March 14, 2003

***2.8 Continuity at the boundary for exponential families**. Let $\{P_\theta,\ \theta \in \Theta\}$ be an exponential family where Θ is the natural parameter space and for some σ-finite measure μ,

$$(dP_\theta/d\mu)(x) \ = \ C(\theta)\exp(\theta \cdot T(x)).$$

Then $1/C(\theta) = K(\theta) = \int \exp(\theta \cdot T(x))d\mu(x)$. In Theorem 2.5.8 we saw that $K(\theta)$ is a highly regular (analytic) function on the interior of Θ. Points on the boundary of Θ may or may not be in Θ. If they are, then differentiability properties would not hold in directions leading outside of Θ, although they might hold in directions leading into Θ. This section will treat the question whether the function K (and so C) is at least continuous at boundary points that happen to be in Θ. There will be one positive result, then a negative one and a counter-example.

Recall that the *convex hull* of a set A of points in a vector space is the smallest convex set including A. If A is a finite set, $A = \{a_1, ..., a_k\}$, then it is easily seen that the convex hull of A is the set of all sums $\sum_{j=1}^k \lambda_j a_j$ where $\lambda_j \geq 0$ for all j and $\sum_{j=1}^k \lambda_j = 1$. The convex hull of a finite set is a closed interval in $\mathbb{R}^1$, a convex polygon in $\mathbb{R}^2$, and a convex polyhedron in $\mathbb{R}^d$ for $d \geq 3$, whose vertices are some of the a_j (other a_i may be contained in the convex hull of a_j for $j \neq i$).

A convex function on a closed interval is not necessarily continuous at the endpoints: for example let $f(0) = f(1) = 1$ and $f(x) = 0$ for $0 < x < 1$. Then f is convex on $[0, 1]$. It turns out, though, that this kind of discontinuity on an interval can't happen for functions $K(\cdot)$ (a closed interval is the convex hull of its endpoints):

2.8.1 Theorem. Whenever a finite set of points, $F = \{\theta_1, \ldots, \theta_k\}$, is included in the natural parameter space Θ, $K(\cdot)$ and $C(\cdot)$ are both continuous on the convex hull H of F.

Proof. By Theorem 2.5.6, Θ is convex, so it includes H. By Jensen's inequality (RAP, 10.2.6), for any $\theta \in H$, so $\theta = \sum_{j=1}^k \lambda_j \theta_j$ where $\lambda_j \geq 0$ and $\sum_{j=1}^k \lambda_j = 1$,

$$\exp(\theta \cdot T) \ \leq \ \sum_{j=1}^k \lambda_j \exp(\theta_j \cdot T) \ \leq \ \exp(\theta_1 \cdot T) + \cdots + \exp(\theta_k \cdot T).$$

By assumption, the function on the right is integrable for μ, while the function on the left is continuous in θ. It follows by dominated convergence that $K(\cdot)$ is continuous on H. Since $K(\theta) > 0$ for all θ, $C(\cdot)$ is also continuous on H. □

If F is in the interior of Θ, then we have even stronger regularity properties of $K(\cdot)$ on H by Theorem 2.5.8. So Theorem 2.8.1 is of interest only when at least one θ_i is on the boundary of Θ. For example, if two of the θ_i are on the boundary, then the line segment joining them is included in the boundary and $K(\cdot)$ is continuous along such a line segment and also as the segment is approached from within H.

For any exponential family of order 1, Θ is an interval (which may be open or closed at each end). If either endpoint is in Θ, $K(\cdot)$ is continuous there, so $K(\cdot)$ is continuous everywhere on Θ. The situation can be different for families of order 2 or higher, as follows.

2.8.2. Proposition. For any exponential family with natural parameter space Θ, if θ_0 is a point in Θ, on its boundary, which is a limit point of points ϕ_j on the boundary of Θ which are not in Θ, then $K(\cdot)$ is not continuous at θ_0 on Θ.

