ON THE GENERAL THEORY OF MICROWAVE INTERACTIONS WITH ELLIPSOIDAL FERRIMAGNETIC INSULATORS

by

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ABSTRACT

In this thesis are formulated the general differential equations of motion for the magnetization in a small ferrimagnetic ellipsoid magnetized in an arbitrary direction and excited by spatially uniform microwave magnetic fields of arbitrary frequencies and directions. The magnetization is assumed to consist of the uniform precession plus a typical small-amplitude spin wave.

At first the spin wave terms are neglected and various solutions obtained. The approach used is to first assume that all applied microwave fields are zero but that at some initial instant the magnetization is nonparallel to the internal field. The resulting transient of the magnetization is obtained and yields the natural precession frequency as well as information about modulation products that are created by the internal modulating fields. The forced or steady-state solutions when various driving fields are applied are next obtained for both linear responses, in which the magnetization precesses chiefly at the driving frequency, and nonlinear responses, in which it does not. A mechanism of second harmonic parametric coupling is analyzed, which is strongly dependent on sample shape and the minimum threshold is given.

The spin wave terms are now considered and various quantities of interest derived. The generalized spin wave spectrum is found together with the first and second order instability thresholds beyond which certain spin waves go unstable. These thresholds are obtained for both linear and nonlinear responses. Direct parametric coupling between microwave driving fields and certain spin waves is shown to exist. A method is described by which it appears feasible to selectively drive any spin wave that is degenerate with the uniform precession. This would provide a means of measuring the spin wave line width, $2\,\Delta H_k$, as a function of wave number k without changing sample geometry, saturation magnetization and/or frequency.

The transient build up and decay of the uniform precession, in response to a pulsed microwave driving field, is discussed as well as the dynamics of spin wave interaction. In particular it is shown that if the magnetization of a spheroid could suddenly be inverted with respect to the magnetizing field, certain cut-off spin waves would grow very rapidly at the expense of the uniform precession. The spin wave spectrums, appropriate to succeeding stages of the ensuing transient, are derived using a quasi-static approximation and they give at least a qualitative picture of the highly nonlinear loss mechanism. The position of the uniform precession, relative to the spin wave manifold during transient conditions, is also studied.

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F.R.M.

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INTRODUCTION

It is the intent of this thesis to establish a broad theoretical framework suitable for treating many of the linear and nonlinear interaction problems arising in a small ferrimagnetic ellipsoid in a relatively simple and straightforward manner with emphasis on the physical picture. The fundamental principle underlying practically all microwave devices involving ferromagnetic interaction is that of ferromagnetic resonance. In ferrimagnetic insulators, the basic mechanism of resonance is well understood but there are some sophisticated aspects of the problem, generally of nonlinear character that are only beginning to be fully appreciated. These manifest themselves in the realms of parametric amplification and oscillation, harmonic and subharmonic generation, and ferromagnetic resonance loss mechanisms.

The magnetization within a ferrimagnetic solid is made up of an ensemble of magnetic moments, arising from individual electron spins, which are coupled together quantum-mechanically. Their positions are subject to statistical variations and it is customary to resolve these normally small fluctuations into a Fourier expansion both in space and time coordinates and to call an individual member of the expansion a spin wave. The conventional picture of resonance neglects these fluctuations and, for small enough samples, it assumes that all of the magnetic moments are in phase -- cooperating to produce a rigid magnetization vector. A microwave field of the correct form and frequency can drive this vector into resonance, under which conditions the configuration of the magnetization is termed the uniform precession. Nonlinearities complicate the usual small signal theory and arise, broadly, from two sources. First, the equations of motion governing the uniform precession are themselves nonlinear, and, as expected, predict harmonic generation and frequency mixing. Second, certain of the neglected spin wave states are potentially unstable at high enough microwave signal levels, and must therefore be taken into account.

In an effort to simplify the problem, the following approach is utilized. The various components, which go to make up the total internal field within the ellipsoid, are enumerated and discussed. These fields are then resolved into appropriate sets of rotating spherical coordinate components. This is done for the sake of mathematical simplicity and because the physics can be followed rather easily from the resulting geometrical representation of the field vectors. In connection with this, it is worth noting that almost all authors, who deal with elementary expositions of ferromagnetic resonance, immediately point out the analogy of the subject with that of gyroscopic motion. Now almost all texts on the latter subject make use of spherical coordinates and it would appear potentially profitable to do so in the case of ferromagnetic resonance. This has usually not been done, however, for no discernable reason save perhaps one. The concept of demagnetizing factors arises inevitably when an ellipsoid is considered and these quantities are defined in terms of the three principal ellipsoid axes. This fact has undoubtedly influenced the use of cartesian coordinates in such problems.

There is another departure in this thesis from the usual formulation. Although the magnetization and the related field quantities are expanded into a Fourier series of terms in so far as their spatial variation is concerned, the time dependence is not -- at least at the outset. There is a strong temptation to do this for there is a tendency to assume that the nonlinear problem will yield only to an expansion in various orders of the small signal or linearized solution. That this is all too often the case is unfortunate but in this particular problem the differential equations (or close approximations to them) may be integrated directly in many instances.

No major changes of variables occur in the course of solution. All of the formulation is done in terms of the amplitudes and phases of the various quantities of interest -- an obvious advantage when it comes to the physical interpretation of the results.

In Chapter 1, there are derived the general differential equations of motion for the uniform component of magnetization in a small ferrimagnetic ellipsoid magnetized in an arbitrary direction and excited by spatially uniform microwave magnetic fields of arbitrary frequencies and directions. Following this, various solutions are obtained. The approach used is to first assume that all applied microwave fields are zero but that at some initial instant the magnetization is nonparallel to the internal field. The resulting transient of the magnetization is obtained and yields the natural precession frequency as well as information about modulation products that are created by the internal modulating fields. The forced or steady-state solutions when various driving fields are applied are next obtained for both linear responses, in which the magnetization precesses chiefly at the driving frequency, and nonlinear responses, in which it does not. A mechanism of second harmonic parametric coupling is analyzed, which is strongly dependent on sample shape and the minimum threshold is given.

In Chapter 2, the magnetization is assumed to consist of the uniform precession together with a set of spin waves. Since the spin wave amplitudes are normally small, products involving two of them are negligible compared to products of the uniform precession and one of them. This implies that we need consider only a single, typical spin wave. In some cases, this is not sufficient, but for the sake of clarity and simplicity only one such component is considered here. After the basic set of four coupled differential equations is formulated (for the amplitudes and phases of both uniform precession and spin wave) various quantities of interest are derived. The generalized spin wave spectrum is found together with the first and second order instability thresholds beyond which certain spin waves go unstable. These thresholds are obtained for both linear responses, in which the magnetization precesses chiefly at the driving frequency, and nonlinear responses, in which it does not. Direct parametric coupling between microwave driving fields and certain spin waves is shown to exist. A method is described by which it appears feasible to selectively drive any spin wave that is degenerate with the uniform precession. This would provide a means of measuring the spin wave line width, $2\,\Delta H_k$, as a function of wave number k without changing sample geometry, saturation magnetization and/or frequency.

Chapter 3 is devoted to non steady-state aspects of the magnetization and in it are discussed the transient behavior of the uniform precession and the spin waves together with their mutual interaction. The build up and decay of the uniform precession, which follows the switching on or off of a microwave driving field, is considered as well as the dynamic behavior of the spin wave spectrum during certain transients involving sudden changes in the magnetizing field. Those spin waves, which are found to be cut off, are analyzed in some detail.*

*Material abstracted from this thesis has been published in a very brief survey paper. See Reference 1.

CHAPTER 1

THE UNIFORM PRECESSION

1.1. The Basic Equations. A ferrimagnetic ellipsoid is assumed to have its principal axes along x, y, and z as shown in Fig. 1-1. The total internal dc magnetizing field is assumed to lie along the z' axis so that in the absence of any excitation the magnetization vector $\vec{M}$ of the material is also along z'. Demagnetizing factors N_x, N_y, and N_z (whose sum is unity) for the principal directions can be calculated from the physical shape of the ellipsoid in the usual manner,[2] and are assumed known. The externally applied dc magnetic field must in general be nonparallel to the magnetization. In fact it may be shown that (referring to Fig. 1-1) the applied field will be of the form

$$\begin{aligned}\vec{H}_{applied} = \; & \vec{i}' \; \frac{N_x - N_y}{2} M_{z'} \sin\alpha \sin 2\beta \\ & + \vec{j}' \; \frac{N_x \cos^2\beta + N_y \sin^2\beta - N_z}{2} M_{z'} \sin 2\alpha \\ & + \vec{k}' \; H_o\end{aligned} \qquad (1\text{-}1)$$

where $\vec{i}'$, $\vec{j}'$, and $\vec{k}'$, are unit vectors in the primed coordinate system. $M_{z'}$ is the dc component of $\vec{M}$ along z', and H_o is any value of magnetic field sufficient to magnetize the sample. The transverse components of the applied field are needed to cancel out similar components of the demagnetizing field. They are zero only when the ellipsoid is magnetized along a principal axis.

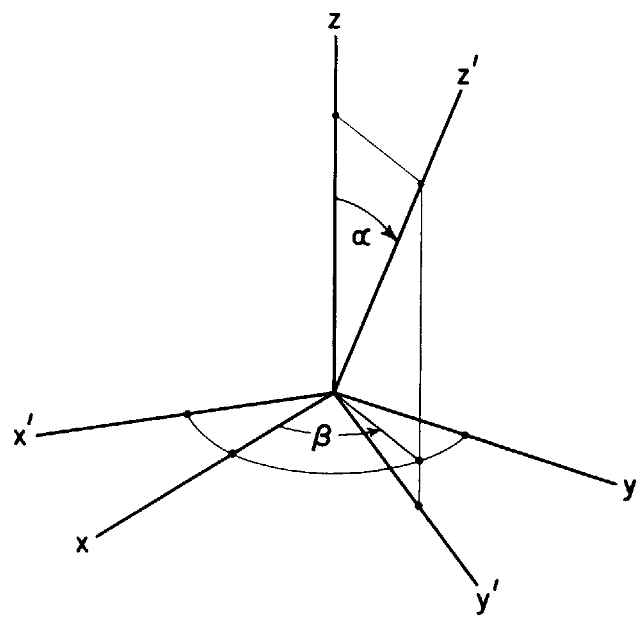

Fig. 1-1

The equation governing the motion of the magnetization is (in mks units)

$$\frac{d\vec{M}}{dt} = \gamma\mu_0 \left(\vec{M} \times \vec{H}\right) + \text{damping term} \tag{1-2}$$

where γ , a negative quantity, is the gyromagnetic ratio that includes g factor, and μ_0 is the permeability of free space; $\vec{H}$ is the internal magnetic field.

The actual form of the magnetization vector depends on the exact boundary conditions and the spatial and temporal dependence of the applied fields, as well as the statistical distribution of spin wave excitation from lattice vibrations, collisions etc. Here we assume, however, that the applied r-f fields are spatially uniform, and that the magnetization is essentially the same, with the exception of small-amplitude wave disturbances. These are neglected for the time being but will be considered in Chapter 2. In terms of primed coordinates, the magnetization vector is

$$\vec{M} = \vec{i}\,' M \sin\theta \cos\phi + \vec{j}\,' M \sin\theta \sin\phi + \vec{k}\,' M \cos\theta \tag{1-3}$$

The total internal magnetic field $\vec{H}$ will consist of the applied field (including time-varying as well as dc components), and the demagnetizing field, due to surface magnetic dipoles. The latter involves specific boundary conditions, accounted for in the values of N_x, N_y, and N_z. This field is given by

$$\vec{H}_{demagnetizing} = -\left(\vec{i}\,' N_x \overline{M}_x + \vec{j}\,' N_y \overline{M}_y + \vec{k}\,' N_z \overline{M}_z\right) \tag{1-4}$$

(in unprimed coordinates) where the bars denote spatial averages. Note that the demagnetizing factors, which evolve strictly from static considerations, have been applied to the case where M_x, M_y, and M_z include time-varying components that are spatially uniform. This magnetostatic formulation, which is permissible as long as the ellipsoid dimensions are small compared to any appropriate wave length, is the usual approximation.[3] By transforming the magnetization into M_x, M_y, and M_z components, multiplying respectively by $-N_x$, $-N_y$, and $-N_z$ and then transforming back into primed coordinates, one has the demagnetizing field expressed in x', y', and z' components. The transformations appropriate to Fig. 1-1 are

$$\begin{aligned} x &= x' \sin\beta + y' \cos\alpha \cos\beta + z' \sin\alpha \cos\beta \\ y &= -x' \cos\beta + y' \cos\alpha \sin\beta + z' \sin\alpha \sin\beta \\ z &= - y' \sin\alpha + z' \cos\alpha \end{aligned} \tag{1-5}$$

and

$$\begin{aligned} x' &= x \sin\beta - y \cos\beta \\ y' &= x \cos\alpha \cos\beta + y \cos\alpha \sin\beta - z \sin\alpha \\ z' &= x \sin\alpha \cos\beta + y \sin\alpha \sin\beta + z \cos\alpha \end{aligned} \tag{1-6}$$

The dc components along x' and y' just cancel the corresponding components of the applied dc field, and leave the net internal dc field along z'. This is given by (1-1) where $M_{z'} = M \cos\theta$. For small values of the cone angle, the $\cos\theta$ is approximately unity.

