

PLANETARY RADAR RANGING ERROR ANALYSIS

by

MICHAEL RICHARD COUSSA

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Signature of Author..... Signature redacted
Department of Earth and Planetary Sciences
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Certified by..... Signature redacted
Thesis Supervisor

Accepted by..... Signature redacted
Chairman, Departmental Committee on
Graduate Students



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ABSTRACT

A semi-analytical investigation of the error in the estimate of the radar time delay to an under-spread target is conducted. The target is assumed to be described by a Gaussian scattering law in both delay and doppler frequency and it is assumed that system noise power exceeds the average echo signal power in a single coherent integration period. The results of the investigation are applied to typical radar ranging for Venus and the standard deviation of the delay error is given in terms of radar receiver and processing parameters. When the total reception time of the signal and average transmitted power is fixed the delay error is seen to decrease as the resolution of the radar system in delay and doppler frequency is increased. As resolution becomes sufficiently fine a point of diminishing return (with respect to the resulting decrease in the delay error) is reached.

Thesis Supervisor: Charles C. Counselman, III
Title: Assistant Professor of
Planetary Science

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I. INTRODUCTION

Planetary radar ranging is done routinely for Mercury, Venus, and Mars. The usual method for both simple ranging and mapping planetary surfaces is the delay-doppler method, first proposed by P. E. Green in 1959 (see Green, 1960).

Green pointed out that the signal reflected from all points along the intersection of the planet's surface with a plane parallel to the plane containing the apparent axis of rotation and line of sight to the planet will have the same doppler shift due to the relative motion of the earth and planet (see figure 1). As we will see later, the doppler shift for points on this line relative to the doppler shift for the sub-radar point is directly proportional to the distance of the plane containing this line of intersection from the plane parallel to it passing through the subradar point (the subradar point is the intersection with the planetary surface of the line joining the radar and the center of mass of the planet). In practice, processing of the received echo signal can never achieve resolution fine enough to tell us how much power was reflected from a "line" on the planet's surface but does indicate how much was reflected by a "strip" of finite width on the surface, called a doppler strip (see figure 1b). It is convenient to express width of a doppler strip in units of frequency since it is the frequency shift of the echo signal that is actually determined by analysis of the receiver output power. It will

be shown later (see section IB) that the width of a strip is inversely proportional to the Coherent Integration Period (CIP) of the receiver output signal so that in order to reduce the width of a strip to zero (i.e., reduce the strip to a line) the CIP would have to be infinitely long. There are practical and statistical reasons for limiting the length of the CIP. They are discussed in section II.

Green also noted that the intersection of the surface of a spherical planet with a plane perpendicular to the line of sight to the planet forms a circle and the signals reflected from all points of the circle will return to the antenna at the same time. In practice, as one might expect, instrumental factors, the uneven nature of the planetary surface, and statistical reasons prevent us from resolving a circle on the surface of the planet and limit us to resolving an annulus or ring of finite width on the surface, called a delay ring, from which we may say an echo signal was reflected. The width of a delay ring is expressed in units of time.

Although planetary targets subtend very small angles compared to typical antenna beamwidths, Green noticed that it is possible to resolve small regions on a planet's surface by resolving the distribution of echo power in both doppler and delay, i.e., as a function of doppler frequency shift and of time of reception of the echo signal. The power density at each point in the delay-doppler coordinate system is related to the power of the signal reflected from the region of the planet's surface corresponding to the intersection of a delay

ring and a doppler strip (see figure 1b). Just how the received power distribution is related to the form of the transmitted signal and to the reflection characteristics of the planet's surface will be discussed when we talk about the ambiguity function (section IB). For a perfectly smooth and spherical planet most of the echo power would be received effectively from the first Fresnel zone (see e.g., Rossi, pl67) so that the distribution of power versus delay and doppler would be very narrowly peaked. However, because of topography and roughness of real planetary surfaces a measurable amount of power is backscattered from other regions on the planet.

Planetary radar ranging is done by sampling the received power density distribution (in delay and frequency) at several different values of delay and at one or more doppler frequencies. To determine the position of the subradar point a least squares fit is made to the observed echo power distribution of a theoretical distribution or "template" (see Rogers et al, 1970). One of the parameters estimated in the fit is the delay to the subradar point. The resulting uncertainty in the delay estimate and the effect on this uncertainty of varying the number and resolution of delay rings and doppler strips will be examined in this paper.

I A. BACKSCATTERED POWER

We begin by obtaining an expression for the distribution in time delay and doppler frequency of power backscattered by

the planet's surface. The delay and doppler shift associated with the motion of the center of mass of the planet will be ignored for the moment; in the radar receiver we attempt to eliminate these effects based upon our a priori knowledge of the relative motion between the target and receiver. The effect on the backscattered power distribution of the signal processing performed in the radar receiver will be discussed in the next section (I B).

A wave reflected from an annulus on the planetary surface subtending an angle between ϕ and $\phi + d\phi$ was incident upon the surface at an angle between ϕ and $\phi + d\phi$ (see figure 1a). This corresponds to a signal delay time between τ and $\tau + d\tau$ (by delay to a point on the planet's surface we mean round trip time of the echo travelling between the antenna and a point on the planetary surface). Because the planet is rotating the signal reflected from different infinitesimal regions along the same annulus will have different values of doppler shift. In particular, the doppler shift lying between f and $f + df$ is due to reflection from an infinitesimal region of the planet's surface formed by the intersection of the surface and the infinitesimal region parallel to the plane containing the line of sight and apparent axis of rotation (see figure 1b).

The width of a delay ring and a doppler strip as discussed in this section and this section only is thus $d\tau$ and df respectively and does not refer to the delay and frequency resolution of the radar system. The delay and doppler

resolution depend upon the shape of the transmitted pulse and the radar receiver system as we will see in sections IIIB and IB.

The Cartesian coordinate system x, y, z (see figure 2) has its origin at the center of mass of the planet with the z axis pointing toward the antenna, the x axis perpendicular to the z axis and the y axis perpendicular to the $x-z$ plane. The apparent angular rotation of the planet projected into the $x-y$ plane, $\vec{\omega}_a$, is along the y axis. The radius of the planet is constant for our purposes and is referred to as ρ in figure 2. The infinitesimal area of the surface corresponding to the n^{th} delay ring is centered at $z = z_n$ and the intersection of the m^{th} doppler strip with this delay ring is centered on the coordinates x_m, y_{mn}, z_n in the (x, y, z) coordinate system.

The time delay (round trip time of the echo reflected from a point on the surface) for power reflected from the subradar point is γ_0 and the time delay for power reflected from the n^{th} delay ring (see figure 2) is γ_n . The differential area of intersection of a delay ring with a doppler strip is dS on the surface of the planet and its projected area on the $x-z$ plane is dA (see figure 2). From figure 2 we see we may write z_n as

$$\begin{aligned} z_n &= \rho - \frac{c}{2} (\gamma_n - \gamma_0) \\ &= \rho - \frac{c}{2} T \end{aligned} \tag{1}$$

where $T \equiv \gamma_n - \gamma_0$
 $c = \text{speed of light}$

solving for x_m

$$x_m = \frac{cF}{2\sqrt{w_0}} \quad (2)$$

The differential area in the x-z plane, dA , is (see figure 2)

$$dA = dx dz = \frac{c^2}{4\sqrt{w_0}} dF dT \quad (3)$$

θ is the angle between the y axis and the vector between dS and the origin in the (x,y,z) coordinate system so that (see figure 2)

$$dA = dS \cos\theta$$

Thus

$$dS = \frac{c^2 dF dT}{4\sqrt{w_0} \cos\theta} \quad (4)$$

To obtain $\cos\theta$ in terms of the delay-doppler position of we define direction cosines as follows

$$\frac{x_m}{\rho} = \frac{cF}{2\rho w_0} \quad (5)$$

$$\frac{z_n}{\rho} = 1 - \frac{c}{2\rho} T \quad (6)$$

$$\frac{y_{mn}}{\rho} = \cos\theta \quad (7)$$

If f_m is the doppler frequency of the signal returning from the doppler strip centered at (x_m, y_m, z_m) , f_0 is the doppler frequency of the signal returning from the subradar point, and ν is the frequency of the radar carrier signal emitted from the antenna, then we have

$$f_m - f_0 \equiv F = 2\nu \frac{x_m}{c} \omega_a$$

Using the fact the sum of the squares of the direction cosines equals 1 we write

$$x_m^2 + y_m^2 + z_m^2 = \rho^2$$

$$y_m^2 = \pm (\rho^2 - x_m^2 - z_m^2)^{1/2} = \pm \rho \cos \theta$$

Using equations (5) and (6) we may write $\cos \theta$ as

$$\cos \theta = \pm \left[1 - \frac{c^2 F^2}{4\nu^2 \rho^2 \omega_a^2} - \left(1 - \frac{c}{2\rho T} \right)^2 \right]^{1/2} \quad (7A)$$

$$= \pm \frac{c}{2\rho \nu \omega_a} \left[\nu^2 \omega_a^2 T^2 \left(\frac{4\rho}{c} - T \right) - F^2 \right]^{1/2} \quad (7B)$$

The negative value of $\cos \theta$ corresponds to the element dS in the Southern hemisphere of the planet (if the direction of $\vec{\omega}_a$ is defined to be North). For the present it is assumed the planet is symmetric about the x-z plane. We shall also assume the planet has rotational symmetry about the **z** axis. Substituting the positive value of $\cos \theta$ into equation (4)

we have

$$dS = \frac{c^2}{4V\omega_a} \frac{dF dT}{\left[1 - \frac{c^2 F^2}{4\rho^2 v^2 \omega_s^2} - \left(1 - \frac{c}{2\rho T}\right)^2\right]^{1/2}} \quad (8)$$

If we call the power scattered back to the antenna per unit area of planetary surface $\frac{P_R}{\sigma} \sigma'(\phi)$ (P_R and σ are defined below; since rotational symmetry has been assumed σ' is a function of only the angle of incidence, ϕ), then the power at the input of the radar receiver is

$$P''(z_n, x_m) dx dz = P_R' \frac{\int \sigma'(\phi) dS}{\int \sigma(\phi) dS} \quad (9)$$

where (see Radar Studies of Mars, 1970)

$$\int \sigma(\phi) dS = \text{radar cross section} \equiv \sigma$$

σ' = cross section per unit area of surface of the target

$$P_R' = B' \cdot P_T$$

$$B' = \left(\frac{G_T L_T L_A}{4\pi R^2} \right) \times \left(\frac{\sigma}{4\pi R^2} \right) \times L_A$$

σ = radar cross section

P_T = transmitted power (peak value)

G_T = antenna gain

L_T = transmitter waveguide attenuation

R = distance to planet

L_A = atmospheric attenuation

We would now like to obtain a relation between the angle of incidence, ϕ , and the relative delay and doppler coordinates T, F, so that $\sigma'(\phi)$ and thus the right hand side of equation (9) may be expressed entirely in terms of T and F. To do this we note, from figure 2

$$\begin{aligned} \rho \sin \phi &= \sqrt{1 - z_n^2} \rho \\ &= \rho \sqrt{1 - \left(\rho - \frac{cT}{2}\right)} \end{aligned}$$

From the above equations we can express ϕ as a function of z or T. We denote these functional relations by writing

$$\begin{aligned} \phi &= \phi_z(z) \\ &= \phi_T(T) \end{aligned}$$

so that σ' may be expressed in terms of z or T: (although, of course, $z \neq T$)

$$\begin{aligned} \sigma' &\equiv \sigma'(\phi) = \sigma'_z(\phi_z(z)) \equiv \sigma'_z(z) \\ &= \sigma'_T(\phi_T(T)) \equiv \sigma'_T(T) \end{aligned}$$

where the different subscripts denote the different functional dependence. Expressing σ' in the form $\sigma'_T(T)$ and using equation (8) we may write equation (9) as

$$P'_R \frac{\sigma'(\phi) ds}{\sigma} = P'(T, F) dT dF$$

where

$$P'(T, F) = K \frac{\sigma'_T(T)}{\left[1 - \left(\frac{cF}{2\rho\sqrt{W_0}}\right)^2 - \left(1 - \frac{cT}{2\rho}\right)^2\right]^{1/2}} \quad (10)$$

$$K = \frac{P'_R}{\sigma} \quad (11)$$

and σ' is called the target scattering function.

I B. THE AMBIGUITY FUNCTION

So far we have only talked about the power scattered by the planet and not the power appearing at the output of our receiver. The connection between the two is made by means of a function $\Psi^2(T,F)$ called the ambiguity function of the radar system.

In this section we will see that $\Psi^2(T,F)$ actually represents the radar system's response to a point non-fluctuating target (examples of this kind of target will be given later in this section). If, for example, the point target is at zero delay and doppler frequency T and F are the delay and frequency variables of the radar output.

The function $\Psi^2(T,F)$ depends on both the transmitted waveform and the receiver impulse response but we will assume that the receiver filter has an impulse response, $h_1(t)$ which is matched to the transmitted waveform. If the transmitted waveform is $\gamma(t)$ then $h_1(t)$ will be $\gamma(t_0 - t)$, the time reversal of $\gamma(t)$ shifted an amount t_0 to make $h_1(t)$ realizable. For this combination of signal and filter $\Psi^2(T,F)$ is given by (see Evans and Hagfors, chapter 1, 1968)

$$\Psi^2(T,F) = \left| \int_{-\infty}^{\infty} dt \underline{\gamma}^*(t) \underline{\gamma}(t+T) e^{j2\pi Ft} \right|^2 \quad (12)$$

where the star indicates the complex conjugate of the function and the line beneath the symbol indicates the function is complex. We use complex numbers to describe both the amplitude and phase of a signal (see, e.g., Bracewell, 1965).

The amplitude of the complex modulated carrier wave $\underline{Y}(t)$ is $|\underline{Y}(t)| \equiv \sqrt{\underline{Y}(t)\underline{Y}(t)^*}$, the magnitude of the complex number $\underline{Y}(t)$. The projection of $\underline{Y}(t)$ onto the imaginary (j) axis is called the imaginary part of $\underline{Y}(t)$ and the projection of $\underline{Y}(t)$ onto the real axis is referred to as the real part of $\underline{Y}(t)$. The phase of $\underline{Y}(t)$ with respect to the positive real axis is the tangent of the angle between $\underline{Y}(t)$ and the positive real axis. For example, the phase of $\underline{Y}(t)$ is 0° relative to the positive real axis and the phase of $jY(t)$ relative to the positive real axis is 90° . Likewise, the phase of $-jY(t)$ and $-Y(t)$ is -90° and 180° , respectively. In this paper, however, we will be concerned only with purely real rectangular pulses which modulate the transmitted carrier wave.

Radar ranging signal processing utilizes both analogue and digital systems (see, for example, Pettengill et al, 1969). The echo signal is first mixed to lower frequencies and passed through a pulse matched filter by analogue systems and then sampled at discrete time intervals, converted to digital form and presented to a digital computer for final processing. This processing scheme will be described in detail later in this section. In this case, $\Psi^2(T,F)$ take the form (see Evans and Hagfors, chapter 1, 1968, and Shapiro, 1967)

$$\Psi^2(T,F) = |H_0(F)|^2 \sum_{r=0}^{R-1} |A_R L_R \int_{-\infty}^{\infty} d\xi X(\xi) m(\xi + T - rT_p)|^2 \quad (13)$$

where

A_R = antenna receiving aperture

L_R = waveguide attenuation in the receiver

and where $m(t)$ is a periodic series of rectangular pulses of duration τ_b (called the baud length), and repetition period t_p (called the pulse repetition period or PRP), $\chi(t)$ is the envelope of one period of $m(t)$ and $H(F)$ is

$$H(f) = \frac{\sin^2 \pi M t_p F}{\sin^2 \pi t_p F} \quad (14)$$

where $M t_p$ is the Coherent Integration Period (defined later in this section) and R is the number of Coherent Integration Periods. The variables just mentioned will be precisely defined as the need arises later in this section. For the present, it is sufficient to note that $\Psi(T, F)$ is separable (i.e., has a part dependent only upon F and a part dependent only upon T). The theoretical radar receiver output as a function of T and F , $P(T, F)$, may be written as a convolution of the ambiguity function with $P'(T, F)$, the target scattering function (which describes the distribution of backscattered power from the target):

$$P(T, F) = \iint d\gamma df \Psi^*(T-\gamma, F-f) P'(\gamma, f) \quad (15)$$

where the integral is taken over all values of relative delay and doppler of the target.

Let us examine the physical meaning of $\Psi^*(T, F)$ and investigate how the ambiguity function contributes to ambiguity in relating points of the target from which the

transmitted signal was reflected to points in the plot of output power vs. T and F . The theoretical model of the receiver output power, $P(T,F)$, may be considered a result of the ambiguity function "probing" the target scattering function. Specifically, from our understanding of the convolution integral we can interpret the output power as the volume under the surface formed by the product of $P'(T',F')$ with $\Psi^2(T',F')$ offset an amount T in relative delay and F in relative doppler shift. If $\Psi^2(T',F')$ were an impulse then there would be a one to one relation between $P(T,F)$ and $P'(T,F)$. In fact, in this case, we would have $P(T,F)=P'(T,F)$. However, because of the extent of $\Psi^2(T,F)$ in the T and F coordinate as shown in figure 4g, $P(T,F)$ at each value of T and F corresponds to $P'(T,F)$ at more than one value of T and F . In other words, the value of the power at the receiver output that appears at, say, relative delay $T=T_0$ and relative doppler $F=F_0$ is due to power reflected from a region on the target having a range of values of T and F about the value $T=T_0$, $F=F_0$ that is determined by the shape of $\Psi^2(T,F)$. If Ψ^2 had only one sharp peak then the region on the target contributing to the value of $P(T,F)$ at $T=T_0$, $F=F_0$ would be narrow, whereas if Ψ^2 was broad in either the T or F dimension (or both) the region on the target contributing to $P(T_0,F_0)$ would be large in extent. Unless $\Psi^2(T,F)$ is an impulse, then, it is impossible to say the output power at any value of (T,F) was reflected from that point on the target having that same value of (T,F) ; we can

only specify a region of finite resolution on the target, whose resolution depends on the shape of $\Psi^2(T,F)$, from which we may say the output power at any point (T,F) was reflected. The effect of this resolution on the ranging error will be examined later in this paper. Also contributing to possible error in ranging is the fact that Ψ^2 is periodic (with period t_p) which indicates $P(T,F)$ is also periodic and thus our estimate of the delay to the target obtained by analyzing the receiver output may be incorrect by an amount nt_p (where $n=0,1,2,\dots$) unless our a priori estimate of delay is in error by less than about t_p . However, the orbits of the earth and the planets for which radar ranging is presently conducted is known sufficiently well to eliminate an error of magnitude t_p .

To gain further insight into the role of $\Psi^2(T,F)$ in the signal processing let us consider a ranging experiment with a stationary (with respect to the receiver) point source. It will be convenient, for instructive purposes, to trace the signal in this case only to the output of the pulse matched filter and to calculate the output power appearing at that point. We will then consider the point target to be moving relative to the receiver and follow the processing all the way through the computer.

