Instability-Induced Transformation of Interfacial Layers in Composites and its Multifunctional Applications

by

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Submitted to the Department of Mechanical Engineering in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

February 2016

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Abstract

Inspired by the undulating patterns found in different plant and biological cells in nature, this thesis explores the deformation-induced wrinkling instability transformation in the microstructure of composites with thin interfacial layers. The hybrid microstructure of a composite material has an essential influence on the effective properties and behavior of the composite. Hence, in this research, the principles and mechanics of interfacial layer instabilities are purposely designed to achieve sudden pattern transformations in the composite structure to generate new controlled multifunctional behavior.

The wrinkling instability transformation is investigated in multilayered and networked composites consisting of relatively stiff interfacial layers or cells embedded in a soft matrix. Through analytical modeling, finite element simulations, and physical experiments, a comprehensive study is performed to elucidate: i) the conditions governing wrinkling, ii) the effect of wrinkling on local deformations, iii) the energy stored and dissipated in wrinkling composites, and iv) the change in the effective mechanical behavior of the composites. It is concluded that these properties are directly dependent on the composite’s material and geometrical properties, and that composites can be designed to exhibit enhanced energy absorption (including energy storage and dissipation), stress mitigation effects, bilinear and multi-linear elastic stress-strain behavior, switchable effective stiffness, and can be used for energy harvesting applications.

Design guidelines are presented to assist the process of deploying instability-transformation to tune, control, and switch the mechanical properties of multifunctional composite materials. Instability principles such as buckling and wrinkling are favorable mechanisms as they can be designed to be elastic and therefore reversible. The ability to alter and transform the microstructure enables on-demand tunability and active control of the composite’s properties and attributes.

Thesis Supervisor: Mary C. Boyce, Ford Professor of Engineering

Thesis Supervisor: Nicholas X. Fang, Associate Professor of Engineering
Acknowledgement

As I finish an important era of my life with this PhD thesis, I would like to express my appreciation to several key people who have been supportive scientifically, mentally, or emotionally through these past four years. First of all, I would like to thank my advisor, Professor Mary C. Boyce, for all her immense support and guidance during my research work. She has been a great inspiration and role model for me both in science and in life, and I deeply appreciate her continuous support, motivation, and encouragement. I am also indebted to my co-advisor, Professor Nicholas X. Fang, for going above and beyond to help me in any way he can. He has always been available for me, and I am thankful for all his technical guidance, attention, and care.

I am also grateful to the rest of my wonderful thesis committee, Professor Brian Wardle and Professor Pedro Reis, who always provided me valuable advice, feedback, and encouragement. I appreciated all my interactions with them, and always felt more inspired by their comments. I would also like to thank Professor David Parks for always welcoming me to meet him to discuss fascinating and challenging areas of mechanics. Also, a special thanks to Professor Yaning Li, who worked closely with me in the beginning of my project, showing patience and providing me guidance and insight. In addition, I would like to express my appreciation to Leslie Regan, Juliette Pickering, and the rest of the mechanical engineering support staff for being amazingly helpful, positive, and making everything run smoothly.

Thank you to all the group members of the Boyce and Fang group for all your help and friendship; it has been a joy working with you all, and thank you for making my PhD journey more complete. A special thanks to Mark Guttag for being a wonderful office mate, and making my every day of work more pleasant and enjoyable. I also wish to express my gratitude to anyone who has been involved with, has helped, and has supported in developing MITxplore. It has been a fantastic experience starting this organization, and it has been a privilege for me to work with so many dedicated MIT students.

Last, but not least, I would like to sincerely thank my family and friends; they have been my unlimited source of inspiration. I am forever thankful to my family for always believing in me, supporting me, and encouraging me to aim higher. I cannot express how grateful I am for having such magnificent parents, Fariba and Amir, who are always there for me ready to help, comfort, support and motivate me. I also thank my sister, Mariam, for being the best role model for me, and I am grateful for being able to follow in her footsteps. A special thanks to Thomas Sayre-McCord for his endless support, encouragement, love, and making every day a great day. I would also like to express my appreciation to all my wonderful friends both at MIT and far away. Thank you for always being there for me, for all the good memories, all the fun, and for making this the best journey so far in my life. Thanks to all you remarkable people in my life; I am very excited for the next steps in my life, and I feel extremely fortunate to have all my dear family and friends along with me.
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Chapter 1

Introduction

1.1 Motivation

Many structured composites in nature possess undulating and wrinkled interfacial layers that are found to regulate mechanical [1], chemical [2], acoustic, adhesive [3], thermal, electrical [4,5] and optical functions [6,7], and also serve to reveal underlying physiological mechanisms in the diagnosis of diseases [8]. As one example, the wavy layers of the silvery reflectors around the eyes of squid [6] (L. Pealeii) facilitate broadband reflectance and result in silvery iridescence, providing inspiration for creating structures of tailored stacks of wavy layers to achieve different optical functions (Figure 1.1a). Another example is the wavy elastica lamellae embedded in smooth muscle in arterial walls which provides a diagnostic signature for disease. The degree of the elastica undulation in the distal section of the right coronary artery was found to be significantly greater in persons with coronary artery disease compared to that found in the non-coronary disease group [8](Figure 1.1b). As a third example, wavy undulating patterns are observed in many plant epidermis cell walls (Figure 1.1c and d) and are shown to affect the mechanical properties of the cell. The plant epidermis is a layer of cells between the outmost cuticle layer and the ground tissue, which has an important architectural control element regulating the growth properties of underlying tissues, and hence also the mechanical properties of the cell.

We were inspired by these wrinkled structures in nature to investigate the wrinkling instability, and a good understanding of the wrinkling instability can provide clues to other underlying structural, bio-chemical, and mechanical attributes and behaviours. A comprehensive investigation of the mechanism governing the wrinkling formation, as well as
the effect of the wrinkling on the macroscopic properties of the composite material, allows us to actively transform the interfacial layer microstructure to achieve new composite behavior, and enables on-demand tunability of the composites' attributes to provide, for example, active control of effective mechanical stiffness and deformation, energy storage and dissipation properties, material swelling and growth, and wave propagation phenomenon (e.g., phononic and photonic).

Figure 1.1: Examples of undulating patterns observed in nature; a) Reflectors on L. Pealeii squid [7]; b) Arterial walls [8]; c) Cells of seed coat of Viscaria vulgaris [2] d) Epidermis cells of Arabidopsis [9].
From a mechanical point of view, cells with wrinkled interfacial layers or cell walls, can be modelled as a planar composite structures where the cell cores are a soft elastic matrix, while the interfacial layer or cell wall is modelled as a much thinner and stiffer elastic material. For example, focusing on one epidermis wall, we can model the structure as a composite composed of a stiff interfacial layer (a beam) embedded in a soft elastic matrix, as shown in Figure 1.2. When the plant cells are growing, there will be compressive stresses created in the cell walls and interfacial layer due to the constrained growth of the cells. Once the compressive stresses in the interfacial layer reaches a critical value, which is determined by the composite’s material and geometrical features, instability occurs in the interfacial layer and the initially straight interfacial layer transforms to a wrinkled interfacial layer. This instability-induced transformation of the interfacial layer provides us the opportunity to design on-demand tunable composite materials.

![Figure 1.2](image.png)

**Figure 1.2:** Schematic showing how the epidermis plant cells can be modeled as a planar composite structure consisting of a straight and relatively stiffer interfacial layer embedded in a soft matrix. Under a critical compressive load the interfacial layer will undergo instability and transform into a wrinkling pattern.

Tunable and active materials have emerged as a promising class of materials which exhibit changes in properties or behavior on-demand based on the external environment, allowing the creation of interesting adaptable multifunctional materials. [10,11,12,13,14,15,16]. The tunability can be governed by the inherent material behavior [17,18,19,20] as well as by proper geometric design of the constituent materials [21,22,23,24,25,26]. An interesting method to achieve switchable and tunable materials is by exploiting mechanical instabilities...
within the structure. It has been shown that one can deliberately deploy the principles and mechanics of instabilities to achieve on-demand ubiquitous pattern transformations in structured materials, e.g. [21,27]. Instability mechanisms, such as buckling and wrinkling, are particularly effective as they can be designed to remain elastic during the loading cycle; hence the instability mechanism can be reversible and repeatable.

Inspired by the wrinkling observed in nature, and the potential for using the wrinkling instability transformation to achieve tunable materials, we were motivated to investigate and develop design guidelines for composite with multi-functional, switchable and mechanomutable properties deploying the wrinkling instability. The ability to actively alter the microstructure of a composite can enable on-demand tunability, and enables us to create multifunctional composites and hybrid materials.
1.2 Objective of thesis

The objective of this thesis is to use instability-induced transformation of interfacial layers in composites to achieve new multifunctional behavior. We explore the effect of the wrinkling in the microstructure of composites with thin interfacial layers, to create composites with on-demand tunability and active control of its effective mechanical properties, behavior, and attributes.

More specifically, the objectives of this thesis are:

- Understand the wrinkling mechanism in the interfacial layers or cell walls of composites and create design guidelines predicting the critical conditions for achieving wrinkling instability.
- Predict the local deformation fields created in the composite due to the wrinkling instability
- Establish the effect of wrinkling on the effective mechanical properties, i.e. macroscopic stress-strain behavior and effective stiffness, of multilayered composites.
- Predict the energy storage properties of elastic multilayered composites undergoing wrinkling.
- Understand the energy absorption and dissipation properties of composites with elastic-perfectly plastic interfacial layers.
- Establish design guidelines for creating tunable materials with controlled macroscopic behavior and effective properties using the wrinkling instability
- Investigate the energy harvesting mechanism created by introducing piezoelectric material to the wrinkling interfacial layers.
1.3 Background

1.3.1 Instabilities and pattern transformation

An interesting method to achieve switchable and tunable materials is by exploiting mechanical instabilities within the structure. It has been shown that one can deliberately deploy the principles and mechanics of instabilities to achieve on-demand ubiquitous pattern transformations in structured materials, e.g. [21,27]. Instability mechanisms, such as buckling and wrinkling, are particularly effective as they can be designed to remain elastic during the loading cycle; hence the instability mechanism can be reversible and repeatable. This opens the opportunity to use instability and pattern transformations in many engineering applications, e.g. for microfluidic systems for biological research, sensing and diagnostics, elastic and electromagnetic wave filtering and wave guiding, and controllable adhesive surfaces and antireflective coatings.

Figure 1.3: a) The pattern transformation in a square lattice due to instability. Experimental pictures from macroscopic strain of 2% and 6%. Image adapted from [21]; b) Compression of a stiff thin layer on a soft substrate demonstrate the formation of the surface wrinkling pattern. Image adapted from [28]; c) Mode shapes that form under equi-biaxial compression of a surface layer; 1D mode, square checkerboard mode, hexagonal mode, triangular mode, and herringbone mode. Image adapted from [29].
Wrinkling of thin surface layers has been widely studied by many investigators due to its potential in numerous application areas as well as due to its prevalence in natural materials [22,30,31]. The formation of wrinkles can be explained using an energy minimization argument. Under axial compressive stress, after a certain critical load, it is energetically favorable for the stiff film to buckle. When the stiff film is not attached to a soft substrate, the film will buckle into the shape of a single half sine wave. However, with a soft substrate attached, when the stiff film buckles it causes stretching in the substrate. Because of this, the total energy of the system is a combination of the bending energy of the film and the stretching energy of the substrate. The bending energy in the film is given by $U_{Bending} \propto \frac{E_1 t^3 A^2}{\lambda^4}$ and the stretching energy in the matrix is given by $U_{Stretching} \propto \frac{E_0 A^2}{\lambda}$, where $E_1$ and $E_0$ are the Young’s modulus of the stiff film and the soft matrix respectively, $t$ is the thickness of the film, $A$ is the amplitude and $\lambda$ is the wavelength of the wrinkling pattern. The total energy of the system, i.e. the film and substrate, is hence given by: $U_{Total} = U_{Bending} + U_{Stretching}$. Buckling of the film into a single half sine wave would minimize the energy of the film, but it would induce a large stretching energy in the substrate. For the energy of the entire system to be minimized the film buckles into a higher mode with many sine waves (Figure 1.3b); i.e. a wrinkling pattern. This causes a higher bending energy in the film, but greatly reduces the stretching energy in the substrate, thus minimizing the total energy of the system. Energy minimization of the total energy of the system leads to the scaling law for the wavelength of the initial wrinkling pattern: $\lambda_{cr} \propto t \left( \frac{E_1}{E_0} \right)^{\frac{1}{3}}$. It is clear that the wavelength of the wrinkled film and surface can be varied by changing different material and geometric parameters in the film and substrate, while the critical load causing the wrinkling of the stiff film has been established to be a function of only the material parameters of the film and the substrate.

In addition, biaxial compression of these structures consisting of a thin stiff surface layer on a soft substrate has also been studied. It was shown that there are a number of possible stable instability modes (Figure 1.3c) [29]. Results also show that under equi-biaxial compression the sequence of loading can affect the surface topography. More specifically, it was shown
that for a case where the biaxial loading occurs simultaneously the surface takes a disordered labyrinth pattern. However, when the biaxial loading occurs sequentially the surface takes an ordered herringbone pattern [31].

The instability in multilayered composites have been investigated, including both for concentrated [32,33,34] and dilute composites [34,35]. Extensive research has shown that concentrated composites will buckle in a macroscopic mode with a long wave mode, which leads to the well-known kinking failure, and the instability conditions depend on the material and geometrical parameters of the composites. Dilute composites have been shown to have microscopic instability modes, where the thin stiff layers buckle in a wrinkling pattern. The instability in dilute composites are further explored in this thesis and analytical models are developed predicting the wrinkling criterions based on the composites’ material and geometrical features.

The instability in surface layers and multilayered composites due to an axial loading can be explained by studying the instability in a beam. Consequently, we will next evaluate the governing equation and instability in a single beam, as well as for a beam resting on a soft substrate.

1.3.1.1 Governing equation and instability in a beam

In linear mechanics of deformable bodies and structures, the displacements of a structure is proportional to the applied load. However, in the case of instabilities and buckling of structures, there is a nonlinear relationship between the applied load and the resulting displacement; a small increase in load gives a disproportionate increase in displacement, as can be seen in Figure 1.4a. [36]
The nonlinearity mentioned here is a purely geometrical nonlinearity; i.e. it is not a material nonlinearity and the material can for example be presented as a linear isotropic elastic material throughout the analyses (neglecting cases where plasticity occurs). The geometric nonlinearity is due to large deformations and enters the theory either in the strain relation or in expressions representing the influence of rotation of a material element on the behavior of the whole structure. The larger deformations and associated rotations result in an extra nonlinear contribution in the strain expression, and the kinematic relations for a straight beam. They are given by:

\[ \mathbf{u} = (u, v, w), \quad \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \quad \kappa = -\frac{\partial^2 w}{\partial x^2}, \quad \theta = -\frac{\partial w}{\partial x} \]  

(1.1)

where \( \mathbf{u} \) is the displacement vector of the beam, \( \varepsilon_x \) and \( \kappa \) are the local axial strain and curvature of the beam, and \( \theta \) is the angle of rotation (Figure 1.4b).

The most general case of evaluating instability is focusing on a straight beam subjected to a centrally applied compressive load, \( P \).

**Figure 1.4:** a) A typical load-displacement curve for a linear and nonlinear (taking rotation into consideration) case. b) Schematics of a beam showing the deflection \( w(x) \) and angle \( \theta \). c) Forces and moments acting on a beam element; \( N \) is axial force, \( Q \) is shear force, and \( M \) is the bending moment.
There are several ways of proceeding to establish the governing equilibrium equation for a beam; in this section we will focus on a finite small material element of the beam and satisfy the force and moment balance to derive the equilibrium equation for the beam. Due to the effects of the moderately large deformations, we account for the influence of rotation of the material element on its overall response. Figure 1.4c shows a deformed material element, the forces and moments acting on the element, and the angle of rotation. Furthermore, in our derivations we will assume small rotation angles, \( \theta \), so that we approximate \( \sin(\theta) \) and \( \cos(\theta) \) to first order terms only.

Balance of forces in the x-direction gives:

\[
-N + (N + dN) - Q \cdot \theta + (Q + dQ)(\theta + d\theta) = 0
\]
\[
N + Q \cdot d\theta + \theta \cdot dQ = 0
\]
resulting in:
\[
\frac{dN}{dx} + Q \frac{d\theta}{dx} + \theta \frac{dQ}{dx} = 0
\] (1.2)

Balance of forces in the z-direction gives:

\[
-Q + (Q + dQ) - N \cdot \theta + (N + dN)(\theta + d\theta) = 0
\]
\[
Q - N \cdot d\theta - \theta \cdot dN = 0
\]
resulting in:
\[
\frac{dQ}{dx} + N \frac{d\theta}{dx} + \theta \frac{dN}{dx} = 0
\] (1.3)

Balance of moments gives:

\[
M - (M + dM) + Q \cdot dx = 0
\]
resulting in:
\[
Q = \frac{dM}{dx}
\] (1.4)

It is established that the shear stresses, and consequently also shear forces, on a slender beam are small compared with the other stresses; this allows us to neglect the quadratic terms of the shear force multiplied with the rotation angle, as done in the above equations. Hence, the Equations 1.2-1.4 are reduced to:

\[
\frac{dN}{dx} = 0, \quad \frac{dQ}{dx} + N \frac{d\theta}{dx} = 0, \quad Q = \frac{dM}{dx}
\] (1.5)
Next, we use Equation 1.5 together with: 
\[ EI\kappa = -EI\frac{\partial^2 w}{\partial x^2} \]
\[ \Theta = -\frac{\partial w}{\partial x} \]
and assume a constant flexural rigidity for the beam \((EI=\text{constant})\), to obtain the two equations of equilibrium for the beam:

\[ \frac{dN}{dx} = 0 \]

\[ \frac{d^2 M}{dx^2} - N \frac{d\Theta}{dx} = 0 \]

which can be rewritten to:

\[ EI \frac{d^4 w}{dx^4} - N \frac{d^2 w}{dx^2} = 0 \]  \hspace{1cm} (1.6)

Equations 1.6 are the well-known governing equilibrium equation of a beam under moderate deflections. The equilibrium equation is quadratic with respect to the dependent variables \(N\) and \(w\), which makes the differential equation nonlinear. However, \(N\) is usually constant in \(x\)-direction and through boundary conditions found to be: \(N=-P\). This results in the governing equilibrium equation for a beam to be given by:

\[ EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = 0 \]  \hspace{1cm} (1.7)

This equilibrium equation is linear, but takes into account the influence of geometric nonlinearity. The solution to the equilibrium equation (Equation 1.7) can be found by assuming a trigonometric function for \(w(x)\), and by imposing the boundary conditions, the solutions will give the mode-shapes for the beam upon buckling.

1.3.1.2 Instability in a beam on elastic foundation

In the previous section the instability of a straight beam was studied and the governing equations were derived determining the trigonometric mode- and buckling shapes for when the beam buckles. However, it is important to note that for real applications the minimum energy of the system will arise from the first mode-shape; i.e. there is only one buckling shape physically possible. Interestingly though, once the beam is resting on an elastic material (a foundation), higher order mode shapes can be obtained. The total potential energy of the system, which now includes energy contributions from both the beam and the elastic foundation (bending and stretching strain energy), will have a minimum when there are
several waves in the buckling mode of the beam; that is to say, a wrinkling pattern is observed
in the beam. This is known as the instability of a beam on a Winkler foundation [37]. Below
we develop the modified governing equations of equilibrium for a beam on elastic foundation.

The elastic foundation can be modeled as an infinite set of parallel springs with no shear
coupling between the springs [37], as can be seen in Figure 1.5a. The foundation’s behavior
at any point is only a function of the lateral displacement of the beam at that point. Hence, the
following force-displacement relationship can be used to describe the elastic foundation’s
response:

\[ q(x) = -k \cdot w(x) \] \hspace{1cm} (1.8)

where \( q \) = force per length, \( w \) = lateral displacement, \( k \) = foundation/spring elastic modulus.

The forces and moments acting on a small finite material element of the beam are shown
schematically in Figure 1.5b. It can be seen that compared with the simple straight beam
considered in previous section, there is now also an additional force “\( q \, dx \)” acting on the
material element due to the elastic foundation. Following the same procedure and derivation
as done in the Section 1.3.1.1, we obtain the following equilibrium equation:

\[ EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = q \] \hspace{1cm} (1.9)

Inserting for “\( q \)” by using Equation 1.8, we obtain the governing equilibrium equation for a
beam on an elastic foundation:

\[ EI \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} + k \cdot w(x) = 0 \] \hspace{1cm} (1.10)

A trigonometric function for the deflection \( w(x) \) can be assumed and together with the
appropriate boundary conditions for the problem, the exact solution can be found. It is
important to note that the minimum axial force, \( P_{cr} \), that results in the beam buckling will
result in solutions for \( w(x) \) where the wavelength of deformation is smaller than the length
of the beam. This leads to a wrinkling pattern of the beam. Furthermore, it is clear that the
elastic foundation modulus, \( k \), has an important effect on the buckling and wrinkling pattern
created, as shown in Figure 1.5c.
Figure 1.5: a) Schematic of a beam on elastic foundation. b) Forces and moments on a finite beam element. c) A wrinkling pattern is achieved when the elastic foundation modulus, $k$, is larger than the critical value at which initial buckling occurs. The elastic foundation can be modeled as an infinite set of parallel springs with no shear coupling between the springs [37]. This a good approximation for small deformations of a linear elastic material.
1.3.2 Effective behavior in composites

The macroscopic stress-strain behavior and effective stiffness of a multilayered composite with straight layers can, in the small linear strain range, trivially be found through the simple Voigt model [38], for cases where the applied loading is aligned with the layers. In the case where the loading is normal to the layers, the Reuss model [39] can be used to find the effective behavior of the multilayered composite.

When more complicated microstructures are introduced in a composite, very interesting macroscopic behavior and properties can be obtained. An example of such composites is cellular materials (Figure 1.6a). Cellular materials are composites in which there are two phases; one phase is solid, and the other phase is either gas/air, fluid, or a soft matrix. The synthetic cellular materials tend to have either a two-dimensional (honeycomb) or three-dimensional (foam) structure, and are used for a wide range of engineering applications due to their interesting macroscopic behavior and properties. Honeycombs is one of the most extensively used cellular structures and are commonly constructed of metals (steel, aluminum) or polymers (polycarbonate and polypropylene). The honeycomb is stiff in the out-of-plane direction, and relatively softer in the in-plane direction [40]. The performance of honeycombs is optimized by varying the geometrical and material parameters of the composite.

Cellular materials characteristically possess an effective compressive stress-strain curve that exhibits an initial linear elastic response, followed by a plateau region that is due to the initiation and propagation of localized bands of either elastic buckling, plastic buckling, yielding, or fracture in the microstructure. Following the plateau region is a further densification region, where the effective stiffness of the material increases dramatically due to contact of the cellular walls, before ultimate failure occurs in the cellular structure (Figure 1.6b). [41] More specifically, for in-plane compression of empty honeycombs, the cell walls first bend, giving linear elastic deformation. Beyond a critical applied strain the cells collapse by elastic buckling, plastic yielding, creep or brittle fracture depending on the cell wall material. The collapsing proceeds in bands where a horizontal or a diagonal shear band of collapse occurs which then proceeds to another band and another band; finally, opposing side
walls start touching each other; then the structure densifies and the stiffness increases rapidly. These three deformation mechanisms are displayed as separate regimes in the stress-strain curves of cellular materials (Figure 1.6b). For different materials in compression, the curves have broadly similar shapes, with a linear-elastic regime, followed by a plateau of roughly constant stress, leading into a final regime of steeply rising (densification) stress. Increasing the relative density of the honeycomb increases the relative thickness of the cell walls, and thus the resistance to cell wall bending and cell collapse goes up, resulting in a higher effective tangent modulus and plateau stress.

![Figure 1.6: a) Examples of cellular structures with different honeycomb structure; square, hexagonal, and triangular. Image adapted from [42] b) Typical shape of the effective stress-strain curve for an elastomeric honeycomb in compression. Plot adapted from [41].](image)

As a cellular structure is statically or dynamically compressed, energy is absorbed. Cellular structures can be designed to enable the storage and/or dissipation of energy under lower loads and at lighter weight due to the local deformation mechanisms occurring in the microstructure (i.e. the cell walls) and the propagation of deformation and/or failure in the cellular structure. The energy absorbed by the cellular material per unit volume corresponds to the area under the effective stress-strain curve as shown in Figure 1.6b. Most of the energy is absorbed in the long plateau of the curve, allowing large energy absorption at near constant load. Hence, honeycomb composites are attractive for use as energy absorption devices in many aerospace (e.g. helicopter under-floor structures), and mechanical engineering applications, and are also commonly used as the core material in sandwich structure due to their high stiffness to weight ratio and high transverse shear resistance. They are widely applied for shock mitigation, packaging, as well as damping.
1.3.3 Energy absorption mechanisms

An important area of research is controlling and enhancing energy absorption of materials, as the knowledge can be used for many engineering applications, such as: sensors and actuators, damping, packaging, energy conversion, and so on. It is common to define energy absorption to be the sum of energy stored and dissipated in a material during loading. The stored energy will be released upon unloading, while the dissipated energy is not recoverable. The energy storage capabilities of a homogeneous material under compression can easily be found using its constitutive model and elasticity theory. By introducing microstructural features and designs into a homogeneous material, the effective properties and the energy absorption of the materials can be greatly altered and enhanced. [41,43] For example, light-weight structures such as cellular materials, with honeycomb or foam microstructures, provide dramatically altered effective behavior and properties compared with their constituent homogeneous material. [43,44,45,46,47,48] As mentioned in Chapter 1.3.2, cellular materials characteristically possess an effective compressive stress-strain curve that exhibits an initial linear elastic response, followed by a plateau region that is due to the initiation and propagation of localized bands of either elastic buckling, plastic buckling, yielding or fracture in the microstructure. Following the plateau region is a further densification region, where the effective stiffness of the material increases dramatically due to contact of the cellular walls, before ultimate failure occurs in the cellular structure. [41] Hence, cellular structures can be designed to enable the storage and/or dissipation of energy under lower loads and at lighter weight due to the local deformation mechanisms occurring in the microstructure (i.e. the cell walls) and the propagation of deformation and/or failure in the cellular structure. [43,44,46,48] Furthermore, it has been shown that the design of the cellular microstructure, such as the geometry and size of the cells or a hierarchical microstructure of the cells, can be used to tailor the effective properties of the cellular structure. [49,50,51]

Previous studies have shown that the energy absorption properties of cellular structures can be enhanced by filling the voids of the cellular structure with gas or fluid. [41,52,53,54] The
gas and fluid will oppose the deformation of the cell walls, functioning as a form of reinforcement to the cellular structure. In addition, energy will be consumed as the fluid flows in and through the cells, and more viscous and advanced fluids can be used to increase the energy absorption effect. More recently, the pursuit of achieving higher energy absorption has led to adding soft materials, such as polymers, as fillings to the cellular structures. The soft matrix fillings in the cellular structures increase the reinforcement effect, and the created composite materials demonstrate new and interesting effective properties. [55,56,57,58,59]

Experiment and simulation based studies of in-plane compression loading of honeycomb structures filled with a soft matrix material show that there is a synergistic effect occurring, and the overall energy absorption of the composite, built up by the cellular network and the matrix fillings, is greater than each of the two constituent materials alone (Figure 1.7). [56,58,60] However, when the results are presented as energy absorption density, per unit mass, the increase is not always as significant. Additionally, other structural designs such as filling hollow metal cylinders with a core solid or foam also provided powerful energy absorption mechanisms by capitalizing on the wrinkling behavior of the cylindrical walls under compression [52,61,62,63].

Figure 1.7: Load vs. displacement plot of the compression of a circular honeycomb made of polycarbonate, filled with PDMS matrix. The energy absorption is shown to be enhanced by adding the matrix to the honeycomb. The experiments were conducted on several samples showing consistent response (Sample #1 and #2). Results are adapted from [56].
1.3.4 Review of energy harvesters

Piezoelectric material have become increasingly important as sensors and actuators in active structural control applications. In recent years piezoelectric elements have been widely used as actuators in adaptive structures from beams and plates, to trusses and more complicated structures, and as distributed sensors for both modal and wave control [64]. Piezoelectric materials possess interesting properties which make them very useful as sensors or control elements for structures. They produce voltage or electric field under an applied strain; this is known as the direct piezoelectric effect and makes piezoelectric materials well suited for sensor applications (Figure 1.8). The other property of the piezoelectric material is that they strain when an electric field is applied across them; this is known as the inverse piezoelectric effect, making piezoelectric material useful for actuator applications (Figure 1.8).

![Diagram showing the direct and indirect piezoelectric effect of converting mechanical strain to electric field and vice versa.](image)

**Figure 1.8**: a) Schematic showing the direct and indirect piezoelectric effect of converting mechanical strain to electric field and vice versa. b) Schematic showing the reaction of poled piezoelectric material to an applied load. [65]
This direct and inverse piezoelectric effect makes piezoelectric materials very important for many engineering applications, such as: actuators, sensors, transducers, energy harvesters, motors, speakers, microphones, vibration controllers, ultrasounds, and for other medical applications.

Most piezoelectric materials have crystal structures which can be polarized by applying an electric field [66]. Furthermore, some piezoelectric materials have the property that an applied stress or strain causes an electric field in the material. Polar piezoelectric materials have a spontaneous polarization (a net dipole moment) associated with the crystal asymmetry, even in the absence of applied electric field or stress/strain. These piezoelectric materials are often called pyroelectric because the degree of polarization changes with temperature. Next, some polar piezoelectric material are called ferroelectrics. For ferroelectric materials, the dipole moment or spontaneous polarization can be permanently changed by applying a high electric field or a mechanical stress or strain. The permanent polarization process of ferroelectrics are called the poling process [65], shown schematically in Figure 1.9a. Poly-vinylidene fluoride (PVDF) is a piezoelectric polymer. As the piezoelectric polymer is subjected to a mechanical load, positive and negative charges develop on the material surfaces. This allows of the polymer material to convert mechanical energy into electrical energy, and vice versa. This piezoelectric effect originates from induced polarization. To induce polarization, the dipoles in the semi-crystalline PVDF polymer must be reoriented through the application of a strong electric field (Figure 1.9b) at elevated temperatures. The temperature is then lowered in the presence of the electric field so that the domains are locked in the polarized states. The material’s piezoelectric effect is directly related to the degree of polarization achieved. The two most common techniques to induce polarization in piezoelectric polymers are electrode and corona poling [67].
Interest in the application of piezoelectric energy harvesters for converting mechanical energy into electrical energy has increased dramatically in recent years [68]. Using piezoelectric materials for energy harvesting purposes has gained much attention due to its high efficiency with direct conversion, compatibility with established fabrication processing at macro-, micro-, and even nano-scales [69], and availability of widely known piezoelectric materials. Moreover, with the decrease in power requirements for many electronic applications, the use of piezoelectricity to energy harvesting has become more feasible. Extensive research has been conducted on the energy harvesting properties of piezoelectric elements of several geometries. The most common is the cantilever beam configuration [70,71,72,73,74,75,76]. Uni-morph energy harvesters (Figure 1.10a) consist of a cantilever beam with one layer of piezoelectric material, vibrated through a translational and small rotational motion of the base [77,78,79]. Bi-morph structures (Figure 1.10b) consist of a cantilever beam with two piezoelectric layers; one on either side of the beam [80]. A great deal of the latest research

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**Figure 1.9:** a) Schematic showing the poling process of ferroelectrics, and the resulting poling direction. [65] b) Schematic of the reorientation of dipoles in PVDF via poling. [67]
has focused on improving the performance of piezoelectric energy harvesters by exploring various piezoelectric materials [68,81,82], device configurations [83,84,85,86], and optimization of circuit designs [87,88].

**Figure 1.10:**

a) Uni-morph harvesters with translational and small rotational base motion [77].

b) Uni-morph and Bi-morph harvesters designed in series and parallel circuit [80].

c) The direction of the material poling relative to the applied load, can be controlled and designed for [80].
1.4 Overview of thesis

This thesis investigates the deformation-induced wrinkling instability transformation in the microstructure of composites consisting of thin interfacial layers. The hybrid microstructure of a composite material has an essential influence on the effective properties and behavior of the composite. Hence, in this research, the principles and mechanics of instabilities are purposely used for the interfacial layer and designed to achieve sudden pattern transformations in the composite structure to generate new controllable multifunctional behavior.

![Diagram](image)

**Figure 1.11:** The content and structure of this thesis.
The main content of this thesis is organized into seven chapters, where we systematically investigate different aspects of the instability-induced transformation of interfacial layers in composites and its multifunctional applications, through analytical modeling, finite element simulations, and mechanical experiments. Figure 1.11 schematically shows the structure of this thesis, and the structure is described in more detail below.

In Chapter 2 the methods deployed in this investigation are presented. Details on the finite element models and simulations executed in the following chapters of this thesis are described. Moreover, the procedure and specifics of the physical experiments done on specimens are explained. The analytical models developed in this thesis are explained and derived in each respective chapter.

Chapter 3 investigates the wrinkling instability transformation in multilayered and networked composites consisting of relatively stiff interfacial layers or cells embedded in a soft matrix. Analytical models are developed predicting the wrinkling conditions, based on the composite’s material and geometrical parameters. We also demonstrate that as the multilayered composite becomes more concentrated, the instability mode will transition to a long-wave instability mode instead.

In Chapter 4, we focus on the wrinkling composites, and derive analytical models predicting the local stresses and strains developed in the interfacial layers and the matrix layers of the multilayered composites, as the composite is compressed from pre- to post-buckling strains. The models are verified with finite element simulations.

Chapter 5 investigates the mechanical effective behavior of the transforming elastic composites. Analytical models are developed for the strain energy density in the elastic composites as they are compressed through the pre-buckling region, the point of wrinkling
instability, and into the post-buckling region. Moreover, the macroscopic stress-strain behavior of the composite are analytically expressed, and a bilinear and multi-linear elastic stress-strain behavior is observed. Finally, the effective stiffness of the composite is studied and a switchable behavior is observed.

Chapter 6 introduces plasticity in the previously all elastic composites. The thin relatively stiff interfacial layers are modeled as elastic-perfectly plastic materials, and we derive simplified models predicting the energy absorbed, stored and dissipated as the composites are loaded. The macroscopic stress-strain behavior is also studied. It is shown that the energy absorption density and energy dissipation density are enhanced by deploying and designing the wrinkling instability to occur in the composites.

Chapter 7 combines the results from the previous chapters and presents design guidelines to assist the process of deploying instability-transformation to tune, control, and switch the mechanical properties of multifunctional composite materials. In addition, we demonstrate that we can create tunable elastic composites with enhanced energy storage properties, and with interesting new effective mechanical behavior. For the composites with plastic interfacial layers we show that the energy absorption and dissipated by the composite can be controlled and designed for based on the material and geometrical combinations in the composite.

In Chapter 8, we introduce the concept of adding piezoelectric material in the interfacial layers of the composite to extract voltage/electricity. By using the knowledge provided in earlier chapters, we can systematically add piezoelectric patches at the locations in the interfacial layer with the largest deformation. By using the direct piezoelectric effect we demonstrate and present a new energy harvesting mechanism. In addition, by strategically adding the piezoelectric patches at the center of the interfacial layers, we demonstrate a new switching mechanism.
Chapter 2
Methodology

2.1 Introduction

This thesis investigates instability transformations in the interfacial layers of multilayered and networked composites, its effect on the composite's macroscopic properties, and its multifunctional application. The investigations are done by a combination of analytical-, numerical-, and experimental studies. The analytical models are developed in detail in each respective chapter of this thesis, and numerical simulations and physical experiments are done to verify and further explore the analytical results. In this chapter we present the methods and details behind the numerical simulations (finite element simulations) conducted in the subsequent chapters of this thesis. We also describe the details, setup, and procedure for the physical experiments executed in this research.

2.2 Finite Element (FE) simulations

2.2.1 Details of finite element simulations

In this thesis, finite element (FE) based micromechanical simulations are performed using the ABAQUS software. Representative volume elements (RVEs) of multilayered or networked composites are constructed by modeling thin and relatively stiff interfacial layers of thickness \( t \), embedded between softer matrix layers of depth \( d \) on both sides (Figure 2.1).

The interfacial and matrix layers in the composites are modeled using 2D plane strain quadratic elements (ABAQUS, CPE8R) and allowing for non-linear geometric deformations. At least three elements are set across the thickness of the interfacial layer to ensure that the
interfacial layer can deform freely due to the instability transformation. Linear elastic constitutive material model are assigned to the interfacial layer in Chapters 3-5, while an elastic-perfectly plastic material model is used in Chapter 6. The matrix layers are modeled as neo-Hookean in Chapter 3 (to be mimic the mechanical experiments), while it was modeled as linear elastic material in Chapter 4-6. The material parameters are $E_l$ and $v_l$ for the interfacial layer and $E_0$ and $v_0$ for the matrix layers. Fully general periodic boundary conditions are used to introduce the periodic nature of these composites, as shown in Figure 2.1 (explained further in Section 2.2.2). To simulate the plane strain compressive loading on the periodic structures, displacement gradients are imposed in the field of the RVEs by prescribing displacements at the vertexes, while applying the periodic boundary conditions [89] to the edges.

**Figure 2.1:** a) Schematics of the FE models of layered structure under compression with the repetitive unit cell, the RVE, marked up. b) Details of the RVE modelled in our simulations with key material and geometric parameters shown. Moreover, periodic boundary conditions are applied to all external sides of the RVE, and an in-plane displacement loading, $\delta_1$, is applied at vertex-4.

The FE simulations are conducted in two steps to study the initial instability and to analyze the post-buckling process with finite deformation (further explained in Section 2.2.3). First, the FE models are solved as eigenvalue problems via ABAQUS/BUCKLE, to find the instability deformations (eigenmodes) and the critical strain (eigenvalues). Second, to investigate the post-buckling behavior, the first eigenmode is introduced as an initial
geometric imperfection (the perturbed amplitude is 1% of $t'$) to the FE model and then a post-buckling analysis is performed in ABAQUS/STANDARD using an implicit algorithm. Our finite element models have up to 1,875,561 nodes, 6,240,000 elements, and take up to 10 hours to run (smaller models take 1-2 hours to run).

2.2.2 Periodic boundary conditions in FE analysis

In performing finite element simulations where there is a periodic geometry or a structure repeating itself, it is very convenient to use periodic boundary conditions to only simulate and analyze a representative volume element (RVE). This method can save us a lot of computation time and more importantly it will evaluate the RVE as a part of a bigger structure so that there will be no artificial effects from the boundary condition on the RVE's response. To develop equations to represent the periodic boundary conditions, we consider a 2D-solid with periodic boundary conditions in the x- and y-direction undergoing a deformation (Figure 2.2).

![Figure 2.2: Periodic boundary conditions. The abbreviations stand for: BL= node bottom left, BR= node bottom right, TL= node top left, TR= node top right, SL= node set left, SR= node set right, SB= node set bottom, ST= node set top](image)

1 Finite element simulations of models with imperfections of 0.2%~5% thickness of the interfacial layer were conducted. Result show that as long as the initial geometry imperfection is less than 5% thickness of the layer, $t$, the critical buckling strain is not influenced by the imperfection.
For a given 2D solid under deformation, we define the deformation gradient tensor $\mathbf{F}$ and the displacement tensor $\mathbf{H}$ is defined as:

$$
\mathbf{F} = \frac{\partial x}{\partial x} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad \mathbf{H} = \mathbf{F}^{-1} = \begin{bmatrix} F_{11} - 1 & F_{12} \\ F_{21} & F_{22} - 1 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}
$$

In order to fulfill the periodicity conditions of symmetry in the x- and y-direction, the following relations exists for displacement $u$:

- Symmetry in x-direction:
  
  \begin{align*}
  u_x^{SR} - u_x^{SL} &= H_{11} \cdot L_1 \\
  u_y^{SR} - u_y^{SL} &= H_{21} \cdot L_1 \\
  u_x^{BR} - u_x^{BL} &= H_{11} \cdot L_1 \\
  u_y^{BR} - u_y^{BL} &= H_{21} \cdot L_1 \\
  u_x^{TR} - u_x^{TL} &= H_{11} \cdot L_1 \\
  u_y^{TR} - u_y^{TL} &= H_{21} \cdot L_1 
  \end{align*}

- Symmetry in y-direction:
  
  \begin{align*}
  u_x^{ST} - u_x^{SB} &= H_{12} \cdot L_2 \\
  u_y^{ST} - u_y^{SB} &= H_{22} \cdot L_2 \\
  u_x^{TR} - u_x^{BR} &= H_{12} \cdot L_2 \\
  u_y^{TR} - u_y^{BR} &= H_{22} \cdot L_2 \\
  u_x^{TL} - u_x^{BL} &= H_{12} \cdot L_2 \\
  u_y^{TL} - u_y^{BL} &= H_{22} \cdot L_2 
  \end{align*}

The above equations create the periodicity in both x- and y-direction. However, in order to prevent pure translation and rotation of the RVE it is necessary to fix one node of the system in both x- and y-direction. That is to say that we can for example fix the bottom left node, such that:

$$
u_x^{BL} = 0 \quad \text{and} \quad u_y^{BL} = 0 \quad (2.4)$$

In problems where the FE analysis is considering the effects of having an externally applied displacement/load on the RVE, it is necessary to transform the external stimulus to a strain first, then the strains are presented as the displacement gradient tensor. Implementing this in the FE analysis requires first creating two fake nodes (a and b) in the model that will have components representing the components of the displacement gradient tensor. Hence, the following relation exists between the fake nodes and the displacement gradients:
$a_x = H_{11} \quad b_x = H_{12}$

$\begin{align*}
    a_y &= H_{21} \\
    b_y &= H_{22}
\end{align*}$

Consequently, any displacement specified to these fake nodes as a boundary condition actually means defining the components of the displacement gradient tensor affecting the equations building up the periodic boundary condition. Hence, the method described in this section allows us to create 2D-structures that have periodic boundary conditions in both x- and y- direction and that are under an externally applied load/strain.

2.2.3 Buckling and post-buckling analysis

The finite element simulations conducted in this thesis consists of both a buckling analysis and a post-buckling analysis. The buckling analysis is a procedure in ABAQUS/BUCKLE which takes in a FE model with an applied load, performs an instability analysis calculating the critical load and the corresponding mode shapes, and outputs only the number of eigenvalues requested and their respective buckling mode shapes.

Executing a post-buckling analysis is required to obtain information such as stress-strain distribution, energy densities, reaction forces and etc. In order to execute a post-buckling analysis it is necessary to impose an initial imperfection to the beam so that when the load is applied the beam gets affected by the instability and consequently buckles or wrinkles. Hence, in the post-buckling analysis we start with the same FE model and then add a very small imperfection to the model, which is taken to be the first buckling mode shape obtained from the pure buckling analysis. Then the load applied needs to be adjusted to make sure it reaches the critical load at which the system gets instable and wrinkling occurs; this is done by assuring that the applied load is bigger than the first eigenvalue multiplied with the load applied in the pure buckling analysis. Then the model is executed in ABAQUS/STANDARD and the results from the post-buckling analysis output all information needed in order to obtain
all results needed to be compare with the analytical solutions. Figure 2.3 schematically shows the major steps of the buckling and post-buckling analyses.

2.2.4 Data post-processing for macroscopic behavior of RVE

The post-buckling analysis in ABAQUS/STANDARD provides results for the local stresses, local strains, the strain energy density, and energy dissipated for each element in the modelled representative volume element (RVE). In this thesis, in addition to these information, the effective and macroscopic behavior of the RVE models are also of interest. Due to the periodic boundary conditions imposed, the effective behavior and response of the RVE is found by using the principle of virtual work [89]. It has been shown that the macroscopic (average) first Piola-Kirchhoff stresses can be found as a function of the applied strain, by studying the reaction forces at the virtual nodes introduced in the periodic boundary conditions for the RVE [89] (the periodic boundary conditions are described in Section 2.2.2). The engineering macroscopic stresses are found by calculating the reaction forces at the virtual nodes (found
through the post-buckling analysis in ABAQUS), and averaging it over the initial volume of the RVE. Once the macroscopic engineering stress-strain behavior is established, the tangent Young’s modulus, i.e. the effective stiffness of the modeled RVE, is found by numerically calculating the slope of the engineering stress-strain curves. Finally, the macroscopic axial engineering stress-strain results can be transformed into axial true stress-true strain results through the well-known relations:

\[ \sigma_{x}^{\text{comp}} = \bar{\sigma}_{x}^{\text{comp}} (1 + |\varepsilon|) \quad \text{and} \quad \bar{\varepsilon} = \ln(1 + |\varepsilon|) \]  

\[ (2.6) \]

where \( \bar{\sigma}_{x} \) and \( \bar{\varepsilon} \) are the macroscopic engineering axial stress and strain of the RVE, respectively, and \( \bar{\sigma}_{x} \) and \( \bar{\varepsilon} \) are the corresponding true stress and true strains of the RVE. This method of obtaining the macroscopic response and behavior of the RVE are used for the composites investigated in Chapter 5 and 6.

