbf{b} - \mathbf{b}_0}{\lambda} + \mathbf{b}_0 \in S$ . Therefore,  $\mathbf{x}$  and  $\mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q}$  is a feasible solution for  $\Pi_{Adapt}(S)$ .

$$\begin{aligned}
& \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q}) \geq \hat{\mathbf{b}} \\
& \Rightarrow \mathbf{A}\mathbf{x} + \mathbf{B}\left(\frac{1}{\lambda}\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}\right) \geq \frac{\mathbf{b} - \mathbf{b}_0}{\lambda} + \mathbf{b}_0 \text{ (by multiplying with } \alpha) \\
& \Rightarrow \alpha(\mathbf{A}\mathbf{x} + \mathbf{B}\left(\frac{1}{\lambda}\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}\right)) \geq \frac{\alpha}{\lambda}(\mathbf{b} - \mathbf{b}_0) + \alpha\mathbf{b}_0 \\
& \Rightarrow \mathbf{A}\alpha\mathbf{x} + \mathbf{B}\left(\frac{\alpha}{\lambda}\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \alpha\mathbf{q}\right) \geq \frac{\alpha}{\lambda}(\mathbf{b} - \mathbf{b}_0) + \alpha\mathbf{b}_0 \\
& \text{By the definition of } \hat{\mathbf{x}} \text{ and } \hat{\mathbf{y}}(\mathbf{b}) \text{ and Lemma 6} \\
& \Rightarrow \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\hat{\mathbf{y}}(\mathbf{b}) \geq \mathbf{b}.
\end{aligned}$$

It only remains to show

$$z_{\text{Aff}}(\mathcal{U}) \leq \alpha \cdot z_{\text{Adapt}}(\mathcal{U}).$$

Recall that for any  $\mathbf{b} \in \mathcal{U}$ ,  $\hat{\mathbf{b}} = \frac{\mathbf{b} - \mathbf{b}_0}{\lambda} + \mathbf{b}_0 \in \mathcal{S}$ .

$$\begin{aligned}
& z_{\text{Aff}}(\mathcal{U}) \\
& \leq \min \mathbf{c}^T \hat{\mathbf{x}} + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{d}^T \hat{\mathbf{y}}(\mathbf{b}) \\
& = \min \mathbf{c}^T (\alpha\mathbf{x}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{d}^T \left(\frac{\alpha}{\lambda}\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \alpha\mathbf{q}\right) \\
& = \alpha \left[ \min \mathbf{c}^T (\mathbf{x}) + \max_{\mathbf{b} \in \mathcal{U}} \mathbf{d}^T \left(\frac{1}{\lambda}\mathbf{P}(\mathbf{b} - \mathbf{b}_0) + \mathbf{q}\right) \right] \text{ (replacing } \hat{\mathbf{b}} = \frac{\mathbf{b} - \mathbf{b}_0}{\lambda} + \mathbf{b}_0 \in \mathcal{S}) \\
& = \alpha \left[ \min \mathbf{c}^T (\mathbf{x}) + \max_{\hat{\mathbf{b}} \in \mathcal{S}} \mathbf{d}^T (\mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q}) \right] \\
& \leq \alpha \cdot z_{\text{Aff}}(\mathcal{S}) \text{ (because } \mathbf{x} \text{ and } \mathbf{P}(\hat{\mathbf{b}} - \mathbf{b}_0) + \mathbf{q} \text{ is an optimal affine solution for } \mathcal{S}) \\
& = \alpha \cdot z_{\text{Adapt}}(\mathcal{S}) \text{ From Theorem 2) } \\
& \leq \alpha \cdot z_{\text{Adapt}}(\mathcal{U}) \text{ (because } \mathcal{S} \subset \mathcal{U}).
\end{aligned}$$

■

#### 4 On the Power of the New Bound

In this section, we compare the bound obtained in this paper with the bound in Bertsimas et al. (2011) for a variety of uncertainty sets. We first introduce two widely used uncertainty sets.

