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APPENDIX A. UNIQUENESS OF LIKELIHOOD RATIOS

Let $(X, \mathcal{B})$ be a measurable space. For two measures μ and ν on $\mathcal{B}$, μ is said to be *absolutely continuous* with respect to ν if for each $B \in \mathcal{B}$, $\nu(B) = 0$ implies $\mu(B) = 0$. Recall that the measure ν is called *σ-finite* if X is the union of a sequence of sets $B_n \in \mathcal{B}$ with $\nu(B_n) < \infty$. Recall also that by the Radon-Nikodym theorem (e.g. RAP, Theorem 5.5.4), if μ is finite (i.e. $\mu(X) < \infty$), ν is σ-finite , and μ is absolutely continuous with respect to ν, then there is an integrable function $d\mu/d\nu := f \geq 0$ such that

$$\int_B f \, d\nu = \mu(B) \quad \text{for all} \quad B \in \mathcal{B}.$$

Then for any integrable function g for μ,

$$\text{(A.1)} \qquad \int gf \, d\nu = \int g \, d\mu;$$

this is true for indicator functions $f = 1_B$, thus for simple functions, then by monotone convergence for nonnegative measurable functions, then by linearity for any μ-integrable function.

Now, if κ is a finite measure absolutely continuous with respect to μ, then letting $g := 1_B d\kappa/d\mu$ for any $B \in \mathcal{B}$, which is μ-integrable, we get from (A.1)

$$\int_B \frac{d\kappa}{d\mu}\frac{d\mu}{d\nu} d\nu = \int_B \frac{d\kappa}{d\mu} d\mu = \kappa(B).$$

It follows that a.e. ν, i.e. almost everywhere for ν,

$$\text{(A.2)} \qquad \frac{d\kappa}{d\nu} = \frac{d\kappa}{d\mu}\frac{d\mu}{d\nu}.$$

Let P and Q be probability measures on $(X, \mathcal{B})$ and for a σ-finite measure ν such that P and Q are both absolutely continuous with respect to ν, a form of likelihood ratio of Q to P is defined by

$$R_{Q/P,\nu}(x) := \begin{cases} \frac{dQ/d\nu}{dP/d\nu}(x), & \text{if } (dP/d\nu)(x) > 0 \\ +\infty, & \text{if } (dQ/d\nu)(x) > 0 = (dP/d\nu)(x) \\ 0, & \text{if } (dQ/d\nu)(x) = (dP/d\nu)(x) = 0. \end{cases}$$

The likelihood ratio actually doesn't depend on ν:

A.3 Theorem. If P and Q are probability measures on $(X, \mathcal{B})$, both absolutely continuous with respect to σ-finite measures ν and τ on $(X, \mathcal{B})$, then

$$R_{Q/P,\nu}(x) = R_{Q/P,\tau}(x)$$

for $(P+Q)$-almost all x.

Proof. We can assume $\tau = P+Q$, which is absolutely continuous with respect to ν. Then by (A.2),

$$\frac{dP}{d\nu} = \frac{dP}{d\tau}\frac{d\tau}{d\nu} \quad \text{and} \quad \frac{dQ}{d\nu} = \frac{dQ}{d\tau}\frac{d\tau}{d\nu}, \quad \nu\text{-almost everywhere.} \tag{A.4}$$

Thus

$$R_{Q/P,\nu} = \frac{dQ/d\nu}{dP/d\nu} = \frac{dQ/d\tau}{dP/d\tau} = R_{Q/P,\tau}$$

on the set where $d\tau/d\nu > 0$ and $dP/d\tau > 0$; ν-almost everywhere on this set, $dP/d\nu > 0$ by (A.4). On the set where $d\tau/d\nu > 0 = dP/d\tau$, we have almost everywhere for ν, $dQ/d\nu > 0 = dP/d\nu$, so $R_{Q/P,\nu} = +\infty = R_{Q/P,\tau}$. The set where $d\tau/d\nu = 0$ has $(P+Q)$-measure 0, so $R_{Q/P,\nu} = R_{Q/P,\tau}$ almost everywhere for $P+Q$. □

Recall that $R_{Q/P}$ was defined in Sec. 1.1 as $R_{Q/P,\tau}$ for $\tau = P+Q$. Thus Theorem A.3 gives $R_{Q/P,\nu} = R_{Q/P}$ $(P+Q)$-almost everywhere, for any σ-finite ν dominating P and Q.