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ORIGINAL PAPER

Finding an optimal strategy of incorporating renewable sources of energy and electricity storing systems in a regional electrical grid

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Abstract A game with a finite (more than three) number of players on a polyhedron of connected player strategies is studied. This game describes the interaction among (a) the base load power plant (the generator), (b) all the large customers of a regional electrical grid that receive electric energy from the generator, as well as from the available renewable sources of energy, both directly and via electricity storing facilities, and (c) the transmission company. An auxiliary three-person game on polyhedra of disjoint player strategies that is associated with the initial game is also considered. It is shown that an equilibrium point in the auxiliary game is an equilibrium point in the above game with connected player strategies. Verifiable necessary and sufficient conditions of an equilibrium in the auxiliary three-person game are proposed, and these conditions allow one to find equilibria in (the auxiliary) solvable game by solving three linear programming problems two of which form a dual pair.

Keywords Electrical grid · Equilibrium points · Linear programming · n-Person game · Saddle points

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1 Introduction

In [1], the functioning of a part of a country's electrical grid (called a regional electrical grid in [1] and further in this paper) in which

- m is the number of industrial customers within the grid,
- n is the number of utility companies that have access to the (low voltage) distribution lines via which individual end users of the grid receive electricity,
- r is the number of groups of advanced customers that have licences to operate the existing (low voltage) distribution lines directly, rather than via utility companies,
- $Y^g(l)$ is the volume of electric energy produced by the generator in the period of time from hour $l-1$ to hour l, $l \in \overline{1,24}$,
- $y_i^g(l)$ is the volume of electric energy produced by the generator that is bought by industrial customer i, $i \in \overline{1,m}$ in the period of time from hour $l-1$ to hour l, $l \in \overline{1,24}$,
- $z_j^g(l)$ is the volume of electric energy produced by the generator that is bought by utility company j, $j \in \overline{1,n}$ in the period of time from hour $l-1$ to hour l, $l \in \overline{1,24}$,
- $u_k^g(l)$ is the volume of electric energy produced by the generator that is bought by group of advanced customers k, $k \in \overline{1,r}$ in the period of time from hour $l-1$ to hour l, $l \in \overline{1,24}$,

was considered within a 24 h period of time.

It was shown that the interaction among (a) the generator, (b) all the above large customers that receive electric energy from the generator (i.e., with utility companies, industrial customers, and groups of advanced customers), as well as from the available renewable sources of energy both directly and via electricity storing facilities, and (c) the transmission company can be described in the form of a $(m+n+r+2)$-person game [1] with connected player strategies. Finding Nash equilibria in this game presents considerable difficulties due to the structure of both the set of player strategies (since they are connected) and the size of the mathematical problems to be solved to this end. So developing methods for effectively finding Nash equilibrium strategies (in solvable games) presents both theoretical and practical interest.

An approach to analyzing and solving the above $(m+n+r+2)$-person game that is based on establishing verifiable sufficient conditions for Nash equilibrium points in this game is proposed in the present paper. These sufficient conditions allow one to reduce finding Nash equilibrium points (in solvable games) to solving three auxiliary linear programming problems, two of which form a dual pair. The established possibility of finding equilibrium strategies in the game under consideration by linear programming techniques, which have high computational potential, makes the (proposed in [1]) game-theoretic approach to estimating the scale of incorporating renewable sources of energy and electricity storing systems in a regional electrical grid of any country an effective quantitative analysis means for studying large-scale electrical grids.

2 The problem statement and mathematical formulation

Let us consider the $(m+n+r+2)$-person game from [1]

$$
\begin{aligned}
&\sum_{i=1}^{m}\langle \tilde{p}^{y},\tilde{y}\rangle_i+\sum_{j=1}^{n}\langle \tilde{p}^{z},\tilde{z}\rangle_j+\sum_{k=1}^{r}\langle \tilde{p}^{u},\tilde{u}\rangle_k \qquad \text{(Game 1)}\\
&\quad-\theta\sum_{l=1}^{24}\max_{\lambda_l\in\overline{1,\Lambda_l}}\left(a_{\lambda_l}+b_{\lambda_l}Y^g(l)\right)-\sum_{l=1}^{24}\max_{\mu_l\in\overline{1,\Gamma_l}}\left(c_{\mu_l}+d_{\mu_l}Y^g(l)\right)\\
&\quad-\left(\sum_{i=1}^{m}\langle \tilde{\epsilon}^{y},\tilde{y}\rangle_i\theta_i^y+\sum_{j=1}^{n}\langle \tilde{\epsilon}^{z},\tilde{z}\rangle_j\theta_j^z+\sum_{k=1}^{r}\langle \tilde{\epsilon}^{u},\tilde{u}\rangle_k\theta_k^u\right)\to\max_{(\tilde{p}^y,\tilde{p}^z,\tilde{p}^u)},\\
&\left(\sum_{i=1}^{m}\langle \tilde{\epsilon}^{y},\tilde{y}\rangle_i\theta_i^y+\sum_{j=1}^{n}\langle \tilde{\epsilon}^{z},\tilde{z}\rangle_j\theta_j^z+\sum_{k=1}^{r}\langle \tilde{\epsilon}^{u},\tilde{u}\rangle_k\theta_k^u\right)+\theta\sum_{l=1}^{24}\max_{\lambda_l\in\overline{1,\Lambda_l}}\left(a_{\lambda_l}+b_{\lambda_l}Y^g(l)\right)\\
&\quad+\left(\sum_{i=1}^{m}\langle \tilde{\epsilon}^{y},\tilde{y}\rangle_i s_i^y+\sum_{j=1}^{n}\langle \tilde{\epsilon}^{z},\tilde{z}\rangle_j s_j^z+\sum_{k=1}^{r}\langle \tilde{\epsilon}^{u},\tilde{u}\rangle_k s_k^u\right)\to\max_{(\theta^y,\ \theta^z,\ \theta^u,\ \tilde{s}^y,\tilde{s}^z,\tilde{s}^u)},\\
&\langle \tilde{p}^{y},\tilde{y}\rangle_i+\langle \tilde{\epsilon}^{y},\tilde{y}\rangle_i s_i^y+\langle \tilde{q}^{y},\tilde{y}\rangle_i\to\min_{(\tilde{y})_i},\quad i\in\overline{1,m},\\
&\langle \tilde{p}^{z},\tilde{z}\rangle_j+\langle \tilde{\epsilon}^{z},\tilde{z}\rangle_j s_j^z+\langle \tilde{q}^{z},\tilde{z}\rangle_j\to\min_{(\tilde{z})_j},\quad j\in\overline{1,n},\\
&\langle \tilde{p}^{u},\tilde{u}\rangle_k+\langle \tilde{\epsilon}^{u},\tilde{u}\rangle_k s_k^u+\langle \tilde{q}^{u},\tilde{u}\rangle_k\to\min_{(\tilde{u})_k},\quad k\in\overline{1,r},\\
&(\tilde{y},\ \tilde{z},\ \tilde{u},\ Y^g)\in\Omega,\ (\tilde{p}^y,\ \tilde{p}^z,\ \tilde{p}^u)\in M,\ (\theta_1^y,\dots,\theta_m^y,\ \theta_1^z,\dots,\theta_n^z,\ \theta_1^u,\dots,\theta_r^u)\in T,\\
&\qquad\qquad(s_1^y,\dots,s_m^y,\ s_1^z,\dots,s_n^z,\ s_1^u,\dots,s_r^u)\in S,
\end{aligned}
$$

where [1]

