***1.4 Realizable rules**. This section, giving sufficient conditions for randomized decision rules to be realizable, is relatively abstract and can be passed over on first reading.

A measurable space $(Y, \mathcal{V})$ will be called a *Lusin space* if there exists a metric on Y for which (Y, d) is separable, $\mathcal{V}$ is the Borel σ-algebra and Y is a Borel subset of its completion. This is, evidently, a rather large class of measurable spaces. So the following shows that Theorem 1.3.1 applies rather generally.

1.4.1 Theorem. If the action space $(A, \mathcal{E})$ is a Lusin space, then every decision rule d with values in $D_{\mathcal{E}}$ (i.e., any randomized decision rule) is realizable.

Proof. By the Borel isomorphism theorem (RAP, Theorem 13.1.1), we can assume that A is the real line $\mathbb{R}$ or a countable subset of it, with $\mathcal{E}$ the Borel σ-algebra. If A is countable, then $\mathcal{E}$ is the class of all subsets of A and we can assume that A is the set $\mathbb{N}$ of nonnegative integers, or a finite subset of it.

Then, for each x in X, $d(x)$ is a law on $\mathbb{R}$. Let $F(x, \cdot)$ be its (cumulative) distribution function, $F(x,t) := d(x)((-\infty, t])$. It will be shown that F is jointly measurable. For $k = 1, 2, \cdots$, let

$$F_k(x,t) := \sum_{n=-\infty}^{\infty} F(x, n/2^k) 1\{(n-1)/2^k < t \leq n/2^k\},$$

where $1\{\cdots\}$ is 1 when $\cdots$ holds and 0 otherwise. For each k and t, there will be just one n for which the nth term of the sum is non-zero, so the series converges. Clearly each F_k is jointly measurable. Now $F_k \downarrow F$, so F is jointly measurable.

Let Ω be the open unit interval $(0, 1)$ with Borel σ-algebra $\mathcal{F}$. For $0 < s < 1$ let $\delta(x, s) := \inf\{t : F(x,t) \geq s\}$. Note that $\delta(\cdot, \cdot)$ takes values in A, also when $A \subset \mathbb{N}$. If s and t are any real numbers, then

$$(1.4.2) \qquad F(x,t) \geq s \quad \text{if and only if} \quad \delta(x, s) \leq t,$$

where "only if" follows from the definition of δ and "if" from the right-continuity of the distribution function $F(x, \cdot)$ (see also RAP, proof of Proposition 9.1.2). To show δ is jointly measurable here is:

1.4.3 Lemma. For any measurable space $(Y, \mathcal{U})$ and real-valued function h on Y, h is measurable if and only if $S(h) := \{\langle y, s \rangle : s \leq h(y)\}$ is jointly measurable in $Y \times \mathbb{R}$ (for the Borel σ-algebra in $\mathbb{R}$).

Proof. To prove "if," for each s, $\{y : h(y) \geq s\}$ is measurable, which implies that h is measurable. For "only if" the conclusion holds if h is a simple function, in other words a finite sum $\sum_i c_i 1_{A(i)}$ for disjoint measurable sets $A(i)$ and constants c_i. If h is bounded above, there exist simple h_n decreasing down to h, and then $S(h_n) \downarrow S(h)$, so the conclusion holds for h. Now for any measurable h let $H_n := \min(h, n)$. Then each H_n is bounded above and measurable, so the sets $S(H_n)$ are measurable, and they increase up to $S(h)$, so it is measurable. □

Now applying the Lemma for any fixed t and with $h(x) := F(x, t)$, it follows by (1.4.2) that δ is jointly measurable. Let μ be Lebesgue measure on $\Omega = (0, 1)$. Then for each x, $\delta(x, \cdot)$ is a random variable such that

$$\mu(\delta(x, \cdot)^{-1}(C)) = d(x)(C)$$

if C is any half-line $(-\infty, t]$ and thus if C is any Borel set, since a distribution function uniquely determines a probability measure (RAP, Theorem 9.1.1). Thus $\delta(\cdot, \cdot)$ has the required properties and Theorem 1.4.1 is proved. □

PROBLEM

1. Prove directly (without applying Theorem 1.4.1) that every randomized decision rule for $(A, \mathcal{E})$ is realizable if $\mathcal{E}$ is the σ-algebra of all subsets of A and
   (a) A is finite, or
   (b) A is countable.