It is convenient to go one step further and express the transverse demagnetizing fields in terms of components that are parallel (H_1) and perpendicular (H_1^*) to the transverse magnetization as shown in Fig. 1-2. When the transformations are actually carried out

$$(H_1)_{demag.} = (A + B\cos 2\phi + W \sin 2\phi) M \sin\theta \tag{1-7}$$

$$(H_1^*)_{demag.} = (B \sin 2\phi - W \cos 2\phi) M \sin\theta \tag{1-8}$$

where

$$2A = N_x(\sin^2\beta + \cos^2\alpha \cos^2\beta) + N_y(\cos^2\beta + \cos^2\alpha \sin^2\beta) + N_z \sin^2\alpha$$

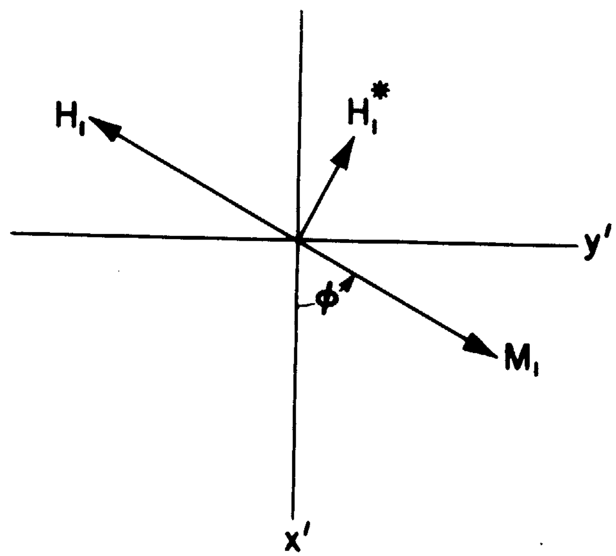

Fig. 1-2

$$2B = N_x(\sin^2\beta - \cos^2\alpha\cos^2\beta) + N_y(\cos^2\beta - \cos^2\alpha\sin^2\beta) - N_z\sin^2\alpha$$

$$2W = (N_x - N_y)\cos\alpha\sin 2\beta$$

In addition, the resultant magnetic field component along z' is

$$(H_{z'})_{\substack{\text{applied dc} \\ \text{plus demag.}}} = H_o - ZM\cos\theta - CM\sin\theta\cos\phi - DM\sin\theta\sin\phi \tag{1-9}$$

where

$$Z = (N_x\cos^2\beta + N_y\sin^2\beta)\sin^2\alpha + N_z\cos^2\alpha$$

$$C = \frac{N_x - N_y}{2}\sin\alpha\sin 2\beta$$

$$D = (N_x\cos^2\beta + N_y\sin^2\beta - N_z)\frac{\sin 2\alpha}{2}$$

Notice that if the magnetizing field coincides with a principal axis of the ellipsoid, the time-varying components of the z' demagnetizing field vanish. Previous analyses were restricted to such geometries, and certain interesting solutions were therefore missed.

Various damping terms, which are phenomenological and roughly equivalent to one another, may be used in (1-2). We choose a Landau-Lifshitz term which is proportional to $\vec{M} \times (\vec{M} \times \vec{H})$; hence, (1-2) may be written

$$\frac{d\vec{M}}{dt} = \gamma\mu_o\left[\vec{M} \times (\vec{H} + \vec{H}_{damping})\right] = \gamma\mu_o(\vec{M} \times \vec{H}_t) \tag{1-10}$$

where $\vec{H}_{damp.} = \frac{-\gamma\mu_o\Delta H_o}{M\omega}(\vec{M} \times \vec{H})$, ω is the precession frequency, and ΔH_o is half the ferromagnetic resonance line width. In general, the vector $\vec{H}$ will consist of the applied dc magnetic field,* the demagnetizing field and applied

*Additional components will arise from the fields due to anisotropy, inhomogeneities, etc. These are neglected in this thesis but may be treated in a similar way.

time-varying fields. Since the latter will be very small compared to the other two fields, we can approximate $\vec{H}_{damping}$ by $\frac{\Delta H_o}{MH_{z'}}(\vec{M} \times \vec{H}_{z'})$. This field has a component perpendicular to the transverse magnetization given by

$$(H_1^*)_{damp.} = -\Delta H_o \sin\theta \tag{1-11}$$

Any applied r-f magnetic fields are assumed to be broken up into circularly-polarized components in the transverse plane and into linear components along the z' axis. A typical set of components is

$$\vec{h}_{rf} = \vec{i}' h_i \cos(\omega_i t + \alpha_i) + \vec{j}' h_i \sin(\omega_i t + \alpha_i) + \vec{K}' h_{z_j} \sin(\omega_j t + \beta_j) \tag{1-12}$$

where positive and negative values of ω_i represent, respectively, positive and negative cicular polarization.

The complete expressions for the magnitudes of the various vectors, due to all the sources enumerated above (including the applied microwave fields resolved into H_1, H_1^*, and $H_{z'}$ components), are

$$M_1 = M\sin\theta \tag{1-13}$$

$$M_{z'} = M\cos\theta \tag{1-14}$$

$$H_1 = (A + B\cos 2\phi + W\sin 2\phi)M\sin\theta - \sum_i h_i \cos(\omega_i t - \phi + \alpha_i) \tag{1-15}$$

$$H_1^* = (B\sin 2\phi - W\cos 2\phi)M\sin\theta + \sum_i h_i \sin(\omega_i t - \phi + \alpha_i) - \Delta H_o \sin\theta \tag{1-16}$$

$$H_{z'} = H_o - ZM\cos\theta - CM\sin\theta\cos\phi - DM\sin\theta\sin\phi + \sum_j h_{z_j} \sin(\omega_j t + \beta_j) \tag{1-17}$$

The equations of motion for this rotating set of vectors may now be written, by inspection, from $\dot{\vec{M}} = \gamma\mu_o(\vec{M} \times \vec{H}_t)$. They are

$$M_1 \dot{\phi} = -\gamma\mu_o \left(M_1 H_{z'} + M_{z'} H_1 \right) \tag{1-18}$$

$$\dot{M}_1 = -\gamma\mu_o\, M_{z'} H_1^* \tag{1-19}$$

and may be expanded into

$$\dot{\phi} = -\gamma\mu_o \Big\{ H_o - (Z-A)M\cos\theta + BM\cos\theta\cos 2\phi + WM\cos\theta\sin 2\phi - CM\sin\theta\cos\phi - DM\sin\theta\sin\phi - \cot\theta \sum_i h_i \cos(\omega_i t - \phi + \alpha_i) + \sum_j h_{z_j} \sin(\omega_j t + \beta_j) \Big\} \tag{1-20}$$

and

$$\dot{\theta} = -\gamma\mu_o \Big\{ \left(BM\sin 2\phi - WM\cos 2\phi - \Delta H_o \right) \sin\theta + \sum_i h_i \sin(\omega_i t - \phi + \alpha_i) \Big\} \tag{1-21}$$

where A, B, C, D, W, and Z have been given. An alternate form is

$$\dot{\phi} = \frac{\omega_{x'} + \omega_{y'}}{2} + \frac{\omega_{x'} - \omega_{y'}}{2}\cos 2\phi + \omega_W \sin 2\phi - C\omega_M \sin\theta\cos\phi - D\omega_M\sin\theta\sin\phi - \cot\theta \sum_i \omega_{h_i}\cos(\omega_i t - \phi + \alpha_i) + \sum_j \omega_{h_{z_j}} \sin(\omega_j t + \beta_j) \tag{1-22}$$

and

$$\dot{\theta} = \left(\frac{\omega_{x'} + \omega_{y'}}{2\cos\theta} \sin 2\phi - \frac{\omega_W}{\cos\theta}\cos 2\phi - \omega_{lo} \right) \sin\theta + \sum_i \omega_{h_i} \sin(\omega_i t - \phi + \alpha_i) \tag{1-23}$$

where

$$\omega_M = -\gamma\mu_0 M \qquad \omega_{x'} = \omega_H - (Z - X)\,\omega_M \cos\theta$$

$$\omega_W = W\,\omega_M \cos\theta \qquad \omega_{y'} = \omega_H - (Z - Y)\,\omega_M \cos\theta$$

$$\omega_{h_i} = -\gamma\mu_0 h_i \qquad \omega_H = -\gamma\mu_0 H_0$$

$$\omega_{h_{z_j}} = -\gamma\mu_0 h_{z_j} \qquad X = A + B$$

$$\omega_{\ell_0} = -\gamma\mu_0 \Delta H_0 \qquad Y = A - B$$

1.2. The Transient Solution. Let the applied r-f fields be zero and assume that θ is small. Equations (1-22) and (1-23) become

$$\dot{\phi} = \frac{\omega_{x'} + \omega_{y'}}{2} + \frac{\omega_{x'} - \omega_{y'}}{2} \cos 2\phi + \omega_W \sin 2\phi - C\,\omega_M \theta \cos\phi - D\,\omega_M \theta \sin\phi \tag{1-24}$$

and

$$\dot{\theta} = \left(\frac{\omega_{x'} - \omega_{y'}}{2} \sin 2\phi - \omega_W \cos 2\phi - \omega_{\ell_0} \right) \theta \tag{1-25}$$

The Generalized Resonance Frequency. When θ is small, the C and D terms in (1-24) contribute only second order terms to the resonant frequency. They, therefore, may be neglected and (1-24) integrated directly to give

$$\tan\left(\phi + \frac{\pi}{4} - \xi_0\right) = \sin\eta \tan(\omega_0 t + k) + \cos\eta \tag{1-26}$$

where

$$\xi_0 = \frac{1}{2} \tan^{-1}\left(\frac{2\,\omega_W}{\omega_{x'} - \omega_{y'}} \right)$$

$$\cos\eta = -\frac{2\sqrt{\left(\frac{\omega_{x'} - \omega_{y'}}{2}\right)^2 + \omega_W^2}}{\omega_{x'} + \omega_{y'}}$$

and k is an arbitrary phase constant. The generalized Kittel frequency is

$$\omega_0^2 = \omega_{x'}\omega_{y'} - \omega_W^2 \tag{1-27}$$

which may also be written as

$$\omega_0^2 = \left\{ \left[\omega_H - (Z-X)\omega_M\right]\left[\omega_H - (Z-Y)\omega_M\right] - W^2\omega_M^2 \right\} \tag{1-28}$$

with

$$X = N_x \sin^2\beta + N_y \cos^2\beta$$
$$Y = (N_x \cos^2\beta + N_y \sin^2\beta)\cos^2\alpha + N_z \sin^2\alpha$$
$$Z = (N_x \cos^2\beta + N_y \sin^2\beta)\sin^2\alpha + N_z \cos^2\alpha$$
$$W = \frac{N_x - N_y}{2} \cos\alpha \sin 2\beta$$

The quantities X, Y, and Z are effective demagnetizing factors in the x', y', and z' directions. W will be non-zero if the magnetization does not lie in a principal plane of the ellipsoid. It is apparent that $X + Y + Z = N_x + N_y + N_z = 1$, independent of the orientation. The net magnetizing field along the z' axis of the ellipsoid must be positive in order for (1-28) to be valid, since the derivation assumes that the magnetization is saturated. The applied field H_o must therefore be greater than ZM.