- $\tilde{y}$ is the vector of volumes of electric energy bought by all the industrial customers from the generator and those received from both the renewable sources of energy and storage facilities,
- $\tilde{z}$ is the vector of volumes of electric energy bought by all the utility companies from the generator and those received from both the renewable sources of energy and storage facilities,
- $\tilde{u}$ is the vector of volumes of electric energy bought by all the groups of advanced customers from the generator and those received from both the renewable sources of energy and storage facilities,
- $\tilde{p}^y$ is the vector of prices whose non-zero component $p_i^y(l)$ is the price at which a unit volume of electric energy is sold by the generator to industrial customer $i,\ i\in\overline{1,m}$ in the period of time from hour $l-1$ to hour l, $l\in\overline{1,24}$,
- $\tilde{p}^z$ is the vector of prices whose non-zero component $p_j^z(l)$ is the price at which a unit volume of electric energy is sold by the generator to utility company $j,\ j\in\overline{1,n}$ in the period of time from hour $l-1$ to hour l, $l\in\overline{1,24}$,
- $\tilde{p}^u$ is the vector of prices whose non-zero component $p_k^u(l)$ is the price at which a unit volume of electric energy is sold by the generator to group of advanced customers $k,\ k\in\overline{1,r}$ in the period of time from hour $l-1$ to hour l, $l\in\overline{1,24}$,

θ_i^y is the hourly price for a unit volume of electric energy that the transmission company charges the generator for transmitting electricity to industrial customer i, $i \in \overline{1,m}$,

θ_j^z is the hourly price for a unit volume of electric energy that the transmission company charges the generator for transmitting electricity to utility company j, $j \in \overline{1,n}$,

θ_k^u is the hourly price for a unit volume of electric energy that the transmission company charges the generator for transmitting electricity to group of advanced customers k, $k \in \overline{1,r}$,

θ be the hourly price for a unit volume of the electric energy lost in transmitting electricity to the large grid customers via the (high voltage) transmission line that the transmission company charges the generator,

s_i^y is the hourly price that industrial customer i of the electrical grid pays the transmission company for a unit volume of electric energy transmitted to this industrial customer from the generator, $i \in \overline{1,m}$,

s_j^z is the hourly price that utility company j pays the transmission company for a unit volume of electric energy transmitted to this utility company from the generator, $j \in \overline{1,n}$,

s_k^u is the hourly price that group of advanced customers k of the electrical grid pays the transmission company for a unit volume of electric energy transmitted to this group of advanced customers from the generator, $k \in \overline{1,r}$.

As in [1], here

$$
\begin{aligned}
\tilde{y} &= \left(y^g; y_1^w, y_2^w, \ldots, y_m^w; y_1^s, y_2^s, \ldots, y_m^s; y_1^{st}, y_2^{st}, \ldots, y_m^{st}\right),\\
\tilde{z} &= \left(z^g; z_1^w, z_2^w, \ldots, z_n^w; z_1^s, z_2^s, \ldots, z_n^s; z_1^{st}, z_2^{st}, \ldots, z_n^{st}\right),\\
\tilde{u} &= \left(u^g; u_1^w, u_2^w, \ldots, u_r^w; u_1^s, u_2^s, \ldots, u_r^s; u_1^{st}, u_2^{st}, \ldots, u_r^{st}\right),\\
\tilde{\epsilon}^y &= (\epsilon^y; 0, \ldots, 0),\ \tilde{\epsilon}^z = (\epsilon^z; 0, \ldots, 0),\ \tilde{\epsilon}^u = (\epsilon^u; 0, \ldots, 0),
\end{aligned}
$$

where $\langle \tilde{\epsilon}^y, \tilde{y}\rangle_i$, $\langle \tilde{\epsilon}^z, \tilde{z}\rangle_j$ and $\langle \tilde{\epsilon}^u, \tilde{u}\rangle_k$ are parts of the scalar products $\langle \tilde{\epsilon}^y, \tilde{y}\rangle$, $\langle \tilde{\epsilon}^z, \tilde{z}\rangle$, and $\langle \tilde{\epsilon}^u, \tilde{u}\rangle$, respectively, relating to industrial customer i, utility company j, and group of advanced customers k, $i \in \overline{1,m}$, $j \in \overline{1,n}$, $k \in \overline{1,r}$, respectively, whereas, as before, ϵ^y, ϵ^z, and $\epsilon^u$are vectors of corresponding dimensions with all the components equalling 1. (See the description of all the parameters of Game 1 in [1].) The polyhedron Ω is a set of feasible volumes of electricity produced by the generator and those received by all the large customers of the grid from both the generator and the renewable sources of energy (wind and solar) within the grid (directly and via the electricity storage systems). The polyhedron M is a set of feasible prices for electricity sold by the generator to all the large customers of the grid, whereas the polyhedra T and S are sets of feasible prices for transmitting electricity to the customers that are to be paid to the transmission company by the generator and by the customers, respectively. Also, it is assumed that the prices for electricity received from renewable sources of energy (wind and solar) supplied by producers of electric energy from these sources and the prices for storing electricity for each of the large customers are known real numbers, reflecting some average values of these prices. All the above polyhedra are formed by linear inequalities that include two-sided constraints on each variable, along with linear constraints on sums of subsets of these variables and/or on the sum of all the variables relating to each legal entity being a player in the game. The constraints reflect, in particular, certain "caps" on the consumption volumes and on the electricity prices that may be in force within the grid. Also, constraints on particular sums of weighted variables can be present in the systems of constraints describing the polyhedra.

Let us consider an auxiliary three-person game with the following payoff functions:

$$\sum_{i=1}^{m} \langle \tilde{p}^y, \tilde{y} \rangle_i + \sum_{j=1}^{n} \langle \tilde{p}^z, \tilde{z} \rangle_j + \sum_{k=1}^{r} \langle \tilde{p}^u, \tilde{u} \rangle_k \tag{Game 2}$$

$$- \theta \sum_{l=1}^{24} \max_{\lambda_l \in \overline{1, \Lambda_l}} \left(a_{\lambda_l} + b_{\lambda_l} Y^g(l) \right) - \sum_{l=1}^{24} \max_{\mu_l \in \overline{1, \Gamma_l}} \left(c_{\mu_l} + d_{\mu_l} Y^g(l) \right)$$

$$- \left(\sum_{i=1}^{m} \langle \tilde{\epsilon}^y, \tilde{y} \rangle_i \theta_i^y + \sum_{j=1}^{n} \langle \tilde{\epsilon}^z, \tilde{z} \rangle_j \theta_j^z + \sum_{k=1}^{r} \langle \tilde{\epsilon}^u, \tilde{u} \rangle_k \theta_k^u \right) \to \max_{(\tilde{p}^y, \tilde{p}^z, \tilde{p}^u)},$$

$$\left(\sum_{i=1}^{m} \langle \tilde{\epsilon}^y, \tilde{y} \rangle_i \theta_i^y + \sum_{j=1}^{n} \langle \tilde{\epsilon}^z, \tilde{z} \rangle_j \theta_j^z + \sum_{k=1}^{r} \langle \tilde{\epsilon}^u, \tilde{u} \rangle_k \theta_k^u \right) + \theta \sum_{l=1}^{24} \max_{\lambda_l \in \overline{1, \Lambda_l}} \left(a_{\lambda_l} + b_{\lambda_l} Y^g(l) \right)$$