**Budgeted uncertainty set.** The budgeted uncertainty set is defined as,

$$\Delta_k := \left\{ \mathbf{b} \in [0, 1]^m \mid \sum_{i=1}^m b_i \leq k \right\}, \quad \text{for integer } k \in \{1, \dots, m\}. \quad (7)$$

Bertsimas et al. (2011) show that

$$\text{sym}(\Delta_k) = \frac{k}{m}, \quad \mathbf{b}_0(\Delta_k) = \frac{k}{m+k} \mathbf{e}. \quad (8)$$

**Demand uncertainty set.** We define the following set,

$$\mathbf{DU}(\mu, \Gamma) := \left\{ \mathbf{b} \in \mathbb{R}_+^m \mid \left| \frac{\sum_{i \in S} \mathbf{b}_i - \text{card}(S)\mu}{\sqrt{\text{card}(S)}} \right| \leq \Gamma, \forall S \subseteq N := \{1, \dots, m\} \right\}. \quad (9)$$

Such a set can model the demand uncertainty where  $\mathbf{b}$  is the demand of  $m$  products;  $\mu$  and  $\Gamma$  are the center and the span of the uncertain range.

The set  $\mathbf{DU}$  has different symmetry properties, depending on the relation between  $\mu$  and  $\Gamma$ . If  $\mu \geq \Gamma$ , the set  $\mathbf{DU}$  is in fact symmetric. Intuitively,  $\mathbf{DU}$  is the intersection of an  $L_\infty$  ball centered at  $\mu \mathbf{e}$  with  $(2^m - 2m)$  halfspaces that are symmetric with respect to  $\mu \mathbf{e}$ . If  $\mu < \Gamma$ ,  $\mathbf{DU}$  is not symmetric any more. Bertsimas et al. (2011) show that

**Proposition 1** *Assume the uncertainty set is  $\mathbf{DU}$ .*

1. If  $\mu \geq \Gamma$ , then

$$\mathbf{sym}(\mathbf{DU}) = 1, \quad \mathbf{b}_0(\mathbf{DU}) = \mu \mathbf{e}. \quad (10)$$

2. If  $\frac{1}{\sqrt{m}}\Gamma < \mu < \Gamma$ , then

$$\mathbf{sym}(\mathbf{DU}) = \frac{\sqrt{m}\mu + \Gamma}{(1 + \sqrt{m})\Gamma}, \quad \mathbf{b}_0(\mathbf{DU}) = \frac{(\sqrt{m}\mu + \Gamma)(\mu + \Gamma)}{\sqrt{m}\mu + (2 + \sqrt{m})\Gamma} \mathbf{e}. \quad (11)$$

3. If  $0 \leq \mu \leq \frac{1}{\sqrt{m}}\Gamma$ , then,

$$\mathbf{sym}(\mathbf{DU}) = \frac{\sqrt{m}\mu + \Gamma}{\sqrt{m}(\mu + \Gamma)}, \quad \mathbf{b}_0(\mathbf{DU}) = \frac{(\sqrt{m}\mu + \Gamma)(\mu + \Gamma)}{2\sqrt{m}\mu + (1 + \sqrt{m})\Gamma} \mathbf{e}. \quad (12)$$

Define  $\mathbf{conv}\{\Delta_1, \{\mathbf{e}\}\}$  to be the convex hull of the standard  $m$ -simplex  $\Delta_1 := \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{e}^T \mathbf{b} \leq 1, \mathbf{b} \geq 0\}$  and a point  $\mathbf{e} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ .

In Table 1, we calculate the symmetry and simplex dilation factor for several uncertainty sets.