Consider transmitting a real signal whose envelope, $m(t)$, is shown in figure 4a. We see $m(t)$ is a periodic series of rectangular pulses of unit height and duration τ_b called the 'baud length'. Let the function $\chi(t)$ (see figure 4b) represent a single pulse so that in terms of $\chi(t)$ we may

write $m(t)$ as

$$\begin{aligned} m(t) &= \sum_{n=0}^{N_s-1} \chi(t - nt_p) \\ &= \chi(t) * \sum_{n=0}^{N_s-1} u_0(t - nt_p) \end{aligned} \quad (16)$$

where

$$\chi(t) = \begin{cases} 1 & 0 \leq t \leq \tau_p \\ 0 & \text{otherwise} \end{cases}$$

and where the asterisk denotes convolution, t_p is the pulse repetition period, and N_s is the number of pulse periods transmitted. (For a typical Mars ranging experiment, for example, the number of pulses transmitted $N_s = 600 \text{ sec} / 10^{-3} \text{ sec} = 6 \times 10^5$ although the signal is processed coherently over only M pulses where M is $\ll N_s$; typically, $1 \leq M \leq 50$). In this example and in the rest of this paper the receiver and transmitter are assumed co-located. The target we will consider first is a stationary, non-fluctuating point target, i.e., the target scattering function, call it $P'(T, F)$ is an impulse at constant relative delay and zero doppler shift. The position of $P'(T, F)$ in figure 4c is at $T=T_s$ and $F=F_s$ (recall that T and F are measured relative to a priori estimate of the delay and doppler shift of the signal reflected from the subradar point). The point non-fluctuating target has negligible depth in delay and doppler and may be contrasted with the point fluctuating target whose target scattering function (figure 4c) has negligible depth in relative delay but a spread of relative doppler frequencies due, for example, to rapid rotation about some axis (see Evans and Hagfors, 1968). The receiver that will

develop maximum peak signal to noise ratio for the non-fluctuating point target employs a filter matched to the signal that enters the receiver. Since we are considering a point nonfluctuating target the reflected signal will be the same as the transmitted signal, neglecting the factor P'_R . (Except for a time delay and frequency shift due to relative motion of earth and target: we will assume in this example that both these effects are compensated for by the radar system before the signal enters the filter so that the target delay and doppler values appear fixed for the duration of the transmitted signal at the values they had when the signal was first reflected from the target). The reflected signal received by the antenna and appearing at the input of the filter whose impulse response is $h_1(t)$ depends upon the receiving parameters A_R and L_R , in addition to the factor P'_R . Specifically, the reflected signal at the input of $h_1(t)$ is dependent upon the factor $P_R \equiv A_R L_R P'_R$. The factor P_R will be omitted for the present, but will be accounted for at the end of this section to yield the proper theoretical output power distribution $P(T,F)$. In fact, as we will see, when convolving Ψ^2 with P' to obtain P we need only account for the factor $A_R L_R$ since P' contains the factor P'_R .

In light of the above discussion, the receiver will use a filter whose impulse response, $h_1(t)$, is equal to $\chi(t_0-t)$, the time reversal of one of the pulses in the transmitted pulse train shifted an amount t_0 to make the filter realizable (no output before input - see figure 4d). We see from

figures 4b and 4d that if $t_0 = \gamma_B$ then $h_1(t)$ will be realizable. We will choose, then $t_0 = \gamma_B$ so that $h_1(t) = \chi(\gamma_B - t)$.

Let us assume we know the precise delay to the target at all times so that we may begin receiving at the precise time the reflected signal first enters the receiver (In practice one allows a brief interval of time between the end of transmission of the signal and the time the reflected signal returns to the antenna). In this case the output of $h_1(t)$ due to the signal reflected by the point target will begin at zero delay relative to the a priori "estimate", which means our value of T_s is zero. We will also assume we know exactly the relative motion between the antenna and the target so that $F_s = 0$. The output voltage of $h_1(t)$ is the convolution of the reflected signal, $m(t)$, (as we explained above the reflected signal entering $h_1(t)$ is the same as the transmitted signal) with $h_1(t)$. Since $m(t)$ is a periodic train of rectangles each of which have the same dimensions as $h_1(t)$ (compare figure 4a and figure 4d) the output of $h_1(t)$ will be a periodic series of triangles of width $2\gamma_B$ and height γ_B with the same period (t_p) as $m(t)$. (Note that although the a priori estimate of delay was correct [$T_s = 0$] the peak of the triangle occurs at an offset of γ_B . This is due to the fact that $h_1(t)$ is the time reversal of the transmitted signal offset an amount $t_0 = \gamma_B$. We may compensate for this effect by offsetting the train of impulses that sample the receiver output by an amount $t_0 = \gamma_B$). We may therefore represent the

output of $h_1(t)$ as the convolution of one period of $m(t)$ with $h_1(t)$ (which will yield one triangle of height γ_B and width $2\gamma_B$) convolved with a periodic series of unit impulses. This will produce the periodic series of triangles just described. Calling $\mathcal{L}(T, F)$ the output voltage of the filter $h_1(t)$ (see figure 4e) we have:

$$\mathcal{L}(T, 0) = \int_0^{t_p} dt \chi(t) h_1(T-t) * \sum_{n=0}^{N_s-1} u_n(T-nt_p) \quad (17)$$

where we have used

$$\text{one period of } m(t) = \begin{cases} \chi(t) & 0 \leq t \leq \gamma_B \\ 0 & \gamma_B < t \leq t_p \end{cases} \quad (18)$$

The second argument of \mathcal{L} corresponds to F ; it is zero because the doppler shift due to the relative motion of the earth and the point nonfluctuating target has been (assumed) correctly eliminated by the time the echo signal is presented to $h_1(t)$.

Using $h_1(t) = \chi(\gamma_B - t)$ equation (17) becomes

$$\begin{aligned} \mathcal{L}(T-\gamma_B, 0) &= \int_0^{t_p} \chi(t) \chi(t+\gamma_B-T) dt * \sum_{n=0}^{N_s-1} u_n(T-nt_p) \\ &= \int_0^{t_p} dt m(t) \chi(t+\gamma_B-T) \end{aligned} \quad (19)$$

The output power is

$$\Psi_p^2(T-\gamma_B, 0) = |\mathcal{L}(T, 0)|^2$$

Since $\mathcal{L}(\tau-\tau_B, 0)$ is a series of triangles (see figure 4e)

$\Psi_p^2(\tau-\tau_B, 0)$ is a series of triangles - squared (see figure 4f).

The function $\Psi_p^2(\tau-\tau_B, 0)$ may therefore be expressed in the form

$$\Psi_p^2(\tau-\tau_B, 0) = \left| \int_0^{t_p} dt \chi(t) \chi(t+\tau_B-\tau) \right|^2 * \sum_{n=0}^{N_s-1} u_n(\tau-n t_p) \quad (20)$$

$$= \left| \int_0^{t_p} dt m(t) \chi(t+\tau_B-\tau) \right|^2 \quad (21)$$

Next, we assume the nonfluctuating point target has a known constant component of motion relative to the receiver along the line of sight of the target to the receiver. We will assume that this component of relative motion is directed toward the receiver. Also, we assume that we are able to compensate for the change in delay between the receiver and target due to the motion of the target toward the receiver so that the target delay appears fixed for the duration of the transmitted signal at the value it had when the signal was first reflected from the target (see figure 4ff). Therefore the signal that is presented to $h(t)$ is the same as the transmitted signal but with a doppler shift in frequency, F , due only to the motion of the point target toward the receiver (F is positive for relative motion of the target toward the receiver and negative for relative motion away from the receiver). Assume we begin receiving at the instant the reflected signal enters $h_1(t)$.

The description of the signal processing that follows will consider the general case of a target with finite extent in both delay and doppler frequency. Application to the specific

instance of the point nonfluctuating target just described will be made along the way and in the end we will demonstrate that $\Psi^2(T,F)$ is simply the output power distribution in the case of a point nonfluctuating target.

The signal that returns to the antenna is doppler shifted (due to the relative motion of the earth and target) in frequency and is mixed (by the radar system) to an intermediate frequency plus doppler frequency $f_{IF} + F$ where F is the doppler shift relative to the subradar point (by this stage the radar system has removed the frequency shift corresponding to the subradar point). The signal is then presented to phase quadrature detectors (see figure 5) where it is separately mixed with a sine and cosine wave of frequency f_{IF} and low pass filtered (the frequency f_{IF} of the signal that is mixed with the signal of frequency $f_{IF} + F$ will produce a signal with frequency components $2f_{IF} - F$ and F . The low pass filter is designed such that the frequency components $2f_{IF} - F$ is in the stop band). The output of the "cosine" mixer, $x(t)$ is the real part and the output of the "sine" mixer, $y(t)$, is the imaginary part of a complex number $\underline{z}(t) = z(t)e^{j2\pi Ft} = x(t) + jy(t)$. The complex number $\underline{z}(t)$ may be considered as a two dimensional vector (whose projections onto the two orthogonal axis are $x(t)$ and $y(t)$) rotating at the relative angular doppler frequency $2\pi F$. That is, $\frac{d\theta(t)}{dt} = 2\pi F$, where $\theta(t) = \tan^{-1} [y(t)/x(t)]$. For a point nonfluctuating target with a component of relative motion directed toward the receiver $z(t) = m(t)$.

The sine and cosine components, $y(t)$ and $x(t)$ respectively, are then presented the pulse matched filter $h_1(t)$ (see figure 5c), and sampled. The sine and cosine components after sampling are labeled $a(T+lt_s)$ and $b(T+lt_s)$ respectively. We define the complex voltage sample A as

$$A(T+lt_s) = b(T+lt_s) + ja(T+lt_s) \quad (22)$$

Where l takes on integer values ≥ 0 and t_s is the sampling interval (which must be $\leq \frac{1}{2} \gamma_B$, as will be explained later). Note that when $lt_s = nt_p$ (i.e., lt_s is an integer multiple of the pulse repetition period) $A(T)$ and $A(T+nt_p)$ will correspond to reflection from the same annular region on the planet's surface where T is the relative delay of the signal reflected from this region. For convenience, when reference is made to a and b in the text as the sampled sine and cosine components of the output per se the argument will contain only the "T" term.

Before saying anything more specific about the echo signal processing we should examine what values of the PRF would be best to use for studying a spread (i.e., finite extent in delay and doppler dimension) target.

If the delay depth of the planet is T_D , to avoid sampling at one point in time the signal reflected from more than one delay ring on the target it is necessary to have

$$\frac{1}{\text{PRF}} = t_p > T_D \quad (23)$$

If the PRF were less than T_D "self noise" would be present in the sampled signal. Later we will show that the power at any doppler frequency for any delay ring may be obtained by sampling the rotating sine - cosine vector at integer multiples of the pulse repetition period, t_p . Let B_{ll} equal the limb doppler spread of the target $= 2 \times B_{cl}$ (where B_{cl} is the center to limb doppler spread). Then, if we are to avoid aliasing the sampling theorem tells us we must have

$$2B_{cl} < \frac{1}{t_p} = \text{PRF} \quad (24)$$

Thus, the bounds on the PRF are

$$2B_{cl} < \text{PRF} < \frac{1}{T_D} \quad (25)$$

If we sample at the PRF no aliasing will occur as long as the PRF satisfies equation (25). We see it is possible to sample such that equation (25) is satisfied if

$$T_D (2B_{cl}) < 1 \quad (26)$$

in which case the target is said to be "underspread" and it is possible to sample such that no aliasing or self noise will be present. If $T_D (2B_{cl})$ was > 1 then the target would be "overspread" and, strictly speaking, it would not be possible to avoid the problem of self noise.

For the underspread planet (the only case that will be examined in this paper), let us choose $2B_{cl}$, for example, to be 100 cps and delay depth $T_D = 10$ msec. Equation (25) is

just satisfied because

$$T_D (2B_{ce}) = (100 \text{ cps})(10^{-2} \text{ sec}) = 1$$

(actually the product should be less than 1, but this may be accomplished by considering T_D to be infinitesimally less than 10 msec). The baud length, γ_B , is chosen to be $10 \mu\text{sec}$ and the sampling period, t_s , is then $5 \mu\text{sec}$. The sampling period is made $\frac{1}{2} \gamma_B$ for the following reason: The frequency response of the filter in figure 5c whose impulse response is $h_1(t)$ is $H(f) = e^{-j2\pi f \frac{\gamma_B}{2}} \cdot \frac{\sin 2\pi f \frac{\gamma_B}{2}}{\pi f}$, a function whose central peak is $2(1/\gamma_B) = 200 \text{ kHz}$ wide between the first nulls. We would like to sample the output of the filters $h_1(t)$ such that there is no aliasing of the sampled signal in the frequency domain. We may accomplish this by sampling at the Nyquist rate. Assuming $H(f)$ is essentially zero outside $\pm 100 \text{ kHz}$ the output of $H(f)$ is thus also essentially zero outside $\pm 100 \text{ kHz}$ and so the Nyquist rate is 200 kHz which means the sampling period must be $\leq \frac{1}{200 \text{ kHz}} = 5 \mu\text{sec} = \frac{1}{2} \gamma_B$. Because the PRF is $< \frac{1}{T_D}$ sampling at intervals of $\frac{1}{2} \gamma_B$ will yield a power output free of self noise and aliasing whose envelope is of the form shown in figure 5d (i.e., each of the values in the set $\{a(T+lt_s)\}$ is independent of the other values; the same is true for $\{b(T+lt_s)\}$). The total number of samples in each period is seen to be

$$\frac{2t_p}{\gamma_B}$$

The processing of $a(T)$ and $b(T)$, the sampled sine and cosine components of the output, can be understood in terms of rotating vectors, as mentioned earlier. Let the vector $A(T+lt_s) = b(T+lt_s) + ja(T+lt_s)$ be rotating at, say, the angular doppler frequency $\omega_F = 2\pi F$. The pulse repetition period is t_p , so referring to figure 5d we see points of equal delay relative to the subradar point are separated in time by t_p seconds. By the n^{th} period the vector at doppler frequency F and delay T (corresponding to the delay ring at relative delay T) will have rotated $\omega_F n t_p$ radians. To obtain the voltage at doppler frequency F for the delay ring at relative delay T we must rotate the vector from each period back the amount it has advanced relative to the vector from the first period and then sum all the vectors. Thus $V(T, F)$, the voltage corresponding to the signal reflected from the region of the targets surface defined by the intersection of the delay ring at relative delay T and doppler strip at relative doppler frequency shift F is

$$V(T, F) = \sum_{n=0}^{N_s-1} A(T+nt_p) e^{-j2\pi F n t_p}$$

where the total number of periods received is N_s . Thus $V(T, F)$ is simply the Discrete Fourier Transform (DFT) of $A(T+nt_p)$. However, we should note that in practice multiplying by $e^{j2\pi F n t_p}$ will not in general yield the relative doppler frequency component precisely at relative doppler frequency F . This is because the relative motions between

the target and the antenna are a priori imperfectly known and thus the radar system does not remove precisely the doppler frequency shift corresponding to the subradar point. This means $A(T + lt_s)$ would not be rotating at, say, F but rather at some relative doppler frequency F' whose value is nearly F if our a priori information was nearly correct.

In practice, for certain statistical reasons which are soon to be discussed, when calculating $P(T, F)$, the output power, the sum over N_s interpulse periods is broken up into R sums over M periods ($R \cdot M = N_s$), and $M \cdot t_p$ is called the Coherent Integration Period (CIP). The total output power is obtained by summing the power calculated from each of the R CIP's:

$$\begin{aligned}
 P_s(T, F) &= \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} A(T + [m+r]t_p) e^{-j2\pi F m t_p} \right|^2 \\
 &= \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} A(T + [m+r]t_p) h_0(m t_p) \right|^2 = \sum_{r=0}^{R-1} |V_r(T, F)|^2 \quad (27)
 \end{aligned}$$

where

$$h_0(m t_p) = e^{-j2\pi F m t_p} \quad (27A)$$

$$V_r(T, F) = \sum_{m=0}^{M-1} A(T + [m+r]t_p) h_0(m t_p) \quad (27B)$$

Let us now show that the resolution of each doppler strip is inversely proportional to the length of the CIP and also derive the form of the ambiguity function for the receiver shown in figure 5a. From figure 5 we see that the complex

voltage sample $A(T + lt_s)$ may be expressed

$$\begin{aligned} A(T + lt_s) &= z(t) e^{j2\pi F't} * h_1(t) \cdot \sum_k u_0(t - \tau_B - lt_s) \\ &= \sum_{\xi} \int d\xi h_1(\xi) z(t - \xi) e^{j2\pi F'(t - \xi)} u_0(t - \tau_B - T - lt_s) \end{aligned}$$

where $z(t) e^{j2\pi F't} = x(t) + jy(t)$ and $z(t)$ is multiplied by $e^{j2\pi F't}$ because, as explained previously, $z(t)$ is a vector rotating at relative doppler frequency F' .