### 2.3 Physical Experiments

A set of physical experiments are conducted on multilayered composites in this thesis. Below we describe details about the experimental setup and procedure, the materials used, and the samples fabricated.

#### 2.3.1 Experimental procedure and setup

Multilayered composites specimens are fabricated using a 3D multi-material printer (details described below). Plane strain compression tests are performed on the physical samples using a screw-driven Zwick Mechanical Tester. The specimens are placed in a fixture restricting the deformation in the out-of-plane direction so that the plane strain conditions are imposed. Mineral oil is used as a lubricant between the specimens and the fixture to provide near frictionless conditions. The fixture is transparent to enable direct monitoring during testing; this also allows measurement of the axial strain using Digital Image Correlation (DIC). The
tests are performed quasi-statically with a nominal engineering strain rate of 0.00015/s for the softer composites and 0.000057/s for stiffer composites. The steps of the experiment is schematically shown in Figure 2.4.

![3D printed samples](image)

**Figure 2.4:** Schematic showing procedure of: 3D printing specimens, using a plane strain fixture, and mechanical testing and the loading conditions.

### 2.3.2 Physical specimens and materials

Single- and multi-layered composite specimens are fabricated with a 3D multi-material printer (Objet Connex500 at MIT). Two base polymers are used in printing: an acrylic-based photo-polymer, VeroWhite (Young’s Modulus 1.2 GPa, referred to here as VW), and a rubbery material, TangoPlus (initial Young’s Modulus 0.9 MPa, referred to here as TP). The transparent matrix is printed in TangoPlus, and the interfacial layer is printed in VeroWhite and also as a mixture of the two base materials, a so-called “Digital” Material (Young’s Modulus 18 MPa, referred to here as DM). Figure 2.5 shows examples of the engineering stress-strain curve obtained through the material testing for each of the three materials used.
Figure 2.5: Examples of engineering stress-strain curves from material testing of the three materials used to make the physical specimens (TangoPlus, “TP”, Digital Material with shore 95, “DM”, and VeroWhite, “VW”). The initial tangent Young’s modulus, \( E \), of each material is found.

Single-layered specimens:
Composites consisting of one centrally located single interfacial layers of either VW or DM embedded in TP are fabricated. The planar dimensions of specimens with VW interfacial layers are 81mm x 88mm (width x height in Figure 2.4), chosen to minimize any influence of boundaries on the resulting instabilities. The specimens with DM interfaces are 41mm x 33mm (width x height). The interfacial layers are of thickness \( t_0 = [0.5, 1.0] \)mm, and the out-of-plane dimension (depth) is 12mm for all specimens.

Multilayered specimens:
Specimens with several interfacial layers of either VW or DM embedded in TP are fabricated. The interfacial layer thickness is kept constant at \( t_0 = 0.5 \)mm, while the distance between the layers are set to be \( d = 10 \)mm for the VW interface specimens, and \( d = [3, 10] \)mm for the DM interface specimens. The height of the VW samples are 88mm and the height of the DM samples are 33mm. The out-of-plane dimension is again 12mm for all specimens.
Chapter 3

Instability in multilayered and networked composites

3.1 Introduction

In this chapter we will study the instability in a multilayered composites consisting of thin interfacial layers embedded in a soft matrix. Instabilities have been a subject of study in a number of composite material systems where structural mechanics approaches [90,91,92], energy methods [33], and Bloch wave analyses [27,93] have been found to predict this complex phenomenon.

We will explore the mechanisms of the formation of wrinkled interfacial layers in multilayered composites. Predictive analytical models will be presented providing the material and microstructural conditions which govern the occurrence of wrinkling, and the changes in the deformation in the post-buckling region. Moreover, we will establish how the instability mode for multilayered composites transitions from a wrinkling mode to a macroscopic long wave instability mode, as the composite becomes more concentrated.

Finite element simulations are conducted to verify the analytical models. In addition, mechanical experiments are performed on 3D printed prototypes to confirm and explore the mechanisms further.

The ability to actively alter the structure of the interfacial layers of a composite can enable on-demand tunability of the underlying physical, bio-chemical and mechanical attributes. In addition, it can provide active control of the composite’s macroscopic effective properties, making it useful for many engineering applications.
3.2 Wrinkling instability in a stiff interfacial layer embedded in soft matrix

Let's consider compression of a multilayered composite consisting of an interfacial layer with thickness $t$, and stiffness $E_i$, supported by soft compliant matrix on both sides with thickness $d$ and stiffness $E_0$ (Figure 3.1b). The compressive macroscopically applied strain, $\varepsilon$, can be directly applied through a compression load, or indirectly through constrained contraction of the adjacent matrix and/or other internally generated turgor pressure. As the applied loading on the composite is increased, the compressive stress in the interfacial layer will reach a critical value where bending will be energetically favoured over simple compression. This results in the interfacial layer undergoing instability into a higher mode of deformation, which is the wrinkling pattern seen in Figure 3.1b. Due to the periodic nature of the wrinkling, we assume the lateral displacement of the interfacial layer to be given by a sinusoidal function in the x-direction, namely:

$$w(x) = w_{\text{max}} \cdot \sin \left( \frac{2\pi x}{\lambda} \right)$$

(3.1)

where $w_{\text{max}}$ is the amplitude of the wrinkling pattern, and $\lambda$ is the wavelength of the wrinkling pattern.

The local bending energy (per unit depth) of the interfacial layer scales then as $U_b \propto \frac{E_i t^3 w_{\text{max}}^2}{\lambda^4}$, while the local strain energy in the surrounding matrix layers scales as $U_s \propto \frac{E_0 w_{\text{max}}^2}{\lambda}$ (Figure 3.1c). Energy minimization results in scaling laws for the critical wavelength $\lambda_{cr} \propto t \left( \frac{E_i}{E_0} \right)^{\frac{1}{3}}$ and the critical strain $\varepsilon_{cr} \propto \left( \frac{E_i}{E_0} \right)^{\frac{2}{3}}$. The fact that the critical wavelength of the wrinkling pattern scales with $t$ and $E_i/E_0$, is similar to the well-studied case of wrinkling of a

---

2 In this thesis, $E_i$, $v_i$ are the stiffness and Poisson's ratio of the interfacial layer, respectively. $t$ is thickness of the interfacial layer. When the matrix is one material, $E_0$ and $v_0$ are the stiffness and the Poisson's ratio of the matrix material; when the matrix on the two sides of the interfacial layer are two different materials, $E_{01}$ and $E_{02}$ are the stiffness of the two materials, and $v_{01}$ and $v_{02}$ are the Poisson's ratio of the two materials.
stiff coating on a compliant substrate [30, 94, 95, 96, 97, 98]. Beyond the above mentioned scaling laws, more accurate closed-form analytical models predicting the critical instability load and pattern are developed and presented in the next sections. Moreover, the progression of these wrinkling deformations under post-buckling strains are also presented.

**Figure 3.1:** a) Schematic of a multilayered composite consisting of soft matrix layers and stiff interfacial layers. Under an applied compressive load of $\bar{\varepsilon}$, the interfacial layers will undergo instability; b) Schematic of one interfacial layer supported by soft matrix on both sides. The initial mode and material and geometric parameters are shown on the left. The instability mode will be one where there is a wrinkling pattern in the interfacial layer as shown in the figure to the right; c) Schematics of the matrix stretching strain energy and interfacial layer bending strain energy. They add up to become the total strain energy of the composite, with a clear minimum energy for a given wrinkling wavelength, $\lambda$. 

4.1.1 Scaling Laws for wrinkled layers. The wrinkling instability load $P_{cr}$ can be modeled as a non-dimensional parameter $\frac{P_{cr} d}{E_0 t^3}$, where $E_0$ is the Young’s modulus of the interfacial layer. Recent scaling laws [30, 94, 95, 96, 97, 98] have shown that $P_{cr} \propto d^{1/3} t^{1/6}$ for moderately thick interfacial layers and $P_{cr} \propto d^{1/4} t^{1/8}$ for very thin interfacial layers. We refer to these scaling laws as “scaling laws” without loss of generality. The parameter $\frac{P_{cr} d}{E_0 t^3}$ has units of a non-dimensional constant $\lambda$.
3.2.1 Analytical model of wrinkling instability

We analytically evaluate a representative volume element, RVE, that consists of a composite with a stiff interfacial layer supported by soft matrix on both sides. The thickness of the interfacial layer is \( t \), and the Young’s modulus and the Poisson’s ratio of the interfacial layer are \((E_1, \nu_1)\) and those of the two surrounding matrix regions are \((E_0, \nu_0)\) (Figure 3.2a). The interfacial layer can be modelled as a beam under an axial compression \( \bar{\sigma} = E_1 \bar{\varepsilon} \), and distributed normal and shear tractions, \( \sigma_{yy} \) and \( \sigma_{xy} \). We can find the critical stress at which instability occurs, \( \bar{\sigma}_{cr} \), and the wavelength of the wrinkling pattern, \( \lambda_{cr} \), by solving the governing equation of the interfacial layer together with the boundary value problem [90,99].

![Figure 3.2: a) Schematic of the wrinkling of a single interfacial layer with material properties \((E_i, \nu_i)\) supported by soft matrix on both sides with material properties \((E_0, \nu_0)\). The wrinkling wavelength, \( \lambda \), and deformation \( w(x) \) are shown; b) Free body diagram of the axial stress, \( \bar{\sigma} \), and the tractions, \( \sigma_{yy} \), acting on the interfacial layer; c) The axial stress is a function of the non-dimensional wrinkling wavelength. The minimum value of the axial stress corresponds to the critical stress that causes the instability, \( \bar{\sigma}_{cr} \) and gives the wrinkling wavelength, \( \lambda_{cr} \).](image)
For a dilute composite where the matrix layers on both sides of the interfacial layer are made of the same material, \((E_0, \nu_0)\), and the matrix layers are wide enough for there to be no interaction between the interfacial layers (the critical separation distance, \(D\), will be studied in Section 3.3), the instability is governed by the normal tractions on the interfacial layer, such that the governing equation per depth for the interfacial layer is given by:

\[
\frac{E_1 t^3}{12} \frac{d^4 w}{dx^4} + \bar{\sigma} t \frac{d^2 w}{dx^2} = -\sigma_{yy}
\]

(3.2)

where \(\frac{E_1 t^3}{12}\) is the bending stiffness of the interfacial layer, \(w(x)\) is the deflection of the interfacial layer, \(\bar{\sigma}\) is the applied axial stress, and \(\sigma_{yy}\) is the normal traction component from the matrix layers and onto the interfacial layer.

Figure 3.2 shows schematically the variables and the local tractions on the interfacial layer and the matrix layers. In order to find the stresses from the matrix normal to the interfacial layer, \(\sigma_{yy}\), we will use the method developed by [90]. An Airy’s stress function, \(\Phi(x, y)\), is defined for the matrix, such that it satisfies the bi-harmonic condition: \(\nabla^4 \Phi = 0\). Moreover, it is periodic in the \(x\)-direction, due to the periodic wrinkling pattern, and diminishing in the \(y\)-direction as we move away from the interfacial layer.

\[
\Phi(x, y) = A \cdot \sin \left(\frac{2\pi x}{\lambda}\right) \cdot (1 - B|y|) \cdot e^{-\frac{2\pi |y|}{\lambda}}
\]

(3.3)

The constants \(A\) and \(B\) are found through two boundary conditions under plane stress at \(y=0\):

\((i)\) \(\varepsilon_x = 0 \rightarrow B = -\frac{\pi}{\lambda} (1 + \nu_0)\)

\((ii)\) \(w(x = \frac{1}{4}) = w_0 \rightarrow A = \frac{w_{max} \lambda E_0}{\pi(3-\nu_0)(1+\nu_0)}\)

Hence the Airy stress function for the matrix layers under plane stress conditions becomes:

\[
\Phi(x, y) = \frac{w_{max} \lambda E_0 \left[\frac{\pi}{2} + \frac{\pi \nu_0}{\lambda}\right]|y|}{\pi(3-\nu_0)(1+\nu_0)} \cdot \sin \left(\frac{2\pi x}{\lambda}\right) \cdot e^{-\frac{2\pi |y|}{\lambda}}
\]

(3.4)

The \(\sigma_{yy}\) traction applied from the matrix and on to the interfacial layer, can be found by taking derivative of the Airy stress function, \(\sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}\):
\[
\sigma_{yy}(y=0) = -\frac{8w_{\max} \pi E_0}{\lambda(3-\nu_0)(1+\nu_0)} \cdot \sin \left(\frac{2\pi x}{\lambda}\right)
\]  

(3.5)

Next, the expression for the \( w(x) \) and \( \sigma_{yy}(y=0) \) are used together with the governing equation of the interfacial layer, Equation 3.2, to solve for the axial applied stress, \( \bar{\sigma} \):

\[
\bar{\sigma} = \frac{\pi^2 E_1}{3} \cdot \left(\frac{t}{\lambda}\right)^2 + \frac{2E_0}{\pi(3-\nu_0)(1+\nu_0)} \cdot \left(\frac{t}{\lambda}\right)
\]

(3.6)

As shown in Figure 3.2c there is a minimum in the value of the axial applied stress, \( \bar{\sigma} \), as a function of the wrinkling wavelength of the interfacial layer normalized with respect to its thickness, \( \frac{\lambda}{t} \). Hence, instability occurs at the minimum applied axial stress, which we define as the critical stress: \( \sigma_{cr} = \min\{\bar{\sigma}\} \). This corresponds to the critical axial applied strain: \( \varepsilon_{cr} = \frac{\sigma_{cr}}{E_1} \), and the wavelength of the wrinkling instability pattern is given by: \( \frac{\lambda_{cr}}{t} \).

For \textit{plane stress} conditions, the applied axial strain causing the wrinkling instability in the interfacial layer is and the wavelength of the wrinkling pattern are given by:

\[
\varepsilon_{cr} = -2.08[(3-\nu_0)(1+\nu_0)]^{-\frac{2}{3}} E_1^{-\frac{2}{3}} \]  

(3.7)

\[
\frac{\lambda_{cr}}{t} = 2.18[(3-\nu_0)(1+\nu_0)]^{\frac{1}{3}} E_1^{\frac{1}{3}} \]

Similarly under \textit{plane strain} conditions, the applied axial strain causing the wrinkling instability, and the non-dimensional wavelength of the wrinkling pattern are given by:

\[
\varepsilon_{cr} = -2.08 \left[\frac{(3-4\nu_0)}{(1-\nu_0)^2}\right]^{-\frac{2}{3}} E_1^{-\frac{2}{3}} \]  

(3.8)

\[
\frac{\lambda_{cr}}{t} = 2.18 \left[\frac{(3-4\nu_0)}{(1-\nu_0)^2}\right]^{\frac{1}{3}} E_1^{\frac{1}{3}} \]
The critical strain $\xi_{cr}$ and the nondimensionalized wavelength $\frac{\lambda_{cr}}{t}$ show, respectively, the $-\frac{2}{3}$ and $\frac{1}{3}$ power law scaling with the stiffness ratio $\frac{E_1}{E_0}$, as was presented in the beginning of this section. Using Equation 3.7 and 3.8, the critical strain and wavelength can be expressed directly in terms of one another, for both plane stress and plane strain:

$$\xi_{cr} = -\pi^2 \left(\frac{\lambda_{cr}}{t}\right)^{-2} \quad \text{or} \quad \frac{\lambda_{cr}}{t} = \frac{\pi}{\sqrt{\xi_{cr}}}.$$  \hspace{1cm} (3.9)

When the onset of instability occurs at small strains, Equation 3.7 and 3.8 for the critical wavelength are valid. However, when the critical strain is modest (for example, a 5% critical strain will decrease the critical wavelength by approximately 5%); the increased thickness of the interface layer due to axial strain should be taken into account. When the compressive strain $\bar{\xi}$ is increased beyond $\xi_{cr}$, the post-buckling wavelength $\lambda$ decreases due to the decrease in effective end-to-end length of the interface. Assuming the overall contour length of the interface is preserved, the post-buckling wavelength $\lambda$ and amplitude $w_{max}$ are obtained from kinematics [22] as a function of the applied strain, $\bar{\xi}$:

$$\lambda(\bar{\xi}) = \lambda_{cr} e^{-|\bar{\xi}|} \quad \text{for} \quad |\bar{\xi}| > |\xi_{cr}| \hspace{1cm} (3.10)$$

$$w_{max}(\bar{\xi}) = \frac{\lambda(\bar{\xi})}{\pi} \sqrt{|\bar{\xi} - \xi_{cr}|} \quad \text{for} \quad |\bar{\xi}| > |\xi_{cr}|$$
Figure 3.3: Comparison of the results of analytical-, FE-, and mechanical experimental results for different dilute composites; a) FE simulation results of the instability wrinkling modes for various stiffness ratios and geometries; b) comparison of the analytical (Eqs. 3.8a) and FE results of critical strain vs. stiffness ratio; c) comparison of the analytical (Eqs. 3.8b) and FE results of the non-dimensional wrinkling wavelength vs. stiffness ratio.
The results of the analytical derivations were further examined by comparison to FE simulations for dilute composites. FE simulation results of the wavy patterns of the interfacial layer are shown in Figure 3.3a, in which the deformed images are shown at a strain larger than the critical strain, \(|\varepsilon| > |\varepsilon_{cr}|\). The map of deformed images (shown in Figure 3.3a) depicts the strong dependence of wavelength on stiffness ratio and thickness. Figures 3.3b and c compare the analytical results with finite element results, confirming the linear dependence on \(t\) and the exponential dependence on stiffness ratio. These results show that the new analytical model is accurate for a large range of stiffness ratios. The results are further compared to the case for a coating on a compliant substrate [90,22,31] (dashed line), showing the coating structures wrinkle at a lower critical strain, \(\varepsilon_{cr}^{coating}/\varepsilon_{cr}^{int.layer} = (1/4)^{\frac{1}{3}}\), and the coating structures produce a longer instability wrinkling wavelength, \(\lambda_{cr}^{coating}/\lambda_{cr}^{int.layer} = (2)^{\frac{1}{3}}\).

Physical experiments were conducted to further verify the analytical and FE results, for different composites with varying interfacial layer thickness and material stiffness. Four different physical prototypes were fabricated using a 3D printer, representing different dilute composites with a single interfacial layer supported by matrix on both sides. The interfacial layer has a thickness of \(t\) and Young’s modulus of \(E_i\) (\(t\in [0.5,1]\)mm and \(E_i\in [18,1200]\)MPa), while the matrix layers have thickness of \(d\) and Young’s modulus of \(E_0\) (\(d\in [40, 80]\)mm and \(E_0= 0.9\)MPa). Experimental details can be found in Chapter 2. Plane strain compression of the four different single layered composites, show the onset and evolution of wrinkling of the interfacial layers during loading, and also its reversibility upon unloading (Figure 3.4a). The wrinkle onset, wavelength, and amplitude are proven to strongly depend on interfacial thickness, \(t\), and the ratio of elastic moduli of the interfacial material to that of the matrix, \(E_i/E_0\). The critical strain initial wrinkling wavelength are compared with our analytical and finite element results in Figure 3.3b and c. The post-buckling wrinkling wavelengths and post-buckling wrinkling amplitudes of wrinkling of the interfacial layer in the 3D-printed specimens were compared with the analytical prediction using Equations 3.8b and Equation 3.10, shown in Figure 3.4b. It can be seen that Equation 3.8 and 3.10 capture the on-set of wrinkling and the mechanism of wrinkle developments during the post-buckling process accurately.
Figure 3.4: Experimental results showing the onset and development of the wrinkling in a composite composed of a single interfacial layer supported by soft matrix on both sides; a) Specimen 1 (row 1) with interface thickness $t_0=0.5\text{mm}$ and $E_I = 18\text{MPa}$, Specimen 2 (row 2) with interface thickness $t_0=1\text{mm}$ and $E_I = 18\text{MPa}$, Specimen 3 (row 3) with interface thickness of $t_0=0.5\text{mm}$ and $E_I = 1200\text{MPa}$, Specimen 4 (row 4) with interface thickness of $t_0=1\text{mm}$ and $E_I = 1200\text{MPa}$; b) Comparison of the experimental results with our analytical predictions of the non-dimensional post-wrinkling wavelength (Equation 3.10a) and post-wrinkling amplitude (Equation 3.10b)
3.3 Instability in multilayered composites

We now consider a multilayered composite where interfacial layers of thickness $t$ and modulus $E_I$, are separated by a distance $D$ by soft matrix with modulus $E_i$, as shown in Figure 3.5. The loadings of a multilayered composite with distance $D$ between the interfacial layers, under a far-field compressive loading, are shown in Figure 3.5a. A free body diagram of an isolated interfacial layer is shown in Figure 3.5b, where $\sigma_{yy}$ and $\sigma_{xy}$ are the interfacial normal and shear traction components from the matrix layers and onto the interfacial layer.

The interfacial layer can be regarded as a beam under an axial compression load and under distributed normal and shear load along its length. The governing equation of the interfacial layer per unit depth thus becomes:

$$\frac{E_I t^3}{12} \frac{d^4 w}{dx^4} + \sigma t \frac{d^2 w}{dx^2} - \frac{t}{2} \frac{d \sigma_{yy}}{dx} = -\sigma_{yy}. \quad (3.11)$$

where $\frac{E_I t^3}{12}$ is the bending stiffness of the interfacial layer and $w(x)$ is the deflection of the interfacial layer. Solving the governing equation together with the boundary value program gives the critical applied load that will cause instability and the deformation mode of the instability.

We will now show that for a multilayered composite, the distance-to-thickness ratio, $D/t$, is found to govern two different modes of instability:

Case I: The dilute composite giving wrinkling-instability. When $D/t$ is large (specified by Equation 3.13), the matrix stress fields emanating from neighboring layers do not interact, hence the shear term in Equation 3.11 is negligible and interfacial layers behave independently of one another. Therefore, the instability deformation is wrinkling of the interfacial layer, and follows the analysis of an isolated interfacial layer as was given in Section 3.2.
Case II: The concentrated case (long-wave instability): when $D/t$ is small, the matrix stress fields overlap and the matrix shear stress is no longer negligible and governs the instability. In this case, the instability induced deformations are not localized at the interfaces and a cooperative long-wave instability mode occurs. The long-wave mode leads to the well-known localized kinking failure in composites under compression.

**Figure 3.5:** a) Schematics of a multilayered composites with interfacial layers of thickness $t$, separated by distance $D$ of soft matrix (measured centre-to-centre line), compressed axially by a strain, $\bar{\varepsilon}$; b) The free body diagram of an isolated interfacial layer, and the stresses acting on the interfacial layer. The interfacial layer will have wrinkling instability if $D/t$ is sufficiently large (Equation 3.13).
For completeness, we note that the term ‘micro-buckling’ has often been used differently in the composite literature [91,33] and refers to what we term here as the cooperative long-wave instability. To avoid any potential confusion, we will refer to the two interfacial instability conditions as ‘wrinkling’ and cooperative ‘long-wave mode’.

For the concentrated case, the long-wave mode for a multilayered composite where the two phases buckle together was derived by Rosen [33] using the energy method and assuming a shear mode deformation. In the literature [90,91], further studies were also reported which provide a more general solution taking into account both the normal stress and shear stress terms in Equation 3.11. For plane strain case, the critical strain at which the long-wave mode occurs as predicted by Rosen [33] is given by:

\[
\varepsilon_{cr}^{Longwave} = \frac{1}{2(1+v_o)} \frac{D^2}{t(D-t)} \left( \frac{E_1}{E_0} \right)^{-1}
\]  

(3.12)

where \( t \) is the thickness of the interfacial layers, \( D \) is the distance between the interfacial layers, and \((E_I, v_I)\) and \((E_o, v_0)\) are the stiffness and Poisson’s ratio of the interfacial layer and matrix layers, respectively.

By equating the critical strain in Equation 3.8a and Equation 3.12, \( \varepsilon_{cr}^{Wrinkling} = \varepsilon_{cr}^{Longwave} \), the critical layer distance to thickness ratio \( D^*/t \) at which the instability transitions from a wrinkling form of instability to a cooperative long-wave instability is found to be a function of the stiffness ratio and the Poisson’s ratio of the matrix:

\[
\frac{D^*}{t} = \left\{ 0.5 - \sqrt{0.25 - 0.24(3 - v_o)^2(1 + v_o)^{-\frac{1}{3}}(E_1/E_0)^{-\frac{1}{3}}} \right\}^{-1}
\]  

(3.13)

Alternatively, this equation for the critical distance between the layers determining the instability mode, can be rewritten and gives a relationship between the critical distance between the layers, \( D^* \), and the critical buckling wavelength \( \lambda_{cr} \) (given in Equation 3.8b):

\[
D^* = \frac{1.92}{3-v_o} \lambda_{cr}^* \approx 0.77 \lambda_{cr}^*
\]  

(3.14)
Figure 3.6: Mode transition mechanisms for multilayered composites from a dilute case to a concentrated case; a) FE results of the eigenmodes in the layers (wrinkling or long-wave mode) for different composites with varying distance-to-thickness ratio and stiffness ratios. In the transition zone, the wavelengths of the wrinkling pattern increase. b) Normal and shear stress contours for the case of $E_0/E = 100$ and changing $D/t$ shows the transition from local wrinkling to macroscopic long-wave instability. As the distance between the layers decreases, shear stresses overlap and the long-wave mode of instability occurs.
Equation 3.14 shows that if the distance between layers $D<0.77\lambda_{cr}$, then the layers start to interact and the shear stress contribution becomes significant such that a cooperative long-wavelength mode of instability occurs. In contrast, if the distance between the interfacial layers $D>0.77\lambda_{cr}$ then there is no interaction between the different interfacial layers, and they act and undergo instability as if they were one single interfacial layer supported by matrix on both sides; i.e. wrinkling in the interfacial layer is the mode of instability.

The transition from the wrinkling mode to the cooperative long-wave instability mode is shown in the FE simulation results of Figure 3.6. The FE simulation results for different multilayered composites, Figure 3.6a, illustrate the transition from wrinkling to long-wave mode as the composite’s geometrical features change, $D/t=[7.5, 15, 30, 60]$, or the material properties, the stiffness ratio $E/E_0$, change. It can be seen that when the stiffness ratio increases and/or the distance between layers decreases, the instability mode transitions from a local wrinkling mode to a long-wave mode with an infinite long wavelength (i.e. the long-wave mode is deformation of the interfacial layer such that there is one wavelength no matter the RVE length). More specifically, the wrinkling versus long-wave mode can be explained by studying the local stresses in the matrix layers of the composite. Focusing on the case of $E_1/E_0 = 100$, Figure 3.6b shows the transition in the normal- and shear stress-fields for $D/t=[7.5, 15, 30]$. We observe that as $D/t$ decreases, the stress fields begin to overlap and the matrix shear stresses govern the mode of instability; this leads to the transition in the instability of the interfacial layer from the wrinkling mode and to the long wavelength instability mode.

Figure 3.7a shows that wrinkling becomes the more prevalent instability mode in the interfacial layers of multilayered composites as the stiffness ratio between the interfacial layer and soft matrix decreases. For smaller stiffness ratios only very small intervals of $D/t$ will result in long-wavelength modes of instability. The general transition between wrinkling and macroscopic long-wave modes, as shown in Figure 3.7a, give a design guideline for choosing material properties and layer spacing to provide transforming wrinkling interface structures upon deformation.
Figure 3.7: a) Design diagram indicating the condition for mode transition. Lines are from analytical models, empty symbols are FE results, and filled symbols are experimental results (solid lines represent the case of wrinkling via Equation 3.8, dashed lines represent the case of long-wave mode via Equation 3.12, the darker parts of these lines represent when the mode occurs, while the lighter parts of these lines represent when that mode is superseded by the other mode). b) Instability transition phase diagram showing the transition from wrinkling to longwave mode as a function of $E_l/E_0$ and $D/t$ for multilayered composites; comparing analytical prediction (solid line) with FE results (square symbols) and mechanical experiments (circular symbols). Solid line is plotted from Equation 3.13, while the dashed line with bar symbols is from the Bloch wave analysis [34], and the range of the transition zone are denoted by the bar symbols.
More generally, the results of the transition in instability mode from wrinkling to long wave mode in multilayered composites are plotted in a wrinkling-transition phase diagram, Figure 3.7b. Results from our analytical predictions (Equation 3.12), FE simulations, mechanical experiments, and a Bloch wave [34] analysis are all shown in the instability phase diagram, and they are all in great agreement. Based on the geometric feature, $D/t$, and material properties, $E_i/E_o$, of the composite, this phase diagrams shows the instability mode that will occur as the composite is being compressed. Furthermore, we can see that for a given $E_i/E_o$, as the distance between the layers are decreased, the instability mode transitions from wrinkling and over to a long-wave mod. The FE and experimental results fall within the correct sections of the phase diagram, showing excellent agreement and supporting the predictive design guidelines.

Multilayered physical prototypes were fabricated using a multi-material 3D-printer, and plane strain compression tests were conducted (Experimental details can be found in Chapter 2). Images were taken during the loading and unloading of the different composites, and the results are show in Figure 3.8. The dependence of the wrinkling and long-wave instabilities on $E_i/E_0$ and $D/t$, as well as the transition from the wrinkling mode to the cooperative long-wave instability mode are demonstrated in the experimental results. Specimen 1 [$D/t=20; E_i/E_0 =20$] shows a wrinkling mode which is in agreement with the model predictions. Specimen 2 [$D/t=6; E_i/E_0 =20$] shows a case with the same stiffness ratio as Specimen 1, but reduced spacing: model results predict this case to be located at the transition zone, meaning it could either wrinkle or buckle in a long wave mode and is sensitive to perturbation (imperfection, boundary conditions etc.); here a cooperative wrinkle is observed. Finally, Specimen 3 [$D/t=20; E_i/E_0=1300$] shows the influence of a much greater stiffness ratio at a fixed spacing and shows the transition to a long-wave mode which is also in agreement with the model results. These results demonstrate the predictive capability of the model and hence its use as a guideline for materials design.
Figure 3.8: Results of the mechanical experiments of the formation and development of the wrinkling and long-wave instability modes in multilayered composites. Specimen 1 (row 1) $E_1/E_0=20$ and $D/t=20$ undergoes wrinkling instability; Specimen 2 (row 2) $E_1/E_0=20$ and $d/t=6$ undergoes a modified wrinkling instability due to being in the transition zone; Specimen 3 (row 3) $E_1/E_0=1300$ and $D/t=20$ undergoes long-wave mode instability.
3.4 Instability in multi-material and networked composites

Building upon the findings and results of instabilities in dilute and concentrated multilayered composites, we propose to investigate the instability in more complex composite networks and geometries. This is of interest as it allows more applications of the wrinkling mechanism and also opens a possibility of observing new phenomena. In this section we will extend the concepts covered in Chapter 3.2-3.3 to study the instability and wrinkling in more complex composites. We will look at the wrinkling in the stiff interfacial layers in multi-material composites, as well as the wrinkling in the network walls of composites created by a stiff networks embedded in a soft matrix.

3.4.1 Wrinkling in multi-material multilayered composite

We will now study the instability in a dilute multi-material composite where an interfacial layer of thickness \( t \), Young’s modulus \( E_1 \), and Poisson ratio \( \nu_1 \), is supported by different soft matrix layers on either sides. The different material properties for the matrix layers on either side of the interfacial layer are \( (E_{01}, \nu_{01}) \) and \( (E_{02}, \nu_{02}) \), as shown in Figure 3.9a. Using the approach of Chapter 3.2 and 3.3 we can find the critical strain that causes wrinkling instability and the wrinkling wavelength in the interfacial layer of this multi-material composite.

For this multi-material dilute composite, the governing differential equation of the interfacial layer per unit depth is given by (we will assume that the system is governed by the normal tractions):

\[
\frac{E_1 t^3}{12} \frac{d^4 w}{dx^4} + \bar{\sigma} \frac{d^2 w}{dx^2} = -\sigma_{yy}
\]

(3.15)

where \( D = \frac{E_1 I_1}{12} \) is the bending stiffness of the interfacial layer per unit depth and \( w(x) = w_{max} \cdot \sin \frac{2\pi x}{\lambda} \) is the deflection of the interfacial layer; \( \bar{\sigma} \) is the applied compressive stress in the interfacial layer, and \( \sigma_{yy} \) is the interfacial normal traction component (as shown in Figure 3.9b).
As done in Section 3.2, by defining an Airy stress function for each of the matrix layers and then solving the boundary value problem, the normal traction from the matrix layers and onto the interfacial layer, $\sigma_{yy}$, under plane strain conditions, are found to be:

$$\sigma_{yy}\bigg|_{y=0} = 2\frac{w_{max}}{\lambda_{cr}} \left[ \frac{2\pi E_{01}(1-v_{01})}{(3-4v_{01})} + \frac{2\pi E_{02}(1-v_{02})}{(3-4v_{02})} \right] \sin \left( \frac{2\pi x}{\lambda} \right) \quad (3.16)$$

Next, we solve the governing equation (Equation 3.15) to find expression for the applied normal axial stress, $\sigma$, and find the minimum stress that will cause instability (procedure described in Chapter 3.2). Consequently, under plane strain conditions, the critical strain causing wrinkling instability in the interfacial layer of a multi-material composite, as well as the initial wrinkling wavelength are given by:

**Figure 3.9:** a) Schematic of an interfacial layer with thickness $t$, and material properties $(E_1, v_1)$, supported by different matrix on either side, $(E_{01}, v_{01})$ and $(E_{02}, v_{02})$. As this multi-material composite is compressed, a critical strain will be reached that causes the wrinkling instability in the interfacial layer with a periodic deformation of $w(x)$ and an initial wavelength of $\lambda_{cr}$; b) The axially applied compressive stress $\sigma_{cr} = \bar{E}_1 \bar{\varepsilon}_{cr}$, and the normal and shear tractions, $\sigma_{yy}$ and $\sigma_{xy}$, acting on the interfacial layer.
\[
\bar{\varepsilon}_{cr} = \left[ \frac{3}{2} \frac{E_{01}(3-4v_{02})(1-v_{01})^2 + E_{02}(3-4v_{01})(1-v_{02})^2}{E_1(3-4v_{01})(3-4v_{02})} \right]^{\frac{2}{3}}
\]

\[
\frac{\lambda_{cr}}{t} = 2 \left[ \frac{\pi^3}{12} \frac{E_{01}(3-4v_{01})(3-4v_{02})}{E_{01}(3-4v_{02})(1-v_{01})^2 + E_{02}(3-4v_{01})(1-v_{02})^2} \right]^{\frac{1}{3}}
\]

It is important to note that by taking \( E_{02} = 0 \) and \( v_{02} = 0 \), we effectively eliminate one side of the matrix and the critical conditions reduce to those for the wrinkling of a coating on a soft substrate [22].

If the two matrix layers have equal Poisson ratio, \( v_{01} = v_{02} = v_0 \), Equation 3.17 becomes:

\[
\bar{\varepsilon}_{cr} = 1.31 \left[ \frac{(3-4v_0)}{(1-v_0)^2} \right]^{\frac{2}{3}} \left( \frac{E_1}{E_{01}+E_{02}} \right)^{\frac{2}{3}}
\]

\[
\frac{\lambda_{cr}}{t} = 2.74 \left[ \frac{(3-4v_0)}{(1-v_0)^2} \right]^{\frac{1}{3}} \left( \frac{E_1}{E_{01}+E_{02}} \right)^{\frac{1}{3}}
\]

To further verify these analytical models under plane strain conditions, FE simulations of an interfacial layer of thickness \( t \) and stiffness \( E_l \) bonded on either side to different matrix mediums were conducted and shown in Figure 3.10. For the case of \( v_{01} = v_{02} = 0.48 \), Figure 3.10a shows the different wrinkling deformations of the interfacial layer based on just changing the stiffness of the two matrix layers: \( E_{01} \) and \( E_{02} \).

More generally, if \( v_{01} \neq v_{02} \), Equation 3.17 shows that the critical strain and wrinkling wavelength are related to two independent non-dimensional parameters \( \frac{E_1}{E_{01}} \) and \( \frac{E_{02}}{E_{01}} \). To make quantitative evaluation, both of the non-dimensional parameters are varied such that \( \frac{E_1}{E_{01}} \in [10 - 500] \) and \( \frac{E_{02}}{E_{01}} \in [0 - 1] \) to create different composites. The analytical and FE results for the critical strain causing instability and the initial wrinkling wavelength for these different multi-material composites, are compared in Figure 3.10b and c for when \( v_{01} = v_{02} = 0.48 \).
Figures 3.10b and c show that the critical strain and wrinkling wavelength is mainly determined by $\frac{E_1}{E_{01}}$ where $E_{01} > E_{02}$. Moreover, for a certain value of $\frac{E_1}{E_{01}}$, by only varying $\frac{E_{02}}{E_{01}} \in [0 - 1]$, the critical strain can be tuned by a factor of 1~1.6, while the critical wavelength can be tuned by a factor of 1~0.8, as shown in Figure 3.10b and c, respectively.

It can be seen that the critical strain is mainly determined by the stiffness ratio $\frac{E_1}{E_{01}}$, but can also be further tuned by $\frac{E_{02}}{E_{01}}$. The influence of $\frac{E_{02}}{E_{01}}$ on the critical strain increases when $\frac{E_1}{E_{01}}$ increases (as shown in Figure 3.10b). At the same time, the influence of $\frac{E_{02}}{E_{01}}$ on the non-dimensional critical wavelength also increases when $\frac{E_1}{E_{01}}$ increases (as shown in Figure 3.10c).

The study on the dissimilar matrix layers around the interfacial layer builds a foundation to design functionally graded materials, and hence interesting for many multi-purpose applications.
Figure 3.10: a) FE results showing the wrinkling modes of instability for different composites with varying materials for the matrix layers. b) Comparison of the analytical prediction (Eqs. 3.18a) and FE results of critical strain vs. stiffness ratio of the two matrix layers (the parameters are $E_1/E_{01}$ and $E_{02}/E_{01}$); c) Comparison of the analytical prediction (Eqs. 3.18b) and FE results of non-dimensional wrinkling wave length vs. stiffness ratio of the two matrix mediums of the two matrix layers (the parameters are $E_1/E_{01}$ and $E_{02}/E_{01}$).
3.4.2 Instability in networked composites

Finite element models of rectangular-, hexagonal- and diamond- cellular networks embedded in a soft matrix were built to study the instability of networked composites (Figure 3.11). Equivalent bi-axial stresses were generated in the finite element models of the networked composites by using constrained expansion of the network, done by using thermal expansion. Both the network and matrix material are assumed to be linear elastic and isotropic with thermal expansion coefficient \( \alpha = 1 \). To simulate the constrained expansion and the periodicity of the structure, the vertexes of the RVE were pinned and periodic boundary conditions were applied. Thus, the RVE is under bi-axial compression. We note that the network is not only constrained by the matrix, but is also subjected to transverse compressive stress due to the expansion of the matrix. However, since the stiffness ratio is large in our analyses, this compressive stress due to the matrix expansion can still be neglected.

![Figure 3.11: Schematic of a RVE of a rectangular -, hexagonal-, and diamond- shaped network embedded in soft matrix. The network has material properties \((E_1, v_1)\), while the material properties of the matrix is \((E_0, v_0)\). The thickness of the network walls is \( t \).](image)

**Figure 3.11**: Schematic of a RVE of a rectangular -, hexagonal-, and diamond- shaped network embedded in soft matrix. The network has material properties \((E_1, v_1)\), while the material properties of the matrix is \((E_0, v_0)\). The thickness of the network walls is \( t \).

*Rectangular and brick-shaped networked composites*

We first start by showing the pure axial compression of rectangular networked composites with different stiffness ratios between the network and the supporting matrix. Figure 3.12a shows finite element simulation results of the RVE of a rectangular network being compressed...
in the y-direction, and the instabilities created in the network walls. As the stiffness ratio between the materials, \( \frac{E_I}{E_0} \), increase, so does the wavelength of the wrinkling pattern observed. This is in agreement with the predictions for the multilayered composites presented in Chapter 3.2. Moreover, for these rectangular networked composites, we observe that the corners of the rectangular network result in irregularities in the wrinkling pattern. As the stiffness ratio between the network and the matrix increases, we see that the instability mode transitions from a microscopic instability and wrinkling to a macroscopic instability where the matrix and network together experience a long-wave mode of instability. This can be explained by evaluating the shear stress distribution in the matrix; for small stiffness ratios there are wrinkling patterns and hence only local shear stresses in the matrix, but as the wavelength increases the shear stresses start evolving and overlapping. When the shear

![Graphs showing wrinkling and instability patterns with different material stiffness ratios.](image)

**Figure 3.12:** Wrinkling and instability patterns, and shear stresses in the RVE of rectangular networked composites, with \( \frac{L_y}{L_x} = 0.4 \) and \( t = 0.5\text{mm} \) and with different material stiffness ratios, \( \frac{E_I}{E_0} \). Uniaxial compressive loading in the **a)** y-direction and **b)** x-direction.
stresses are fully overlapped, the whole structure is governed by shear deformation and a macroscopic instability is observed giving a long-wave form of instability (Figure 3.12a). Static compression of the composite structure in the x-direction (Figure 3.12b) results in wrinkling in the network walls for small stiffness ratios. However, as the stiffness ratio increases, the shear stresses never overlap due to the long distance between consecutive horizontal layers and the long-wave mode instability will no longer occur. Instead the networked composite undergoes a macro-instability only in the network with one half-wave pattern within the length of the RVE in the x-direction.

Biaxial compression of rectangular networked composites were also studied for geometries where wrinkling was the guaranteed mode of instability in the network walls. The results for eight different composites are shown in Figure 3.13a. The length aspect ratio of each rectangular cell is defined as the ratio \( R = \frac{L_x}{L_y} \), where \( L_i \) is the cell length in the \( i \)-direction. The cases of \( R = [0.5, 1, 2, 4] \) are considered and compared in this study. To rule out the influence of cell size, the area of each cell in all models is kept constant (30\( \mu \)m x 30\( \mu \)m). Furthermore, a network thickness of \( t = 1 \mu \)m and stiffness ratio of \( \frac{E_i}{E_0} = 100 \) were kept fixed for all the FE simulations. Finally, the eight different composites evaluated can be divided into two groups: \( i \) assembly of rectangular cells with corner junctions, and \( ii \) assembly of rectangular cells with mid junctions. As shown in Figure 3.13a, for all \( R \) values, the deformation of cells with corner junctions is localized in the corners while other part of the interfacial segments remain almost straight. However, the deformation of the cells with mid junctions is distributed nicely along the longer segments with uniform wavelength. Figure 3.13b shows that the analytical model is accurate in predicting the critical strain causing instability for all the different networks and cell assemblies considered. The critical strains of square assemblies with corner junctions (group-\( i \)) are slightly larger than strains of that with mid junctions (group-\( ii \)). Moreover, as shown in Figure 3.13c, the influence of length aspect ratio \( R \) has only little influence on the critical strain, especially when \( R > 1 \). The main influence of the assembly lies in the instability modes.
Figure 3.13: a) Instability modes in rectangular networked composites with corner and mid junctions and varying $R=L_x/L_y$. b) Critical strain for composites with fixed $R=1$, but varying stiffness between the network and the soft matrix, $E_y/E_0$. c) Critical strain for composites with fixed stiffness between the network and the soft matrix $E_y/E_0=100$, but varying geometry and length ratio of rectangular cells, $R$.

**Hexagonal- and diamond-shaped networked composites**

Biaxial compression of more complex composites composed of a hexagonal- or diamond-shaped cellular network embedded in soft matrix were also studied. The networks have a stiffness $E_I$, and are composed of regular hexagonal- or diamond-shaped cells with a uniform size (the cell height is kept constant at $H=30\mu$m, as shown in Figure 3.14a and Figure 3.15a).
Finite element simulations were done on representative volume elements (RVE) containing two units. The wall thickness of the network was varied, \( t = [0.5, 1, 2, 4] \mu m \), as well as the stiffness ratio in the composites, \( \frac{E_1}{E_0} = [50, 100, 1000] \).

**Figure 3.14:**

a) RVE of hexagonal networked composite (\( H = 30 \mu m \)),

b) Analytical and FE results for the instability critical strain (left axis) and wrinkling wavelength (right axis) for composites with different stiffness ratios and thicknesses of the network wall, \( t \); \( t = 0.5 \mu m \) (circular symbols), \( t = 1 \mu m \) (diamond symbols), \( t = 4 \mu m \) (star symbols), \( t = 0.5 \) (circular symbols);

c) FE results for the instability modes in the network and composite for composite with varying stiffness ratio and geometric ratio.
FE results of the instability mode for biaxial compression of composites with varying geometry, \( t/H \), and material composition, \( E_t/E_0 \), are shown in Figure 3.14 (hexagonal networks) and Figure 3.15 (diamond networks). Due to the symmetric geometry of these composites, the number of wrinkling wave in each segment of the network walls are the same for each of the different composites evaluated. The non-dimensional wrinkling wavelength, \( \frac{\lambda_{cr}}{t} \), as well as the critical strain causing the instability, \( \varepsilon_{cr} \), found from the FE simulations are compared with our analytical models (Equation 3.8) in Figure 3.14b and Figure 3.15b. It can be seen that the agreement between the analytical and FE results are great for the composites’ critical strains, while the wrinkling wavelength is accurate as long as each segment of the network walls is long enough to include several wrinkling waves within its length. Hence, future work on this research topic can study how to include the influence of the finite length of the interfacial layer or network walls into the analytical model for the instability predictions.

