Hints on 18.466 PS4, due Friday, March 7, 2003

1. Problem 2.1.1. For $n = 1$, $x_1 = 0$ or 1, $f(p, x_1) = p$ if $x_1 = 1$, $1 - p$ if $x_1 = 0$, so $f(p, x_1) = p^{x_1}(1-p)^{1-x_1}$. For n observations, $f(p, \{x_j\}_{j=1}^n) = \Pi_{j=1}^n f(p, x_j)$. For the Poisson case $f(\theta, x) = G(\theta, x)h(x)$, $\theta = \lambda$, and $x = (x_1, \ldots, x_n) = (k_1, \ldots, k_n)$ for integers $k_i \geq 0$. Find G and h.

2. Problem 2.1.4. For n observations the likelihood function is $1/(b-a)^n$ if $a \leq x_j \leq b$ for all $j = 1, \ldots, n$ and 0 otherwise. The order statistics are $x_1, \ldots, x_n$ arranged in order, $x_{(1)} \leq \cdots \leq x_{(n)}$.

3. Problem 2.2.5. For any $(n+1)$-vector $\{a_k\}_{k=0}^n$,

$$\sum_{k=0}^{n} a_k p^k (1-p)^{n-k} \;=\; \sum_{j=0}^{n} c_j p^j$$

for some $c_0, c_1, \ldots, c_n$. The mapping from $\{a_k\}_{k=0}^n$ to $\{c_j\}_{j=0}^n$ is linear. If it's one-to-one then it's onto $\mathbb{R}^{n+1}$. If the left hand side of the display is 0 for $0 \leq p \leq 1$, to show that $a_k = 0$ for all $k = 0, 1, \ldots, n$, write the expression as $(1-p)^n \sum_{k=0}^n a_k (p/(1-p))^k$ (for $p \neq 1$) and let $x = p/(1-p)$. A non-zero polynomial of degree at most n has at most n roots.

4. Problem 2.2.7. Since $X_1, \ldots, X_5$ are i.i.d., their joint distribution is preserved by any permutation of the indices $1, 2, 3, 4, 5$, and so is S_5. Thus $E(X_j|S_5)$ doesn't depend on $j = 1, \ldots, 5$. Since $E(S_5|S_5) = S_5$, solve for $E(X_j|S_5)$.

5. Problem 2.3.3. (a) If $h^{-n} 1_{\theta \leq X_{(1)} \leq X_{(n)} \leq \theta + h} \equiv f(\theta, x) \equiv G(\theta, T(x))H(x)$, where for each θ, $G(\theta, T(x))$ is a measurable function of $T(x) = T(X_1, \ldots, X_n)$, to show that $X_{(1)}$ and $X_{(n)}$ are also, consider $\sup\{\theta : \theta \text{ rational}, \ f(\theta, x) > 0\}$ and $\inf\{\theta + h : \theta \text{ rational}, f(\theta, x) > 0\}$.

(b) has a hint; apply Theorem 2.3.2 about non-unique unbiased estimates.