No.	Uncertainty set	Symmetry	Upper bound for $\lambda$
1	$\{\mathbf{b} : \ \mathbf{b}\ _p \leq 1, \mathbf{b} \geq \mathbf{0}\}$	$m^{1/p}$	$m^{\frac{p-1}{p}}$
2	$\{\mathbf{b} : \ \mathbf{E}(\mathbf{b} - \bar{\mathbf{b}})\ _2 \leq 1\} \subset \mathbb{R}_+^m$ where $\mathbf{E}$ is invertible	1	$m$
3	$\{\mathbf{b} : \ \mathbf{b}\ _{p_1} \leq 1, \ \mathbf{b}\ _{p_2} \leq r, \mathbf{b} \geq \mathbf{0}\}$	$\frac{1}{rm^{1/p_1}}$	$\max\left\{m^{\frac{p_1-1}{p_1}}, rm^{\frac{p_2-1}{p_2}}\right\}$
4	$\Delta_k$	$\frac{k}{m}$	$k$
5	$\mathbf{DU}(\mu, \Gamma), \mu \geq \Gamma$	1	$\frac{m\mu}{\Gamma + \mu} + \frac{\Gamma\sqrt{m}}{\mu + \Gamma}$
6	$\mathbf{DU}(\mu, \Gamma), \frac{1}{\sqrt{m}}\Gamma < \mu < \Gamma$	$\frac{\sqrt{m}\mu + \Gamma}{(1 + \sqrt{m})\Gamma}$	$\frac{m\mu}{\Gamma + \mu} + \frac{\Gamma\sqrt{m}}{\mu + \Gamma}$
7	$\mathbf{DU}(\mu, \Gamma), \mu \leq \frac{1}{\sqrt{m}}\Gamma$	$\frac{\sqrt{m}\mu + \Gamma}{\sqrt{m}(\mu + \Gamma)}$	$\frac{m\mu}{\Gamma + \mu} + \frac{\Gamma\sqrt{m}}{\mu + \Gamma}$
8	$\mathbf{conv}\{\Delta_1, \{\mathbf{e}\}\}$	$\frac{1}{m-1}$	$\frac{m}{m-1}$
9	$\{\mathbf{B} \in \mathbb{S}_+^m \mid \mathbf{I} \bullet \mathbf{B} \leq 1\}$	$\frac{1}{m}$	1

**Table 1** Bounds on the symmetry and the simplex dilation factor for various uncertainty sets. In uncertainty set 9,  $\mathbb{S}_+^m$  is the cone of positive semidefinite matrices, and  $\mathbf{I}$  is the identity matrix.  $\mathbf{I} \bullet \mathbf{B}$  is the trace of the multiplication of  $\mathbf{I}$  and  $\mathbf{B}$ . For more details, see Bertsimas et al. (2011).

In Table 2, we compare the proposed bound on the suboptimality of affine policies to the bound on the suboptimality of robust policies for  $\Pi_{Adapt}(\mathcal{U})$ , which were obtained in Bertsimas et al. (2011). In the following tables  $k$  is an integer. In Table 2, we define Eqs (13), (14) and (15):

$$\frac{\left(\frac{\Gamma}{\mu} + 1\right) \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}}{\frac{\Gamma}{\mu} + 1 + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \quad (13)$$

$$\frac{\left(\frac{\Gamma + \Gamma\sqrt{m}}{\Gamma + \mu\sqrt{m}} + 1\right) \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}}{\frac{\Gamma + \Gamma\sqrt{m}}{\Gamma + \mu\sqrt{m}} + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \quad (14)$$

$$\frac{\left(\frac{\sqrt{m}(\mu + \Gamma)}{\sqrt{m}\mu + \Gamma} + 1\right) \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}}{\frac{\sqrt{m}(\mu + \Gamma)}{\sqrt{m}\mu + \Gamma} + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \quad (15)$$