Two limiting cases of (1-24) are of interest. One case, when the $\sin\phi$ and $\cos\phi$ terms are zero, occurs when the ellipsoid is magnetized along a principal axis; the other, when the $\sin 2\phi$ and $\cos 2\phi$ terms are zero, occurs when the ellipsoid is magnetized in a principal plane and X = Y.

<u>The Magnetization Along a Principal Axis (C = D = 0).</u> The complete transient solution for the case of magnetization along a principal axis

has already been given by the author,[4] but the results will be repeated and extended. There is no loss, in generality, if $\alpha = 0$ and $\beta = \pi/2$, so that the primed and unprimed coordinates coincide. If θ is small so that the $\cos\theta$ is approximately unity (1-22) may be integrated to give

$$\tan\left(\phi + \frac{\pi}{4}\right) = \sin\eta \tan(\omega_o t + K_o) + \cos\eta \tag{1-29}$$

where

$$\cos\eta = \frac{\omega_y - \omega_x}{\omega_y + \omega_x} \qquad\qquad \omega_o^2 = \omega_x \omega_y$$

For convenience, choose $k_o = \tan^{-1}\left(\frac{1-\cos\eta}{\sin\eta}\right)$ so that $\phi(0) = 0$. Then

$$\sin\phi = \frac{\sqrt{1-\cos\eta}\ \sin\omega_o t}{\sqrt{1+\cos\eta\cos 2\omega_o t}} \tag{1-30}$$

$$\cos\phi = \frac{\sqrt{1+\cos\eta}\ \cos\omega_o t}{\sqrt{1+\cos\eta\cos 2\omega_o t}} \tag{1-31}$$

and

$$\sin 2\phi = \frac{\sin\eta\ \sin 2\omega_o t}{1+\cos\eta\cos 2\omega_o t} \tag{1-32}$$

This latter result may be substituted into (1-25) which may be integrated to give

$$\theta = \theta(o)e^{-\omega_{\ell o}t}\left[\frac{1+\cos\eta\cos 2\omega_o t}{1+\cos\eta}\right]^{\frac{1}{2}} \tag{1-33}$$

where $\theta(0)$ is the value of θ, when $t = 0$. Thus the magnetization components are

$$M_x = M\theta(0)\, e^{-\omega_{\ell 0} t} \cos\omega_0 t \tag{1-34}$$

$$M_y = M\theta(0)\, e^{-\omega_{\ell 0} t} \sqrt{\frac{\omega_x}{\omega_y}}\, \sin\omega_0 t \tag{1-35}$$

and

$$M_z = M - \frac{M}{2}\,\theta(0)^2 e^{-2\omega_{\ell 0} t}\left[\frac{\omega_x+\omega_y}{2\omega_y} + \frac{\omega_y-\omega_x}{2\omega_y}\cos 2\omega_0 t\right] \tag{1-36}$$

When $\omega_{\ell 0} = 0$, we conveniently replace $\dfrac{\theta(0)}{\sqrt{1+\cos\eta}}$ by $\theta_{rms} = \sqrt{\overline{\theta^2}}$.

<u>The Magnetization in a Principal Plane with X = Y (B = W = 0).</u>

The general case (subject to the constraint that B = W = 0) is represented if $\beta = \frac{\pi}{2}$ and $N_x = N_y \cos^2\alpha + N_z \sin^2\alpha$. Then (if $\omega_{\ell 0} = 0$)

$$\theta = \theta_0 \tag{1-37}$$

and

$$\dot{\phi} = \omega_H - (N_z - N_y)\,\omega_M \cos 2\alpha \cos\theta_0 + \frac{N_z - N_y}{2}\,\omega_M \sin 2\alpha \sin\theta_0 \sin\phi \tag{1-38}$$

which may be integrated to give

$$\tan\frac{\phi}{2} = \sin\eta' \tan\left(\frac{\omega_0 t}{2} + K\right) + \cos\eta' \tag{1-39}$$

where

$$\cos\eta' = \frac{(N_y - N_z)\,\omega_M \sin 2\alpha \sin\theta_0}{\omega_H - (N_z - N_y)\,\omega_M \cos 2\alpha \cos\theta_0} \tag{1-40}$$

and

$$\omega_0^2 = \left[\omega_H - (N_z - N_y)\omega_M \cos 2\alpha \cos\theta_0\right]^2 - \left(\frac{N_z - N_y}{2}\right)^2 \omega_M^2 \sin^2 2\alpha \sin^2\theta_0 \quad (1\text{-}41)$$

If tan k + cot η = 0, so that $\phi(t = 0) = 0$, the primed coordinate components become

$$M_{x'} = M \sin\theta_0 \frac{\sin\eta \sin(\omega_0 t + \eta)}{1 - \cos\eta \cos(\omega_0 t + \eta)} \quad (1\text{-}42)$$

$$M_{y'} = M \sin\theta_0 \frac{\cos\eta - \cos(\omega_0 t + \eta)}{1 - \cos\eta \cos(\omega_0 t + \eta)} \quad (1\text{-}43)$$

and

$$M_{z'} = M \cos\theta_0 \quad (1\text{-}44)$$

It is evident that a second harmonic component will be found in the transverse components of magnetization instead of in the longitudinal component, as in the previous case.

<u>The General Case for Small Cone Angle.</u> Let us now consider the general case in which all terms are non-zero. Let $\phi = \hat{\phi} + \xi_0$ where $\xi_0 = \frac{1}{2}\tan^{-1}\left(\frac{2\omega_W}{\omega_{x'} - \omega_{y'}}\right)$. This transformation corresponds, geometrically, to a rotation in the x'-y' plane as shown in Fig. 1-3. It is convenient to express the transverse components of the magnetization in the double primed coordinates. Furthermore, assume that $\phi(0) = \xi_0$. Then (1-24) and (1-25) become

$$\dot{\hat{\phi}} = A_0 + B_0 \cos 2\hat{\phi} + C_0 \cos\hat{\phi} + D_0 \sin\hat{\phi} \quad (1\text{-}45)$$

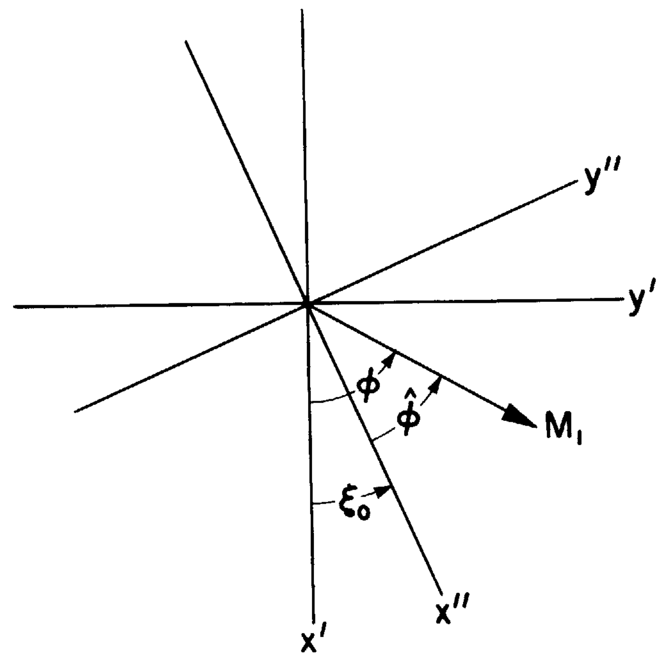

Fig. 1-3

and

$$\dot{\theta} = \left(B_o \sin 2\hat{\phi} - \omega_{\ell o} \right) \theta \tag{1-46}$$

where

$$A_o = \frac{\omega_{x'} + \omega_{y'}}{2}$$

$$B_o = \sqrt{\left(\frac{\omega_{x'} - \omega_{y'}}{2}\right)^2 + \omega_W^2}$$

$$C_o = -\left(C \cos \xi_o + D \sin \xi_o \right) \omega_M \theta$$

$$D_o = \left(C \sin \xi_o - D \cos \xi_o \right) \omega_M \theta$$

Assume that C_o and D_o perturb the solution already obtained when they are zero.* Then

$$\sin \hat{\phi} = \frac{\sqrt{1 - \cos\eta}\ \sin(\omega_o t + \Delta)}{\sqrt{1 + \cos\eta \cos 2(\omega_o t + \Delta)}} \tag{1-47}$$

where $\cos \eta = -\frac{B_o}{A_o}$ and $\omega_o^2 = A_o^2 - B_o^2$. It may be shown that Δ satisfies the equation

$$\dot{\Delta} = \frac{\sqrt{1 + \cos\eta \cos 2u}}{\sin \eta} \left[C_o \sqrt{1 + \cos\eta}\ \cos u + D_o \sqrt{1 - \cos\eta}\ \sin u \right] \tag{1-48}$$

with $u = \omega_o t + \Delta$. If Δ is small it is permissible to replace u by $\omega_o t$, and since $\cos\eta$ is normally small the square root may be expanded

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*The perturbed solution may be assumed in any of several forms. It is often convenient to write it $\hat{\phi} = \hat{\phi}^o + \Delta\hat{\phi}$ where $\hat{\phi}^o$ is the unperturbed phase.

keeping the first two terms. It follows that

$$\dot{\Delta} \simeq \frac{\left(1 + \frac{\cos\eta}{2}\cos 2\omega_0 t\right)}{\sin\eta}\left[C_0\sqrt{1+\cos\eta}\,\cos\omega_0 t + D_0\sqrt{1-\cos\eta}\,\sin\omega_0 t\right] \tag{1-49}$$

and the equation for θ becomes (if $\omega_{\ell 0}=0$)

$$\theta \simeq \theta_{rms}\sqrt{1+\cos\eta\cos 2u}\; e^{-\frac{B_0\sin\eta}{\omega_0}\int\frac{\sin 2u}{1+\cos\eta\cos 2u}\dot{\Delta}\,dt} \tag{1-50}$$

Again, replace u by $\omega_0 t$ and the exponential by the first two terms in its expansion. Then, since $\frac{1}{1+\cos\eta\cos 2u} \simeq 1 - \cos\eta\cos 2u$, the cone angle is approximately

$$\theta \simeq \theta_{rms}\sqrt{1+\cos\eta\cos 2u}\left[1 - \frac{B_0\sin\eta}{\omega_0^2}\int\sin 2\omega_0 t\left(1-\cos\eta\cos 2\omega_0 t\right)\dot{\Delta}\,d(\omega_0 t)\right] \tag{1-51}$$

The integrations may now be performed without further complication, and the double primed components of magnetization $M\theta\sin\hat{\phi}$, $M\theta\cos\hat{\phi}$, and $M(1 - \frac{1}{2}\theta^2)$ found. As expected, second harmonic terms occur in all three components. (When expanding to a given order, great care should be taken to include all terms that are pertinent.) It is interesting to observe that since the longitudinal and transverse harmonics have different angular dependences, it is possible to experimentally separate the modulation effects. The longitudinal second harmonic power is proportional to $\frac{B_o^2}{A_o^2}$, and that of the transverse second harmonic to $\frac{C^2 + D^2}{A_o^2}$. Bennett[5] has studied the case of a thin disc and found reasonable agreement with theory.

It is apparent from the previous results that in general the precession phase, ϕ , is modulated by internal fields. The demagnetizing field was considered here, but fields due to anisotropy, inhomogeneities, etc. will have similar effects. When the ellipsoid is magnetized along a principal axis, the modulation rate (due to the demagnetizing field) is at twice the average precession frequency since there are two planes of symmetry in the transverse plane through which the surface magnetic dipoles rotate. If the cross section is circular, the transverse demagnetizing field (H_1) and the transient precession frequency are constant. The Kittel frequency is the geometric mean of the two instantaneous frequencies, which correspond to the torques produced by the total internal field that is present when the transverse component of magnetization is aligned in turn with the two principal axes of the ellipsoid. It is the average or steady component of the resultant motion.