Figure 3.14c and Figure 3.15c illustrate that as long as \( \frac{E_1}{E_0} \) or \( t/H \) are small (soft and dilute composites), the matrix provide resistance to the network such that the network walls in the composite undergo a microscopic wrinkling instability. However, a macroscopic instability mode, the long-wave mode, is seen in the composites as \( \frac{E_1}{E_0} \) or \( t/H \) increases.

Moreover, the composites with the wrinkling instability in the network walls, \( \frac{E_1}{E_0} \) and \( t/H \) small, are also increasingly more sensitive to the initial imperfection during the post-buckling analysis due to the local nature of the stresses and deformations. For example, due to a light asymmetry in the mesh, small asymmetry and deviation is observed in the results for the case of \( \frac{E_1}{E_0} = 50 \) and \( t/H=1/60 \) (Figure 3.14c and 3.15c). On the other hand, as \( \frac{E_1}{E_0} \) and \( t/H \) increases, the influences of the matrix decreases and the system approaches the case of compression of an empty cellular network. For example, for the hexagonal networked composites, as \( \frac{E_1}{E_0} \) and \( t/H \) increases, the instability modes are asymptotic to the classical results of hexagonal honeycombs under biaxial compression [41].
Figure 3.15: a) RVE of diamond shaped networked composite \((H=30 \, \mu m)\), b) Analytical and FE results for the instability critical strain (left axis) and wrinkling wavelength (right axis) for composites with different stiffness ratios and thicknesses of the network wall, \(t\); \(t=0.5 \, \mu m\) (circular symbols), \(t=1 \, \mu m\) (diamond symbols), \(t=4 \, \mu m\) (star symbols), \(t=0.5\) (circular symbols); c) FE results for the instability modes in the network and composite for composite with varying stiffness ratio and geometric ratio.
3.5 Conclusion

In this chapter, analytical and finite element based micromechanical models were developed to explore and predict instabilities in the interfacial layers of multilayered composites. The models provide quantitative tools and guidelines for the design of reversible, tunable, and multi-functional composites. Additionally, physical prototypes of multilayered polymer composites were fabricated, using a multi-material 3D printer, and mechanically tested, confirming the analytical and numerical predictions.

More specifically, the analytical models developed predict the critical compressive strain causing instability, the initial wrinkling wavelength, the post-buckling wavelength and amplitude for thin interfacial layers embedded in a compliant matrix. The models were shown to be functions of the geometry (thickness of the interfacial layer) and material properties (Young’s moduli and Poisson’s ratios) of the composite. In addition, models were derived to also reveal the conditions which govern the transition from a wrinkling pattern to a macroscopic long-wave instability mode, in multilayered composites.

Moreover, the instability in networked composites where studied, where rectangular-, diamond-, or hexagonal networks were embedded in soft matrix. Wrinkling patterns in the network walls were observed as an instability mode, as well as macroscopic instabilities with long wave modes. The critical strain causing instability was mainly affected by the stiffness ratio of the material in the composite and the thickness to cell size ratio of the network. The shape and aspect ratio of the cell has only a small influence on the critical strain.

The analytical models together with the numerical approaches presented in this Chapter, have a great potential to be used to derive biomimetic principles for bio-inspired active and multi-functional hybrid materials or actuating devices, or functionally graded materials. In the next chapters of this thesis, we will investigate further the effect of the wrinkling instability on the composites macroscopic behavior and properties, and show some interesting new applications.
Chapter 4

Modelling local deformations due to wrinkling instability

4.1 Introduction

The ability to predict the deformation mechanisms and corresponding local strains and stresses at any point in the composite is essential for designing a composite that can be tailored to meet a designer’s requirements. In this chapter we develop analytical expressions for the local strains and stresses in a composite where the stiff thin interfacial layers undergo wrinkling. Looking at Figure 4.1, the thin interfacial layers in the multilayered structure or in the rectangular network can be modeled as a multilayered composite with a representative volume element (RVE) that contains a thin interfacial layer with stiffness $E_I$, Poisson ratio $v_I$, thickness $t$, and length $L$, supported by soft matrix on both sides with stiffness $E_o$, Poisson ratio $v_o$, depth $d$, and length $L$. Furthermore, the distance between successive interfacial layers is given by $D=(2d+t)$. Here we take the case of plane strain, and develop analytical models describing the local deformations and behavior of the composite as the composite material is being compressed in the pre-buckling region, undergoes instability, and then further compressed into the post-buckling region.

Chapter 3 evaluated the compression of a multilayered composite consisting of a thin stiff interfacial layer surrounded by soft matrix on both side, and the modes of instability occurring based on the composite’s geometry and material combinations. It was established that for dilute composites once a critical global strain, $\varepsilon_{cr}$, is applied, the interfacial layer in the composite will undergo wrinkling instability (Figure 4.1). A composite is regarded as dilute
as long as \( D/t \) is large enough, or in other words as long as the volume fraction of the interfacial layers, \( t/D \), is small. The wrinkling instability will occur as long as:

\[
\frac{t}{D} < 0.5 - \sqrt{0.25 - 0.24(3 - v_0)^3(1 + v_0)^{-\frac{1}{3}} \left( \frac{E_1}{E_0} \right)^{\frac{1}{3}}}
\]  

(4.1)

Figure 4.1: Representative volume element (RVE) of a composite containing a stiff interfacial layer with length \( L \), thickness \( t \), Young’s modulus of \( E_1 \) and Poisson ratio \( v_1 \), supported by soft matrix on both side with length \( L \), thickness \( d \), Young’s modulus of \( E_0 \) and Poisson ratio \( v_0 \) (\( D=2d+t \) is the full composite RVE thickness). Upon a global compressive strain \( \varepsilon \), greater than the buckling strain \( \varepsilon_{cr} \), \( |\varepsilon| > |\varepsilon_{cr}| \), the interfacial layer will undergo instability and deform into a wrinkling pattern with a wavelength of \( \lambda \).

In this chapter we will only focus on dilute composites where Equation 4.1 is satisfied, and wrinkling is the mode of instability. In Chapter 3.2 it was shown that, when the macroscopically applied compressive strain, \( \varepsilon \), reaches a critical value \( \varepsilon_{cr} \), then the interfacial layer will take on a wrinkling waveform. After wrinkling, as the composite is compressed further into the post-buckling region, \( |\varepsilon| > |\varepsilon_{cr}| \), the number of wrinkles in the interfacial layer will remain constant, but the amplitude of the deformations will increase, while the length of
the wrinkling wavelengths will decrease. Figure 4.2 schematically shows the post-buckling behavior, both right at buckling strain, and at a larger post-buckling strain.

The lateral displacement of the interfacial layer due to the wrinkling pattern, \( w(x, \varepsilon) \), in the is given by:

\[
w(x, \varepsilon) = w_{\text{max}}(\varepsilon) \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right) = \frac{\lambda(\varepsilon)}{\pi} \sqrt{|\varepsilon| - |\varepsilon_{\text{cr}}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right)
\]  

(4.2)

where \( w_{\text{max}}(\varepsilon) \) is the post-buckling amplitude and \( \lambda(\varepsilon) \) is the post-buckling wavelength of the wrinkling pattern, given by:

\[
w_{\text{max}}(\varepsilon) = \frac{\lambda(\varepsilon)}{\pi} \sqrt{|\varepsilon| - |\varepsilon_{\text{cr}}|} ,
\]  

(4.3)

\[\lambda(\varepsilon) = \lambda_{cr} e^{-|\varepsilon|}\]

where the critical strain causing instability, \( \varepsilon_{\text{cr}} \), and the initial wrinkling wavelength, \( \lambda_{cr} \), is given by:

\[
\varepsilon_{\text{cr}} = -2.08 \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{2/3} \left( \frac{E_1}{E_0} \right)^{2/3}
\]  

(4.4)

\[
\lambda_{cr} = \frac{t}{2.18} \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{1/3} \left( \frac{E_1}{E_0} \right)^{1/3}
\]

**Figure 4.2:** Schematic of the composite RVE at; a) Buckling initiation, \( \varepsilon = \varepsilon_{\text{cr}} \), with the critical wavelength, \( \lambda_{cr} \), and negligible wrinkling amplitude, \( w_{\text{max}} \). b) Post-buckling region, \( |\varepsilon| > |\varepsilon_{\text{cr}}| \), with post-buckling wavelength, \( \lambda(\varepsilon) \), and post-buckling wrinkling deformation, \( w(x, \varepsilon) \), and amplitude, \( w_{\text{max}}(\varepsilon) \).
In the next sub-chapters we will first develop the local strains and stresses in the interfacial layer and the matrix for pre-buckling strains, and then move on to analytically express the local strain and stresses for post-buckling strains. Finite element simulations are used to verify the analytical models.

4.2 Analytical models of local deformations in pre-buckling

As long as the applied global strain, $|\varepsilon|$, is below the critical buckling strain, $|\varepsilon_{cr}|$, there will be a homogeneous compression in the composite, in both the interfacial layers and matrix layers, and linear elasticity applies. For an interfacial layer and matrix that have a linear elastic constitutive material model, the local strains developed in the interfacial layer or matrix under an uniaxial compression of $\bar{\varepsilon}$, and under plane strain conditions, are given by:

$$\varepsilon_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = \bar{\varepsilon}$$
$$\varepsilon_{y}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = -\bar{v}_1 \bar{\varepsilon}$$
$$\varepsilon_{xy}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = 0$$
$$\varepsilon_{z}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = 0$$

The corresponding local stresses in the interfacial layer and matrix, are found by:

$$\sigma_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = \bar{E}_1 \bar{\varepsilon}$$
$$\sigma_{y}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = 0$$
$$\sigma_{xy}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = 0$$
$$\sigma_{z}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = \bar{v}_1 \bar{E}_1 \bar{\varepsilon}$$

$$\sigma_{x}^{\text{matrix}}(x, y, \bar{\varepsilon}) = \bar{E}_0 \bar{\varepsilon}$$
$$\sigma_{y}^{\text{matrix}}(x, y, \bar{\varepsilon}) = 0$$
$$\sigma_{xy}^{\text{matrix}}(x, y, \bar{\varepsilon}) = 0$$
$$\sigma_{z}^{\text{matrix}}(x, y, \bar{\varepsilon}) = \bar{v}_0 \bar{E}_0 \bar{\varepsilon}$$

where $x$ is the location along the interfacial layer $x \in [0, L]$, $h$ is the distance away from the central line of the interfacial layer such that $h \in \left[-\frac{t}{2}, \frac{t}{2}\right]$, and $y$ is the distance into the depth.
of the matrix away from the interfacial layer such that \( y \in [-d, d] \). Furthermore, \( E_1 \) and \( E_0 \) are the plane strain Young’s modulus, and \( \bar{\nu}_1 \) and \( \bar{\nu}_0 \) are the plane strain Poisson’s ratio, for the interfacial layer and the matrix. They are given by:

\[
E_1 = \frac{E_1}{1-\nu_1}, \quad E_0 = \frac{E_0}{1-\nu_0}, \quad \bar{\nu}_1 = \frac{\nu_1}{1-\nu_1}, \quad \bar{\nu}_0 = \frac{\nu_0}{1-\nu_0}
\]   \hspace{1cm} (4.7)

### 4.3 Local deformations in the interfacial layer during post-buckling

#### 4.3.1 Analytical model predicting interfacial layer deformations

The interfacial layer is approximated as a thin beam and shear deformations are neglected, such that the Timoshenko beam theory is simplified to the Euler-Bernoulli beam theory. Using the Euler-Bernoulli assumptions\(^3\), the local strains and stresses at any point in the interfacial layer can be found. In the post-buckling region, the local strain in the interfacial layer, \( \varepsilon_{\text{int.,layer}}(x, h, \bar{\varepsilon}) \), will consist of the strain at the center line of the beam, plus strains due to the bending and curving of the interfacial layer. This we can written as:

\[
\varepsilon_{\text{int.,layer}}(x, h, \bar{\varepsilon}) = \varepsilon_{\text{mid}}^m(x, \bar{\varepsilon}) + h \cdot \kappa(x, \bar{\varepsilon})
\]   \hspace{1cm} (4.8)

where \( \varepsilon_{\text{mid}}^m \) is the strain at the mid-plane/center of the interfacial layer, \( h \) is the distance away from the center line, and \( \kappa \) is the curvature due to the bending and wrinkling of the interfacial layer (Figure 4.3). Equation 4.9 shows that \( \varepsilon_{\text{mid}}^m \) can be expressed as linear term plus a non-linear term that captures large deformation effects (\( u \) and \( w \) are the deflections in the \( x- \) and \( y- \)direction, respectively). We approximate the beam to be inextensible in the post-buckling region, which means that the length of the beam remains constant, and compression mainly deforms the beam through bending and increased curvature. Hence, \( du/dx \) will stay constant.

\(^3\) The assumption of a thin structure needs to be satisfied, i.e., \( \frac{\lambda_{\text{ef}}}{t} > 8-10 \), thus \( E/E_0 \) should be larger than 12.5-25.
at $\bar{\varepsilon}_{cr}$, in the post-buckling process. Using Equation 4.2 and 4.8, the strain at the mid-plane of the interfacial layer is found to be:

$$\varepsilon_{x}^{mid}(x, \bar{\varepsilon}) = \frac{du}{dx} + \frac{1}{2} \left( \frac{d\varepsilon}{dx} \right)^2 = \bar{\varepsilon}_{cr} + \frac{1}{2} \left( \frac{d\varepsilon}{dx} \right)^2 = \bar{\varepsilon}_{cr} + 2 \cdot (|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|) \cdot \cos^2 \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right)$$ (4.9)

where $\bar{\varepsilon}_{cr}$ is the critical strain at which instability occurs in the composites (Equation 4.4a), $\lambda(\bar{\varepsilon})$ is the wavelength for the wrinkling pattern of the interfacial layer at an applied strain of $\bar{\varepsilon}$ (Equation 4.3b), and $x$ is the location along the interfacial layer.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{beam_element.png}
\caption{Schematic of a beam element under bending with a curvature, $\kappa(x, \bar{\varepsilon})$. $x$ is along the beam, $h$ is across the beam thickness, and the center line is the mid-plane at $h=0$.}
\end{figure}

The curvature, $\kappa$, of the wrinkling pattern of the interfacial layer can be found at each point along the interface, $x$, for a global applied strain, $\bar{\varepsilon}$, by using $w(x)$ from Equation 4.2, according to:

$$\kappa(x, \bar{\varepsilon}) = \frac{\frac{d^2w}{dx^2}}{\left[ 1 + \left( \frac{dw}{dx} \right)^2 \right]^{3/2}} = \frac{-4\pi \sqrt{|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right)}{\lambda(\bar{\varepsilon}) \cdot \left[ 1 + 4 \cdot (|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|) \cdot \cos^2 \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right) \right]^{3/2}}$$ (4.10)

By taking small deformation approximation, we can neglect non-linear deformation effects. Hence, we can linearize the deformations through a first order approximation:

$$\frac{dw}{dx} \ll 1 \quad \rightarrow \quad \left( \frac{dw}{dx} \right)^2 \approx 0$$ (4.11)
The small deformation approximation, reduces Equation 4.9 and Equation 4.10 to:

\[ \varepsilon_{x}^{\text{mid}}(x, \bar{\varepsilon}) \approx \frac{du}{dx} \approx \bar{\varepsilon}_{cr} \quad (4.12) \]

\[ \kappa(x, \bar{\varepsilon}) \approx \frac{d^{2}w}{dx^{2}} \approx \frac{-4\pi}{\lambda(\bar{\varepsilon})} \sqrt{|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right) \]

Consequently, the linearized expression for the local strains in the interfacial layer, \( \varepsilon_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) \), for any macroscopic applied strain, \( \bar{\varepsilon} \), is given by:

\[ \varepsilon_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) \approx \bar{\varepsilon}_{cr} - h \cdot \frac{4\pi}{\lambda(\bar{\varepsilon})} \sqrt{|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right) \quad (4.13) \]

Moreover, for the plane strain case and assuming a linear elastic constitutive material model, the axial stresses in the interfacial layer are approximated by:

\[ \sigma_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = \bar{E}_{1} \cdot \varepsilon_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) \quad (4.14) \]

\[ \sigma_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) = \bar{E}_{1} \bar{\varepsilon}_{cr} - h \cdot \frac{4\pi \bar{E}_{1}}{\lambda(\bar{\varepsilon})} \sqrt{|\bar{\varepsilon}| - |\bar{\varepsilon}_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\bar{\varepsilon})} \right) \]

where \( \bar{E}_{1} = E_{1}/(1 - \nu_{1}^{2}) \) is the plane strain modulus.
4.3.2 Analytical and FE results for interfacial layer deformations

Finite element (FE) simulations were performed verifying our analytical predictions for the local strains in the interfacial layer in the composite for a range of different geometries and material combinations. Figure 4.4 shows the deformation, $w(x)$, for a few composites with different $E_i/E_0$ at $\bar{\varepsilon} = -0.07$, found from Equation 4.2 compared with FE results. We can see that the post-buckling wavelength and amplitude is good, but there are small deviations between the analytical and FE results, which can consequently also affect the calculated interfacial layer curvature and strains.

![Graphs](image)

**Figure 4.4:** Analytical results (solid line) and FE results (dashed line) for the wrinkling deformation of the interfacial layer, $w(x)$, in composite based on position along x-axis. The composite has $E_0 = 1\, MPa$, $t/D=0.02$, $t=0.5$, and:

a) $E_i/E_0=400$, at $\bar{\varepsilon} = -0.07 = 4.6 \, \bar{\varepsilon}_{cr}$;  
b) $E_i/E_0=300$, at $\bar{\varepsilon} = -0.07 = 3.8 \, \bar{\varepsilon}_{cr}$;  
c) $E_i/E_0=200$, at $\bar{\varepsilon} = -0.07 = 2.9 \, \bar{\varepsilon}_{cr}$;  
d) $E_i/E_0=100$, at $\bar{\varepsilon} = -0.07 = 1.8 \, \bar{\varepsilon}_{cr}$
Figure 4.5 shows the strain contour in the interfacial layer, $\varepsilon_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon})$, calculated through our simplified analytical model (Equation 4.13) and FE results, for the composite with $t/D=0.02$ and $E_i/E_0=200$, at $\bar{\varepsilon}=-0.07$.

![strain contour](image)

**Figure 4.5:** Analytical and FE strain contour for a composite with $E_i/E_0=200$, $E_0 = 1$ MPa, $t/D=0.02$, $t=0.5$. We can see the wrinkled interfacial layer and the local axial strains created in the interfacial layer due to the applied compressive strain, $\bar{\varepsilon} = -0.07 = 3\varepsilon_{cr}$.

More specifically, Figure 4.6a and b shows the good agreement between the FE and analytical results for the strain at the center line of the interfacial layer ($h=0$) and the linear dependence of the strain in the interfacial layer through the thickness of the layer. Figure 4.6a shows the comparison between the analytical (solid line) and FE (dashed line) results for the local strain in the interfacial layer at $h/t=0$ (the center line) and at $h/t=0.5$ for different positions along the interfacial layer, $x$. Figure 4.6b shows the agreement between the analytical (solid line) and FE (symbols) results for the local strain across the interfacial layer $-0.5 \leq h/t \leq 0.5$, at six different locations along a wrinkling wavelength, $x \in [p_1 - p_6]$. Finally, the minimum and maximum strain and stress from the FE simulations and our analytical results are compared in Figure 4.6c for a wide range of different stiffness ratios; the deviation between the results from the two methods are less than 10%.
Figure 4.6: Results for a composite with $E_0 = 1\text{MPa}$, $t/D = 0.02$, and $t = 0.5$, at $\bar{\varepsilon} = -0.07 = 3\varepsilon_{cr}$.

a) When $E_1/E_0 = 200$: The local strain in the interfacial layer at $h/t = 0$ (the center line) and at $h/t = 0.5$ for different positions along the interfacial layer, $x$. At $h/t = 0$ the local strain is the predicted $\varepsilon_{cr} = -0.024$, but our simplified linearized model does not capture the small oscillations seen from the FE results. This is explained in Figure 4.7.

b) The local strain across the interfacial layer $-0.5 \leq h/t \leq 0.5$, at six different locations along a wrinkling wavelength, $x \in [p_1 - p_6]$. The linear dependence of local strain on the distance away from the center line of the interfacial layer, $h$, is verified, as well as the strain at the center line ($h/t = 0$) being constant.

c) Our analytical results for the max and min local strains and stresses in the wrinkled interfacial layer are compared with FE results for a wide range of stiffness ratios $E_1/E_0 (\bar{\varepsilon} = -0.07)$. 

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Looking more carefully at Figure 4.6a, it shows the local strain, $\varepsilon_x^{\text{int.layer}}(x, h, \bar{\varepsilon})$, along the interfacial layer, $x$, at the center line ($h=0$), and at the largest distance away from the neutral axis ($h/t=0.5$), for both FE simulation results and analytical model results (Equation 4.13). The comparison between the analytical model and FE results shows a great agreement for when $h/t = 0.5$. However, the comparison at $h/t = 0$ shows that there is a small oscillation in the results from the FE along the x-axis, that is not captured by our analytical model. The oscillation in the FE results for the $\varepsilon_x^{\text{int.layer}}(x, h = 0, \bar{\varepsilon} = -0.07)$ is due to the fact that at this post-buckling strain $\bar{\varepsilon} \approx 3\varepsilon_{cr}$, the interfacial layer has wrinkled sufficiently so that the neutral axis (N.A.) is no longer always at the same place and no longer at the center of the interfacial layer. The wrinkled interfacial layer can now be regarded as a curved beam, and beam theory gives the expression for the deviation in the position of the neutral axis from the centerline, the eccentricity $\delta_{\text{N.A.}}$, for a curved beam with rectangular cross section:

$$
\delta_{\text{N.A.}} = r_{\text{center}} - \frac{A}{\int_A (rA) \, dr} = r_{\text{center}} - \frac{r_o-r_i}{\ln\left(\frac{r_o}{r_i}\right)} = \left(r_i + \frac{t}{2}\right) - \frac{t}{\ln\left(\frac{r_o}{r_i}\right)}
$$

(4.15)

where $A$ is the area of the rectangular cross section, $t$ is the thickness of the beam, $r_{\text{center}}$ is the distance to the centroid of the beam, and $r_o$ and $r_i$ is the distance to the outer and inner edge of the beam, respectively (Figure 4.7a shows the variables schematically). Studying the strain contours in the interfacial layer obtained from the FE simulations (Figure 4.7b), we can in fact see that the neutral axis, where $\varepsilon_x^{\text{int.layer}} \approx \varepsilon_{cr}$, moves slightly up or down relative to the center line ($h=0$) based on whether it is a wrinkling peak or dip at that specific x-location. Hence, when evaluating the local strains at the center line, $h=0$, as done in Figure 4.6a, the local strain will slightly oscillate around $\varepsilon_{cr}$ due to the minor change in the position of the N.A. compared with the center line. Nonetheless, the local strains increase as the distance from the center line increases ($h/t \to 0.5$), and hence the small deviation becomes insignificant further away from the center of the interfacial layer. Moreover, we often focus on the absolute maximum local strains, which occur at the largest distance away from the center line of the interfacial layer ($h/t = \pm 0.5$). Therefore, in this thesis, we accept that our simplified analytical model does not cover the fact that the neutral axis moves slightly, and focus on the average strain at the center of the interfacial layer instead.
Figure 4.7: a) Schematic of a curved beam with a rectangular cross section, showing how the neutral axis, N.A., will move compared to the centerline of the cross section. b) FE results showing the local strain distribution in just the interfacial layer section of the composite. Zooming in on three locations along the interfacial layer, we can see that the neutral axis, N.A., moves slightly; it is either below, coinciding, or above the centerline, $h=0$, of the interfacial layer based on whether it is a wrinkling peak, inflection point or dip.
4.4 Local deformations in the matrix layers during post-buckling

4.4.1 Analytical model predicting matrix layers deformations

The local strain fields in the matrix layers in the post-buckling region are found by evaluating the effect of the macroscopic compression load together with the effects due to the interfacial layer wrinkling (Figure 4.8). The wrinkling of the interfacial layer mainly influences the deformation of the matrix close to the interfacial layer, $|y| \leq \lambda(\bar{e})$, and as the distance away from the interfacial layer increases, $|y| \geq \lambda(\bar{e})$, this influence diminishes and the deformation is essentially that of the macroscopic compression. We use a small strain assumption, constructing a linear model and use superposition to evaluate the local strains in the matrix to be the sum of the local strains from the macroscopic compression, $\bar{e}$, together with the local strains in the matrix due to the wrinkling of the interfacial layer. In this section we will evaluate the plane strain case, and first develop the local stress and strains created in the matrix due to the macroscopic compression and wrinkling deformations separately, and then add them up to get the overall analytical model for the local stresses and strain the matrix layers.

Deformation 1 - Compression:

The macroscopically applied strain, $\bar{e}$, will provide a uniform compression of the matrix layers (Figure 4.8). The local strains from this applied compression strain, $\bar{e}$, will be:

$$
\varepsilon_{x}^{\text{comp}}(x, y, \bar{e}) = \bar{e} \quad , \quad \varepsilon_{y}^{\text{comp}}(x, y, \bar{e}) = -\bar{\nu}_{0}\bar{e} \quad , \quad \varepsilon_{xy}^{\text{comp}}(x, y, \bar{e}) = 0 \quad (4.16)
$$

Consequently, for a linear elastic material under plane strain conditions, the local planar stresses in the matrix due to the uniform compression, is given by:

$$
\sigma_{x}^{\text{comp}}(x, y, \bar{e}) = \bar{E}_{0}\bar{e} \quad , \quad \sigma_{y}^{\text{comp}}(x, y, \bar{e}) = 0 \quad , \quad \sigma_{xy}^{\text{comp}}(x, y, \bar{e}) = 0 \quad (4.17)
$$

where $\bar{E}_{0}$ and $\bar{\nu}_{0}$ are the plane strain Young's modulus and Poisson ratio (Equation 4.7).
Deformation 2 - Wrinkling:

The wrinkling instability of the interfacial layer only causes large deformations in the matrix that is close to the interfacial layer (matrix within the area of $|y| \leq \lambda(\bar{\varepsilon})$). Following the method suggested previously [33,90], an Airy’s stress function can be defined for the matrix layers that satisfies the bi-harmonic equation of plane elasticity, is periodic in x-direction, and satisfies the relevant boundary conditions. This Airy’s stress function was developed in Chapter 3.2 for the matrix layer in our wrinkling composites, and given by:

$$\Phi(x, \pm y, \bar{\varepsilon}) = \frac{w_{\text{max}} \lambda E_0}{\pi^2(3-v_0)(1+v_0)} \cdot \left[1 + \left(\frac{\pi}{\lambda(\bar{\varepsilon})} \cdot |y|\right)^2\right] \cdot \sin\left(\frac{2\pi x}{\lambda(\bar{\varepsilon})}\right) \cdot e\left(-\frac{2\pi |y|}{\lambda(\bar{\varepsilon})}\right)$$  \hspace{1cm} (4.18)

where $w_{\text{max}}(\bar{\varepsilon})$ and $\lambda(\bar{\varepsilon})$ are the post-buckling wrinkling amplitude and wavelength (Equation 4.3), $E_0$ and $v_0$ are the Young’s modulus and Poisson ratio of the matrix material, $x$...
is the location along the matrix, and \( y \) is the location across the matrix away from the interfacial layer (\(|y|>0\)). As required, this Airy stress function is periodic in \( x \), while also being diminishing with \(|y|\).

With the Airy's stress function in the matrix layers now defined (Equation 4.18), we can take derivatives to find expressions for the local stresses at any point in the matrix due to the wrinkling of the interfacial layer, at any specific applied strain, \(|\bar{\varepsilon}| \geq |\bar{\varepsilon}_{cr}|\). For a plane strain scenario where \( E_0 = \frac{E_0}{1-\nu_0} \) and \( \bar{v}_0 = \frac{v_0}{1-v_0} \), the local stresses in the matrix are given by:

\[
\sigma_{x,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = \frac{\partial^2 \Phi}{\partial y^2} = \frac{\mp 4\pi w_{max} E_0}{\lambda(3-\bar{v}_0)(1+\bar{v}_0)} \left[ \bar{\varepsilon}_0 - \frac{\pi(1+\bar{v}_0)}{\lambda} \cdot |y| \right] \sin \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]

\[
\sigma_{y,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = \frac{\partial^2 \Phi}{\partial x^2} = \frac{\mp 4\pi w_{max} E_0}{\lambda(3-\bar{v}_0)(1+\bar{v}_0)} \left[ 1 + \frac{\pi(1+\bar{v}_0)}{\lambda} \cdot |y| \right] \sin \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]

\[
\sigma_{xy,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = -\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\mp 2\pi w_{max} E_0}{\lambda(3-\bar{v}_0)(1+\bar{v}_0)} \left[ 1 - \bar{\varepsilon}_0 + \frac{2\pi(1+\bar{v}_0)}{\lambda} \cdot |y| \right] \cos \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]

Using the constitutive relations for a linear elastic material under plane strain conditions, we get the following expression for the local strain at any point in the matrix due to the wrinkling of the interfacial layer, at any specific global strain, \(|\bar{\varepsilon}| \geq |\bar{\varepsilon}_{cr}|\):

\[
\varepsilon_{x,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = \pm \frac{4\pi^2 w_{max}(1+\bar{v}_0)}{\lambda^2(3-\bar{v}_0)} \cdot |y| \cdot \sin \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]

\[
\varepsilon_{y,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = \mp \frac{4\pi^2 w_{max}}{\lambda^2(3-\bar{v}_0)} \left[ \frac{\lambda(1-\bar{v}_0)}{\pi} + (1+\bar{v}_0) \cdot |y| \right] \sin \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]

\[
\varepsilon_{xy,\text{winkle}}(x, \pm y, \bar{\varepsilon}) = \mp \frac{4\pi w_{max}}{\lambda(3-\bar{v}_0)} \left[ 1 - \bar{\varepsilon}_0 + \frac{2\pi(1+\bar{v}_0)}{\lambda} \cdot |y| \right] \cos \left( \frac{2\pi x}{\lambda} \right) e^{-\frac{2\pi|y|}{\lambda}}
\]
Total matrix deformation - compression and wrinkling:

The total post-buckling deformation of the matrix will be a superposition of the local deformations from the macroscopic compression and the local deformations due to the wrinkling interfacial layer (adding Equations 4.16+4.20 and Equations 4.17+4.19), as shown in Figure 4.8. Therefore, the local total strains developed in the matrix due to a macroscopically applied strain in the post-buckling region, $\bar{\varepsilon}$, are given by:

$$
\varepsilon_{x\text{matrix}}(x, \pm y, \bar{\varepsilon}) = \bar{\varepsilon} \mp \frac{4\pi^2 w_{\text{max}}(1 + \bar{\nu}_0)}{\lambda^2(3 - \bar{\nu}_0)} \cdot |y| \cdot \sin\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

(4.21)

$$
\varepsilon_{y\text{matrix}}(x, \pm y, \bar{\varepsilon}) = -\bar{\nu}_0 \bar{\varepsilon} \mp \frac{4\pi^2 w_{\text{max}}}{\lambda^2(3 - \bar{\nu}_0)} \left[\frac{\lambda(1 - \bar{\nu}_0)}{\pi} + (1 + \bar{\nu}_0) \cdot |y|\right] \sin\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

$$
\varepsilon_{xy\text{matrix}}(x, \pm y, \bar{\varepsilon}) = \mp \frac{4\pi w_{\text{max}}}{\lambda(3 - \bar{\nu}_0)} \left[1 - \bar{\nu}_0 + \frac{2\pi(1 + \bar{\nu}_0)}{\lambda} \cdot |y|\right] \cos\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

Similarly, the local stresses at any point in the matrix in the post-buckling region, are given by:

$$
\sigma_{x\text{matrix}}(x, \pm y, \bar{\varepsilon}) = \bar{E}_0 \bar{\varepsilon} \mp \frac{4\pi w_{\text{max}} \bar{E}_0}{\lambda(3 - \bar{\nu}_0)(1 + \bar{\nu}_0)} \left[\bar{\nu}_0 - \frac{\pi(1 + \bar{\nu}_0)}{\lambda} \cdot |y|\right] \sin\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

(4.22)

$$
\sigma_{y\text{matrix}}(x, \pm y, \bar{\varepsilon}) = \mp \frac{4\pi w_{\text{max}} \bar{E}_0}{\lambda(3 - \bar{\nu}_0)(1 + \bar{\nu}_0)} \left[1 + \frac{\pi(1 + \bar{\nu}_0)}{\lambda} \cdot |y|\right] \sin\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

$$
\sigma_{xy\text{matrix}}(x, \pm y, \bar{\varepsilon}) = \mp \frac{2\pi w_{\text{max}} \bar{E}_0}{\lambda(3 - \bar{\nu}_0)(1 + \bar{\nu}_0)} \left[1 - \bar{\nu}_0 + \frac{2\pi(1 + \bar{\nu}_0)}{\lambda} \cdot |y|\right] \cos\left(\frac{2\pi x}{\lambda}\right) e\left(-\frac{2\pi |y|}{\lambda}\right)
$$

where $x$ is the location along the matrix layer $x \in [0, L]$, $y$ is the distance across the matrix layer such that $\pm y \in [0, \pm d]$, $\bar{\varepsilon}$ is the macroscopic strain applied in the post-buckling regime, and $w_{\text{max}}(\bar{\varepsilon})$ and $\lambda(\bar{\varepsilon})$ are the post-buckling wrinkling amplitude and wavelength.
4.4.2 Analytical and FE results for interfacial layer deformations

Figure 4.9 shows the strain and stress contours in the matrix layers (only shown for one of the two matrix layers) calculated from our analytical models (Equation 4.21 and 4.22) together with results from FE simulations for a macroscopic applied load of $\bar{\varepsilon} = -0.07$.

![Image showing strain and stress contours](image)

**Figure 4.9:** The a) strain- and b) stress- contours in the matrix, for a composite with $E_r/E_0=200$, $E_0 = 1MPa$, $t/D=0.02$, $t=0.5$, at compressive strain $\bar{\varepsilon} = -0.07 = 3\varepsilon_{cr}$. 
More specifically, the results from comparing the strain distribution along the matrix (x-direction), are presented in Figure 4.10a and exhibit great agreement. Similarly, evaluating the strain distributions across the matrix (y-direction), are shown in Figure 4.10b and reveal similar trends between analytical and FE results. Figure 4.10b also demonstrates that the strains remain constant for $y \geq \lambda(\bar{e})$, which verifies the assumption made in our analytical prediction that the wrinkling deformation only effects the areas within a wavelength-depth from the interfacial layer.

However, Figure 4.10b at $x/\lambda = 0.676$ shows some deviation between our analytical model predictions and the FE results, in the area closest to the interfacial layer. The analytical $\varepsilon_x^{\text{matrix}}$ and $\varepsilon_y^{\text{matrix}}$ strains converge to the FE results when $y/\lambda(\bar{e}) > 0.2$, while the analytical shear strain, $\varepsilon_{xy}^{\text{matrix}}$, approaches the FE results when $y/\lambda(\bar{e}) > 0.6$. This is due to the constant term in our analytical expression (Equation 4.21c) which does not allow for $\varepsilon_{xy}^{\text{matrix}} \sim 0$ as $y \sim 0$. However, this deviation in our analytical predictions is only an issue in the region of $y/\lambda(\bar{e}) < 0.6$ and only at the exact peak of each wrinkling amplitude, where perfect symmetry leads to no shear stresses.

We conclude that our simplified linear model for the local strains in the matrix in the post-buckling region are valid and predict the behavior of the matrix well, and our analytical models are more complete and accurate as the distance from the interfacial layer is increased.
Figure 4.10: FE results (dashed line) and analytical results (solid line) of the strain distribution in the matrix, $\varepsilon_{ij}^{\text{matrix}}$, for a composite with $E_t/E_0=200$, $E_0=1\text{MPa}$, $t/D=0.02$, $\lambda=0.5$, at global applied strain $\varepsilon = -0.07 = 3\varepsilon_{cr}$. 

a) Comparison of strains along the matrix (x-direction) at 3 different depths across the matrix, y-locations. 

b) Comparison of strains across the matrix (y-direction) at 3 different positions along the matrix, x-locations.
Chapter 5

Effective behavior of elastic composites undergoing wrinkling instability

5.1 Introduction

Many engineering applications require knowledge about the composite material’s effective stress-strain behavior, the effective composite stiffness, and the composite’s energy storage abilities. Significant research has previously been conducted within this area, but in this chapter we will present a new method of deliberately using the wrinkling mechanism presented in Chapter 3 and 4 of this thesis, to achieve a composite with enhanced energy storage properties, effective stress mitigation effects, and a switchable effective stiffness.

The energy storage capabilities of a homogeneous elastic material under compression can easily be found using its constitutive model and elasticity theory. By introducing microstructural features and designs into a homogeneous material, the effective properties and the energy absorption of the materials has been previously proven to be greatly altered and enhanced [41,43]. For example, light-weight structures such as cellular materials, with honeycomb or foam microstructures, provide dramatically altered effective behavior and properties compared with their constituent homogeneous material [43,44,45,46,47,48]. Moreover, cellular structures can be designed to enable the storage and/or dissipation of energy under lower loads and at lighter weight due to the local deformation mechanisms occurring in the microstructure (i.e. the cell walls) and the propagation of deformation and/or failure in the cellular structure [43,44,46,48]. Previous studies have also shown that the energy absorption properties of cellular structures can be enhanced by filling the voids of the cellular structure with gas, fluid, or another soft material [41,52,53,54] [55,56,57,58,59]. The gas,
fluid or soft material will oppose the deformation of the cell walls, functioning as a form of reinforcement to the cellular structure, and the created composite materials demonstrate new and interesting effective properties. However, when the results are presented as energy absorption density, per unit mass, the increase is not always as significant.

Herein, we present a new concept of intentionally inducing a wrinkling instability in the interfacial layers within a composite microstructure to create a reversible energy-storage mechanism. This tunable composite material exhibits a change in the energy storage mechanism upon reaching a critical condition, which gives an increase in energy storage, while also providing a mitigation of the load transfer and a decrease in the effective macroscopic stiffness. The switch in the composite’s effective properties and behavior occur at a critical threshold of compressive loading which can be tailored by the geometric features of the structure and the relative contrast in material properties, as studied in Chapter 3. The analytical models developed to predict the deformation mechanisms in Chapter 4, are used to find: i) the energy storage of the composite and its breakdown into contributions from the different constituent materials as a function of strain, ii) the effective macroscopic stress-strain behavior of the highly nonlinear elastic composites, and iii) the evolution in effective macroscopic stiffness of these composite materials. Finite element simulations (FE) and mechanical experiments are performed to validate the analytical model presented in this chapter and further explore the mechanisms.

Figure 5.1 shows the effective true stress-strain curve for: 1) a set of straight stiff layers, 2) a homogeneous soft matrix, and 3) a composite consisting of embedding the stiff straight layers in the soft matrix, computed through finite element (FE) analysis. It is evident that when the straight interfacial layers are embedded into a matrix, the effective composite’s behavior differs dramatically from the sum of the simple matrix and the empty layers, and a nonlinear elastic behavior is observed. Indeed, a bilinear elastic behavior is found. The nonlinear behavior is due to the wrinkling instability in the straight stiff interfacial layers (Figure 5.1b). The presence of the matrix surrounding the straight layers delays the instability point in the stiff layers, and results in a higher mode of instability giving the wrinkling pattern. Prior to wrinkling, the composite’s modulus is enhanced by the stiff layers, and after wrinkling the
tangent modulus decreases to be more that of the matrix. The total stored strain energy density for any material during loading is given by: \( U_{Total} = \frac{\sigma_{Total}}{\varepsilon_{Total}} = \int \sigma \, d\varepsilon \), where \( \sigma \) and \( \varepsilon \) are the macroscopic effective stress and strain, respectively. At a macroscopic applied strain of \( \varepsilon = -0.07 \), the strain energy density of this layered composite is a factor of 2 greater than if we had considered a simple additive contribution of the matrix and the interfacial layer, i.e. the wrinkling behavior drastically enhances the energy storage.

**Figure 5.1:** a) The true stress-strain curves for three periodic structures; structure-1: straight layers (Young’s modulus of \( E_1 = 100\text{MPa} \), thickness \( t = 0.5\text{mm} \), length \( l = 60\text{mm} \), separation \( D = 24\text{mm} \)), structure-2: simple matrix (Young’s modulus of \( E_0 = 1\text{MPa} \)), and structure-3: composite composed of stiff straight layers embedded in soft matrix (stiffness ratio \( E_1/E_0 = 100 \)). At a compression strain of \( \varepsilon = -0.07 \), the strain energy density for the composite is 1.9 times greater than the sum of the straight layers and the simple matrix when in isolation. b) Schematic of a representative volume element (RVE) of a multilayered periodic composite structure consisting of thin stiff interfacial layers separated by distance \( D \), supported by soft matrix on both sides. When a macroscopic compressive strain greater than the critical strain is applied, \( |\varepsilon| > |\varepsilon_{cr}| \), the interfacial layers undergo instability and make wrinkling patterns with wavelength \( \lambda(\varepsilon) \). The geometric parameters \( (t, L, d, D) \) and material parameters \( (E_1, E_0, v_1, v_0) \) are shown in the schematic.
5.2 Analytical models for composite energy storage, non-linear stress-strain behavior, and effective tangent stiffness

In this section we develop analytical models for multifunctional composites with tunable energy storage, nonlinear stress-strain behavior, stress mitigation behavior, and also a switchable effective stiffness.

5.2.1 Analytical models for composite strain energy density

As the layered composite is compressed through the pre-buckling, buckling (initiation of wrinkling instability), and post-buckling region, it is storing energy. The total energy stored, \( u_{\text{Total}} \), in the composite material at any macroscopic applied loading, \( \vec{\varepsilon} \), is the sum of the energy stored in the interfacial layer and the supporting matrix layers:

\[
\begin{align*}
U_{\text{Total}}(\vec{\varepsilon}) &= U_{\text{int.layer}}(\vec{\varepsilon}) + U_{\text{matrix}}(\vec{\varepsilon}) \\
\text{with } \quad u_k(\vec{\varepsilon}) &= \int \frac{1}{2} \bar{\sigma}^k : \bar{\varepsilon}^k \, dV^k
\end{align*}
\]  

(5.1)

where \( \bar{\sigma}^k \) and \( \bar{\varepsilon}^k \) are the two-dimensional stress and strain tensor in a volume element \( dV^k \), for phase \( k \) of the composites, \( k \in \{\text{int.layer}, \text{matrix}\} \), where we are taking conditions of plane strain. Using the local stress and strain expressions developed for the interfacial layer and the matrix, in Chapter 4, we can find the analytical predictions of the total energy stored in the composite material when compressed into the post-buckling region. More interestingly, the strain energy density function, \( U_k \) for each phase \( k \) and for the total composite is found by calculating the total strain energy per unit volume of initial geometry: 

\[
U_k = \frac{u_k(\vec{\varepsilon})}{V_{\text{initial}}}.
\]
**Strain energy density in composite in pre-buckling region:**

We use the expressions developed for the local stresses and strains in the interfacial and matrix layers (Equations 4.5 and 4.6) to find the strain energy of a composite with interfacial layer thickness $t$, matrix layer thickness $2d$, and length $L$. The stiffness of the interfacial layer and matrix are $\bar{E}_1 = \frac{E_1}{1-\nu_1^2}$ and $\bar{E}_0 = \frac{E_0}{1-\nu_0^2}$, respectively.