No.	Uncertainty set	Bound on $\frac{z_{Rob}}{z_{Adapt}}$	Bound on $\frac{z_{Aff}}{z_{Adapt}}$
1	$\{\mathbf{b} : \ \mathbf{b}\ _p \leq 1, \mathbf{b} \geq \mathbf{0}\}$	$1 + m^{\frac{1}{p}}$	$\frac{m^{\frac{p-1}{p}} + m}{m^{\frac{1}{p}} + m}$
2	$\{\mathbf{b} : \ \mathbf{E}(\mathbf{b} - \bar{\mathbf{b}})\ _2 \leq 1\} \subset \mathbb{R}_+^m$ where $\mathbf{E}$ is invertible	$1 + \max_{i \in \{1, \dots, m\}} \frac{E_{ii}^{-1}}{b_i}$	$\frac{m(1 + \max \frac{E_{ii}^{-1}}{b_i})}{m + \max \frac{E_{ii}^{-1}}{b_i}}$
3	$\{\mathbf{b} : \ \mathbf{b}\ _{p_1} \leq 1, \ \mathbf{b}\ _{p_2} \leq r, \mathbf{b} \geq \mathbf{0}\}$	$\frac{1}{rm^{1/p_1}}$	$\frac{s(1 + rm^{\frac{1}{p_1}})}{s + rm^{\frac{1}{p_1}}}$ , $s = \max \left\{ m^{\frac{p_1-1}{p_1}}, rm^{\frac{p_2-1}{p_2}} \right\}$
4	$\Delta_k$	$1 + \frac{m}{k}$	$\frac{k+m}{k + \frac{m}{k}}$
5	$\mathbf{DU}(\mu, \Gamma), \mu \geq \Gamma$	$1 + \frac{\Gamma}{\mu}$	Eq.(13)
6	$\mathbf{DU}(\mu, \Gamma), \frac{1}{\sqrt{m}}\Gamma < \mu < \Gamma$	$1 + \frac{(1 + \sqrt{m})\Gamma}{\sqrt{m}\mu + \Gamma}$	Eq.(14)
7	$\mathbf{DU}(\mu, \Gamma), \mu \leq \frac{1}{\sqrt{m}}\Gamma$	$1 + \frac{\sqrt{m}(\mu + \Gamma)}{\sqrt{m}\mu + \Gamma}$	Eq.(15)
8	$\mathbf{conv}\{\Delta_1, \{\mathbf{e}\}\}$	$m$	$\frac{m^2 + m}{m^2}$
9	$\{\mathbf{B} \in \mathbb{S}_+^m \mid \mathbf{I} \bullet \mathbf{B} \leq 1\}$	$1 + m$	1

**Table 2** The ratios  $\frac{z_{Rob}}{z_{Adapt}}$  and  $\frac{z_{Aff}}{z_{Adapt}}$  for various uncertainty sets.  $\mathbf{I} \bullet \mathbf{B}$  is the trace of the multiplication of  $\mathbf{I}$  and  $\mathbf{B}$ .

In Table 3, we calculate the ratios  $\frac{z_{Aff}}{z_{Adapt}} / \frac{z_{Rob}}{z_{Adapt}}$ , which demonstrates that the improvement of the performance of the affine solution compared to the robust solution can be significant.

No.	Uncertainty set	Bound on the ratio $\frac{z_{Aff}}{z_{Adapt}} / \frac{z_{Rob}}{z_{Adapt}}$
1	$\{\mathbf{b} : \ \mathbf{b}\ _p \leq 1, \mathbf{b} \geq \mathbf{0}\}$	$\frac{m}{m^{\frac{1}{p}} + m}$
2	$\{\mathbf{b} : \ \mathbf{E}(\mathbf{b} - \bar{\mathbf{b}})\ _2 \leq 1\} \subset \mathbb{R}_+^m$ where $\mathbf{E}$ is invertible	$\frac{m}{m + \max_i \frac{E_{ii}^{-1}}{b_i}}$
3	$\Delta_k$	$\frac{k^2}{k^2 + m}$
4	$\mathbf{DU}(\mu, \Gamma), \mu \geq \Gamma$	$\frac{\left(\frac{\Gamma}{\mu} + 1\right)}{\frac{\Gamma}{\mu} + 1 + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \leq \frac{4}{4 + m + \sqrt{m}}$
5	$\mathbf{DU}(\mu, \Gamma), \frac{1}{\sqrt{m}}\Gamma < \mu < \Gamma$	$\frac{\frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}}{\frac{\sqrt{m}(\mu + \Gamma)}{\sqrt{m}\mu + \Gamma} + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \leq \frac{4m}{5m + 2\sqrt{m} + 1}$
6	$\mathbf{DU}(\mu, \Gamma), \mu \leq \frac{1}{\sqrt{m}}\Gamma$	$\frac{\frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}}{\frac{\Gamma + \Gamma\sqrt{m}}{\Gamma + \mu\sqrt{m}} + \frac{m\mu + \Gamma\sqrt{m}}{\Gamma + \mu}} \leq \frac{4m}{5m + 2\sqrt{m} + 1}$
7	$\text{conv}\{\Delta_1, \{\mathbf{e}\}\}$	$\frac{m}{(m-1)^2 + m}$
9	$\{\mathbf{B} \in \mathbb{S}_+^m \mid \mathbf{I} \bullet \mathbf{B} \leq 1\}$	$\frac{1}{m+1}$

**Table 3** Upper bound on the ratio  $\frac{z_{Aff}}{z_{Adapt}} / \frac{z_{Rob}}{z_{Adapt}}$ , or simply  $z_{Aff}/z_{Rob}$ .  $\mathbf{I} \bullet \mathbf{B}$  is the trace of the multiplication of  $\mathbf{I}$  and  $\mathbf{B}$ .