We would expect efficient odd harmonics to be created in the transverse plane because of this process, since the effective modulating field is essentially independent of θ . This does not occur because the cone angle θ is also modulated by the same field and in such an interrelated manner that products like $\theta \cos\phi$ and $\theta \sin\phi$ contain only the fundamental frequency.* The longi-

---

*Actually a factor $\left(1 + \cos\eta \cos 2\omega t\right)^{\frac{\cos\theta - 1}{2}}$ should be included in the transverse components where $\cos\eta$ also depende on $\cos\theta$. This is normally a very small effect.

tudinal component of magnetization contains a component of second harmonic proportional to θ^2. When the ellipsoid is magnetized at an angle, time-varying components of demagnetizing field at the fundamental frequency occur in the longitudinal direction. These modulate the precession phase without modulating the cone angle, and second harmonic (and higher) components are formed in the transverse plane. Since the modulating field is proportional to θ, the second harmonic is again proportional to θ^2. In general, both transverse and longitudinal modulation processes may occur so that a superposition of results exist; in addition, cross modulation between the two may be expected. Either of these two modulating processes may be obtained (normally less efficiently) by utilizing applied r-f fields as the modulating agents.[4,6,7] If the ellipsoid is a spheroid magnetized along its axis of rotation, any modulation effects must come from applied r-f fields. These must serve double duty, so to speak, for the resonance must be driven, and the resulting precession modulated. For example, if longitudinal second harmonic is desired, a transverse positive circularly-polarized field is needed to open the cone angle, and a transverse negative circularly-polarized field is needed to modulate it (making the transverse precession path an ellipse). Both fields are equally important and the optimum condition calls for a linearly-polarized excitation.[6] By using an ellipsoid instead of a spheroid and letting the internal demagnetizing field perform the modulation, the entire driving field may be utilized for exciting the resonance.

1.3. Forced Linear Steady-State Solutions.

Transverse Microwave Excitation. If the general ellipsoid is subjected to a transverse microwave field of elliptical polarization, two driving terms are needed in (1-22) and (1-23). The positive circularly- polarized component is assumed to be of amplitude h_1, frequency ω, and phase α_1; the negative circularly-polarized component is assumed to be of amplitude h_2, frequency $-\omega$, and phase α_2. If θ is assumed to be small, the longitudinal modulating terms may be neglected.* It is convenient then to transform the equations to the double-primed coordinate system, as before.

$$\dot{\hat{\phi}} = A_0 + B_0 \cos 2\hat{\phi} - \frac{1}{\theta}\left[\omega_{h_1}\cos\left(\omega t - \hat{\phi} + \alpha_1\right) + \omega_{h_2}\cos\left(-\omega t - \hat{\phi} + \alpha_2\right)\right] \quad (1\text{-}52)$$

and

$$\dot{\theta} = \left(B_0 \sin 2\hat{\phi} - \omega_{\ell 0}\right)\theta + \omega_{h_1}\sin\left(\omega t - \hat{\phi} + \alpha_1\right) + \omega_{h_2}\sin\left(-\omega t - \hat{\phi} + \alpha_2\right) \quad (1\text{-}53)$$

The phases α_1 and α_2 are understood to be measured from the x" axis. Let us assume that $\hat{\phi}$ and θ have in the steady state the same form as the lossless transient solution discussed previously. Then

$$\sin\hat{\phi} = \frac{\sqrt{1-\cos\eta}\,\sin\omega t}{\sqrt{1+\cos\eta\,\cos 2\omega t}} \quad (1\text{-}54)$$

*The perturbation analysis discussed previously is of course still valid.

$$\cos\hat{\phi} = \frac{\sqrt{1+\cos\eta}\ \cos\omega t}{\sqrt{1+\cos\eta\cos 2\omega t}} \qquad (1\text{-}55)$$

and

$$\theta = \theta_{rms}\sqrt{1+\cos\eta\cos 2\omega t} \qquad (1\text{-}56)$$

where the parameter $\cos\eta$ is, as yet, unspecified. If $\alpha_1 = \alpha_2$,and the field amplitudes satisfy

$$\omega_{h1} + \omega_{h2} = \omega_h\sqrt{1+\cos\eta} \qquad (1\text{-}57)$$

and

$$\omega_{h1} - \omega_{h2} = \omega_h\sqrt{1-\cos\eta} \qquad (1\text{-}58)$$

it may be demonstrated that (1-52) and (1-53) reduce to

$$\dot{\hat{\phi}} = A_o - \omega_{\ell o}\cot\alpha_1 + B_o\cos 2\hat{\phi} \qquad (1\text{-}59)$$

and

$$\dot{\theta} = B_o\sin 2\hat{\phi}\ \theta \qquad (1\text{-}60)$$

provided that $\theta_{rms} = \frac{\omega_h}{\omega_{\ell o}}\sin\alpha_1$. Equations (1-59) and (1-60) are similar to those for the transient case.*

*The chosen driving field is the only one capable of exciting the resonance optimumly. For example, a pure positive circularly-polarized field will not, in general, lead to the maximum response even if its frequency coincides exactly with that of resonance.

It follows then that the initial assumptions are justified, and that

$$\cos\eta = \frac{-B_o}{A_o - \omega_{lo}\cot\alpha_1} \tag{1-61}$$

and

$$\omega^2 = \left(A_o - \omega_{lo}\cot\alpha_1\right)^2 - B_o^2 \tag{1-62}$$

Since the resonant frequency is given by $\omega_o^2 = A_o^2 - B_o^2$ it is apparent that

$$\cot\alpha_1^{\pm} = \frac{A_o \mp \sqrt{A_o^2 - (\omega_o^2 - \omega^2)}}{\omega_{lo}} \tag{1-63}$$

The values of α_1^+ and α_1^- correspond to positive and negative values of ω.

Corresponding to these are two values each for $\cos\eta$ and θ

$$\cos\eta^{\pm} = \frac{\mp B_o}{\sqrt{A_o^2 - (\omega_o^2 - \omega^2)}} \tag{1-64}$$

and

$$\theta_{rms}^{\pm} = \frac{\omega_h}{\omega_{lo}} \sin\alpha_1^{\pm} \tag{1-65}$$

The transverse components of the magnetization, and the driving field are, respectively

$$M_x^{\pm} = M\theta_{rms}^{\pm}\sqrt{1+\cos\eta^{\pm}}\,\cos\omega t \tag{1-66}$$

$$M_y^{\pm} = \pm M\theta_{rms}^{\pm}\sqrt{1-\cos\eta^{\pm}}\,\sin|\omega|t \tag{1-67}$$

and

$$h_x^{\pm} = h\left[\sqrt{1+\cos\eta^{\pm}}\cos\alpha_1^{\pm}\cos\omega t \mp \sqrt{1-\cos\eta^{\pm}}\sin\alpha_1^{\pm}\sin|\omega|t\right] \tag{1-68}$$

$$h_y^{\pm} = h\left[\sqrt{1+\cos\eta^{\pm}}\sin\alpha_1^{\pm}\cos\omega t \pm \sqrt{1-\cos\eta^{\pm}}\cos\alpha_1^{\pm}\sin|\omega|t\right] \tag{1-69}$$

In general $\left|\cos\eta^{\pm}\right| = \frac{B_o}{\sqrt{B_o^2 + \omega^2}}$ so that $0 \leq \left|\cos\eta^{\pm}\right| \leq 1$. If $B_o = 0$, it is apparent that $\cos\eta^{\pm} = 0$, independent of frequency; then $h_2 = 0$, and both the driving field and the response are circularly polarized. If $B_o \neq 0$, the value of $\cos\eta^{\pm}$ is a function of frequency. The driving field approaches linear polarization if $\frac{\omega}{B_o} << 1$, and circular polarization at frequencies where $\frac{\omega}{B_o} >> 1$. At resonance the maximum value of $\cos\eta^{+}$ occurs when $N_y = 1$, and the ellipsoid is magnetized along the z axis. Then

$$\cos\eta^{+} = \frac{M}{2H_o + M} \tag{1-70}$$

This geometry maximizes the production of longitudinal second harmonic. The path of the transverse magnetization and the driving field is sketched in Figs. 1-4 and 1-5 for both positive and negative elliptical polarizations. Notice the phase relationships of the vectors and the relative orientation of the principal axes of the ellipsoids in both cases. When $\omega = \omega_o$, then $\alpha^{+} = \frac{\pi}{2}$, and $\alpha^{-} \simeq 0$.

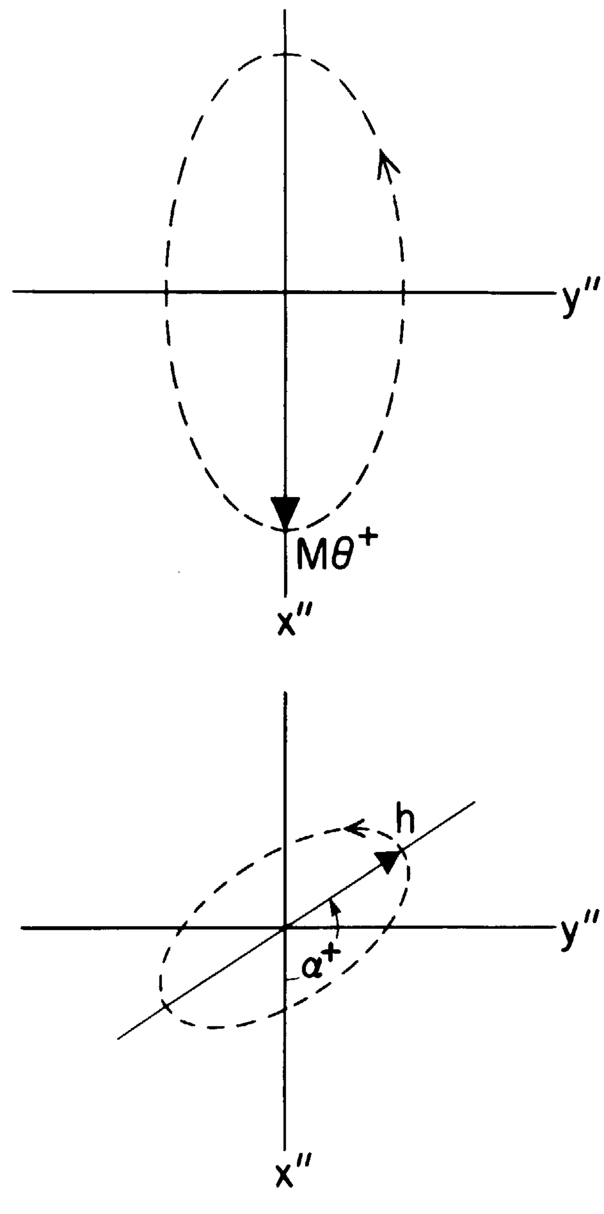

Fig. 1-4

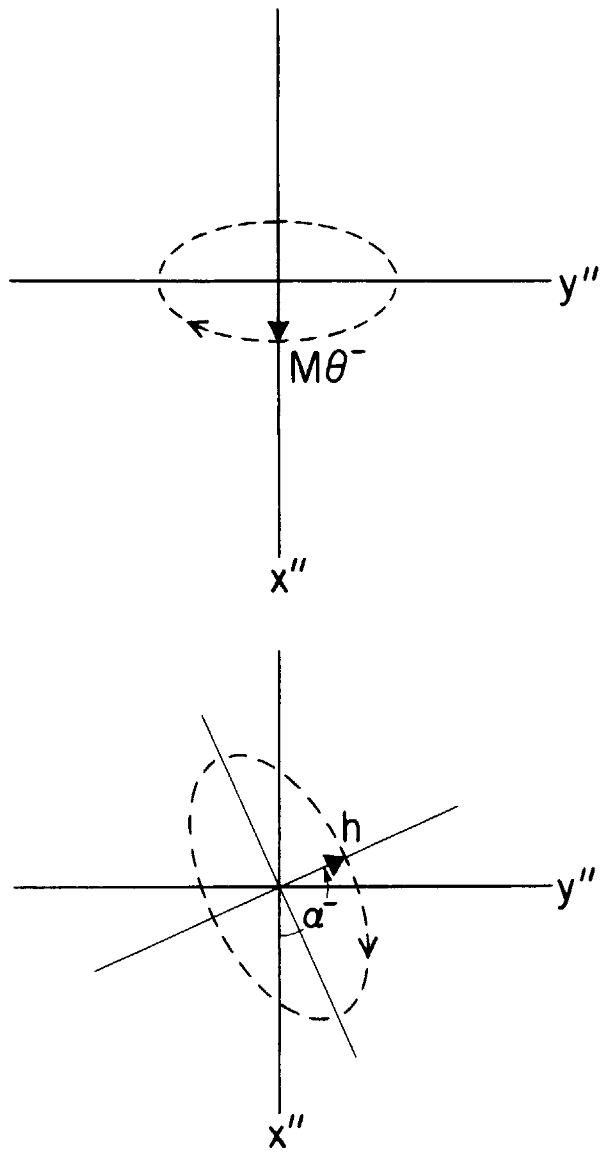

Fig. 1-5

<u>The Tensor Permeability.</u> The most general form of the linearized external permeability tensor is (in double-primed coordinates)

$$\overleftrightarrow{\mu}_{x''y''z'} = \begin{bmatrix} \mu_1^* & jK_1^* & 0 \\ jK_2^* & \mu_2^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \mu_0 \tag{1-71}$$

where

$$\mu_1^* = \mu_1' - j\mu_1'' \qquad K_1^* = K_1' - jK_1''$$
$$\mu_2^* = \mu_2' - j\mu_2'' \qquad K_2^* = K_2' - jK_2''$$