$$+ \left(\sum_{i=1}^{m} \langle \tilde{\epsilon}^y, \tilde{y} \rangle_i s_i^y + \sum_{j=1}^{n} \langle \tilde{\epsilon}^z, \tilde{z} \rangle_j s_j^z + \sum_{k=1}^{r} \langle \tilde{\epsilon}^u, \tilde{u} \rangle_k s_k^u \right) \to \max_{(\theta^y,\ \theta^z,\ \theta^u,\ \tilde{s}^y, \tilde{s}^z, \tilde{s}^u)},$$

$$\left(\sum_{i=1}^{m} \langle \tilde{p}^y, \tilde{y} \rangle_i + \sum_{j=1}^{n} \langle \tilde{p}^z, \tilde{z} \rangle_j + \sum_{k=1}^{r} \langle \tilde{p}^u, \tilde{u} \rangle_k \right) + \left(\sum_{i=1}^{m} \langle \tilde{q}^y, \tilde{y} \rangle_i + \sum_{j=1}^{n} \langle \tilde{q}^z, \tilde{z} \rangle_j \right.$$

$$\left. + \sum_{k=1}^{r} \langle \tilde{q}^u, \tilde{u} \rangle_k \right) + \left(\sum_{i=1}^{m} \langle \tilde{\epsilon}^y, \tilde{y} \rangle_i s_i^y + \sum_{j=1}^{n} \langle \tilde{\epsilon}^z, \tilde{z} \rangle_j s_j^z + \sum_{k=1}^{r} \langle \tilde{\epsilon}^u, \tilde{u} \rangle_k s_k^u \right) \to \min_{(\tilde{y}, \tilde{z}, \tilde{u})},$$

for which the inclusions from Game 1

$$(\tilde{y},\ \tilde{z},\ \tilde{u},\ Y^g) \in \Omega,\ (\tilde{p}^y,\ \tilde{p}^z,\ \tilde{p}^u) \in M,\ (\theta^y,\ \theta^z,\ \theta^u) \in T, (\tilde{s}^y,\ \tilde{s}^z,\ \tilde{s}^u) \in S,$$

where $\theta^y = (\theta_1^y, \ldots, \theta_m^y)$, $\theta^z = (\theta_1^z, \ldots, \theta_n^z)$, $\theta^u = (\theta_1^u, \ldots, \theta_r^u)$, $\tilde{s}^y = (s_1^y, \ldots, s_m^y)$, $\tilde{s}^z = (s_1^z, \ldots, s_n^z)$, $\tilde{s}^u = (s_1^u, \ldots, s_r^u)$, also hold.

As in [1], let the equalities

$$\tilde{q}^y = \left(0; \lambda_1^{yw}(av), \ldots, \lambda_m^{yw}(av); \lambda_1^{ys}(av), \ldots, \lambda_m^{ys}(av), \pi_1^y, \ldots, \pi_m^y\right),$$
$$\tilde{q}^z = \left(0; \lambda_1^{zw}(av), \ldots, \lambda_n^{zw}(av); \lambda_1^{zs}(av), \ldots, \lambda_n^{zs}(av), \pi_1^z, \ldots, \pi_n^z\right),$$
$$\tilde{q}^u = \left(0; \lambda_1^{uw}(av), \ldots, \lambda_r^{uw}(av); \lambda_1^{us}(av), \ldots, \lambda_r^{us}(av), \pi_1^u, \ldots, \pi_r^u\right),$$

hold. Here

λ_i^{yw} means the (average hourly) expenses of industrial customer i, $i \in \overline{1, m}$ that are associated with receiving a unit volume of electric energy from wind energy,

λ_i^{ys} means the (average hourly) expenses of industrial customer i, $i \in \overline{1,m}$ that are associated with receiving a unit volume of electric energy from solar energy, and

π_i^y means the (average hourly) expenses of industrial customer i, $i \in \overline{1,m}$ that are associated with operating its storage system per unit volume of electricity available to this customer,

and $\lambda_j^{zw}(av)$, $\lambda_k^{uw}(av)$, $\lambda_j^{zs}(av)$, $\lambda_k^{us}(av)$, and π_j^z, π_k^u, $j \in \overline{1,n}$, $k \in \overline{1,r}$ have the same meaning that do λ_i^{yw}, λ_i^{ys}, and π_i^y, respectively [1].

In both Games 1 and 2, the polyhedra Ω, M, T and S are sets of the player strategies, and it is assumed that these polyhedra are described by compatible systems of linear inequalities (see Concluding Remark 3) so that both games are those on polyhedra of player strategies.

One should notice that Game 1 is a $(m+n+r+2)$-person game in which strategies of $m+n+r+1$ players are connected in virtue of constraints from the system that models the functioning of the generator [1]

$$\begin{aligned}
&\langle \epsilon, Y^g \rangle - \big(\langle \epsilon^y, y^g \rangle + \langle \epsilon^z, z^g \rangle + \langle \epsilon^u, u^g \rangle\big) - \langle \epsilon, MAX_{loss}(Y^g) \rangle = 0,\\
&H_{min} \le \langle \epsilon, Y^g \rangle \le H_{max},\\
&\langle y^g, p^y \rangle + \langle z^g, p^z \rangle + \langle u^g, p^u \rangle - \langle \epsilon, MAX_{expen}(Y^g) \rangle\\
&\quad - \Psi(Y^g, y^g, z^g, u^g) \to \max_{(p^y, p^z, p^u)},
\end{aligned} \tag{1}$$

where $Y^g = (Y^g(1), \ldots, Y^g(24))$, ϵ, ϵ^y, ϵ^z, ϵ^u are vectors of corresponding dimensions whose all components equal 1, H_{min} and H_{max} are the minimal and the maximal technologically possible production capacities of the generator within 24 h, respectively, and

$$\begin{aligned}
y^g &= \big(y_1^g(1), \ldots, y_1^g(24); y_2^g(1), \ldots, y_2^g(24); \ldots; y_m^g(1), \ldots, y_m^g(24)\big),\\
z^g &= \big(z_1^g(1), \ldots, z_1^g(24); z_2^g(1), \ldots, z_2^g(24); \ldots; z_n^g(1), \ldots, z_n^g(24)\big),\\
u^g &= \big(u_1^g(1), \ldots, u_1^g(24); u_2^g(1), \ldots, u_2^g(24); \ldots; u_r^g(1), \ldots, u_r^g(24)\big),
\end{aligned}$$

and the function $\Psi(Y^g, y^g, z^g, u^g)$ describes the generator expenses associated with transmitting electric energy to the grid customers [1]

$$\begin{aligned}
\Psi(Y^g, y^g, z^g, u^g) = \Psi(Y^g, \tilde{y}, \tilde{z}, \tilde{u})) &= \theta \sum_{l=1}^{24} \max_{\lambda_l \in \overline{1,\Lambda_l}} \Big(a_{\lambda_l}^l + b_{\lambda_l}^l Y^g(l)\Big)\\
&+ \Bigg(\sum_{i=1}^{m} \langle \tilde{\epsilon}^y, \tilde{y} \rangle_i \theta_i^y + \sum_{j=1}^{n} \langle \tilde{\epsilon}^z, \tilde{z} \rangle_j \theta_j^z + \sum_{k=1}^{r} \langle \tilde{\epsilon}^u, \tilde{u} \rangle_k \theta_k^u\Bigg),
\end{aligned}$$

which are present in the description of the polyhedron Ω.

