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APPENDIX C - SEPARABILITY OF PROCESSES

Let $(X, \mathcal{A}, P)$ be a probability space and Θ a set (parameter space). A function h from $X \times \Theta$ into $[-\infty, \infty]$ is called a *stochastic process* if for each $\theta \in \Theta$, $h(\cdot, \theta)$ is a measurable function on X.

Suppose $(\Theta, \mathcal{T})$ is a topological space. The process $h(\cdot, \cdot)$ is called *separable* (relative to the closed sets) if there is a countable set $S \subset \Theta$ (called a *separant*) and a set $A \subset X$ with $P(A) = 0$ such that for every closed set $J \subset [-\infty, \infty]$ and every open set $U \subset \Theta$,

$$\{x : \ h(x, \theta) \in J \ \text{ for all } \ \theta \in S \cap U\} \subset A \cup \{x : \ h(x, \theta) \in J \ \text{ for all } \ \theta \in U\}.$$

For Sec. 3.3 the following is needed:

C.1 Theorem. If $(\Theta, \mathcal{T})$ is separable, i.e. has a countable dense set Y, a process $h(\cdot, \cdot)$ is separable relative to the closed sets if and only if there is a countable dense set $T \subset \Theta$ and a set $B \subset X$ with $P(B) = 0$ such that for all $x \notin B$, the graph of $h(x, \cdot)$ restricted to T is dense in the whole graph.

Proof. First, to check "if", take $A = B$ and $S = T$. If $x \notin A$ and $h(x, \theta) \in J$ for all $\theta \in S \cap U$, then for each $\phi \in U$, the point $(\phi, h(x, \phi))$ is in the closure of the set of points $(\theta, h(x, \theta))$ for $\theta \in S \cap U$, which is included in the closed set $\Theta \times J$ by assumption, so $h(x, \phi) \in J$ as desired.

To prove "only if", suppose $h(\cdot, \cdot)$ is separable for a set $A \subset X$ and separant S. Let $T := S \cup Y$, which is countable, dense in Θ, and still a separant. Let $x \notin A$ and suppose the graph G_T of $h(x, \cdot)$ restricted to T is not dense in the whole graph. Then take $\phi \in \Theta$ such that $(\phi, h(x, \phi))$ is not in the closure of G_T in $\Theta \times [-\infty, \infty]$. By definition of product topology, take an open neighborhood U of ϕ in Θ and an open neighborhood V of $h(x, \phi)$ such that G_T is disjoint from $U \times V$. Let $J := [-\infty, \infty] \setminus V$, a closed set. Then $h(x, \theta) \in J$ for all $\theta \in U \cap T$ but $h(x, \phi) \notin J$, contradicting separability. Q.E.D.

To put the next fact in a notation more familiar in probability theory, we will have $\Theta = T \subset \mathbb{R}$, the probability space X will be written as Ω, and for $t \in T$ and $\omega \in \Omega$ we will write $x_t(\omega)$ instead of $h(\omega, t)$. A main fact about separability, proved by Doob, is the following:

C.2 Theorem. Let $x_t, \ t \in T \subset \mathbb{R}$, be any stochastic process with values in $[-\infty, \infty]$, defined over a probability space $(\Omega, \mathcal{A}, P)$. Then there exists another stochastic process $(t, \omega) \mapsto y_t(\omega)$, also for $t \in T$ and $\omega \in \Omega$, and also with values in $[-\infty, \infty]$, such that for each $t \in T$, $P(x_t = y_t) = 1$, and such that y_t is separable relative to the class of closed subsets of $[-\infty, \infty]$.

Remarks. Usually, x_t will be a real-valued stochastic process. Then if the paths $t \mapsto x_t(\omega)$ for almost all ω can be taken to have some regularity property such as continuity or right-continuity, for such a choice, the process will already be separable. For a general process, however, y_t may need to take infinite values even if x_t does not.

NOTE

The notion of separability for stochastic processes is due to J. L. Doob. I am thankful to Donald L. Cohn for telling me Theorem C.1. Theorem C.2 is given in Doob (1953) Theorem 2.4 of Chap. 2, p. 57.

REFERENCE

Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.