Appendix B. Preservation of dimension by 1-1 continuous functions. The following classic topological fact was needed in Sec. 2.5:

B.1 Theorem. There is no 1-1, continuous function f from a non-empty open set U in $\mathbb{R}^m$ into $\mathbb{R}^k$ for $k < m$.

Proof. If f is such a function consider the restriction of f to a closed ball $B := \{x : |x - p| \leq r\} \subset U$ for some $p \in U$ and small enough $r > 0$. Then since B is compact, f restricted to B is a homeomorphism, i.e. its inverse is also continuous (RAP, Theorem 2.2.11). The interior of B is an open set in $\mathbb{R}^m$.

The *dimension* of a separable metric space is defined recursively as follows. The dimension of the empty set is -1. A set has dimension $\leq n$ if and only if every point has arbitrarily small neighborhoods whose boundaries have dimension $\leq n - 1$. Clearly, if a space S has dimension $\leq k$, so does any subset of S. An open set in $\mathbb{R}^m$ has dimension m (Hurewicz and Wallman, 1941, p. 44 Theorem IV.3), which yields a contradiction. □

NOTES

Brouwer (1911) proved that $\mathbb{R}^m$ and $\mathbb{R}^k$ are not homeomorphic for $k \neq m$. For a modern treatment see Munkres (1984), p. 109. Then Brouwer (1913) invented the notion of dimension, or topological dimension (there are other, different definitions of dimension, for example Hausdorff dimension of totally bounded metric spaces, which can have non-integer values). Hurewicz and Wallman (1941) gave an exposition of the theory of topological dimension.

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