However, Game 2 is a three-person game on polyhedra of disjoint player strategies, since (a) the vectors $\tilde{y}$, $\tilde{z}$, $\tilde{u}$ "compete" neither with each other nor with the vector Y^g within any quadruple of vectors $(\tilde{y}, \tilde{z}, \tilde{u}, Y^g)$ from the polyhedron Ω, (b) the polyhedron Ω does not intersect with the polyhedra M, T and S, and (c) player 1 (the generator) chooses only the prices $\tilde{p}^y$, $\tilde{p}^z$, $\tilde{p}^u$, whereas the vector Y^g is completely determined by the vectors $\tilde{y}$, $\tilde{z}$, $\tilde{u}$ in virtue of the first two relations from system (1).

Let us show that Nash equilibria in the auxiliary game with disjoint player strategies determine Nash equilibria in the initial game with connected player strategies.

Let

$$\tilde{\tilde{y}} = (\tilde{y}, \tilde{z}, \tilde{u}),\ \tilde{\tilde{x}} = (\tilde{p}^y, \tilde{p}^z, \tilde{p}^u),\ \delta = (\tilde{q}^y, \tilde{q}^z, \tilde{q}^u),$$

and let

$$\tilde{\tilde{t}} = \left(\theta^y; 0, 0, \ldots, 0, \theta^z; 0, 0, \ldots, 0, \theta^u; 0, 0, \ldots, 0\right),$$
$$\tilde{\tilde{s}} = \left(s^y; 0, 0, \ldots, 0, s^z; 0, 0, \ldots, 0, s^u; 0, 0, \ldots, 0\right),$$

where all the zero components of the vectors $\tilde{\tilde{t}}$ and $\tilde{\tilde{s}}$ correspond to the following components of the vectors $\tilde{y}$, $\tilde{z}$, and $\tilde{u}$:

$$\left(y_1^w, y_2^w, \ldots, y_m^w; y_1^s, y_2^s, \ldots, y_m^s; y_1^{st}, y_2^{st}, \ldots, y_m^{st}\right),$$
$$\left(z_1^w, z_2^w, \ldots, z_n^w; z_1^s, z_2^s, \ldots, z_n^s, z_1^{st}, z_2^{st}, \ldots, z_n^{st}\right),$$
$$\left(u_1^w, u_2^w, \ldots, u_r^w; u_1^s, u_2^s, \ldots, u_r^s; u_1^{st}, u_2^{st}, \ldots, u_r^{st}\right),$$

respectively. Further, let

$$\hat{y} = \left(\tilde{\tilde{y}}, Y^g\right),\ \hat{x} = \left(\tilde{\tilde{x}}, 0_x\right),\ \hat{t} = \left(\tilde{\tilde{t}}, 0_t\right),\ \hat{s} = \left(\tilde{\tilde{s}}, 0_s\right),\ \Delta = (\delta, 0),$$

where 0_x, 0_t, 0_s, and 0 are zero vectors of the same dimension as that of the vector Y^g.

Then Game 2 can be rewritten as

$$\begin{aligned} \langle \hat{y},\ \hat{x}\rangle - \langle \hat{y}, \hat{t}\rangle - f_1(\hat{y}) - f_2(\hat{y}) &\to \max_{\hat{x}\in\hat{M}} \\ \langle \hat{y}, \hat{t}\rangle + \langle \hat{y}, \hat{s}\rangle + f_1(\hat{y}) &\to \max_{(\hat{t},\hat{s})\in\hat{T}\times\hat{S},} \\ \langle \hat{y}, \hat{x}\rangle + \langle \hat{y}, \hat{s}\rangle + \langle \Delta, \hat{y}\rangle &\to \min_{\hat{y}\in\hat{\Omega}}, \end{aligned} \qquad \text{(Game 3)}$$

where the sets $\hat{M}$, $\hat{T}$, $\hat{S}$, $\hat{\Omega}$ are formed by the same kind of constraints that form the sets M, T, S, Ω, respectively (though the constraints describing the polyhedra $\hat{M}$, $\hat{T}$, $\hat{S}$, $\hat{\Omega}$ bind the variables $\hat{x}$, $\hat{t}$, $\hat{s}$ and $\hat{y}$, respectively),

$$f_1(\hat{y}) = \langle \theta\epsilon, MAX_{loss}(\hat{y})\rangle,\ f_2(\hat{y}) = \langle \epsilon, MAX_{expen}(\hat{y})\rangle,$$

where [1]

$$MAX_{loss}(Y^g) = \left(\max_{\lambda_1\in\overline{1,\Lambda_1}} \left(a_{\lambda_1} + b_{\lambda_1} Y^g(1)\right), \ldots, \max_{\lambda_{24}\in\overline{1,\Lambda_{24}}} \left(a_{\lambda_{24}} + b_{\lambda_{24}} Y^g(24)\right)\right),$$

$$MAX_{expen}(Y^g) = \left(\max_{\mu_1 \in \overline{1,\Gamma_1}} \left(c_{\mu_1} + b_{\mu_1} Y^g(1)\right), \ldots, \max_{\mu_{24} \in \overline{1,\Gamma_{24}}} \left(c_{\mu_{24}} + b_{\mu_{24}} Y^g(24)\right) \right),$$

and, as before, $\epsilon \in R^{24}_+$ is the vector with all the components equalling 1, whereas $MAX_{loss}(Y^g)$ and $MAX_{expen}(Y^g)$ can be viewed as the vector functions whose components depend on the vector $\hat{y}$ due to the above relations between the vectors $\hat{y}$ and Y^g.

Finally, let us consider an auxiliary two-person game on the polyhedra $\hat{\Omega}$ and $\hat{M} \times \hat{S}$ of the player strategies

$$(\hat{y}^*,\ (\hat{x}^*, \hat{s}^*)) \in Sp_{(\hat{y},\ (\hat{x},\ \hat{s})) \in \hat{\Omega} \times (\hat{M} \times \hat{S})} \left(\langle \hat{y}, \hat{x} + \hat{s} \rangle + \langle \Delta, \hat{y} \rangle \right), \qquad \text{(Game 4)}$$

which is a game with disjoint player strategies, where the payoff function is minimized with respect to $\hat{y}$ and is maximized with respect to $(\hat{x}, \hat{s})$.

3 Basic assertions

Assertion 1 A quadruple of the vectors $(\hat{y}^*,\ \hat{x}^*,\ \hat{t}^*, \hat{s}^*)$ forms a Nash equilibrium point in Game 3 if and only if the triple of the vectors $(\hat{y}^*, \hat{x}^*, \hat{s}^*)$ forms the saddle point $\left(\hat{y}^*, (\hat{x}^*, \hat{s}^*)\right)$ in Game 4, and the inclusion $\hat{t}^* \in Argmax_{\hat{t} \in \hat{T}} \langle \hat{y}^*, \hat{t} \rangle$ holds.

