# HEAT SOURCE IN A COMPRESSIBLE FLUID

by

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Prof. Joseph S. Newell
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Dear Professor Newell:

In accordance with the regulations of the faculty, I hereby submit a thesis entitled, 'HEAT SOURCE IN A COMPRESSIBLE FLUID', in partial fulfillment of the requirements for the degree of Master of Science in Aeronautical Engineering.

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# OBJECT

The object of this thesis is to develop the linearized equations necessary to solve for the forces acting on a heat source in a two dimensional viscous, heat-conducting fluid.

The simplified problem of the forces acting on a heat source in a non-viscous, non-heat conducting fluid is to be solved.

## INTRODUCTION

In this thesis a study is made to determine the forces acting on a heat source in a compressible fluid. By considering that the presence of the source causes only small disturbances it is possible to linearize the differential equations which characterize the flow. This assumption greatly simplifies the equations involved.

The solution obtained is for a point source of heat at the origin, in a two-dimensional flow. The solution for a source of heat at any other point $(\xi,\eta)$, in the plane, is obtained by a change in variables from $(x,y)$ to $(x-\xi,\ y-\eta)$. It is possible, therefore, to solve for the forces acting on a source of heat of any shape if a source distribution is assumed over the area concerned, by integrating with respect to $\xi,\eta$.

The results of this thesis can be of value in the field of combustion by affording an insight into the disturbances produced, and the forces acting, on a flame front of known shape.

## NOTATION

The following notation will be used throughout this thesis.

$a$ = velocity of sound.

$C_p$ = specific heat at constant pressure.

$C_v$ = specific heat at constant volume.

$M$ = Mach number.

$p$ = static pressure.

$q$ = local rate of heat addition.

$Q$ = total rate of heat addition.

$R$ = gas constant.

$T$ = absolute temperature.

$u$ = velocity in x-direction.

$U$ = free stream velocity.

$v$ = velocity in y-direction.

$\alpha = \sqrt{1-M^{o2}}$

$\beta = \sqrt{M^{o2}-1}$

$\gamma$ = ratio of specific heats $C_p/C_v$

$\lambda$ = coefficient of heat conductivity.

$\mu$ = coefficient of viscosity.

$\mu'$ = second viscosity coefficient.

$\sigma_x$ = normal stress in x-direction.

$\sigma_y$ = normal stress in y-direction.

$\tau_{xy}$ = shear stress in xy-plane.

## GENERAL EQUATIONS FOR TWO-DIMENSIONAL FLOW WITH HEAT ADDITION

The exact equations governing the flow of compressible viscous fluids with heat addition can be written as follows:

Equation of State

$$p = \rho RT \tag{1}$$

Continuity Equation

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 . \tag{2}$$

Dynamic Equations

$$\rho \left[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{3}$$

$$\rho \left[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \tag{4}$$

Energy Equation

$$\frac{DQ}{Dt} + \frac{1}{\rho} \phi = \frac{Dh}{Dt} - \frac{1}{\rho} \frac{Dp}{Dt} \tag{5}$$

In the equations above we have the following relationships:

$$\sigma_x = 2\mu \frac{\partial u}{\partial x} - \frac{2}{3} (\mu - \mu') \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - p$$

$$\sigma_y = 2\mu \frac{\partial v}{\partial y} - \frac{2}{3} (\mu - \mu') \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - p$$

$$\frac{DQ}{Dt} = \frac{1}{\rho} \left[ \frac{\partial}{\partial x} \left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y}\right) + q \right]$$

$$\phi = -\frac{2}{3}(\mu - \mu')\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]^2 + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$+ \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]^2$$

$$h = C_p T$$

$$\tau_{xy} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

Let us consider that a uniform flow is approaching a source of heat with a velocity U, in the positive x-direction. If it is now assumed that the presence of the source of heat creates only small disturbances we may write,

$$u = U + u'$$

$$v = v'$$

$$p = p^{\circ} + p'$$

$$\rho = \rho^{\circ} + \rho'$$

$$T = T^{\circ} + T'$$

where the superscript zero stands for the undisturbed conditions and the prime indicates the perturbation caused by the addition of heat.

We will now insert the new variables into equations (4) through (5), after eliminating T from equation (5) with the aid of equation (1). Neglecting second order terms we arrive at the following simplified, and linearized, equations for a steady state viscous flow with heat addition:

Continuity Equation

$$\rho^\circ \left[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right] + U \frac{\partial \rho'}{\partial x} = 0 \tag{6}$$

Dynamic Equations

$$\rho^\circ U \frac{\partial u'}{\partial x} = \mu \left[ \frac{4}{3} \frac{\partial^2 u'}{\partial x^2} + \frac{1}{3} \frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial^2 u'}{\partial y^2} \right] - \frac{\partial p'}{\partial x} \tag{7}$$

$$\rho^\circ U \frac{\partial v'}{\partial x} = \mu \left[ \frac{4}{3} \frac{\partial^2 v'}{\partial y^2} + \frac{1}{3} \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial^2 v'}{\partial x^2} \right] - \frac{\partial p'}{\partial y} \tag{8}$$

Energy Equation

$$\lambda \frac{T^\circ}{p^\circ} \left[ \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \right] - \lambda \frac{T^\circ}{\rho^\circ} \left[ \frac{\partial^2 \rho'}{\partial x^2} + \frac{\partial^2 \rho'}{\partial y^2} \right] - U \left[ \frac{Cp}{R} - 1 \right] \frac{\partial p'}{\partial x}$$

$$+ CpUT^\circ \frac{\partial \rho'}{\partial x} = - q \tag{9}$$

In deriving equations (6) through (9) a constant heat transfer coefficient has been assumed. In addition, terms containing the second viscosity coefficient $\mu'$ have been omitted as experiments to determine $\mu'$ indicate that its value is substantially less than the value of $\mu$. The omission of $\mu'$, therefore, does not materially change the problem under consideration.

