18.466 PS7, due Friday, April 11, 2003

1. Problem 3.3-1.

2. Problem 3.3-2. *Hint*: Proposition 3.4.1 relates to this.

The remaining three problems relate to what are called *location-scale* families of densities on one-dimensional space. Namely, if f is a probability density on the line (with respect to Lebesgue measure, $dF(x) = f(x)dx$), then for $-\infty < \mu < \infty$ and $0 < \sigma < \infty$, letting θ be the 2-dimensional parameter (μ, σ), we have the family of laws P_θ with densities $f(\theta, x) = \sigma^{-1} f((x - \mu)/\sigma)$.

3. (a) Show that if f is a probability density then so is $f(\theta, x)$ for any $\theta = (\mu, \sigma)$ with $0 < \sigma < \infty$.
(b) If the distribution with density f has mean 0 and variance 1, show that the distribution with density $f(\theta, \cdot)$ has mean μ and variance σ^2.
(c) If the distribution with density f has a unique median at 0, show that the one with density $f(\theta, \cdot)$ has a unique median at μ.

4. Recall the double-exponential distribution with density $f(x) = e^{-|x|}/2$ for all real x, and that we form h functions in the "log likelihood case" (Sec. 3.3, p.1) as $h(\theta, x) = -\log f(\theta, x)$. So we get the h function from the location-scale family of this distribution

$$h(\theta, x) \;=\; c + \frac{1}{\sigma}\left|\frac{x - \mu}{\sigma}\right|$$

for a constant c. In the pure location case ($\sigma \equiv 1$) this h function was useful and robust and gave us the median location estimator. Show however that in this location-scale case, it is not adjustable for any f such that $\int |x| f(x) dx = +\infty$, such as the Cauchy density in the next problem (assumption (A-3) doesn't hold). *Hint*: see Lemma 3.3.8.

5. Begin with the Cauchy density $f(x) = 1/(\pi(1 + x^2))$ for all real x and form a location-scale family from it as mentioned above.
(a) Show that in this case the corresponding h function is adjustable for an arbitrary probability law P on the real line. *Hints*: again use Lemma 3.3.8. It is enough to show that for each $\theta = (\mu, \sigma)$, $h(\theta, x) - a(x)$ is a bounded function of x. Show that the partial derivatives of h with respect to μ and σ are bounded when μ is bounded and σ is bounded away from 0 and ∞.
(b) Show however that if $P(\{x_0\}) > 1/2$ for some x_0, then (A-4) fails: there is no pseudo-true θ_0 in the parameter space. *Hint*: consider $\mu = x_0$ and σ decreasing toward 0.