Proof To simplify the notation, in the reasoning associated with the proof of Assertion 1, variables $y,\ x,\ t,\ s$ are considered instead of the variables $\hat{y},\ \hat{x},\ \hat{t},\ \hat{s}$, and the polyhedra Ω, M, T, S are considered instead of the polyhedra $\hat{\Omega}, \hat{M}, \hat{T}, \hat{S}$, respectively.

1. *Necessity.* Let the quadruple of the vectors $(y^*,\ x^*,\ t^*,\ s^*)$ form a Nash equilibrium point in Game 3. Then the following inequalities

$$\begin{aligned}
&\langle y^*, x \rangle - \langle y^*, t^* \rangle - f_1(y^*) - f_2(y^*) \\
&\quad \le \langle y^*, x^* \rangle - \langle y^*, t^* \rangle - f_1(y^*) - f_2(y^*), \quad \forall x \in M,\ s = s^*, \\
&\langle y^*, t \rangle + \langle y^*, s \rangle + f_1(y^*) \le \langle y^*, t^* \rangle + \langle y^*, s^* \rangle + f_1(y^*), \quad \forall (t, s) \in T \times S,\ x = x^*, \\
&\langle y^*, x^* \rangle + \langle y^*, s^* \rangle + \langle \Delta, y^* \rangle \le \langle y, x^* \rangle + \langle y, s^* \rangle + \langle \Delta, y \rangle, \quad \forall y \in \Omega,\ t = t^*
\end{aligned} \qquad (2)$$

hold. Since the polyhedra Ω and T are disjoint, holding the third inequality from (2) means that the right part of the pair of inequalities

$$\langle y^*, x + s \rangle + \langle \Delta, y^* \rangle \le \langle y^*, x^* + s^* \rangle + \langle \Delta, y^* \rangle \le \langle y, x^* + s^* \rangle + \langle \Delta, y \rangle$$

holds $\forall y \in \Omega$.
Holding the first inequality from (2) means that the inequality

$$\langle y^*, x \rangle \le \langle y^*, x^* \rangle, \quad \forall x \in M,\ s = s^*$$

holds, and since the polyhedra M and S are disjoint, one can conclude that the inequality

$$\langle y^*, x\rangle \le \langle y^*, x^*\rangle, \quad \forall x \in M$$

also holds.
Since the second inequality from (2) holds for any pair of the vectors $(t, s),\ t \in T,\ s \in S$, in particular, for the pairs (t^*, s) when $x = x^*$, and since the polyhedra $S,\ T$ and M are disjoint, one can conclude that the inequality

$$\langle y^*, s\rangle \le \langle y^*, s^*\rangle, \quad \forall s \in S$$

also holds. Holding the inequalities

$$\langle y^*, x\rangle \le \langle y^*, x^*\rangle, \quad \forall x \in M, \quad \langle y^*, s\rangle \le \langle y^*, s^*\rangle, \quad \forall s \in S$$

means that the left inequality from the above pair of the inequalities

$$\langle y^*, x + s\rangle + \langle \Delta, y^*\rangle \le \langle y^*, x^* + s^*\rangle + \langle \Delta, y^*\rangle \le \langle y, x^* + s^*\rangle + \langle \Delta, y\rangle$$

holds $\forall x \in M,\ \forall s \in S$ and, consequently, $\forall (x, s) \in M \times S$. Thus, the triple of the vectors (y^*, x^*, s^*) forms the saddle point $\big(y^*,\ (x^*,\ s^*)\big)$ in Game 4.
Since the second inequality from (2) holds for any pair of the vectors $(t, s),\ t \in T,\ s \in S$, in particular, for the pairs (t, s^*) when $x = x^*$, and since the polyhedra $T,\ S$ and M are disjoint, one can conclude that the inequality

$$\langle y^*, t\rangle \le \langle y^*, t^*\rangle, \quad \forall t \in T$$

also holds, which means that the inclusion

$$t^* \in Argmax_{t \in T} \langle y^*, t\rangle$$

holds.

2. *Sufficiency.* Let the pair of inequalities

$$\langle y^*, x + s\rangle + \langle \Delta, y^*\rangle \le \langle y^*, x^* + s^*\rangle + \langle \Delta, y^*\rangle \le \langle y, x^* + s^*\rangle + \langle \Delta, y\rangle$$

hold $\forall y \in \Omega,\ \forall (x, s) \in M \times S$, along with the inclusion $t^* \in Argmax_{t \in T} \langle y^*, t\rangle$. The right part of this pair of inequalities means that the inequality

$$\langle y^*, x^*\rangle + \langle y^*, s^*\rangle + \langle \Delta, y^*\rangle \le \langle y, x^*\rangle + \langle y, s^*\rangle + \langle \Delta, y\rangle$$

holds $\forall y \in \Omega$ and, consequently, $\forall y \in \Omega,\ t = t^*$ (since the polyhedra Ω and T are disjoint).
From the left part of the above pair of inequalities, it follows that the inequality

$$\langle y^*, x\rangle \le \langle y^*, x^*\rangle$$

holds $\forall x \in M,\ s = s^*$, and, consequently, the inequality

$$\begin{aligned}&\langle y^*, x\rangle - \langle y^*, t^*\rangle - f_1(y^*) - f_2(y^*)\\&\quad \le \langle y^*, x^*\rangle - \langle y^*, t^*\rangle - f_1(y^*) - f_2(y^*), \quad \forall x \in M,\ s = s^*\end{aligned}$$

also holds, whereas the inequality

$$\langle y^*, s\rangle \le \langle y^*, s^*\rangle$$

holds $\forall s \in S,\ x = x^*$. So, since the polyhedra S and T are disjoint, from the inclusion

$$t^* \in Argmax_{t \in T} \langle y^*, t\rangle,$$

it follows that the inequality

$$\langle y^*, t\rangle + \langle y^*, s\rangle \le \langle y^*, t^*\rangle + \langle y^*, s^*\rangle,$$

holds $\forall t \in T,\ \forall s \in S,\ x = x^*$ and, consequently, $\forall (t, s) \in T \times S,\ x = x^*$. Thus, the three inequalities

$$\begin{aligned}&\langle y^*, x\rangle - \langle y^*, t^*\rangle - f_1(y^*) - f_2(y^*)\\&\quad \le \langle y^*, x^*\rangle - \langle y^*, t^*\rangle - f_1(y^*) - f_2(y^*),\ \forall x \in M,\ s = s^*,\\&\langle y^*, t\rangle + \langle y^*, s\rangle + f_1(y^*) \le \langle y^*, t^*\rangle + \langle y^*, s^*\rangle + f_1(y^*), \quad \forall (t, s) \in T \times S,\ x = x^*,\\&\langle y^*, x^*\rangle + \langle y^*, s^*\rangle + \langle \Delta, y^*\rangle \le \langle y, x^*\rangle + \langle y, s^*\rangle + \langle \Delta, y\rangle, \quad \forall y \in \Omega,\ t = t^*\end{aligned}$$

hold, which means that the quadruple of the vectors $(y^*,\ x^*,\ t^*,\ s^*)$ forms a Nash equilibrium point in Game 3.