As illustrated in Table 3, the performance of the affine policy is significantly better than the robust solution for  $\Pi_{Adapt}(\mathcal{U})$  for many widely used uncertainty sets  $\mathcal{U}$ . This includes the budgeted uncertainty set  $\Delta_k$ , the demand uncertainty set  $\mathbf{DU}(\mu, \Gamma)$ ,  $\text{conv}\{\Delta_1, \{\mathbf{e}\}\}$  and  $\{\mathbf{B} \in \mathbb{S}_+^m \mid \mathbf{I} \bullet \mathbf{B} \leq 1\}$ .

## 5 Comparison with A Posteriori Bounds

In this numerical study, we consider a multi item two-stage inventory control model similar to the one discussed in Bertsimas and Georghiou (2013). The computational methods for solving the optimization problem and calculating the affine policy performance have been described in Hadjiyiannis et al. (2011b) and Kuhn et al. (2011). All of our numerical results are carried out using the IBM ILOG CPLEX 12.5 optimization package.

The inventory problem can be described as follows. At the beginning of the second period, the decision maker observes all product demands  $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathcal{U}$ .

Each product demand can be served by either placing an order  $x_i$ , with unit cost  $c_{x_i}$  in the first period, which will be delivered at the beginning of the second period, or by placing an order  $z_i(\mathbf{b})$  at the beginning of the second period which has a more expensive unit cost  $c_{z_i} < c_{x_i}$  and it is delivered immediately. If the ordered quantity is greater than the demand, the excess units are stored in a warehouse, incurring a unit holding cost  $c_{h_i}$ . If there is a shortfall in the available quantity, then the orders are backlogged incurring a unit cost  $c_{t_i}$ . The level of available

inventory for each item is given by  $I_i(\mathbf{b})$ . The decision maker wishes to determine the ordering levels for all products  $x_i$  and  $z_i(\mathbf{b})$  that minimize the total ordering, backlogging and holding costs. The problem can be formulated as the following two-stage adaptive optimization problem.

$$\begin{aligned} \min \sum_{i=1}^m c_{x_i} x_i + \max_{\mathbf{b} \in \mathcal{U}} \left( \sum_{i=1}^m c_{z_i} z_i(\mathbf{b}) + \sum_{i=1}^m \max\{c_{t_i} I_i(\mathbf{b}), c_{h_i} I_i(\mathbf{b})\} \right) \\ I_i(\mathbf{b}) = x_i + z_i(\mathbf{b}) - b_i \\ x_i \geq 0, z_i \geq 0, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in \mathcal{U}. \end{aligned} \quad (16)$$

We define  $\mathcal{U}^* := \bigcup_{\mathbf{b} \in \mathcal{U}} \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \\ \mathbf{0}_m \end{bmatrix}$  where  $\mathbf{0}_m := \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ . So  $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \\ \mathbf{0}_m \end{bmatrix} \in \mathcal{U}^*$  if and only if  $\mathbf{b} \in \mathcal{U}$ .

**Lemma 7** *The inventory control problem (16) can be reformulated to the standard form of Problem (1) as follows:*

$$\begin{aligned} \min \sum_{i=1}^m c_{x_i} x_i + \max_{\bar{\mathbf{b}} \in \mathcal{U}^*} \left( \sum_{i=1}^m c_{z_i} z_i(\bar{\mathbf{b}}) + \sum_{i=1}^m o_i(\bar{\mathbf{b}}) \right) \\ \frac{o_i(\bar{\mathbf{b}})}{2c_{h_i}} - x_i + s_i(\bar{\mathbf{b}}) - z_i(\bar{\mathbf{b}}) \geq 0 \\ - \frac{o_i(\bar{\mathbf{b}})}{c_{t_i}} + x_i + z_i(\bar{\mathbf{b}}) \geq b_i \\ 2s_i(\bar{\mathbf{b}}) - x_i - z_i(\bar{\mathbf{b}}) \geq b_i \\ o_i \geq 0, s_i \geq 0, x_i \geq 0, z_i \geq 0, \bar{\mathbf{b}} \in \mathcal{U}^*. \end{aligned} \quad (17)$$

**Proof** To bring the inventory control problem (16) into the standard form of Problem (17), we provide a reformulation with the following steps. We start with Problem (16).