The total strain energy storage, $u_{\text{Total}}(\varepsilon)$, for strains that are below the critical buckling strain, $|\varepsilon| < |\varepsilon_{\text{cr}}|$, are given by:

\[
u_{\text{int.layer}}(\varepsilon) = \int \frac{1}{2} \left[ \sigma_x \varepsilon_x \right]_{\text{int.layer}}^{\text{int.layer}} dv_{\text{int.layer}} = \int_0^L \int_{-d/2}^{d/2} \frac{1}{2} \left[ \sigma_x \varepsilon_x \right]_{\text{int.layer}}^{\text{int.layer}} dh \, dx
\]

\[= \frac{1}{2} \bar{E}_1 \varepsilon^2 \, tL \quad (5.2)\]

\[
u_{\text{matrix}}(\varepsilon) = \int \frac{1}{2} \left[ \sigma_x \varepsilon_x \right]_{\text{matrix}}^{\text{matrix}} dv_{\text{matrix}} = \int_0^L \int_{-d/2}^{d/2} \frac{1}{2} \left[ \sigma_x \varepsilon_x \right]_{\text{matrix}}^{\text{matrix}} dy \, dx = \bar{E}_0 \varepsilon^2 \, dL
\]

\[\Rightarrow u_{\text{Total}}(\varepsilon) = \nu_{\text{int.layer}}(\varepsilon) + \nu_{\text{matrix}}(\varepsilon) = \frac{1}{2} \bar{E}_1 \varepsilon^2 \, tL + \bar{E}_0 \varepsilon^2 \, dL
\]

The strain energy density in the interfacial layer, $U_{\text{int.layer}}$, and matrix layers, $U_{\text{matrix}}$, and the full composite, $U_{\text{Total}}$, is then given by:

\[U_{\text{int.layer}}(\varepsilon) = \frac{\nu_{\text{int.layer}}(\varepsilon)}{V_{\text{int.layer}}} = \frac{1}{2} \bar{E}_1 \varepsilon^2 \, t \frac{L}{t} = \frac{1}{2} \bar{E}_1 \varepsilon^2
\]

\[U_{\text{matrix}}(\varepsilon) = \frac{\nu_{\text{matrix}}(\varepsilon)}{V_{\text{matrix}}} = \frac{\bar{E}_0 \varepsilon^2 \, dL}{2d \, L} = \frac{1}{2} \bar{E}_0 \varepsilon^2 \quad (5.3)
\]

\[U_{\text{Total}}(\varepsilon) = \frac{u_{\text{Total}}(\varepsilon)}{V_{\text{Total}}} = f_m U_{\text{matrix}}(\varepsilon) + f_{\text{int.l}} U_{\text{int.layer}}(\varepsilon) = \frac{(\bar{E}_1 \, t + \bar{E}_0 \, 2d) \, \varepsilon^2}{2(t+2d)}
\]

where $f_m = 1 - f_{\text{int.l}} = 2d/(2d + t)$ and $f_{\text{int.l}} = t/(2d + t)$ are the volume fraction of the matrix and the interfacial layer in the composite, respectively.
Strain energy density in composite in post-buckling region:

We use the expressions developed for the local stresses and strains in the interfacial and matrix layers (Equations 4.13 and 4.14, and Equations 4.21 and 4.22) to find the strain energy of a composite with interfacial layer thickness $t$, matrix layer thickness $2d$, and length $L = \lambda_{cr}$. The plane strain stiffness of the interfacial layer and matrix are $E_1 = \frac{E_1}{1 - \nu_1^2}$ and $E_0 = \frac{E_0}{1 - \nu_0^2}$.

The total strain energy storage, $u_{\text{Total}}(\varepsilon)$, for strains that are above the critical buckling strain, $|\varepsilon| > |\varepsilon_{cr}|$, are given by:

$$u_{\text{Total}}(\varepsilon) = u_{\text{int.layer}}(\varepsilon) + u_{\text{matrix}}(\varepsilon)$$

(5.4)

where:

$$u_{\text{int.layer}}(\varepsilon) = \int \frac{1}{2} \left[ \sigma_x \varepsilon_x \right]_{\text{int.layer}} dV_{\text{int.layer}}$$

$$= \frac{1}{2} \overline{E}_1 \varepsilon_{cr}^2 t \lambda_{cr} + \frac{\overline{E}_0 \pi^2 (|\varepsilon| - |\varepsilon_{cr}|) t^3 \lambda_{cr}}{12 \lambda} - \frac{\overline{E}_1 \pi (|\varepsilon| - |\varepsilon_{cr}|) t^3}{12 \lambda} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right)$$

$$u_{\text{matrix}}(\varepsilon) = \int \frac{1}{2} \left[ \sigma_{ij} \varepsilon_{ij} \right]_{\text{matrix}} dV_{\text{matrix}}$$

$$= \overline{E}_0 \varepsilon_2 \lambda_{cr} d + \frac{\overline{E}_0 \varepsilon \lambda \wmax (1 + \bar{\nu}_0)}{\pi (3 - \bar{\nu}_0)} \cdot \left[ \cos \left( \frac{2\pi \lambda_{cr}}{\lambda} \right) - 1 \right]$$

$$+ \frac{\overline{E}_0 \pi \wmax (5 - 2 \bar{\nu}_0 + \bar{\nu}_0^2)}{2 \lambda (3 - \bar{\nu}_0)^2 (1 + \bar{\nu}_0)} \left[ \lambda_{cr} + \frac{\lambda}{4\pi} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right) \right]$$

$$+ \frac{\overline{E}_0 \pi \wmax (1 - \bar{\nu}_0)}{\lambda^2 (3 - \bar{\nu}_0)^2} \left[ \lambda_0 + \frac{2\lambda}{(1 + \bar{\nu}_0)} + \frac{\lambda (1 + \bar{\nu}_0)}{2 (1 - \bar{\nu}_0)} \right] \left[ \lambda_{cr} - \frac{\lambda}{4\pi} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right) \right]$$

$$- \frac{\overline{E}_0 \varepsilon \wmax (\lambda + 2d \pi)(1 + \bar{\nu}_0)}{\pi (3 - \bar{\nu}_0)} \cdot \left[ \cos \left( \frac{2\pi \lambda_{cr}}{\lambda} \right) - 1 \right] e^{-\frac{2d\pi}{\lambda}}$$

$$- \frac{4\overline{E}_0 \pi^2 \wmax (1 - \bar{\nu}_0)}{\lambda^2 (3 - \bar{\nu}_0)^2} \left[ d + \frac{\lambda}{4\pi} + \frac{\lambda}{2 \pi (1 + \bar{\nu}_0)} + (1 + \bar{\nu}_0) (d + \frac{\lambda}{8\pi} + \frac{d^2 \pi}{\lambda}) \right] \left[ \lambda_{cr} - \frac{\lambda}{4\pi} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right) \right] e^{-\frac{4d\pi}{\lambda}}$$

$$- \frac{\overline{E}_0 \pi \wmax (5 - 2 \bar{\nu}_0 + \bar{\nu}_0^2)}{2 \lambda (3 - \bar{\nu}_0)^2 (1 + \bar{\nu}_0)} \left[ \lambda_{cr} + \frac{\lambda}{4\pi} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right) \right] e^{-\frac{4d\pi}{\lambda}}$$

$$- \frac{2d \overline{E}_0 \pi^2 \wmax}{\lambda^2 (3 - \bar{\nu}_0)^2 (1 + \bar{\nu}_0)} \left[ \lambda_0 (3 - \bar{\nu}_0)(1 + \bar{\nu}_0) + 2 d \pi (1 + \bar{\nu}_0)^2 \right] \left[ \lambda_{cr} + \frac{\lambda}{4\pi} \sin \left( \frac{4\pi \lambda_{cr}}{\lambda} \right) \right] e^{-\frac{4d\pi}{\lambda}}$$

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The strain energy density in the interfacial layer, $U_{\text{interlayer}}$, and matrix layers, $U_{\text{matrix}}$, and the full composite, $U_{\text{Total}}$, is then given by:

$$U_{\text{Total}}(\varepsilon) = f_m U_{\text{matrix}}(\varepsilon) + f_{\text{int.l}} U_{\text{interlayer}}(\varepsilon)$$  \hspace{1cm} (5.5)$$

where $f_m = 1 - f_{\text{int.l}} = 2d/(2d + t)$ and $f_{\text{int.l}} = t/(2d + t)$ are the volume fraction of the matrix and the interfacial layer in the composite, respectively. The strain energy density in the interfacial and matrix layers of the composite are given by:

$$U_{\text{interlayer}}(\varepsilon) = \frac{1}{2} E_1 \varepsilon_{cr}^2 + \frac{E_1 \pi^2 (\varepsilon - |\varepsilon_{cr}|)}{3 \lambda^2} - \frac{E_1 \pi (|\varepsilon| - |\varepsilon_{cr}|)}{12 \lambda \lambda_{cr}} \sin \left(\frac{4\pi \lambda_{cr}}{\lambda}\right)$$  \hspace{1cm} (5.6)$$

$$U_{\text{matrix}}(\varepsilon) = \frac{1}{2} E_0 \varepsilon^2 + \frac{E_0 \lambda \bar{w}_{\text{max}}(1+\bar{v}_0)}{2d \pi (3-\bar{v}_0)} \left[\cos \left(\frac{2\pi \lambda_{cr}}{\lambda}\right) - 1\right]$$

$$+ \frac{E_0 \lambda \bar{w}_{\text{max}}(5-2\bar{v}_0 + \bar{v}_0)}{4d \pi (3-\bar{v}_0)^2} \left[1 + \frac{2}{(1+\bar{v}_0)} + \frac{(1+\bar{v}_0)}{2(1-\bar{v}_0)} \left[1 - \frac{\lambda}{4\pi \lambda_{cr}} \sin \left(\frac{4\pi \lambda_{cr}}{\lambda}\right)\right]\right]$$

$$- \frac{E_0 \bar{w}_{\text{max}}(\lambda + 2d \pi)(1+\bar{v}_0)}{2d \pi (3-\bar{v}_0)} \left[1 + \frac{2}{(1+\bar{v}_0)} + \frac{(1+\bar{v}_0)}{2(1-\bar{v}_0)} \left[1 - \frac{\lambda}{4\pi \lambda_{cr}} \sin \left(\frac{4\pi \lambda_{cr}}{\lambda}\right)\right]\right] e^{-\frac{4d \pi}{\lambda}}$$

$$+ \frac{E_0 \pi \bar{w}_{\text{max}}(5-2\bar{v}_0 + \bar{v}_0)}{4d \pi (3-\bar{v}_0)^2(1+\bar{v}_0)} \left[1 + \frac{\lambda}{4\pi \lambda_{cr}} \sin \left(\frac{4\pi \lambda_{cr}}{\lambda}\right)\right] e^{\frac{4d \pi}{\lambda}}$$

$$- \frac{E_0 \pi \bar{w}_{\text{max}}}{(3-\bar{v}_0)(1+\bar{v}_0)^2(1+\bar{v}_0)} \left[(3-\bar{v}_0)(1+\bar{v}_0) + \frac{2d \pi}{\lambda} (1+\bar{v}_0)^2 \right] \left[1 + \frac{\lambda}{4\pi \lambda_{cr}} \sin \left(\frac{4\pi \lambda_{cr}}{\lambda}\right)\right] e^{-\frac{4d \pi}{\lambda}}$$

where the wavelength, $\lambda = \lambda(\varepsilon)$, wrinkling amplitude, $w_{\text{max}} = w_{\text{max}}(\varepsilon)$, and the critical strain, $\varepsilon_{cr}$, are defined by Equation 4.3 and 4.4. The plane strain stiffness and Poisson ratio of the interfacial layer and matrix material are given by:

$$E_1 = \frac{E_1}{1-\nu_1^2} \hspace{1cm} E_0 = \frac{E_0}{1-\nu_0^2} \hspace{1cm} \bar{v}_1 = \frac{\nu_1}{1-\nu_1} \hspace{1cm} \bar{v}_0 = \frac{\nu_0}{1-\nu_0}$$

Furthermore, the non-dimensional variables used in the equations are defined as:

$$\tilde{\lambda}_{cr} = \lambda_{cr}/t \hspace{1cm} \tilde{\lambda} = \lambda(\varepsilon)/t \hspace{1cm} \tilde{w}_{\text{max}} = w_{\text{max}}(\varepsilon)/\lambda_{cr} \hspace{1cm} \bar{w}_{\text{max}} = w_{\text{max}}(\varepsilon)/\lambda(\varepsilon)$$
5.2.2 Analytical models for stress-strain behavior, and effective tangent stiffness

Other important composite properties are the macroscopic effective stress and the effective stiffness, which are both dependent on the microstructure and the material combination in the composite.

The macroscopic axial engineering stress-strain behavior of the composite is obtained by differentiating the strain energy density expression:

$$\bar{\sigma}_x^{\text{comp}}(\bar{\varepsilon}) = \frac{d U_{\text{Total}}(\bar{\varepsilon})}{d \bar{\varepsilon}}$$  \hspace{1cm} (5.7)

The true stress-strain relation for the composite can then be found by:

$$\bar{\sigma}_x^{\text{comp}} = \bar{\sigma}_x^{\text{comp}}(1 + |\bar{\varepsilon}|) \text{ and } \bar{\varepsilon} = \ln(1 + |\bar{\varepsilon}|).$$

The effective stiffness of the composite, $E_x^{\text{comp}}(\bar{\varepsilon})$, can be calculated at any applied strain, $\bar{\varepsilon}$, by differentiating the effective engineering stress:

$$E_x^{\text{comp}}(\bar{\varepsilon}) = \frac{d \bar{\sigma}_x^{\text{comp}}(\bar{\varepsilon})}{d \bar{\varepsilon}} = \frac{d^2 U_{\text{Total}}(\bar{\varepsilon})}{d \bar{\varepsilon}^2}$$  \hspace{1cm} (5.8)

**Macroscopic stress and stiffness in composite in pre-buckling region, $|\bar{\varepsilon}| < |\bar{\varepsilon}_{cr}|$:**

Prior to wrinkling, the macroscopic axial stress-strain behavior of the composite, given by Equation 5.7, gives the simple Voigt model:

$$\bar{\sigma}_x^{\text{comp}}(\bar{\varepsilon}) = (f_m \bar{E}_0 + f_{\text{int.l}} \bar{E}_1) \cdot \bar{\varepsilon}$$  \hspace{1cm} (5.9)

where $f_m = 1 - f_{\text{int.l}} = 2d/(2d + t)$ and $f_{\text{int.l}} = t/(2d + t)$ are the volume fraction of the matrix and the interfacial layer in the composite, respectively.

Similarly, the effective stiffness of the composite can be found through the single Voigt model:

$$E_x^{\text{comp}}(\bar{\varepsilon}) = f_m \bar{E}_0 + f_{\text{int.l}} \bar{E}_1$$  \hspace{1cm} (5.10)
Macroscopic stress and stiffness in composite in post-buckling region, $|\bar{\varepsilon}| > |\bar{\varepsilon}_{cr}|$:

When wrinkling has occurred, the macroscopic axial stress-strain behavior of the composite is obtained by using Equation 5.7 together with Equation 5.5 and 5.6. The macroscopic engineering stress in the composite at any macroscopic applied strain, $\bar{\varepsilon}$, is hence given by:

$$\bar{\sigma}_{x}^{\text{comp}}(\bar{\varepsilon}) = f_{\text{int. l}} \cdot \frac{d \tilde{u}_{\text{int layer}}(\bar{\varepsilon})}{d \bar{\varepsilon}} + f_{\text{matrix}} \cdot \frac{d \tilde{u}_{\text{matrix}}(\bar{\varepsilon})}{d \bar{\varepsilon}} \quad (5.11)$$

The full expression for $\bar{\sigma}_{x}^{\text{comp}}(\bar{\varepsilon})$ can be found in Appendix A (Equation A-6).

The macroscopic true stress-strain relation for the composite can be found by:

$$\bar{\sigma}_{x}^{\text{comp}} = \bar{\sigma}_{x}^{\text{comp}}(1 + |\bar{\varepsilon}|) \quad \text{and} \quad \bar{\varepsilon} = \ln(1 + |\bar{\varepsilon}|) \quad (5.12)$$

The effective stiffness of the composite, $E_{x}^{\text{comp}}(\bar{\varepsilon})$, can be calculated at any applied strain, $\bar{\varepsilon}$, by using Equation 5.8 together with Equation 5.5 or 5.11. The composite’s effective stiffness is hence given by:

$$E_{x}^{\text{comp}}(\bar{\varepsilon}) = f_{\text{int. l}} \cdot \frac{d^2 \tilde{u}_{\text{int layer}}(\bar{\varepsilon})}{d \bar{\varepsilon}^2} + f_{\text{matrix}} \cdot \frac{d^2 \tilde{u}_{\text{matrix}}(\bar{\varepsilon})}{d \bar{\varepsilon}^2} \quad (5.13)$$

The full expression for $E_{x}^{\text{comp}}(\bar{\varepsilon})$ can be found in Appendix A (Equation A-8).
5.3 Results for macroscopic behavior of elastic wrinkling composites

To verify the analytical models of the strain energy densities (Section 5.2.1), the effective stress-strain behavior, and the effective stiffness (Section 5.2.2), finite element (FE) simulations were conducted on multilayered composites for different macroscopic strains, $|\varepsilon| \in [0 - 0.07]$.

Figure 5.3 shows these FE and analytical results for composites varying stiffness ratio $E_i/E_o$, while keeping the interfacial layer concentration constant at $f_{int.l} = t/D = 0.021$, to ensure that wrinkling is the mode of instability in all the composites evaluated (as explained in Chapter 3.3). Figure 5.3a shows that the strain energy density in the interfacial layers for all the composites evaluated; the strain energy density is calculated per unit volume of interfacial layer $\frac{u_{int.layer}}{tL}$ (plotted on the left axis and with corresponding solid graph lines), and also calculated per unit volume of the total composite $\frac{u_{int.layer}}{DL} = \frac{u_{int.layer}}{(2d+t)L}$ (plotted on the right axis with corresponding dashed graph lines). The analytical and FE results show a clear bifurcation point where the slope of the graph changes; this change occurs at a macroscopic strain equal to the critical strain at which the wrinkling instability occurs in the composites, $\varepsilon_{cr}$, according to Equation 4.4. In the pre-buckling region, $|\varepsilon| < |\varepsilon_{cr}|$, the strain energy is due to the axial compression of the interfacial layer, and so the strain energy density in the interfacial layer is quadratic with respect to the applied compressive strain, $\varepsilon$, based on elasticity theory. In the post-buckling region, $|\varepsilon| > |\varepsilon_{cr}|$, the slope is reduced since the strain energy in the interfacial layer is now a result of the bending of the wrinkling pattern which now accommodates the applied strain. In contrast, the strain energy density in the matrix layers of the composite grows dramatically in the post-buckling region, $\varepsilon > \varepsilon_{cr}$, as seen in Figure 5.3b. This increase in the matrix strain energy density in the post-buckling region is

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4 FE simulations results of $E_i/E_o=500$ with the current $t/D=48$ indicate that the instability mode is in the beginning of the transition-zone from the wrinkling instability mode into the longwave instability mode. Hence, to ensure pure wrinkling instability, only stiffness ratios up to $E_i/E_o=400$ are presented in this paper for $t/D=48$.  

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largely due to the large local deformations occurring in the matrix layers due to the wrinkling of the interfacial layer. Comparing the strain energy density in the matrix layers of the composites with the strain energy density of a pure matrix being compressed without any wrinkling deformation occurring (dash-dot line in Figure 5.3b), it is clear that the overall matrix strain energy density of the composites in the post-buckling region is governed by the strain energy from the local deformations in the matrix due to the wrinkling. For example, at post-buckling strain of $\varepsilon = -0.07 = 4.6\varepsilon_{cr}$, the strain energy density in the matrix layers of the composite with $E_1/E_0=400$ is approximately 2.5 times greater than for a pure matrix that is undergoing only compression.

Figure 5.3c shows the composite's total strain energy density for composites with varying, $E_1/E_0$. The energy stored in the composite grows more rapidly as the applied strain exceeds the critical strain of the composite, i.e. in the post-buckling region ($|\varepsilon| > |\varepsilon_{cr}|$). This is a direct effect of the strain energy contribution from the matrix layers increasing in the post-buckling region as well. Hence, we see that the overall strain energy density of the composite in the post-buckling region is governed by the strain energy density in the matrix layers. Figure 5.2 shows the contours of the strain energy in the matrix layers and the interfacial layers of the composite at $\varepsilon = -0.07$. Figure 5.2a and b show the composite with $E_1/E_0 = 400$ and $E_1/E_0 = 100$, and we can see that the higher stiffness ratio leads to much higher local strain energies, as a result of the deformations and being further into the post-buckling region. Moreover, it is clear that due to the large areas of the matrix that is undergoing large local deformation, the overall strain energy is mainly stored in the matrix layers. Consequently, by increasing the stiffness of the matrix layers, $E_0$, while keeping the stiffness ratios between the two constructing materials in the material fixed, $E_1/E_0 \in [25 - 400]$, the total energy absorbed by the composite can be immensely amplified. By evaluating the total composite strain energy density in a log-log plot, and using regression we see that the strain energy density in the pre-buckling region is proportional to: $U_{Total} \sim |\varepsilon|^2$, while in the post-buckling region the strain energy density can be estimated by: $U_{Total} \sim |\varepsilon|^{1.4}$ (Figure 5.3d). The reduction in slope is due to the wrinkled interfacial layer contributing less to the energy...
storage than prior to wrinkling. However, at the same time, the wrinkling of the interfacial layer has increased the energy storage in the matrix over that of a homogeneous matrix material. Hence, the composite provides enhanced energy storage with the effective tangent stiffness of its soft matrix (as shown in the effective stress-strain curves next).

![Strain energy in matrix layers](image1)

![Strain energy in interfacial layers](image2)

**Figure 5.2:** FE contours of the strain energy in the matrix layers and in the interfacial layers of the composites at macroscopic applied strain of $|\varepsilon| = 0.07$, for composites with: a) material properties, $E_1=400\text{MPa}$ and $E_0=1\text{MPa}$, and interfacial layer concentration of $t/D=0.02$; b) material properties, $E_1=100\text{MPa}$ and $E_0=1\text{MPa}$, and interfacial layer concentration of $t/D=0.02$; c) material properties, $E_1=400\text{MPa}$ and $E_0=1\text{MPa}$, and interfacial layer concentration of $t/D=0.01$. 

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As the composites are compressed macroscopically through the pre- and into the post-buckling region, the wrinkling acts to mitigate the stress level as evidenced by the decrease in slope. Figure 5.3e shows FE results and analytical predictions (Equation 5.12) of the effective axial true stress of the composite normalized by the stiffness of the matrix layers, $\bar{\sigma}_x^{comp}/E_0$, as a function of the true applied macroscopic strain, $\bar{\varepsilon}$. In the pre-buckling region, the straight interfacial layers contribute significantly to the stiffness of the composite via the Voigt model, and the effective true stress increases rapidly as a consequence of being proportional to the stiffness of the stiff interfacial layers. When the applied strain equals to $\bar{\varepsilon} = \bar{\varepsilon}_{cr}$, a clear bifurcation point is observed in the effective true stress-strain curve due to the wrinkling instability occurring in the interfacial layer of the composite. In the post-buckling region, a post-wrinkling stiffness is observed in the effective stress-strain curves which is due to the soft matrix still providing stiffness to the composite. Hence, the effective stress increases slower with the applied strain in the post-buckling region, which means that these composites can be used to control and mitigate load transfer. Moreover, these composite materials exhibit a bilinear elastic behavior.

Figure 5.3f shows the effective stiffness normalized with respect to the matrix stiffness, $\bar{E}_x^{comp}/E_0$, for composites with different $E_i/E_0$ at $f_{int,t}=t/D=0.021$, as a function of the applied macroscopic strain, $\bar{\varepsilon}$ (showing analytical and FE simulation results). It is apparent that the effective stiffness drops promptly as the applied strain exceed the critical strain, $|\bar{\varepsilon}| > |\bar{\varepsilon}_{cr}|$, and the composite undergoes wrinkling instability, as discussed above. From Figure 5.3f, we can also observe that although the initial normalized effective stiffness is higher for composites with high stiffness ratio in the composite, $E_1/E_0$, the higher stiffness ratios also lead to the composite’s effective stiffness decreasing at a lower applied macroscopic strain due to the critical instability macroscopic strain being inversely proportional to $(E_1/E_0)^{2/3}$.

Interestingly, in the post-buckling region, the normalized effective stiffness is very close for all composites since the composites’ effective stiffness approaches the stiffness of the matrix layers after the interfacial layer wrinkles. The average normalized effective stiffness in the post-buckling region is $\bar{E}_x^{comp}/E_0=1.4$, as well as $\bar{E}_x^{comp}/E_0=1.08$. 

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Figure 5.3: Analytical (line) and FE (symbols) results at different macroscopic strains, $\varepsilon$, for different composites with varying stiffness ratios between the interfacial layer and the matrix, $E_1/E_0 \in [25 - 400]$, with $E_0 = 1 MPa$, with interfacial layer concentration of $f_{int.i} = 0.021$, and with a interfacial layer thickness $t = 0.5$ and composite length, $L = \lambda_{cr}$; a) Strain energy density in the interfacial layer; calculated based on volume of interfacial layer on left axis (with solid graph lines), and calculated based on volume of total composite on right axis (with dashed graph lines). b) Strain energy density in the matrix
layers of the composites. Dashed-dot line shows compression of a pure matrix layer without any wrinkling deformations occurring. e) Strain energy density in the full composite. d) The composite’s strain energy density is proportional to \( \sim |\bar{\varepsilon}|^2 \) as long as \( |\bar{\varepsilon}| < |\varepsilon_{cr}| \), and proportional to \( \sim |\bar{\varepsilon}|^{1.4} \) as long as \( |\bar{\varepsilon}| > |\varepsilon_{cr}| \). e) The effective true stress-strain curves\(^5\) normalized with respect to the stiffness of the matrix, \( \bar{\sigma}_x^{\text{comp}} / E_0 \). f) The effective stiffness normalized with respect to the stiffness of the matrix, \( \bar{E}_x^{\text{comp}} / E_0 \).

\(^5\) The effective composite stresses were found from the FE simulations by using the method proposed by [122]. The reaction forces at the virtual nodes were used to find the engineering effective stress. Then Equation 5.12 was used to transform the engineering stress-strain results into true stress-strain results instead.
The effect of the volume fraction of the interfacial layers was studied keeping $E_i/E_0 = 400$ and varying the interfacial layer concentration $f_{int.l} = t/D \in [0.007, 0.01, 0.014, 0.017]$. Figure 5.4 shows the analytical and FE results for the strain energy densities, the effective true stress-strain behavior, and effective stiffness for these composites. The result show that the energy storage density and effective properties are governed by the concentration of the interfacial layers and the $t/D$ ratio; results are independent on the specific values for $t$ and $D$, as long as $t/D$ is such that wrinkling occurs in the composite. Note that the critical strain at which the interfacial layer will undergo wrinkling instability is independent of geometric features and only depends on the stiffness ratio, $E_i/E_0$ (Equation 4.4a). As a consequence, since the stiffness ratio is kept constant at $E_i/E_0 = 400$ for all composites, Figure 5.4 shows clear bifurcation points in the energy density plots and effective property plots that are all at the same applied strain, $\varepsilon = \varepsilon_{cr} = -0.015$. The higher the concentration of interfacial layers is in the composite, i.e. the higher $t/D$ ratio, the higher the energy density and effective properties of the composite will be in the pre- and post-buckling region. This indicates that by only changing the geometric features of the composite, we can control and tailor the pre- and post-buckling behavior of multilayered composites to get interesting new features.

More specifically, Figure 5.4a demonstrates that the strain energy density in the interfacial layer, calculated based on the volume of the interfacial layer, $\frac{u_{int.layer}}{tL}$ (left axis of the curve), only varies little with the change in the geometry of the composite. However, the strain energy density in the interfacial layer calculated based on the total volume of the composite, $\frac{u_{int.layer}}{DL} = \frac{u_{int.layer}}{(2d+t)L}$ (right axis of the curve), shows a clear deviation between the different composites where the more concentrated composites have a higher strain energy density as expected. Comparing the contribution of the strain energy density of the interfacial layer to the full composite with that of the matrix layers (Figure 5.4a-c), it can be concluded that the composites’ strain energy density has contributions from both the matrix and the interfacial layers. As the concentration of interfacial layers increases, the relative contribution of matrix to the relative contribution of the matrix to the composite’s strain energy also increases. Figure 5.2a and c show the strain energy contours in the matrix and interfacial layers of two
composites where the interfacial layer concentrations for Figure 5.2a is double that of Figure 5.2c ($t/D=0.02$ versus 0.01). It visible that the large deformations and strain energy in the matrix are occurring close to the interfacial layer, and hence the composite with $t/D=0.01$ have large regions in the matrix with low strain energy. Consequently, the matrix strain energy density and total composite strain energy density is lower as well (Figure 5.4b and c). The total strain energy density for all composites are low in the pre-buckling region, but as the macroscopically applied strain exceed the critical strain, $|\varepsilon| > |\varepsilon_{cr}|$, the large strain energy produced by the local deformations in the matrix from the wrinkling pattern, lead to a remarkable increase in the total strain energy density of the composite.

The effective true stress-strain curves for these composites with constant material properties, but changing concentration of interfacial layers, Figure 5.4e, illustrate that, in the pre-buckling region, the slope of the effective true stress curve is highest for composites with the highest volume fraction of interfacial layers, following the simple Voigt model. A clear bifurcation is seen in the effective stress of the composite as the applied strain reaches the critical strain, $\tilde{\varepsilon} = \tilde{\varepsilon}_{cr} = -0.015$, and the interfacial layers wrinkle (Figure 5.4e). A post-wrinkling tangent stiffness is exhibited in the effective true stress-strain curves in the post-buckling region, and the slope of the curve is the same for all composites despite the different concentrations. The tangent stiffness is a result of the matrix layers still providing stiffness in the post-buckling region, and hence similar for all the composites evaluated here since $E_0=$constant. Figure 5.4f illustrates a step-wise behavior in the effective stiffness of the composite; the effective stiffness drops rapidly down to a constant value once the instability occurs at $\tilde{\varepsilon} = \tilde{\varepsilon}_{cr} = -0.015$. These result are promising as they open for a new way of designing materials with controllable and tunable pre-buckling effective stress and stiffness behavior, just by changing the volume fraction of the interfacial layer, $t/D$. Indeed, these composite materials also exhibit a bilinear elastic behavior. Furthermore, it is possible to tune and mitigate the stress level by these composites in both the pre- and post-buckling region.
Figure 5.4: Analytical (lines) and FE (symbols) results at different macroscopic strains, $\varepsilon$, for 8 different composites with varying geometric features such that a total of 4 different interfacial layer concentrations, $f_{\text{int.l}} = t/D \in [0.007 - 0.017]$, with material properties: $E_1 = 400\,\text{MPa}$ and $E_0 = 1\,\text{MPa}$, and composite length, $L = \lambda_{\text{cr}}$; a) Strain energy density in the interfacial layer; calculated based on volume of interfacial layer on left axis (with solid graph lines), and calculated based on volume of total composite on right axis (with
dashed graph lines). b) Strain energy density in the matrix layers of the composites. Dashed-dot line shows compression of a pure matrix layer without any wrinkling deformations occurring. c) Strain energy density in the full composite. d) The composite’s strain energy density is proportional to $\sim|\bar{\varepsilon}|^2$ as long as $|\bar{\varepsilon}| < |\bar{\varepsilon}_{cr}|$, and proportional to $\sim|\bar{\varepsilon}|^{1.4}$ as long as $|\bar{\varepsilon}| > |\bar{\varepsilon}_{cr}|$. e) The effective true stress-strain curves\(^6\) for the different composites normalized with respect to the stiffness of the matrix, $\bar{\sigma}_{x}^{comp} / E_0$. f) The effective stiffness normalized with respect to the stiffness of the matrix, $E_{x}^{comp} / E_0$.

\(^6\) The effective composite stresses were found from the FE simulations by using the method proposed by [122]. The reaction forces at the virtual nodes were used to find the engineering effective stress. Then Equation 5.12 was used to transform the engineering stress-strain results into true stress-strain results instead.
5.4 Experimental validation

Physical experiments were performed on a set of different single- and multilayered composites to verify the analytical and FE results. The physical samples were fabricated using a multimaterial 3D printer, and the compression of these samples were conducted as described in Chapter 2.3. For all samples, the matrix was fabricated using the material TangoPlus (TP) with Young’s modulus of $E_0=0.6\pm0.1$MPa. The thin stiffer interfacial layers were fabricated using either the material VeroWhite (VW) with Young’s modulus of $E_i=600\pm100$MPa, or a Digital Material (DM) with Shore95 and Young’s modulus of $E_i=23\pm1$MPa. In addition to exploring the effects of different materials, the changes in geometrical features were also studied; the concentration of interfacial layers, $t/D$, the thickness of the interfacial layers, $t$, and the thickness of a single unit cell $D$ ($D=2d+t$ where $d$ is the matrix thickness) were all evaluated.

Figure 5.5-5.7 show the engineering stress-strain curves for the different composites being compressed using a Zwick mechanical tester during the physical experiments (solid red line). These figures also include the theoretical engineering stress-strain curves for the different composites calculated using Equation 5.9 and 5.11 derived earlier in this chapter (these analytical results are marked as a grey zone, where the zone includes the uncertainty in the material properties of the materials). In all cases evaluated, there are very good agreement between the analytical predictions and the results from the physical experiments.

Figure 5.5 show the composites’ effective stress-strain curves for two different composites where the stiffness ratio, $E_i/E_0$, is different. The interfacial layer is VeroWhite for the composite in Figure 5.5a, while it is Digital Material for the composite in Figure 5.5b. In Figure 5.6a and b, the materials are kept constant (TangoPlus for the matrix, and Digital Material for the interfacial layer). However, the interfacial layer concentration, $t/D$, is changed ($t/D=0.024$ and $t/D=0.0062$) between the two cases evaluated. Finally, in Figure 5.7a and b, the effective stress-strain curves for two different multilayered composites are presented where the geometry ($t$ and $D$) is varied, but the concentration of interfacial layers are kept constant: $t/D=0.047$. The materials are again kept constant for both composites evaluated.
Figure 5.5: Effective stress-strain curves for compression of two different composites with varying stiffness ratio, $E_i/E_o$. The following variables are fixed: $t/D=0.0062$ ($D=80.5\,\text{mm}$, $t=0.5\,\text{mm}$), $L=76\,\text{mm}$, and matrix is TP with $E_o=0.6\pm0.1\,\text{MPa}$, and a) Interfacial layer is VW with $E_i=600\pm100\,\text{MPa}$, b) Interfacial layer is DM (Shore95) with $E_i=23\pm1\,\text{MPa}$.
Figure 5.6: Effective stress-strain curves for compression of two different composites with varying interfacial layer concentration, $t/D$. The following variables are fixed: interfacial layer is DM with $E_I=23\pm1$MPa, and matrix is TP with $E_0=0.6\pm0.1$MPa. 

a) $t/D=0.024$ ($D=41$mm, $t=1$mm), $L=36$mm. 
b) $t/D=0.0062$ ($D=80.5$mm, $t=0.5$mm), $L=76$mm.
Figure 5.7: Effective stress-strain curves for compression of two different multilayered composites with varying geometry, \( t \) and \( D \). The following variables are fixed: \( t/D=0.047 \), \( L=36\text{mm} \), interfacial layer is DM with \( E_I=23\pm1\text{MPa} \), and matrix is TP with \( E_0=0.6\pm0.1\text{MPa} \); a) \( D=21\text{mm}, t=1\text{mm}, \) and 4 layers, b) \( D=10.5\text{mm}, t=0.5\text{mm}, \) and 5 layers.
5.5 Conclusion

A novel concept was presented in this chapter of intentionally inducing the wrinkling instability in the interfacial layers of a multilayered composite to achieve interesting new composite effective properties and behavior. It was shown that as these tunable composite materials are compressed, upon reaching a critical condition, the stored energy properties are changed, and the composites’ provide a mitigation of the load transfer, and a decrease in the effective macroscopic stiffness is observed. The switch in the composite’s effective properties and behavior occur at a critical threshold of compressive loading which can be tailored by the geometric features of the structure and the relative contrast in material properties.

Analytical models were developed predicting the total strain energy density and its breakdown into contributions from the different constituent materials, the effective macroscopic stress-strain behavior of the highly nonlinear elastic composite, and the evolution in the effective macroscopic stiffness of these composite materials as a function of strain. Finite element simulations were conducted verifying the analytical predictions and models.

The strain energy density was shown to be greatly enhanced by deploying the wrinkling instability in these composites, and this method can be used to create composites with reversible energy storage mechanisms. It is evident that when straight interfacial layers are embedded into a matrix to create a composite, the effective composite’s behavior differs dramatically from the sum of the simple matrix and the empty layers, and a nonlinear elastic behavior is observed. Indeed, these composite materials exhibit a bilinear elastic behavior.

These results are promising as they open for a new way of designing materials with controllable and tunable energy storage properties, and pre- and post-buckling effective stress and stiffness behavior. This can be done just by changing the volume fraction of the interfacial layer in the
composite, or the stiffness ratio between the interfacial layer and the matrix layers. Furthermore, it is possible to tune and mitigate the stress level by these composites in both the pre- and post-buckling region.

Future work should include extending this concept and results for multilayered composites to evaluate more complex networked composites or 3D structures. This also opens the door to new interesting loadings situations, new interesting effective behavior, and novel tunable materials.
Chapter 6

Energy absorption in elastic-perfectly plastic wrinkling composites

6.1 Introduction

In previous chapters we considered the instability, local deformation, energy storage, and effective stress-strain behavior, and properties of multilayered composites composed of linear elastic materials. In this chapter we will evaluate the case of introducing plasticity in the composite by modeling the interfacial layer as an elastic-perfectly plastic material. We will study the effect of the elastic-perfectly plastic thin interfacial layer on the composite energy absorption (storage and dissipation) and macroscopic stress, and the composite can be designed to dissipate a desired amount of energy as the applied compressive load is increased.

Energy absorption, which we define to include both energy storage and dissipation, is a very important area of research as it has many essential engineering applications. Energy storage is of great use in applications where we wish to have the energy stored fully released upon removing the applied load, and the material will then return to its initial shape. Hence, energy storage is useful in for example springs, sensors, actuators, packaging etc. Energy dissipation is desired in applications where we do not wish to have all the energy released as the load is removed, and the material will then not fully return to its original shape. Hence, energy dissipation is useful in for example damping, protection, energy conversion etc. Energy dissipation systems use the concept of permanently deforming a material and structure through material plasticity, making it an irreversible process. Engineers have researched on designing more optimal geometries and structures that give increased plastic dissipation [56,100]. Cellular networked structures and foams are currently effective methods of inducing
instabilities, plasticity or plastic hinges, and thereby increasing energy dissipation while also providing a light weight material. Another powerful energy absorption mechanism used today for damping and dissipating crash energy, is manufacturing metal cylinders with a porous/foam core. Under compression the cylinder-walls will buckle and fold in several waves creating a wrinkling pattern. Plasticity occurs at each peak of the folding pattern as well as in the porous core material. The core will also change the wave mode of the metal cylinder buckling. This combination will result in increased energy absorption [52,101,102].

In this chapter, we focuses on the energy absorption properties of multilayered plastic composites being macroscopically compressed through the pre-buckling region, the wrinkling instability, and into the post-buckling region. We define energy absorption density as the sum of the energy dissipated and energy stored in the composite per unit volume, as it is being macroscopically compressed under plane strain conditions. We present analytical models for the total energy absorbed, energy dissipated, and energy stored in the composite as function of the constituent materials, geometry (thickness and spacing of interfacial layers), and the applied loading. In addition, we develop expressions for the macroscopic stress for these plastic composites. Finite element simulations are conducted to verify and further explore the mechanisms. The results in this chapter show that by introducing plasticity in the interfacial layers of the composite, we can purposefully achieve new energy absorption and dissipation properties, making these composites useful for many scientific and engineering applications.

As shown in Figure 6.1a, the composites evaluated can be modeled as a multilayered structure with a representative volume element (RVE) that contains a thin interfacial layer embedded in soft matrix. The interfacial layer has a thickness $t$, length $L$, and composed of an elastic-perfectly plastic material with stiffness $E_i$, Poisson ratio $v_i$, and yield stress, $\sigma_{\text{yield}}$. The interfacial layer is supported by soft matrix on both sides with thickness $d$, length $L$, and composed of linear elastic material with stiffness $E_0$, and Poisson ratio $v_0$. Figure 6.1b shows the elastic-perfectly plastic constitutive relation and the local material behavior of a material element in the interfacial layer. For plane strain conditions, we use the plane strain modulus, $\bar{E}_1$ and $\bar{E}_0$, and plane strain Poisson ratio, $\bar{\nu}_1$ and $\bar{\nu}_0$:

$$
\bar{E}_1 = \frac{E_i}{1-v_i^2}, \quad \bar{E}_0 = \frac{E_0}{1-v_0^2}, \quad \bar{\nu}_1 = \frac{v_i}{1-v_i}, \quad \bar{\nu}_0 = \frac{v_0}{1-v_0}
$$
Figure 6.1: a) Representative volume element (RVE) of a composite containing a stiff interfacial layer with thickness \( t \), yield stress \( \sigma_{\text{yield}} \), Young’s modulus \( E_l \), and Poisson ratio \( v_l \), supported by soft matrix on both side with thickness \( d \), Young’s modulus of \( E_0 \) and Poisson ratio \( v_0 \) (\( D=2d+t \) is the full composite RVE thickness). Upon a macroscopic compressive strain \( \bar{\varepsilon} \), greater than the buckling strain \( \bar{\varepsilon}_{\text{cr}} \), \( |\bar{\varepsilon}| > |\bar{\varepsilon}_{\text{cr}}| \), the interfacial layer will undergo instability and deform into a wrinkling pattern with a wavelength of \( \lambda(\bar{\varepsilon}) \). b) The interfacial layer’s material constitutive relation is chosen to be elastic-perfectly plastic. During loading, the local stress-strain behavior will initially be linear elastic, but once yield stress is reached, the local stresses will remain constant at \( \sigma_{\text{yield}} \) as the local strains increases (the local strains will now all be plastic strain, which will be the residual strain if the material is unloaded).
6.2 Yielding versus wrinkling criterion

In previous chapters we studied compression of the periodic multilayered composite shown in Figure 6.1a, for the case where the interfacial layer does not yield or it has a very high yield stress, such that the interfacial layer is regarded to be linear elastic for the smaller deformations we were considering. It was established that as the composite is being macroscopically compressed, $\varepsilon$, once a critical strain is exceeded, $|\varepsilon| > |\varepsilon_{cr}|$, the interfacial layer will undergo instability resulting in a wrinkling pattern. The critical macroscopic applied strain causing wrinkling instability, $\varepsilon_{cr}$, and the initial wavelength of the wrinkling pattern, $\lambda_{cr}$, are given by:

$$\varepsilon_{cr} = -2.08 \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{2/3} \left( \frac{E_1}{E_0} \right)^{2/3}$$

$$\lambda_{cr} = t \cdot 2.18 \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{1/3} \left( \frac{E_1}{E_0} \right)^{1/3}$$

Alternatively, we can express the critical macroscopic applied strain as an applied macroscopic stress instead in the interfacial layer, according to:

$$\bar{\sigma}_{cr} = \bar{E}_1 \cdot \varepsilon_{cr} = -2.08 \cdot \bar{E}_1 \cdot \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{2/3} \left( \frac{E_1}{E_0} \right)^{2/3} = -2.08 \cdot \frac{(E_1)^{1/3}}{1-v_1^2} \left[ \frac{(1-v_0)^2 \cdot E_0}{3-4v_0} \right]^{2/3}$$

where $\bar{E}_1 = \frac{E_1}{1-v_1^2}$ is the plane strain modulus of the interfacial layer.

In this chapter we introduce a yield stress to the interfacial layer, $\sigma_{yield}$, such that plasticity can occur locally anywhere the local stresses exceed the yield stress. We use an elastic-perfectly plastic material model for the interfacial layer, as shown in Figure 6.1b, which can be written in general form as:

$$\sigma_{local} = \begin{cases} 
\sigma_{local} & \text{if no yielding occurs} - \text{linear elastic range} \\
\sigma_{yield} & \text{if yielding does occur} - \text{plasticity range}
\end{cases}$$

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For our composites, we will use the Von-Mises yield criterion to determine if yielding and plasticity has occurred locally at any point in the interfacial layer, as the composite is being compressed. The Von-Mises stress is calculated by using the local stresses in the interfacial layer, \( \sigma_{ij} \), as shown in Equation 6.3. Using the Von-Mises stress, the yielding criterion can be expressed according to Equation 6.4.

\[
\sigma_{\text{Mises}} = \sqrt{\frac{1}{2}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2] + 3[\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2]}
\]  

(6.3)

Von-Mises yield criterion:

\[
\sigma_{\text{Mises}} = \begin{cases} 
\sigma_{\text{Mises}} & \text{if } \sigma_{\text{Mises}} \leq \sigma_{\text{yield}} \\
\sigma_{\text{yield}} & \text{if } \sigma_{\text{Mises}} \geq \sigma_{\text{yield}} 
\end{cases}
\]  

(6.4)

We consider our well-studied composite under plane strain conditions, such that the local stresses at any point in the interfacial layer is given by:

\[
\sigma_{\text{int.layer}}^x = \bar{E}_1 \cdot \varepsilon_{\text{int.layer}}^x \\
\sigma_{\text{int.layer}}^y = 0 \\
\sigma_{\text{int.layer}}^z = v_1 \sigma_{\text{int.layer}}^x = v_1 \bar{E}_1 \varepsilon_{\text{int.layer}}^x \\
\tau_{\text{int.layer}}^{xy} = \tau_{\text{int.layer}}^{yz} = \tau_{\text{int.layer}}^{zx} = 0
\]  

(6.5)

where \( \bar{E}_1 = \frac{E_1}{1 - v_1^2} \) is the plane strain modulus of the interfacial layer, and \( \varepsilon_{\text{int.layer}}^x \) is the local strain in the interfacial layer, derived and given in Equation 4.5 or 4.13.