Assertion 1 is proved. □

Assertion 2 Any Nash equilibrium point

$$\Big((\tilde{y}^*, \tilde{z}^*,\ \tilde{u}^*, (Y^g)^*), ((\tilde{p}^y)^*,\ (\tilde{p}^z)^*,\ (\tilde{p}^u)^*),\ ((\tilde{\theta}^y)^*,\ (\tilde{\theta}^z)^*,\ (\tilde{\theta}^u)^*),\ ((\tilde{s}^y)^*,\ (\tilde{s}^z)^*,\ (\tilde{s}^u)^*)\Big)$$

in Game 2 is a Nash equilibrium point in Game 1.

Proof Let

$$\Big((\tilde{y}^*, \tilde{z}^*,\ \tilde{u}^*, (Y^g)^*), ((\tilde{p}^y)^*,\ (\tilde{p}^z)^*,\ (\tilde{p}^u)^*),\ ((\tilde{\theta}^y)^*,\ (\tilde{\theta}^z)^*,\ (\tilde{\theta}^u)^*)\ (\tilde{s}^y)^*,\ (\tilde{s}^z)^*,\ (\tilde{s}^u)^*)\Big)$$

be a Nash equilibrium point in Game 2. To show that this point is an equilibrium point in Game 1, it is sufficient to show that all the inequalities

$$\begin{aligned}
&\langle(\tilde{p}^y)^*, \tilde{y}\rangle_i + \langle\tilde{\epsilon}^y, \tilde{y}\rangle_i (s_i^y)^* + \langle\tilde{q}^y, \tilde{y}\rangle_i \geq \langle(\tilde{p}^y)^*, \tilde{y}^*\rangle_i + \langle\tilde{\epsilon}^y, \tilde{y}^*\rangle_i (s_i^y)^* \\
&\quad + \langle\tilde{q}^y, \tilde{y}^*\rangle_i, \quad \forall i \in \overline{1,m}, \ \forall(\tilde{y}, \tilde{z}^*, \tilde{u}^*) \colon (\tilde{y}, \tilde{z}^*, \tilde{u}^*, (Y^g)^*) \in \Omega, \\
&\langle(\tilde{p}^z)^*, \tilde{z}\rangle_j + \langle\tilde{\epsilon}^z, \tilde{z}\rangle_j (s_j^z)^* + \langle\tilde{q}^z, \tilde{z}\rangle_j \geq \langle(\tilde{p}^z)^*, \tilde{z}^*\rangle_j + \langle\tilde{\epsilon}^z, \tilde{z}^*\rangle_j (s_j^z)^* \\
&\quad + \langle\tilde{q}^z, \tilde{z}^*\rangle_j, \quad \forall j \in \overline{1,n}, \ \forall(\tilde{y}^*, \tilde{z}, \tilde{u}^*) \colon (\tilde{y}^*, \tilde{z}, \tilde{u}^*, (Y^g)^*) \in \Omega, \\
&\langle(\tilde{p}^u)^*, \tilde{u}\rangle_k + \langle\tilde{\epsilon}^u, \tilde{u}\rangle_k (s_k^u)^* + \langle\tilde{q}^u, \tilde{u}\rangle_k \geq \langle(\tilde{p}^u)^*, \tilde{u}^*\rangle_k + \langle\tilde{\epsilon}^u, \tilde{u}^*\rangle_k (s_k^u)^* \\
&\quad + \langle\tilde{q}^u, \tilde{u}^*\rangle_k, \quad \forall k \in \overline{1,r}, \ \forall(\tilde{y}^*, \tilde{z}^*, \tilde{u}) : (\tilde{y}^*, \tilde{z}^*, \tilde{u}, (Y^g)^*) \in \Omega,
\end{aligned} \tag{3}$$

hold.

Let us assume that, for instance, the inequality

$$\begin{aligned}
&\langle(\tilde{p}^y)^*, \tilde{y}^0\rangle_{i^0} + \langle\tilde{\epsilon}^y, \tilde{y}^0\rangle_{i^0} (\tilde{s}_{i^0}^y)^* + \langle\tilde{q}^y, \tilde{y}^0\rangle_{i^0} \\
&\quad < \langle(\tilde{p}^y)^*, \tilde{y}^*\rangle_{i^0} + \langle\tilde{\epsilon}^y, \tilde{y}^*\rangle_{i^0} (\tilde{s}_{i^0}^y)^* + \langle\tilde{q}^y, \tilde{y}^*\rangle_{i^0}
\end{aligned}$$

holds for $i^0 \in \overline{1,m}$, along with the rest of the inequalities from system (3) and from the system of constraints of Game 1, where $(\tilde{y}^0,\ \tilde{z}^*,\ \tilde{u}^*, (Y^g)^*) \in \Omega$, and the vector $\tilde{y}^0$, differs from the vector $\tilde{y}^*$ only by the group of coordinates corresponding to number i^0. Since all the inequalities from system (3), except for the inequality corresponding to number i^0, hold as equalities, the inequality

$$\begin{aligned}
&\sum_{i=1, i\neq i^0}^{m} \langle(\tilde{p}^y)^*, \tilde{y}^0\rangle_i + \sum_{j=1}^{n} \langle(\tilde{p}^z)^*, \tilde{z}^*\rangle_j + \sum_{k=1}^{r} \langle(\tilde{p}^u)^*, \tilde{u}^*\rangle_k \\
&\quad + \sum_{i=1, i\neq i^0}^{m} \langle\tilde{q}^y, \tilde{y}^*\rangle_i + \sum_{j=1}^{n} \langle\tilde{q}^z, \tilde{z}^*\rangle_j + \sum_{k=1}^{r} \langle\tilde{q}^u, \tilde{u}^*\rangle_k \\
&\quad + \left(\sum_{i=1, i\neq i^0}^{m} \langle\tilde{\epsilon}^y, \tilde{y}^*\rangle_i (s_i^y)^* + \sum_{j=1}^{n} \langle\tilde{\epsilon}^z, \tilde{z}^*\rangle_j (s_j^z)^* + \sum_{k=1}^{r} \langle\tilde{\epsilon}^u, \tilde{u}^*\rangle_k (s_k^u)^* \right) \\
&\quad + \langle(\tilde{p}^y)^*, \tilde{y}^0\rangle_{i^0} + \langle\tilde{\epsilon}^y, \tilde{y}^0\rangle_{i^0} (s_{i^0}^y)^* + \langle\tilde{q}^y, \tilde{y}^0\rangle_{i^0} \\
&\quad < \sum_{i=1}^{m} \langle(\tilde{p}^y)^*, \tilde{y}^*\rangle_i + \sum_{j=1}^{n} \langle(\tilde{p}^z)^*, \tilde{z}^*\rangle_j + \sum_{k=1}^{r} \langle(\tilde{p}^u)^*, \tilde{u}^*\rangle_k \\
&\quad + \sum_{i=1}^{m} \langle\tilde{q}^y, \tilde{y}^*\rangle_i + \sum_{j=1}^{n} \langle\tilde{q}^z, \tilde{z}^*\rangle_j + \sum_{k=1}^{r} \langle\tilde{q}^u, \tilde{u}^*\rangle_k \\
&\quad + \left(\sum_{i=1}^{m} \langle\tilde{\epsilon}^y, \tilde{y}^*\rangle_i (s_i^y)^* + \sum_{j=1}^{n} \langle\tilde{\epsilon}^z, \tilde{z}^*\rangle_j (s_j^z)^* + \sum_{k=1}^{r} \langle\tilde{\epsilon}^u, \tilde{u}^*\rangle_k (s_k^u)^* \right)
\end{aligned}$$

must hold. This means that there exist a vector from the set Ω for which the value of the payoff function of the third player in Game 2 turns out to be smaller than that for the vector

$$\Big((\tilde{y}^*, \tilde{z}^*,\ \tilde{u}^*, (Y^g)^*), ((\tilde{p}^y)^*,\ (\tilde{p}^z)^*,\ (\tilde{p}^u)^*),\ ((\tilde{\theta}^y)^*,\ (\tilde{\theta}^z)^*,\ (\tilde{\theta}^u)^*),\ (\tilde{s}^y)^*,\ (\tilde{s}^z)^*,\ (\tilde{s}^u)^*)\Big),$$

which is a Nash equilibrium point in Game 2, and this contradicts the definition of a Nash equilibrium point. Assertion 2 is proved. □