We now place equations (6) through (9) in operator form. Then, treating them as four simultaneous equations involving the unknowns $u'$, $v'$, $p'$, and $\rho'$ in terms of q, we may solve for $u'$ and $v'$. We have then:

$$\frac{\partial}{\partial x}\left\{ -\frac{4\lambda\mu^2 UT^\circ}{3\rho^\circ p^\circ}\frac{\partial}{\partial x}\nabla^6 - \frac{\lambda\mu T^\circ}{\rho^\circ}\nabla^6 + \left[\frac{7\lambda\mu U^2 T^\circ}{3p^\circ} + \frac{4\mu^2 U^2}{3\rho^\circ(\gamma-1)}\right]\frac{\partial^2}{\partial x^2}\nabla^4\right.$$

$$+\left[C_p\mu UT^\circ + \lambda UT^\circ\right]\frac{\partial}{\partial x}\nabla^4 - \left[\frac{\lambda U^3\rho^\circ T^\circ}{p^\circ} + \frac{7\mu U^3}{3(\gamma-1)}\right]\frac{\partial^3}{\partial x^3}\nabla^2$$

$$\left. + \left[\frac{\rho^\circ U^4}{(\gamma-1)} - C_p U^2\rho^\circ T^\circ\right]\frac{\partial^4}{\partial x^4} - C_p U^2\rho^\circ T^\circ\frac{\partial^4}{\partial x^2\partial y^2}\right\} u'$$

$$= \frac{\partial}{\partial x}\left[\frac{\mu U}{\rho^\circ}\frac{\partial^2}{\partial x^2}\nabla^2 - U^2\frac{\partial^3}{\partial x^3}\right] q \qquad (10)$$

Similarly

$$\frac{\partial}{\partial x}\left\{ -\frac{4\lambda\mu^2 UT^\circ}{3\rho^\circ p^\circ}\frac{\partial}{\partial x}\nabla^6 - \frac{\lambda\mu T^\circ}{\rho^\circ}\nabla^6 + \left[\frac{7\lambda\mu U^2 T^\circ}{3p^\circ} + \frac{4\mu^2 U^2}{3\rho^\circ(\gamma-1)}\right]\frac{\partial^2}{\partial x^2}\nabla^4\right.$$

$$+\left[Cp\mu UT^\circ + \lambda UT^\circ\right]\frac{\partial}{\partial x}\nabla^4 - \left[\frac{\lambda U^3\rho^\circ T^\circ}{p^\circ} + \frac{7\mu U^3}{3(\gamma-1)}\right]\frac{\partial^3}{\partial x^3}\nabla^2$$

$$\left. + \left[\frac{\rho^\circ U^4}{(\gamma-1)} - CpU^2\rho^\circ T^\circ\right]\frac{\partial^4}{\partial x^4} - CpU^2\rho^\circ T^\circ\frac{\partial^4}{\partial x^2\partial y^2}\right\} v'$$

$$= \frac{\partial}{\partial x}\left[\frac{\mu U}{\rho^\circ}\frac{\partial^2}{\partial x\partial y}\nabla^2 - U^2\frac{\partial^3}{\partial x^2\partial y}\right] q \qquad (11)$$

In equations (10) and (11) the following notation has been used:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

$$\nabla^6 \equiv \frac{\partial^6}{\partial x^6} + 3\frac{\partial^6}{\partial x^4 \partial y^2} + 3\frac{\partial^6}{\partial x^2 \partial y^4} + \frac{\partial^6}{\partial y^6}$$

Equations (6), (7) or (8), (10) and (11) fully describe the flow of a viscous, heat conducting fluid past a source of heat.

## METHOD OF SOLUTION

In order to obtain solutions for equations (10) and (11) we replace u', v', and q by their respective Fourier integral representations.

$$u'(x,y) = \int_0^\infty \int_0^\infty A(\xi,\eta)\sin\xi x \cos \eta y \, d\xi d\eta \tag{12}$$

$$v'(x,y) = \int_0^\infty \int_0^\infty B(\xi,\eta)\cos \xi x \sin \eta y \, d\xi d\eta \tag{13}$$

$$q(x,y) = \int_0^\infty \int_0^\infty C(\xi,\eta)\cos \xi x \cos \eta y \, d\xi d\eta \tag{14}$$

In the resulting integral equations we can equate the coefficients of similar terms. Then if $C(\xi,\eta)$ is known, expressions for $A(\xi,\eta)$ and $B(\xi,\eta)$ are obtained. Using these expressions in equations (12) and (13) we perform the integrations indicated as the final step in the solution for u' and v'.