- (a) First,  $\max\{c_{t_i} I_i(\mathbf{b}), c_{h_i} I_i(\mathbf{b})\}$  can be replaced by introducing auxiliary decision variables  $o_i(\mathbf{b})$ , serving as over-estimators for the max functions. Therefore, the objective function can be replaced with

$$\min \sum_{i=1}^m c_{x_i} x_i + \max_{\mathbf{b} \in \mathcal{U}} \left( \sum_{i=1}^m c_{z_i} z_i(\mathbf{b}) + \sum_{i=1}^m o_i(\mathbf{b}) \right).$$

and the following set of linear constraints are added to Problem (16)

$$\begin{aligned} o_i(\mathbf{b}) &\geq c_{t_i} I_i(\mathbf{b}) \\ o_i(\mathbf{b}) &\geq c_{h_i} I_i(\mathbf{b}) \\ o_i &\geq 0. \end{aligned} \quad (18)$$



- (b) Second, since for  $i = 1, \dots, m$ ,  $c_{t_i}$  is negative and  $c_{h_i}$  is positive, we replace Constraints (18) with the following

$$\begin{aligned} \frac{o_i(\mathbf{b})}{c_{h_i}} - x_i - z_i(\mathbf{b}) &\geq -b_i \\ -\frac{o_i(\mathbf{b})}{c_{t_i}} + x_i + z_i(\mathbf{b}) &\geq b_i \\ o_i &\geq 0. \end{aligned} \quad (19)$$

- (c) Third, we introduce new auxiliary decision variables  $s_1(\mathbf{b}), \dots, s_m(\mathbf{b})$  and we replace Constraints (19) with the following. Since  $s_1(\mathbf{b}), \dots, s_m(\mathbf{b})$  do not appear in the objective function, the optimal value of the optimization problem will remain the same.

$$\begin{aligned} \frac{o_i(\mathbf{b})}{2c_{h_i}} - \frac{x_i}{2} - \frac{z_i(\mathbf{b})}{2} &\geq -\frac{b_i}{2} \\ -\frac{o_i(\mathbf{b})}{c_{t_i}} + x_i + z_i(\mathbf{b}) &\geq b_i \\ 2s_i(\mathbf{b}) - x_i - z_i(\mathbf{b}) &\geq b_i \\ o_i, s_i &\geq 0. \end{aligned} \quad (20)$$

- (d) In the last step, we replace Constraints (20) with Constraints (21).

$$\begin{aligned} \frac{o_i(\mathbf{b})}{2c_{h_i}} - x_i + s_i(\mathbf{b}) - z_i(\mathbf{b}) &\geq 0 \\ -\frac{o_i(\mathbf{b})}{c_{t_i}} + x_i + z_i(\mathbf{b}) &\geq b_i \\ 2s_i(\mathbf{b}) - x_i - z_i(\mathbf{b}) &\geq b_i \\ o_i, s_i &\geq 0. \end{aligned} \quad (21)$$

It is clear that the optimal value of the optimization problem (16) do not change during the steps (a), (b) and (c). For step (d), every  $o_i(\mathbf{b}), s_i(\mathbf{b}), x_i, z_i(\mathbf{b})$  which satisfy Constraints (20), also satisfy Constraints (21).

In addition, we can always find  $o_i(\hat{\mathbf{b}}), s_i(\hat{\mathbf{b}}), x_i, z_i(\hat{\mathbf{b}})$  satisfy Constraints (21) and take the optimal value and also satisfy Constraints (20). So replacing Constraints (20) with Constraints (21) does not change the optimal value of the

optimization problem. Moreover, since  $\bar{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ \mathbf{b} \\ \mathbf{0}_m \end{bmatrix} \in \mathcal{U}^*$  if and only if  $\mathbf{b} \in \mathcal{U}$ .

