18.466 PS10, due Friday, May 9, 2003

1. In n independent trials with probability p of success on each, with $0 < p < 1$, let X be the observed number of successes, so X has possible values $0, 1, \ldots, n$.
(a) Find or recall the Fisher information $I(p)$ for this family at each p with $0 < p < 1$. *Hint*: the second derivative form may be easier to compute with than the squared first derivative form.
(b) What is the asymptotic lower bound as $n \to \infty$ for the mean-square error of estimates T_n of $g(p) = p^2$, as a function of p?
(c) What is the MLE (maximum likelihood estimator) of p^2? *Hint*: you can just take g of the MLE of p.
(d) Find exactly the mean-square error of the MLE of p^2 for each p.
(e) Check if the results of parts (b) and (d) are compatible in light of Theorem 3.8.1.

2. Let $N(\mu, \sigma^2)$ be the normal law on the line with mean μ and variance σ^2 with $0 < \sigma < \infty$. Find the Fisher information matrix of this family with respect to its two parameters μ and σ. Recall that the MLEs of μ and σ^2 are $\overline{X} = (X_1 + \cdots + X_n)/n$ and $(s'_X)^2 := (n-1)s_X^2/n$ where $s_X^2 := (n-1)^{-1} \sum_{j=1}^{n} (X_j - \overline{X})^2$ for $n \geq 2$. Verify the assumptions in sections 3.7 and 3.8 to see that the MLEs of (μ, σ) are efficient in this case.

3. Write the family $N(\mu, \sigma^2)$ instead in the form of an exponential family with densities

$$\exp(\theta_1 T_1(x) + \theta_2 T_2(x) - j(\theta)).$$

(a) Evaluate θ_1 and θ_2 in terms of μ and σ^2 (it's arbitrary which one is called θ_1 or θ_2, so don't worry about that). Evaluate $j(\theta)$ in terms of θ.
(b) What are the possible values of θ_1 and θ_2?
(c) Then what are $T_1(x)$ and $T_2(x)$?
(d) Find the Fisher information matrix $I(\theta) = I(\theta_1, \theta_2)$ and its inverse $I^{-1}(\theta)$. *Hint*: It may be easier to find these directly rather than in terms of the original parameters μ and σ^2.

4. Recall the gamma family of distributions on the line with densities $f((\alpha, \lambda), x) = \lambda^\alpha x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha)$ for $0 < x < \infty$ and 0 for $x \leq 0$, where $0 < \lambda < \infty$ and $0 < \alpha < \infty$. The mean of this distribution is $\mu = \alpha/\lambda$ and its variance is $\sigma^2 = \alpha/\lambda^2$. The MLEs for this family are hard to compute. Some easily computed estimates are the *method-of-moments* estimators $(\tilde{\alpha}, \tilde{\lambda})$ found by solving $\tilde{\alpha}/\tilde{\lambda} = \overline{X}$, $\tilde{\alpha}/\tilde{\lambda}^2 = (s'_X)^2$.
(a) What is the mean-square error of $\overline{X}$ as an estimator of μ?
(b) What is the mean-square error of $(s'_X)^2$ as an estimator of σ^2?
(c) Write the gamma family as an exponential family. For any number n of observations, what is a two-dimensional sufficient statistic $T_n(X^{(n)})$?
(d) Write the likelihood equations in terms of $T_n(X^{(n)})$.
(e) *Extra credit*. Can you determine whether the method-of-moments estimators $\tilde{\alpha}, \tilde{\lambda}$ are asymptotically efficient?