Using the right hand side of the above equation to substitute for the value of $A(T + [Lm+r]t_p)$ in the expression for $P_s(T, F)$

$$P_s(T, F) = \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} \int d\xi h_1(\xi) z(t - \xi) e^{-j2\pi F'\xi} u_0(t - \tau_B - T - [Lm+r]t_p) e^{j2\pi F't} e^{-j2\pi F m t_p} \right|^2$$

the term $u_0(t - \tau_B - T - [Lm+r]t_p)$ picks out the values $t = \tau_B + T + [Lm+r]t_p$ in the exponent of $e^{j2\pi F't}$ and also picks out the same values of t in the argument of $z(t - \xi)$ so the above equation may be written

$$\begin{aligned} P_s(T, F) &= \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} \int d\xi h_1(\xi) z(\tau_B + T + [Lm+r]t_p - \xi) e^{-j2\pi F'\xi} e^{j2\pi F'(\tau_B + T + [Lm+r]t_p)} e^{-j2\pi F m t_p} \right|^2 \\ &= \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} \int d\xi h_1(\xi) z(\tau_B + T + [Lm+r]t_p - \xi) e^{-j2\pi F'\xi} e^{-j2\pi (F - F') m t_p} \right|^2 \end{aligned}$$

With $R \gg M$ we may set (see Shapiro, 1967, pp.1316-7)

$$z(T + \tau_B - \xi - [Lm+r]t_p) \doteq z(T + \tau_B - \xi - r t_p)$$

so that

$$P_s(T, F) = \sum_{r=0}^{R-1} \left| \sum_{m=0}^{M-1} e^{-j2\pi (F - F') m t_p} \int d\xi h_1(\xi) z(T + \tau_B - \xi - r t_p) e^{-j2\pi F'\xi} \right|^2$$

we may now easily perform the summation over m

$$\left| \sum_{m=0}^{M-1} e^{-j2\pi(F-F')t_p} \right|^2 = \frac{\sin^2 \pi M t_p (F-F')}{\sin^2 \pi t_p (F-F')} \equiv |H_0(F-F')|^2 \quad (27c)$$

so that

$$P_s(T, F) = \sum_{r=0}^{R-1} \left| \int d\xi h_1(\xi) z(T+\gamma_B - \xi - r t_p) e^{-j2\pi F' \xi} \right|^2 |H_0(F-F')|^2 \quad (27D)$$

The function $|H_0(F-F')|^2$ has nulls occurring at

$$F-F' = \pm \frac{1}{M t_p}, \pm \frac{2}{M t_p}, \dots$$

maxima between the nulls (see figure 6). The value of the maxima at $F-F'=0$ is much larger than the value of the other maxima so we may consider $|H_0(F-F')|^2$ as a filter whose bandwidth is essentially $\frac{1}{M t_p}$ or $\frac{1}{CIP}$ since $M t_p$ is the time interval of one CIP. Thus, the longer the CIP the narrower the bandwidth centered about F. In the case of a point nonfluctuating target with a component of relative motion along the line of sight to the antenna $z(t) = m(t)$ and, recalling $h_1(\xi) = \chi(\gamma_B - \xi)$, equation (27D) becomes

$$\begin{aligned} P_{s_POINT}(T, F) &= \Psi'^2(T, F, F') = \sum_{r=0}^{R-1} |H_0(F-F')|^2 \left| \int_{-\infty}^{\infty} d\xi \chi(\gamma_B - \xi) m(T+\gamma_B - \xi - r t_p) e^{-j2\pi F' \xi} \right|^2 \\ &= \sum_{r=0}^{R-1} |H_0(F-F')|^2 \left| \int_{-\infty}^{\infty} d\xi \chi(\xi) m(\xi + T - r t_p) e^{-j2\pi F'(\gamma_B - \xi)} \right|^2 \end{aligned}$$

and, assuming the radar receiver correctly eliminates the doppler shift due to relative motion between the point

nonfluctuating target and the antenna we set $F' = 0$ and

$$\Psi^2(T; F, 0) \equiv \Psi'^2(T, F) = |H_0(F)|^2 \sum_{r=0}^{R-1} \left| \int_{-\infty}^{\infty} d\xi \chi(\xi) m(\xi + T - r t_p) \right|^2$$

a sketch of a plot of the above equation is shown in figure 4g.

We see from the preceding analysis that $\Psi'(T, F)$ gives the distribution as a function of relative delay and doppler of power reflected from a nonfluctuating point target that appears at the output of a ("noiseless") receiver that uses a pulse matched filter (see figures 4g and 4ff). The power appearing at the output of $h_1(t)$ that was reflected from a target with a finite delay depth and a distribution of doppler frequencies is given by the convolution of $\Psi^2(T, F)$, the receiver output for a nonfluctuating point target, with the description of the target given by the target scattering function, $P'(T, F)$

$$P(T, F) = \iint d\tau df \Psi^2(T-\tau; F-f) P'(\tau, f) \quad (15)$$

where, as stated earlier, the integral is taken over all values of relative delay and doppler and where

$$\Psi^2(T, F) = A_R L_R \Psi'^2(T, F).$$

The factor $A_R L_R$ has been inserted to account for the factor P_R that was omitted earlier in this section; since P' contains the factor P'_R we need only insert $A_R L_R$ to account for the factor $P_R (= A_R L_R P'_R)$.

From equation (15) we see the quantity $\Psi^2(T-\tau, F-f)P'(\tau, f)d\tau df$ is the power reflected from the surface of the planet corresponding to a relative delay between τ and $\tau + d\tau$ and a frequency offset between f and $f + df$ and contributing to $P(T, F)$, the output power at relative delay T and relative doppler shift F . The total contribution to $P(T, F)$ is the sum of $\Psi^2(T-\tau, F-f)P'(\tau, f)d\tau df$ over all values of f and τ , as we see from equation (15). The sum just described is a sum of powers. One may wonder why we do not sum over voltages and then take the square of the this sum to obtain the output power. To explain this we must consider the statistical nature of the surface of a target with a rough surface, of finite extent in delay and doppler (i.e., a spread target).

A rough surface is often described in terms of the rms (root mean square) surface slope. If we knew the relative surface height at each point on the target we could calculate the rms slope by taking the spatial derivative of the height at each point and calculating the mean and variance for this set of measurements. For most targets studied by radar astronomy a description of the height at each point, even if it could be made, would be much too complex to be useful. In practice, an estimate of the rms slope due to surface height variations roughly on the order of the wavelength of the transmitted

signal or larger is used in developing a target scattering function, which is a statistical description of the surface as opposed to a complex description of every individual point on the surface. In practice the rms slope is one of the parameters estimated in a least squares fit of theoretical to the observed echo power.

We will now demonstrate that for a target with a rough surface, such as one of the terrestrial planets, it turns out that a sum of powers yields a lower variance of the output power than a sum of voltages.

Because the surface is rough as the planet rotates the signal reflected from a position on the planet fixed relative to the subradar point will have a random value of magnitude and phase corresponding to random variations in the surface structure (for a detailed description of the reflection of electromagnetic waves from many different models of rough surfaces see Beckmann, 1963). This leads to a random variation in the sampled receiver output values $a(T + \Delta t_s)$ and $b(T + \Delta t_s)$ and thus $P_s(T, F)$ will be a random variable whose values are distributed according to some probability density $\rho(P_s)$.

We will now determine the density $\rho(P_{SA})$ so that we may calculate the expected value and variance of $P_{SA}(T, F)$, the average observed output power $\bar{P}_s(T, F) = P_s(T, F) / M^2 R$. After we obtain an expression for the variance of $P_{SA}(T, F)$ we will obtain an expression for the variance of the average output power as obtained by squaring a sum

of voltages and compare the relative value of these variances.

The quantities $b(T+nt_p)$ and $a(T+nt_p)$ are the real and imaginary parts of a signal reflected from the area of the target's surface corresponding to the relative delay T . The signal reflected from different regions along this delay ring will have different values of doppler shift so that $b(T+nt_p)$ and $a(T+nt_p)$ are due to contributions of the signal reflected from all regions comprising the delay ring at relative delay T , each region having a different value of doppler shift. If we define $U(T,F)$ to be the complex voltage signal (at the output of a filter whose impulse response $*h_f(t)$) that is the contribution to $A(T+nt_p)$ from the region on the planet corresponding to the relative doppler frequency shift between F and $F+dF$ then we may write

$$A(T+nt_p) = \int_{-B_T}^{B_T} U(T+nt_p, F) dF \quad (28)$$

where $+B_T$ and $-B_T$ are the maximum and minimum values of doppler shift associated with the region of equal delay on the target from which the signal contributing to $A(T)$ was reflected. As we have shown, rotating $A(T+nt_p)$ back by multiplying by $e^{-j2\pi F m t_p}$ and summing over all values of m will yield the voltage reflected from that region of the delay ring (at relative delay T) corresponding to

the doppler frequency F; i.e., from equation (27B)

$$\begin{aligned} \frac{1}{M} V_r(T, F) &= \frac{1}{M} \sum_{m=0}^{M-1} A(T+[m+r]t_p) e^{-j2\pi F m t_p} \\ &= \frac{1}{M} \sum_{m=0}^{M-1} \{b(T+[m+r]t_p) + ja(T+[m+r]t_p)\} e^{-j2\pi F m t_p} \end{aligned} \quad (28A)$$

$$= \frac{1}{M} [d_r(T, F) + j c_r(T, F)] \quad (28B)$$

$$= \frac{1}{M} U_r(T, F) e^{-j\theta_r} \quad (28C)$$

where

$$U_r(T, F) = \sqrt{c_r^2(T, F) + d_r^2(T, F)}$$

$$\theta_r = \tan^{-1} \frac{c_r}{d_r}$$

(28D)

$$d_r(T, F) = \sum_{m=0}^{M-1} [b(T+[m+r]t_p) \cos(2\pi F m t_p) + a(T+[m+r]t_p) \sin(2\pi F m t_p)]$$

(28E)

$$c_r(T, F) = \sum_{m=0}^{M-1} [a(T+[m+r]t_p) \cos(2\pi F m t_p) - b(T+[m+r]t_p) \sin(2\pi F m t_p)]$$

Because of the rough surface $A(T+nt_p)$ will have a random value of magnitude and phase. In particular, $b(T+nt_p)$ and $a(T+nt_p)$, the real and imaginary parts of the signal reflected from the surface of the planet corresponding to relative delay T are independent random variables and each have a Gaussian distribution with an expected value of zero and a variance which we will

call $\frac{1}{2} \sigma_0^2(T)$. Since $A(T+nt_p) = b(T+nt_p) + ja(T+nt_p)$ and $b(T+nt_p)$ and $a(T+nt_p)$ are independent $A(T+nt_p)$ will have a zero expected value and variance $\sigma_0^2(T)$. In addition, as the planet rotates the annular region of the surface corresponding to a fixed value of relative delay will present different small scale (i.e., on the order of a wavelength) structure to the incident radar beam and therefore it is reasonable to assume $A(T+lt_p)$ and $A(T+st_p)$ are independent when $l \neq s$ (l and s are integers). We may relate $\sigma_0^2(T)$ to $U(T,F)$ through the integral expression of $A(T)$ (given in equation (28)):

$$\text{Var}[A(T)] = E[|A(T)|^2] = |A(T)|^2 = \int_{-\theta_r - \delta_r}^{\theta_r + \delta_r} \int \phi_{UU}(T; F, \nu) dF d\nu = \sigma_0^2(T) \quad (28F)$$

where $\phi_{UU}(T; F, \nu)$ is

$$\phi_{UU}(T; F, \nu) = E[U(T, F)U(T, \nu)]$$

Using equation (28a) we may calculate the auto-correlation function of $\frac{1}{M} V_R(T, F)$ from which we may obtain the mean square value of $\frac{1}{M} V_R$ which is equal to the variance of $\frac{1}{M} V_r$ since $\frac{1}{M} \bar{V}_R = 0$ (this follows from equations (28B)(28D), and (28E) and the fact $E[A] = E[b] = 0$):

$$\phi_{V_r V_r}(T; F, F') = \frac{1}{M^2} \overline{V_r(T, F)V_r(T, F')} = \frac{1}{M^2} \sum_{m=0}^{M-1} \overline{|A(T+[m+r]t_p)|^2} e^{-j2\pi(F-F')mt_p} \quad (28G)$$

where we have made use of the fact $E[A(T+[l+r]t_p)A(T+[l+r]t_p)] = 0$ because these values of A are independent when $l \neq r$ and $E[A(T+[m+r]t_p)] = 0$ for any integers m and r . Because

the complex voltage sample $A(T+[m+r]t_p)$ corresponds to the relative delay coordinate T for all values of m and r we have

$$E[|A(T+[m+r]t_p)|^2] = E[|A(T)|^2]$$

so now we write equation (28G) as

$$\phi_{V_r V_r}(T; F, F') = \frac{1}{M^2} \sum_{m=0}^{M-1} E[|A(T)|^2] e^{-j2\pi(F-F')mt_p}$$

and using equation (28F) the above becomes

$$\phi_{V_r V_r}(T; F, F') = \frac{1}{M^2} \sum_{m=0}^{M-1} \iint_{-B_T}^{B_T} \phi_{VV}(T; F, \nu) dF d\nu e^{-j2\pi(F-F')mt_p}$$

Since

$$\sum_{m=0}^{M-1} e^{-j2\pi(F-F')mt_p} = \begin{cases} M & F = F' \\ 0 & \text{otherwise} \end{cases}$$

$$\equiv \delta_{FF'}$$

we have

$$\phi_{V_r V_r}(T; F, F') = \frac{1}{M^2} \iint_{-B_T}^{B_T} dF d\nu \phi_{VV}(T; F, \nu) M \delta_{FF'}$$

$$\phi_{V_r V_r}(T; F, F') = \frac{1}{M^2} \overline{|V_r(T, F')|^2} = \frac{2B_T}{M} \int_{-B_T}^{B_T} d\nu \phi_{VV}(T; F', \nu) \quad (28H)$$

we define

$$\sigma_V^2(T, F') \equiv \int_{-B_T}^{B_T} d\nu \phi_{VV}(T; F', \nu)$$

so that equation (28H) may be written

$$\frac{1}{M^2} \overline{|V_r(T, F)|^2} = \frac{2B_T}{M} \sigma_v^2(T, F) \quad (28I)$$

From equations (28D) and (28E) and the fact $\text{Var}[a(t)] = \text{Var}[b(t)]$ it is easily verified $E[d_r] = E[c_r] = 0$ and $E[d_r c_r] = E[d_r]E[c_r] = 0$ so from equation (28B)

$$\text{Var}\left[\frac{1}{M} V_r(T, F)\right] = \frac{1}{M^2} \overline{|V_r(T, F)|^2} = \text{Var}\left[\frac{1}{M} c_r(T, F)\right] + \text{Var}\left[\frac{1}{M} d_r(T, F)\right] = \frac{2B_T}{M} \sigma_v^2(T, F)$$

Also, from equation (28D) and (28E) we may show d_r and c_r have equal values of variance (although using (28D) and (28E) the explicit form of the variance is not easily obtained) and thus from the above equations we have

$$\text{Var}\left[\frac{1}{M} d_r(T, F)\right] = \text{Var}\left[\frac{1}{M} c_r(T, F)\right] = \frac{B_T}{M} \sigma_v^2(T, F) \equiv \sigma^2(T, F) \quad (28J)$$

We may use equations (27) and (28B) to write $P_{SA}(T, F)$, the average output power, as

$$\begin{aligned} P_{SA}(T, F) &= \frac{1}{R} \sum_{r=0}^{R-1} \left| \frac{1}{M} \sum_{m=0}^{M-1} A(T + [m+r]t_p) e^{-j2\pi F m t_p} \right|^2 = \frac{1}{M^2 R} P_S(T, F) \\ &= C(T, F) + D(T, F) \end{aligned}$$

where

$$\begin{aligned} C(T, F) &= \frac{1}{R} \sum_{r=0}^{R-1} \frac{c_r(T, F)}{M^2} \\ D(T, F) &= \frac{1}{R} \sum_{r=0}^{R-1} \frac{d_r(T, F)}{M^2} \end{aligned} \quad (29)$$

Using the Central Limit Theorem we will assume $\frac{C_r}{M}$ and $\frac{d_r}{M}$ both have a density that is well approximated by a Gaussian density of zero expected value and variance $\sigma^2(T,F)$ (see equations (28D), (28E), and (28J)). Thus, $C(T,F)$ and $D(T,F)$ have a χ^2 density that is given by (see Gnedenko, p174)

$$p(D) = p(C) = \frac{(R/2)^{R/2}}{\Gamma(\frac{R}{2})\sigma^2} \left(\frac{C}{\sigma^2}\right)^{R/2-1} e^{-\frac{RC}{2\sigma^2}}$$

so that \bar{C} and \bar{D} , the expected values of C and D respectively, is

$$\bar{C}(T,F) = \int_0^{\infty} C p(C) dC = \sigma^2(T,F) = \bar{D}(T,F)$$

and since C and D are independent, $E[P_{SA}] = \bar{P}_{SA}$, the expected value of P_{SA} is

$$\bar{P}_{SA}(T,F) = 2\sigma^2(T,F)$$

Since $P_{SA}(T,F)$ is the average value of the output power in the presence of random fluctuations due only to reflection from a rough surface (note we have not mentioned, thus far, the effects of the noise due to the radar system) the expected value of $P_{SA}(T,F)$ should be equal to $P(T,F)$, the theoretical model of the echo power distribution (given by equation (15)) so we will write

$$\bar{P}_{SA}(T,F) = P(T,F) \tag{29A}$$

To obtain $\overline{P_{SA}^2}(T, F)$ we calculate

$$\overline{C^2}(T, F) = \int_0^{\infty} c^2 p(c) dc = \overline{D^2}(T, F) = 2\sigma^4 \left(\frac{1}{R} + 1\right)$$

and since $C(T, F)$ and $D(T, F)$ are independent

$$\overline{P_{SA}^2}(T, F) = \overline{C^2}(T, F) + \overline{D^2}(T, F) = 4\sigma^4(T, F) \left(\frac{1}{R} + 1\right)$$

so

$$\text{Var}[P_{SA}(T, F)] = \overline{P_{SA}^2}(T, F) - \overline{P_{SA}}(T, F)^2 = \frac{4\sigma^4(T, F)}{R} \quad (30)$$

According to equation (30) the variance of the average output power $P_{SA}(T, F)$ is inversely proportional to the number of Coherent Integration Periods used in decoding $P_{SA}(T, F)$ at each value of F .

To answer the question posed earlier to why we calculate $P_{SA}(T, F)$ by summing powers instead of summing voltages and squaring the resulting voltage let us calculate the variance of

$$\frac{1}{R} \left| \sum_{r=0}^{R-1} \frac{1}{M} V_r(T, F) \right|^2 = \frac{1}{M^2 R} P_V(T, F) \equiv Z \quad (31)$$

and compare the result with equation (30). Let us first consider the sum

$$\rho e^{i\psi} = \frac{1}{M} \sum_{r=0}^{R-1} V_r(T, F) = \sum_{r=0}^{R-1} \frac{v_r(T, F)}{M} e^{j\theta_r} \quad (31A)$$

and determine $p(\rho)$, the probability density of ρ , the amplitude of the resultant voltage. This is done in

Beckmann (p128) and the result is

$$p(\rho) = \frac{2\rho M^2}{R(\overline{v_r^2})} e^{-M^2 \rho^2 / R(\overline{v_r^2})} \quad (32)$$

Since the real and imaginary parts of $\frac{1}{M} V_r(T, F)$ have a Gaussian density of zero mean and variance $\sigma^2(T, F)$ the probability density of $\frac{1}{M} V_r$, the magnitude of $\frac{1}{M} V_r$ has the Rayleigh density

$$p\left(\frac{v_r}{M}\right) = \begin{cases} \frac{v_r}{M\sigma^2} e^{-v_r^2/2M^2\sigma^2} & v_r > 0 \\ 0 & v_r < 0 \end{cases}$$

which is simply the χ^2 distribution for a sum of only two terms (see Gnedenko, p173). We may now calculate $\overline{\left(\frac{v_r}{M}\right)^2}$:

$$\overline{\left(\frac{v_r}{M}\right)^2} = \int_0^{\infty} \left(\frac{v_r}{M}\right)^2 p\left(\frac{v_r}{M}\right) d\left(\frac{v_r}{M}\right) = 2\sigma^2(T, F)$$

so that equation (32) becomes

$$p(\rho) = \frac{\rho}{R\sigma^2} e^{-\rho^2/2R\sigma^2}$$

Comparing equations (31) and (31A) we see

$$Z = \frac{1}{R} \rho^2.$$

$p(Z)$, the probability density of $Z = \frac{1}{R} \rho^2$, is easily obtained from $p(\rho)$ (see Lee, pp.191-193); the result is

$$p(Z) = \begin{cases} \frac{1}{\sigma^2} e^{-Z/2\sigma^2} & Z > 0 \\ 0 & Z < 0 \end{cases}$$

so that

$$\bar{Z} = \int_0^{\infty} Z p(Z) dZ = 2\sigma^2$$

$$\bar{Z}^2 = \int_0^{\infty} Z^2 p(Z) dZ = 16\sigma^4$$

and $\text{Var}[Z]$, the variance of Z , is

$$\text{Var}[Z] = \text{Var}\left[\frac{1}{M^2 R} P_v(T, F)\right] = \bar{Z}^2 - \bar{Z}^2 = 12\sigma^4(T, F) \quad (33)$$

which is independent of the number of GIP's used in decoding relative doppler frequencies. This is understandable, since the sum in equations (31) actually corresponds to only one GIP of duration $(R \cdot M)T_p$.