Using (1-66) through (1-69), it is easily shown that these components satisfy the following relations:

$$a\frac{\omega_M}{\omega_{\ell 0}}\sin\alpha^+ = \left[(\mu_1'-1)a + K_1'b\right]\cos\alpha^+ + \left[(\mu_1''-1)b + K_1''a\right]\sin\alpha^+ \tag{1-72a}$$

$$b\frac{\omega_M}{\omega_{\ell 0}}\sin\alpha^- = \left[(\mu_1'-1)b - K_1'a\right]\cos\alpha^- - \left[(\mu_1''-1)a - K_1''b\right]\sin\alpha^- \tag{1-72b}$$

$$\left[(\mu_1'-1)b + K_1'a\right]\sin\alpha^+ = \left[(\mu_1''-1)a + K_1''b\right]\cos\alpha^+ \tag{1-72c}$$

$$\left[(\mu_1'-1)a - K_1'b\right]\sin\alpha^- = \left[-(\mu_1''-1)b + K_1''a\right]\cos\alpha^- \tag{1-72d}$$

$$b \frac{\omega_M}{\omega_{\ell o}} \sin\alpha^+ = \left[(\mu_2'-1)b - K_2'a\right]\cos\alpha^+ + \left[(\mu_2''-1)a - K_2''b\right]\sin\alpha^+ \qquad (1\text{-}73a)$$

$$a \frac{\omega_M}{\omega_{\ell o}} \sin\alpha^- = \left[(\mu_2'-1)a + K_2'b\right]\cos\alpha^- - \left[(\mu_2''-1)b + K_2''a\right]\sin\alpha^- \qquad (1\text{-}73b)$$

$$\left[(\mu_2'-1)a - K_2'b\right]\sin\alpha^+ = \left[(\mu_2''-1)b - K_2''a\right]\cos\alpha^+ \qquad (1\text{-}73c)$$

$$\left[(\mu_2'-1)b + K_2'a\right]\sin\alpha^- = -\left[(\mu_2''-1)a + K_2''b\right]\cos\alpha^- \qquad (1\text{-}73d)$$

where $a = \sqrt{1 + \cos\eta^+}$ and $b = \sqrt{1 - \cos\eta^+}$. There are two sets of four linear equations, each in four unknowns. These values together with (1-63) will allow (1-72) and (1-73) to be solved and generalize the results of Hogan[8] and Jepsen[7], as well as the author.[4]

If $B_o = 0$, both a and b are unity and a great simplification results. It follows that $\mu_1^* = \mu_2^* \equiv \mu^*$, $K_1^* = -K_2^* \equiv -K^*$, and (1-72) and (1-73) reduce to

$$(\mu'-1) \mp K' = \frac{\omega_M}{\omega_{\ell o}} \sin\alpha^\pm \cos\alpha^\pm \qquad (1\text{-}74a)$$

and

$$K'' \mp (\mu''-1) = -\frac{\omega_M}{\omega_{\ell o}} \sin^2\alpha^\pm \qquad (1\text{-}74b)$$

A straightforward rotation transformation yields the tensor components for the unprimed coordinates. Thus

$$\overleftrightarrow{\mu}_{xyz} = T^t \cdot \overleftrightarrow{\mu}_{x''y''z'} \cdot T \tag{1-75}$$

where

$$T = \begin{bmatrix} (\sin\beta\cos\xi_0 + \cos\alpha\cos\beta\sin\xi_0) & (\cos\alpha\sin\beta\sin\xi_0 - \cos\beta\cos\xi_0) & (-\sin\alpha\sin\xi_0) \\ (\cos\alpha\cos\beta\cos\xi_0 - \sin\beta\sin\xi_0) & (\cos\alpha\sin\beta\cos\xi_0 + \cos\beta\sin\xi_0) & (-\sin\alpha\cos\xi_0) \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{bmatrix} \tag{1-76}$$

and T^t is the transpose of T.

<u>1.4. Forced Nonlinear Steady-State Solutions.</u> A longitudinal r-f field (h_2) of frequency ω_2, in conjunction with a transverse driving field (h_1) at frequency ω_1 can produce nonlinear resonance.[4,7,9] The longitudinal field frequency modulates the uniform precession (assumed to be excited at a frequency ω) and forms sidebands at frequencies $\omega \pm K\omega_2$ in the transverse plane. If the kth sideband is equal to ω_1 and the dc magnetic field is adjusted for resonance at ω, it may be shown that resonance occurs at $\omega_0 = \omega$, then given by

$$\omega = \omega_1 \pm K\omega_2 \qquad K = 0, 1, 2, \cdots \tag{1-77}$$

The cone angle of the resonance is approximately

$$\theta_{\omega_K} = \frac{\omega_{h1}}{\omega_{\ell o}} J_K\left(\frac{\omega_{h2}}{\omega_2}\right) \tag{1-78}$$

where $J_k(\delta)$ is a Bessel function of order k. The resonance is dominant and the above equation correct provided that

$$\theta_{\omega_K} >> \frac{\omega_{h1}}{\sqrt{(K\omega_2)^2 + \omega_{\ell o}^2}} \tag{1-79}$$

If h_2 is not large enough to insure that the above condition is satisfied, it follows that the response must be chiefly at the transverse driving frequency ω_1. If we assume for simplicity that the sample is a spheroid, the cone angle of this linear response is (from (1-63) and (1-65), with $B_o = 0$)

$$\theta_{\omega_{K=o}} = \frac{\omega_{h1}}{\sqrt{(\omega_o - \omega_1)^2 + \omega_{\ell o}^2}} \tag{1-80}$$

Since $\omega_o = \omega_1 \pm k\,\omega_2$, it follows that

$$\theta_{\omega_{K=o}} = \frac{\omega_{h1}}{\sqrt{(K\omega_2)^2 + \omega_{\ell o}^2}} \tag{1-81}$$

Thus the transition from linear to nonlinear resonance occurs when $\theta_{\omega_K} > \theta_{\omega_{K=o}}$.

We would expect similar behavior from the time-varying longitudinal demagnetizing field with two important exceptions. First, the modulation field will be at a frequency ω rather than at ω_2, and second it is, as we have seen, proportional to the cone angle θ. The first consideration leads to

$$\omega = \frac{\omega_1}{K+1} \qquad k = 0, 1, 2, \ldots \tag{1-82}$$

which means that subharmonic generation should be expected when the sample is biased to the subharmonic frequency, and the second consideration leads to the obvious conclusion that a feedback effect is present. This is due to the fact that the cone angle θ, is determined by the strength of the modulation, which in turn is proportional to θ. Because of the feedback, we may expect the subharmonic resonances to exist only under certain restrictive conditions. Since this nonlinear process depends on the longitudinal component of the time-varying demagnetizing field, the geometry $N_x = 1/3$, $N_y = 2/3$, $N_z = 0$, $\alpha = \frac{\pi}{4}$, and $\beta = \frac{\pi}{2}$ is chosen so that the effect may be studied independently.

Subharmonic Resonance. For the above mentioned geometry, and a single transverse circularly-polarized driving field, (1-22) and (1-23) are

$$\dot{\phi} = \omega_0 - \frac{\omega_M}{3}\theta \sin\phi - \frac{\omega_h}{\theta}\cos(\omega_1 t - \phi + \alpha_1) \tag{1-83}$$

and

$$\dot{\theta} = \omega_h \sin(\omega_1 t - \phi + \alpha_1) - \omega_{\ell o}\theta \tag{1-84}$$

where $\omega_o = -\gamma\mu_o H_o$. Assume that

$$\phi = \omega t + \sum_{n=1}^{\infty}(a_n \sin n\omega t + b_n \cos n\omega t) \tag{1-85}$$

and

$$\theta = \theta_0\left[1 + \sum_{n=1}^{\infty}\left(c_n \sin n\omega t + d_n \cos n\omega t\right)\right] \tag{1-86}$$

where $\omega = \frac{\omega_1}{2}$. The quantities a_n, b_n, c_n, and d_n are all assumed much less than unity. When (1-85) and (1-86) are substituted into (1-83) and (1-84), and the latter are expanded to include third order terms, there results

$$\sin\alpha_1 = \frac{\omega_0 - \omega}{\omega_h\,\omega_M}\cdot 6\omega \tag{1-87}$$

At resonance, the $\sin\alpha_1$ is zero and the maximum cone angle is given by

$$\left(\frac{\omega_{10}}{\omega_h} + \frac{\omega_M}{6\omega}\cos\alpha_1\right)\theta_0 = \frac{1}{4}\left(\frac{\omega_M}{3\omega}\,\theta_0\right)^3\cos\alpha_1 \tag{1-88}$$

where α_1 is 0 or π radians. The former value leads to a coefficent b_1 (for non-zero θ_0), which is always greater than unity; and hence, in contradiction to the assumptions. The non-zero solution for θ_0 with $\alpha_1 = \pi$ is

$$\theta_0 = \frac{6\omega}{\omega_M}\sqrt{\frac{1}{2} - \frac{3\omega\,\omega_{10}}{\omega_h\,\omega_M}} \tag{1-89}$$

In order that (1-89) be real, the driving field must exceed a certain threshold, given by

$$h^c = \frac{6\omega}{\omega_M}\,\Delta H_0 \tag{1-90}$$

Above the threshold, θ_0 increases very rapidly.

If we consider the general ellipsoid, the results are essentially of the same form. Thus

$$\theta = \frac{h^c}{\Delta H_0} \sqrt{\frac{1}{2}\left(1 - \frac{h^c}{h}\right)} \tag{1-91}$$

where

$$h^c = \frac{\omega_1}{\sqrt{C^2+D^2}\ \omega_M} \Delta H_0{}^{*} \tag{1-92}$$

A plot of θ versus h is given in Fig. 1-6. The curve is valid only in the region of small θ. Notice that the slope is infinite at the threshold. The minimum threshold occurs for a thin disc magnetized at 45° to its plane. This geometry yields the maximum value of $\sqrt{C^2 + D^2}$ --equal to one half. Below the threshold the magnetization precesses at the driving frequency ω_1 in a linear manner, but the amplitude of the precession is small since the sample is biased to the half frequency. The reader should be warned at this point that the over-all picture is complicated by certain spin wave instabilities that will be developed in Chapter 2. It appears that there will be a premature saturation of the resonance.

<u>Parametric Coupling.</u> Under certain conditions it is possible to have

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*This threshold is approximately correct as long as $\omega > \sqrt{B^2 + W^2}\ \omega_M$; otherwise cross modulation effects between the transverse and longitudinal modulating fields should be taken into consideration.