Using the expressions given in Equation 6.5, the Von-Mises stress (Equation 6.3) can be rewritten to be expressed as a function of the axial stress only:

\[
\sigma_{\text{Mises}} = \sqrt{v_1^2 - v_1 + 1} \cdot |\sigma_{\text{int.layer}}^x|
\]  

(6.6)
Next, using the Von-Mises criterion (Equation 6.4), yielding occurs at any point in the interfacial layer, where the following condition is satisfied:

\[
\sqrt{\nu_1^2 - \nu_1 + 1} \cdot |\sigma_{x}^{\text{int.layer}}| \geq \sigma_{\text{yield}} \quad \rightarrow \quad |\sigma_{x}^{\text{int.layer}}| \geq \frac{\sigma_{\text{yield}}}{\sqrt{\nu_1^2 - \nu_1 + 1}}
\]  

(6.7)

It is crucial for most engineering applications to know if the interfacial layer in the composite will undergo wrinkling instability first, or if it will yield first. We will now determine the criterions for whether the interfacial layer in the composite will wrinkle or yield first, as the composite is being compressed. Consider the multilayered composite (Figure 6.1a) being compressed uniformly by a macroscopic strain, $\varepsilon$, or stress, $\sigma$, such that the initial pre-buckling axial stresses in the interfacial layer is given by:

\[
\sigma_{x}^{\text{int.layer}} = E_1 \cdot \varepsilon_{x}^{\text{int.layer}} \quad \rightarrow \quad \sigma_{x}^{\text{int.layer}} = E_1 \cdot \varepsilon = \sigma
\]  

(6.8)

Equation 6.2 gives us the expression predicting the critical stress that will cause wrinkling of the interfacial layer, while Equation 6.7 gives us the expression predicting when yielding will occur in the interfacial layer. We summarize the two different conditions as follows:

1. Wrinkling occurs when:

\[
\sigma_{x}^{\text{int.layer}} = \sigma_{\text{cr-wrinkling}} = -2.08 \cdot \frac{(E_1)^{\frac{1}{3}}}{1-v_1^2} \cdot \left[\frac{(1-v_0)^2}{3-4v_0}\right]^{\frac{2}{3}}
\]  

(6.9)

We create a design plot (Figure 6.2) showing the critical stress causing wrinkling in the interfacial layer of the composite, as a function of the composite’s material properties; the stiffness of the interfacial layer, $E_1 \in [10 - 1000] MPa$, and the matrix layers, $E_0 \in [1 - 100] MPa$. We have in earlier chapters established that the stiffness ratio between the interfacial layer and the matrix, $\frac{E_1}{E_0}$, should be greater than 10, to ensure that the wrinkling wavelength is long enough compared to the thickness of the interfacial layer, $\frac{\lambda_{cr}}{t} > 8-10$, such that our assumptions of the interfacial layer behaving as a thin beam is valid. Hence, the design plot in Figure 6.2 only shows the wrinkling initiation stress for composites with $\frac{E_1}{E_0} > 10$. 

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Figure 6.2: Design plot showing the critical stress causing wrinkling in the interfacial layer of the composite (Equation 6.9), as a function of the composite’s material properties; the stiffness of the interfacial layer, $E_1$, and the stiffness of the matrix layers, $E_0$. To ensure that the interfacial layer can be approximated and modeled as a wrinkling beam, we only evaluate composites with stiffness ratio $\frac{E_1}{E_0} > 10$ (grey region corresponds to $\frac{E_1}{E_0} < 10$).

2. Yielding occurs when:

$$\sigma_{x}^{\text{int.layer}} = \sigma_{\text{cr-yielding}} = \frac{-\sigma_{\text{yield}}}{\sqrt{\nu_1^2 - \nu_1 + 1}}$$ (6.10)

It is clear that the critical applied stress causing yielding is directly related to the yield stress of the material chosen for the interfacial layer. Hence, designers have great flexibility and control over whether the composite will yield or wrinkle, just by choosing different materials. The well-known Ashby plot (Figure 6.3) can be of great help for designers in picking a material that gives the desired outcome, since it plots different common material’s yield stress as well the material’s material stiffness.
Figure 6.3: The Ashby material plot can be used to select a material with desired elastic modulus and yield stress [103]. Red symbols have been added in the plot representing the material properties chosen for the elastic-perfectly plastic interfacial layers of the composites evaluated in Section 6.4.1 (red region) and Section 6.4.2 (red dot).

In this work, we wish to mainly study composites where wrinkling occurs prior to yielding, so that we can use the large microstructural deformations and changes that occur in the composite due to wrinkling, to achieve new interesting effective properties and phenomena. Hence, by combining the conditions in Equations 6.9 and 6.10, we find a general expression for the criterion, based on only the material properties, which must be satisfied in order for the interfacial layers of the composite to wrinkle prior to yielding:

\[ |\tilde{\sigma}_{cr-wrinking}| < |\tilde{\sigma}_{cr-yielding}| \]  

\[ 2.08 \cdot \frac{\left(E_1\right)^{\frac{1}{3}}}{1-v_1^2} \cdot \frac{\left[(1-v_0)^2 \cdot E_0\right]^{\frac{2}{3}}}{3-4v_0} < \frac{\sigma_{yield}}{\sqrt{v_1^2-v_1+1}} \]

\[ \sigma_{yield} > 2.08 \cdot \frac{\left(v_1^2-v_1+1\right)^{\frac{4}{3}}}{\left(1-v_1^2\right)\left(3-4v_0\right)^{\frac{2}{3}}} \cdot \left(E_1\right)^{\frac{1}{3}} \cdot \left(E_0\right)^{\frac{2}{3}} \]
We define the stiffness ratio to be \( m = \frac{E_i}{E_0} \), and insert for \( E_0 \) as an ratio of \( E_1 \), i.e. \( E_0 = \frac{E_1}{m} \) where \( m \in [10 - 1000] \). Hence, the criterion for ensuring that wrinkling occurs prior to yielding can be written as the following expression:

\[
\sigma_{\text{yield}} > 2.08 \cdot \frac{\sqrt{v_1^2 - v_1 + 1 \cdot (1 - v_0)^3}}{(1 - v_1^2) \cdot (3 - 4v_0)^{\frac{2}{3}}} \cdot m^{\frac{2}{3}} \cdot E_1
\]

(6.12)

\[
\frac{\sigma_{\text{yield}}}{E_1} > 2.08 \cdot \frac{\sqrt{v_1^2 - v_1 + 1 \cdot (1 - v_0)^3}}{(1 - v_1^2) \cdot (3 - 4v_0)^{\frac{2}{3}}} \cdot m^{\frac{2}{3}}
\]

where \( m = \frac{E_i}{E_0} \in [10 - 1000] \)

Figure 6.4a evaluates composites with different material properties (changing \( \sigma_{\text{yield}}, E_1, E_0 \)), and shows based on Equation 6.12a how different composites transition from having the interfacial layer first wrinkle under compression to it yielding first. Figure 6.4a shows that composites will have yielding occurring in the interfacial layers prior to wrinkling, as one of the following changes are made to the material properties of the composite:

1) The stiffness of the interfacial layer, \( E_1 \), is increased.
2) The yield stress of the interfacial layer, \( \sigma_{\text{yield}} \), is decreased.
3) The stiffness ratio between the interfacial layer and the matrix layers, \( \frac{E_i}{E_0} \), is decreased.

A general non-dimensional phase diagram is shown in Figure 6.4b, by using Equation 6.12b. This phase diagram shows based on the stiffness ratio between the interfacial layer and the matrix in the composite, \( \frac{E_i}{E_0} \), and based on the yield stress to stiffness ratio in the interfacial layer, \( \frac{\sigma_{\text{yield}}}{E_1} \), whether the interfacial layer in the composite will first yield or wrinkle under a macroscopic applied compression. It is evident that wrinkling occurs first in the composite as the stiffness ratio, \( \frac{E_i}{E_0} \), increases or as the yield stress to stiffness ratio of the interfacial layer, \( \frac{\sigma_{\text{yield}}}{E_1} \), increases.
Figure 6.4: a) The transition from wrinkling to yielding occurring first in the interfacial layer of a composite, based on the composite’s material properties. b) Phase diagram showing how wrinkling will occur prior to yielding in the interfacial layers of a composite under compression, as the material property ratios $\frac{E_1}{E_0}$ or $\frac{\sigma_{yield}}{E_1}$ is increased.
6.3 Analytical model for energy absorption

In the remaining parts of this chapter we will study the energy absorption in a composite being compressed where the interfacial layers of the composite is composed of an elastic-perfectly plastic material with a given yield stress, $\sigma_{yield}$. More specifically, we will study the effect of the wrinkling of the interfacial layer on the energy absorption properties of the composite, as well as its effect on the macroscopic effective stress of the composite. The energy absorbed by a structure consists of the energy stored elastically in the structure as it is being loaded, as well as the energy dissipated in the structure during loading due to plasticity, viscoelasticity, or other material dissipation mechanisms. Hence, all the energy stored will be released upon unloading of the structure, while the dissipated energy will not be retrievable. We will be studying the energy absorption of our composites by studying the stored energy density and dissipated energy density as the composite is being macroscopically compressed.

The total energy absorbed by the composite is the sum of the energy absorbed by the matrix and the interfacial layers:

$$u_{Total} = u_{matrix} + u_{int.layer}$$

(6.13)

Since the matrix layers are considered to be linear elastic materials, the energy absorption will only consist of the energy stored. On the other hand, since the interfacial layers are elastic-perfectly plastic materials, the interfacial layer’s energy absorption consists of both elastic energy stored and plastic energy dissipated. Equation 6.13 can be rewritten as:

$$u_{Total} = u_{matrix}^{\text{elastic}} + u_{int.layer}^{\text{elastic}} + u_{int.layer}^{\text{plastic}}$$

(6.14)

Stored energy

Dissipated energy

$$u_{Total} = \iint \sigma_{ij}^m \varepsilon_{ij}^m \, dV^m + \iint \sigma_{ij}^{int.l} \varepsilon_{ij}^{int.l}^{\text{elastic}} \, dV^{int.l} + \iint \sigma_{ij}^{int.l} \varepsilon_{ij}^{int.l}^{\text{plastic}} \, dV^{int.l}$$

Stored energy

Dissipated energy
where $\sigma_{ij}^m$ and $\varepsilon_{ij}^m$ are the local stresses and strains in the matrix, $\sigma_{ij}^{\text{int.l}}$ is the local stresses in the interfacial layer, while $[\varepsilon_{ij}^{\text{int.l}}]^{\text{elastic}}$ and $[\varepsilon_{ij}^{\text{int.l}}]^{\text{plastic}}$ are the local elastic and plastic strain in the interfacial layer, respectively.

Similarly, by averaging the total energy of the composite with respect to the total volume of the composite, the energy densities can be found:

$\triangleright$ Total absorbed energy density:

$$U_{\text{Total}} = \frac{1}{DL} \left\{ \iint \sigma_{ij}^m d\varepsilon_{ij}^m \, dV^m + \iint \sigma_{ij}^{\text{int.l}} [d\varepsilon_{ij}^{\text{int.l}}]^{\text{elastic}} \, dV^{\text{int.l}} + \iint \sigma_{ij}^{\text{int.l}} [d\varepsilon_{ij}^{\text{int.l}}]^{\text{plastic}} \, dV^{\text{int.l}} \right\}$$

$$= f_m U_{\text{matrix}}^{\text{stored}}(\bar{\varepsilon}) + f_{\text{int.l}} U_{\text{int.l}}^{\text{stored}}(\bar{\varepsilon}) + f_{\text{int.l}} U_{\text{int.l}}^{\text{dissipated}}(\bar{\varepsilon})$$

$\triangleright$ Total stored energy density:

$$U_{\text{Storage}} = \frac{1}{DL} \left\{ \iint \sigma_{ij}^m d\varepsilon_{ij}^m \, dV^m + \iint \sigma_{ij}^{\text{int.l}} [d\varepsilon_{ij}^{\text{int.l}}]^{\text{elastic}} \, dV^{\text{int.l}} \right\}$$

$$= f_m U_{\text{matrix}}^{\text{stored}}(\bar{\varepsilon}) + f_{\text{int.l}} U_{\text{int.l}}^{\text{stored}}(\bar{\varepsilon})$$

(6.15)

$\triangleright$ Total dissipated energy density:

$$U_{\text{Dissipated}} = \frac{1}{DL} \left\{ \iint \sigma_{ij}^{\text{int.l}} [d\varepsilon_{ij}^{\text{int.l}}]^{\text{plastic}} \, dV^{\text{int.l}} \right\} = f_{\text{int.l}} U_{\text{int.l}}^{\text{dissipated}}(\bar{\varepsilon})$$

where $f_m = 1 - f_{\text{int.l}} = 2d/(2d + t)$ and $f_{\text{int.l}} = t/(2d + t)$ are the volume fraction of the matrix and the interfacial layer in the composite, respectively. Furthermore, $U_{\text{matrix}}^{\text{stored}}(\bar{\varepsilon})$, $U_{\text{int.l}}^{\text{stored}}$, and $U_{\text{int.l}}^{\text{dissipated}}$ are the stored- and dissipated energy densities in the interfacial or matrix layers of the composite.

The total energy stored in the elastic matrix layers was previously calculated in Chapter 5.2.1, for any applied macroscopic strain, $\bar{\varepsilon}$, in the pre-buckling or post-buckling region. The energy density is found for a representative volume element which is one wavelength of the wrinkling
pattern, \( L = \lambda_{cr} \). Using the analytical models developed earlier, Equation 5.3b and 5.6b, the elastic strain energy density stored in the matrix layers of the wrinkling composite during pre- and post-buckling, are given by:

In the pre-buckling region, \( |\varepsilon| < |\varepsilon_{cr}| \):

\[
U_{\text{matrix}}^{\text{stored}}(\varepsilon) = \frac{1}{2} \int_{L} \sigma_{ij}^{\text{matrix}} d\varepsilon_{ij}^{\text{matrix}} dV_{\text{matrix}} = \frac{1}{2} \frac{E_{0}}{\lambda_{cr}} \left[ \bar{E}_{0} \varepsilon^{2} d\lambda_{cr} \right] = \frac{\bar{E}_{0} \varepsilon^{2}}{2} \quad (6.16a)
\]

In the post-buckling region \( |\varepsilon| > |\varepsilon_{cr}| \):

\[
U_{\text{matrix}}^{\text{stored}}(\varepsilon) = \frac{1}{2} \frac{E_{0}}{\lambda_{cr}} \int \left[ \varepsilon \sigma_{ij}^{\text{matrix}} + \frac{1}{2} [\sigma, \varepsilon]^{\text{matrix}} dV_{\text{matrix}} \right] = \frac{1}{2} \frac{E_{0}}{\lambda_{cr}} u_{\text{matrix}}(\varepsilon) = \frac{\bar{E}_{0} \varepsilon^{2}}{2} \quad (6.16b)
\]

where the wavelength, \( \lambda = \lambda(\varepsilon) \), wrinkling amplitude, \( w_{\text{max}} = w_{\text{max}}(\varepsilon) \), and the critical strain, \( \varepsilon_{cr} \), are defined by Equation 6.18 and 6.19. The plane strain stiffness and Poisson ratio of the matrix material are given by: \( \bar{E}_{0} = \frac{E_{0}}{1-v_{0}^{2}} \) and \( \bar{v}_{0} = \frac{v_{0}}{1-v_{0}} \). Furthermore, the non-dimensional variables are defined as:

\[
\tilde{\lambda}_{cr} = \lambda_{cr}/t \quad , \quad \tilde{\lambda} = \lambda(\varepsilon)/t \quad , \quad \tilde{w}_{\text{max}} = w_{\text{max}}(\varepsilon)/\tilde{\lambda}_{cr} \quad , \quad \tilde{w}_{\text{max}} = w_{\text{max}}(\varepsilon)/\lambda(\varepsilon)
\]
In order to calculate the total stored-, dissipated-, and absorbed- energy densities in the composite (Equation 6.15a-c), we also need to know the local stresses and strains in the interfacial layers of the composite as the composite is being compressed macroscopically by \( \bar{\varepsilon} \) under plane strain conditions. The local stresses developed in the interfacial layer during the pre- and post-buckling loading of the composite were expressed in Chapter 4.2-4.3, for the case where the interfacial layer is composed of a linear elastic material. Now that the interfacial layer is composed of elastic-perfectly plastic material, the local stresses given by Equation 4.6 and 4.14 must be modified to account for the yield stress and plasticity that can occur locally in the interfacial layer as the composite is being compressed and wrinkling instability occurs. We will use the Von-Mises yield criterion (as described in Chapter 6.2) to determine when yielding occurs in the interfacial layer and what the local stresses and strains become after yielding. We will also simplify the plasticity model, by making the approximation that the plasticity is captured by the effective plane strain axial compression plus bending of the interfacial layer, and hence we avoid unnecessarily using full tensorial expressions. Figure 6.5 shows a schematic of the steps conducted in the procedure of including yielding into the analytical models for the interfacial layer local stresses and strains, and then the procedure to calculate the stored and dissipated energy in the interfacial layer. The procedure shown in Figure 6.5 is described in detail below.
Figure 6.5: Procedure of how to use an elastic-perfectly plastic material model for the interfacial layer’s to find the local stresses and strains after yielding in the interfacial layer. Then the stored and dissipated energy in the interfacial layer is calculated.
We first start by determining if the macroscopically applied strain, $\varepsilon$, is in the pre-buckling region, $|\varepsilon| < |\varepsilon_{cr}|$, or in the post-buckling region, $|\varepsilon| > |\varepsilon_{cr}|$, (Step 1). In the pre-buckling region we know that the interfacial layer is being uniformly compressed, while in the post-buckling region we know that the interfacial layer has wrinkled due to instability and the wrinkling pattern, $w(x, \varepsilon)$, is given by (Figure 6.1):

$$w(x, \varepsilon) = w_{\text{max}}(\varepsilon) \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right) = \frac{\lambda(\varepsilon)}{\pi} \sqrt{|\varepsilon| - |\varepsilon_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right)$$

(6.17)

where $w_{\text{max}}(\varepsilon)$ is the post-buckling amplitude and $\lambda(\varepsilon)$ is the post-buckling wavelength of the wrinkling pattern, given by:

$$w_{\text{max}}(\varepsilon) = \frac{\lambda(\varepsilon)}{\pi} \sqrt{|\varepsilon| - |\varepsilon_{cr}|},$$

$$\lambda(\varepsilon) = \lambda_{cr} e^{-|\varepsilon|}$$

where the critical strain causing instability, $\varepsilon_{cr}$, and the initial wrinkling wavelength, $\lambda_{cr}$, is given by:

$$\varepsilon_{cr} = -2.08 \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{2/3} \left( \frac{E_1}{E_0} \right)^{2/3}$$

(6.19)

$$\lambda_{cr} = t \cdot 2.18 \left[ \frac{3-4v_0}{(1-v_0)^2} \right]^{1/3} \left( \frac{E_1}{E_0} \right)^{1/3}$$

Next, in Step 2, the local stresses, $\sigma_{ij}^{\text{int.layer}}(x, h, \varepsilon)$, and strains, $\varepsilon_{ij}^{\text{int.layer}}(x, h, \varepsilon)$, in the interfacial layer are defined according to the results presented in Chapter 4.2-4.3, at any macroscopically applied strain, $\varepsilon$:

**In pre-buckling region, $|\varepsilon| < |\varepsilon_{cr}|$:**

- $\varepsilon_x^{\text{int.layer}}(x, h, \varepsilon) = \varepsilon$
- $\varepsilon_y^{\text{int.layer}}(x, h, \varepsilon) = -\bar{v}_1 \varepsilon$
- $\varepsilon_z^{\text{int.layer}}(x, h, \varepsilon) = 0$
- $\varepsilon_{xy}^{\text{int.layer}}(x, h, \varepsilon) = 0$

**In post-buckling region, $|\varepsilon| > |\varepsilon_{cr}|$:**

- $\varepsilon_x^{\text{int.layer}}(x, h, \varepsilon) \approx \varepsilon_{cr} - h \cdot \frac{4\pi}{\lambda} \sqrt{|\varepsilon| - |\varepsilon_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda} \right)$
- $\varepsilon_y^{\text{int.layer}}(x, h, \varepsilon) \approx -\bar{v}_1 \varepsilon_x^{\text{int.layer}}(x, h, \varepsilon)$
- $\varepsilon_z^{\text{int.layer}}(x, h, \varepsilon) = 0$
- $\varepsilon_{xy}^{\text{int.layer}}(x, h, \varepsilon) \approx 0$

(6.20)
In pre-buckling region, $|\bar{\varepsilon}| < |\bar{\varepsilon}_{cr}|$:

\[
\begin{align*}
\sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon}) &= \bar{E}_1 \bar{\varepsilon} \\
\sigma_y^{\text{int.layer}}(x, h, \bar{\varepsilon}) &= 0 \\
\sigma_z^{\text{int.layer}}(x, h, \bar{\varepsilon}) &= \bar{v}_1 \bar{E}_1 \bar{\varepsilon} \\
\sigma_{xy}^{\text{int.layer}}(x, h, \bar{\varepsilon}) &= 0
\end{align*}
\]

In post-buckling region, $|\bar{\varepsilon}| > |\bar{\varepsilon}_{cr}|$:

\[
\begin{align*}
\sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon}) &\approx \bar{E}_1 \bar{\varepsilon}_{x}^{\text{int.layer}}(x, h, \bar{\varepsilon}) \\
\sigma_y^{\text{int.layer}}(x, h, \bar{\varepsilon}) &\approx 0 \\
\sigma_z^{\text{int.layer}}(x, h, \bar{\varepsilon}) &\approx \bar{v}_1 \sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon}) \\
\sigma_{xy}^{\text{int.layer}}(x, h, \bar{\varepsilon}) &\approx 0
\end{align*}
\] (6.21)

Based on the local stresses we will calculate the local Von-Mises stress all points in the interfacial layer (Step 3). Under plane-strain uniaxial loading, the Von-Mises criterion were shown to be (Equation 6.7):

\[
\sigma_{\text{Mises}}(x, h, \bar{\varepsilon}) = \sqrt{\nu_1^2 - \nu_1 + 1} \cdot |\sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon})| \geq \sigma_{\text{yield}}
\] (6.22)

where $\sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon})$ is given in Equation 6.21.

If the Von-Mises stress is less than the yield stress (Step 4 - No), there is no plasticity occurring, and so the process is still fully elastic. This means that the local stresses in the interfacial layer remain as calculated in Equation 6.21. Furthermore, the local strains are all elastic strains (there is no plastic strain developed). That is to say:

\[
\varepsilon_{ij}^{\text{int.l}}(x, h, \bar{\varepsilon}) = \left[\varepsilon_{ij}^{\text{int.l}}(x, h, \bar{\varepsilon})\right]^{\text{elastic}} + \left[\varepsilon_{ij}^{\text{int.l}}(x, h, \bar{\varepsilon})\right]^{\text{plastic}} = \varepsilon_{ij}^{\text{int.layer}}(x, h, \bar{\varepsilon})
\] (6.23)

\[
\left[\varepsilon_{ij}^{\text{int.l}}(x, h, \bar{\varepsilon})\right]^{\text{elastic}} = \varepsilon_{ij}^{\text{int.layer}}(x, h, \bar{\varepsilon})
\]

\[
\left[\varepsilon_{ij}^{\text{int.l}}(x, h, \bar{\varepsilon})\right]^{\text{plastic}} = 0
\]

On the other hand, if the Mises stress is higher than the yield stress (Step 4 - Yes), plasticity is occurring locally in the material. We first find the local elastic strain, which is the local strain corresponding to the initial first yield in the material. Using Equation 6.7, the axial elastic strain can be found by:
Moreover, since the total local strain is the sum of the elastic and plastic strain, \( \varepsilon_{ij}^{\text{total}} = \varepsilon_{ij}^{\text{elastic}} + \varepsilon_{ij}^{\text{plastic}} \), we can easily find the local plastic strain developed in the interfacial layer for a given applied loading. The local plastic strain is given by:

\[
[\varepsilon_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})]^{\text{plastic}} = \varepsilon_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon}) - \left[ \varepsilon_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon}) \right]^{\text{elastic}}
\]

\[
= \varepsilon_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon}) - \frac{\sigma_{\text{yield}}}{E_1\sqrt{v_1^2 - v_1 + 1}}
\]

In the case where the Mises stress exceeds the yield stress and plasticity occurs, the local stresses in the interfacial layer must be modified to account for the yielding that occurs (Step 5-Yes). Since, we are considering the plane strain case, and an elastic-perfectly plastic material for the interfacial layer, the local stresses due to the yielding are given by:

\[
[\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})] = \frac{\sigma_{\text{yield}}}{\sqrt{k^2 - k + 1}}
\]

\[
[\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})] = k[\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})]
\]

where \( k = \nu_1 \) at the initial point of yielding. Under plane strain compression, it has been shown [104] that \( k \) increases towards a constant value of \( 1/2 \) as the local strains are increased. Hence, we will in this study consider the upper limit and set \( k = 1/2 \), such that the interfacial local stresses become constant after yielding and given by:

\[
[\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})] \approx \frac{2}{\sqrt{3}} \sigma_{\text{yield}}
\]

\[
[\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})] \approx \frac{1}{2} [\sigma_{ij}^{\text{int,l}}(x, h, \bar{\varepsilon})] = \frac{1}{\sqrt{3}} \sigma_{\text{yield}}
\]

Finally, in Step 6, we will use the expressions developed for the local stresses, local elastic strains, and local plastic strains, to calculate the energy stored, dissipated and absorbed by the interfacial layer during the loading process. Due to the plane strain condition, the only
contribution to the energy will be due to the axial stress and stains in the composite:
\[ \sigma_{x}^{\text{int.1}}(x, h, \varepsilon), \quad [\varepsilon_{x}^{\text{int.1}}(x, h, \varepsilon)]^{\text{elastic}}, \quad \text{and} \quad [\varepsilon_{x}^{\text{int.1}}(x, h, \varepsilon)]^{\text{plastic}}. \]

The total energy absorbed, energy stored, and energy dissipated by the interfacial layers are hence given by:

\[ u_{\text{int.layer}} = \int \sigma_{x}^{\text{int.1}} d\varepsilon_{x}^{\text{int.1}} dV^{\text{int.1}} \]

where the local strains and stresses are given by Equation 6.20 and 6.21 for material elements that do not yield, and by Equation 6.24, 6.25 and 6.27 for material elements where yielding does occur. The energy densities are calculated by dividing Equation 6.28 by the initial volume of the interfacial layer:

\[ U_{\text{int.layer}} = \frac{1}{V_{\text{int.layer}}} \left[ u_{\text{int.layer}}^{\text{stored}} + u_{\text{int.layer}}^{\text{dissipated}} \right] = U_{\text{int.layer}}^{\text{stored}} + U_{\text{int.layer}}^{\text{dissipated}} \]

In addition, the macroscopic axial stress-strain behavior of the composite during loading can be obtained by differentiating the total energy absorption of the composite, \( U_{\text{Total}}(\varepsilon) \), with respect to the applied macroscopic load, \( \varepsilon \). By combining Equations 6.15, 6.16 and 6.29, we can calculate the composite's macroscopic engineering stress, according to:

\[ \sigma_{x}^{\text{comp}}(\varepsilon) = \frac{d U_{\text{Total}}(\varepsilon)}{d \varepsilon} \]

The true stress-strain relation for the composite can then be found by: \( \sigma_{x}^{\text{comp}} = \sigma_{x}^{\text{comp}}(1 + |\varepsilon|) \) and \( \varepsilon = \ln(1 + |\varepsilon|) \).

The analytical method described in this chapter is provided in Appendix C as a MATLAB code.
6.4 Results of energy absorption

In Chapter 6.3 we established the method for calculating the energy absorbed, stored and dissipated in a multilayered composite being compressed under plane strain conditions, where wrinkling is the mode of instability, and the interfacial layer is composed of an elastic-perfectly plastic material. Dissipation occurs at locations where yielding and plasticity occurs in the material, which is found through the Von-Mises yield criterion, Equation 6.22.

Figure 6.6: Analytical and FE results for a composite with $E_i/E_o=200$, $E_o = 1MPa$, $\sigma_{yield} = 10 MPa$, $t/D=0.02$, $t=0.5$, at applied compressive strain of $\varepsilon = -0.07 = 3\varepsilon_{cr}$. FE and analytical results of the wrinkled interfacial layer and; a) the local Von Mises stresses created in the interfacial layer due to the compression and wrinkling mechanism, b) the local axial stresses created in the interfacial layer due to the compression and wrinkling mechanism.
Figure 6.6a show our analytical results compared with finite element (FE) simulations results for the Von-Mises stresses developed in the interfacial layer of a composite compressed with a macroscopic applied strain of $|\varepsilon| = 0.07$, where $E_i/E_0 = 200$ (with $E_i = 200\text{MPa}$). For this composite, the yield stress of the interfacial layer is set to be: $\sigma_{yield} = 10\text{MPa}$, such that at any location where the Von-Mises stress exceeds 10MPa, yielding is occurring locally. The local axial stress in the interfacial layer, $\sigma_x^{\text{int}}(x, h, \varepsilon)$, is also found from our analytical results (Equation 6.21 and 6.27) and compared with finite element simulations, Figure 6.6b. Figure 6.6 shows great agreement between the analytical and FE results.

Moreover, Figure 6.7a shows the energy absorbed by the composite, the energy stored in the matrix and interfacial layers of the composite, as well as the energy dissipated in the composite from our analytical models (Equation 6.14 and 6.15), compared with FE simulations. The composite’s overall absorbed energy density, stored energy density and dissipated energy density is shown as a function of the macroscopic strain, $|\varepsilon| \in [0 - 0.07]$ in Figure 6.7b.

**Figure 6.7:** Analytical and FE results for the a) total energy absorbed, stored and dissipated in a composite; b) the total absorbed -, stored-, and dissipated- energy densities for the composite. The composite has $E_i/E_0 = 200$, $E_0 = 1\text{MPa}$, $t/D = 0.02$, $t = 0.5$, and $L = 5\lambda_{cr} = 60\text{mm}$, and the yield stress in the interfacial layer is $\sigma_{yield} = 10\text{MPa}$. 

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To verify the analytical models presented in Chapter 6.3 for the absorbed-, stored-, and dissipated energy densities (Equation 6.15), finite element (FE) simulations were run on multilayered composites at different macroscopic strains, $|\varepsilon| \in [0 - 0.07]$. Figure 6.8 shows the composites evaluated and discussed in Section 6.4.1 and 6.4.2 marked up in the phase diagram indicating if there will be yielding or wrinkling occurring first in the interfacial layer of the composite.

Figure 6.8: The green symbols in the phase diagram represent the cases studied in Section 6.4.1, while the blue symbols represent the cases studied in Section 6.4.2. We can see based on the material properties of the composite if wrinkling or yielding occur first.
6.4.1 Composites with varying interfacial layer stiffness, $E_I$

Figure 6.9 and 6.11 shows the FE and analytical results for different composites’ dissipated-, stored- and absorbed- energy densities for composites with varying stiffness ratio $E_I/E_0 \in [25 - 400]$, while keeping $E_0=1\text{MPa}$, $\sigma_{\text{yield}}=10\text{ MPa}$, and also keeping the interfacial layer concentration at $t/D=0.021$, to ensure that wrinkling is the mode of instability in all the composites evaluated (as seen in Chapter 3.2-3.3). Moreover, the macroscopic stress-strain behavior of the composites are included as well.

Figure 6.9a shows that the dissipated energy density in the interfacial layers for all the composites evaluated; the dissipated energy density is calculated per unit volume of interfacial layer $\frac{u_{\text{int.layer}}^\text{dissipated}}{\text{int.layer}}$ (plotted on the left axis and with corresponding solid graph lines), and also calculated per unit volume of the total composite $\frac{u_{\text{int.layer}}^\text{dissipated}}{DL} = \frac{u_{\text{int.layer}}^\text{dissipated}}{(2d+t)L}$ (plotted on the right axis with corresponding dashed graph lines), where $L=\lambda_{cr}$. The analytical and FE results show a clear bifurcation point where the graphs grow suddenly from zero; this change occurs once the local strain at any point in the interfacial layer exceeds the local strain corresponding to when yielding occurs, as given by Equation 6.24. From that point on, since material elements have yielded, the local strains will have an elastic and plastic component as indicated by Equation 6.24 and 6.25. Due to the local plasticity and plastic strain, energy will be dissipated in the interfacial layer of the composite, and this is demonstrated by the sudden growth in the dissipated energy density of the composite as seen in Figure 6.9a. It can be seen that plastic dissipation occurs at a lower macroscopic strain, $|\varepsilon|$, for composites with higher stiffness, $E_I$ (and consequently also stiffness ratio $E_I/E_0$ since $E_0$ is kept constant). This is due to the elastic and plastic local strains being directly related to the stiffness of the interfacial layer; i.e. a higher stiffness will lead to a lower strain at which plasticity starts, as seen in Equation 6.24 and 6.25.
**Figure 6.9:** FE and analytical results for the a) dissipated energy density, b) total stored energy density, and stored energy density in c) matrix and d) in interfacial layer, for composites with varying stiffness ratio $E_1/E_0 \in [25 - 400]$, while keeping $E_0 = 1 \text{MPa}$, and $t/D = 0.021$. The interfacial layer's yield stress is: $\sigma_{yield} = 10 \text{MPa}$. The dot-dashed line is the strain energy density for pure compression of a homogenous matrix.
Figure 6.9a also shows some deviation between our analytical model and FE simulations as we move further into the post-buckling region. This deviation is small compared with the composite’s total stored energy density (Figure 6.9b) or absorption energy density (Figure 6.11a), but should still be acknowledged. The deviation is due to two factors:

i) The first factor is related to the concept of plastic hinges; as we move further into the post-buckling region, the plastic zones in the thickness of the interfacial layer will grow bigger and hence affect the deformation behavior at the location of the wrinkling peaks. Figure 6.10a shows that at the larger macroscopic strains, the curvature at the wrinkling peaks has gotten sharp and plasticity is through the material (seen in the Von Mises stresses), i.e. the interfacial layer is hinged at the wrinkling peaks. When the post-buckling compression has led to full plasticity across the thickness of the interfacial layer at the wrinkling peaks, the wrinkling peaks will function as a plastic hinges. The interfacial layer can then conceptually be presented as a set of shorter beams connected between the hinges, which are fully plastic, as schematically shown in Figure 6.10b.

ii) Our simplified analytical model approximates the lateral- and shear stresses in the interfacial layer to be negligible, \( \sigma_y^{\text{int.layer}}(x, h, \bar{\varepsilon}) \approx 0 \) and \( \sigma_{xy}^{\text{int.layer}}(x, h, \bar{\varepsilon}) \approx 0 \), compared with the axial stress, \( \sigma_x^{\text{int.layer}}(x, h, \bar{\varepsilon}) \). As shown in Figure 6.10a, this approximation is valid right after the wrinkling instability, but as we move further into the post-buckling region and the plastic hinges occur, the deformation of the interfacial layer is no longer fully sinusoidal and the shear and lateral stresses start growing.

The total stored energy density in the composite as a function of the macroscopically applied strain, \( \bar{\varepsilon} \), is shown in Figure 6.9b, illustrating very good agreement between the analytical and FE results for all different composites. This total stored energy density in the composite is found by adding the contribution from the stored energy density for the matrix layers (Figure 6.9c) and the stored energy density for the interfacial layer (Figure 6.9d), according to:

\[
U_{\text{Storage}} = f_m U_{\text{matrix}}^{\text{storage}}(\bar{\varepsilon}) + f_{\text{int.l}} U_{\text{int.l}}^{\text{storage}}(\bar{\varepsilon}) \tag{6.29}
\]

where \( f_m = 1 - f_{\text{int.l}} = 2d/(2d+t) \) and \( f_{\text{int.l}} = t/(2d+t) \) are the volume fraction of the matrix and interfacial layer in the composite, respectively.
Figure 6.10: FE results from the composite with $E_t/E_0=400$; **a)** The local stresses in the interfacial layer for smaller strains, $\bar{\epsilon} = -0.034$ compared with for larger strains, $\bar{\epsilon} = -0.07$. In addition, at $\bar{\epsilon} = -0.07$, it is clear that the curvature at the wrinkling peaks has gotten sharp, and plasticity is through the material (as seen in the Mises stresses). A plastic hinge is created at these sharp wrinkling peaks. **b)** In the extreme case, when the yielding zones have developed through the whole thickness of the interfacial layer at the wrinkling peaks, the interfacial layer can be considered as a set of beams with length $\lambda/2$ separated by plastic hinges.
The stored energy density of the matrix layers is exactly the strain energy density of the matrix, since it is composed of a linear elastic material. This means that just as discussed in detail in Chapter 5, the stored energy density in the matrix layers of the composite grows dramatically in the post-buckling region, $|\varepsilon| > |\varepsilon_{cr}|$, as seen in Figure 6.9c. This increase in the matrix strain energy density in the post-buckling region is largely due to the large local deformations occurring in the matrix layers due to the wrinkling of the interfacial layer.

Figure 6.9d shows the stored energy density in the interfacial layer calculated per volume of interfacial layer $\frac{U_{int.layer}}{tL}$ (plotted on the left axis and with corresponding solid graph lines), and also per volume of the total composite $\frac{U_{int.layer}}{DL} = \frac{U_{int.layer}}{(2d+t)L}$ (plotted on the right axis with corresponding dashed graph lines), where $L = \lambda_{cr}$. The analytical and FE results show a clear bifurcation point where the slope of the graph changes; this change occurs at a macroscopic strain, $\bar{\varepsilon}$, which leads to local plasticity occurring locally in the interfacial layer. Due to the elastic-perfectly plastic material behavior of the interfacial layer, once plasticity has occurred for a material element, the local elastic strain remains constant in that material element as $\bar{\varepsilon}$ is increased, while the local plastic strain increases. The stored energy density is found by taking volume averaged integral of the elastic strain energy in each material element of the interfacial layer. Consequently, since the elastic strain remains constant (at value given by Equation 6.24) for material elements where plasticity has occurred, the strain energy density of that specific material element will also remain constant. This means that the slope of the total stored energy density in the interfacial layer decreases in the post-plasticity region compared to pre-plasticity. This can be seen in Figure 6.9d; composites with higher stiffness ratio, $E_1/E_0$, will undergo plasticity first due to their higher stiffness, and so the stored energy density graph has a bifurcation point and reduction in slope earlier (for a smaller $\bar{\varepsilon}$). As the interfacial layers are compressed towards full plasticity, the stored energy density graphs should exhibit a plateau region and an asymptotic constant value.

The composites' total absorbed energy densities, which consist both of the stored and dissipated energy, are show in Figure 6.11a for composites with varying, $E_1/E_0$, and
macroscopic applied strain, $|\varepsilon|$. By studying Figure 6.9 it is clear that the absorbed energy density in the composite is governed by the stored energy density in the matrix layers of the composite. The absorbed energy in the composite grows more rapidly as the applied strain exceeds the critical strain of the composite, i.e. in the post-buckling region ($|\varepsilon| > |\varepsilon_{cr}|$) where wrinkling has occurred. This is a direct effect of the stored energy contribution from the matrix layers increasing in the post-buckling region as well, as was studied in detail in Chapter 5.

Moreover, since the matrix layers with the large local deformations are the main source of energy to the energy absorbed in the composites, by increasing the stiffness of the matrix layers, $E_0$, while keeping the stiffness ratios between the two constructing materials in the material fixed, $E_1/E_0 \in [25 - 400]$, the total energy absorbed by the composite can be immensely amplified.

By evaluating the total composite absorbed energy density in a log-log plot, and using regression we see that the strain energy density in the pre-buckling region is proportional to: $U_{Total} \sim |\varepsilon|^2$, while in the post-buckling region the strain energy density can be estimated by: $U_{Total} \sim |\varepsilon|^{1.3}$ (Figure 6.11b). This reduction in slope is due to the wrinkled interfacial layers contributing less to the energy absorption, than prior to wrinkling. However, at the same time, the wrinkling of the interfacial layers has increased the energy storage in the matrix over that of a homogeneous matrix material. Hence, the composite provides enhanced energy absorption in the post-buckling region, and the case of single homogeneous materials.

The macroscopic true stress-strain behavior of the composites are shown in Figure 6.11c. A clear bifurcation point is seen at the critical strain where instability occurs, and the effective true stress levels out. The plasticity occurring during the post-buckling will reduce the slope of the stress-strain curve, and the main stiffness is provided by the soft matrix layers. Hence, it is clear that this plastic composites can be used to mitigate stress transfer, similar to the elastic composites (Chapter 5.3). An excellent agreement is observed between the analytical and FE results. The macroscopic true stress-strain behavior during a loading-unloading cycle is shown for three of the composites in Figure 6.11d. Due to the local plasticity occurring and dissipating energy, the unloading curve does not fall on top of the loading curve, and there is a permanent offset at zero stress showing the residual strain in the composites upon unloading.
Figure 6.11: a) FE and analytical results for the absorbed energy density, for composites with varying stiffness ratio $E_i/E_0 \in [25 - 400]$, while keeping $E_0=1\text{MPa}$, and $t/D=0.021$. The interfacial layer’s yield stress is set to: $\sigma_{\text{yield}} = 10\ \text{MPa}$. The dot-dashed line is the strain energy density for pure compression of a homogenous matrix. b) Log-log plot of the absorbed energy density showing the slope change in the pre- and post-buckling region. c) The macroscopic true stress-strain behavior for the different composites, showing a clear change is slope as wrinkling and plasticity occurs in the interfacial layers. d) The macroscopic true stress-strain behavior for three of the composites under a loading-unloading cycle.
6.4.2 Composites with varying matrix stiffness, $E_0$

In the previous section we studied the energy absorption, storage and dissipation, as well as the macroscopic behavior of different composites where the stiffness of the interfacial layer, $E_I$, was varied between the different composites. In this section we will further explore the mechanism and our analytical models by evaluating the absorbed-, stored-, and dissipated energy densities for composites where $E_I$ is kept constant and $E_0$ is varied. Finite element (FE) simulations were run on multilayered composites with varying stiffness ratio $E_I/E_0 \in [25 - 400]$, while keeping $E_I=1000 \, MPa$, and the yield stress in the interfacial layer constant at $\sigma_{yield} = 50 \, MPa$. Again, the interfacial layer concentration was kept constant at $t/D=0.021$, to ensure wrinkling mode of instability in all the composites evaluated (as seen in Chapter 3).

Figure 6.12a shows the absorbed energy density for the different composites as a function of the macroscopically applied strain, $\varepsilon$. We can see that in this case where $E_I$ is kept constant while ratio $E_I/E_0 \in [25 - 400]$, the energy absorbed increases as the stiffness ratio between the interfacial layer and the matrix decreases (in contrast with our previous case in Figure 6.11). This can be explained by the fact that as the stiffness ratio is decreased, the stiffness of the matrix layers, $E_0$, is increased. Furthermore, since the volume fraction of the matrix is much greater than the interfacial layer, the increase in the stiffness of the matrix layers has a direct positive effect on the energy absorption properties of the composite as well. Moreover, we established earlier that the critical strain causing wrinkling instability (Equation 6.1) increases with a decrease in the stiffness ratio $E_I/E_0$. This means that as the stiffness ratio $E_I/E_0$ is decreased, the pre-buckling region where the energy absorption is growing the fastest, $U_{Total} \sim |\varepsilon|^2$, is also increasing, which leads to higher absorbed energy densities (Figure 6.12b).

The macroscopic true stress-strain behavior of these composites are shown in Figure 6.12c. A clear bifurcation point is seen at the critical strain where instability occurs, and the effective true stress levels out (explained in Chapter 6.4.1). The effective stress is highest for the
composites with the lowest stiffness ratio, $E_{i}/E_{0}$. This is due to the fact that the critical strain causing instability increases for the composites with lower stiffness ratio, and also, the matrix layers will have a higher stiffness causing a greater slope to the stress-strain curve. Again, great agreement is observed between the analytical and FE results.

The macroscopic true stress-strain behavior during a loading-unloading cycle is shown for three of the composites in Figure 6.12d. Due to the local plasticity occurring and dissipating energy, the unloading curve deviate more from the loading curve the lower the stiffness ratio in the composite. At zero macroscopic stress there is a permanent offset showing the residual strain in the composites upon unloading.