Comparing equation (30) with equation (33) we see that the fluctuations in the average output power $P_{SA}(T, F)$ are on the order of a factor R smaller than the fluctuations in $\frac{1}{M^2 R} P_v(T, F)$. It is for this reason that power outputs from each GIP rather than voltage outputs are summed over the R GIP's.

II. ANALYTICAL INVESTIGATION OF UNDERSPREAD

GAUSSIAN TARGET

The real and imaginary parts of the sampled receiver at time T relative to the a priori estimate of the time of arrival of the echo signal, $b(T)$ and $a(T)$, are due to reflection of the signal from the delay ring on the planet corresponding to relative delay T (for an underspread target). We have assumed that the independent random variables $a(T)$ and $b(T)$ have a Gaussian density of zero expected value and variance $\frac{1}{2} \sigma^2(T)$. We will now consider the effect of adding to $x(t)$ and $y(t)$ (see figure 5) the random signal $n(t)$ which is due to receiving system noise and determine the resultant noise signal in $a(T)$ and $b(T)$; $n(t)$ is independent of $x(t)$ and $y(t)$ and is assumed to be white noise with an expected value of zero and variance $\frac{1}{2} N_0^2$ watts per unit bandwidth. The expected value of $\eta(t)$, the noise at the output of the filters $h_i(t)$ in figure 5 due to the noise $n(t)$ at the input, is zero and the variance of $\eta(t)$ is the convolution of $\phi_{hh}(\gamma)$, the autocorrelation function of $h_i(t)$, with $\phi_{nn}(\gamma)$, the autocorrelation function of $n(t)$, evaluated at $\gamma=0$. The functions $\phi_{hh}(\gamma)$, $\phi_{nn}(\gamma)$, and $\phi_{\eta\eta}(\gamma)$ appear as shown in figure 7, where $\phi_{\eta\eta}$ is the autocorrelation function of $\eta(t)$:

$$\phi_{\eta\eta}(\gamma) = \int d\gamma' \phi_{hh}(\gamma') \phi_{nn}(\gamma - \gamma')$$

Because $n(t)$ is white its autocorrelation function is an impulse and we assume the area of the impulse to be $\frac{1}{2} N_0^2$.

Thus,

$$\phi_{\eta\eta}(\tau) = \frac{1}{2} N_0^2 \phi_{hh}(\tau)$$

and (see figure 7)

$$\text{Var}[\eta(t)] = \phi_{\eta\eta}(0) = \frac{1}{2} N_0^2 \gamma_B$$

so that the sampled output of the sine and cosine mixers in the presence of additive system noise is $a(T) + \eta(T)$ and $b(T) + \eta(T)$ respectively, where the variance of $\eta(T)$ is $\frac{1}{2} N_0^2 \gamma_B$. The expected value of the sum $a(T) + \eta(T)$ is the sum of the individual expected values $\bar{a}(T)$ and $\bar{\eta}(T)$ (which is zero) and because $a(T)$ and $\eta(T)$ are independent the variance of the sum is the sum of the variance:

$$\text{Var}[a(T) + \eta(T)] = \frac{1}{2} [\sigma^2(T) + N_0^2 \gamma_B]$$

The same is true for the sum $b(T) + \eta(T)$.

Let us call $P_{sNA}(T, F)$ the average power calculated when the effect of system noise is considered in addition to the effects of a rough surface (i.e., P_{sNA} is the average echo power calculated from the samples $a(T) + \eta(T)$ and $b(T) + \eta(T)$). The expected value and variance of $P_{sNA}(T, F)$ may be calculated in the same manner in which the expected value and variance of $P_{sA}(T, F)$ was calculated by replacing $\sigma^2(T)$ with $\sigma^2(T) + N_0^2 \gamma_B$. When this is done equation (28G)

becomes

$$\overline{V_{A_r}(T,F)V_{A_r}(T,F')} = \frac{1}{M^2} \sum_{m=0}^{M-1} [\sigma^2(T) + N_0^2 \gamma_B] e^{-j2\pi(F-F')mT_p}$$

where $V_{A_r}(T,F)$ is the average value of the complex voltage from the r^{th} CIP calculated from the samples $a(T) + \eta(T)$ and $b(T) + \eta(T)$. Calculating the variance of V_{A_r} in the same manner in which the variance of $\frac{1}{M^2} V_r$ was calculated will lead to the result

$$\text{Var}[V_{A_r}(T,F)] = \overline{|V_{A_r}(T,F)|^2} = \frac{1}{M} [2B_T \sigma_v^2(T,F) + N_0^2 \gamma_B]$$

which may be compared with equation (28I). We may go on to calculate $\text{Var}[P_{SNA}]$ in the same manner that led to equation (30) and find

$$\text{Var}[P_{SNA}(T,F)] = \frac{4\sigma_v^4(T,F)}{R} \equiv \sigma_{SNA}^2(T,F) \quad (33A)$$

where

$$\sigma_{SNA}^2(T,F) = \frac{1}{M} [2B_T \sigma_v^2(T,F) + N_0^2 \gamma_B]$$

and using equation (28J) and (29A) σ_{SNA}^2 may be written

$$\begin{aligned} \sigma_{SNA}^2(T,F) &= \overline{P_{SA}}(T,F) + \frac{N_0^2 \gamma_B}{M} \\ &= P(T,F) + \frac{N_0^2 \gamma_B}{M} \end{aligned} \quad (33B)$$

and the expected value of $P_{SNA}(T,F)$ is

$$\overline{P_{SNA}}(T,F) = 2\sigma_{SNA}^2(T,F) = 2P(T,F) + \frac{2N_0^2 \gamma_B}{M} \quad (33C)$$

Let us examine the effect of the variance σ_{SNA}^2 of the output echo power P_{SNA} on the implied uncertainty in the range (delay) estimate. Consider first the limiting case of no system noise and no random fluctuations in the amplitude and phase of the reflected signal (e.g., a target with a smooth surface).

In this case the average observed echo power is the same as $P(T,F)$, the theoretical model of the echo power distribution and one could estimate the delay to the sub-radar point by calculating the cross correlation in delay of the theoretical echo power distribution with the average observed echo power distribution (which is the same as the theoretical distribution except for a possible delay and doppler frequency offset due to a priori uncertainty with respect to these parameters). The value of trial offset at which the maximum value of the cross correlation occurs (say it is T'_S) is used to obtain the delay to the subradar point (the a priori estimate of delay is corrected by adding to it T'_S).

When system noise is added to the sampled receiver output and we observe a target with a rough surface so that $a(T)$ and $b(T)$ are random variables the average observed echo power will be $P_{SNA}(T,F)$. In this case we will cross correlate (in delay) $P_{SNA}(T,F)$ with $P(T,F)$ to obtain the delay to the subradar point. The value of trial offset, $T' = T'_S$, at which the maximum value of cross correlation

occurs is used to obtain the delay to the subradar point. However, the maximum value of the cross correlation of P_{SNA} with P will have an uncertainty associated with it due to the variance σ_{SNA}^2 associated with P_{SNA} and we will now determine this uncertainty.

The average cross correlation function, $R_A(T', F)$ will be defined as the cross correlation in delay of $P(T, F)$ with the average echo power $P_{SNA}(T, F)$: (in practice $P_{SNA}(T, F)$ is observed for only a finite number of values of relative delay T , say T_k $k = 0, 1, 2, \dots, K$. However, we may obtain $P_{SNA}(T, F)$ by interpolation of $P_{SNA}(T_k, F)$ $k = 0, 1, 2, \dots, K$.):

$$R_A(T', F) = \int_{-\infty}^{\infty} dT P(T, F) P_{SNA}(T+T', F) \quad (34)$$

The expected value of R_A is

$$E[R_A(T', F)] = \overline{R_A}(T', F) = \int_{-\infty}^{\infty} dT P(T, F) E[P_{SNA}(T+T', F)] \quad (34A)$$

The cross correlation in delay, $R(T', F)$, may be calculated for each value of F at which $P_{SNA}(T, F)$ is observed (i.e., it may be calculated for each doppler strip that is "decoded". See figure 8a). Sometimes only one doppler strip, centered on the subradar point (the "zero frequency component") is decoded. Suppose, however, we observe N_T doppler strips at each value of T corresponding to values of relative doppler shift $F = F_n$, $n = 0, 1, 2, \dots, N_T$. Then summing the average cross correlation $R_A(T', F)$ overall values of F_n at each

value of T' (see figure 8b) yields

$$R_s[T'] = \sum_{n=0}^{N_{T'}-1} R_A(T', F_n) \quad (35)$$

$$E[R_s(T')] = \sum_{n=0}^{N_{T'}-1} E[R_A(T', F_n)] \quad (35A)$$

Qualitatively, when investigating the terrestrial planets, one might guess that summing $R_A(T', F_n)$ over all values of n tends to increase SNR, the signal to noise ratio, as long as one is careful not to sum $R_A(T', F_n)$ over many doppler values at large values of relative delay, for at large values of delay the noise in the received power (reflected from terrestrial planets) is relatively strong compared to the received power. We will now examine **quantatatively** a scheme for determining the error in the observed value of delay. This scheme, as we will see, depends upon knowledge of $\phi_{R_s R_s}(\gamma)$, the autocorrelation function of $R_s(T')$, which we will use to determine statistical properties of $R_s(T')$ such as its variance and covariance.

The value of T' at which the maximum value in $R_s(T')$ occurs, call it T_s' (which corresponds to the error in the a priori delay to the subradar point) may be obtained by taking the first derivative of $\overline{R_s(T')}$ and setting it equal to zero. The reason for using this procedure to determine the effect of the variance of $R_s(T')$ in determining the delay to the target is that we now have a linear problem.

That is assuming $\bar{R}_s(T')$ is parabolic in the vicinity of T'_s the first derivative of $\bar{R}_s(T')$ will be linear in this vicinity and one standard deviation in $\frac{dR_s(T')}{dT'}$ will be proportional to one standard deviation in T' , and since the value of $T' = T'_s$ is a measure of the observed delay to the planet, one standard deviation in T'_s is a measure of the error in the observed delay to the planet, which is what we would like to know.

From equations (35) and (35A) we may write

$$\frac{dR_s(T')}{dT'} = \sum_n \frac{d}{dT'} R_A(T', F_n) \quad (36)$$

$$\frac{d\bar{R}_s(T')}{dT'} = \sum_n \frac{d}{dT'} \bar{R}_A(T', F_n) \quad (36A)$$

and furthermore, using equation (34A),

$$\begin{aligned} \frac{dR_A(T', F_n)}{dT'} &= \frac{d}{dT'} \int_{-\infty}^{\infty} dT P(T, F_n) P_{SNA}(T+T', F_n) = \int_{-\infty}^{\infty} dT P(T, F_n) \frac{\partial P_{SNA}(T+T', F_n)}{\partial T'} \\ &= \int_{-\infty}^{\infty} dT \frac{\partial P(T, F_n)}{\partial T'} P_{SNA}(T+T', F_n) \end{aligned}$$

so that

$$\frac{dR_s(T')}{dT'} = \sum_n \frac{d}{dT'} \int dT P(T, F_n) P_{SNA}(T+T', F_n) \quad (36B)$$

and

$$\frac{d\bar{R}_s(T')}{dT'} = \sum_n \frac{d}{dT'} \int dT P(T, F_n) \overline{P_{SNA}(T+T', F_n)} \quad (36C)$$

As mentioned earlier, in the vicinity of $\frac{d\bar{R}_s(T')}{dT'} = 0$ we will approximate $\frac{d\bar{R}_s(T')}{dT'}$ with a straight line, e.g.,

$$\frac{d\bar{R}_s(T')}{dT'} = -aT' + b \quad (37)$$

where $-a$ is the slope and b the intercept (on the $\frac{dR_s}{dT}$ axis) of the "best" line (in the least squares sense) that fits the actual curve formed by the plot of $\frac{dR_s(T')}{dT'}$ vs. T' . We will assume the least squares fit of $\frac{d\bar{R}_s(T')}{dT'}$ vs. T' to a straight line is so close that the variance of the parameters a and b (that are obtained from the fit) is zero and thus a and b will be treated as constants. From the above equation we see the maximum value of $R_s(T')$ occurs at $T' = \frac{b}{a} \equiv T'_s$, the value of T' for which $\frac{d\bar{R}_s}{dT'} = 0$. Therefore, determining the error in the observed value of T'_s due to the variance of $\left. \frac{d\bar{R}_s(T')}{dT'} \right|_{T'=T'_s}$ (which, ultimately, is a function of the processing parameters and nature of the target's surface and motion relative to the earth) is sufficient to tell us the error in the observed delay to the planet, and this is the object of this paper.

In order to calculate the standard deviation of the error in T'_s we express $\frac{dR_s(T')}{dT'}$ as

$$\frac{dR_s(T')}{dT'} = -a\xi' + b \quad (37A)$$

where ξ' represents the random value of the variable T' that results from taking the first derivative of $R_S(T')$ and solving for T' (e.g., $\xi' = -\frac{1}{a} \left[\frac{dR_S(T')}{dT'} - b \right]$) using the values of a and b determined from the plot of $\frac{d\bar{R}_S(T')}{dT'}$ vs. T' . (see figure 9A)).

The motivation behind this procedure and how it is used to obtain the standard deviation of the error in T'_S is given in the following argument, with reference to figure (9). Suppose we use the plot of $\frac{d\bar{R}_S(T')}{dT'}$ vs. T' to obtain T'_S and then calculate the derivative of $R_S(T')$ with respect to T' and evaluate it at $T' = T'_S$. From the plot of $\frac{d\bar{R}_S(T')}{dT'}$ vs. T' we assumed there would be a linear relationship between $\frac{dR_S(T')}{dT'}$ and T' so using the values a and b we solve for ξ' , the random value of T' corresponding to the random value $\left. \frac{dR_S(T')}{dT'} \right|_{T'=T'_S}$:

$$\xi'_S = \frac{1}{a} \left[b - \left. \frac{dR_S(T')}{dT'} \right|_{T'=T'_S} \right]$$

which is equation (37A) evaluated at T'_S . Next we want to know what the variance of ξ'_S would be if we made many observations of $\left. \frac{dR_S(T')}{dT'} \right|_{T'=T'_S}$ at $T' = T'_S$. This information is obtained from the above equation, which is equivalent

to equation (37A):

$$\text{Var}[\xi'_s] = \frac{1}{a^2} \text{Var} \left[\frac{dR_s(T')}{dT'} \right]_{T'=T'_s} \equiv \Delta D \quad (38)$$

and so the standard deviation of the error in the a priori delay (i.e., the standard deviation of the uncertainty in T'_s) is $\frac{1}{a}$ times the standard deviation of $\frac{dR_s(T')}{dT'}$ evaluated at $T' = T'_s$:

$$\Delta D = \sqrt{\text{Var}[\xi'_s]} = \frac{1}{a} \left\{ \text{Var} \left[\frac{dR_s(T')}{dT'} \right]_{T'=T'_s} \right\}^{1/2} \quad (38A)$$

Let us now calculate the variance of $\frac{dR_s(T')}{dT'}$. It can be shown (see 6.573 class notes, spring 1971, p126)

$$\frac{\partial^2}{\partial T_1 \partial T_2} E[R_s(T_1)R_s(T_2)] = E[\dot{R}_s(T_1)\dot{R}_s(T_2)] \quad (39)$$

$$\frac{d}{dT'} E[R_s(T')] = E[\dot{R}_s(T')] \quad (40)$$

where the dot above the function R_s signifies total differentiation with respect to the argument of R_s . If, in addition, $R_s(T')$ is wide sense stationary (and we will assume this is the case), then we may write $T_2' = T_1' + \gamma$ and

$$E \left[\frac{d}{dT_1} R_s(T_1) \frac{d}{dT_2} R_s(T_2) \right] = E [\dot{R}_s(T_1) \dot{R}_s(T_1 + \gamma)]$$

Using equation (39) and the relation $T_2' = T_1' + \gamma$

$$= \frac{\partial \gamma}{\partial T_1'} \frac{\partial \gamma}{\partial T_2'} \frac{d^2}{d\gamma^2} E [R_S(T_1') R_S(T_1' + \gamma)]$$

$$= - \frac{d^2}{dT_1'^2} E [R_S(T_1') R_S(T_1' + \gamma)]$$

Defining

$$\phi_{R_S R_S}(\gamma) = E [R_S(T_1') R_S(T_1' + \gamma)]$$

$$\phi_{\dot{R}_S \dot{R}_S}(\gamma) = E [\dot{R}_S(T_1') \dot{R}_S(T_1' + \gamma)]$$

we have

$$\phi_{\dot{R}_S \dot{R}_S}(\gamma) = - \frac{d^2}{d\gamma^2} \phi_{R_S R_S}(\gamma) \quad (41)$$

which relates $\phi_{R_S R_S}(\gamma)$, the autocorrelation function of $R_S(T_1')$, to $\phi_{\dot{R}_S \dot{R}_S}(\gamma)$, the autocorrelation function of the derivative of $R_S(T_1')$ with respect to T_1' .

The variance of $\frac{dR_S(T_1')}{dT_1'}$ may now be written in terms of equation (40) and (41):

$$\text{Var}[\dot{R}_S(T_1')] = \overline{[\dot{R}_S(T_1')]^2} - \left(\overline{\dot{R}_S(T_1')} \right)^2$$

$$= - \frac{d^2}{dT_1'^2} \phi_{R_S R_S}(\gamma) \Big|_{\gamma=0} - \left(\frac{d}{dT_1'} R_S(T_1') \right)^2$$

substituting the right hand side of the above equation into the right hand side of equation (38) yields

$$\Delta D = \frac{1}{a} \left[-\frac{d^2}{d\gamma^2} \phi_{R_s R_s}(\gamma) \Big|_{\gamma=0} - \left(\frac{d}{dT'} R_s(T') \right)^2 \right]_{T'=T_s'}^{1/2}$$

By definition, however, $\frac{d\bar{R}_s(T')}{dT'} \Big|_{T'=T_s'}$ equals zero, so that ΔD reduces to

$$\Delta D = \frac{1}{a} \left[-\frac{d^2}{d\gamma^2} \phi_{R_s R_s}(\gamma) \right]_{\gamma=0}^{1/2} \Big|_{T'=T_s'} \quad (42)$$

In order to evaluate the above we must calculate $\phi_{R_s R_s}(\gamma)$.