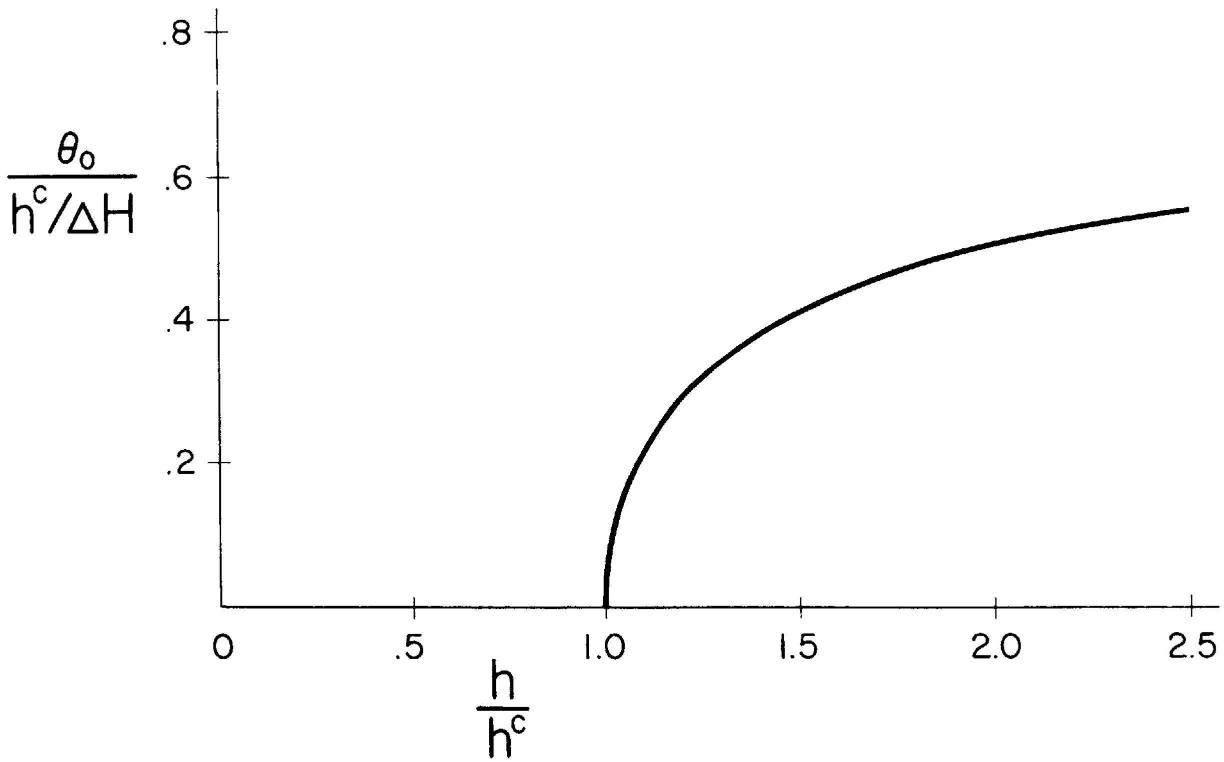


Fig. 1-6

direct parametric coupling between a longitudinal modulating field (of amplitude h_z and frequency ω_z) and the uniform precession. This coupling is due to unequal transverse demagnetizing factors. Consider an ellipsoid magnetized along a principal axis (such as $\alpha = 0$, $\beta = \frac{\pi}{2}$). From (1-22) and (1-23) we have

$$\dot{\phi} = \frac{\omega_x + \omega_y}{2} + \frac{\omega_x - \omega_y}{2} \cos 2\phi + \omega_{h_z} \cos(\omega_z t + \beta_z) \tag{1-93}$$

and

$$\dot{\theta} = \left(\frac{\omega_x - \omega_y}{2} \sin 2\phi - \omega_{\ell o} \right) \theta \tag{1-94}$$

It may be shown that if $\frac{\omega_{hz}}{\omega_z} \ll 1$, the phase of the uniform precession is approximately

$$\dot{\phi} = \phi^{o} + \frac{\omega_{hz}}{\omega_z} \sin(\omega_z t + \beta_z) \tag{1-95}$$

where ϕ^{o} is the unperturbed solution given by (1-30). If $\omega_z = 2\omega_o$, it follows that the average value of $\sin 2\phi$ is $\overline{\sin 2\phi} \simeq \frac{\omega_{hz}}{2\omega_o} \sin \beta_z$. The effective line width is therefore

$$\Delta H_{eff} = \Delta H_o \left[1 - \frac{(N_x - N_y)\omega_M}{4\omega_o} \frac{h_z}{\Delta H_o} \sin \beta_z \right] \tag{1-96}$$

If $N_x > N_y$ and $\sin \beta_z = 1$, the threshold for instability ($\Delta H_{eff} = 0$) is

$$h_z = \frac{4\omega_o}{(N_x - N_y)\omega_M} \Delta H_o \tag{1-97}$$

which is obviously minimized when $N_x = 1$. Then

$$h_z = \frac{4\sqrt{H_o(H_o+M)}}{M} \Delta H_o \tag{1-98}$$

This coupling may occur simultaneously with any of the preceding ones. It is worth noting that the self-generated longitudinal second harmonic field has a phase $\beta_z = 0$, so that it does not parametrically couple.

CHAPTER 2

SPIN WAVE INTERACTIONS

2.1. A Continuum Model of Spin Waves. Spin waves arise because the discrete magnetic moments, within the ferrimagnetic solid, are not necessarily in phase. The enormous number of these, in any macroscopic volume, makes it possible to use a continuous function to describe the magnetization. It is assumed that

$$\vec{\delta M}_k = \vec{i}'\,\delta M \cos\phi_k \cos\vec{k}\cdot\vec{r} + \vec{j}'\,\delta M \sin\phi_k \cos\vec{k}\cdot\vec{r} + \vec{k}'\,M_{z'} \qquad (2\text{-}1)$$

(where $|\vec{k}| = k = \dfrac{2\pi}{\lambda_k}$) represents a standing spin wave of wave length λ_k, directed along $\vec{k}$.* The unit vectors $\vec{i}'$, $\vec{j}'$, and $\vec{k}'$ refer to the cartesian coordinates x', y', and z', and the magnetizing field is assumed to be in the direction of the latter. It is possible to resolve any disturbance of the magnetization into a set of such waves so that

---

*It might be expected that

$\vec{i}'\ \delta M \cos(\phi_k - \vec{k}\cdot\vec{r}) \quad + \quad \vec{j}'\ \delta M \sin(\phi_k - \vec{k}\cdot\vec{r}) \quad + \quad \vec{k}' M_{z'}$, would

represent a traveling spin wave; it does not, except when $\vec{k}$ is directed along z'. The correct representation for progressive waves will be given later on in the text.

$$\vec{M} = \vec{M}_o + \sum_K \delta\vec{M}_K \tag{2-2}$$

where $\vec{M}_o$ is the uniform precession. Following Suhl,[10,11] we assume that the spin wave length is very small compared to the dimensions of the ellipsoid so that the exact boundary conditions can be ignored. As discussed by him, many of the pertinent results derived under such assumptions may be extrapolated to hold, qualitatively, in the region of small k.

If the spin wave amplitudes are all assumed to be much smaller than the uniform precession, the magnetization in any small volume will be conserved as in (2-3) –

$$M^2 = \left[M_{ox'} + \sum_K (\delta M_K)_{x'}\right]^2 + \left[M_{oy'} + \sum_K (\delta M_K)_{y'}\right]^2 + \left[M_{oz'} + \sum_K (\delta M_K)_{z'}\right]^2 \tag{2-3}$$

If products involving two spin waves are neglected compared to those involving the uniform precession and one spin wave, and the relation $M_{ox'}^2 + M_{oy'}^2 + M_{oz'}^2 = M^2$ is used, there will result

$$\sum_K \left[M_{ox'}(\delta M_K)_{x'} + M_{oy'}(\delta M_K)_{y'} + M_{oz'}(\delta M_K)_{z'} \right] = 0 \tag{2-4}$$

This equality will hold for each separate term as well as for the sum, so that

$$(\delta M_K)_{z'} = \frac{-1}{M_{oz'}}\left[M_{ox'}(\delta M_K)_{x'} + M_{oy'}(\delta M_K)_{y'} \right] \tag{2-5}$$

As long as this approximation is valid, it is sufficient to consider a single spin wave and the uniform precession when setting up the internal field components. Later on, as many spin waves as are desired may be included in the interaction equations; however, for clarity and simplicity, only one such typical mode will be considered in this thesis.

The Magnetization in an Ellipsoid. In terms of the primed coordinates of Fig. 1-1, the magnetization vectors are as shown in Fig. 2-1. Since the uniform precession is given by

$$\vec{M}_o = \vec{i}' M\sin\theta\cos\phi_o + \vec{j}' M\sin\theta\sin\phi_o + \vec{k}' M\cos\theta \tag{2-6}$$

it follows from (2-1) and (2-5) that

$$\begin{aligned}\vec{\delta M}_k = {} & \vec{i}'\,\delta M\cos\phi_k\cos\vec{k}\cdot\vec{r} + \vec{j}'\,\delta M\sin\phi_k\cos\vec{k}\cdot\vec{r} \\ & - \vec{k}'\,\delta M\tan\theta\cos\chi\cos\vec{k}\cdot\vec{r}\end{aligned} \tag{2-7}$$

where $\chi = \phi_k - \phi_o$. The correctness of (2-5) depends on δM satisfying $\delta M << 2\,M\cos\theta$ or $2\,M\,\dfrac{\cos^2\theta}{\sin\theta}$. The propagation vector of $\vec{\delta M}_k$, which is depicted in Fig. 2-2 is

$$\vec{k} = k\left[\sin\psi\left(\vec{i}'\cos\xi + \vec{j}'\sin\xi\right) + \vec{k}'\cos\psi\right] \tag{2-8}$$

The Internal Field. The total internal field, including terms due to the spin wave, will consist of five terms*

*We continue to neglect fields due to anisotropy and inhomogeneities. Their inclusion, however, poses no particular difficulties.

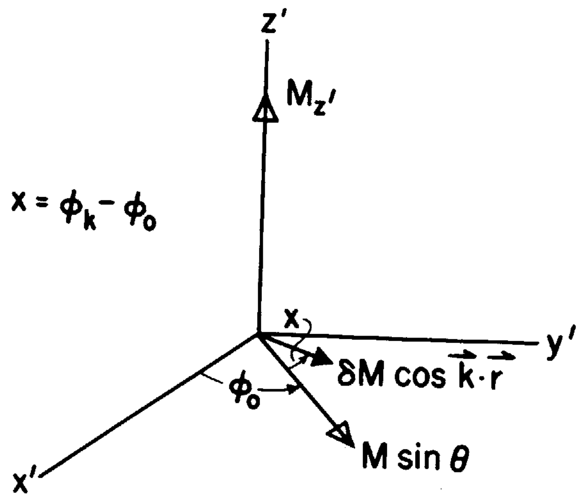


Fig. 2-1

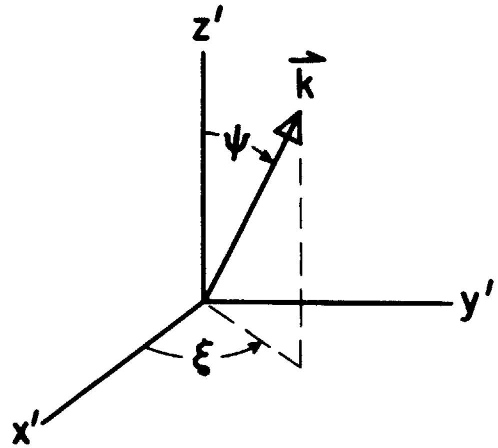


Fig. 2-2

$$\vec{H}_t = \vec{H}_{applied} + \vec{H}_{demagnetizing} + \vec{H}_{volume\ dipolar} + \vec{H}_{exchange} + \vec{H}_{damping}$$

The first and second terms were discussed in Chapter 1 and are uneffected by the inclusion of the spin wave. (The latter does not contribute to the surface demagnetizing field since its spatial variation is assumed to be very rapid.)

The volume dipolar field arises because there is a divergence of magnetization in the sample, and $\nabla \cdot \vec{H}_{vol.\ dip.} = -\nabla \cdot \vec{M}$. Since the ellipsoid is assumed to be small enough for propagation effects to be neglected, a magnetostatic approximation may be used. Then $\vec{H}_{vol.\ dip.} = \nabla \phi'$, where ϕ' is a scaler potential satisfying $\nabla^2 \phi' = -\nabla \cdot \vec{M}$. It is easily shown that

$$\phi' = \frac{-1}{K^2}\left(K_{x'} \cos\phi_k + K_{y'} \sin\phi_k - K_{z'} \tan\theta \cos\chi\right) \delta M \cos \vec{K} \cdot \vec{r} \tag{2-9}$$

where $k_{x'} = k \sin\psi \cos\xi$, $k_{y'} = k \sin\psi \sin\xi$, and $k_{z'} = k \cos\psi$.