Proof. By Fatou's Lemma (RAP, 4.3.3), $K(\theta)$ must approach $+\infty$ as θ approaches ϕ_j from within Θ for each j. Let θ_j for $j \geq 1$ be a point of Θ at distance less than $1/j$ from ϕ_j such that $K(\theta_j) > j$. Then $\theta_j \to \theta_0$, so $K(\cdot)$ is not continuous at θ_0. □

Now, it will be shown that the situation in the last proposition can actually occur.

2.8.3 Proposition. There is an exponential family of order 2 with natural parameter space $\Theta = \{(\psi, \phi) \in \mathbb{R}^2 : \psi > 0, \phi > 0\} \cup \{(0,0)\}$. Thus the function $K(\cdot, \cdot)$ for the family is not continuous at $(0,0)$.

Proof. Let $P_{(\psi,\phi)}$ have density $C(\psi,\phi)e^{-\psi x - \phi y}$ with respect to the measure $\mu := \mu_1 + \mu_2$ on $\mathbb{R}^2$ defined by $d\mu_1(x,y) = dx/(1+x^2)$ on the curve $x \geq 0$, $y = -\sqrt{x}$ and $d\mu_2(x,y) = dy/(1+y^2)$ on the curve $y \geq 0$, $x = -\sqrt{y}$. Evidently the family is exponential as in (2.5.3), with $T(x,y) \equiv (-x,-y)$. The measure μ is finite, so $(0,0) \in \Theta$. For any $\psi > 0$ and $\phi > 0$,

$$\int_0^\infty e^{-\psi x + \phi\sqrt{x}} dx/(1+x^2) \; < \; \infty,$$

and likewise for μ_2. For $\psi = 0 < \phi$, $\int_0^\infty e^{\phi\sqrt{x}} dx/(1+x^2) = +\infty$, and likewise for $\phi = 0 < \psi$ and μ_2. Clearly (ψ, ϕ) with $\psi < 0$ or $\phi < 0$ are not in Θ, so it is as described. Since $(0,0)$ is in Θ and is a limit of points in the boundary of Θ not in Θ, $K(\cdot,\cdot)$ is not continuous on Θ at $(0,0)$ by Proposition 2.8.2, completing the proof. □

Note that for the example in the last proof, for any $\delta > 0$, $K(\psi, \phi)$ is continuous as $\psi \downarrow 0$ and $\phi \downarrow 0$ through a region $\delta\psi < \phi < \psi/\delta$, by Theorem 2.8.1, but K is not continuous at $(0,0)$ along some curves tangent to the vertical or horizontal axis.

PROBLEM

1. If the natural parameter space Θ of an exponential family of order 2 is the open unit square together with parts of its boundary as follows, at what points of Θ is $K(\cdot)$ restricted to Θ continuous? Explain.
   (a) $\Theta = \{0 \le \theta_1 \le 1,\ 0 \le \theta_2 \le 1\}$.
   (b) $\Theta = \{0 \le \theta_1 < 1,\ 0 \le \theta_2 < 1\}$.
   (c) $\Theta = \{0 < \theta_1 < 1, 0 < \theta_2 < 1\} \cup \{0 \le \theta_1 \le 1, \theta_2 = 0\}$.

2. (a) For any exponential family, show that the natural parameter space is an F_σ, that is, a countable union of closed sets. *Hint*: show that for any $n = 1, 2, ...,$ $\{\theta :\ K(\theta) \le n\}$ is closed.
   (b) For the natural parameter space Θ as in Proposition 2.8.3, find explicitly a sequence of closed sets F_n whose union is Θ (not by way of part (a)).
   (c) Show that any convex set in $\mathbb{R}^2$ whose interior is the same as that of Θ in Proposition 2.8.3 must necessarily be an F_σ.

NOTES

Barndorff-Nielsen (1978) calls an exponential family "regular" if its natural parameter space Θ is open. It is not difficult to give examples of exponential families for which Θ has non-empty boundary and is closed (an example of order 1 is given in Section 3.2 below). I assume that the results of this section are known but at this writing I do not have literature references for them.

REFERENCE

Barndorff-Nielsen, O. (1978). See Sec. 2.5.