Using equation (35)

$$\phi_{R_s R_s}(\gamma) = R_s(T') R_s(T'+\gamma) = \sum_{n=0}^{N_s-1} \overline{R_A(T', F_n) R_A(T'+\gamma, F_n)} + \sum_{\substack{n=1 \\ k \neq l}} \overline{R_A(T', F_n) R_A(T'+\gamma, F_l)} \quad (43)$$

Let us work with the first term on the right hand side of the above equation; it may be written

$$\sum_n E [R_A(T', F_n) R_A(T'+\gamma)] = \sum_n \phi_{R_A R_A}(\gamma, F_n) = \sum_n \int_{-\infty}^{\infty} dT' R_A(T', F_n) R_A(T'+\gamma, F_n)$$

using equation (34) to substitute for R_A in the preceding:

$$\begin{aligned} \sum_n \phi_{R_A R_A}(\gamma, F_n) &= \sum_n \int_{-\infty}^{\infty} dT' \left[\int_{-\infty}^{\infty} dT P(T, F_n) P_{SNA}(T+T', F_n) \cdot \int_{-\infty}^{\infty} dV P(V, F_n) P_{SNA}(V+T'+\gamma, F_n) \right] \\ &= \sum_n \int_{-\infty}^{\infty} dV \int_{-\infty}^{\infty} dT P(T, F_n) P(V, F_n) \int_{-\infty}^{\infty} dT' P_{SNA}(T'+T, F_n) P_{SNA}(V+T'+\gamma, F_n) \end{aligned}$$

writing $\nu+T'+\gamma = \nu+\gamma-T+[T+T']$ in the argument of the second P_{SNA} in the last integral on the right above:

$$= \sum_n \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} dT P(T, F_n) P(\nu, F_n) \phi_{P_{SNA} P_{SNA}}(\nu+\gamma-T, F_n)$$

where

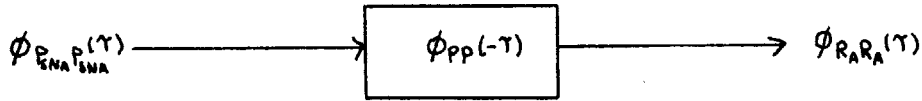
$$\phi_{P_{SNA} P_{SNA}}(\nu+\gamma-T, F_n) = \int_{-\infty}^{\infty} dT' P_{SNA}(T+T', F_n) P_{SNA}(\nu+\gamma-T+[T+T'], F_n)$$

making the substitution $\nu-T=l$, $d\nu = dl$:

$$\begin{aligned} \sum_n \phi_{R_A R_A}(\gamma, F_n) &= \sum_n \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dT P(T, F_n) P(T+l, F_n) \phi_{P_{SNA} P_{SNA}}(l+\gamma, F_n) \\ &= \sum_n \int_{-\infty}^{\infty} dl \phi_{PP}(l, F_n) \phi_{P_{SNA} P_{SNA}}(l+\gamma, F_n) \end{aligned} \quad (44)$$

$$\text{where } \phi_{PP}(l, F_n) = \int_{-\infty}^{\infty} dT P(T, F_n) P(T+l, F_n)$$

Equation (44) suggests the following block diagram:



The second term on the right hand side of equation (43) may similarly be expressed

$$\sum_{\substack{k, m \\ k \neq m}} \overline{R_A(T', F_n) R_A(T'+\gamma, F_m)} = \sum_k \sum_m \int_{-\infty}^{\infty} dl \phi_{PP}(l; F_n, F_m) \phi_{P_{SNA} P_{SNA}}(l+\gamma; F_n, F_m) \quad (45)$$

$$\text{where } \phi_{PP}(l; F_n, F_m) = \int_{-\infty}^{\infty} dT P(T, F_n) P(T+l, F_m) \quad (45A)$$

$$\phi_{P_{SNA} P_{SNA}}(l+\gamma; F_n, F_m) = \int_{-\infty}^{\infty} d\zeta P_{SNA}(\zeta, F_n) P_{SNA}(\zeta+l+\gamma, F_m) \quad (45B)$$

In general, the function $\phi_{xy}(t; F_1, F_2)$ will be defined as

$$\phi_{xy}(t; f_1, f_2) = \int_{-\infty}^{\infty} dt' x(t', f_1) y(t+t', f_2) \quad (45c)$$

and also

$$\phi_{xy}(t; f_1, f_1) = \int_{-\infty}^{\infty} dt' x(t', f_1) y(t+t', f_1) = \phi_{xy}(t, f_1) \quad (45d)$$

Now, in terms of equations (44) and (45) we may write

$$\phi_{R_s R_s}(\gamma) \quad \text{as} \quad (46)$$

$$\begin{aligned} \phi_{R_s R_s}(\gamma) &= \sum_{n=-\infty}^{\infty} \int dl \phi_{pp}(l, F_n) \phi_{P_{SNA} P_{SNA}}(l+\gamma, F_n) + \sum_{\substack{k, m \\ k \neq m}} \int dl \phi_{pp}(l; F_n, F_m) \phi_{P_{SNA} P_{SNA}}(l+\gamma; F_n, F_m) \\ &= \sum_k \sum_m \int_{-\infty}^{\infty} dl \phi_{pp}(l; F_n, F_m) \phi_{P_{SNA} P_{SNA}}(l+\gamma; F_n, F_m) \end{aligned} \quad (46A)$$

where the above equality was obtained by removing the restriction $k \neq m$ in the second term on the right hand side of equation (46) and recognizing from equation (45D) that $\phi_{pp}(l; F_n, F_n) = \phi_{pp}(l, F_n)$ and $\phi_{P_{SNA} P_{SNA}}(l+\gamma; F_n, F_n) = \phi_{P_{SNA} P_{SNA}}(l+\gamma, F_n)$.

At this point we would like to express $\phi_{R_s R_s}(\gamma)$ in terms of functions such as $\bar{P}_{SNA}(T, F)$, $P(T, F)$, and $\sigma_{P_{SNA}}^2(T, F)$. Therefore we recall from equations (33A), (33B), and (33C)

$$\bar{P}_{SNA}(T, F) = P(T, F) + \frac{N_0^2 \gamma_0}{M} \quad (47)$$

$$\text{Var} [P_{SNA}(T, F)] = \sigma_{SNA}^2(T, F) = \frac{1}{R} \left[P(T, F) + \frac{N_0^2 \gamma_0}{M} \right]^2 \quad (48)$$

It will be convenient to write $P_{SNA}(T,F)$ in terms of a constant part (namely, the expected value $\overline{P_{SNA}}(T,F)$) and a random "noise" component which we shall call $N(T,F)$:

$$P_{SNA}(T,F) = \overline{P}(T,F) + N(T,F) \quad (49)$$

In order for equation (49) to satisfy equations (47) and (48) $N(T,F)$ must be independent of $\overline{P_{SNA}}$ and must have the following properties:

$$E[N(T,F)] = 0 \quad (49A)$$

$$\text{Var}[N(T,F)] = \frac{4}{R} \left[P(T,F) + \frac{N_0^2 \gamma_B}{M} \right]^2 \quad (49B)$$

and we will furthermore assume $N(T,F)$ to be "white". The autocorrelation function of $P_{SNA}(T,F)$ may now be written

$$\phi_{P_{SNA}P_{SNA}}(T',F) = \phi_{\overline{P_{SNA}}\overline{P_{SNA}}}(T',F) + \phi_{NN}(T',F) \quad (50)$$

since P_{SNA} and $N(T,F)$ are independent and $\overline{N}(T,F) = 0$.

The function $\phi_{P_{SNA}P_{SNA}}(T'; F_R, F_m)$ which will be required to evaluate the integral in equation (46A), is

$$\phi_{P_{SNA}P_{SNA}}(T'; F_R, F_m) = \phi_{\overline{P_{SNA}}\overline{P_{SNA}}}(T'; F_R, F_m) + \phi_{NN}(T'; F_R, F_m) \quad (50A)$$

Recognizing the fact the system noise, in practice, is much greater than the echo signal we will neglect

$E[P_{SA}(T,F)] = P(T,F)$, the expected value of the average echo power reflected from the rough surface (assuming zero system noise), with respect to $\frac{N_0^2 \gamma_B}{M}$, the expected value

of the system noise, and thus equation (48) becomes

$$\sigma_{SNA}^2(T, F) \doteq \frac{4N_0^4 \gamma_B^2}{RM^2} .$$

Later on we will require an explicit value (in terms of \bar{P}_{SNA} , P , and $\frac{N_0^2 \gamma_B}{M}$) of $\phi_{NN}(T'; F_R, F_M)$, defined in accordance with equation (45D) as

$$\phi_{NN}(T'; F_R, F_M) = \int_{-\infty}^{\infty} dT N(T, F_R) N(T+T', F_M) .$$

Since $N(T, F)$ is of zero expected value, variance σ_{SNA}^2 , and is assumed "white", the above becomes

$$\phi_{NN}(T'; F_R, F_M) = \frac{4N_0^4 \gamma_B^2}{M^2} U_0(T') \delta_{F_R F_M} \quad (51)$$

where

$$\delta_{F_R F_M} = \begin{cases} 1 & F_R = F_M \\ 0 & \text{otherwise} \end{cases}$$

and $\delta_{F_R F_M}$ is merely a statement of the fact the noise $N(T, F)$ corresponding to different doppler strips (different values of F) is independent so that, for example, $E[N(T, F_R) N(T, F_M)] = 0$ if $F_R \neq F_M$. This follows from the assumption $N(T, F)$ is "white".

Under the assumption of dominating system noise $\bar{P}_{SNA}(T, F)$ is, approximately, (see equation (47) and figure 10A)

$$\bar{P}_{SNA}(T, F) \doteq \frac{N_0^2 \gamma_B}{M}$$

To avoid obtaining an infinite value for the autocorrelation of $\overline{P}_{SNA}(T, F)$ we may consider $\overline{P}_{SNA}(T, F)$ to be a periodic series of rectangles of arbitrary length (period) T_p and height $\frac{N_0^2 \gamma_B}{M}$ placed side by side (see figure 10B) so, using the definition of the autocorrelation of a periodic signal of period T

$$\begin{aligned} \phi_{\overline{P}_{SNA} \overline{P}_{SNA}}(T', F) &= \frac{1}{T_p} \int_0^{T_p} dT \overline{P}_{SNA}(T, F) \overline{P}_{SNA}(T+T', F) = \frac{1}{T_p} \int_0^{T_p} dT \frac{N_0^2 \gamma_B^2}{M^2} \\ &= \frac{N_0^4 \gamma_B^2}{M^2} \end{aligned} \quad (52)$$

and now we may substitute equations (52) and (51) into the right hand side of equation (50A) to obtain

$$\phi_{P_{SNA} P_{SNA}}(T'; F_h, F_m) = \frac{N_0^4 \gamma_B^2}{M^2} \left[1 + \frac{4}{R} U_0(T') \delta_{F_h F_m} \right] \quad (53)$$

From equation (46) we see we must cross correlate

$\phi_{P_{SNA} P_{SNA}}$ with ϕ_{PP} to obtain $\phi_{R_S R_S}$; this is done below:

$$\begin{aligned} \int_{-\infty}^{\infty} dl \phi_{PP}(l; F_h, F_m) \phi_{P_{SNA} P_{SNA}}(l+T; F_h, F_m) &= \int_{-\infty}^{\infty} dl \phi_{PP}(l; F_h, F_m) \frac{N_0^4 \gamma_B^2}{M^2} \left[1 + \frac{4}{R} U_0(l+T) \delta_{F_h F_m} \right] \\ &= \frac{N_0^4 \gamma_B^2}{M^2} \left[A_{PP}(F_h, F_m) + \frac{4}{R} \phi_{PP}(T; F_h, F_m) \delta_{F_h F_m} \right] \end{aligned} \quad (54)$$

$$\text{where } A_{PP}(F_h, F_m) = \int_{-\infty}^{\infty} dl \phi_{PP}(l; F_h, F_m)$$

and we have used the fact ϕ_{pp} must be symmetric so that $\phi_{pp}(-\gamma; F_n, F_m) = \phi_{pp}(\gamma; F_n, F_m)$. Substituting the right hand side of equation (54) into the right hand side of equation (46A) we have finally,

$$\phi_{R_s R_s}(\gamma) = \frac{N_0^4 \gamma_B^2}{M^2} \sum_k \sum_m [A_{pp}(F_k, F_m) + \frac{4}{R} \phi_{pp}(\gamma; F_k, F_m) \delta_{F_k F_m}] \quad (55)$$

and substituting the above expression of $\phi_{R_s R_s}(\gamma)$ into the right hand side of equation (42) yields

$$\begin{aligned} \Delta D &= \frac{1}{a} \left[- \frac{4N_0^4 \gamma_B^2}{RM^2} \sum_k \sum_m \frac{d^2}{d\gamma^2} \phi_{pp}(\gamma; F_k, F_m) \delta_{F_k F_m} \right]_{\substack{\gamma=0 \\ T'=T_s'}}^{1/2} \\ &= \frac{1}{a} \left[- \frac{4N_0^4 \gamma_B^2}{RM^2} \sum_n \frac{d^2}{d\gamma^2} \phi_{pp}(\gamma, F_n) \right]_{\substack{\gamma=0 \\ T'=T_s'}}^{1/2} \end{aligned} \quad (56)$$

where we have used the identity $\phi_{pp}(\gamma; F_k, F_m) = \phi_{pp}(\gamma, F_m)$ (see equation 45D)).

The value of a is obtained by differentiating equation (37) with respect to T' :

$$\frac{d^2 \bar{R}_s(T')}{dT'^2} = -a$$

and using equation (36C) the above may be written

$$-a = \sum_n \frac{d^2}{dT'^2} \int_{-\infty}^{\infty} dT P(T, F_n) \overline{P_{sNA}(T+T', F_n)}$$

Substituting the right hand side of equation (47) for the value of \bar{P}_{sNA} in the above integral:

$$a = - \sum_n \frac{d^2}{dT'^2} \int_{-\infty}^{\infty} dT P(T, F_n) P(T+T', F_n) \quad (57)$$

where we have made use of the fact

$$\frac{d^2}{dT'^2} \int_{-\infty}^{\infty} P(T, F_n) \frac{N_0^2 \gamma_B}{M} dT = 0$$

We may now replace the d that appears in equation (56) by the right hand side of equation (57) so that

$$\begin{aligned} \Delta D &= \frac{\left[-\frac{4N_0^4 \gamma_B^2}{M^2 R} \sum_n \frac{d^2}{dT^2} \phi_{PP}(\gamma, F_n) \right]_{\gamma=0}^{1/2}}{-\sum_n \frac{d^2}{dT'^2} \int_{-\infty}^{\infty} dT P(T, F_n) P(T+T', F_n) \Big|_{T'=T'_s}} \\ &= \frac{\left[-\frac{4N_0^4 \gamma_B^2}{M^2 R} \sum_n \frac{d^2}{dT^2} \phi_{PP}(\gamma, F_n) \right]_{\gamma=0}^{1/2}}{-\sum_n \frac{d^2}{dT'^2} \phi_{PP}(T', F_n) \Big|_{T'=T'_s}} \end{aligned} \quad (58)$$

Equation (58) expresses ΔD , the standard deviation in the observed value of delay for an underspread target, in a convenient form because, presuming the preceding assumptions are made, it can be evaluated as a function of the radar system parameters with a knowledge of only $P(T, F)$, the theoretical model of the received echo power distribution (which, as we saw in equation (15), is the convolution of the ambiguity function with the target scattering function).

At this point we will verify that the dimension of ΔD is seconds if all times are expressed in seconds. If P is measured in Watts then the dimension of ϕ_{PP} is $W^2 \cdot \text{sec}$

and $\frac{d^2 \phi_{pp}}{d\gamma^2}$ has the dimension $\frac{W^2 \cdot \text{sec}}{\text{sec}^2}$. From the equation above equation (54) we see the dimension of

$$\int_{-\infty}^{\infty} d\ell \phi_{pp}(\ell, F_h, F_m) \frac{N_0^4 \gamma_B^2}{M^2} \frac{4}{R} U_0(\ell + \gamma) \delta_{F_h, F_m} = \frac{4N_0^4 \gamma_B^2}{M^2 R} \phi_{pp}(\ell, F_h)$$

is $W^4 \cdot \text{sec}^2$ since the dimension of $d\ell$ is sec, the dimension of $N_0^2 \gamma_B$ is W (per unit bandwidth [cps]; see page 48; if $P(T, F)$ is dimensioned as W then so is $N_0^2 \gamma_B$; i.e., they must have the same dimension), and the dimension of ϕ_{pp} is $W^2 \cdot \text{sec}$. Therefore, the dimension of $\frac{4N_0^4 \gamma_B^2}{M^2 R} \sum_n \frac{d^2}{d\gamma^2} \phi_{pp}(\gamma, F_n)$ is $\frac{W^4 \cdot \text{sec}^2}{\text{sec}^2} = W^4$ and the dimension of equation (58) is $\frac{[W^4]^{1/2}}{(W^2 \cdot \text{sec})/\text{sec}} = \text{sec}$, as it should be.

Let us assume a simple form of the target scattering function in order to determine whether equation (58) will yield reasonable results. In particular, suppose the target scattering function is Gaussian in both the delay and doppler dimension, e.g., suppose σ' is

$$\sigma' = \pi R_p^2 e^{-T^2/T_c^2 - F^2/B_c^2}$$

where πR_p^2 is the geometrical cross section of the planet, T_c is the "characteristic" delay depth and B_c is the "characteristic" doppler width, i.e., at $T = T_c, F = B_c, \sigma'$ has diminished to $\frac{1}{e}$ times the value of σ' at $T = F = 0$. We will assume $T_c = 0.1 \times T_D$ and $B_c = 0.1 \times B_{ll}$ where T_D is the delay depth of the target and B_{ll} is the limb-to-limb

relative doppler spread. The radar cross section, σ , may be expressed as

$$\begin{aligned}\sigma &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dT dF \pi R_p^2 e^{-T^2/T_c^2 - F^2/B_c^2} \\ &= (\pi^2 T_c B_c) \pi R_p^2\end{aligned}$$

where R_p is the radius of the target. In the case of Venus, for example, $T_D = 40 \text{ msec}$, $B_{dB} = 10 \text{ cps}$, and

$$\begin{aligned}\sigma &= \pi^2 (0.1 \times 40 \times 10^{-3} \text{ sec})(0.1 \times 10 \text{ cps}) \pi R_p^2 \\ &= 0.04 \pi R_p^2.\end{aligned}$$

The average power received from the cell corresponding to delay-doppler coordinate (T, F) is

$$\begin{aligned}P(T, F) &= P_R \frac{\iint dT df \Psi^2(T-\tau, F-f) \sigma'(T, F)}{\Psi_{PK}^2 \cdot \sigma} \\ &= \frac{1}{\Psi_{PK}^2} \iint dT df \Psi^2(T-\tau, F-f) P'(T, F)\end{aligned}$$

where Ψ_{PK}^2 is the peak value of $\Psi^2(T, F)$ (see Figure 4g).