It is possible to represent the heat addition from a source by an impulse function. Thus if we let the total rate of heat addition be a constant Q, then

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} q(x,y)\,dxdy = Q$$

Therefore, a possible solution for the local rate of heat addition q(x,y) is

$$q(x,y) = \begin{cases} 0 \text{ when } |x| \text{ or } |y| > \epsilon \\ \\ \dfrac{Q}{4\epsilon^2} \text{ when } |x| \text{ or } |y| \leqq \epsilon \end{cases} \tag{15}$$

We also have the relationship for a Fourier integral,

$$C(\xi,\eta) = \frac{4}{\pi}\int_0^\infty \int_0^\infty q(\mu,\lambda) \cos \xi\mu \cos \eta\lambda d\mu d\lambda \tag{16}$$

Hence solving for $C(\xi,\eta)$ we have:

$$C(\xi,\eta) = \frac{4}{\pi}\int_0^\epsilon \int_0^\epsilon \frac{Q}{4\epsilon^2} \cos \xi\mu \cos \eta\lambda \, d\mu \, d\lambda$$

and

$$C(\xi,\eta) = \frac{Q}{\pi^2}$$

We now have for the Fourier integral representation of the heat added

$$q(x,y) = \frac{Q}{\pi^2}\int_0^\infty \int_0^\infty \cos x \xi \cos y \eta \, d\xi d\eta \tag{17}$$

In order to determine the forces, if any, acting on the source of heat, an integration around a convenient contour surrounding the source is performed: For the force (X) in the x-direction:

$$X = -\int_c p\,dy - \int_c \rho u(u\,dy - v\,dx) \tag{18}$$

The boundary conditions to be imposed on the linearized solution is that all disturbances must approach zero, for all values of y, at large distances in front of the heat source. Therefore

$$\lim_{x \to -\infty} \left[ u', \; v', \; p', \; \rho' \right] = 0 \tag{19}$$

## EQUATIONS FOR THE SIMPLIFIED PROBLEM OF NO VISCOSITY OR HEAT CONDUCTION

The linearized differential equations which characterize a two-dimensional flow with heat addition are greatly simplified if viscosity and heat conduction are absent. Hence, when $\mu = \lambda = 0$, equations (7) and (8) reduce to

$$\rho^{\circ} U \frac{\partial u'}{\partial x} = - \frac{\partial p'}{\partial x} \qquad (20)$$

$$\rho^{\circ} U \frac{\partial v'}{\partial x} = - \frac{\partial p'}{\partial y} \qquad (21)$$

Equation (20) may be integrated, giving

$$\rho^{\circ} U u' = - p' + f(y)$$

However, on applying the boundary conditions (19) it is evident that $u'$ and $p'$ are zero for all values of y when $x = - \infty$. Therefore we conclude that

$$f(y) \equiv 0$$

and

$$\rho^{\circ} U u' = - p' \qquad (22)$$

If equation (22) is differentiated with respect to y and the value of $\frac{\partial p'}{\partial y}$ inserted into equation (21) we have

$$\frac{\partial v'}{\partial x} = \frac{\partial u'}{\partial y} \qquad (23)$$

The flow is therefore irrotational and a velocity potential $\phi$ may be introduced. Let

$$u' = \frac{\partial \phi}{\partial x} \tag{24a}$$

$$v' = \frac{\partial \phi}{\partial y} \tag{24b}$$

Using the velocity potential the continuity equation (6) becomes

$$\rho^\circ \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right] + U \frac{\partial \rho'}{\partial x} = 0 \tag{25}$$

Returning now to the energy equation (9), we have for $\mu = \lambda = 0$

$$U\left[\frac{Cp}{R} - 1\right] \frac{\partial p'}{\partial x} - CpUT^\circ \frac{\partial \rho'}{\partial x} = q$$

or

$$\frac{1}{\gamma - 1}\left[U \frac{\partial p'}{\partial x} - a^{\circ 2} U \frac{\partial \rho'}{\partial x}\right] = q \tag{26}$$

Then, by eliminating $\frac{\partial p'}{\partial x}$ and $\frac{\partial \rho'}{\partial x}$ from equation (26) by using equations (20), (24a) and (25) we obtain

$$\frac{1}{(\gamma-1)}\left[-\rho^{\circ}U^{2}\frac{\partial^{2}\phi}{\partial x^{2}}+\rho^{\circ}a^{\circ 2}\left(\frac{\partial^{2}\phi}{\partial x^{2}}+\frac{\partial^{2}\phi}{\partial y^{2}}\right)\right]=q$$

or

$$\frac{\rho^{\circ}a^{\circ 2}}{(\gamma-1)}\left[(1-M^{\circ 2})\frac{\partial^{2}\phi}{\partial x^{2}}+\frac{\partial^{2}\phi}{\partial y^{2}}\right]=q \qquad (27)$$

where $M^{\circ}$ represents the Mach number of the undisturbed fluid.

The differential equations involving $u'(x,y)$ and $v'(x,y)$ in terms of $q(x,y)$ can be obtained from equation (27) by differentiation. Therefore

$$\frac{\rho^{\circ}a^{\circ 2}}{(\gamma-1)}\left[(1-M^{\circ 2})\frac{\partial^{2}u'}{\partial x^{2}}+\frac{\partial^{2}u'}{\partial y^{2}}\right]=\frac{\partial q}{\partial x} \qquad (28)$$

and

$$\frac{\rho^{\circ}a^{\circ 2}}{(\gamma-1)}\left[(1-M^{\circ 2})\frac{\partial^{2}v'}{\partial x^{2}}+\frac{\partial^{2}v'}{\partial y^{2}}\right]=\frac{\partial q}{\partial y} \qquad (29)$$

The solutions for $u'$ and $v'$ are obtained by applying the method outlined in Section IV of this paper.