To further explore the energy absorption properties for these composites, the energy dissipated and stored by the composite is evaluated separately. Figure 6.13a shows that the dissipated energy density in the interfacial layers for all the composites evaluated; the strain energy density is calculated per unit volume of interfacial layer also calculated per unit volume of the total composite. The analytical and FE results show that the dissipated energy increases with the decrease in the stiffness ratio, $E_{i}/E_{0}$. For the high-stiffness ratio composites, the wrinkling instability occurs at a low $|\varepsilon|$, and as the wrinkling pattern is created in the interfacial layer there are local areas in the interfacial layer where the deformation is large enough for plasticity to occur. The plasticity will occur at the peaks of the wrinkling patterns, and start initially at the edge of the interfacial layer, but as the composite is being compressed further, the plastic zones will move inwards toward the center of the interfacial layer. For the low-stiffness ratio composites the pre-buckling region is bigger, which means that the interfacial layer is compressed uniformly by $|\varepsilon|$ until larger strains. If the local Von-Mises stresses in the interfacial layer reaches the yield stress before wrinkling instability has occurred, the interfacial layer will at once yield and start dissipating energy through its volume, instead of at local specific points. This will lead to a much higher energy dissipation, as seen in Figure 6.13a. Moreover, we can see that for $E_{i}/E_{0}=25$, where the interfacial layer yields fully before wrinkling, the dissipated energy is proportional to $U_{\text{dissipated int.layer}} \sim |\varepsilon|$, which is expected since the material model for the interfacial layer is elastic-perfectly plastic.
Figure 6.12: a) FE and analytical results for the absorbed energy density, for composites with varying stiffness ratio $E_1/E_0 \in [25 \text{ - } 400]$, while keeping $E_1=1000\text{MPa}$, and $t/D=0.021$. The interfacial layer’s yield stress is set to: $\sigma_{\text{yield}} = 50 \text{MPa}$. b) Log-log plot of the absorbed energy density showing the slope change in the pre- and post-buckling region. c) The macroscopic true stress-strain behavior for the different composites. d) The macroscopic true stress-strain behavior for three of the composites under a loading-unloading cycle.
The total stored energy density in the composite as a function of $\bar{\varepsilon}$, is shown in Figure 6.13b, illustrating very good agreement between the analytical and FE results for all different composites. This total stored energy density in the composite is found by adding the contribution from the stored energy density for the matrix layers (Figure 6.13c) and the stored energy density for the interfacial layer (Figure 6.13d) based on their volume fraction:

$$U_{storage} = f_m U_{\text{matrix}}^{storage}(\bar{\varepsilon}) + f_{\text{int.t}} U_{\text{int.t}}^{storage}(\bar{\varepsilon}),$$

where $f_m = k/(2d + t)$ and $f_{\text{int.t}} = t/(2d + t)$ are the volume fraction of the matrix and interfacial layer in the composite, respectively.

The stored energy density of the matrix layers is exactly the strain energy density, since the matrix is a linear elastic material. As discussed in detail in Chapter 5, the stored energy density in the matrix layers of the composite grows quadratically as long as wrinkling has not occurred, $U_{\text{matrix}}^{\text{stored}} \sim |\bar{\varepsilon}|^2$. For post-buckling strains, $|\bar{\varepsilon}| > |\varepsilon_{cr}|$, the wrinkling deformation will increase the matrix strain energy density greatly because of the large local deformations occurring in the matrix layers due to the wrinkling of the interfacial layer (Figure 6.13c). However, the matrix stored energy density is still highest for the composites with the lowest $E_1/E_0$ due to the matrix stiffness being higher and directly effecting the energy storage.

Figure 6.13d shows the stored energy density in the interfacial layer as the composite is being compressed. The interfacial layer stored energy is again quadratic with strain initially. Then as the applied macroscopic strain, $|\bar{\varepsilon}|$, reaches the critical strain causing wrinkling or the critical strain causing yielding in the interfacial layer, the slope of the energy curves changes. For the case where wrinkling occurs before plasticity, the slope of the curve will decrease due to the interfacial layer wrinkling and bending of the interfacial layer being the main source of energy storage. On the other hand, for the case where plasticity occurs before wrinkling instability, we see that the stored energy density in the interfacial layer becomes constant. This can be explained by the fact that the interfacial layer has an elastic-perfectly plastic material model, which means that the elastic strain remains constant once yielding has occurred, resulting in a constant stored energy density as well.
Figure 6.13: FE and analytical results for the a) dissipated energy density, b) total stored energy density, and stored energy density in c) matrix and d) in interfacial layer, for composites with varying stiffness ratio $E_i/E_0 \in [25 - 400]$, while keeping $E_i=1000\text{MPa}$, and $\nu/D=0.021$. The interfacial layer's yield stress is: $\sigma_{\text{yield}} = 50 \text{ MPa}$. 

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Figure 6.13: FE and analytical results for the a) dissipated energy density, b) total stored energy density, and stored energy density in c) matrix and d) in interfacial layer, for composites with varying stiffness ratio $E_i/E_0 \in [25 - 400]$, while keeping $E_i=1000\text{MPa}$, and $\nu/D=0.021$. The interfacial layer’s yield stress is: $\sigma_{\text{yield}} = 50 \text{ MPa}$. 

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Figure 6.13: FE and analytical results for the a) dissipated energy density, b) total stored energy density, and stored energy density in c) matrix and d) in interfacial layer, for composites with varying stiffness ratio $E_i/E_0 \in [25 - 400]$, while keeping $E_i=1000\text{MPa}$, and $\nu/D=0.021$. The interfacial layer’s yield stress is: $\sigma_{\text{yield}} = 50 \text{ MPa}$. 

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6.4.3 Composites with varying geometric features, $t/D$

The effect of the volume fraction of the interfacial layer on the composites’ absorbed-, stored- and dissipated- energy densities is studied by keeping the material parameters $E_i/E_0=200$, $E_0=1\text{MPa}$, and $\sigma_{\text{yield}} = 10\text{ MPa}$ constant, while varying the interfacial layer concentration $f_{\text{int}, t} = t/D\varepsilon \in [0.007, 0.01, 0.014, 0.017]$. Figure 6.14a and b show the analytical and FE results for the absorbed energy density for the geometrically different composites, demonstrating great agreement. The result show that the absorbed energy density is governed by the concentration of the interfacial layers and the $t/D$ ratio; the results are independent of the specific values for $t$ and $D$, as long as $t/D$ is such that wrinkling occurs. Note that for all these composites, wrinkling occurs prior to yielding, and the critical strain at which the interfacial layer will undergo wrinkling instability is independent of geometric features and it only depends on the stiffness ratio, $E_i/E_0$ (Equation 6.1). As a consequence, since the stiffness ratio is kept constant at $E_i/E_0=200$ for all composites, Figure 6.14 and Figure 6.15b-d show bifurcation points in the absorbed energy and stored energy plots that are all at the same macroscopically applied strain of $\bar{\varepsilon} = \bar{\varepsilon}_{cr} = -0.024$. Moreover, the higher the concentration of interfacial layers is in the composite, i.e. the higher $t/D$ ratio, the higher the energy density in the composite throughout the pre- and post-buckling region, as well as the elastic- and plastic- region. This indicates that by only changing the geometric features of the composite, we can control and tailor the energy properties of multilayered composites to get interesting new features.

In addition, the macroscopic true stress-strain behavior of the composites are shown in Figure 6.15c. Again, a clear bifurcation point is seen at the critical strain where instability occurs, $\bar{\varepsilon} = \bar{\varepsilon}_{cr} = -0.02$, and the effective true stress levels out. The plasticity occurring during the post-buckling will reduce the slope of the stress-strain curve, and the main stiffness is provided by the soft matrix layers. Hence, it is clear that plastic composites can be used to mitigate stress transfer, and the effective behavior can be controlled and tuned just by changing the concentration of interfacial layers, the $t/D$ ratio.
The macroscopic true stress-strain behavior during a loading-unloading cycle is shown for two of the composites in Figure 6.14d. Due to the local plasticity occurring and dissipating energy, the unloading curve does not fall on top of the loading curve. This also leads to a permanent offset at zero stress, which show the residual strain in the composites upon unloading.

Figure 6.15 shows in more details the dissipated energy density and the stored energy density which together make the composite’s absorbed energy density. More specifically, Figure 6.15a demonstrates that the dissipated energy density in the interfacial layer per volume of the interfacial layer, \( \frac{u_{\text{int.layer}}^{\text{dissipated}}}{tL} \), only varies little with the change in the geometry of the composite. However, the dissipated energy density in the interfacial layer per total volume of composite, \( \frac{u_{\text{int.layer}}^{\text{dissipated}}}{DL} = \frac{u_{\text{int.layer}}^{\text{dissipated}}}{(2d+t)L} \), shows a more clear deviation between the different composites where the more concentrated composites have a higher dissipated energy density, as expected. Figure 6.15a shows that the dissipated energy is obtained once plasticity occurs locally in the wrinkled interfacial layer, and hence the macroscopic strain causing dissipated energy is in the post-buckling region, \( |\dot{\varepsilon}| > |\varepsilon_{cr}| \). Some deviation is seen between the analytical and FE results for the dissipated energy, which are due to the approximations made in our simplified analytical model as discussed in Chapter 6.3 and Chapter 6.4.1. However, the deviation in the dissipated energy is small when evaluated against the composite’s stored energy density and absorbed energy density, such that this deviations are acceptable.
Figure 6.14: a) FE and analytical results for the absorbed energy density, for composites with varying concentration of interfacial layer \( f_{\text{int}} = t/D \in [0.007, 0.01, 0.014, 0.017] \), while keeping \( E_t/E_0 = 200 \) and \( E_0 = 1 \text{MPa} \). The interfacial layer’s yield stress is set to: \( \sigma_{\text{yield}} = 10 \text{MPa} \), such that \( \sigma_{\text{yield}}/E_t = 0.05 \). The dot-dashed line is the strain energy density for pure compression of a homogenous matrix. b) Log-log plot of the absorbed energy density showing the slope change in the pre- and post-buckling region. c) The macroscopic true stress-strain behavior of the composites under loading with different geometrical features. d) The macroscopic true stress-strain behavior for two of the composites with different geometrical features during a loading-unloading cycle.
The total stored energy density in the composites are shown in Figure 6.15b as a function of the macroscopically applied strain, $\varepsilon$, for the different composites with different, $t/D$. Great agreement is observed between the analytical and FE results. As before, the total stored energy density in the composite is found by adding the contribution from the stored energy density in the matrix layers (Figure 6.15c) and the stored energy density in the interfacial layer (Figure 6.15d) based on their volume fraction: 

$$U_{\text{storage}} = f_m U_{\text{storage, matrix}}(\varepsilon) + f_{\text{int,l}} U_{\text{storage, interfacial}}(\varepsilon),$$

where $f_m = 2d/(2d + t)$ and $f_{\text{int,l}} = t/(2d + t)$ are the volume fraction of the matrix and interfacial layer in the composite, respectively. It can be seen that the composite’s stored energy density is governed by the stored energy density in the matrix layers. Since the matrix is composed of a linear elastic material, the stored energy density of the matrix layers is exactly the strain energy density of the matrix. As discussed in detail in Chapter 5, the stored energy density in the matrix layers of the composite grows quadratically as long as wrinkling has not occurred, $U_{\text{matrix}}^{\text{stored}} \sim |\varepsilon|^2$. For post-buckling strains, $|\varepsilon| > |\varepsilon_{cr}|$, the wrinkling deformation will increase the matrix strain energy density greatly because of the large local deformations occurring in the matrix layers due to the wrinkling of the interfacial layer (Figure 6.15c). Furthermore, the higher $t/D$ is, the higher is also the stored energy density in the matrix, because the area with the highest deformations in the matrix will occupy a greater volume fraction of the total matrix.

Figure 6.15d shows the stored energy density in the interfacial layer as the composite is being compressed. The stored energy density is quadratic with the applied strain for $|\varepsilon| < |\varepsilon_{cr}|$, as expected. As the strain exceed the critical wrinkling strain, the energy stored in the interfacial layer grows much slower due to both the curved deformation of the interfacial layer from the wrinkling pattern, as well as the local plasticity that occurs in the interfacial layer. As the composite is compressed further, the local areas in the interfacial layer where plasticity occurs will grow; since elastic-perfectly plastic material model is chosen for the interfacial layer, the elastic stored energy will remain constant in all areas where yielding and plasticity occurs. Hence, as the applied strain is increased and the areas that yield grows, the graph for the stored energy density in the interfacial layer will flatten out and the slope will approach zero.
Figure 6.15: FE and analytical results for the a) dissipated energy density, b) total stored energy density, and stored energy density in c) matrix and d) in interfacial layer, for composites with varying concentration of interfacial layer, \( f_{\text{int,l}} = l/D \in [0.007, 0.01, 0.014, 0.017] \), while keeping \( E_l/E_0=200 \) and \( E_0=1\,\text{MPa} \). The interfacial layer’s yield stress is: \( \sigma_{\text{yield}} = 10\,\text{MPa} \), such that \( \sigma_{\text{yield}}/E_1 = 0.05 \). The dot-dashed line is the strain energy density for pure compression of a homogenous matrix.
6.5 Energy absorption in networked composites

In this last section, we will extend the concept of having multilayered composites with interfacial layers that have a yield stress (Chapter 6.2-6.4), to networked composites. The composite structures composed of a network of a stiff material embedded in soft matrix, permits the use of several energy absorption mechanisms simultaneously, increasing the overall energy absorption of the structure. Based on material choice, the wrinkling in the stiffer network walls allows for plasticity to occur at all wrinkling peaks, and plastic hinges to occur at the corners.

Finite element simulations were performed studying the energy absorption properties of a networked composite versus that of the pure matrix or empty network (Figure 6.16). The network was modeled as an elastic-perfectly plastic material \( (E_l=1.2\text{GPa}, \sigma_{\text{yield}}=25\text{MPa}) \), while the matrix was set to be linear elastic \( (E_o=10\text{MPa}) \). The geometric parameters are shown in Figure 6.16, and the thickness of the network walls, \( t \), was varied between the simulations, \( t \in [0.3, 0.6, 1.0] \text{mm} \), while ensuring that wrinkling is the mode of instability in the network walls for all the composites evaluated.

Figure 6.17 show the FE simulations results for the absorbed energy density and the effective true stress-true strain behavior for the pure matrix and the empty network being loaded and unloaded. Since the pure matrix is composed of linear elastic material, the absorbed energy density (i.e. the strain energy density) and the effective stress-strain response is trivial; it shows that the stress-strain curve for loading and unloading curve coincide, indicating that energy is only stored under compression of the material, and all the strain energy is released when the matrix is unloaded (Figure 6.17a). Studying the effective stress during the loading and unloading of the pure network (Figure 6.17b), we can see that during loading, at a small applied strain, the network buckles elastically in a very long wave-mode. More specifically, the stress-strain curve shows a high initial effective stiffness for the network, and then a clear
bifurcation point where the network buckles and the slope of the stress-strain curve reduces dramatically (corresponding to the region with the constant slope in the absorbed energy density graph). In the post-buckling region, the amplitude of the long wave-mode deformation of the network will increase, but still at no location does the local Von-Mises stresses exceed the yield stress for plasticity to occur. Consequently, the compression of the empty network is fully elastic up to strains of 8%, which means that the unloading curve coincides with the loading curve. Again, strain energy is only stored during compression and released fully during unloading.

Figure 6.16: Schematic of the structures evaluated through FE simulations; a pure matrix, an empty rectangular network, and a networked composite created by submerging the network in the soft matrix. The thickness of the network walls are varied \( t = [0.3, 0.6, 1.0] \text{mm} \), while the other geometric parameters are kept constant: \( h=11 \text{mm} \) and \( l=30 \text{mm} \). The materials are chosen to be: \( E_1 = 1200 \text{ MPa} \), \( \sigma_{\text{yield}} = 25 \text{ MPa} \) and \( E_0 = 10 \text{ MPa} \).
Figure 6.17: a) The absorbed energy density and the effective true stress-strain curve for a pure matrix under loading and unloading. b) The absorbed energy density and the effective true stress-strain curve for an empty network with $t=0.3\text{mm}$ under loading and unloading. At no strains does the deformations in the network become large enough for plasticity to occur. Hence both the pure matrix and empty network are elastically loaded, only storing energy.

The energy absorption (i.e. energy stored and dissipated) and the effective true stress-strain curve for the networked composites illustrates a very different behavior (Figure 6.18 and 6.19). When the stiff network is embedded in the matrix to create a composite, energy will be both stored and dissipated when loading the composite (as shown in Figure 6.18). As the composite is being compressed, at a higher strain than for the empty network, instability occurs in the network walls of the composite. The macroscopic strain causing instability is higher because the presence of the matrix acts as a reinforcement to the network walls in the composite, and hence delaying the point of instability. Furthermore, the presence of the matrix, leads to the instability mode in the composite being wrinkling of the walls of the network structure. The delay in instability and the wrinkling pattern induces larger local
strains in the interfacial layers and the material goes plastic at the locations with the highest
deformation and hence dissipating energy. Moreover, the energy stored in the composite will
increase due to the large local deformations in the matrix due to the wrinkling deformations.
The energy stored in the composite will consist of the strain energy from the matrix layers, as
well as the energy stored in the plastic networks (which will remain constant once plasticity
has occurred), and the contribution from the matrix and the network on the composite’s total
stored energy is shown in Figure 6.18. As a consequence of all these mechanisms, we can see
that the total absorbed energy density is greater for the composite than for the pure matrix and
the empty network.

The macroscopic true stress-strain curve for loading and unloading of two different
composites, are shown in Figure 6.19. As mentioned above, the presence of the matrix delays
the point of bifurcation, from the instability in the network, as well providing stiffness in the
post-buckling region. The Von-Mises stresses in the network walls are shown for the
composites at 8% applied strain, clearly showing the local areas where yielding and plasticity
has occurred (gray zones), because of the delay in the instability and the due to the larger
deformations from the wrinkling pattern created. Upon unloading of the composite, the stored
energy is released, while the plasticity leads to some permanent deformation of the network
walls, which leads to energy dissipated and residual strains. In addition, we note that the
unloading curves for these composites evaluated are very steep (especially in the case where
the network thickness wall is greater and a higher volume fraction of network in the
composite). This can be explained by the fact that for these composites, due to the matrix
delaying the point of instability, the vertical network walls reach the critical load which causes
yielding and plasticity just before the critical strain causing wrinkling occurs. Hence, the
vertical walls of the network, which are providing large amount of stiffness, yield before
wrinkling, and are hence fully plastic as the applied load is increased. Consequently, the
unloading curve of the composite is governed by the linear elastic unloading of these fully
yielded network walls, rather than the geometrical changed due while un-wrinkling the
network walls.
Figure 6.18: The dissipated energy density, stored energy density (showing the contribution form the network and the matrix), as well as the total absorbed energy density for networked composites under loading up to $\varepsilon_{\text{max}} = -0.08$. Yielding occurs in the network walls which will lead to dissipated energy, and wrinkling pattern increases the energy stored in the matrix layers. Hence, the absorbed energy density is enhanced as the composites are compressed. Composite with a) network thickness, $t=0.3\text{mm}$, b) network thickness, $t=1.0\text{mm}$.
Figure 6.19: The effective true stress-strain curve for networked composite under loading and unloading up to different max. strains, $\bar{\varepsilon}_{\text{max}} = [-4\%, -6\% \text{ or } -8\%]$. Yielding occurs in the network walls which will lead to dissipated energy. Hence, energy is stored and dissipated as the networked composites are compressed. Composite with: a) network thickness, $t=0.3\text{mm}$, b) network thickness, $t=1.0\text{mm}$.
We calculate the energy absorbed during the loading of the pure matrix, the empty network, and the different network composites, as well as the energy dissipated as the composites are loaded and unloaded. The absorbed energy density is shown in Table 6.1, while the dissipated energy density is shown in Table 6.2. For the composites we have evaluated, Table 6.1 and 6.2 clearly shows that by creating a networked composite we are significantly enhancing both the absorbed energy density and the dissipated energy density of the structures. The stored energy density increases by a factor of 15 for the composite compared with the empty network. Simultaneously, dissipated energy is introduced to the structures (dissipation increases from zero to maximum 0.11 Joules/mm$^3$).

Finally, by studying the effective stress-strain curves in Figure 6.19, we can see that as the composites are unloaded back to zero stress, the network returns to a geometric configuration that show little permanent deformations, and the unloading curves exhibit small nonlinear effects. This indicates a potential for the system to be partially recoverable. In addition, if the matrix that the network is embedded in is changed to a material with viscoelastic behavior, we can also achieve viscoelastic dissipation during a cyclic load, which means that the energy absorption and dissipation in the composites can be increased.
### Table 6.1: Total Absorbed Energy Density [Joules/mm$^3$]

<table>
<thead>
<tr>
<th>Max applied strain</th>
<th>$\bar{\epsilon}_{\text{max}} = -4%$</th>
<th>$\bar{\epsilon}_{\text{max}} = -6%$</th>
<th>$\bar{\epsilon}_{\text{max}} = -8%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure matrix</td>
<td>0.007</td>
<td>0.017</td>
<td>0.038</td>
</tr>
<tr>
<td>Empty network</td>
<td>0.002</td>
<td>0.004</td>
<td>0.006</td>
</tr>
<tr>
<td>with $t=0.3\text{mm}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Networked composite with $t = 0.3 \text{ mm}$</td>
<td>0.030</td>
<td>0.056</td>
<td>0.088</td>
</tr>
<tr>
<td>Networked composite with $t = 0.6 \text{ mm}$</td>
<td>0.051</td>
<td>0.092</td>
<td>0.137</td>
</tr>
<tr>
<td>Networked composite with $t = 1 \text{ mm}$</td>
<td>0.078</td>
<td>0.137</td>
<td>0.186</td>
</tr>
</tbody>
</table>

### Table 6.2: Total Dissipated Energy Density [Joules/mm$^3$]

<table>
<thead>
<tr>
<th>Max applied strain</th>
<th>$\bar{\epsilon}_{\text{max}} = -4%$</th>
<th>$\bar{\epsilon}_{\text{max}} = -6%$</th>
<th>$\bar{\epsilon}_{\text{max}} = -8%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure matrix</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Empty network</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>with $t=0.3\text{mm}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Networked composite with $t = 0.3 \text{ mm}$</td>
<td>0.012</td>
<td>0.024</td>
<td>0.034</td>
</tr>
<tr>
<td>Networked composite with $t = 0.6 \text{ mm}$</td>
<td>0.029</td>
<td>0.060</td>
<td>0.093</td>
</tr>
<tr>
<td>Networked composite with $t = 1 \text{ mm}$</td>
<td>0.041</td>
<td>0.080</td>
<td>0.109</td>
</tr>
</tbody>
</table>
6.6 Conclusion

In this chapter we have studied the compression of a multilayered composite where the thin interfacial layers are composed by an elastic-perfectly plastic material. A phase diagram was developed to help designers in predicting whether the interfacial layer in the composite will first experience yielding or undergo wrinkling instability, based on only the material properties of the composite material.

General analytical models where developed to calculate local stresses and strains due to plasticity occurring in the interfacial layer of the composite under compression, by purposefully using the wrinkling as the mode of instability. Furthermore, analytical models were developed for the dissipated energy density, the stored energy density, and the absorbed energy density for these multilayered composites. In additions, the macroscopic behavior of the composites were also investigated by considering the macroscopic axial stress-strain behavior. The analytical models were compared with finite element simulations and showed good agreement for strain up to 4.5 times the strain that causes wrinkling instability to occur in the interfacial layers.

A set of different composites were studied in detail where the material parameters and the geometrical parameters, i.e. the concentration of the interfacial layers, were changed. By changing any of these parameters, the energy densities changed significantly, which indicates the power of this mechanism to be versatile and useful for many engineering applications. Moreover, by purposefully designing and using the wrinkling instability as the mode of deformation in these composites, the response and the energy densities in the structures are predicted well by our method. Hence, a designer can achieve a desired amount of stored energy density or dissipated energy density by changing any of the following parameters: $E_I/E_0$, $E_I$ or $E_0$, $\sigma_{yield}$, and $t/D$. 

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The absorbed energy density and dissipated energy density of networked composites under compression were also evaluated. It was shown that by submerging an empty plastic network in a soft matrix, the energy absorption is enhanced greatly, while energy dissipation is introduced to the structure. This is due to the presence of the matrix in the matrix acting as a reinforcement to the network walls delaying the point of instability, causing wrinkling to be the instability mode in the interfacial layer, as well as providing stiffness in the post-buckling region.

Finally, the macroscopic behavior of the composites were shown to be greatly altered between the pre- and post-buckling of the composites. More specifically, the macroscopic stress was shown to level out once wrinkling and/or plasticity occurs. It was shown that by changing the material or geometrical parameters the effective behavior of these plastic composites can be controlled and tuned, and this method can actively be used for mitigating load transfer.

Consequently, we have in this chapter presented an efficient new method for absorbing and dissipating energy that exploits the wrinkling instability. Our analytical models can be used to predict the composites’ energy densities, and offers the designers a large design space and combination options in order to achieve the desired energy absorption or dissipation for their specific application.
Chapter 7

Design of tuneable composites using wrinkling instability

7.1 Introduction

The methods and results presented in earlier chapter of this thesis clearly suggest that by using the wrinkling instability mechanism actively, we can alter and analytically predict the effective behavior of wrinkling composites, and hence also control and design the energy storage, the effective bilinear stress-strain behavior, and effective stiffness of elastic composites. Furthermore, by only varying the material- and geometrical- combinations in these composites, we can create composites where we tune and control its effective properties based on inducing wrinkling with loading. Additionally, by using softer materials such as rubbers, the wrinkling mechanism can be an elastic process, and thus the change in the effective properties can be reversible and switchable. In contrast, by using a plastic material for the interfacial layer, we have demonstrated that the energy absorption can be adjusted by introducing a new energy dissipation mechanism through the wrinkling of interfacial layers.

Recently, tunable and active materials have emerged as a promising class of materials which exhibit changes in properties or behavior on-demand based on the external environment, allowing the creation of interesting tunable multifunctional materials [10,11,12,13,14,15,16]. The tunability can be governed by the inherent material behavior [17,18,19,20] as well as by proper geometric design of the constituent materials [21,22,23,24,25,26]. Likewise, the mechanism and concepts we have presented in this thesis, can be used to create multifunctional composites exhibiting tunable energy storage and dissipation, stress mitigation, and also a switchable effective stiffness.
In this chapter we will combine the results from earlier chapters to create design guides for the stored energy density in elastic composites, as well as the absorbed and dissipated energy density in composites with plastic interfacial layer. In addition, we will demonstrate the ability of creating tuneable composites with interesting new macroscopic behavior and properties, by deploying the wrinkling instability.

7.2 Design plots for energy storage in elastic multilayered composites

The analytical results presented in this thesis (Chapter 5) can be used to create design diagrams for the total strain energy density, i.e. the energy storage capacity, in elastic composites. The total strain energy density is dependent on several variables, which gives the designers great flexibility in their design to achieve the desired energy stored. The variables affecting the composite’s strain energy density are show in Figure 7.1a: the stiffness ratio between the stiff interfacial layer and the soft matrix, $E_i/E_0$, the concentration of interfacial layers, $t/D$, and the applied macroscopic compressive strain, $\bar{\varepsilon}$.

Strain energy density design plots are created to simplify the process of determining the strain energy stored by a composite based on its many design parameters. The contour plot shown in Figure 7.1b provides the stored strain energy density, $U_{\text{Total}}$, as a function of $E_i/E_0$ and $\bar{\varepsilon}$, at a given $t/D=0.013$. This design contour plot shows that the strain energy density increases with both increased $E_i/E_0$ and $\bar{\varepsilon}$.

In order to show the effect of changing the volume fraction of interfacial layer, $f_{\text{int}}=t/D$, on $U_{\text{Total}}$, we plot in Figure 7.1c first the contour lines for $U_{\text{Total}} = [0.004, 0.008, 0.012, 0.016, 0.020]$ Joules/mm$^3$ for when $t/D=0.013$ (solid line), corresponding to Figure 7.1b. Then the same contour lines are added for composites with $t/D= 0.010$ (dashed line) and $t/D=0.02$.
(dash-dot line). Figure 7.1c shows that as $t/D$ decreases (i.e. more dilute composites), the contour lines shift up, which means that the strain energy density value changes later as $E_i/E_0$ and $\bar{\varepsilon}$ are increased. The upward shift for the contour lines for more dilute composites mean that the strain energy density of the composites increase more slowly with $E_i/E_0$ and $\bar{\varepsilon}$ the more dilute the composite is. This is because the strain energy density is lower for more dilute composites where the local large deformations in the neighborhood of the wrinkles does not permeate the matrix layer. Moreover, we can conclude that the effect of changing the volume fraction of interfacial layers, $f_{int} = t/D$, on the strain energy density, is larger as $E_i/E_0$ or $\bar{\varepsilon}$ is increased. Figure 7.1c also shows that as $E_i/E_0 \to 1$, the different contour lines for the different $t/D$ converge to one line, which is due to the energy density of the composite approaching the energy density of a homogeneous material compressed up to $\bar{\varepsilon}$.
Figure 7.1: Design guidelines plots for the stored energy density in multilayered composites with $E_0=1\text{MPa}$; a) Schematic showing the variables. b) The stored energy density, $U_{Total} = \frac{U_{total}}{V_{Total}}$, in the composites for different $E_1/E_0$ and $\varepsilon$, for $t/D = 0.013$. c) The effect of changing $t/D$ on the contour lines for the stored energy density. Higher concentration of interfacial layer ($t/D$ higher) leads to a downwards shift in the contour lines, indicating a more rapid increase in the stored energy density as $E_1/E_0$ or $\varepsilon$ are increased.
7.3 Design of tunable elastic materials

Another advantageous property of the method presented in this thesis, is that a just by changing the elastic composite’s material and geometric features, we can obtain: a specific desired strain energy density, a desired effective multi-linear elastic stress-strain behavior, and/or a switchable tangent stiffness. The properties and behavior of the composite can be engineered and altered based on the following variables: the individual material stiffness and the stiffness ratio $E_1/E_0$, the concentration of the interfacial layers, $t/D$, and the applied macroscopically compressive loads, $\bar{\varepsilon}$, as illustrated in Figure 7.1a. Consequently, our method gives designers the freedom to pick among several materials based on their resources and the specific application, which makes this method more versatile in its applications.

In this chapter we will show how different multilayered composites can be combined to achieve interesting new effective properties. Figure 7.2 and Figure 7.3 evaluates a multilayered composite with a RVE consisting of two interfacial layers with stiffness $E_1$ and $E_2$, Poisson ratio $\nu_1$ and $\nu_2$, thickness $t_1$ and $t_2$, and distance between the layers of $D_1$ and $D_2$, embedded in a matrix with stiffness $E_0$ and Poisson ratio $\nu_0$. Figure 7.3a shows the composite strain energy density, the effective true stress-strain curve, and the composite’s effective stiffness for the case where the geometric features are kept constant, $t_1/D_1 = t_2/D_2 = 0.02$, but the material properties, $E_1$ and $E_2$, are varied. The dashed curves show the results for different composites where $E_1/E_0 = E_2/E_0$, while the solid lines show the result for when the dashes curves are combined such that the composites have different interfacial layers: $E_1/E_0 \neq E_2/E_0$. By having different material properties for the interfacial layers we see that we can tune the stain energy density as a function the applied strain. The effective stress is also modified, and now exhibit several bifurcation points. The effective stiffness of the composite also shows very interesting features; by combining the materials the effective stiffness demonstrate several steps in its value as the applied load is increased. Hence, by creating multi-material and multilayered composites, we can control and mitigate the effective stress level of the composite, as well as
tuning its effective stiffness as a function of the applied strain, $\bar{\varepsilon}$, creating a multi-linear elastic material.

Similarly, the geometric features of the composite can be changed to control and tune the effective behavior of the composite. Figure 7.3b shows how the composite's strain energy density, effective true stress-strain, and effective stiffness changes based on the distance between the interfacial layers. The dashed lines show the results for when the geometry for the two layers are the same such that $\frac{t_1}{D_1} = \frac{t_2}{D_2}$ and equal to 0.02 or 0.005. The solid lines show that by combining these results, such that $\frac{t_1}{D_1} \neq \frac{t_2}{D_2}$, the strain energy density, effective stress-strain behavior, and stiffness can be tuned. Furthermore, as long as the stiffness of the two interfacial layers are the same, $E_1=E_2=1000$ MPa, the bifurcation point in the effective true stress-strain and the strain at which there is a drop in the effective stiffness remains unchanged. But by having different materials so that the stiffness of the two interfacial layers are not the same, $E_1=1000$ MPa and $E_2=2000$MPa, we can see that we can add new bifurcation points to the effective stress-strain behavior, and new steps in the effective stiffness.

**Figure 7.2:** Designing multi-material and multilayered composites with new and tunable effective behavior. For all cases the following are kept constant: $E_0=5MPa$, $t_1=t_2=0.5mm$ and $L=60mm$. The composite's strain energy density, effective true stress-strain behavior, and effective stiffness as function of applied strain, $\bar{\varepsilon}$, are shown in Figure 7.3.
Figure 7.3: Designing multi-material and multilayered composites with new and tunable effective behavior (schematic in Figure 7.2). For all cases the following are kept constant: $E_0=5\text{MPa}$, $t_1=t_2=0.5\text{mm}$ and $L=60\text{mm}$. The composite’s strain energy density, effective true stress-strain curve, and effective stiffness as function of applied strain, $\varepsilon$, for when: a) Constant geometry, $t_1/D_1 = t_2/D_2 = 0.02$, but changing materials, $E_1 \neq E_2$, such that $E_1/E_0$ and $E_2/E_0 \in [100, 200, 400]$; b) Changing geometry, $D_1 \neq D_2$, and concentration of interfacial layers: $t_1/D_1$ and $t_2/D_2 \in [0.02, 0.005]$, for when $E_1/E_0$ and $E_2/E_0 \in [100, 200, 400]$. 
7.4 Design plots for energy absorption and dissipation in plastic multilayered composites

The analytical results presented in Chapter 6 can be used to create design diagrams for the total absorbed energy density and the total dissipated energy density, in composites with an elastic-perfectly plastic interfacial layer (as done in Chapter 7.2 for fully elastic composites). The total absorbed- and dissipated- energy density is dependent on several variables, which gives the designers great flexibility in their design to achieve the desired energy stored or dissipated. The variables affecting the composite’s energy densities are show in Figure 7.4a: the yield stress of the interfacial layer, $\sigma_{\text{yield}}$, the stiffness ratio between the stiff interfacial layer and the soft matrix, $E_1/E_0$, the concentration of interfacial layers, $t/D$, and the applied macroscopic compressive strain, $\varepsilon$.

Design plots are created for the composite’s absorbed energy density, as well as the dissipated energy density, to simplify the process of determining the energy stored and dissipated by a composite based on its many design parameters. The contour plot shown in Figure 7.4b provides the absorbed strain energy density, $U_{\text{Total}}$, as a function of $E_1/E_0$ and $\varepsilon$, at a given $t/D=0.013$ and $\sigma_{\text{yield}} = 10\, MPa$. Similarly, the contour plot in Figure 7.4c provides the dissipated strain energy density, $U_{\text{Dissipated}}$, for the composites. These design contour plot shows that the absorbed and dissipated energy density increases with both increased $E_1/E_0$ and $\varepsilon$. However, the effect of the stiffness ratio, $E_1/E_0$, on the energy absorption is small. This is because the absorbed energy density is governed by the strain energy contribution from the matrix, and for all these composites the stiffness of the matrix $E_0$ was kept constant. In contrast, we see that the dissipated energy density is much more influenced by an increase in stiffness ratio, $E_1/E_0$; this is due to the fact that the dissipated energy is only dependent on the plasticity occurring in the interfacial layer, and hence higher $E_1$ has a direct increasing effect on the energy dissipation.
Figure 7.4: a) Schematic showing the variables of the composite with elastic-perfectly plastic interfacial layer. b) Design guidelines plots for the absorbed energy density in multilayered composites with varying $E_1/E_0$ and $\bar{\varepsilon}$, while keeping $E_0=1MPa$, $\sigma_{yield} = 10MPa$ and $t/D= 0.013$; c) Design guidelines plots for the dissipated energy density in multilayered composites with varying different $E_1/E_0$ and $\bar{\varepsilon}$, while keeping $E_0=1MPa$, $\sigma_{yield} = 10MPa$ and $t/D= 0.013$. 

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Figure 7.5a and b shows the change in the design plots for the absorbed energy density and dissipated energy density of a composite, when the yield stress in the interfacial layer, $\sigma_{yield}$, is changed (the interfacial layer volume fraction is kept constant at $t/D=0.013$). It is obvious that as the yield stress is lowered, there is more areas in the interfacial layer that undergoes plasticity for a given compression load. Hence, the dissipation energy density is higher for lower yield stresses. On the other hand, since the interfacial layer is modeled as an elastic-perfectly plastic material, once a material element has yielded its local stresses will remain constant. This means that the total absorbed energy in the composite levels out as more areas in the interfacial layer yields. As a consequence, Figure 7.5a shows that the absorbed energy density decreases when the yield stress in the interfacial layer is decreased.

We also investigate the effect of the volume fraction of interfacial layer, $f_{int,t}=t/D$, on the total absorbed and dissipated energy in the composite. Figure 7.5c and d shows the change in the design plots for the absorbed- and dissipated- energy density for a composite where $t/D$ is varied, while the yield stress of the interfacial layer is kept constant ($\sigma_{yield} = 10MPa$). As expected, these figures show us that both the absorbed energy density and the dissipated energy density in a composite is decreased as the volume fraction of the interfacial layer is lowered. Moreover, since the absorbed energy in the composite has a large contribution form the matrix strain energy density as well, the absorbed energy density is effected less by the change in $t/D$ than the dissipated energy density (which is directly related to the volume fraction of interfacial layer in the composite).
**Figure 7.5:** a) The change in absorbed energy density as $\sigma_{\text{yield}}$ is varied (at constant $t/D$); b) The change in dissipated energy density as $\sigma_{\text{yield}}$ is varied (at constant $t/D$); c) The change in absorbed energy density as $t/D$ is varied (at constant $\sigma_{\text{yield}}$); d) The change in dissipated energy density as $t/D$ is varied (at constant $\sigma_{\text{yield}}$).
7.5 Design of energy absorption in plastic multilayered composites

As mentioned earlier in Chapter 7.3, an important advantageous property of the method presented in this thesis, is that a just by changing the composite’s material and geometric features, we can obtain a specific desired effective behavior and property. This also applied to the energy absorption, storage or energy dissipation of a multilayered composite with elastic-perfectly plastic interfacial layers. The energy density behavior of the composite can be engineered and altered based on the following variables: the yield stress of the interfacial layer, \( \sigma_{\text{yield}} \), the individual material stiffness and the stiffness ratio, \( E_i/E_0 \), the concentration of interfacial layers, \( t/D \), and the applied macroscopic compressive strain, \( \varepsilon \) (variables shown in in Figure 7.4a).

Here we will show how different multilayered composites can be combined to achieve interesting new energy absorption, storage and dissipation properties. Figure 7.6 and Figure 7.7 considers a multilayered composite with a RVE consisting of two interfacial layers with yield stress \( \sigma_{y,1} \) and \( \sigma_{y,2} \), stiffness \( E_1 \) and \( E_2 \), Poisson ratio \( v_1 \) and \( v_2 \), thickness \( t_1 \) and \( t_2 \), and distance between the layers of \( D_1 \) and \( D_2 \), embedded in a matrix with stiffness \( E_0 \) and Poisson ratio \( v_0 \). Figure 7.7a shows the composite absorbed energy density, dissipated energy density and stored energy density for the case where the geometric features are kept constant, \( \frac{t_1}{D_1} = \frac{t_2}{D_2} = 0.02 \), but the material properties, \( (E_1, \sigma_{y,1}) \) and \( (E_2, \sigma_{y,2}) \) are varied. The dashed curves show the results for different composites where \( \frac{E_1}{E_0} = \frac{E_2}{E_0} \), while the solid lines show the result for when the dashes curves are combined such that the composites have different interfacial layers: \( \frac{E_1}{E_0} \neq \frac{E_2}{E_0} \). By having different material properties for the interfacial layers we see that we can tune the energy densities as a function the applied strain in very interesting new ways. More specifically, for example by studying the dissipated energy densities for these composites, we can see that we are able to control the strain at which energy dissipation starts, as well as the evolution as the applied strain, \( \varepsilon \), increases.
Similarly, the geometric features of the composite can be changed to control and tune the effective behavior of the composite. Figure 7.7b shows how the composite’s absorbed-, dissipated-, and stored-energy density changes based the volume fraction of the interfacial layer. The dashed lines show the results for when the geometry and volume fraction for the two layers are the same such that $\frac{t_1}{D_1} = \frac{t_2}{D_2}$ and equal to 0.02 or 0.005. The solid lines show that by combining these results, such that $\frac{t_1}{D_1} \neq \frac{t_2}{D_2}$, the strain energy densities can be tuned. Hence, by creating multi-material and multilayered composites that undergo wrinkling, we can control and design the absorbed- and dissipated energy density of the composite.

**Figure 7.6:** Designing multi-material and multilayered composites with new energy absorption and dissipation properties. For all cases the following are kept constant: $E_0=10MPa$, $t_1=t_2=0.5mm$ and $L=60mm$. The composite’s absorbed-, dissipated- and stored-energy density as function of applied strain, $\tilde{\varepsilon}$, are shown in Figure 7.7.
Figure 7.7: Designing composites with new and tunable energy density behavior (schematic in Figure 7.6). The following are kept constant: $E_0=10\text{ MPa}$, $t_1=t_2=0.5\text{ mm}$ and $L=60\text{ mm}$. The composite’s absorbed-, dissipated-, and stored- energy density as function of applied strain, $\bar{\varepsilon}$, for when: a) Constant geometry, $t_1/D_1 = t_2/D_2 = 0.02$, but changing material properties of the interfacial layer: Case 1-3 = [$E_1=E_2$ and $\sigma_y,1=\sigma_y,2$], while Case 4 = [$E_1\neq E_2$ and $\sigma_y,1=\sigma_y,2$] and Case 5 = [$E_1\neq E_2$ and $\sigma_y,1\neq\sigma_y,2$]. b) Changing geometry and material: Case 1-3 = [$D_1=D_2$, $E_1=E_2$, $\sigma_y,1=\sigma_y,2$], while Case 4 = [$D_1\neq D_2$ and $E_1=E_2$, $\sigma_y,1=\sigma_y,2$] and Case 5 = [$D_1\neq D_2$ and $E_1\neq E_2$ and $\sigma_y,1\neq\sigma_y,2$].
7.6 Conclusion

In this chapter, we have studied and explored further the opportunities and advantages given to a designer by using our method of creating multilayered composites where instability induced transformation of the interfacial layer is actively deployed to achieve interesting new effective composite properties. The effective composite behavior can be predicted through our analytical models in Chapter 5 and 6, and the energy densities, effective stress-strain behavior, and the effective stiffness depend on several material and geometrical variables. This gives the designers great flexibility in their design to achieve the desired energy stored, energy dissipated, effective stress level, or effective composite stiffness.

Efficient strain energy density design plots were created for elastic multilayered composites to simplify the process of determining the strain energy stored by a composite based on its many design parameters. For multilayered composites with a plastic interfacial layer, we have shown examples of design plots for the absorbed energy density and the dissipated energy density in the composites as a function of the material- and geometrical parameters, as well as the macroscopic applied strain.

In addition, we have demonstrated that we can use our method to design tunable materials. For elastic composites, the strain energy density, the effective stress-strain behavior, and the effective stiffness of the composite were tuned and altered to exhibit new desirable features, just by varying the material and geometrical combinations. By creating multi-material and multilayered composites, we can control and mitigate the effective stress level of the composite, as well as tuning its effective stiffness as a function of the applied strain, $\bar{\varepsilon}$, creating a multi-linear elastic material. When plasticity is introduced in the composites, by having an elastic-perfectly plastic interfacial layer, we can design the energy dissipated and absorbed by the composite. It was established that by having different material and geometrical combinations for the interfacial layers, we can tune the energy densities as a function of the applied strain in very interesting new ways, such as the ability to control the applied strain at which energy dissipation starts, as well as its evolution with the applied strain, $\bar{\varepsilon}$. 
Chapter 8
Piezoelectric wrinkling composites

8.1 Introduction

Piezoelectric materials are useful for many engineering applications due to their ability to convert mechanical deformation or load to electricity, as well as the inverse; namely converting electrical energy to mechanical energy [66]. This interesting property of the piezoelectric materials has been directly deployed to create: actuators, sensors, transducers, energy harvesters, motors, speakers, microphones, vibration controllers, ultrasounds, and for other medical applications. In this chapter we will design and add piezoelectric material patches in selected parts of the interfacial layers in a multilayered composite, and use the wrinkling instability to generate electricity such that the composite can be used for energy harvesting and/or sensor applications.