Note that as $\Psi(T, F)$ approaches an impulse of volume, say, $\Psi_{PK}^2 \delta T \delta F$ $P(T, F)$ becomes

$$P(T, F) = P_R \cdot \frac{\sigma'(T, F) \delta T \delta F}{\sigma}$$

which is the correct behavior (see Evans and Hagfors, 1968, p66).

The convolution of Ψ^2 with P' can be performed, with some difficulty, over the range of T but cannot be performed analytically over the range of F (recall, as a function of F , $\Psi^2 = \frac{\sin^2 \pi M t_p F}{\sin^2 \pi t_p F}$). Even if the convolution in F could be obtained analytically the resulting

power distribution P would be rather complicated. We would still need to obtain an expression for $\phi_{pp}(T,F)$ in order to examine the behavior of equation (58) and this would be a difficult if not impossible task to do exactly. Therefore, we will begin at this time to make some simplifying assumptions.

Assume first of all the extent of $\Psi^2(T,F)$ is very sharply peaked compared to $P'(T,F)$. (Later we will assume the opposite is true in order to obtain the behavior of ΔD when P' is sharply peaked compared to Ψ^2). Approximating one period of $\Psi^2(T,F)$ as an impulse of volume V_{Ψ^2} we have

$$P(T,F) \doteq P_R \frac{\sigma'(T,F) V_{\Psi^2}}{\sigma \Psi_{pk}^2}$$

We have convolved $P'(T,F)$ with only one period of $\Psi^2(T,F)$ rather than with the entire periodic expression $\Psi^2(T,F)$ because we have assumed the delay to the target is known to within one period (see the discussion in Section IB). In the delay dimension Ψ^2 is a series of triangles squared and as a function of F it is given by $|H_o(F)|^2$ (see equation (15)). The peak height of the triangles-squared is P_T (i.e., the peak height of the triangles is $\sqrt{P_T}$). Ψ_{pk}^2 , the peak value of $\Psi^2(T,F)$, occurs at $T=\tau_0, F=0$ (see, for example, figure 4g) and in this example its value is

$$\Psi_{pk}^2 = \Psi^2(\tau_0, 0) = P_T \cdot M^2$$

Since the base of the triangles squared is $2\gamma_B$ the area under each triangle-squared, A_T , is

$$A_T = 2 \cdot \int_0^{\gamma_B} \left(\frac{\sqrt{P_T}}{\gamma_B} T \right)^2 dT = \frac{2}{3} P_T \gamma_B$$

The area under one period of the curve described by $|H_0(F)|^2$, A_F , is equal to the area under one period of $\frac{\sin^2 \pi M t_p F}{\sin^2 \pi t_p F}$. Because it is difficult to integrate this function we will make the approximation

$$\begin{aligned} A_F &\doteq 2 \cdot \left(\frac{1}{2} \times \frac{1}{M t_p} \times M^2 \right) && \text{(see Kraut, p208)} \\ &= \frac{M}{t_p} \end{aligned}$$

Because $\Psi^2(T,F)$ is separable the volume under Ψ^2 is $A_T A_F$ and now we may write $P(T,F)$ as

$$P(T,F) = P_R \frac{\sigma^2(T,F) A_T A_F}{\sigma \Psi_{PK}^2}$$

Before we continue we should note dividing by Ψ_{PK}^2 has the effect of normalizing $\Psi^2(T,F)$ so that, for example, the peak height of the triangles-squared is unity and so the peak height of each of the triangles is also unity. Thus the height of ϕ_{hh} (the autocorrelation of the pulse matched filter) at $\gamma=0$ is unity and the variance of $\eta(t)$ ($= \phi_{nn}(\gamma) * \phi_{hh}(\gamma)$) evaluated at $\gamma=0$) will be $\frac{1}{2} N_0^2 \cdot 1$ as opposed to $\frac{1}{2} N_0^2 \gamma_B$ (see figure 7). The dimension of $N_0^2 \cdot 1$ is watts · sec or watts per unit bandwidth (see p39)

and is equal to kT_N where k is Boltzman's constant and T_N is the system temperature.

Let us now calculate

$$\begin{aligned}\phi_{PP}(\gamma, F_n) &= \int_{-\infty}^{\infty} dT P(T, F_n) P(T+\gamma, F_n) \\ &= P_R^2 A_T^2 A_F^2 \frac{(\pi R_P^2)^2}{\sigma^2} \int_{-\infty}^{\infty} dT e^{-2F_n^2/B_c^2} e^{-T^2/T_c^2 - (T+\gamma)^2/T_c^2} \\ &= (P_R A_T A_F \frac{\pi R_P^2}{\sigma})^2 e^{-2F_n^2/B_c^2} \frac{T_c}{\sqrt{2}} e^{-\gamma^2/2T_c^2}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dT^2} \phi_{PP}(\gamma, F_n) \Big|_{\gamma=0} &= (P_R A_T A_F \frac{\pi R_P^2}{\sigma})^2 e^{-2F_n^2/B_c^2} \frac{e^{-\gamma^2/2T_c^2}}{\sqrt{2} T_c} \left[\frac{2T^2}{T_c^2} - 1 \right] \Big|_{\gamma=0} \\ &= - \frac{(P_R A_T A_F \frac{\pi R_P^2}{\sigma})^2}{\sqrt{2} T_c} e^{-2F_n^2/B_c^2}\end{aligned}\tag{61}$$

Assuming T_s' , the position of the peak value of $\bar{R}_s(T')$ is zero or very close to zero

$$\frac{d^2}{dT'^2} \phi_{PP}(T', F_n) \Big|_{T'=T_s'=0} = - \frac{(P_R A_T A_F \frac{\pi R_P^2}{\sigma})^2}{\sqrt{2} T_c} e^{-2F_n^2/B_c^2}\tag{62}$$

and substituting the right hand side of equation (61) and (62) into equation (58) yields (replacing $N_0^2 \gamma_0$ with $N_0^2 = kT_N$)

$$\Delta D = \frac{2 \cdot 2^{1/4} k T_N \sqrt{T_c}}{\sqrt{M^2 R}} \frac{\Psi_{PR}^2}{P_R A_T A_F \frac{\pi R_P^2}{\sigma}} \frac{1}{\left[\sum_{n=0}^{N_s'-1} e^{-2F_n^2/B_c^2} \right]^{1/2}}\tag{63}$$

We may express R, the number of CIP's, as the total duration of reception of the echo signal, t_T (we will assume t_T is fixed in this paper), divided by Mt_p , the duration of one CIP:

$$R = \frac{t_T}{Mt_p} .$$

The maximum number of doppler strips decoded is $B_{D1}Mt_p$, the limb-to-limb doppler spread of the target divided by $\frac{1}{Mt_p}$, the frequency resolution of each doppler cell. We denote $[B_{D1}Mt_p]$ as the largest integer value $\leq B_{D1}Mt_p$ so that the sum in the denominator of equation (63) is limited to $[B_{D1}Mt_p]$ terms. Let us substitute into the right hand side of equation (63) the values of A_T, A_F , and Ψ_{PR}^2 given above and also make use of the relations $R = \frac{t_T}{Mt_p}$ and $N_{T3} = [B_{D1}Mt_p]$ to yield

$$\Delta D = \frac{3/2 \cdot 2 \cdot 2^{1/4} k T_N \sqrt{T_c}}{P_R \frac{\pi R P^2}{\sigma}} \frac{t_p}{\gamma_B} \frac{\sqrt{Mt_p}}{\sqrt{t_T}} \frac{1}{\left[\sum_{n=0}^{[B_{D1}(Mt_p)]-1} e^{-2F_n^2/B_c^2} \right]^{1/2}} \quad (63A)$$

In examining the behavior of ΔD as certain parameters are varied we will fix the value of P_A , the average transmitter power and t_T , the reception time. Because we are assuming P_A is constant the peak transmitted power, P_T , must necessarily change as γ_B and t_p are varied. The

relation between P_T, P_A, γ_B and t_p is

$$P_T = P_A \frac{t_p}{\gamma_B} \quad (63B)$$

Writing $P_R = B \cdot P_T$ (where $B \approx A_R L_R B'$; see section II A) and substituting the right hand side of the above equation for P_T equation (63A) becomes

$$\Delta D = \frac{3/2 \cdot 2 \cdot 2^{1/4} k T_N \sqrt{T_c}}{B \cdot \frac{\pi R^2}{\sigma} \cdot P_A} \frac{\sqrt{M t_p}}{\sqrt{t_T}} \frac{1}{\left[\sum_{n=0}^{[B/M t_p]-1} e^{-2 F_n^2 / B^2} \right]^{1/2}} \quad (64)$$

We should recall that the above equation applies only when the delay and doppler resolution of the radar receiver is very fine so that $\Psi^2(T, F)$ may be approximated by an impulse compared to $P'(T, F)$. This means we must have

$$\begin{aligned} \frac{1}{M t_p} &\ll 0.1 \times B_{df} = B_c \\ \gamma_B &\ll 0.1 \times T_D = T_c \end{aligned} \quad (64A)$$

We now consider the case in which $P'(T, F)$ may be approximated by an impulse compared to $\Psi^2(T, F)$. In this instance we find

$$P(T, F) = P_R \frac{\Psi^2(T, F) \cdot \sigma}{\Psi_{PK}^2 \cdot \sigma}$$

Again assuming T_s' , the position of the peak value of $\bar{R}_s(T')$ is zero or very close to zero equation (58)

becomes (replacing $N_0^2 \gamma_B$ with kT_N)

$$\Delta D = \frac{2kT_N}{\sqrt{M^2 R}} \frac{1}{\left[\sum_{n=0}^{N_T-1} \frac{d^2}{dT'^2} \phi_{PP}(T', F_n) \right]_{T'=0}^{1/2}}$$

$$\text{where } \phi_{PP}(T', F_n) = \frac{P_R^2}{\Psi_{PK}^2} \int dT \Psi^2(T+T', F_n) \Psi^2(T, F_n)$$

Since the autocorrelation of $\Psi^2(T, F_n)$ is to be performed only in the delay dimension we realize this involves autocorrelating the now familiar triangle-squared function. According to the above expression of ΔD we must then calculate the second derivative of this autocorrelation with respect to T' and evaluate it at $T'=0$. A somewhat tedious calculation yields

$$\frac{d^2}{dT'^2} \phi_{PP}(T', F_n) = P_R^2 \frac{|H_0(F_n)|^4}{M^4} \cdot -\frac{5}{3} \frac{1}{\gamma_B}$$

and

$$\Delta D = \frac{2kT_N}{\sqrt{M^2 R}} \frac{1}{\left[P_R^2 \sum_{n=0}^{N_T-1} \frac{|H_0(F_n)|^4}{M^4} \cdot \frac{5}{3} \frac{1}{\gamma_B} \right]^{1/2}}$$

using $R = \frac{t_T}{M t_P}$ and $N_{T=0}$ (number of doppler strips decoded at $T'=0$) = 1

$$\Delta D = \frac{2kT_N}{P_R \sqrt{M t_T}} \frac{\sqrt{\frac{3}{5} t_P \gamma_B}}{\left[|H_0(F_{n=0})|^4 \cdot \frac{1}{M^4} \right]^{1/2}}$$

The effect of this very poor resolution may be considered as nearly a uniform weighting of $P'(T,F)$ over all values of T and F for which the target scattering function is of significant value. Thus, $|H_0(F_{n=0})|^4$ evaluated at any value of $F_{n=0}$ in the range $-\frac{1}{2}B_c \leq F_{n=0} \leq \frac{1}{2}B_c$ is not significantly different from $|H_0(F_{n=0}=0)|^4 = M^4$ so

$$\Delta D = \frac{2kT_N}{P_R \sqrt{Mt_T}} \cdot \sqrt{\frac{3}{5} t_p \gamma_B} = \frac{2kT_N \sqrt{\frac{3}{5} \gamma_B}}{P_R \sqrt{Mt_p} \sqrt{t_T}}$$

and using $P_R = B \cdot P_T$ and equation (63B) the above may be written

$$\Delta D = \frac{2kT_N \cdot \gamma_B}{B \cdot P_A} \frac{\sqrt{\frac{3}{5} \gamma_B}}{\sqrt{t_T}} \frac{1}{\sqrt{Mt_p}} \quad (65)$$

Because we have assumed poor resolution in this case we must limit the application of the above formula to the region

$$\begin{aligned} \frac{1}{Mt_p} &>> 0.1 \times B_{\text{eff}} = B_c \\ \gamma_B &>> 0.1 \times T_D = T_c \end{aligned} \quad (65A)$$

In addition to the restrictions on the values of $\frac{1}{Mt_p}$ and γ_B in equations (64) and (65) we recall, according to the basic premise of this paper that we will confine our attention to underspread targets, we require

$$T_D < t_p < \frac{1}{B_{\text{eff}}}$$

(the desirable implications of this inequality are discussed in section IB)

III. NUMERICAL RESULTS

Let us, for example, apply our results to a typical Venus ranging experiment since Venus is a good practical example of an underspread target. The values of the parameters appearing in equations (64) and (65) will be consistent with the values used in the radar measurements made on Venus by Pettengill et al in 1964 using the facilities of Arecibo Ionospheric Observatory (see table I and Pettengill et al, 1967). (At the operating wavelength of the Arecibo facility the value of B_{ll} for Venus ranges between 6 cps at inferior conjunction to 25 cps at superior conjunction; since T_D for Venus is 40 msec the product $B_{ll}T_D$ is always ≤ 1 and thus Venus is always underspread). Using the values of the parameters listed in table I we calculate

$$\sigma = (\pi^2 T_c B_c) \pi R_p^2 = 0.04 \pi R_p^2 = 4.7 \times 10^{12} \text{ m}^2$$

$$P_R = B \cdot P_T = \left(\frac{P_T G_T L_T L_A}{4\pi R^2} \right) \times \frac{\sigma}{4\pi R^2} \times A_R L_R L_A = 1.83 \times 10^{-24} P_T$$

(at zenith)

$$N_0^2 = kT_N = 2.76 \times 10^{-21} \text{ Watt} \cdot \text{sec} \quad \text{(at zenith)}$$

When the antenna is pointing 20° below the zenith point the gain decreases by nearly a factor of 2 (see table I) and we will assume the effective receiving aperture goes down to 0.4 times the geometrical antenna cross section

and the system temperature increases slightly, say, by a factor of 1.25 so that

$$P_R = 5.6 \times 10^{-25} P_T \quad (\text{at } 20^\circ \text{ zenith angle})$$

$$kT_N = 3.5 \times 10^{-21} \text{ W}\cdot\text{sec} \quad (\text{at } 20^\circ \text{ zenith angle})$$

Using equations (64) and (65) the behavior of ΔD with Mt_p , the length of the CIP, was examined in the case of very fine and very poor resolution. Values of the parameters typical of those for observations of Venus made at Arecibo in 1964 (see table I) were used. The received time (t_r) and P_A , the average transmitter power, were fixed at 800 sec and 100 kW respectively. The zenith angle of Venus was assumed to be 20° . For both fine and poor resolution t_p was restricted to the values $T_D < t_p < \frac{1}{B_{eff}}$ which is

$$0.04 \text{ sec} < t_p < 0.10 \text{ sec}$$

for Venus.

In the case of fine delay-doppler resolution only the values of $\frac{1}{Mt_p} \leq 0.1 \times B_{eff} = 1 \text{ cps}$ were considered (of course, the smaller $\frac{1}{Mt_p}$, i.e., the larger Mt_p , the better the approximation that led to equation (64)). Also, in this region, γ_B is assumed $\leq 10^{-3}$ sec (typical values of γ_B used by Pettengill et al in 1964 were 0.1 - 1 msec) to satisfy the requirement $\gamma_B \ll 0.1 \times T_D = 4 \times 10^{-3} \text{ sec}$ (see equation (64A)).

Using equation (64) with the appropriate parameters listed in table I we find

$$\Delta D = 10.8 \mu\text{sec} \quad \text{for } t_T = 800 \text{ sec} \quad (66)$$

$$P_A = 100 \text{ kW}$$

$$\text{zenith angle} = 20^\circ$$

$$0.04 \leq t_p \leq 0.10$$

$$\gamma_B \leq 1 \text{ msec}$$

independent of Mt_p , the duration of the Coherent Integration Period (as long as $\frac{1}{Mt_p}$ is $\leq 1 \text{ cps}$ to satisfy the assumption made in arriving at equation (64)).

To understand why ΔD is independent of Mt_p in this case let us calculate r_{AVE} , the ratio of the average echo power to the standard deviation of the echo power corresponding to each delay-doppler cell that is decoded. Using equations (33A), (33B), and (33C)

$$r_{\text{AVE}} = \frac{\overline{P_{\text{SNA}}(T,F)}}{[\text{Var}(P_{\text{SNA}}(T,F))]^{1/2}} = \sqrt{NR} \quad (67)$$

$$= \sqrt{\frac{t_T}{Mt_p}}$$

From the above one might expect that ΔD would increase as frequency resolution, or the length of the CIP (Mt_p), is increased. However, as the frequency resolution increases more delay-doppler cells may be decoded and, as it turns out, r_{AVE} decreases at the same rate as the square root of the summation of the average power from each of the $[B_{99}Mt_p]^{-1}$ doppler cells increases (see equation (64)) and

as a result we recover the same signal to noise ratio for all values of Mt_p .

In obtaining ΔD in the case of fine resolution (equation (64)) we have found ΔD is the same for all "allowed" values of t_p , i.e., $0.04 \text{ sec} \leq t_p \leq 0.10 \text{ sec}$ and for all values of γ_B sufficiently small (namely, less than about 1 msec). This is a result of varying P_T , the peak transmitted power in such a way that P_A , the average transmitted power, is constant. Thus, $t_p/P_R\gamma_B = t_p/8P_T\gamma_B = 1/P_A B$ is constant and substituting $1/8P_A$ for $t_p/P_R\gamma_B$ in equation (63A) we find ΔD is independent of t_p and γ_B (as long as the delay-doppler resolution is sufficiently fine so that the approximations that led to equation (63A) are valid).

We may compare the value of ΔD given in equation (66) with the estimated delay error of 50 μsec for Venus ranging reported by Pettengill et al in 1967 using the same values of the appropriate parameters we have used to obtain equation (66) (γ_B was $\leq 1 \text{ msec}$ and t_p was $\approx 0.1 \text{ sec}$). The difference between the two estimates of delay error is due to the simple Gaussian model assumed in this paper and to the not very greatly refined error analysis used by Pettengill et al in 1967 (see Pettengill et al, 1967, and Jurgens and Dyce, 1970). However, this difference is not great if we consider the "99% confidence level", or $3 \cdot \Delta D$ error, which is 33 μsec .