Remark As one can see from the description of Games 2 and 3, any particular forms of the functions describing the loss of electricity in the transmission lines and expenses of the generator associated with producing electricity do not affect the fact that each equilibrium point in Game 2 is an equilibrium in Game 1. One should, however, bear in mind that Assertion 2 suggests that, generally, the set of equilibrium points in Game 2 is only a subset of the set of equilibrium points in Game 1, and the whole set of equilibrium points in Game 1 certainly depends on particular forms of both above-mentioned functions.

4 On calculating equilibrium points in Game 2

Game 4 is a particular case of the two-person game with the payoff function

$$\langle p^1, x^1\rangle + \langle x^1, D^1 y^1\rangle + \langle q^1, y^1\rangle \tag{4}$$

on (generally unbounded) polyhedral sets $M^1,\ \Omega^1$, where M^1 and Ω^1 are described by compatible systems of linear inequalities

$$M^1 = \{x^1 \in R^m_+ \colon A^1 x^1 \geq b^1\},\quad \Omega^1 = \{y^1 \in R^n_+ \colon B^1 y^1 \geq d^1\}, \tag{5}$$

$A^1,\ B^1,\ D^1$ are matrices, $b^1,\ d^1,\ p^1,\ q^1,\ x^1,\ y^1$ are vectors of corresponding dimensions, and function (4) is maximized with respect to y^1 and is minimized with respect to x^1. One can easily be certain about this by considering

$$p^1 = \Delta,\ x^1 = \hat{y},\ y^1 = (\hat{x}, \hat{s}),\ q^1 = (0, 0),$$

by choosing the matrix $(E_m|E_m)$ as the matrix D^1, where E_m is an $m \times m$ unit matrix, i.e., the matrix whose non-zero elements equal 1 and occupy the main diagonal of E_m, and by setting $2m = n$.

Theorem [2] *The solvability of game* (4), (5) *is equivalent to that of two linear programming problems*

$$\langle b^1, z^1\rangle + \langle q^1, y^1\rangle \to \max_{(z^1, y^1)\in Q^1},$$

and

$$\langle -d^1, t^1 \rangle + \langle p^1, x^1 \rangle \to \min_{(t^1, x^1) \in P^1}$$

forming a dual pair, where

$$Q^1 = \{(z^1, y^1) \geq 0 \colon z^1 A^1 \leq p^1 + D^1 y^1,\ B^1 y^1 \geq d^1\},$$
$$P^1 = \{(t^1, x^1) \geq 0 \colon t^1 B^1 \leq -q^1 - x^1 D^1,\ A^1 x^1 \geq b^1\},$$

and t^1, z^1 are vectors of corresponding dimensions.

This theorem allows one to find equilibrium points in solvable Game 1 by finite methods of linear programming for any imaginable numbers of constraints and variables that may appear in practical problems.

5 Concluding remarks

1. As mentioned in [1], components of a Nash equilibrium point that can be calculated for solvable Games 1 and 2, allow one to determine (corresponding to this equilibrium point) (a) the optimal hourly volumes of electricity to be bought by each of the large grid customers from both the generator and the suppliers (i.e., from the companies who transform wind and solar energy into electric energy [1]), (b) the optimal hourly prices to be paid by the large grid customers to the generator, to the suppliers, and to the transmission company, and (c) the optimal hourly prices to be paid by the generator to the transmission company.
2. The proposed game model describes the interaction among the generator, the (large) grid customers, and the transmission company in a regional electrical grid, being part of a country's electrical grid. However, the description of the functioning of a country's electrical grid as a whole usually presents a challenge, first of all, from the viewpoint of the model size. Nevertheless, the proposed approach to modeling seems to be promising, since it allows one to remain within linear programming in calculating optimal values of the above volumes of electricity and the prices. As is known, linear programming techniques have a high computational potential, so the number of customers can substantially be increased in considering the interaction of several parts of a country's electrical grid.
   At the same time, one should bear in mind that in the proposed (regional) model, it is assumed that neither the generating facilities nor the transmission company (if both consist of several legal entities) compete for the regional customers. However, within the whole country's grid, both the generators and the transmission companies serving customers from a particular part of the grid may compete at least for supplying electricity to other parts of the grid. So one should expect that the games describing the interaction of the generating facilities, the large customers of the whole grid, and the transmission companies are likely to be those in which strategies of all the players are connected [3–5].

In the framework of the description of the above interaction (within the whole grid), by solving corresponding games with connected player strategies, one may receive answers to the following two basic questions:

(a) What are the chances of renewable sources of energy to be incorporated in the currently existing grid, as well as in the one that is likely to function in the future, and

(b) under which economic conditions can renewable sources of energy successfully compete with traditional electricity generating facilities or at least successfully supplement them within a country's electrical grid?

(Certainly, in analyzing any answers to the above questions, one should bear in mind that the configuration of a country's electrical grid in general much depends on the corresponding government regulations, the market structure, and the availability of the transmission lines between separate parts of the grid.)

The proposed approach to modeling the interaction among a generating company, a transmission company, and a set of large customers of a regional electrical grid can, however, be extended to cover the cases in which there are either several separate generating companies (acting as different legal entities) or several transmission companies (acting as different legal entities), as well as the cases in which both several separate generating companies and several separate transmission companies compete though only for the customers of the (regional) grid. If this is the case, Game 1 will become an $(m + n + r + \triangle + \eta)$-person game on polyhedra of player strategies some of which are connected, where $\triangle$ and η are the numbers of the generating companies and the transmission companies, respectively. However, one can easily be certain that a mathematical model for this problem may be designed in such a form that the corresponding auxiliary game (similar to Game 2) will remain a three-person game on disjoint player strategies for which assertions similar to Assertions 1 and 2 will hold.

3. It is substantial to notice that the auxiliary three-person game considered in the paper is always (in applied problems) solved on bounded polyhedral sets [6] (which is the case in the game problem that is the subject of study in this paper, as long as the systems of constraints in the model of the grid describing the polyhedra $\hat{\Omega}$, $\hat{M}$, $\hat{T}$, and $\hat{S}$ are compatible). A simple technique that allows one to "correct" the right hand sides of the above systems of (linear) constraints to make these systems compatible while keeping them "close" to the initial ones is proposed in [7].
4. One should bear in mind that the model proposed in [1] reflects mostly economic and technological restrictions imposed on the interaction among the generator, the transmission company, and the (large) customers of a regional electrical grid. However, consumer rights of all the groups of the customers, especially those of the households that receive electricity from the utility companies, must be observed under any solutions (including equilibrium ones) that the electric energy providers may agree upon [8,9]. For instance, in describing the interaction among all the customers of the grid, the generator, and the transmission company in the form of an $(m+n+r+2)$-person game, it was assumed that the revenue of the transmission company comes from both the (large) grid customers and the generator. This means that if certain financial restrictions, mentioned in [1], are eventually imposed on

both the generator and the transmission company, a different description of this interaction in the framework of which, for instance, either only the generator or only the customers of the grid are charged for the electricity transmission should be used. However, a detailed discussion of such matters lies beyond the scope of this paper.