Replacing $u'$ and $q$ in equation (28) by their Fourier integral representations, equations (12) and (17) respectively,

$$\int_{0}^{\infty}\int_{0}^{\infty}A(\xi,\eta)\left[(1-M^{\circ 2})\xi^{2}+\eta^{2}\right]\sin x\xi\cos y\eta\,d\xi\,d\eta$$

$$=\frac{(\gamma-1)Q}{\gamma p^{\circ}\pi^{2}}\int_{0}^{\infty}\int_{0}^{\infty}\xi\sin x\xi\cos y\eta\,d\xi\,d\eta$$

Equating coefficients of similar terms we find that

$$A\ (\xi,\eta) = \frac{(\gamma-1)Q}{\gamma p^{\circ}\pi^{2}} \left[ \frac{\xi}{(1-M^{\circ 2})\xi^{2}+\eta^{2}} \right]$$

Equation (28) can now be written as

$$u'(x,y) = \frac{(\gamma-1)Q}{\gamma p^{\circ}\pi^{2}} \int_{0}^{\infty}\int_{0}^{\infty} \frac{\xi \sin x\xi \cos y\eta}{(1-M^{\circ 2})\xi^{2}+\eta^{2}}\, d\xi d\eta \tag{30}$$

Similarly by replacing v' and q, in equation (29), by their respective Fourier integral representations we have,

$$\int_{0}^{\infty}\int_{0}^{\infty} B(\xi,\eta) \left[ (1-M^{\circ 2})\xi^{2} + \eta^{2} \right] \cos x\xi \sin y\eta\ d\xi d\eta$$

$$\int_{0}^{\infty}\int_{0}^{\infty} = \frac{(\gamma-1)Q}{\gamma p^{\circ}\pi^{2}} \int_{0}^{\infty}\int_{0}^{\infty} \eta \cos x\xi \sin y\eta\ d\xi d\eta$$

From which we find,

$$B(\xi,\eta) = \frac{(\gamma-1)Q}{\gamma p^{\circ}\pi^{2}} \left[ \frac{\eta}{(1-M^{\circ 2})\xi^{2}+\eta^{2}} \right]$$

Therefore equation (28) becomes

$$v'(x,y) = \frac{(\gamma-1)Q}{\gamma p^\circ \pi^2} \int_0^\infty \int_0^\infty \frac{\eta \cos x\xi \sin y\eta}{(1-M^{\circ 2})\xi^2+\eta^2} d\xi d\eta \tag{31}$$

Finally the equations corresponding to a line source of heat, at the origin, can be written as

$$u'(x,y) = \frac{(\gamma-1)Q}{\gamma p^\circ \pi^2} \int_0^\infty \cos y\eta \int_0^\infty \frac{\xi \sin x\xi}{(1-M^{\circ 2})\xi^2+\eta^2} d\eta d\xi \tag{32}$$

and

$$v'(x,y) = \frac{(\gamma-1)Q}{\gamma p^\circ \pi^2} \int_0^\infty \cos x\xi \int_0^\infty \frac{\eta \sin y\eta}{(1-M^{\circ 2})\xi^2+\eta^2} d\eta d\xi \tag{33}$$

## HEAT SOURCE IN A NON-VISCOUS, NON-HEAT CONDUCTING FLUID, $M^{\circ} < 1$

Starting with equation (32), the following integration will be performed first.

$$I_1 = \int_0^{\infty} \frac{\xi \sin x\xi}{(1-M^{\circ 2})\xi^2+\eta^2} d\xi = \frac{1}{\alpha^2} \int_0^{\infty} \frac{\xi \sin x\xi d\xi}{\xi^2+(\frac{\eta}{\alpha})^2}$$

where $\alpha^2 = (1-M^{\circ 2})$

The method of contour integration will be used to evaluate the above integral. Let,

$$I' = \frac{1}{2\alpha^2} \int_c \frac{Ze^{ixz}dZ}{\left[Z+i\frac{\eta}{\alpha}\right]\left[Z-i\frac{\eta}{\alpha}\right]}$$

$$I' = \frac{\pi i}{\alpha} \text{ Res} \left[\frac{Ze^{ixz}}{Z+i\frac{\eta}{\alpha}}\right]_{Z = i\frac{\eta}{\alpha}} \qquad x > 0$$

$$I' = \frac{\pi i}{\alpha^2} \left[\frac{e^{-\frac{x}{\alpha}\eta}}{2}\right]$$

Hence,

$$I_1 = \frac{\pi}{2\alpha^2} e^{-\frac{x}{\alpha}\eta} \qquad x > 0$$

and

$$I_1 = \frac{x}{|x|} \frac{\pi}{2\alpha^2} e^{-\frac{|x|}{\alpha}\eta} \qquad -\infty < x < \infty$$

We now have for equation (32), after the first integration with respect to $\xi$:

$$u'(x,y) = \frac{x}{|x|}\,\frac{(\gamma-1)Q}{2\gamma p^{\circ}\pi\alpha^{2}}\int_{0}^{\infty} e^{-\frac{|x|}{\alpha}\eta}\cos y\eta\, d\eta$$

Performing the final integration, with respect to $\eta$, we obtain,

$$u'(x,y) = \frac{(\gamma-1)Q}{2\pi\gamma p^{\circ}\alpha}\left[\frac{x}{x^{2}+\alpha^{2}y^{2}}\right]$$

or

$$u'(x,y) = \frac{(\gamma-1)Q}{2\pi\gamma p^{\circ}\sqrt{1-M^{\circ 2}}}\left[\frac{x}{x^{2}+(1-M^{\circ 2})y^{2}}\right] \qquad (34)$$

Equation (34) represents the velocity increment in the x-direction caused by a heat source, of strength Q, placed at the origin.