Let us now turn our attention to figure 11, in which we have plotted the duration of the CIP (Mt_p) versus ΔD in the case of poor delay-doppler resolution with the use of equation (65) and table I. In this plot $\frac{1}{Mt_p}$ is always ≥ 5 cps in order to satisfy the requirement $\frac{1}{Mt_p} \gg 0.1 \times B_{dl} = 1$ cps and γ_B was fixed at 40×10^{-4} sec to satisfy the inequality $\gamma_B \gg 0.1 \times T_D = 40 \times 10^{-3}$ sec (see equation (65A)). Again, we have set $P_A = 100$ kW, $t_T = 800$ sec, and the antenna was assumed to be pointing at a zenith angle of 20° . The range of t_p for which figure 11 applies is $0.05 \text{ sec} \leq t_p \leq 0.10 \text{ sec}$. The minimum value of the duration of the CIP for any value of t_p is, of course, t_p sec, and must always be greater than 0.04 sec because the baud length in figure 11 is assumed to be 0.04 sec. Since we are interested in measuring the range to the target t_p must be greater than γ_B . As a somewhat arbitrary lower cutoff, therefore, the length of the CIP versus ΔD was not plotted for a CIP less than 0.05 sec.

As we have mentioned, figure 11 describes the relation between ΔD and Mt_p for values of Mt_p so small that there is essentially no frequency resolution (i.e., the distance between the first nulls of the frequency resolution function $|H_o(F)|^2$, is much greater than the radar doppler spread of the planet). In spite of this, from figure 11, we see ΔD decreases as Mt_p increases. This decrease may be attributed to the fact that as the frequency resolution approaches the radar doppler width of the planet less noise signal contributes to the output power because the doppler frequency

dimension of the target occupies a greater fraction of the resolution bandwidth. The result is ΔD decreases as the square root of the bandwidth, i.e., as $\sqrt{\frac{1}{M t_p}}$, as we see in equation (65). (Also, see Evans and Hagfors, 1968, p393).

III A. SUMMARY

We have examined the behavior of the semi-analytically derived expression for one standard deviation (ΔD) in the estimate of delay to an underspread planetary target (with a rough surface) as a function of the radar receiver and processing parameters. The average transmitted power and duration of reception of the echo signal were fixed (in practice, this is approximately true) and γ_0 (the delay resolution), $\frac{1}{M t_p}$ (the frequency resolution), and t_p were varied. Using a simple Gaussian model for the target scattering function and making some approximations it was found that, in general, ΔD is smaller for fine delay-doppler resolution than for poor resolution. In particular, ΔD was observed to decrease as $M t_p$ increased (i.e., as the frequency resolution increased. For sufficiently high resolution, however, ΔD was found to be independent of the length of the CIP. This was explained by noting that in the method we used to determine the delay to the subradar point the result of summing the power samples from the large number of doppler cells (which high

frequency resolution allowed to be decoded) was cancelled by the large fluctuations in the decoded output power in each cell that occurred when the received time and average transmitter power was fixed. For very poor resolution a plot of ΔD versus Mt_p was obtained (figure 11) using values of the appropriate parameters for a typical Venus ranging experiment and it was observed that ΔD varied as $(Mt_p)^{-1/2}$ (equation (65)). Thus in this case ΔD decreases as Mt_p increases and we saw this was due to the fact the target occupies a greater percentage of the doppler resolution bandwidth as the frequency resolution increases (i.e., as Mt_p increases). Therefore, as Mt_p was increased the signal power corresponding to the region beyond the limb of the target (but within the resolution of the receiver), which contributes only to "noise", was decreased.

IV. SUGGESTIONS FOR FURTHER RESEARCH

There are several obvious, interesting, and useful extensions of the work presented in this paper. As might be expected, future investigation in this area must deal with the problem of evaluating ΔD for more realistic planetary targets and also for overspread planetary targets.

Attention should first be given to evaluating the expression for ΔD as derived in this paper for a planet obeying a scattering law that gives a more accurate description of the observed echo power distribution than that given by a Gaussian scattering function. Scattering functions that yield close agreement with planetary radar observations have been derived by Hagfors (see, e.g., Evans and Hagfors, Chapter 4, 1968; and Jurgens and Dyce, 1970) and Muhleman (see, e.g., Jurgens and Dyce, 1970). (The scattering functions derived by Hagfors and Muhleman correspond to the variable σ' in equation (10)). In this case an analytical evaluation of ΔD is strictly impossible to obtain and the best treatment of the problem of evaluating the behavior of ΔD with respect to the radar system and processing parameters would employ **numerical** methods of analysis.

The next step would be to derive an expression for in the case of overspread targets (with a rough surface), specifically, targets for which $B_{00} T_D > 1$ and for which the

PRF was $< B_{ll}$ and $t_p < T_D$. The severity of the resulting "aliasing" of the observed echo power distribution in both the delay and doppler coordinate (see the discussion in section IE) would depend upon how much greater than unity the product $B_{ll}T_D$ was and on the scattering law that the target obeyed. Both Mercury ($B_{ll}T_D = 4.8$) and Mars ($B_{ll}T_D \approx 520$) are examples of overspread targets.

Finally, the effects of large scale features, such as mountain ranges, on the estimated delay error for both underspread and overspread targets should be investigated.

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LIST OF SYMBOLS

A list of some of the frequently used symbols is given below

T	Time delay relative to delay corresponding to subradar point
F	Doppler frequency shift relative to doppler shift of subradar point
$\Psi^2(T,F)$	Ambiguity function
$P'(T,F)$	Target-scattering function
$P(T,F)$	Theoretical echo power distribution (convolution of $\Psi^2(T,F)$ with $P'(T,F)$)
$P_{SA}(T,F)$	Average output echo power distribution for echo signal reflected from a target with rough surface but assuming zero system noise
$P_{SNA}(T,F)$	Average output echo power distribution in presence of system noise and random fluctuations of sampled echo amplitude due to reflection from rough surface
$R_A(T',F)$	Cross correlation of $P_{SNA}(T,F)$ with $P(T,F)$
$R_S(T')$	$\sum_n R_A(T',F_n)$
$E[x(t,f)] = \overline{x(t,f)}$	Expected value of $x(t,f)$
$\phi_{xy}(t_1, f_1, f_2)$	$\int_{-\infty}^{\infty} dt' x(t', f_1) y(t+t', f_2)$
$\phi_{xy}(t, f_1)$	$\phi_{xy}(t, f_1, f_1) = \int_{-\infty}^{\infty} dt' x(t, f_1) y(t, f_1)$
B_{dl}	Limb-to-limb doppler spread of target
ΔD	One standard deviation in observed delay to subradar point
Mt_p	Duration of 1 CIP
N_S	Number of pulse repetition periods in time t_1
N_T	Number of doppler strips decoded at relative delay τ
R	Number of CIP's used in decoding a doppler cell
γ_B	Baud length

LIST OF SYMBOLS (Continued)

T_D	Delay depth of target
t_p	Pulse repetition period
T_s'	Position of peak value of $\bar{R}_s(\tau')$, corresponds to observed position of subradar point relative to a priori estimate
t_T	Total time duration of reception of echo signal

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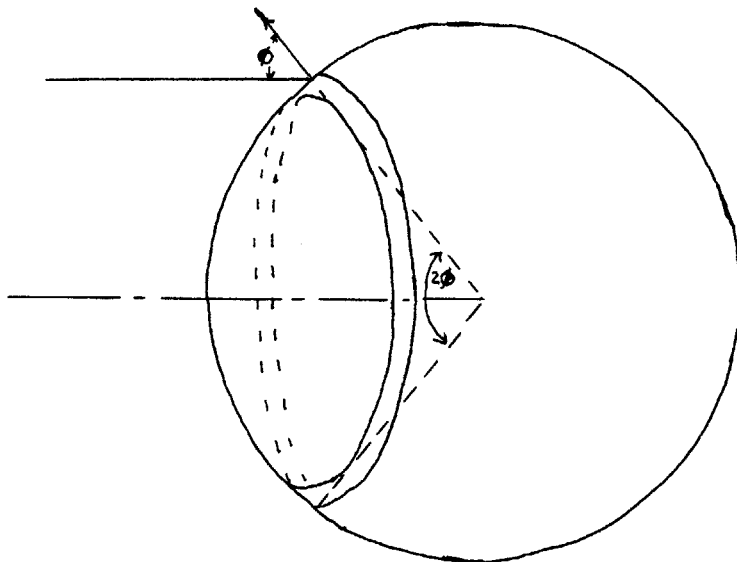
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FIGURE I

a.



b.

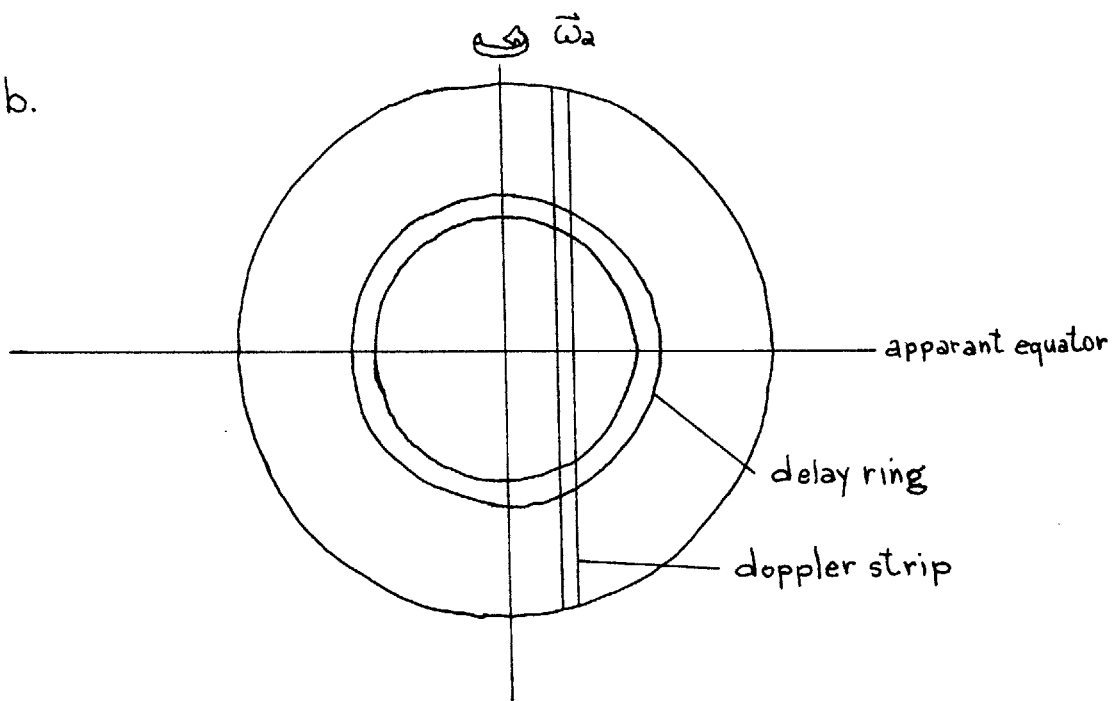


FIGURE 4

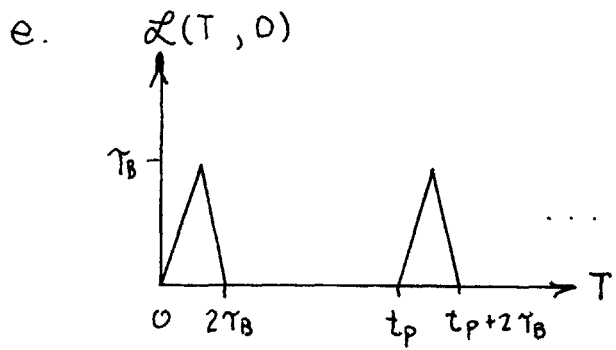
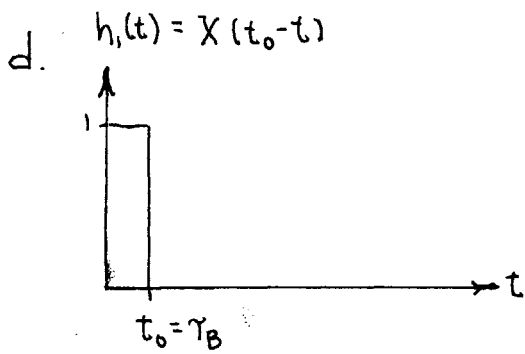
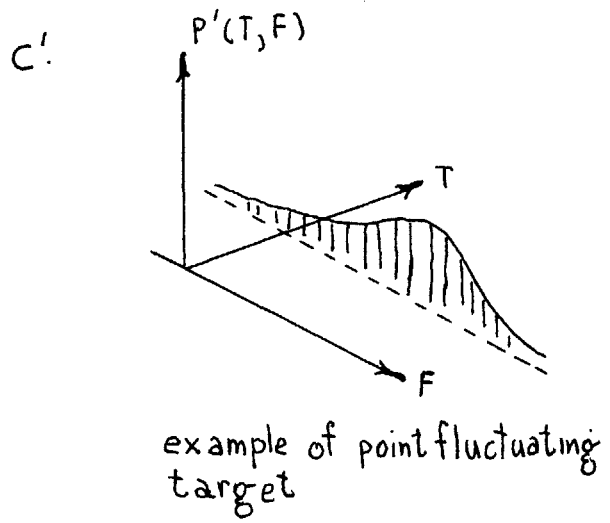
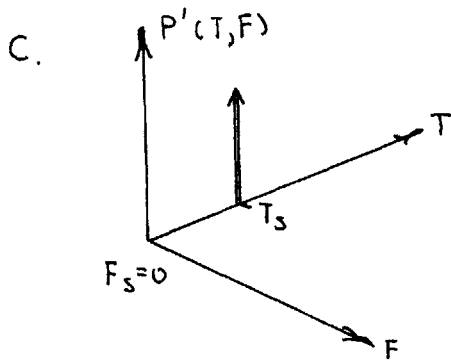
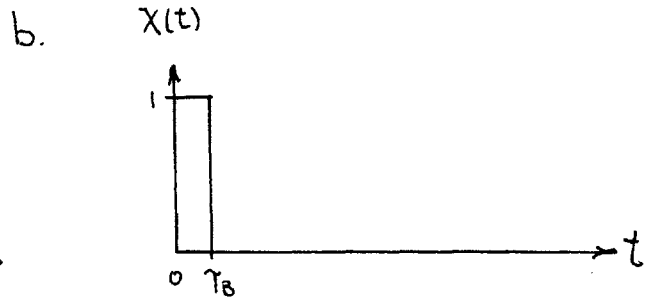
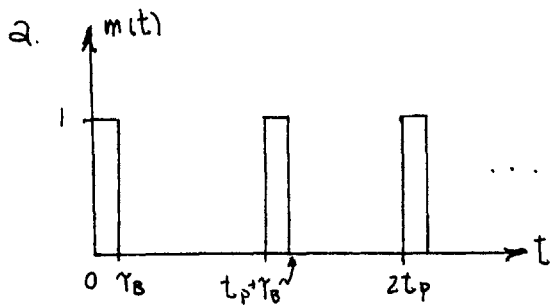
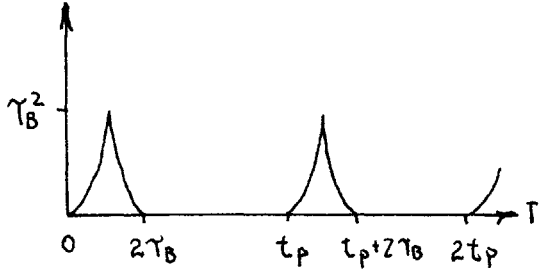
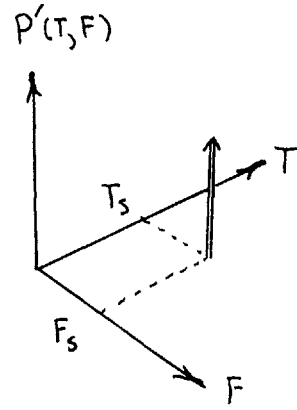


FIGURE 4 (Cont.)

f. $\Psi_p^2(T, \tau_B, 0)$



ff.



gg. $\Psi'^2(T, F)$

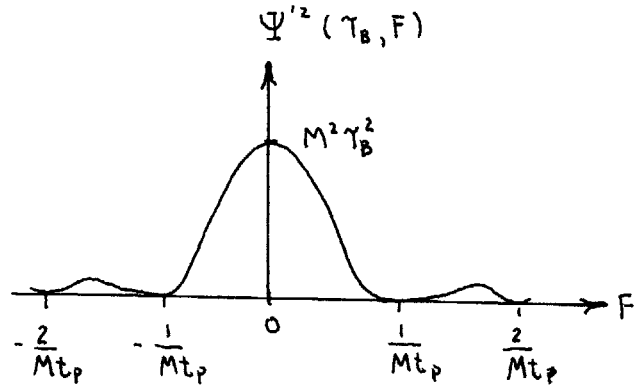
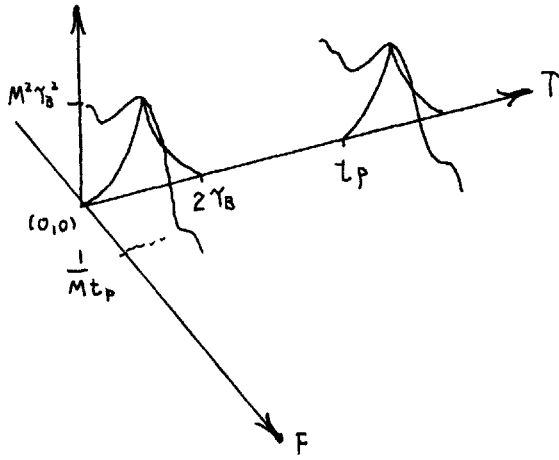
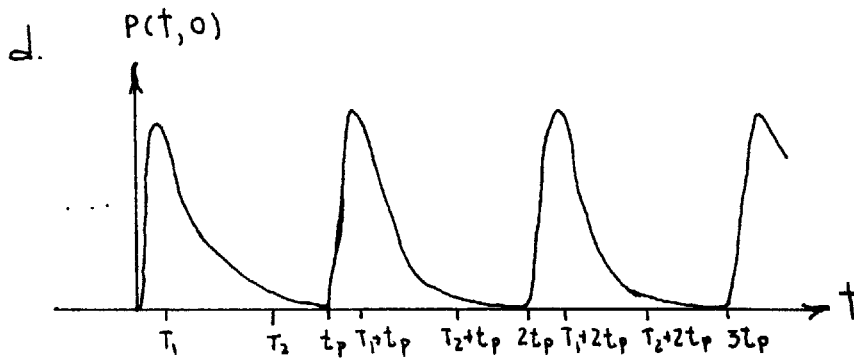
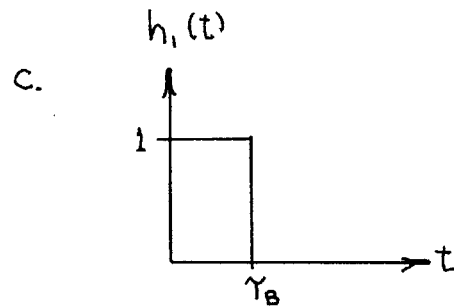
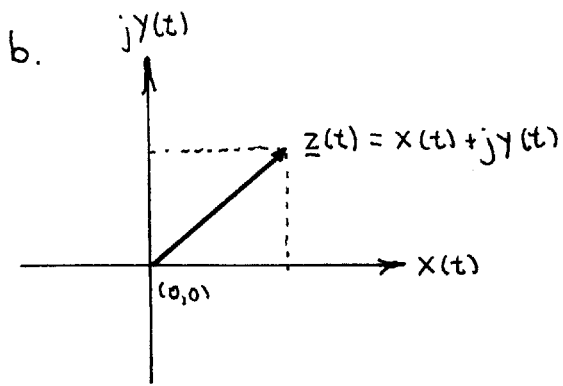
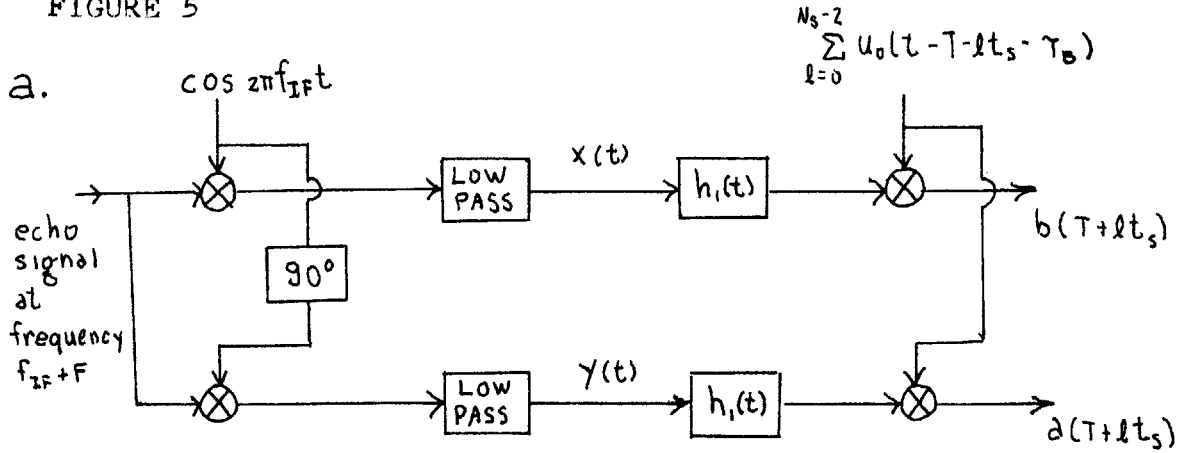