When a piezoelectric polymer, such as PVDF, is subjected to a mechanical load, positive and negative charges develop on the material’s surface. This can be used to achieve current flow by connecting the two surfaces of the material together through a circuit. This ability of the material to convert mechanical energy into electrical energy is known as the direct piezoelectric effect. Conversely, a piezoelectric polymer will deform under an applied electric field. Hence, the inverse piezoelectric effect is the ability of the material to convert electrical energy into mechanical energy. An active area of research is investigating how to generate and harvest energy through the direct piezoelectric effect. The common method is to create uni-morph or bi-morph cantilever beams and vibrate the end of these structures through translations or rotations [68, 78, 105]. As the frequency of vibration is varied, different modes of deformation are achieved along the length of the cantilever beam. The higher modes of
deformation will cause larger local deformation in the piezoelectric layers of the cantilever beams generating electricity through the direct piezoelectric effect. Hence, these structures are used as energy harvesters. Extensive research has been conducted exploring the various design parameters of these energy harvester; e.g. the type of piezoelectric material, the geometry, the loading and boundary conditions, the circuit connections, and the optimal vibration frequency \[68,81,82\] [83,84,85,86]. As mentioned in Chapter 1.3.4, the output response of these energy harvesters are very dependent on the vibration frequency, and optimal outputs are only achieved for a small ranges in the frequency. This can limit the application of these uni-morph and bi-morph structures in many settings, and also reduce their total energy harvesting potentials.

Figure 8.1: Schematic showing that patches of piezoelectric material can be placed at the top and bottom of the interfacial layer at the locations of the wrinkling peaks. When wrinkling occurs in the interfacial layers of a composite compressed at a macroscopic strain of \(|\varepsilon| > |\varepsilon_{cr}|\), the piezoelectric patches will either be in compression (blue) or tension (pink) based on their location along the thickness of the interfacial layer.

In this chapter we will introduce a new energy harvesting mechanism by strategically placing patches of piezoelectric material in the instability-induced wrinkling composites to generate electricity. As shown in Figure 8.1, the piezoelectric patches can be designed to be placed at the peak locations of the wrinkling pattern created in the interfacial layers of a composite, where the local deformations are the greatest. We propose this as a new energy harvesting mechanism. In addition, by placing the patches of the piezoelectric material in the middle of the thickness of the interfacial layer, we propose a new mechanism for creating a switch.
The addition of the piezoelectric patches in the interfacial layer can impact the bending stiffness of the interfacial layer locally in that area. If the piezoelectric material is less stiff than the interfacial layer that it’s replacing, then the highest curvature will indeed occur in the piezoelectric patches. In contrast, if the piezoelectric material is stiffer than the interfacial layer that it’s replacing, then the regions along the interfacial layer in between the piezoelectric patches will experience the bending, and the regions with the piezoelectric patches will essentially remain straight and just rotate (i.e. there will not be much local strain developed in the piezoelectric patches). Hence, if the piezoelectric material’s stiffness is different than the interfacial layer’s stiffness, then the waveform of the wrinkle will no longer be sinusoidal. As a consequence, in this chapter we will always match the stiffness of the interfacial layer to be the same as the piezoelectric material, such that the well-studied wrinkling instability will be the mode of instability.

This chapter will begin with an analytical derivation of the electrical energy created in composites with wrinkling piezoelectric interfacial layers, and then results will be presented for the energy harvesting and switching applications.
8.2 Analytical model for direct piezoelectric effect in wrinkling composites

We have in earlier chapters of this thesis established that for a dilute multilayered composite, a wrinkling instability will occur in the interfacial layers of the composite when the macroscopically applied compressive strain, $|\varepsilon|$, exceeds a critical strain, $|\varepsilon| > |\varepsilon_{cr}|$, which is determined by the material combinations in the composite. Under plane strain conditions, the critical macroscopic strain at which the wrinkling instability occurs, $\varepsilon_{cr}$, and the initial wavelength of the wrinkling pattern, $\lambda_{cr}$, are given by:

$$\varepsilon_{cr} = -2.08 \left( \frac{3 - 4v_0}{(1 - v_0)^2} \right)^{\frac{1}{3}} \left( \frac{E_1}{E_0} \right)^{\frac{2}{3}}$$

$$\lambda_{cr} = t \cdot 2.18 \left( \frac{3 - 4v_0}{(1 - v_0)^2} \right)^{\frac{1}{3}} \left( \frac{E_1}{E_0} \right)^{\frac{1}{3}}$$

where $E_1$ and $E_0$ are the Young’s modulus of the interfacial layer and the matrix, $v_1$ and $v_0$ are the Poisson’s ratios, and $t$ is the thickness of the interfacial layer.

In addition, in Chapter 4 we established the local strains and stresses developed at any point in the composites as it is macroscopically compressed though the pre-buckling, the point of instability, and into the post-buckling region. The local strains in the interfacial layer at any location $(x, h)$, for any given post-buckling applied strain, $|\varepsilon| > |\varepsilon_{cr}|$, are given by:

$$\varepsilon^\text{int.layer}_{x}(x, h, \varepsilon) = \varepsilon_{cr} - h \cdot \frac{4\pi}{\lambda(\varepsilon)} \sqrt{|\varepsilon| - |\varepsilon_{cr}|} \cdot \sin\left(\frac{2\pi x}{\lambda(\varepsilon)}\right)$$

$$\varepsilon^\text{int.layer}_{y}(x, h, \varepsilon) = -\frac{v_1}{1 - v_1} \cdot \varepsilon^\text{int.layer}_{x}(x, h, \varepsilon)$$

$$\varepsilon^\text{int.layer}_{xy}(x, h, \varepsilon) \approx 0, \varepsilon^\text{int.layer}_{z}(x, h, \varepsilon) = \varepsilon^\text{int.layer}_{xz}(x, h, \varepsilon) = \varepsilon^\text{int.layer}_{yz}(x, h, \varepsilon) = 0$$

where $h$ is the distance away from the center line of the interfacial layer, $x$ is the location along the interfacial layer, and $\lambda(\varepsilon)$ is the post-buckling wavelength for the wrinkling pattern (Equation 4.3b).

Using the expressions for the local strains in the interfacial layer, we can proceed to find the electric energy generated if the interfacial layers contain regions of piezoelectric material,
such as PVDF. Figure 8.2a shows schematically the wrinkled interfacial layer of a multilayered composite, where the patches of piezoelectric material are strategically added at the locations of the interfacial layer with the highest local strains. In addition, the variables defining the properties of the interfacial layer are illustrated.

Figure 8.2: a) Schematic showing that patches of piezoelectric material can be placed at the top and bottom of the interfacial layer at the locations of the wrinkling peaks. The variables defining the interfacial layer and the piezo patches are also shown. The piezoelectric material can be aligned such that the poling is in any direction {x, y or z}. b) For piezoelectric material it is common to set a local coordinate system {1,2,3} such that the poling direction is out of the plane, i.e. in the 3-direction.

Next, we use the constitutive relations for piezoelectric materials, that couple deformation and electric field, to capture the direct piezoelectric effect occurring in the patches of piezoelectric material in the interfacial layers. The linear piezoelectric constitutive relations, which can be derived from thermodynamic principles, couple linear elastic relations with linear dielectric relations through the piezoelectric tensor \([106,107,108]\). There are several forms to the piezoelectric constitutive relation, and below we show two of the most common forms, Form 1 and Form 2, used often for the direct piezoelectric effect:
Form 1: Applied electric field and stress

\[
\begin{bmatrix} D \\ S \end{bmatrix} = \begin{bmatrix} \varepsilon^T & d \\ -d^T & s^E \end{bmatrix} \begin{bmatrix} E \\ T \end{bmatrix}
\]  
(8.3)

Form 2: Applied electric field and strain

\[
\begin{bmatrix} D \\ T \end{bmatrix} = \begin{bmatrix} \varepsilon^S & e \\ -e^T & c^E \end{bmatrix} \begin{bmatrix} E \\ S \end{bmatrix}
\]  
(8.4)

where:

\[
c^E = (s^E)^{-1}, \quad e = dc^E, \quad \varepsilon^S = \varepsilon^T - dc^Ed^T
\]

Furthermore, the table below shows examples for typical values for the piezoelectric coefficients and electric constants for a PZT and PVDF piezoelectric material [106]:

<table>
<thead>
<tr>
<th>Young's modulus (GPa)</th>
<th>PZT-5H</th>
<th>PVDF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>71</td>
<td>4-6</td>
</tr>
<tr>
<td>(d_{31}) (pC/N)</td>
<td>-274</td>
<td>18-24</td>
</tr>
<tr>
<td>(d_{32}) (pC/N)</td>
<td>-274</td>
<td>2.5-3</td>
</tr>
<tr>
<td>(d_{33}) (pC/N)</td>
<td>593</td>
<td>-33</td>
</tr>
<tr>
<td>(\varepsilon_{33}) (nF/m)</td>
<td>30.1</td>
<td>0.106</td>
</tr>
</tbody>
</table>

Considering the constitutive equations (Equation 8.3 and 8.4) for our composites with wrinkled interfacial layers that have piezoelectric patches, there are several conditions that must be incorporated into the constitutive equations:

1) There is no applied electric field in our system, such that: \(E = 0\)

2) There are local strains developed in the piezoelectric material in the interfacial layer, which is due to the wrinkling of the interfacial layer. The local strains are known at any point, including in the piezoelectric material, i.e.: \(S = \text{Given by Equation 8.2}\)
3) We are considering a plane strain case, such that the elastic stiffness matrix is set to be the plane strain elastic stiffness matrix for the interfacial layer: $c^E = c^{E*}$

$$c^E = c^{E*} = \frac{E_1}{(1+\nu_1)(1-2\nu_1)} \begin{bmatrix}
1 - \nu_1 & \nu_1 & \nu_1 & 0 & 0 & 0 \\
\nu_1 & 1 - \nu_1 & \nu_1 & 0 & 0 & 0 \\
\nu_1 & \nu_1 & 1 - \nu_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1-2\nu_1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1-2\nu_1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-2\nu_1}{2}
\end{bmatrix}$$

Incorporating the above three conditions into Form 2 of the constitutive relations for the piezoelectric patches (Equations 8.4), we obtain the expression for the electric displacement, $D$, and the local stresses, $T$:

$${\{D\} = \begin{bmatrix} \varepsilon^S & e \\ -e_t & c^E \end{bmatrix}{\{E\}}}_{1-3} \quad \{D\} = [e]{S} \quad \{T\} = [c^E]{S} \quad (8.5)$$

A piezoelectric material is often modeled as a capacitor due to its ability to store electric energy (as shown in Figure 8.3). Consequently, once the electric displacement, $D$, is found, we can calculate other electrical properties for each piezoelectric patch in the interfacial layer. The electrical properties include: the charge generated, $q$, the voltage induced, $V$, the electric field induced, $E$, the electric energy generated, $W$, and the electric energy density generated, $W^*$. Table 8.2 shows the equations used to find these electrical properties.
Figure 8.3: A piezoelectric material layer is often modeled as a capacitor due to its ability to store electric energy. The capacitance for this schematic is: 

\[ C_p = \frac{\varepsilon_{33} A_p}{t_p} \]

where \( \varepsilon_{33} \) is the dielectric permittivity of the material. \( q \) is the charge generated on the two surfaces of the piezoelectric material.
Table 8.2: The electrical properties for the piezoelectric patches due to the local strains

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
</table>
| Electric displacement, $D$   | \[
\begin{pmatrix}
D_1 \\
D_2 \\
D_3
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & e_{15} & 0 \\
0 & 0 & 0 & e_{24} & 0 & 0 \\
e_{31} & e_{32} & e_{33} & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
S_1 \\
S_2 \\
S_3 \\
S_4 \\
S_5 \\
S_6
\end{pmatrix}
\]

where $S$ is the local strain found by Equation 7.2 and $e$ is defined in Equation 7.4.

| Charge, $q$                  | $q = \int\int [D_1 \ D_2 \ D_3] \begin{pmatrix}dA_1 \\
dA_2 \\
dA_3\end{pmatrix}$ |
|------------------------------|--------------------------------------------------|

where $dA_i$ is the area perpendicular to the $i$ axis.

<table>
<thead>
<tr>
<th>Voltage, $V$</th>
<th>$V = \frac{q}{C_p}$</th>
</tr>
</thead>
</table>

where the capacitance is defined as: $C_p = \frac{\varepsilon_{33} A_p}{t_p}$.

<table>
<thead>
<tr>
<th>Electric field, $E$</th>
<th>$E = \frac{V}{t_p}$</th>
</tr>
</thead>
</table>

where $t_p$ is distance between the charged surfaces.

<table>
<thead>
<tr>
<th>Electric energy, $W$</th>
<th>$W = qV$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Electric energy density, $W^*$</th>
<th>$W^* = \frac{qV}{A_p \ t_p}$</th>
</tr>
</thead>
</table>
As mentioned earlier, we define a local coordinate system \(\{1,2,3\}\) for each piezoelectric patch, in such a way that the 3-direction is along the poling direction of the piezoelectric material. The piezoelectric material can be placed and connected in different orientations and direction in the interfacial layer; that means that the local coordinates \(\{1,2,3\}\) can be oriented differently with respect to the global coordinates of the interfacial layer \(\{x,y,z\}\). Figure 8.4a show the three main scenarios:

- **Case A** – The poling direction is across the thickness of the interfacial layer, and is aligned with the global y-axis.
- **Case B** – The poling direction is along the length of the interfacial layer, and is aligned with the global x-axis.
- **Case C** – The poling direction is into the depth of the interfacial layer, and is aligned with the global z-direction.

**Figure 8.4:** a) The different cases of how the poling of the piezoelectric material is compared with the global coordinates \(\{x,y,z\}\). The poling direction is defined as \(\textbf{P}\) and a local coordinate system \(\{1,2,3\}\) is oriented such that the 3-axis is in the poling direction. b) The thickness \(t_p\), the length \(L_p\), and the depth \(b\), of each piezoelectric patch.
Next, we consider all three cases with the different poling directions, and we develop analytical expressions for the electric displacement, $D$, for each piezoelectric patch. The relationships provided in Table 8.2 are used to also find the charge, voltage, and electric energy density created in the piezoelectric patch due to the wrinkling deformations for the different cases with different poling directions. Figure 8.4a shows the poling directions for the three cases, and Figure 8.4b shows the geometrical dimensions for each piezoelectric patch.

**Case A) Poling across the thickness**

Electric displacements:

$$D_1 = 0 \quad , \quad D_2 = 0 \quad , \quad D_3 = e_{31}S_1 + e_{33}S_3$$

(8.6)

where $S_1 = \varepsilon_x^{int\_layer}$ and $S_3 = \varepsilon_y^{int\_layer}$ (averaged across $y$-thickness)

and $e_{31} = \frac{E_1[d_31(1-v_1)+d_32v_1+d_33v_1]}{(1+v_1)(1-2v_1)}$ and $e_{33} = \frac{E_1[d_31v_1+d_32v_1+d_33(1-v_1)]}{(1+v_1)(1-2v_1)}$

Generated charge, $q$, and induced voltage, $V$:

$$q = \int (e_{31}S_1 + e_{33}S_3) \, b \, dx \quad , \quad V = \frac{q}{c_p} = \frac{t_p}{\varepsilon_{33}A_p} q = \frac{t_p}{\varepsilon_{33} b L_p} q$$

(8.7)

Induced electric energy density, $W^*$:

$$W^* = \frac{q V}{A_p L_p} = \frac{1}{\varepsilon_{33}} \left( \frac{q}{b L_p} \right)^2$$

(8.8)

**Case B) Poling along the length**

Electric displacements:

$$D_1 = 0 \quad , \quad D_2 = 0 \quad , \quad D_3 = e_{32}S_1 + e_{33}S_3$$

(8.9)

where $S_2 = \varepsilon_y^{int\_layer}$ and $S_3 = \varepsilon_x^{int\_layer}$ (averaged across $x$-length)

and $e_{32} = \frac{E_1[d_31v_1+d_32(1-v_1)+d_33v_1]}{(1+v_1)(1-2v_1)}$ and $e_{33} = \frac{E_1[d_31v_1+d_32v_1+d_33(1-v_1)]}{(1+v_1)(1-2v_1)}$
Generated charge, \( q \), and induced voltage, \( V \):

\[
q = \int (e_{32} S_2 + e_{33} S_3) \, b \, dy, \quad V = \frac{q}{c_p} = \frac{\epsilon_{33} A_p}{\epsilon_{33} b \, t_p} q = \frac{\epsilon_{33} A_p}{\epsilon_{33} b \, t_p} q
\]  

(8.10)

Induced electric energy density, \( W^* \):

\[
W^* = \frac{q V}{A_p b} = \frac{1}{\epsilon_{33}^2} \left( \frac{q}{b \, t_p} \right)^2
\]  

(8.11)

Case C) Poling into the depth

Electric displacements:

\[
D_1 = 0, \quad D_2 = 0, \quad D_3 = e_{31} S_1 + e_{32} S_2
\]  

(8.12)

where \( S_1 = \epsilon_y^{\text{int.layer}} \) and \( S_2 = \epsilon_x^{\text{int.layer}} \) (constant in z-depth due to plane strain)

and \( e_{31} = \frac{E_i [d_{31} (1-v_i) + d_{32} v_i + d_{33} v_i]}{(1+v_i)(1-2v_i)} \) and \( e_{32} = \frac{E_i [d_{31} v_i + d_{32} (1-v_i) + d_{33} v_i]}{(1+v_i)(1-2v_i)} \)

Generated charge, \( q \), and induced voltage, \( V \):

\[
q = \int (e_{31} S_1 + e_{31} S_2) \, dx \, dy, \quad V = \frac{q}{c_p} = \frac{b}{\epsilon_{33} A_p} q = \frac{b}{\epsilon_{33} b \, t_p} q
\]  

(8.13)

Induced electric energy density, \( W^* \):

\[
W^* = \frac{q V}{A_p b} = \frac{1}{\epsilon_{33}^2} \left( \frac{q}{b \, t_p} \right)^2
\]  

(8.14)

For the all the three poling cases evaluated above (Case A, B, C) the local strains \( \epsilon_{ij}^{\text{int.layer}} \) are given in Equation 8.2, \( E_i \) and \( v_i \) are the Young modulus and Poisson ratio of the piezoelectric material (which is designed to the same as the interfacial layer). Furthermore, \( e_y \) and \( d_y \) are the piezoelectric induced stress coefficients and the piezoelectric coefficients (shown in Equation 8.4). Finally, \( \epsilon_{33}^T \) is the dielectric permeability of the piezoelectric material.
8.3 Results for direct piezoelectric effect in wrinkling composites

In this section we will use the analytical models developed in Chapter 8.2 to study the effect of having piezoelectric patches in the interfacial layer of the wrinkling composite. We will first evaluate the case where the piezoelectric patches are located at the edges of the thickness of the interfacial layer, where the local deformations are highest, mainly for energy harvesting applications. Then, we will study the case where the piezoelectric patches are located at the center of the thickness of the wrinkling interfacial layer, where the local strains remain constant during the wrinkling process, mainly for switching applications.

![Figure 8.5: a) For energy harvesting applications, piezoelectric patches are placed systematically at each wrinling peak and groove; one patch at each edge of the thickness of the interfacial layer. One patch will be in compression and one will be in tension. b) We evaluate just one wrinkling peak in our simulations. The material- and geometrical- parameters for the interfacial layer and the patches of piezoelectric material, on top and bottom of the wrinkling peaks, are shown in this figure.](image)

**8.3.1 Energy harvesting application**

For energy harvesting applications we wish to generate the highest electric energy density possible. Since we know that the largest deformations and local strains due to wrinkling occur at the edges of the interfacial layer (i.e. $h = \pm t/2$), we will consider the case where there is a piezoelectric patch at the top and bottom of each wrinkling peak in the interfacial layer (as shown in Figure 8.5a) We evaluate the charge, voltage, and electric energy density obtained by each piezoelectric patch as the composite is being macroscopically compressed by, $\bar{E}$, in the post-buckling region, where wrinkling has occurred in the interfacial layer. Focusing on
one peak of the wrinkling pattern created in the interfacial layer (i.e. one half-wavelength of the wrinkling wavelength), Figure 8.5b shows the geometrical and material features of the piezoelectric patches. For each wrinkling peak there is one piezoelectric patch in compression and one in tension, corresponding to the bottom and top piezoelectric patches. We model the piezoelectric material as PVDF and use material properties based on Table 8.1; $E_I = 5\text{GPa}$, $d_{31} = 21\text{pC/N}$, $d_{32} = 2.75\text{pC/N}$, $d_{33} = -33\text{pC/N}$ and $\varepsilon_{33} = 0.106\text{nF/m}$.

Moreover, we also consider the effect of the poling direction on the generated charge, voltage, and electric energy density generated in each piezoelectric patch. The poling directions were described in Chapter 8.2 to be: 
- **Case A** – poling direction is across the thickness of the interfacial layer,
- **Case B** – poling direction is along the length of the interfacial layer, and
- **Case C** – the poling direction is into the depth of the interfacial layer.

In order to compare the results for the different poling cases (Case A, B and C), we evaluate the surface charge density, $q^*$, and the electric field, $E$, generated to make the results independent of the geometrical dimensions. The parameters are:

1. **Surface charge density, $q^*$**: $q^* = \frac{q}{A_p}$, where $q$ is given in Equation 8.7a, 8.10a and 8.13a, and $A_p$ is the surface area of the piezoelectric patch (Figure 8.3)
2. **Electric field, i.e. voltage per unit length, $E$**: $E = \frac{V}{t_p}$, where $V$ is given in Equation 8.7b, 8.10b and 8.13b, and $t_p$ is the distance between the surfaces of the piezoelectric material (Figure 8.3).

Figure 8.6 shows the surface charge density, the electric field, and the electric energy density generated in the top and bottom piezoelectric patch in a wrinkling peak, as function of the macroscopic applied strain, $\varepsilon$, in the post-buckling region, $|\varepsilon| > |\varepsilon_c|$. It is evident that the patches in compression produce higher electrical properties for all post-buckling strains and for all poling directions. This can be explained by the fact that the interfacial layers and the piezoelectric patches are in compression during pre-buckling, such that the compression is increased for the bottom patches during post-buckling, while the top patches go from being in compression and over to being in tension during post-buckling. Studying the effect of poling direction, we see that the surface charge density, the electric field, and electrical energy...
density is highest for the case A; when poling is across the thickness of the piezoelectric patch. In addition, the results for case B, where the poling is along the interfacial layer, are just slightly less than for case A. However, the length chosen for the piezoelectric patch can be longer along the interfacial layer than across, $L_p > l_p$, and hence the total charge or voltage obtained can be greater designed to be greater for case B (poling along the interfacial layer).

Figure 8.6: The electrical properties of each single piezoelectric patch in compression and tension in the wrinkling interfacial layer of a composite. The three different poling directions are evaluated for each patch (Case A-C). $E_0$ is kept constant for all simulations; $E_0=E_f/200=25MPa$. a) the surface energy density, $q^*$, b) the electric field, $E$, and c) the electric energy density $W^*$, as a function of the applied macroscopic strain, $|\varepsilon|$. 

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We also investigate the effect of the composite’s material combination on the electric properties of the composites, since the wrinkling wavelength is directly related to the stiffness ratio, $E_i/E_o$, between the relatively stiffer interfacial layer and the supporting matrix. Figure 8.7 shows the surface charge density, the electric field, and the electric energy density generated in the top and bottom piezoelectric patch in a wrinkling peak, as function the stiffness ratio in the composite, $E_i/E_o \in [100 - 450]$, while $E_i=5\text{GPa}$ and the macroscopic strain is kept constant at $\varepsilon=-0.07$.

The graphs in Figure 8.7 clearly show that the electrical properties are highest for the patches that are in compression, due to the same explanation as indicated above. In addition, we can see that surface charge density, the electric field, and the electric energy density created, decreases for the patches that are in compression, while they increase for the patches that are in tension, as the stiffness ratio $E_i/E_o$ increases. The reduction in the electric properties with the stiffness ratio for the patches that are in compression, can be explained by the fact that the wavelength of the wrinkling pattern is greater for higher stiffness ratio composites. This means that the local deformations and strains in the interfacial layer and the piezoelectric patches are smaller, and consequently they are inducing less electric energy.

Moreover, as the stiffness ratio in the composite is reduced the critical strain at which the interfacial layers in the composite wrinkles will increase. This means that higher compressive strains are obtained in the interfacial layers before the wrinkling initiation. Consequently, overall higher compressive strains are obtained in the piezoelectric patches that are in compression, while simultaneously smaller strains are obtained in the patches that are in tension (since the patches in tension have transformed from being in compression and over to tension only during the post-buckling process).

Finally, we observe that the electric properties obtained are highest for the case where the poling is across the thickness of the piezoelectric patches (Case A), or along the interfacial layer (Case B). However, as mentioned earlier, the length chosen for the piezoelectric patch can be longer along the interfacial layer than across it, $L_p > t_p$, and hence the total charge or voltage obtained can be greater for the case B (poling along the interfacial layer) than for Case A.
Figure 8.7: The electrical properties of each single piezoelectric patch in compression and tension in the wrinkling interfacial layers of a composite. The three different poling directions are evaluated for each patch (Case A-C). The applied macroscopic strain is kept constant for all simulations, $|\varepsilon| = 0.07$. a) the surface energy density, $q^*$, b) the electric field, $E$, and c) the electric energy density $W^*$, are shown as a function of the stiffness ratio in the composite, $E_i/E_0$. 
We simulate and evaluate the electric properties obtained in a specific composite, as a practical example. Based on the above results we will evaluate the total possible charge, voltage, electric energy, and electric energy density obtained for an RVE that includes one wrinkling wavelength, as shown in Figure 8.8. There will be two piezoelectric patches in compression and two that are in tension in this RVE. Moreover, we will only evaluate the cases where the poling is across the thickness of the interfacial layer (Case A), or along the interfacial layer (Case B), as these cases were previously shown to have the best performance (in Figure 8.6 and 8.7).

We will again model the piezoelectric material to be composed of PVDF (Table 8.1). The geometrical features of the piezoelectric patches are: \( t_p=0.1 \text{mm} \), \( L_p=\lambda_{cr}/4=2.5 \text{mm} \) (the depth, \( b \), does not influence our results as we are considering plane strain situation and also the poling direction is not into the depth). The material and geometrical properties of the full composite are: stiffness of the matrix layers of the composite are constant at \( E_0=E_i/200=25 \text{MPa} \), the thickness of the interfacial layer is \( t=0.5 \text{mm} \), and the thickness of full composite with the interfacial layer and matrix layers is \( D=24 \text{mm} \) (i.e, \( t/D=0.2 \), which was in previous chapters verified to guarantee a wrinkling mode of instability in the interfacial layers of the composite when a critical compressive load is applied).

Figure 8.8a and b show the maximum possible total charge and voltage generated by this RVE, where the piezoelectric patches are connected in parallel circuit when calculating the highest charge achieved, and the piezoelectric patches are connected in series circuit when we are looking for the highest voltage achieved. It is clear that the optimal poling directions changes based on whether we are interested in the maximum output charge or voltage. The small electric energy generated by this RVE is shown in Figure 8.8c. Moreover, the electric energy densities per unit composite volume and per unit piezoelectric material volume, is shown in Figure 8.8d. These results show that the electric outputs increase with the applied macroscopic strain, \( |\varepsilon| \), as expected. In addition, we want to point out that this RVE size is \( \sim 10 \text{mm} \), such that the total electric energy can be greatly enhanced by picking a bigger RVE with several wrinkling waves included.
Figure 8.8: a) The total charge if patches are connected in parallel circuit. b) The total voltage if patches are connected in serial circuit. c) The total electric energy obtained, d) The electrical energy density per unit composite volume, as well as per unit piezoelectric patch volume. The results are for a composite where: $t_p=0.1\text{mm}$, $L_p=\lambda_{cr}/4=2.5\text{mm}$, $E_I=5\text{GPa}$, $E_0=E_1/200=25\text{MPa}$, $t=0.5\text{mm}$ and $D=24\text{mm}$. The piezoelectric material properties are given in Table 8.1.
However, it is important to note that our generated electric energy is still smaller than many other current prominent methods and mechanisms for energy harvesting [105]. Nevertheless, our system does not depend on any specific frequency of vibrations, which the other methods do. Despite the smaller electric energy harvested by our method, this method it is still competitive with other higher output energy harvesting methods, because it deploys a new mechanism, works under quasi-static loading, and it is independent of the frequency of loading. Hence, we are offering a new energy harvesting mechanism that can be interesting in many situations and applications where the methods of using vibrations is not applicable.

8.3.2 Switching application

In this sub-chapter we will show how our composites with the wrinkling interfacial layers containing piezoelectric patches, can be used for switching applications. We have already shown that by putting the patches of piezoelectric material at the edges of the interfacial layer, where the local strain are the highest, we can create a new mechanism for energy harvesting were the main advantage is that this is a quasi-static method, and does not depend on vibration frequencies. Herein, we will move the patches of piezoelectric material so that they are placed at the middle of the interfacial layer, as shown in Figure 8.9a. We evaluate the charge, voltage, and electric energy density obtained by each piezoelectric patch as the composite is being macroscopically compressed by, $\varepsilon$, in the post-buckling region, where wrinkling has occurred in the interfacial layer. Again, focusing on one peak of the wrinkling pattern created in the interfacial layer (i.e. one half-wavelength of the wrinkling wavelength), Figure 8.9b shows the geometrical and material features of the piezoelectric patches. Both of these piezoelectric patches will be in compression throughout the post-buckling loading (unlike the situation in Chapter 8.3.1). This is because the patches are located near the center of the interfacial layer where the local strains are approximated to remain constant after wrinkling, and equal to the macroscopic critical compressive strain at which the wrinkling was initiated (this approximation was studied in Chapter 4).

Moreover, we will again also consider the effect of the poling direction on the generated charge, voltage, and electric energy density in each piezoelectric patch. The poling directions...
were described in Chapter 8.2 to be: *Case A* – poling direction is across the thickness of the interfacial layer, *Case B* – poling direction is along the length of the interfacial layer, and *Case C* – the poling direction is into the depth of the interfacial layer. Finally, we model the piezoelectric material as PVDF and use material properties based on Table 8.1; $E_1=5 \text{GPa}$, $d_{31}=21 \text{pC/N}$, $d_{32}=2.75 \text{pC/N}$, $d_{33}=-33 \text{pC/N}$ and $\varepsilon_{33}=0.106 \text{nF/m}$.

**Figure 8.9:** a) For switching applications, piezoelectric patches are placed systematically at each wrinkling peak and groove; both patches are placed in the middle of the thickness of the interfacial layer. Both patches will be in compression during post-buckling due to the initial compression load during pre-buckling. b) We evaluate just one wrinkling peak in our simulations. The material- and geometrical- parameters for the interfacial layer and the patches of piezoelectric material at the wrinkling peaks, are shown in this figure.

Figure 8.10 a-c shows the surface charge density, the electric field, and the electric energy density generated in the two piezoelectric patches, as function of the macroscopic applied strain, $\bar{\varepsilon}$, in the post-buckling region, $|\bar{\varepsilon}| > |\bar{\varepsilon}_c|$. The surface charge density, $q^*$, and the electric field $E=V/l_p$ are evaluated to make the results independent of the geometrical
dimensions (as described in more detail in Chapter 8.3.1). Figure 8.10a-c show that designing the poling direction of the piezoelectric patches to be across or along the interfacial layer, have little effect on the electric properties. On the other hand, the electric properties are reduced by choosing the poling direction be into the depth of the interfacial layer.

In additions, the results show that the patches that are slightly on the inner side of the interfacial layer’s center line will have a slight increase in their electric properties with $\bar{\varepsilon}$, due to the small addition of compressive strain from bending in the post-buckling region. On the other hand, the piezoelectric patches that are slightly on the outer side of the interfacial layer’s center line, exhibit a slight decrease in the electric properties with $\bar{\varepsilon}$, due to the small addition of tensile strains in the post-buckling region due to bending. However, it is important to note that by moving the patches from the edges of the interfacial layer (analyzed in Chapter 8.3.1) and towards the center of the interfacial layer, we can control and level the electric properties as function of the macroscopically applied strain. By connecting the two patches together in parallel circuit, the surface charge density can be set to be constant for all post-buckling strains (Figure 8.10d). Similarly, by connecting the patches in a series circuit the electric field in the composites can be designed to be constant for all post-buckling strains (Figure 8.10e). The electric energy density will also stay constant for all strains in the post-buckling region, when adding the two patches together in a circuit (Figure 8.10f). Hence, there is a nonlinear behavior observed in the electrical properties of the composite; for pre-buckling strains the electric properties are a function of the applied strain, but once wrinkling occurs, the electric properties can be designed to be constant for all post-buckling applied strain.

This unique property and ability to tune and control the electrical behavior and response of the composites, allows us to use this method to create switches. The composite can be designed to have a certain electrical behavior pre-buckling, but once a critical load is reached, the composites’ electrical behavior changes to a designed constant value that becomes independent of the applied strain. The critical local switching the electrical behavior of the composite is determined by the critical load causing the wrinkling instability to occur in the composite, which is determined by the material combinations in the composite (studied in Chapter 3.2).
Figure 8.10: The electrical properties of piezoelectric patches in the middle of the wrinkling interfacial layer of the composite, as a function of the applied macroscopic strain, \(|\varepsilon|\). The three different poling directions are evaluated (Case A-C). \(E_0\) is kept constant for all simulations; \(E_0=E_0/200=25\, MPa\). a) surface energy density, \(q^*\), for each patch, b) electric field, \(E\), for each patch, c) electric energy density \(W^*\), for each patch are shown. In addition, the d) total surface energy density when connected in parallel circuit, e) total electric field when connected in series circuit, and f) total energy density achieved in patches, are shown.
8.4 Conclusion

In this chapter we have introduced a new method for energy harvesting and for creating a switch that exploits the wrinkling instability. By adding patches of piezoelectric material in the thin interfacial layers of a composite, we have demonstrated that the mechanical load can generate electric charge, voltage, electric field, and electric energy. This is done through the direct piezoelectric effect. Analytical expressions were derived for the electric properties of these wrinkling transforming composites, including the effect of varying the poling direction in the piezoelectric patches on the electrical properties. It was concluded that the electrical properties are enhanced by having the poling direction in the piezoelectric patches be pointing across the thickness or along the length of the interfacial layer.

The electrical properties of the wrinkling composites were shown to increase with increase in the macroscopically applied strain. On the other hand, an increase in the stiffness ratio between the interfacial layer and the matrix layers in the composite, was show to reduce the electric properties. A specific composite was evaluated as an examples for energy harvesting applications. The composite clearly converted the mechanical applied load to electric charge, voltage, and electric energy, but the obtained electricity is much lower than other previous methods available [105]. However, our method is the only method that does not require vibration of the composite or structure to achieve electric energy. The method presented in this chapter uses the wrinkling instability occurring in a composite to achieve the higher modes of deformation and the large local strain, and hence it can be a fully quasi-static process. Consequently, our method is competitive with other current methods due to the fact that it functions and operates for all loading frequencies, and it is not restricted to specific frequency ranges.

These composites with piezoelectric patches have also been studied for switching applications. We demonstrated that by changing the location of the piezoelectric patches, we can design non-linear electric properties. More specifically, it was shown that the electrical properties can be designed to stay constant for all macroscopic applied strains once the
wringling instability occurs. This controlled behavior makes our method applicable for creating switches.

The results presented in this chapter are preliminary results demonstrating a new concept. Further work needs to be done to experimentally verify the conclusions made in this chapter. Moreover, the effect of varying the geometry of the piezoelectric patches can show some interesting tunable behavior. Finally, in order to use the method presented in this chapter for real applications, the detailed knowledge about the exact location of the wringling peaks and grooves are required. This is due to the fact that the piezoelectric patches are strategically added at the right locations, i.e. at the wringling peaks and grooves. Control over the exact position of the wringling can be achieved by for example introducing geometrical imperfections into the interfacial layers. However, more work needs to be conducted investigating how to optimize and control the exact location of the wringling peaks.
Chapter 9
Conclusion and Future work

9.1 Summary and general conclusion

Inspired by the undulating patterns found in different plant and biological cells in nature, this thesis investigated the instability-induced wrinkling transformation in the microstructure of composites consisting of thin interfacial layers. The hybrid microstructure of a composite material has an essential influence on the effective properties and behavior of the composite. Hence, in this research, the principles and mechanics of interfacial layer instabilities were purposely designed to achieve sudden pattern transformations in the composite microstructure to generate new controlled multifunctional behavior.

Through analytical modeling, finite element simulations, and mechanical experiments, a comprehensive study was performed exploring the instability transformation in multilayered and networked composites consisting of relatively stiff interfacial layers or cells embedded in a soft matrix. A model was established predicting the mode of instability, i.e. wrinkling versus a long wave mode, in multilayered composites based on the material and geometric features of the composites. Moreover, the model predicts the critical strain at which the instability occurs and the wrinkling pattern’s initial and post-buckling wavelength and amplitude. A wide range of different networked composites were evaluated as well, exhibiting wrinkling in the network/cell walls. Next, the local deformations, i.e. the local stresses and strain, in the interfacial layers and the soft matrix were modeled through the pre-buckling region, the point of wrinkling, and into the post-buckling region.
The macroscopic effective behavior of elastic composites were studied for composites undergoing a wrinkling-induced transformation of its microstructure. Analytical models for the macroscopic stress-strain behavior and the effective stiffness of elastic composites were developed as function of the applied strain and the composite’s material and geometrical properties. It was concluded that wrinkling composites can be designed to exhibit stress mitigation effects, bilinear and multi-linear elastic stress-strain behavior, and a switchable effective stiffness.

The strain energy density ability of elastic composites were demonstrated to be enhanced by purposefully introducing the wrinkling instability in the interfacial layers of the composite. This effect is due to the large local deformations created in the matrix layers in the vicinity of the wrinkles. The analytical models established for the strain energy density helps designers with the material and geometrical choice when aiming to design composites with a specific energy storage property. Moreover, we introduced plasticity in the composites, by modeling the thin relatively stiff interfacial layers as elastic-perfectly plastic materials. Simplified models were derived predicting the energy absorbed, stored and dissipated as these composites are compressed. It is shown that the absorbed energy density and dissipated energy density is increased by deploying and designing the wrinkling instability to occur in the composite’s interfacial layers.

Design guidelines were presented to assist the process of deploying instability-transformation to tune, control, and switch the mechanical properties of multifunctional composite materials. In addition, it was demonstrated that we can create tunable elastic composites with enhanced energy storage properties, and interesting new effective mechanical behavior, such as a bilinear and multi-linear elastic stress-strain behavior, and a switchable effective stiffness. For the composites with plastic interfacial layers we show that the energy absorption and dissipation by the composite can be controlled and designed for. Moreover, the effective
stress-strain behavior can be altered and controlled as well. Hence, we conclude that the ability to alter and transform the microstructure enables on-demand tunability and active control of the composite’s properties and attributes.

Finally, we presented and explored a new mechanism to harvest energy, by strategically adding piezoelectric material on to the interfacial layers at locations with the highest deformations. Using the direct piezoelectric effect we can obtain voltage/electric power from the local mechanical deformations arising from the wrinkling. This new method is advantageous as the wrinkling deformations do not depend on vibrating the structure at a specific frequency, and hence this method can simplify the process of extracting electric energy from mechanical deformations and loads. In addition, we demonstrated that by varying the location of the piezoelectric material in the interfacial layers, we can control the electric behavior of the composite, which can for example be used for switching applications.

9.2 Significance and contributions

The contributions of this thesis can be summarized in the following points:

- We presented a comprehensive understanding and model of the instability mechanisms in the interfacial layers or cell walls of composites. It was shown that upon reaching a critical condition, a wrinkling pattern is created in the interfacial layers, determined by the composites material and geometrical combinations. Moreover, the local deformations in the interfacial layers and matrix as a result of the wrinkling instability can be predicted by our analytical models.
- By purposefully using the wrinkling instability, we have demonstrated that we can predict, control, and tune the effective mechanical properties, i.e. macroscopic stress-strain and effective stiffness, of multilayered elastic composites. More specifically, the composites can be designed to exhibit stress mitigation effects, bilinear and multi-linear elastic stress-strain behavior, and a switchable effective stiffness. The fact that we have designed
bilinear and multilinear elastic materials are very interesting and novel results, useful for many engineering applications.

- The energy storage properties of elastic wrinkling composites were established and modeled. In additions, the energy absorption and dissipation properties of composites with an elastic-perfectly plastic interfacial layer were studied as well. The results show enhanced energy absorption, storage and dissipation properties in the composite that wrinkle. Design plots were created for the energy properties of multilayered composites.

- We have designed composite materials that exhibit \(~4.5\) times more energy storage, per volume or per mass, compared with soft matrix alone, just by adding small volume fractions of stiff interfacial layers (interfacial layer volume fraction < 2%).

- We demonstrate that we can use instability-induced transformation of interfacial layers in composites to achieve new tunable multifunctional behavior. The wrinkling mechanism was deployed to create composites with on-demand tunability and active control of its effective mechanical behavior, energy properties, and other attributes.

- A new energy harvesting mechanism was proposed by using piezoelectric material in the interfacial layers of the composite. Upon compression, the wrinkling deformation of the interfacial layer was shown to generate electricity. This technique can also be used to create nonlinear switches.

### 9.3 Future work

Extending the work and results presented in this thesis from the basic multilayered and networked structures to more complex networked composites or three-dimensional structures, opens the door to new interesting loadings situations and tunable materials. In addition, studying the viscoelastic effects of designing the composite to be composed of a viscoelastic matrix is necessary, as the viscoelastic effect will allow for enhanced energy absorption and new composite effective behavior. Moreover, in this section, we propose that the concepts and results presented in this thesis are applied and extended to the following new research areas:
9.3.1 Complex network and 3D-composites

Following the methodology outlined in this thesis, future work should unveiling the relationship between instabilities and the composites effective mechanical behavior for three-dimensional composites under biaxial loading. Furthermore, the influence of the composite’s materials and geometry should be included analyzing the three-dimensional structures.

9.3.2 Bending stiffness and flexibility

The wrinkling induced transformation of the interfacial layers or network walls of a composite can exhibit interesting new behavior and properties under other loading scenarios than the ones studied in this thesis. For example, studying the effective bending stiffness of the composite as the interfacial layers or network transforms from being straight to the wrinkled configuration/topology. The bending stiffness will affect the flexibility of the structure, and consequently, with good understanding of the effect of the wrinkling on the bending stiffness, we can design to tune the flexibility of a composite under different environments.

9.3.3 Active composites and stimulus responsive material

Hydrogels and stimulus responsive materials have attracted considerable attention recently. They can be tuned to respond to only certain conditions, which makes them act as “active materials” [94]. There are many applications of the stimulus responsive materials, such as actuating systems, new optical devices, sensors, smart surfaces and so on [109,110,111].

Hydrogels and stimulus responsive materials can be used in our composite materials to generate the stress transformative properties and behavior via an active matrix instead of by a mechanical loading. By using stimulus responsive material as the matrix of the composite, we can design the composite to change and consequently undergo instability and wrinkling only when specific environments and conditions are satisfied. This means that we can tune and switch the mechanical properties of the composite to be automatic, and this can all be
reversible processes. Hence results presented in this thesis can be combined with stimulus responsive materials to create mechanomutable materials.

9.3.4 Debonding and ultimate composite failure

Future work should also include research within the manufacturing limitations of these networked and multilayered composites. Based on the material choice, there can be delamination occurring between the interfacial layer and the supporting matrix as the applied load increases. Extensive research is done on preventing delamination. Incorporating those results and findings into mechanisms studied in this thesis, can be very useful for many engineering applications. Moreover, future work should include studying the macroscopic behavior of these instability controlled composites’ for larger applied strains, and even until the point of failure.

9.3.5 Wave propagation control

Controlling and filtering wave propagation through materials have many applications including nondestructive material testing, acoustic mirrors, wave-guides, acoustic filters, vibration dampers, and ultrasonic transducers. Currently, great attention is drawn towards acoustic metamaterials [112,113,114,115]. It has been shown that the properties of metamaterials to control and filter acoustic waves originate in their periodic microstructure [116,117,118,119,120,121,122,123].