For the velocity increment in the y-direction equation (33) gives us the following integration to be performed first:

$$I_{2} = \int_{0}^{\infty} \frac{\eta \sin y\eta}{\eta^{2}+(\alpha\xi)^{2}}\, d\eta$$

where again $\alpha^{2} = (1-M^{\circ 2})$

Let

$$I' = \frac{1}{2}\int_c \frac{ze^{iyz}dz}{[z+i\alpha\xi][z-i\alpha\xi]}$$

$$= \pi i \text{ Res.} \left[\frac{ze^{iyz}}{z+i\alpha\xi}\right]_{z=i\alpha\xi} \qquad y > 0$$

$$I' = \pi i \left[\frac{e^{-\alpha y\xi}}{2}\right]$$

We have then

$$I_2 = \frac{\pi}{2} e^{-\alpha y\xi} \qquad y > 0$$

and

$$I_2 = \frac{y}{|y|}\frac{\pi}{2} e^{-\alpha|y|\xi} \qquad -\infty < y < \infty$$

We now have for equation (33) after the first integration with respect to $\eta$:

$$v'(x,y) = \frac{y}{|y|}\frac{(\gamma-1)Q}{2\pi\gamma p^\circ}\int_0^\infty e^{-\alpha|y|\xi} \cos x\xi \, d\xi$$

Performing the final integration with, respect to $\xi$, we obtain:

$$v' \ (x,y) = \frac{(\gamma-1)\alpha Q}{2\pi\gamma p^{\circ}} \left[ \frac{y}{x^2+\alpha^2 y^2} \right]$$

or

$$v'(x,y) = \frac{(\gamma-1)\sqrt{1-M^{\circ 2}}\,Q}{2\pi\gamma p^{\circ}} \left[ \frac{y}{x^2+(1-M^{\circ 2})y^2} \right] \qquad (35)$$

Equation (35) represents the velocity increment in the y-direction, caused by a heat source, of strength Q, placed at the origin.

Using the relationship expressed in equation (22), the pressure perturbation is found to be

$$p'(x,y) = -\frac{(\gamma-1)M^{\circ}Q}{2\pi a^{\circ}\sqrt{1-M^{\circ 2}}} \left[ \frac{x}{x^2+(1-M^{\circ 2})y^2} \right] \qquad (36)$$

In deriving an expression for $\rho'(x,y)$ we return to the linearized energy equation (26), which may be written as

$$-\frac{1}{(\gamma-1)} \frac{\partial}{\partial x} \left[ \rho^{\circ}U^2 u' + a^{\circ 2} U \rho' \right] = q$$

Or integrating both sides with respect to x we have,

$$\rho' \ (x,y) = -\frac{\rho^{\circ}M^{\circ}}{a^{\circ}} u' - \frac{(\gamma-1)}{a^{\circ 2}U} \int_{-\infty}^{x} q \ dx$$

Eliminating $u'$ with equation (34), we have finally,

$$\rho'(x,y) = -\frac{(\gamma-1)M^{\circ}Q}{2\pi a^{\circ 3}\sqrt{1-M^{\circ 2}}} \left[ \frac{x}{x^2+(1-M^{\circ 2})y^2} \right] - \frac{(\gamma-1)}{a^{\circ 2}U} \int_{-\infty}^{x} q \ dx \qquad (37)$$

where

$$\int_{-\infty}^{x} q\,dx = \begin{cases} 0 \text{ for } x < 0 \\ \delta(y)\cdot 1(x)Q \text{ for } x \geqq 0 \end{cases}$$

From equation (18) we obtain the following relationship for the force X, on the heat source, in the x-direction, for a linearized solution,

$$X = -\int_c (p^\circ + p')dy - \int_c (\rho^\circ + \rho')(U+u')\left[ (U+u')dy - v'dx \right]$$

or

$$X = -\int_c p'dy - 2\rho^\circ U \int_c u'dy + \rho^\circ U \int_c v'dx - U^2 \int_c \rho' dy \tag{38}$$

Eliminating $p'$, $u'$, $v'$ and $\rho'$ with equations (34), (35), (36) and (37) we have for equation (38),

$$X = \frac{(\gamma-1)M^\circ Q}{2\pi a^\circ \sqrt{1-M^{\circ 2}}} \int_c \frac{x\,dy}{x^2+(1-M^{\circ 2})y^2} - \frac{(\gamma-1)M^\circ Q}{\pi a^\circ \sqrt{1-M^{\circ 2}}} \int_c \frac{x\,dy}{x^2+(1-M^{\circ 2})y^2}$$

$$+ \frac{(\gamma-1)M^\circ \sqrt{1-M^{\circ 2}}\,Q}{2\pi a^\circ} \int_c \frac{y\,dx}{x^2+(1-M^{\circ 2})y^2} + \frac{(\gamma-1)M^{\circ 3} Q}{2\pi a^\circ \sqrt{1-M^{\circ 2}}} \int_c \frac{x\,dy}{x^2+(1-M^{\circ 2})y^2}$$

$$+ \frac{(\gamma-1)M^\circ}{a^\circ} \int_c \delta(y)\cdot 1(x)Q\,dy$$

Or, after combining terms

$$X = \frac{(\gamma-1)M^\circ Q}{2\pi a^\circ\sqrt{1-M^{\circ 2}}}\left[M^{\circ 2} - 1\right]\int_c \frac{x\ dy}{x^2+(1-M^{\circ 2})y^2}$$

$$+ \frac{(\gamma-1)M^\circ\sqrt{1-M^{\circ 2}}\ Q}{2\pi a^\circ}\int_c \frac{y\ dx}{x^2+(1-M^{\circ 2})y^2} + \frac{(\gamma-1)M^\circ}{\alpha^\circ}\int_c \delta(y)\cdot 1(x)Q dy \qquad (39)$$