The exchange term is due to quantum-mechanical coupling between nearest neighbor spins and may be shown to be equivalent to

$$\vec{H}_{exchange} = w\vec{M} + \lambda \nabla^2 \vec{M} \tag{2-10}$$

when the misalignment between spins is not too great.[12] The parameter λ is a measure of the strength of this coupling and for cubic crystals is often given by $\lambda = \frac{H_{ex} \ell^2}{M}$, where H_{ex} is an effective exchange field and ℓ is the lattice constant. The component $w\vec{M}$ will be dropped since it produces no torque ($\mu_0[\vec{M} \times w\vec{M}] = 0$). Because of the form of $\delta\vec{M}_k$, it follows that

$$\vec{H}_{exchange} = \lambda \nabla^2 \delta\vec{M}_k = -\lambda k^2 \delta\vec{M}_k \qquad (2\text{-}11)$$

The loss field is $\vec{H}_{damp.} \simeq \frac{\Delta H}{MH_{z'}} (\vec{M} \times \vec{H}_{z'})$ as discussed in Chapter 1. In general, however, ΔH might be different for the spin wave and the uniform precession. To include this possibility we let

$$\vec{H}_{damp.} = \frac{\Delta H_o}{MH_{z'}} (\vec{M}_o \times \vec{H}_{z'}) + \frac{\Delta H_k}{MH_{z'}} (\delta\vec{M}_k \times \vec{H}_{z'}) \qquad (2\text{-}12)$$

where $2\,\Delta H_o$ and $2\,\Delta H_k$ are the resonance line widths of the uniform precession and spin wave respectively.

In Chapter 1, the uniform precession fields were resolved into rotating components H_1 and H_1^*. It is worthwhile to treat the spin wave in a similar manner and as shown in Fig. 2-3 its transverse fields are resolved into parallel (H_2) and perpendicular (H_2^*) components with respect to the transverse component of $\delta\vec{M}_k$. In terms of these components, it may be shown that the volume dipolar field is

$$(H_2)_{vol.dip.} = \left[\sin^2\psi \cos^2(\phi_k - \xi) - \frac{\sin 2\psi}{2} \tan\theta \cos\chi \cos(\phi_k - \xi)\right] \delta M \cos \vec{k}\cdot\vec{r} \qquad (2\text{-}13)$$

$$(H_2^*)_{vol.dip.} = \left[\frac{\sin^2\psi}{2} \sin 2(\phi_k - \xi) - \frac{\sin 2\psi}{2} \tan\theta \cos\chi \sin(\phi_k - \xi)\right] \delta M \cos \vec{k}\cdot\vec{r} \qquad (2\text{-}14)$$

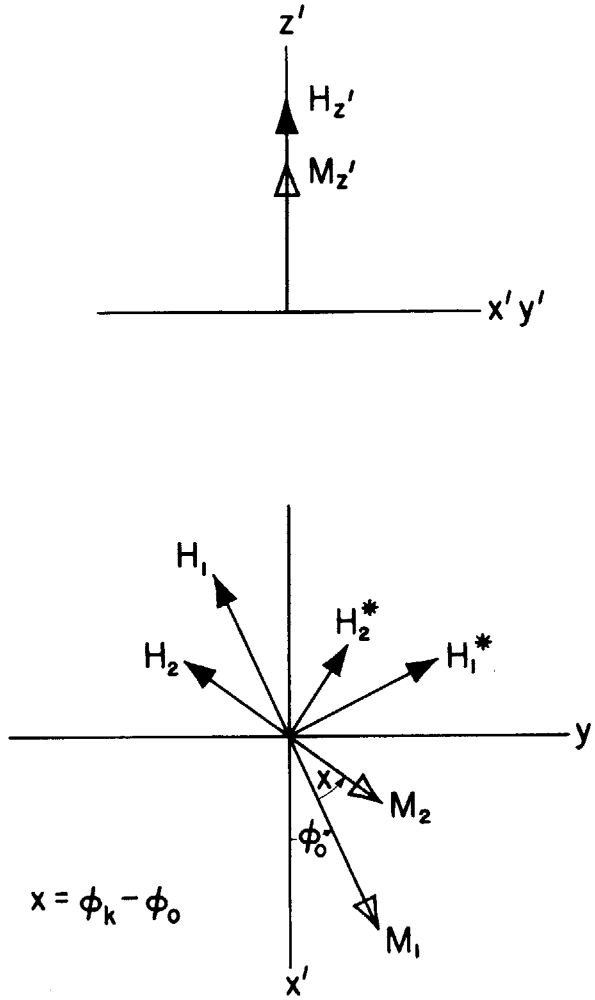


Fig. 2-3

and

$$(H_{z'})_{\text{vol.dip.}} = \left[\cos^2\psi \tan\theta \cos\chi - \frac{\sin 2\psi}{2} \cos(\phi_k - \xi)\right] \delta M \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}15)$$

The components of the exchange field are

$$(H_2)_{\text{exchange}} = \lambda K^2 \delta M \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}16)$$

and

$$(H_{z'})_{\text{exchange}} = \lambda K^2 \delta M \tan\theta \cos\chi \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}17)$$

while the spin wave loss field is given by

$$(H_2^*)_{\text{damp.}} = -\frac{\Delta H_K}{M} \delta M \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}18)$$

The complete expressions for the various vectors due to all the sources enumerated above and including those given in Chapter 1 are

$$M_1 = M \sin\theta \qquad (2\text{-}19)$$

$$M_2 = \delta M \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}20)$$

$$M_{z'} = M\cos\theta - \delta M \tan\theta \cos\chi \cos \vec{K}\cdot\vec{r} \qquad (2\text{-}21)$$

$$H_1 = (A + B\cos 2\phi_0 + W \sin 2\phi_0) M \sin\theta - \sum_i h_i \cos(\omega_i t - \phi_0 + \alpha_i) \qquad (2\text{-}22)$$

$$H_1^* = (B \sin 2\phi_0 - W\cos 2\phi_0) M \sin\theta + \sum_i h_i \sin(\omega_i t - \phi_0 + \alpha_i) - \Delta H_0 \sin\theta \qquad (2\text{-}23)$$

$$H_2 = \left[\lambda K^2 + \sin^2\psi\cos^2(\phi_K - \xi) - \frac{\sin 2\psi}{2}\tan\theta\cos\chi\cos(\phi_K - \xi)\right]\delta M\cos\vec{K}\cdot\vec{r} \quad (2\text{-}24)$$

$$H_2^* = \left[\frac{\sin^2\psi}{2}\sin 2(\phi_K - \xi) - \frac{\sin 2\psi}{2}\tan\theta\cos\chi\sin(\phi_K - \xi) - \frac{\Delta H_K}{M}\right]\delta M\cos\vec{K}\cdot\vec{r} \quad (2\text{-}25)$$

and

$$\begin{aligned} H_{z'} = H_o - ZM\cos\theta - CM\sin\theta\cos\phi_o - DM\sin\theta\sin\phi_o \\ + \sum_j h_{z_j}\sin(\omega_j t + \beta_j) \\ + \left[(\lambda K^2 + \cos^2\psi)\tan\theta\cos\chi - \frac{\sin 2\psi}{2}\cos(\phi_K - \xi)\right]\delta M\cos\vec{K}\cdot\vec{r} \end{aligned} \quad (2\text{-}26)$$

where

$$2A = N_x(\sin^2\beta + \cos^2\alpha\cos^2\beta) + N_y(\cos^2\beta + \cos^2\alpha\sin^2\beta) + N_z\sin^2\alpha$$

$$2B = N_x(\sin^2\beta - \cos^2\alpha\cos^2\beta) + N_y(\cos^2\beta - \cos^2\alpha\sin^2\beta) - N_z\sin^2\alpha$$

$$2C = (N_x - N_y)\sin\alpha\sin 2\beta$$

$$2D = (N_x\cos^2\beta + N_y\sin^2\beta - N_z)\sin 2\alpha$$

$$2W = (N_x - N_y)\cos\alpha\sin 2\beta$$

and

$$Z = (N_x\cos^2\beta + N_y\sin^2\beta)\sin^2\alpha + N_z\cos^2\alpha$$

<u>2.2. The Basic Coupled Equations.</u> The transverse components of Fig. 2-3 may be transformed into those shown in Fig. 2-4, and the equations of motion may be written by inspection from $\frac{d\vec{M}}{dt} = \gamma\mu_o(\vec{M} \times \vec{H}_t)$ and are

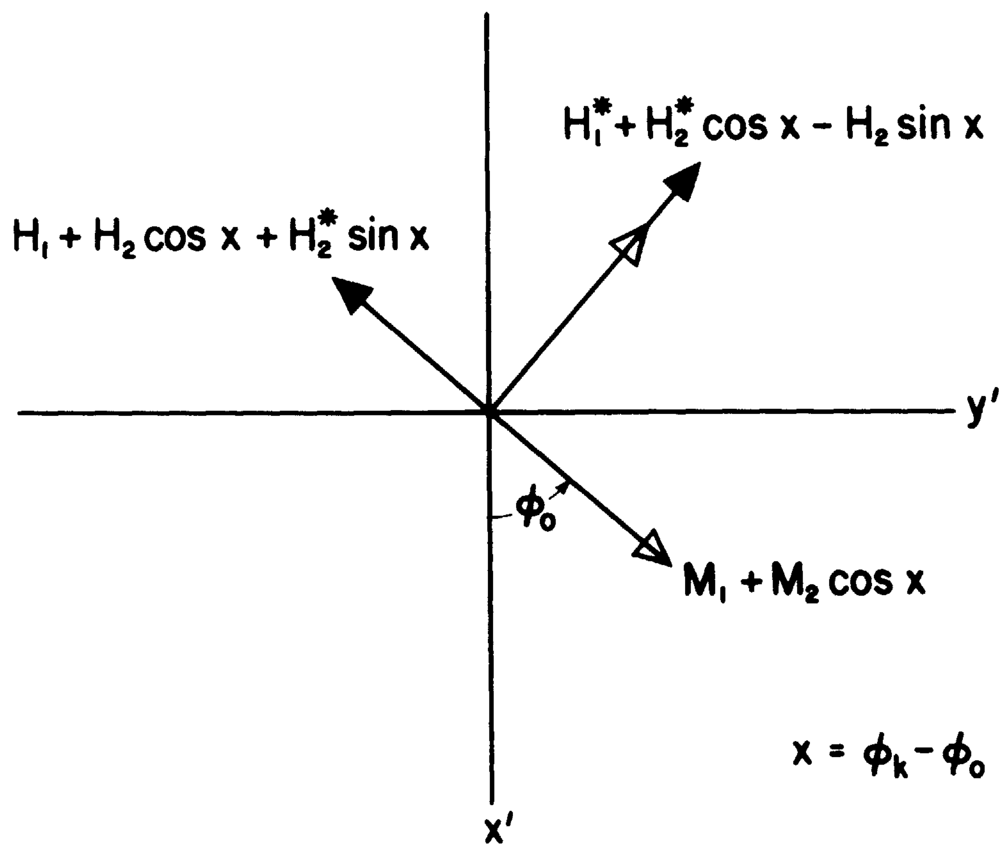

$H_1^* + H_2^* \cos x - H_2 \sin x$
$H_1 + H_2 \cos x + H_2^* \sin x$
y'
ϕ_0
$M_1 + M_2 \cos x$
$x = \phi_k - \phi_0$
x'

Fig. 2-4

$$(M_1 + M_2\cos x)\,\dot{\varphi}_0 + \frac{d}{dt}(M_2 \sin x) = -\gamma\mu_0\Big\{(M_1 + M_2\cos x)\,H_{z'} + M_{z'}(H_1 + H_2\cos x + H_2^*\sin x)\Big\} \tag{2-27}$$

and

$$M_2 \sin x\,\dot{\varphi}_0 - \frac{d}{dt}(M_1 + M_2\cos x) = -\gamma\mu_0\Big\{M_2\sin x\,H_{z'} - M_{z'}(H_1^* + H_2^*\cos x - H_2\sin x)\Big\} \tag{2-28}$$

which become

$$M_1\dot{\varphi}_0 + M_2\cos x\,\dot{\varphi}_K + M_2\sin x = -\gamma\mu_0\Big\{(M_1 H_{z'} + M_{z'}H_1) + (M_2 H_{z'} + M_{z'}H_2)\cos x + M_{z'}H_2^*\sin x\Big\} \tag{2-29}$$

and

$$-\dot{M}_1 - \dot{M}_2\cos x + M_2\sin x\,\dot{\varphi}_K = -\gamma\mu_0\Big\{-M_{z'}H_1^* - M_{z'}H_2^*\cos x + (M_2 H_{z'} + M_{z'}H_2)\sin x\Big\} \tag{2-30}$$

Both $M_{z'}$ and $H_{z'}$ contain spatially invariant terms and terms involving $\cos \vec{k}\cdot\vec{r}$. It is convenient to define $\bar{M}_{z'}$ $(\bar{H}_{z'})$ and $\underline{M}_{z'}$ $(\underline{H}_{z'})$ as the spatially invariant and spatially varying terms respectively.