FIGURE 5



Slice of $P(t, F)$ vs. t, F taken at $F=0$

FIGURE 6

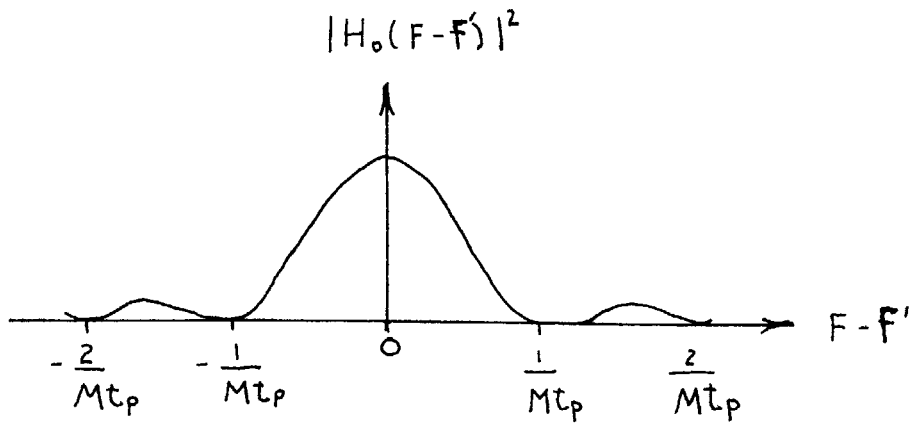


FIGURE 7

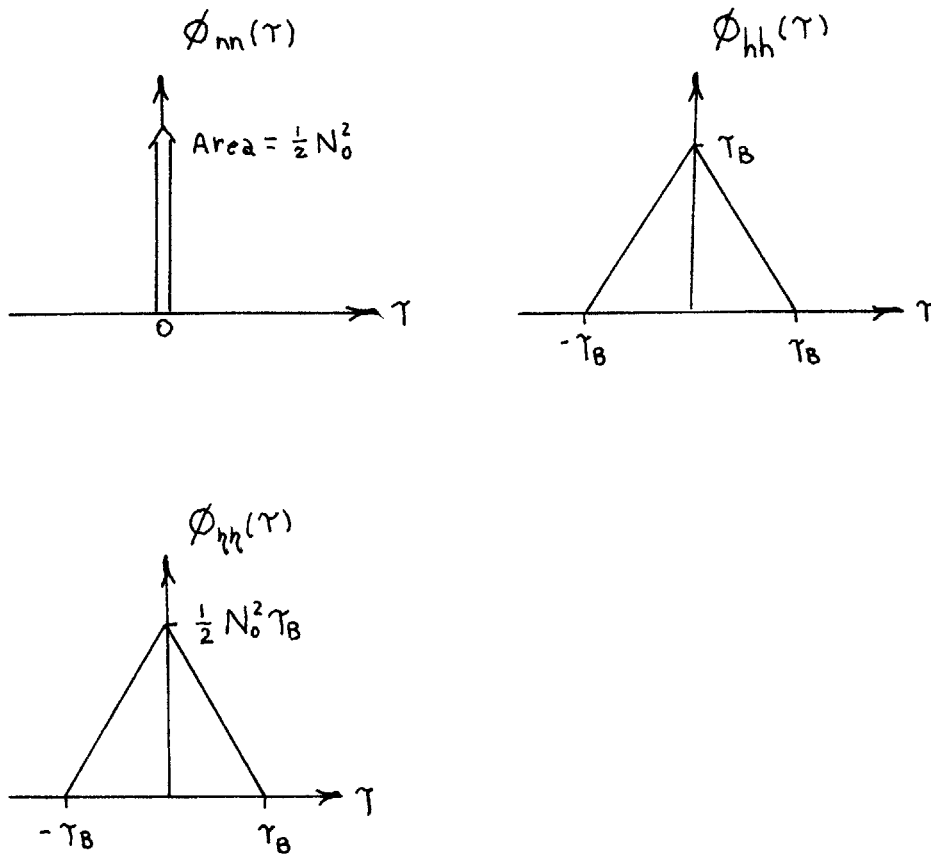


FIGURE 8

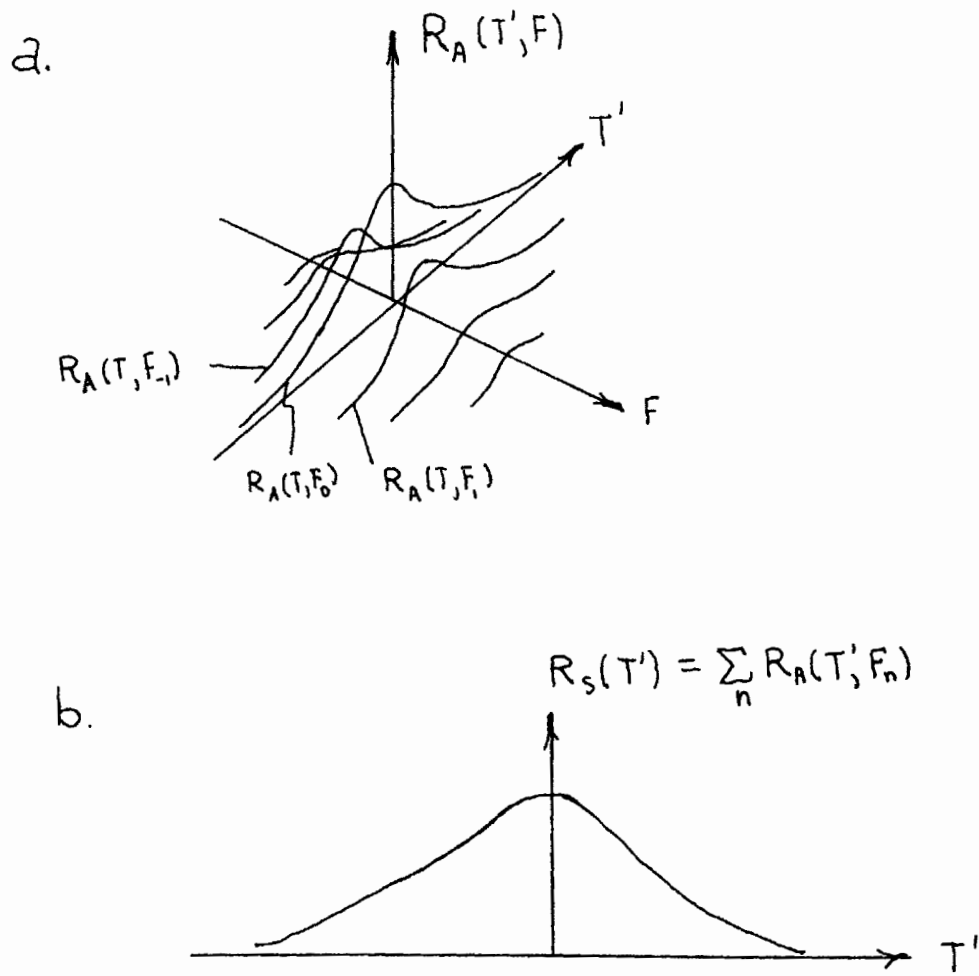


FIGURE 9

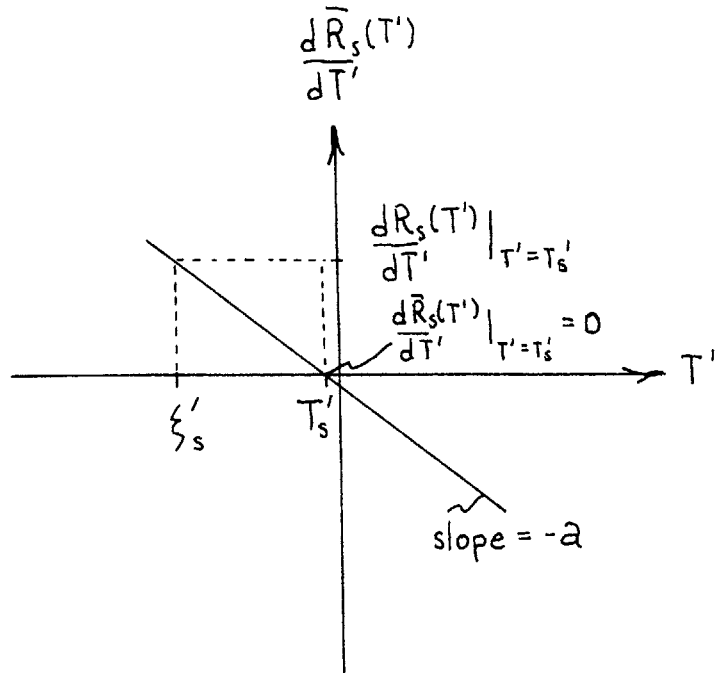
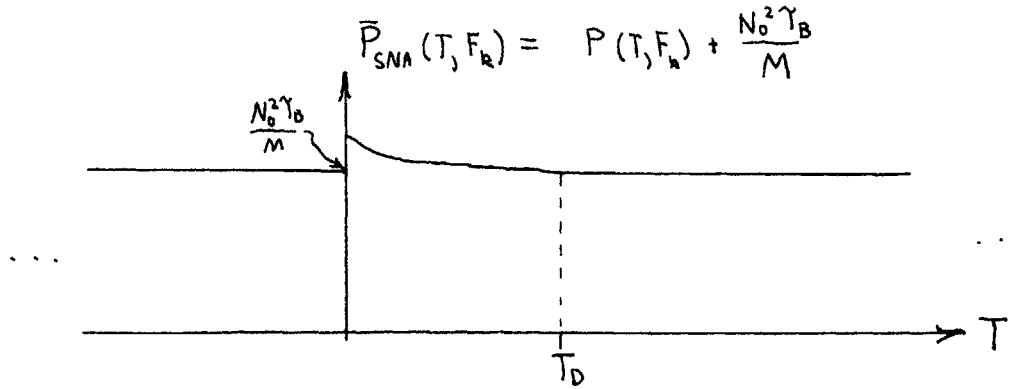
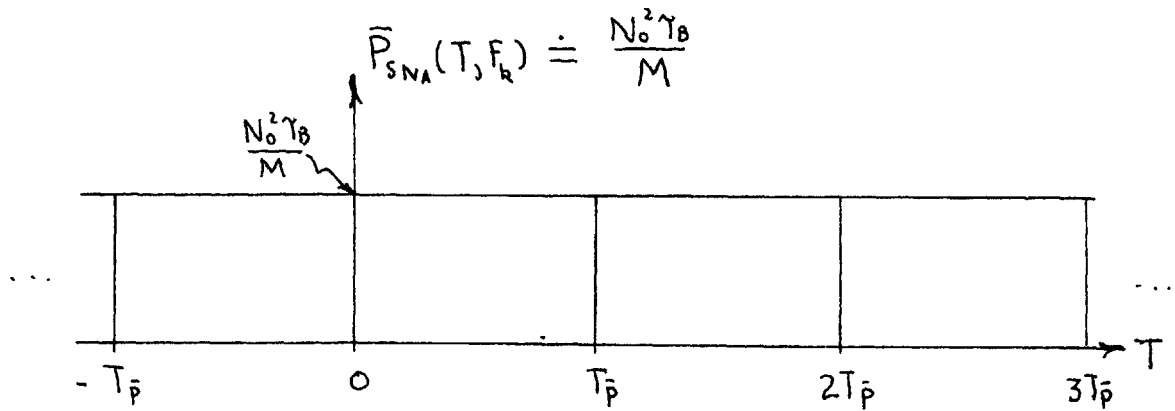


FIGURE 10



Slice of $\bar{P}_{SNA}(T, F)$ vs. T, F taken at $F = F_R$



Slice of $\bar{P}_{SNA}(T, F)$ vs. T, F taken at $F = F_R$ under assumption

$$\frac{N_0^2 \gamma_B}{M} \gg P(T, F)$$

FIGURE 11

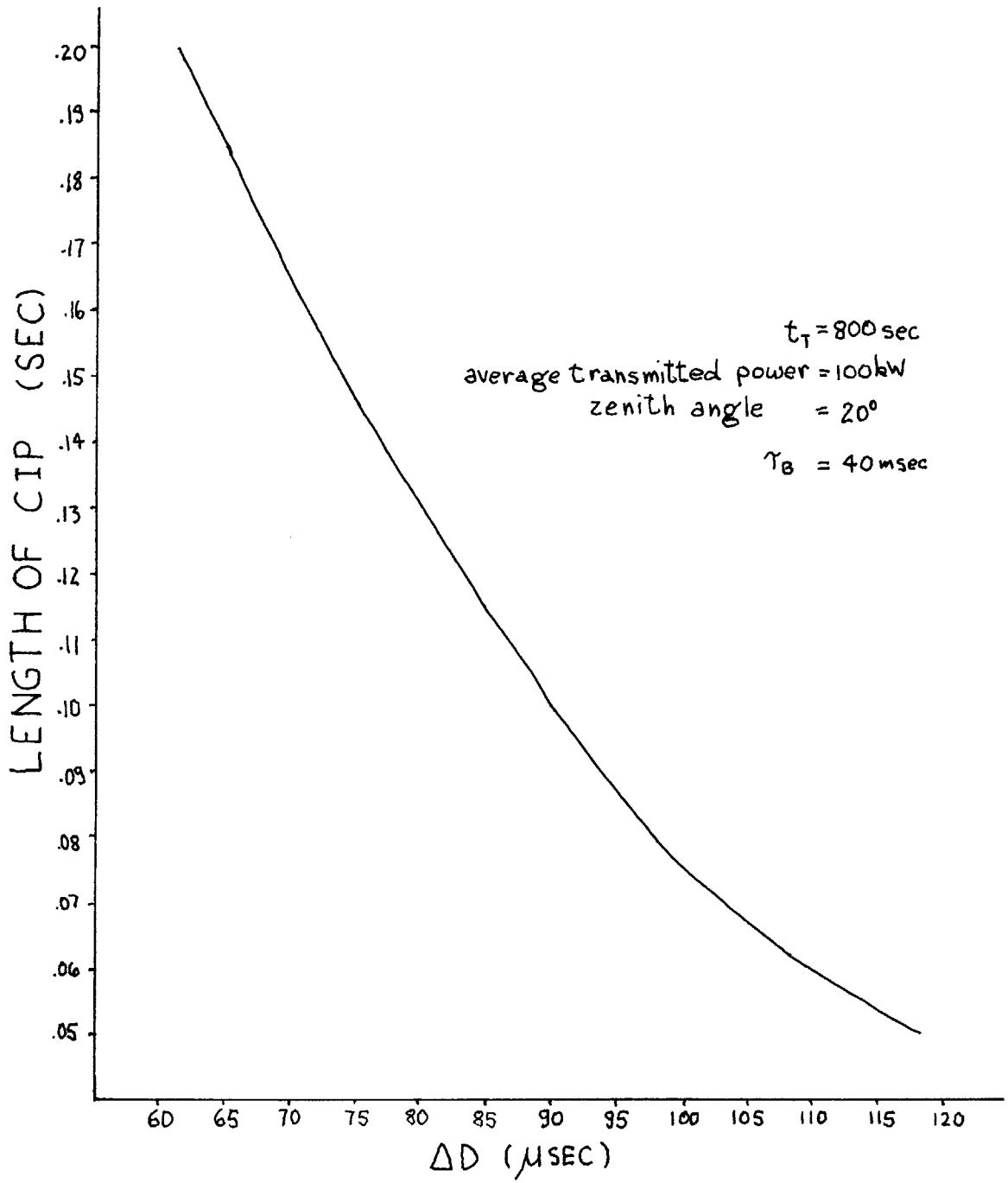


TABLE I

Typical values of relevant parameters corresponding to radar investigation of Venus by Pettengill et al in 1964 (for definition of symbols, see text)

R	=	400 sec · c	=	1.2 x 10 ¹¹ m
t _T	=	800 sec		
ρ	=	6050 km		
λ	=	0.6972 m		
ν	=	430 Mc		
B ₁₀	=	10 cps	when R = 1.2 x 10 ¹¹ m	
ω _a	=	3 x 10 ⁻⁷ sec	when R = 1.2 x 10 ¹¹ m	
T _D	=	40 msec		
G _T	=	4.2 x 10 ⁵	(antenna pointing toward zenith)	
	=	2.4 x 10 ⁵	(antenna pointing 20° below zenith)	
L _T L _R	=	0.7		
L _A ²	=	0.9		
A _R	=	0.6 x π ($\frac{1000}{2}$ ft) ²	=	4.4 x 10 ⁴ m (at zenith)
	=	0.4 x π ($\frac{1000}{2}$ ft) ²	=	2.9 x 10 ⁴ m (at 20° zenith angle)
P _A	=	100 kW		
T _N	=	200° K	(at zenith)	