The integration indicated, to find X, will be greatly simplified if an ellipse with the following characteristics is choosen for the contour;

$$x = R\cos\theta \qquad\qquad 0 \leqq \theta \leqq 2\pi$$
$$\sqrt{1-M^{\circ 2}}\ y = R\sin\theta$$

Hence,

$$dx = -R\cos\theta\ d\theta$$
$$dy = \frac{R\sin\theta}{\sqrt{1-M^{\circ 2}}}\ d\theta$$

Using these relationships we have, for equation (39)

$$X = -\frac{(\gamma-1)M^\circ Q}{2\pi a^\circ}\int_0^{2\pi} \cos^2\theta\ d\theta - \frac{(\gamma-1)M^\circ Q}{2\pi a^\circ}\int_0^{2\pi} \sin^2\theta\ d\theta$$

$$+ \frac{(\gamma-1)M^\circ}{a^\circ}\int_c \delta(y)\cdot 1(x)Q\ dy$$

and

$$X = -\frac{(\gamma-1)M^\circ}{a^\circ}Q + \frac{(\gamma-1)M^\circ}{a^\circ}\int_c \delta(y)\cdot 1(x)Q\ dy$$

However, in going around the contour the integral

$$1(x)Q \int_c \delta(y)\, dy$$

has a value only when $x \geq 0$ and $y = 0$, as

$$\operatorname*{Limit}_{\eta \to 0} \int_{-\eta}^{\eta} \delta(y)\, dy = 1 \quad \text{and} \quad 1(x)Q = Q \text{ for } 0 \leq x$$

Therefore,

$$\int_c \delta(y) \cdot 1(x)\, Q\, dy = Q$$

and

$$X = 0 \qquad \text{for } M^\circ < 1 \tag{40}$$

We have, then, arrived at the following conclusion. If a line source of heat is placed in a non-viscous, non-heat conducting fluid, whose Mach number in the undisturbed fluid is less than unity, no force will be experienced by the source.

## HEAT SOURCE IN A NON-VISCOUS, NON-HEAT CONDUCTING FLUID, $M° > 1$

Before we attempt the solution for the case where $M° > 1$, it will be well to examine the significance of the following integral

$$\delta\ (x) = \frac{1}{\pi}\int_o^\infty \cos x\eta\ d\eta$$

Suppose that we have an impulse function with the following characteristics.

$$\delta\ (x) = 0 \qquad \text{when} \quad |x| > \epsilon$$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \delta\ (x)\ dy = 1$$

A possible solution is, $\delta\ (x) = \dfrac{1}{2\epsilon}$, as

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon}\ dx = 1$$

Now, let this solution be represented by a Fourier integral:

$$\delta\ (x) = \int_o^\infty a(x) \cos x\eta\ d\eta$$

where

$$a\ (x) = \frac{2}{\pi}\int_o^\infty \delta\ (x) \cos \eta x\ dx$$

But $\delta\ (x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}$ between zero and $\epsilon$ and is zero elsewhere.

Hence

$$a\ (x) = \lim_{\epsilon \to 0} \frac{2}{\pi} \int_0^{\epsilon} \frac{\cos \eta x}{2\epsilon}\ dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\sin \eta\epsilon}{\eta\epsilon}$$

$$a(x) = \frac{1}{\pi}$$

Therefore the Fourier integral representation of the impulse function is:

$$\delta\ (x) = \frac{1}{\pi} \int_0^{\infty} \cos x\eta\ d\eta \qquad (41)$$

Returning now to the solution for $M^{\circ} > 1$, we find that equation (30) may be written as:

$$u' = \frac{(\gamma-1)Q}{\gamma p^{\circ} \pi^2} \int_0^{\infty} \xi \sin x\xi \int_0^{\infty} \frac{\cos y\eta}{\eta^2 - (\beta\xi)^2}\ d\eta d\xi \qquad (42)$$

where $\beta^2 = (M^{\circ 2} - 1)$

Once again the method of contour integration will be used to evaluate the following integral

$$I_3 = \int_0^{\infty} \frac{\cos y\eta}{\eta^2 - (\beta\xi)^2} \, d\eta$$

Let,

$$I' = ½ \int_c \frac{e^{iyz} \, dz}{[z+\beta\xi] \; [z-\beta\xi]}$$

$$I' = \frac{\pi i}{2} \text{ Res } \left[ \frac{e^{eyz}}{[z \pm \beta\xi]} \right]_{z = \pm\beta\xi}$$

$$I_3 = -\frac{\pi}{2i} \left[ \frac{e^{i\beta y\xi} - e^{-i\beta y\xi}}{2\beta\xi} \right]$$

Finally

$$I_3 = -\frac{\pi \sin \beta y\xi}{2\beta\xi} \tag{43}$$

Therefore, equation (42) becomes after the integration with respect to $\eta$:

$$u'(x,y) = -\frac{(\gamma-1)Q}{2\xi p^{\circ}\pi\beta} \int_0^{\infty} \sin x\,\xi \, \sin \beta y\,\xi \, d\xi \tag{44}$$