5. Calculating the value of an optimal hourly electricity supply for each large customer by the generator, along with the optimal prices per unit volume for produced and transmitted electricity, by solving Game 2 presents interest under (mentioned in [1]) two approaches to operating the generating facilities. (These approaches consist of selling as much electricity produced as possible via auctions (approach 1) and of selling as much electricity produced as possible via direct long-term contracts with the grid customers while selling via auctions only the remaining part of the electricity produced (approach 2) [1].) Moreover, the description of the interaction among the generator, a transmission company, and all the (large) customers within a regional electrical grid, proposed in the present paper, allows one (a) to calculate the optimal (equilibrium) prices per unit volume of electricity produced by the generator in the framework of the double-sided agreements with these customers on any time basis (hourly, weekly, monthly, yearly) by solving Game 2, and (b) to compare these prices with the market prices in the electricity auctions. However, while it would be interesting to compare the expenses that all the parties involved are to bear, under both approaches, such a comparison would require data some of which is not public, and acquiring this data may present certain difficulties.
   Since (as mentioned earlier) the proposed model allows one to analyze equilibria under various scenarios of the functioning of a regional electrical grid, one may be interested, for instance, in finding (a) what equilibrium hourly prices for each customer should be, provided these prices are the same for each customer (though different for different customers) or are the same for certain hours during 24 h (e.g., from 5 p.m. to 11 p.m.), (b) what electricity prices for each customer should be in the absence of electricity storage systems at customers' disposal, etc.
6. Solutions to Game 1 (which is formulated on the basis of the model proposed in [1]) allow one to quantitatively evaluate the economic expediency of the use of renewable sources of energy and electricity storage systems in any part of the grid under any particular electricity prices associated with the use of these sources of energy and under those for storing electricity and determined by components of the vectors $\tilde{q}^y$, $\tilde{q}^z$, $\tilde{q}^u$. In particular, the components $y_1^{st}, \ldots, y_m^{st}$, $z_1^{st}, \ldots, z_n^{st}$, and $u_1^{st}, \ldots, u_r^{st}$ of the vectors $\hat{y}$, $\hat{z}$, and $\hat{u}$ in solutions to Game 3, respectively, determine the optimal strategy of using the storage systems by the grid customers. These strategies are determined for each large customer of the grid under any particular sets of (average) prices for receiving a unit volume of electricity from renewable sources of energy and under any particular sets of (average) prices for operating the storage system per unit volume of electricity available to each large customer of the grid. The other components of the last three vectors determine, in particular, the volumes of electricity consumed by the large grid customers from the renewable sources of energy that are optimal for each large customer of the grid. However, finding the minimum prices at which the use of renewable sources of energy in a particular part of the grid becomes profitable for a particular customer of the part

of the grid leads to studying games that are more complicated than Game 1 (if one uses the model proposed in [1]).

7. Though the proposed model is designed to describe the interaction among all the customers of a regional electrical grid, the existing base load power plants, picking power plants, transmission companies, and available renewable sources of energy and storage facilities, one can show that the idea underlying this model can be extended to incorporate the option of developing or acquiring all types of new generating facilities (base load power plants, picking power plants, wind and solar power stations and devices). In this case, in fact, one can use the model for calculating investment strategies that, in particular, all the groups of the customers and the generator may eventually be interested in considering.
An approach to developing optimal investment strategies for all the players with the use of the (appropriately extended) model consists of (a) considering a month or a year as the length of the time intervals (instead of hour-length intervals in the model considered in this paper and in [1]), (b) choosing the horizon of the analysis that is not smaller than the period of time in which the investment is expected to be recuperated (under a certain rate of the return on investment), and (c) choosing the manner of incorporating an additional (fixed) cost in the electricity prices to be paid by the customers. The solution to the corresponding game problem will allow one to determine at which prices (a) for the equipment needed to operate facilities transforming wind and solar energy into electricity, and (b) for electricity storages the incorporation of both renewable sources of energy and electricity storage systems in the grid is economically justifiable. One should, however, bear in mind that developing facilities, for instance, for transforming wind and solar energy into electricity is costly so that it is likely to require some government participation in financing the project, for instance, in the form of a public-private partnership between the government and the private sector [5,10].
8. The model used for describing the interaction among the generator, the transmission company, and all the customers of a part of a country's electrical grid is the one with continuous variables, which allows one to remain within the realm of continuous optimization in calculating Nash equilibrium points in Game 1. This is a result of the assumption that the expenses associated with the operation of electricity storage facilities and those associated with transforming wind and solar energy into electricity are linear functions of volumes of the electricity available from them, and the proportionality coefficients determine particular forms of these functions. This assumption should be acceptable as long as these proportionality coefficients are some average expenses per unit volume of electricity received from each of the devices of the same type to be operated within these facilities. Though in the proposed model, these expenses are considered to be constant during each hour within 24 h, one can modify the model to cover the other cases, for instance, the one in which these expenses while being constant during each hour, are different for different hours.
9. Though both the public sector and the private sector of economy in every developed country considers wind and solar energy as alternative sources of energy that could be part of the country's electrical grid, at least under the current technology, transforming these types of energy into electric energy looks quite expensive [11–14]. Nevertheless, numerous businesses develop power stations for transforming wind

and solar energy into electric energy. The proposed approach to modeling the interaction among the generator, the transmission company, and the large grid customers of a regional electrical grid allows one to estimate at which prices for a unit volume of electric energy from wind and solar energy produced by the already functioning stations receiving electricity from these stations is financially reasonable. The moving forces behind a great deal of public interest to both (wind and solar) sources of energy are (a) an attempt to reach a higher degree of independence from the generator than currently exists for the large grid customers, (b) the unwillingness to overpay the utility companies for their services, and (c) the existing incentives to reduce the CO_2 emission, which are financially supported by governments in some countries. The use of devices like energy boxes [15], advising about expected electricity prices in the framework of a "smart" grid [16], can substantially affect decisions on incorporating renewable sources of energy in any part of the grid.

10. The piece-wise linear form of the regularities describing losses of energy in transmission lines reflects further simplifications of the above-mentioned grid regularities, in addition to those made in [1] in developing the mathematical model underlying the structure of Game 1 (see Assumptions 4, 6 in [1]). The same is true for the description of the functioning of a storage facility in the proposed model in the form of a system of linear equations and inequalities, which is a particular simplification of the reality [1].

    At the same time, as it usually takes place in modeling phenomena in nature and in society, one should proceed from the goal that a particular model is expected to serve [17–19]. From this viewpoint, the use of the proposed model of storage facilities looks justifiable, since it does not affect either the form or the features of the game underlying the approach to analyzing a part of a country's electrical grid while allowing one to remain within linear programming in calculating Nash equilibrium points in the "interaction" game under consideration for any estimating purposes.

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