We can define  $s_i(\bar{\mathbf{b}}) := s_i(\mathbf{b})$ ,  $o_i(\bar{\mathbf{b}}) := o_i(\mathbf{b})$  and  $z_i(\bar{\mathbf{b}}) := z_i(\mathbf{b})$  and reformulate Problem (16) to Problem (17). ■

For our computational experiments we randomly generate instances of Problem (17) for various uncertainty sets, and compare the a posteriori bounds for the optimal affine policy with our geometric a priori bound. The parameters are randomly chosen using a uniform distribution from the following sets: advanced and

No.	Uncertainty set	A posteriori bound for affine policy	Geometric bound for affine policy	Geometric bound for robust policy
1	$\Delta_{(5,25)}^*$	2.78	3	5
2	$\Delta_{(6,36)}^*$	3.23	3.5	6
3	$\Delta_{(7,49)}^*$	2.67	4	7
4	$\Delta_{(8,64)}^*$	2.64	4.5	8
5	$\Delta_{(9,81)}^*$	3.21	5	9
6	$\Delta_{(10,100)}^*$	3.1	5.5	10
7	$\mathbf{DU}^*(2, 4, 4)$	1.48	1.6	2.5
8	$\mathbf{DU}^*(3, 9, 9)$	2.06	2.07	3
9	$\mathbf{DU}^*(2, 10, 10)$	1.58	2.15	3.32

**Table 4** Comparing Geometric Bounds and Computational Bounds. The bounds come from randomly generated data for  $c_{h_i}, c_{t_i}, c_{x_i}, c_{z_i}$  as explained in the beginning of the section. For interested readers, we put this data at the following address: [http://web.mit.edu/bidkhori/www/index/Hoda.Bidkhori\\_files/data.xls](http://web.mit.edu/bidkhori/www/index/Hoda.Bidkhori_files/data.xls).

instant ordering costs are chosen from  $c_{x_i} \in [0, 5]$  and  $c_{z_i} \in [0, 10]$ , respectively, such that  $c_{x_i} < c_{z_i}$ . Backlogging and holding costs are elements of  $c_{t_i} \in [-10, 0]$  and  $c_{h_i} \in [0, 5]$ , respectively.

We consider  $\mathbf{DU}^*(\mu, \Gamma, m)$ , where

$$\mathbf{DU}(\mu, \Gamma, m) := \left\{ \mathbf{b} \in \mathbb{R}_+^m \mid \left| \frac{\sum_{i \in S} \mathbf{b}_i - \text{card}(S)\mu}{\sqrt{\text{card}(S)}} \right| \leq \Gamma, \forall S \subseteq N := \{1, \dots, m\} \right\}.$$

Also we consider  $\Delta_{(k,m)}^*$ , where

$$\Delta_{(k,m)} := \left\{ \mathbf{b} : \mathbf{b} \in [0, 1]^m \mid \sum_{i=1}^m b_i \leq k \right\}, \quad \text{for } k \in \{1, \dots, m\}. \quad (22)$$

Following the definition of symmetry and simplex dilation factor, we have  $\mathbf{sym}(\mathcal{U}^*) = \mathbf{sym}(\mathcal{U})$  and also  $\lambda(\mathcal{U}^*) = \lambda(\mathcal{U})$ .

We compare the a posteriori bounds for the optimal affine policy with our geometric a priori bound for randomly generated instances of Problem (17) in Table 4. We find that the geometric a priori bound is quite close to the a posteriori bounds for random instances of the inventory control problem. This suggests that our geometric a priori bound is indeed informative.

## 6 Concluding Remarks

In this paper, we evaluate the performance of the affine policy for two stage adaptive optimization problems based on the geometry of the uncertainty sets. Bertsimas et al. (2011) has linked the performance of adaptive and robust policies based

on the symmetry of the uncertainty set. In this work we link the performance of adaptive and affine policies based on the notion of the simplex dilation factor of the uncertainty set. We found (see Table 4) that our a priori bounds for affine policies significantly improve on the symmetry bounds for robust policies, and are close to the a posteriori bounds. Natural extensions of this work include the study of the performance of affine and piecewise affine policies for multi-stage adaptive optimization problems.

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