An examination of equation (44) reveals that it represents an odd function in x. However, the character of the flow under consideration ($M^\circ > 1$) is such that a disturbance in the fluid is not propagated upstream from the point of disturbance. It is therefore necessary, in order to satisfy this boundary condition, to add a particular solution of the form $\cos x\xi \cos y\eta$. We therefore have for equation (44), after adding a term of the form $\cos x\xi \cos \beta y\xi$,

$$u'(x,y) = -\frac{(\gamma-1)Q}{2\gamma p^\circ \pi\beta}\int_0^\infty \cos (x-\beta y)\xi \, d\xi \tag{45}$$

But, as has been shown, the integral in equation (45) represents an impulse function.

$$\delta(x-\beta y) = \frac{1}{\pi}\int_0^\infty \cos(x-\beta y)\xi \, d\xi$$

Therefore

$$u'(x,y) = -\frac{(\gamma-1)Q}{2\gamma p^\circ\sqrt{M^{\circ 2}-1}} \delta(x-\sqrt{M^{\circ 2}-1}\, y) \tag{46}$$

To solve for $v'(x,y)$ equation (33) is rewritten as,

$$v'(x,y) = -\frac{(\gamma-1)Q}{\gamma p^\circ \pi^2 \beta^2}\int_0^\infty \eta \sin y\eta \int_0^\infty \frac{\cos x\xi}{\xi^2-(\frac{\eta}{\beta})^2} d\xi \, d\eta \tag{47}$$

Applying the method of contour integration to the following integral, we have:

$$I_4 = \int_0^\infty \frac{\cos x\xi}{\xi^2 - (\frac{\eta}{\beta})^2} d\xi$$

Let

$$I' = \frac{1}{2}\int_c \frac{e^{ixz}dz}{(z-\frac{\eta}{-\beta})(z-\frac{\eta}{\beta})}$$

$$I' = \frac{\pi i}{2} \text{ Res} \left[ \frac{e^{ixz}}{[z \pm \frac{\eta}{\beta}]} \right]_{z = \pm\frac{\eta}{\beta}}$$

$$I' = -\frac{\pi i}{2} \left[ \frac{e^{i\frac{x}{\beta}\eta} - e^{-i\frac{x}{\beta}\eta}}{2\frac{\eta}{\beta}} \right]$$

Hence

$$I_4 = -\frac{\pi \sin \frac{x}{\beta}\eta}{2\frac{\eta}{\beta}} \qquad (48)$$

Therefore equation (47) becomes after the integration with respect to $\xi$:

$$v'(x,y) = \frac{(\gamma-1)Q}{2\gamma p^\circ \pi\beta} \int_0^\infty \sin \frac{x}{\beta}\eta \sin y\eta \, d\eta \qquad (49)$$

Once again we must add a term to satisfy the condition of no disturbance propagation upstream from the heat source. After adding a term of the form $\cos \frac{x}{\beta} \eta \cos y\eta$ we have for equation (49):

$$v'(x,y) = \frac{(\gamma-1)Q}{2\gamma p^\circ \pi \beta} \int_0^\infty \cos\left(\frac{x}{\beta} - y\right) \eta \, d\eta$$

Changing variables so that $\eta = \beta\eta'$ we have:

$$v'(x,y) = \frac{(\gamma-1)Q}{2\gamma p^\circ \pi} \int_0^\infty \cos\,(x-\beta y)\eta' \, d\eta' \tag{50}$$

Once again the integral in equation (50) is recognized as the Fourier representation of an impulse function. We have then,

$$v'(x,y) = \frac{(\gamma-1)Q}{2\gamma p^\circ} \; \delta(x-\sqrt{M^{\circ 2}-1}\; y) \tag{51}$$

To find $p'(x,y)$ we return to equation (22). Eliminating $u'$ from (22) and (46),

$$p'(x,y) = \frac{(\gamma-1)M^\circ Q}{2a^\circ\sqrt{M^{\circ 2}-1}} \; \delta(x-\sqrt{M^{\circ 2}-1}\; y) \tag{52}$$

The relationship for $\rho'(x,y)$ is agin found by rewriting equation (26) as,

$$\rho'(x,y) = -\frac{\rho^\circ M^\circ}{a^\circ} u' - \frac{(\gamma-1)}{a^{\circ 2} U} \int_{-\infty}^{x} q \, dx$$

Hence

$$\rho' \ (x,y) = \frac{(\gamma-1)M^oQ}{2a^{o2}\sqrt{M^{o2}-1}} \ \delta(x-\sqrt{M^{o2}-1}\ y) - \frac{(\gamma-1)}{a^{o2}U}\int_{-\infty}^{x} q\ dx \qquad (53)$$

where

$$\int_{-\infty}^{x} q\ dx = \begin{cases} 0 \text{ for } x < 0 \\ \delta(y)\cdot 1(x)Q \text{ for } x \geq 0 \end{cases}$$

The impulse function $\delta\ (x-\sqrt{M^{o2}-1}\ y)$ is of special significance as it indicates that the entire disturbance can be considered to occur in the vicinity of a straight line determined by

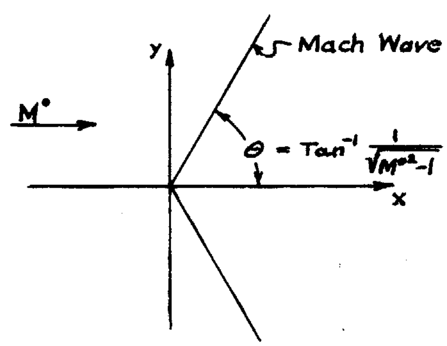
$$x - \sqrt{M^{o2}-1}\ y = 0$$

or

$$\frac{y}{x} = \frac{1}{\sqrt{M^{o2}-1}} \qquad (54)$$

If we examine the characteristics of a Mach wave originating at the origin of coordinates we find that:

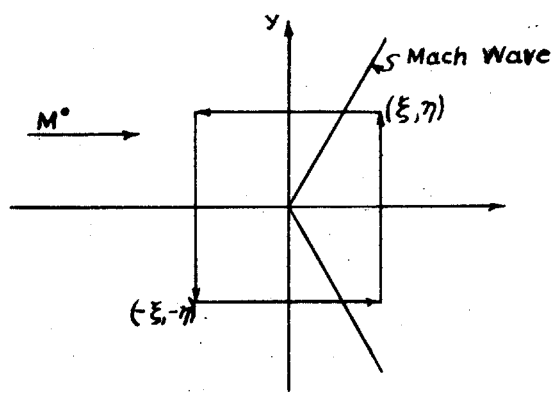
$$\text{Tan}\ \theta = \frac{1}{\sqrt{M^{o2}-1}} = \frac{y}{x}$$




Hence we note that equation (58) is the relationship determining the position of a Mach wave in a supersonic stream.

The disturbance caused by a heat source in a fluid with, $M^{\circ} > 1$, is therefore zero everywhere except on a Mach line originating at the source. Along the Mach line the changes from free stream conditions appear as impulses.

To determine the forces acting on the heat source in the supersonic stream we again apply equation (38). However, as a change from free stream conditions occures only on the Mach line it will be found convenient to integrate around the rectangular contour shown in the sketch below.




$$X = -\int_c p' \, dy - \int_c \left[ (2\rho^{\circ}Uu' + U^2\rho') \, dy - \rho Uv' dx \right]$$

By choosing the above path of integration we find that the only contributions to X come from the following terms

$$X = -U^2 \int_{-\eta}^{\eta} \rho' \, dy - 2\rho^{\circ}U \int_{-\xi}^{\xi} v' dx$$

The quantities p' and u', having values only along the Mach line, will not contribute to X. However $\rho'$, will contribute to X at the point $(\xi, 0)$ because of the term involving, $\delta(y) \cdot 1(x)Q$. We therefore have

$$X = \frac{(\gamma-1)M^{\circ}}{a^{\circ}} \int_{-\eta}^{\eta} \delta(y) \cdot 1(x) Q \, dy - \frac{(\gamma-1)M^{\circ}Q}{a^{\circ}} \int_{-\xi}^{\xi} \delta \left(x - \sqrt{M^{\circ 2} - 1}\, y\right) dx$$

But by definition the integral of the impulse function $\delta(x-\sqrt{M^{\circ 2}-1}\ y)$ with respect to x is equal to unity. Also as we have shown previously

$$\int_c \delta(y)\cdot 1(x)Q\ dy = Q \qquad 0 \leq x$$

Hence

$$X = 0 \qquad M^\circ > 1 \tag{55}$$

SUMMARY OF EQUATIONS

$M^\circ < 1$

$$u' = \frac{(\gamma-1)Q}{2\pi\gamma p^\circ\sqrt{1-M^{\circ 2}}}\left[\frac{x}{x^2+(1-M^{\circ 2})y^2}\right]$$

$$v' = \frac{(\gamma-1)\sqrt{1-M^{\circ 2}}\,Q}{2\pi\gamma p^\circ}\left[\frac{y}{x^2+(1-M^{\circ 2})y^2}\right]$$

$$p' = -\frac{(\gamma-1)M^\circ Q}{2\pi a^\circ\sqrt{1-M^{\circ 2}}}\left[\frac{x}{x^2+(1-M^{\circ 2})y^2}\right]$$

$$\rho' = -\frac{(\gamma-1)M^\circ Q}{2\pi a^{\circ 3}\sqrt{1-M^{\circ 2}}}\left[\frac{x}{x^2+(1-M^{\circ 2})y^2}\right] - \frac{(\gamma-1)}{a^{\circ 2}U}\int_{-\infty}^{x} q\, dx$$

$X = 0$

SUMMARY OF EQUATIONS

$$M^\circ > 1$$
$$x > 0$$

$$u' = \frac{(\gamma-1)Q}{2\gamma p^\circ\sqrt{M^{\circ 2}-1}} \left[\delta(x-\sqrt{M^{\circ 2}-1}\ y\right]$$

$$v' = \frac{(\gamma-1)Q}{2\gamma p^\circ} \left[\delta(x-\sqrt{M^{\circ 2}-1}\ y)\right]$$

$$p' = \frac{(\gamma-1)M^\circ Q}{2a^\circ\sqrt{M^{\circ 2}-1}} \left[\delta(x-\sqrt{M^{\circ 2}-1}\ y)\right]$$

$$\rho' = \frac{(\gamma-1)M^\circ Q}{2a^{\circ 3}\sqrt{M^{\circ 2}-1}} \left[\delta(x-\sqrt{M^{\circ 2}-1}\ y)\right] - \frac{(\gamma-1)}{a^{\circ 2}U}\int_{-\infty}^{x} q\ dx$$

$$X = 0$$

$$\text{where} \int_{-\infty}^{x} q\ dx = \begin{cases} 0 \text{ for } x < 0 \\ \delta(y)\cdot 1(x)Q \text{ for } x \geqq 0 \end{cases}$$

## CONCLUSIONS

The results of this thesis indicate that there are no forces acting on a heat source placed in a non-viscous, non-heat conducting, compressible fluid. The lack of such forces has been demonstrated for both subsonic and supersonic free-stream velocities.