One promising area of research is the effect of wave propagation of electromagnetic, elastic or acoustic/phononic waves in soft composites with wrinkled interfacial layers. Soft metamaterials that are designed and tuned based on reversible wrinkled interfacial layers indicate interesting opportunities of manipulating acoustic properties due to their capability to sustain large elastic deformations. Research has been conducted on deploying our composites with wrinkled interfacial layers to transform elastic wave propagation in the plane of the wrinkles [112]. The wrinkling of the interfacial layers, and its post-buckling
deformation, has interesting effects on the elastic wave propagation. The wrinkling transformation give rise to a system of periodic scatterers, which reflect and interfere with the propagation of waves. It has been shown that the composite can be designed in such a way that the wrinkled interfacial layers produce band-gaps and can filter certain frequencies of the elastic waves. Remarkably, the mechanism of frequency filtering is effective even for composites with similar or identical densities, such as polymer composites. Since the wrinkling transformation can be reversible, the mechanism can be used for tuning, switching, and controlling wave propagation by deformation control. Figure 9.1 illustrates band-gaps created for different composites that undergo wrinkling instability. It is clear that when the wrinkling interfaces are formed, the band-gaps appear, and we can see how the band-gaps evolve with increased compressive strain and deformation of the interfacial layers.

Future work should investigate the possibility of creating an effective medium model to capture the effect of the wrinkled interfacial layer on the wave propagation. In addition, the properties of acoustic and elastic waves through two- and three-dimensional composites with wrinkled or buckled networks and plates should be studied. Complex networks and other geometries introduce more interesting wave propagation phenomena to be detected, and permits more control over transmittance of waves and the band gaps created. Out of plane wave propagation in composites undergoing instabilities, can be used not only to filter waves, but also to guide the wave propagation in desired directions. Furthermore, the wrinkling and instability patterns can create wave-guides that guide energy, EM or acoustic waves in desired paths, making it very useful for many engineering applications.
Figure 9.1: Dispersion diagrams showing the normalized frequency of wave propagation, $\omega$, as a function of wave number for three composites with different concentrations (column 1-3). Composites 1 and 2 undergo wrinkling, while composite 3 exhibits a long-wave mode of instability. The top row is the dispersion diagrams for the undeformed states, while second and third row show the results at deformed states as the applied strain is increased. The wave propagates in an initial direction along the interface, i.e. $\varphi = \pi/2$ ($r_{\text{modulus}}$ is the stiffness ratio between the interfacial layers and the soft matrix, and $r_{\text{density}}$ is the density ratio between two materials). [112]
9.4 Areas of applications

The ability to control and transform a composite material’s microstructure through wrinkling can be used to regulate mechanical, chemical, thermal, photonic, phononic, electrical and optical functions of the material. Consequently, the results in this paper can be useful for many engineering applications, especially with the rapid advancement seen in manufacturing technologies. Herein, we will propose some specific applications of the switchable and tuneable composites by using the wrinkling-induced transformation.

The wrinkling pattern’s wavelength created in the interfacial layers of a composite can directly be studied and used to predict the material properties of the composite (the stiffness ratio between the interfacial layers and the soft matrix). Furthermore, by also evaluating the wrinkling amplitude, the applied load can be predicted as well. This method can hence be used in biological and medical situations where a direct measurement of the materials or loading cannot easily be done. Hence it can be applied for, for example, disease detection and recording changes in biological systems and cells.

Another application of these composites is to create mechanical sensor where once a critical load is reached, the effective composite stiffness switches such that the initiation of the critical load can easily be detected. Furthermore, the effective composite stress-strain behavior can be used to predict the applied load. Electric sensors can also be created by using piezoelectric material in the wrinkling interfacial layer, or by deploying electro-mechanical dielectric composites materials.

Moreover, the enhanced, controlled and tuneable energy absorption (i.e. energy dissipation and storage) in these composites can be used for damping and packaging applications, as well as being used as actuators. Additionally, the ability to control and design the macroscopic stress-strain behavior of these composites can be used to mitigate load transfer. This is useful in many applications where controlling the transferred load, despite the applied load, is crucial; such as protecting soft tissue or working with sensitive electronics.
Another important area of application is metamaterials and wave propagation control (elastic, EM and acoustic waves). Photonic and/or phononic crystals with wrinkling interfacial layers have attributes that are switchable and tunable, affecting relevant wave propagation properties of the composite. This can be used to create novel design for: phononic and photonic mirrors and filters, creating waveguides, and obtaining structural coloring, camouflage and cloaking.

Finally, we have in this thesis proposed a new energy harvesting mechanism using these wrinkling composites. By adding these composites consisting of the wrinkling interfacial layers with patches of piezoelectric material, in places that are frequently under loading and unloading we can harvest energy. In the future, these energy harvesting materials can for examples be placed in floorings, roads, and tires, and be a source generating electricity.
References

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Appendix A

Equations for composite effective behavior during post-wrinkling

In Chapter 5, the strain energy density in the interfacial layer, $U_{\text{int.layer}}$, and matrix layers, $U_{\text{matrix}}$, and the full composite, $U_{\text{Total}}$, during post-wrinkling, were derived to be:

$$U_{\text{Total}}(\varepsilon) = f_m U_{\text{matrix}}(\varepsilon) + f_{\text{int}} U_{\text{int.layer}}(\varepsilon)$$  \hspace{1cm} (A.1)

where $f_m = 1 - f_{\text{int},l} = 2d/(2d + t)$ and $f_{\text{int},l} = t/(2d + t)$ are the volume fraction of the matrix and the interfacial layer in the composite, respectively. The strain energy density in the interfacial and matrix layers of the composite are given by:

$$U_{\text{int.layer}}(\varepsilon) = \frac{1}{2} \bar{E}_1 \varepsilon_1^2 + \frac{E_1 \pi^2 (|\varepsilon| - |\varepsilon_{\text{cr}}|)}{3 \lambda^2} - \frac{E_1 \pi (|\varepsilon| - |\varepsilon_{\text{cr}}|)}{12 \lambda \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)$$  \hspace{1cm} (A.2)

$$U_{\text{matrix}}(\varepsilon) = \frac{1}{2} \bar{E}_0 \varepsilon_0^2 + \frac{E_0 \varepsilon_0 \lambda \bar{w}_{\text{max}} (1 + \bar{v}_0)}{2d \pi (3 - \bar{v}_0)^2} \left[\cos \left(\frac{2\pi \lambda_{\text{cr}}}{\lambda}\right) - 1\right]$$

$$+ \frac{E_0 \pi \lambda \bar{w}_{\text{max}} (5 - 2\bar{v}_0 + \bar{v}_0^2)}{4d (3 - \bar{v}_0)^2 (1 + \bar{v}_0)} \left[1 + \frac{\lambda}{4\pi \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right]$$

$$+ \frac{E_0 \pi \lambda \bar{w}_{\text{max}} (1 - \bar{v}_0)}{2d (3 - \bar{v}_0)^2} \left[1 + \frac{4\pi \lambda_{\text{cr}}}{\lambda} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right]$$

$$+ \frac{E_0 \pi \lambda \bar{w}_{\text{max}} (1 + \bar{v}_0)}{2d (3 - \bar{v}_0)^2} \left[1 + \frac{\lambda}{4\pi \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right] e^{-\frac{4\pi \lambda_{\text{cr}}}{\lambda}}$$

$$\frac{2E_0 \pi^2 \bar{w}_{\text{max}} (1 - \bar{v}_0)}{d (3 - \bar{v}_0)^2} \left[d + \frac{\lambda}{4\pi} + \frac{\lambda}{2\pi (1 + \bar{v}_0)} + \frac{(1 + \bar{v}_0) (d + \lambda)}{8\pi} + \frac{d^2 \pi}{\lambda}\right] \left[1 - \frac{\lambda}{4\pi \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right] e^{-\frac{4\pi \lambda_{\text{cr}}}{\lambda}}$$

$$\frac{E_0 \pi \lambda \bar{w}_{\text{max}} (5 - 2\bar{v}_0 + \bar{v}_0^2)}{4d (3 - \bar{v}_0)^2 (1 + \bar{v}_0)} \left[1 + \frac{\lambda}{4\pi \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right] e^{-\frac{4\pi \lambda_{\text{cr}}}{\lambda}}$$

$$\frac{E_0 \pi \lambda \bar{w}_{\text{max}}}{(3 - \bar{v}_0)^2 (1 + \bar{v}_0)} \left[(3 - \bar{v}_0) (1 + \bar{v}_0) + \frac{2d \pi}{\lambda} (1 + \bar{v}_0)^2 \right] \left[1 + \frac{\lambda}{4\pi \lambda_{\text{cr}}} \sin \left(\frac{4\pi \lambda_{\text{cr}}}{\lambda}\right)\right] e^{-\frac{4\pi \lambda_{\text{cr}}}{\lambda}}$$

where the wavelength, $\lambda = \lambda(\varepsilon)$, wrinkle amplitude, $w_{\text{max}} = w_{\text{max}}(\varepsilon)$, and the critical strain, $\varepsilon_{\text{cr}}$, are defined by Equation A.3 and A.4:
The plane strain stiffness and Poisson ratio of the interfacial layer and matrix material are given by:

\[ \bar{E}_1 = \frac{E_1}{1-v_1^2} \quad , \quad \bar{E}_0 = \frac{E_0}{1-v_0^2} \quad , \quad \bar{\nu}_1 = \frac{\nu_1}{1-v_1} \quad , \quad \bar{\nu}_0 = \frac{\nu_0}{1-v_0} \]

Furthermore, the non-dimensional variables used in the equations are defined as:

\[ \bar{\lambda}_{cr} = \lambda_{cr}/t \quad , \quad \bar{\lambda} = \lambda(\bar{\varepsilon})/t \quad , \quad \bar{w}_{max} = w_{max}(\bar{\varepsilon})/\lambda_{cr} \quad , \quad \bar{w}_{max} = \bar{w}_{max}(\bar{\varepsilon})/\lambda(\bar{\varepsilon}) \]

The composite properties effective properties, i.e. the macroscopic effective stress and the effective stiffness, are very important for many engineering applications. The composite effective properties are both dependent on the microstructure and the material combination in the composite.

The macroscopic axial engineering stress-strain behavior of the composite is obtained by differentiating the strain energy density expression:

\[ \sigma_x^{\text{comp}}(\bar{\varepsilon}) = \frac{d U_{\text{Total}}(\bar{\varepsilon})}{d \bar{\varepsilon}} \]

\[ \sigma_x^{\text{comp}}(\bar{\varepsilon}) = \int_{\text{int.}} \cdot \frac{d U_{\text{int.layer}}(\bar{\varepsilon})}{d \bar{\varepsilon}} + \int_{\text{matrix}} \cdot \frac{d U_{\text{matrix}}(\bar{\varepsilon})}{d \bar{\varepsilon}} \]
\[ \sigma_{\text{comp}}^x (\xi) \]

\[ = \int_{\mathbb{R}} \left( \frac{E_0 \pi t^2}{3 \lambda^2} + \frac{2E_0 \pi t^2}{3 \lambda^2} \left( |\xi| - |\xi_e| \right) - \frac{E_0 \pi t^2}{12 \lambda_e \lambda} \left[ 1 + \left( |\xi| - |\xi_e| \right) \right] \sin \left( \frac{4\pi \lambda_e}{\lambda} \right) \right) \cos \left( \frac{4\pi \lambda_e}{\lambda} \right) \right) \]

\[ + \text{matrix} \left\{ \frac{E_0 \lambda \bar{w}_{\text{max}} (1+\bar{\nu}_0) \left( -1 + \cos \frac{2\lambda_e \pi}{\lambda} \right)}{2d \pi (3-\bar{v}_0)} \right\} - e^{\frac{2d \pi}{\lambda} \xi E_0 \bar{w}_{\text{max}} (1+\bar{\nu}_0) \left( -1 + \cos \frac{2\lambda_e \pi}{\lambda} \right)} \left( 2d \pi + \lambda \right) \]

\[ e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right) \]

\[ + \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]

\[ \frac{2e^{\frac{4d \pi}{\lambda} \xi E_0 \lambda \bar{w}_{\text{max}} (1-\bar{\nu}_0) \left( 2d \pi + \lambda \right)} \times \left( 1 + \frac{2}{1+\bar{\nu}_0} + \frac{1+\bar{\nu}_0}{2(1-\bar{\nu}_0)} \right) \left( 1 - \frac{\lambda \sin \frac{4\lambda_e \pi}{\lambda}}{4\lambda_e \pi} \right)}{2d \pi (3-\bar{v}_0)} \]
\[
\begin{align*}
&\frac{e^{-4d\pi}}{x} \cdot \frac{E_0\pi^2 W_{\text{max}}^2}{(3-v_0^2)(1+v_0)} \left( (3-v_0)(1+v_0) + \frac{2d(1+v_0)^2}{\lambda} \right) \left( \cos \frac{4\alpha_{\text{cr}}\pi}{\lambda} - \frac{\sin \frac{4\alpha_{\text{cr}}\pi}{\lambda}}{4\lambda_{\text{cr}}^2} \right) + \\
&\frac{E_0\pi(5-2v_0+v_0^2)\lambda W_{\text{max}}^2}{4d(1+v_0)(3-v_0)^2} \left( \cos \frac{4\alpha_{\text{cr}}\pi}{\lambda} - \frac{\sin \frac{4\alpha_{\text{cr}}\pi}{\lambda}}{4\lambda_{\text{cr}}^2} \right) + 2e^{-4d\pi} \cdot \frac{E_0\pi(1+v_0)\lambda}{d(3-v_0)^2} (1 - \\
&\frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) \left( \frac{\lambda}{4\pi} + \frac{1}{2(1+v_0)^2} + (1+v_0)^2 \left( \frac{\lambda}{4\pi} - \frac{d^2\pi\lambda}{\lambda^2} \right) \right) \\
&+ \frac{E_0\lambda\pi W_{\text{max}}^2}{d(3-v_0)^2} (1 - \bar{v}_0)(1 + \frac{2}{1+v_0} + \frac{1+v_0}{2(1-v_0)^2})(1 - \frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) - \\
&\frac{e^{-4d\pi}}{x} \cdot \frac{E_0\lambda\pi W_{\text{max}}^2}{d(3-v_0)^2} (1 + \frac{2}{1+v_0} + \frac{1+v_0}{2(1-v_0)^2})(1 - \frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) - \\
&\frac{4e^{-4d\pi}}{x} \cdot \frac{E_0\lambda\pi W_{\text{max}}^2}{d(3-v_0)^2} (d + \frac{\lambda}{4\pi} + \frac{1}{2(1+v_0)^2} + (1+v_0)^2 \left( \frac{\lambda}{4\pi} - \frac{d^2\pi\lambda}{\lambda^2} \right) (1 - \frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) - \\
&\frac{1}{(3-v_0^2)(1+v_0)^2} 2e^{-4d\pi} \cdot \frac{E_0\pi^2}{\lambda} (5-2v_0+v_0^2) \left( (3-v_0)(1+v_0) + \frac{2d(1+v_0)^2}{\lambda} \right) (1 + \frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) + \frac{E_0\lambda\pi W_{\text{max}}^2}{2d(1+v_0)(3-v_0)^2} (1 + \frac{\lambda\sin \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)}{4\lambda_{\text{cr}}^2}) + \frac{E_0\lambda\pi W_{\text{max}}^2}{2d^2(1+v_0)(3-v_0)^2} (1 + \cos \left( \frac{4\alpha_{\text{cr}}\pi}{\lambda} \right)) \right) \\
\end{align*}
\]

where we have in addition to the variables in Equation A.5, we introduce the variables:

\[
\bar{W}_{\text{max}} = \frac{\lambda(\bar{v})}{2\pi\lambda_{\text{cr}} \sqrt{|\bar{v}| - |\bar{v}_{\text{cr}}|}} = \frac{\lambda(\bar{v})}{\pi\lambda_{\text{cr}} \sqrt{|\bar{v}| - |\bar{v}_{\text{cr}}|}} \quad \text{and} \quad \bar{W}_{\text{max}} = \frac{1}{2\pi \sqrt{|\bar{v}| - |\bar{v}_{\text{cr}}|}} \quad \text{(A.7)}
\]
Similarly, the effective stiffness of the composite, $E_{x}^{\text{comp}}(\varepsilon)$, can be calculated at any applied strain, $\varepsilon$, by differentiating the effective engineering stress:

$$E_{x}^{\text{comp}}(\varepsilon) = \frac{d^{2}U_{\text{Total}}(\varepsilon)}{d\varepsilon^{2}}$$

$$E_{x}^{\text{comp}}(\varepsilon) = f_{\text{int.l}} \cdot \frac{d^{2}U_{\text{int.layer}}(\varepsilon)}{d\varepsilon^{2}} + f_{\text{matrix}} \cdot \frac{d^{2}U_{\text{matrix}}(\varepsilon)}{d\varepsilon^{2}}$$

$$E_{x}^{\text{comp}}(\varepsilon) = f_{\text{int.l}} \left\{ \left[ \frac{4E'\pi^{2}}{3\lambda^{2}} [1 + (\varepsilon - \varepsilon_{cT})] - \frac{E'\pi^{2}}{3\lambda^{2}} [2 + 3(\varepsilon - \varepsilon_{cT})] \cos \left( \frac{4\pi\lambda_{cr}}{\lambda} \right) - \frac{E_{\lambda} \pi}{12\lambda_{cr} A} \right] + \frac{16\pi^{2}\lambda_{cr}^{2}}{\lambda^{2}} (\varepsilon - \varepsilon_{cT}) \sin \left( \frac{4\pi\lambda_{cr}}{\lambda} \right) \right\}$$

$$+ f_{\text{matrix}} \left\{ E' - \frac{\varepsilon^{2}E' \lambda_{cr}(1+\nu)(-1+\cos[2E'\pi])}{8d(\varepsilon - E_{cr})^{2}\pi^{2}(3-\nu)} + \frac{\varepsilon^{2}E' \lambda_{cr}(1+\nu)(-1+\cos[2E'\pi])}{2d\sqrt{E - E_{cr}}\pi^{2}(3-\nu)} \right.$$
where the new variables represent:

\[ E' = \bar{E}_0, \quad v' = \bar{v}_0, \quad E = |\bar{e}|, \quad \text{Ecr} = |\bar{e}_{cr}|, \quad \lambda = \lambda(\bar{e}), \quad \text{and} \quad \lambda cr = \lambda_{cr} \]
Appendix B

Post-wrinkling behavior of composites

B-I: Effective behavior of composites

In Chapter 5 and 6 of this thesis we evaluated the macroscopic and effective behavior of elastic and elastic-plastic composites. It is concluded that in the pre-wrinkling region, the macroscopic behavior of the composites are directly dependent on the material combinations, material properties, and geometrical features (i.e. the volume fraction of interfacial layers) of the composites. Next, as the applied strain is increased, the critical applied strain that will cause wrinkling is directly a function of the material combination, i.e. the stiffness ratio, in the composites. Interestingly, in the post-wrinkling region, the variation between the different composites are reduced and similar behavior is observed between the different composites. The sensitivity of the macroscopic behavior are less affected by the material and geometrical parameters during the post-wrinkling region. This Appendix studies these results further.

To study the post-wrinkling behavior we create finite element (FE) models with initial geometries corresponding to the geometries due to wrinkling composites. That is to say, we create FE models were the interfacial layer has a sinusoidal shape, i.e. the wrinkling pattern, as its initial configuration. The lateral displacement of the interfacial layer, \( w(x, \varepsilon) \), at any applied strain \( \varepsilon \), is given by (details in Chapter 4):

\[
w(x, \varepsilon) = w_{max}(\varepsilon) \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right) = \left( \frac{\lambda(\varepsilon)}{\pi} \right) \sqrt{|\varepsilon| - |\varepsilon_{cr}|} \cdot \sin \left( \frac{2\pi x}{\lambda(\varepsilon)} \right)
\]  
(B.1)

where \( w_{max}(\varepsilon) \) is the post-buckling amplitude and \( \lambda(\varepsilon) \) is the post-buckling wavelength of the wrinkling pattern, given by:

\[
w_{max}(\varepsilon) = \frac{\lambda(\varepsilon)}{\pi} \sqrt{|\varepsilon| - |\varepsilon_{cr}|}
\]  
(B.2)

\[
\lambda(\varepsilon) = \lambda_{cr} e^{-|\varepsilon|}
\]

where the critical strain causing instability, \( \varepsilon_{cr} \), and the initial wrinkling wavelength, \( \lambda_{cr} \), is given by:

\[
\varepsilon_{cr} = -2.08 \left[ \frac{3-4\nu_0}{(1-\nu_0)^2} \right]^{2/3} \left( \frac{E_1}{E_0} \right)^{2/3}
\]  
(B.3)

\[
\lambda_{cr} = t \cdot 2.18 \left[ \frac{3-4\nu_0}{(1-\nu_0)^2} \right]^{1/3} \left( \frac{E_1}{E_0} \right)^{1/3}
\]
where $E_I$ and $E_0$ is the Young’s modulus of the interfacial layer and matrix layers, respectively, $v_0$ is the Poisson ratio of the matrix layers, and $t$ is the thickness of the interfacial layer.

We use Equation B.1 to create the sinusoidal patterns of the interfacial layer and then we run standard finite element simulations of compression of the composites with the sinusoidal interfacial layers. This means that in this case, the sinusoidal wrinkling pattern of the interfacial layer is not due to instability; the wrinkling pattern is a pure geometrical feature of the composite.

We evaluate three composites as shown in Figure B-1a:

**Composite 1:** The interfacial layer’s material properties are $(E_I, v_I) = (200\, MPa, 0.48)$, and the material properties of the matrix layers are $(E_0, v_0) = (1\, MPa, 0.48)$. The periodic structure has an interfacial layer thickness of $t=0.5\, mm$, and the thickness of the matrix layers are $d=11.75\, mm$, such that the volume fraction of interfacial layers is: $t/(2d+t)=0.021$. The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, $w(x, \bar{\varepsilon})$, at $\bar{\varepsilon}=1.5\bar{\varepsilon}_{cr}=-0.035$, such that: $w(x) = 0.31\, mm \cdot \sin\left(\frac{2\pi x}{9.78\, mm}\right)$.

**Composite 2:** The interfacial layer’s material properties are $(E_I, v_I) = (400\, MPa, 0.48)$, and the material properties of the matrix layers are $(E_0, v_0) = (1\, MPa, 0.48)$. The periodic structure has an interfacial layer thickness of $t=0.5\, mm$, and the thickness of the matrix layers are $d=11.75\, mm$, such that the volume fraction of interfacial layers is: $t/(2d+t)=0.021$. The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, $w(x, \bar{\varepsilon})$, at $\bar{\varepsilon}=1.5\bar{\varepsilon}_{cr}=-0.02$, such that: $w(x) = 0.302\, mm \cdot \sin\left(\frac{2\pi x}{12.48\, mm}\right)$.

**Composite 3:** The interfacial layer’s material properties are $(E_I, v_I) = (400\, MPa, 0.48)$, and the material properties of the matrix layers are $(E_0, v_0) = (1\, MPa, 0.48)$. The periodic structure has an interfacial layer thickness of $t=0.5\, mm$, and the thickness of the matrix layers are $d=11.75\, mm$, such that the volume fraction of interfacial layers is: $t/(2d+t)=0.021$. The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, $w(x, \bar{\varepsilon})$, at $\bar{\varepsilon}=4.5\bar{\varepsilon}_{cr}=-0.068$, such that: $w(x) = 0.87\, mm \cdot \sin\left(\frac{2\pi x}{11.1mm}\right)$.

The three different finite element models are compressed under plane strain conditions, by applying a compressive load to the representative volume element, RVE (method described in Chapter 2). The RVEs are compressed up to an applied strain of -0.07, and the deformations are shown in Figure B-1b.

The macroscopic behavior of the two different composites are studied. The strain energy density of the composites as a function of the applied compressive strain are shown in Figure B-2a and b. The macroscopic true stress-strain behavior and effective stiffness of the composites are shown in Figure B-2c and d.
The composites' strain energy density results show, as expected, that the energy density is higher for composite 2 than composite 1, because the interfacial layer is made of a stiffer material which will then provide more stiffness and higher strain energy (Figure B-2a and b). Simultaneously, when the initial wrinkling pattern is enhanced, composite 3, we see that the strain energy density is less than the other two composites. This shows that once the geometric configuration of the wrinkling pattern is enhanced sufficiently, i.e. the wrinkling amplitude has increased largely while the wrinkling wavelength has decreased, the energy density grows slower with the applied strain. This is because the contribution from the stiffer interfacial layers are reduced due to the enhanced wrinkling pattern and smaller local deformations in the interfacial layer caused as the applied strain is increased.

The macroscopic true stress-strain behavior of these composites (Figure B-2c) demonstrate that for composite 1 and 2 there is a change in the slope of the curves as a function of the macroscopically applied strain, while for composite 3, the slope remains nearly constant. This is due to the fact that for composite 1 and 2 with initial wrinkling patterns has small-amplitude such that the interfacial layers provide more stiffness to the composite, and with compression the wrinkling pattern are enhanced to the point where the interfacial layers provide little stiffness and the wrinkling pattern changes little with the applied macroscopic strain. On the other hand, composite 3 has an initial wrinkling pattern that is already high-amplitude such that further compression changes the wrinkling pattern very little, and hence the slope of the graph remains constant.
Figure B-2d show that the different composites approach the same effective stiffness as they are compressed and the wrinkling pattern is enhanced, i.e. the wrinkling amplitude increases and the wrinkling wavelength decreases. Composite 3 has an initial configuration where the wrinkling pattern is already enhanced enough such that the effective stiffness stays relatively constant at the asymptotic value as the composite is being compressed. The asymptotic behavior also indicates that once the wrinkling pattern is enhanced and dramatic enough, the effective behavior is governed by the small geometrical changes and not the stiffness ratio between the materials used.

**Figure B-2:** The strain energy density and effective behavior of three composites as function of the macroscopically applied compressive strain. The matrix material properties are \((E_0, v_0) = (1MPa, 0.48)\), and the interfacial layer concentration is \(t/(2d+t) = 0.02\) for all composites. The three composites have different material properties for their interfacial layer and different initial wrinkling patterns (defined in the legends); a) the composite strain energy density, b) the composite strain energy density in log-log graph indicating the power relationship lines, c) the macroscopic true stress-strain behavior, and d) the effective stiffness of the composites showing an asymptotic value.
B-II: Stability of wrinkling modes

In this thesis we have established that upon compression of dilute multilayered composites, a critical compressive strain is reached at which the thin and relatively stiffer interfacial layers of the composite will undergo instability and transform into a wrinkling pattern due to seeking a minimum energy state (Chapter 3). Upon further compression of these composites we have studied and developed models for the post-buckling behavior where the number of wrinkles remain constant, such that the amplitude of the wrinkling pattern increases while the wavelength of the wrinkling pattern decreases.

During post-buckling of any structure there can be special cases where the deformations and strain energies developed in the structure are such that there is another energy minimum state developed for the system as the applied load is increased. This means that there can be another instability occurring at a certain post-buckling strain that changes the deformation of the structure suddenly. In this section we show that for the multilayered wrinkling composites evaluated in this thesis, there is no other minimum energy state causing a secondary instability, such that compression of these composites into the post-buckling region will just lead to increased deformation and not another secondary instability.

Finite element (FE) simulations were conducted to explore the minimum energy state and the instability mode of composites where wrinkling has already occurred. As described in section Appendix B-I, FE models were generated with initial geometries corresponding to the geometries of wrinkling composites. That is to say, FE models were created such that the interfacial layer had an initial sinusoidal shape, i.e. the wrinkling pattern. Then a buckling analysis would be performed using ABAQUS as described in Chapter 2.2, and the buckling modes of the composite was compared with its initial configuration.

We evaluate the same three composites that were described in section Appendix B-I:

**Composite 1:** The interfacial layer’s material properties are \((E_1, v_1) = (200 MPa, 0.48)\), and the material properties of the matrix layers are \((E_0, v_0) = (1 MPa, 0.48)\). The periodic structure has an interfacial layer thickness of \(t=0.5 mm\), and the thickness of the matrix layers are \(d=1.75 mm\), such that the volume fraction of interfacial layers is: \(v/(2d+t)=0.021\). The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, \(w(x, \varepsilon)\), at \(\varepsilon=1.5 \varepsilon_{cr} = -0.035\), such that: \(w(x) = 0.31 mm \cdot \sin \left( \frac{2\pi x}{9.78 mm} \right) \).

**Composite 2:** The interfacial layer’s material properties are \((E_1, v_1) = (400 MPa, 0.48)\), and the material properties of the matrix layers are \((E_0, v_0) = (1 MPa, 0.48)\). The periodic structure has an interfacial layer thickness of \(t=0.5 mm\), and the thickness of the matrix layers are \(d=11.75 mm\), such that the volume fraction of interfacial layers is: \(v/(2d+t)=0.021\). The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, \(w(x, \varepsilon)\), at \(\varepsilon=1.5 \varepsilon_{cr} = -0.02\), such that: \(w(x) = 0.302 mm \cdot \sin \left( \frac{2\pi x}{12.48 mm} \right) \).
Composite 3: The interfacial layer’s material properties are \((E_1, v_1) = (400 \text{MPa}, 0.48)\), and the material properties of the matrix layers are \((E_0, v_0) = (1 \text{MPa}, 0.48)\). The periodic structure has an interfacial layer thickness of \(t=0.5\text{mm}\), and the thickness of the matrix layers are \(d=11.75\text{mm}\), such that the volume fraction of interfacial layers is: \(t/(2d+t)=0.021\). The sinusoidal pattern of the interfacial layer is set to be the wrinkling pattern, \(w(x, \bar{\varepsilon})\), at \(\bar{\varepsilon} = 4.5\bar{\varepsilon}_{cr} = -0.068\), such that: \(w(x) = 0.87\text{mm} \cdot \sin\left(\frac{2\pi x}{11.1\text{mm}}\right)\).

Figure B-3 shows the initial configuration of the three composites evaluated, as well as the FE results from the buckling analysis showing the buckling shape (first eigenmode) and the corresponding eigenvalue. For all cases we can see that the number of wrinkles remain constant between the initial configuration and the buckling shape result from the FE analysis. That is to say that the buckling analysis gives the same results as the initial shape of the composite. Hence, these FE results verify that the minimum energy state is always configuration obtained from pure simple compression of the composite, and there is no other sudden minimum energy state that will lead to a secondary instability changing the buckling shape of the wrinkling composite.
Composite 1

Initial configuration:
No. of wrinkles = 5

Buckling shape (1st eigenmode)
No. of wrinkles = 5
Eigenvalue = -0.043

Composite 2
No. of wrinkles = 4
Eigenvalue = -0.027

Composite 3
No. of wrinkles = 4
Eigenvalue = -0.122

Figure B-3: The RVE of the composites evaluated through finite element simulations using ABAQUS. a) The interfacial layers have an initial sinusoidal pattern. b) The results from the buckling analysis in ABAQUS showing the first buckling mode. The first buckling mode is same wrinkling pattern as the initial configuration, which means that there is no secondary instability occurring during the post-buckling compression of these wrinkling composites.
Appendix C
MATLAB Code

Below is the MATLAB code for calculating the macroscopic behavior and properties of the elastic-plastic composites (Chapter 6 in thesis). It calculates the energy absorbed, -dissipated, and -stored when an elastic-plastic composite is being loaded. The macroscopic stress-strain behavior and effective stiffness is also calculated.

```matlab
m=1;

Phaseshift=[3.49] % at what point does sin(0)
Maxstrain=0.075    % compressing up to this macroscopic strain
Strains=[0:0.001:Maxstrain];

t=0.5     % Thickness of the interfacial layer
L=60;     % Length of composite. Want to make sure length is long enough for several peaks
d=24-t;   % Matrix layer thickness

% Material properties of Interfacial layer (*1) and Matrix (*0)
v1=0.48;   % Poisson ratio
v0=0.48;
EO=1;     % Young's modulus
E1=1200;
YieldStress=25

% Plane strain material properties
El_plane=E1/(1-v1^2);
EO_plane=EO/(1-v0^2);
v1_plane=v1/(1-v1);
v0_plane=v0/(1-v0);
G0_plane=EO_plane/(2*(1+v0_plane));

x_point=Phaseshift;

for round=1:size(Strains,2)
    Applied_strain=Strains(1,m);

    % Calculating critical wrinkling strain and wrinkling wavelength
    Constant_strain=2.08*(((3-4*v0)/(1-v0))^2)^(-2/3);
    Constant_lambda=2.18*(((3-4*v0)/(1-v0))^2)^((1/3));
    Cr_strain=Constant_strain*((El/EO))^(-2/3);
    Cr_lambda=t*Constant_lambda*((El/EO))^(1/3);
```

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Meshing the interfacial layer

```matlab
step_x=30;
step_y=40;
x_i=linspace(0,L,L*step_x);
y_i=linspace(0,0.5*t,(0.5*t)*step_y);
[Xi,Yi]=meshgrid(x_i,y_i);
```

%% applied strain is less than critical wrinkling strain, assume the interface is only being axially compressed

```matlab
if Applied_strain <= Cr_strain;
    Epsilon_x_top=-Applied_strain;
    Epsilon_x_bottom=-Applied_strain;
    Epsilon_y_top=-v1_plane*(-Applied_strain);
    Epsilon_y_bottom=-v1_plane*(-Applied_strain);

    for k=1:size(y_i,2);
        for l=1:size(x_i,2);
            Sigma_x_top(k,l,m)=-E1_plane*Applied_strain;
            Sigma_z_top(k,l,m)=v1*Sigma_x_top(k,l,m);
            Sigma_x_bottom(k,l,m)=-E1_plane*Applied_strain;
            Sigma_z_bottom(k,l,m)=v1*Sigma_x_bottom(k,l,m);
        end
    end

    Sigma_Mises_top=sqrt(1+v1^2-v1).*abs(Sigma_x_top);
    Yield_Strain=YieldStress/(E1_plane*sqrt(1+v1^2-v1));
    value=0.5;
    Constant_stress=YieldStress/(sqrt(1+value^2-value));

    Epsilon_x_plastic_top(1,m)=-(Applied_strain-Yield_Strain);
    Epsilon_x_plastic_bottom(l,m)=-(Applied_strain-Yield_Strain);
    Epsilon_x_elastic_top(1,m)=-Yield_Strain;
    Epsilon_x_elastic_bottom(l,m)=-Yield_Strain;
```

% Defining the different energy sources

```matlab
Matrix_Energy_Pre(1,m)=0.5*E0_plane*d*L*Applied_strain^2;
Interface_Energy_Pre(1,m)=0.5*E1_plane*t*L*Yield_Strain^2+
    (Constant_stress*(Applied_strain-Yield_Strain))'*t*L;
Graph_Energy(1,m)=Interface_Energy_Pre(1,m)+Matrix_Energy_Pre(1,m)
```
Graph Energy Volume(l,m) = (Graph Energy(l,m)) / ((d+t) * L);
Graph Energy Plastic(l,m) = (Constant stress * (Applied strain - Yield Strain)) * t * L;
Graph Energy Elastic Interface(l,m) = 0.5 * El plane * t * L * Yield Strain^2;
Graph Energy Elastic Matrix(l,m) = Matrix Energy Pre(l,m);

else
   Sigma x top(:,:,m) = -El plane * Applied strain;
   Sigma z top(:,:,m) = -vl * El plane * Applied strain;
   Epsilon x elastic_top(l,m) = -Applied strain;
   Epsilon x plastic_top(l,m) = 0;

   Sigma x bottom(:,:,m) = -El plane * Applied strain;
   Sigma z bottom(:,:,m) = -vl * El plane * Applied strain;
   Epsilon x elastic_bottom(l,m) = -Applied strain;
   Epsilon x plastic_bottom(l,m) = 0;

end

m=m+1

else

% Finding postbuckling wavelength
Post lambda = Cr lambda * exp(-Applied strain);
% Calculating the amplitude
A = (Post lambda / pi()) * sqrt(Applied strain - Cr strain);
% Calculating the phase shift between Theory and FE
phase = -(2 * pi() * x point) / Post lambda;

% Finding stresses in Interfacial layer

% Calculating deformation
w0 = v0 * Applied strain * (d+t);
W = w0 + A * sin(((2 * pi() * x_i) / Post lambda) + phase);

% Calculating local strains
Epsilon x top = Cr strain + Yi. * (4 * pi^2 * A / Post lambda^2)
              . * sin(((2 * pi() * x_i) / Post lambda) + phase));
Epsilon x bottom = -Cr strain - Yi. * (4 * pi^2 * A / Post lambda^2)
                  . * sin(((2 * pi() * x_i) / Post lambda) + phase));
Epsilon y top = -vl plane * (-Cr strain) - Yi
                  . * (vl plane * 4 * pi^2 * A / Post lambda^2)
               . * sin(((2 * pi() * x_i) / Post lambda) + phase));
Epsilon y bottom = vl plane * (-Cr strain) + Yi
                  . * (vl plane * 4 * pi^2 * A / Post lambda^2)
               . * sin(((2 * pi() * x_i) / Post lambda) + phase));

% Saving results
% Saving results
Epsilon_x_top_2(:,:,m)=Epsilon_x_top;
Epsilon_y_top_2(:,:,m)=Epsilon_y_top;
Epsilon_x_bottom_2(:,:,m)=Epsilon_x_bottom;
Epsilon_y_bottom_2(:,:,m)=Epsilon_y_bottom;

% Calculating stresses
Sigma_x_top_1=El_plane*Epsilon_x_top;
Sigma_z_top_1=vl*Sigma_x_top_1;
Sigma_x_bottom_1=El_plane*Epsilon_x_bottom;
Sigma_z_bottom_1=vl*Sigma_x_bottom_1;

% Saving results
Sigma_x_top_2(:,:,m)=Sigma_x_top_1;
Sigma_z_top_2(:,:,m)=Sigma_z_top_1;
Sigma_x_bottom_2(:,:,m)=Sigma_x_bottom_1;
Sigma_z_bottom_2(:,:,m)=Sigma_z_bottom_1;

%% Calculate Mises stresses for each element and check if yielding has occurred or not
Sigma_Mises_top=sqrt(1+vl^2-vl).*abs(Sigma_x_top_1);
Sigma_Mises_bottom=sqrt(1+vl^2-vl).*abs(Sigma_x_bottom_1);
Yield_Strain=YieldStress/(El_plane*sqrt(1+vl^2-vl));
value=0.5;
Constant_stress=YieldStress/(sqrt(1+value^2-value));

%% If yielding has occurred for an element, set stresses constant and define plastic local strain
for k=1:size(y_i,2);
    for l=1:size(x_i,2);
        if Sigma_Mises_top(k,l) > YieldStress
            Sigma_x_top(k,l,m)=sign(Sigma_x_top_1(k,l))*Constant_stress;
            Sigma_z_top(k,l,m)=value.*Sigma_x_top(k,l,m);
            Epsilon_x_plastic_top(k,l,m)=sign(Epsilon_x_top_2(k,l,m))*(abs(Epsilon_x_top_2(k,l,m))-Yield_Strain);
            Epsilon_x_elastic_top(k,l,m)=sign(Epsilon_x_top_2(k,l,m))*Yield_Strain;
        else
            Sigma_x_top(k,l,m)=Sigma_x_top_1(k,l);
            Sigma_z_top(k,l,m)=Sigma_z_top_1(k,l);
            Epsilon_x_elastic_top(k,l,m)=Epsilon_x_top_2(k,l,m);
            Epsilon_x_plastic_top(k,l,m)=0;
        End
    End
if Sigma_Mises_bottom(k,l) > YieldStress
    Sigma_x_bottom(k,l,m)=sign(Sigma_x_bottom_1(k,l))*Constant_stress;
    Sigma_z_bottom(k,l,m)=value.*Sigma_x_bottom(k,l,m);
    Epsilon_x_plastic_bottom(k,l,m)=sign(Epsilon_x_bottom_2(k,l,m))*(abs(Epsilon_x_bottom_2(k,l,m))-Yield_Strain);
    Epsilon_x_elastic_bottom(k,l,m)=sign(Epsilon_x_bottom_2(k,l,m))*Yield_Strain;
else

% size for elements used for integration
dx=1/step_x;
dy=1/step_y;

%Calculating the energies for each element in the interface layer
Interface_Energy_Integral_Elastic_1 = 0.5*(Sigma_x_top_integral.*Epsilon_x_elastic_top(:,:,m));
Interface_Energy_Integral_Plastic_1 = (Sigma_x_top_integral.*Epsilon_x_plastic_top(:,:,m));
Interface_Energy_Integral_Elastic_2 = 0.5*(Sigma_x_bottom_integral.*Epsilon_x_elastic_bottom(:,:,m));
Interface_Energy_Integral_Plastic_2 = (Sigma_x_bottom_integral.*Epsilon_x_plastic_bottom(:,:,m));

%Calculating the total energies for the interfacial layer by integrating
Interface_Energy_Post_Elastic(l,m) = (sum(Interface_Energy_Integral_Elastic_1(:)))*dx*dy + (sum(Interface_Energy_Integral_Elastic_2(:)))*dx*dy;
Interface_Energy_Post_Plastic(l,m) = (sum(Interface_Energy_Integral_Plastic_1(:)))*dx*dy + (sum(Interface_Energy_Integral_Plastic_2(:)))*dx*dy;

Interface_Energy_Post_Total(l,m) = Interface_Energy_Post_Elastic(l,m) + Interface_Energy_Post_Plastic(l,m);

%Calculating the energies in the matrix layer
%Mesh the matrix layers
x_m=linspace(0,L,L*step_x);
y_m=linspace(0.5*d,(0.5*d)*step_y);
[Xm,Ym]=meshgrid(x_m,y_m);

%Calculating energies for each element in the matrix layers
Matrix_Energy_Integral = 0.5.*(E0_plane*Applied_strain^2) - 
(((8*pi()^2)*E0Plane*Applied_strain*(1+v0_plane)*A)/(Post_lambda^2*(3-v0_plane))).*Ym.*((sin((2.*pi().*Xm)./Post_lambda)+phase)).*exp(-4.*pi()*.Ym./Post_lambda) + (((32*pi(A)^3*E0_plane*A^2*(1-v0_plan)))/(Post_lambda^3*(3-v0_plan)^2)).*(Ym+(pi()*(1+v0_plane).*Ym.^2))./(Post_lambda*(1-v0_plan)+Post_lambda/(2*pi()*(1+v0_plan))).*(sin((2.*pi().*Xm)/Post_lambda)+phase).^2).*exp(-4.*pi()*.Ym./Post_lambda) + (((8*pi()^2)*E0Plane*A^2)/(Post_lambda^2*(3-v0_plane)^2*(1+v0_plan))).*(1-v0_plan+((2*pi()*(1+v0_plan).*Ym)./(Post_lambda)+phase).^2).*(cos((2.*pi().*Xm)/(Post_lambda)+phase).^2).*exp(-4.*pi()*.Ym./Post_lambda));

% Element size used for integration
dx=1/step_x;
dy=1/step_y;
Calculating the total energies for the matrix layer by integrating
\[
\text{Matrix\_Energy\_Post}(1,m) = 2 \times (\text{sum(Matrix\_Energy\_Integral(::)))} \times dx \times dy;
\]

Adding up the energies - TOTAL ENERGY IN COMPOSITE

\[
\begin{align*}
\text{Graph\_Energy}(1,m) &= \text{Interface\_Energy\_Post\_Total}(1,m) + \text{Matrix\_Energy\_Post}(1,m) \\
\text{Graph\_Energy\_Volume}(1,m) &= \frac{\text{Interface\_Energy\_Post\_Total}(1,m) + \text{Matrix\_Energy\_Post}(1,m)}{(d+t)\times L}; \\
\text{Graph\_Energy\_NonDim}(1,m) &= \frac{\text{Interface\_Energy\_Post\_Total}(1,m) + \text{Matrix\_Energy\_Post}(1,m)}{(E0\times d\times L) + (E1\times t\times L)}; \\
\text{Graph\_Energy\_Plastic}(1,m) &= \text{Interface\_Energy\_Post\_Plastic}(1,m); \\
\text{Graph\_Energy\_Elastic\_Interface}(1,m) &= \text{Interface\_Energy\_Post\_Elastic}(1,m); \\
\text{Graph\_Energy\_Elastic\_Matrix}(1,m) &= \text{Matrix\_Energy\_Post}(1,m);
\end{align*}
\]

\[
m = m + 1
\]

end

Taking derivative of the energy density expression to find the macroscopic stress-strain, and the effective stiffness

for \( s = 1 : \text{size(Stiffnesses,2)} \)

% Macroscopic stress
\[
\begin{align*}
\text{derivative\_1}(s,:) &= (\text{Graph\_Energy\_Volume}(s,3:end) - \text{Graph\_Energy\_Volume}(s,1:end-2)) \div (\text{Strains}(1,3:end) - \text{Strains}(1,1:end-2)); \\
d\text{strain\_1} &= \text{Strains}(1,2:end-1);
\end{align*}
\]

% Effective stiffness stress
\[
\begin{align*}
\text{derivative\_2}(s,:) &= (\text{derivative\_1}(s,3:end) - \text{derivative\_1}(s,1:end-2)) \div (d\text{strain\_1}(1,3:end) - d\text{strain\_1}(1,1:end-2)); \\
d\text{strain\_2} &= d\text{strain\_1}(1,2:end-1);
\end{align*}
\]

end