If we separate (2-29) and (2-30) into constant and $\cos \vec{k}\cdot\vec{r}$ components and then uncouple those equations involving derivatives of both amplitudes and phases, we obtain

$$M_1\dot{\phi}_0 = -\gamma\mu_0\left\{\left(M_1\overline{H}_{z'}+\overline{M}_{z'}H_1\right)+\left(M_2\underline{H}_{z'}+\underline{M}_{z'}H_2\right)\cos\chi+\underline{M}_{z'}H_2^{*}\sin\chi\right\} \tag{2-31}$$

$$\dot{M}_1 = -\gamma\mu_0\left\{\overline{M}_{z'}H_1^{*}+\underline{M}_{z'}H_2^{*}\cos\chi-\left(M_2\underline{H}_{z'}+\underline{M}_{z'}H_2\right)\sin\chi\right\} \tag{2-32}$$

$$M_2\dot{\phi}_K = -\gamma\mu_0\left\{\left(M_2\overline{H}_{z'}+\overline{M}_{z'}H_2\right)+\left(M_1\underline{H}_{z'}+\underline{M}_{z'}H_1\right)\cos\chi-\underline{M}_{z'}H_1^{*}\sin\chi\right\} \tag{2-33}$$

and

$$\dot{M}_2 = -\gamma\mu_0\left\{\overline{M}_{z'}H_2^{*}+\underline{M}_{z'}H_1^{*}\cos\chi+\left(M_1\underline{H}_{z'}+\underline{M}_{z'}H_1\right)\sin\chi\right\} \tag{2-34}$$

It is assumed that products involving $\cos^2 \bar{k}\cdot\bar{r}$ refer to the average spatially invariant component so that a factor of one-half is involved. These are expanded with the aid of (2-19) through (2-26) to give

$$\begin{aligned}
\dot{\phi}_0 = {} & \omega_H-(Z-A)\omega_M\cos\theta+\left(B\cos 2\phi_0+W\sin 2\phi_0\right)\omega_M\cos\theta \\
& -\left(C\cos\phi_0+D\sin\phi_0\right)\omega_M\sin\theta-\cot\theta\sum_i\omega_{h_i}\cos(\omega_i t-\phi_0+\alpha_i) \\
& +\sum_j\omega_{h_{z_j}}\sin(\omega_j t+\beta_j)+\frac{\delta M^2}{2M^2}\omega_M\Big[\cos^2\psi\ \frac{\cos^2\chi}{\cos\theta} \\
& +\frac{\Delta H_K}{M\cos\theta}\sin\chi\cos\chi+\frac{\sin 2\psi}{2}\ \frac{\sin\theta}{\cos^2\theta}\cos^2\chi\cos(\phi_0-\xi) \\
& -\frac{\sin 2\psi}{2}\ \frac{\cos\chi}{\sin\theta}\cos(\phi_K-\xi)-\sin^2\psi\cos^2(\phi_K-\xi)\frac{\cos^2\chi}{\cos\theta} \\
& -\frac{\sin^2\psi}{2}\ \frac{\sin\chi\cos\chi}{\cos\theta}\sin 2(\phi_K-\xi)\Big]
\end{aligned} \tag{2-35}$$

$$
\begin{aligned}
\dot{\theta} = {} & (B \sin 2\phi_0 - W \cos 2\phi_0)\, \omega_M \sin\theta - \omega_{\ell 0} \sin\theta \\
& + \sum_i \omega_{h_i} \sin(\omega_i t - \phi_0 + \alpha_i) + \frac{\delta M^2}{2M^2}\, \omega_M \Bigg[\frac{\Delta H_K}{M} \frac{\sin\theta}{\cos^2\theta} \cos^2\chi \\
& + \frac{\sin 2\psi}{2} \frac{\tan^2\theta}{\cos\theta} \sin(\phi_0 - \xi) \cos^2\chi - \frac{\sin^2\psi}{2} \frac{\sin\theta}{\cos^2\theta} \sin 2(\phi_K - \xi) \cos^2\chi \\
& - \cos^2\psi \frac{\tan\theta}{\cos\theta} \sin\chi \cos\chi + \frac{\sin 2\psi}{2\cos\theta} \sin\chi \cos(\phi_K - \xi) \\
& + \sin^2\psi \frac{\tan\theta}{\cos\theta} \cos^2(\phi_K - \xi) \sin\chi \cos\chi \Bigg]
\end{aligned}
\tag{2-36}
$$

$$
\begin{aligned}
\dot{\phi}_K = {} & \omega_H - (Z - \lambda K^2)\, \omega_M \cos\theta + \sin^2\psi\, \omega_M \cos\theta \cos^2(\phi_K - \xi) \\
& - (C \cos\phi_0 + D \sin\phi_0)\, \omega_M \sin\theta + \sum_j \omega_{h_{z_j}} \sin(\omega_j t + \beta_j) \\
& - \sin 2\psi\, \omega_M \sin\theta \cos\chi \cos(\phi_K - \xi) + (\lambda K^2 + \cos^2\psi - A)\, \omega_M \sin\theta \tan\theta \cos^2\chi \\
& + \tan\theta \Big\{ \cos^2\chi \sum_i \omega_{h_i} \cos(\omega_i t - \phi_0 + \alpha_i) \\
& + \sin\chi \cos\chi \Big[\sum_i \omega_{h_i} \sin(\omega_i t - \phi_0 + \alpha_i) - \omega_{\ell 0} \sin\theta \Big] \Big\} \\
& - \omega_M \sin\theta \tan\theta \cos\chi \, \big[B \cos(\phi_0 + \phi_K) + W \sin(\phi_0 + \phi_K) \big]
\end{aligned}
\tag{2-37}
$$

and

$$\dot{\delta M} = \Big\{ \frac{\sin^2\psi}{2} \cos\theta \sin 2(\phi_k - \xi) - \frac{\Delta H_K}{M} \cos\theta$$
$$- \frac{\sin 2\psi}{2} \sin\theta \sin(2\phi_k - \phi_o - \xi) + \frac{\sin\theta \tan\theta}{2} (\lambda K^2 + \cos^2\psi - A) \sin 2X$$
$$+ \frac{\tan\theta}{2} \sin 2X \sum_i \frac{h_i}{M} \cos(\omega_i t - \phi_o + \alpha_i) \qquad (2\text{-}38)$$
$$+ \tan\theta \cos^2 X \left[\frac{\Delta H_o}{M} \sin\theta - \sum_i \frac{h_i}{M} \sin(\omega_i t - \phi_o + \alpha_i) \right]$$
$$- \tan\theta \sin\theta \cos X \left[B \sin(\phi_o + \phi_k) - W \cos(\phi_o + \phi_k) \right] \Big\} \omega_M \delta M$$

We recall that a quantity ω_q is defined equal to $(-\gamma\mu_o q)$ with the exception of $\omega_{\ell o} = -\gamma\mu_o \Delta H_o$. It is also convenient to define a spin wave loss frequency $\omega_{\ell K} = -\gamma\mu_o \Delta H_k$.

If both δM and $\sin\theta$ are very small, the cross coupling terms between the uniform precession and the spin wave are negligible, and each mode may be considered separately. Equations (2-35) and (2-36), without the spin wave terms, were discussed extensively in Chapter 1. Those results are valid unless an unstable condition is created among one or more of the spin waves. A similar analysis for the spin wave equations (first neglecting the uniform precession) is given in the next section.

2.3. The Generalized Spin Wave Spectrum.

Standing Waves. For simplicity, we will start by assuming that $\sin\theta$ is very small, and that $h_{z_j} = 0$. Equations (2-37) and (2-38) simplify to

$$\dot{\phi}_K = \omega_H - (Z - \lambda K^2)\omega_M \cos\theta + \frac{\sin^2\psi}{2}\omega_M \cos\theta\left[1 + \cos 2(\phi_K - \xi)\right] \tag{2-39}$$

and

$$\dot{\delta M} = \left[\frac{\sin^2\psi}{2}\omega_M \sin 2(\phi_K - \xi) - \omega_{\ell K}\right]\delta M \cos\theta \tag{2-40}$$

Equations of this form were met during the study of the uniform precession in Chapter 1 and may be integrated in a similar manner. It follows that

$$\cos(\phi_K - \xi) = \frac{\sqrt{1 + \cos\eta_K}\,\cos(\omega_K t + \alpha_K - \xi)}{\sqrt{1 + \cos\eta_K \cos(2\omega_K t + 2\alpha_K - 2\xi)}} \tag{2-41}$$

$$\sin(\phi_K - \xi) = \frac{\sqrt{1 - \cos\eta_K}\,\sin(\omega_K t + \alpha_K - \xi)}{\sqrt{1 + \cos\eta_K \cos(2\omega_K t + 2\alpha_K - 2\xi)}} \tag{2-42}$$

and

$$\delta M = \delta M_0\, e^{-\omega_{\ell K}\cos\theta\, t}\sqrt{1 + \cos\eta_K \cos(2\omega_K t + 2\alpha_K - 2\xi)} \tag{2-43}$$

where α_k is an arbitrary phase constant,

$$\cos\eta_K = \frac{-\frac{\sin^2\psi}{2}\omega_M \cos\theta}{\omega_H - (Z - \lambda K^2)\omega_M \cos\theta + \frac{\sin^2\psi}{2}\omega_M \cos\theta} \tag{2-44}$$

and

$$\omega_K^2 = \left(\omega_H - Z\,\omega_M \cos\theta + \lambda K^2 \omega_M \cos\theta\right)$$
$$\times\left(\omega_H - Z\,\omega_M \cos\theta + \lambda K^2 \omega_M \cos\theta + \sin^2\psi\,\omega_M \cos\theta\right) \tag{2-45}$$

The latter equation for $\cos\theta = +1$ and $Z = N_z$, is the small amplitude spin wave spectrum first derived by Suhl.[10] It is shown in Fig. 2-5 for the case of a sphere ($Z = 1/3$) when $\omega_H = \omega_M$.

Some effective thermal driving field, which could have been explicitly written into the equations of motion but was not, is assumed to balance out the spin wave loss term. This implies $\omega_{\ell K}$ can be set equal to zero provided δM_o is replaced by the appropriate rms value.

Typical transverse components of the spin wave are

$\delta M \cos(\phi_k - \xi) \cos \vec{k}\cdot\vec{r}$ and $\delta M \sin(\phi_k - \xi) \cos \vec{k}\cdot\vec{r}$, which become

$\delta M_{rms} \sqrt{1 + \cos\eta_k}\ \cos(\omega_k t - \xi) \cos \vec{k}\cdot\vec{r}$ and

$\delta M_{rms} \sqrt{1 - \cos\eta_k}\ \sin(\omega_k t - \xi) \cos \vec{k}\cdot\vec{r}$ respectively. Except for z' directed waves ($\psi = 0$), the phase ϕ_k is modulated in a manner analagous to that by which the uniform precession is modulated if the transverse demagnetizing factors are unequal. In the latter case the transverse surface demagnetizing field acts as the modulating agent; it is the transverse component of the volume dipolar field in the former case. The field is independent of the sample shape (except for very long spin wave lengths) and is a maximum for $\psi = \pi/2$. The spin wave frequency corresponds to the geometric mean of two

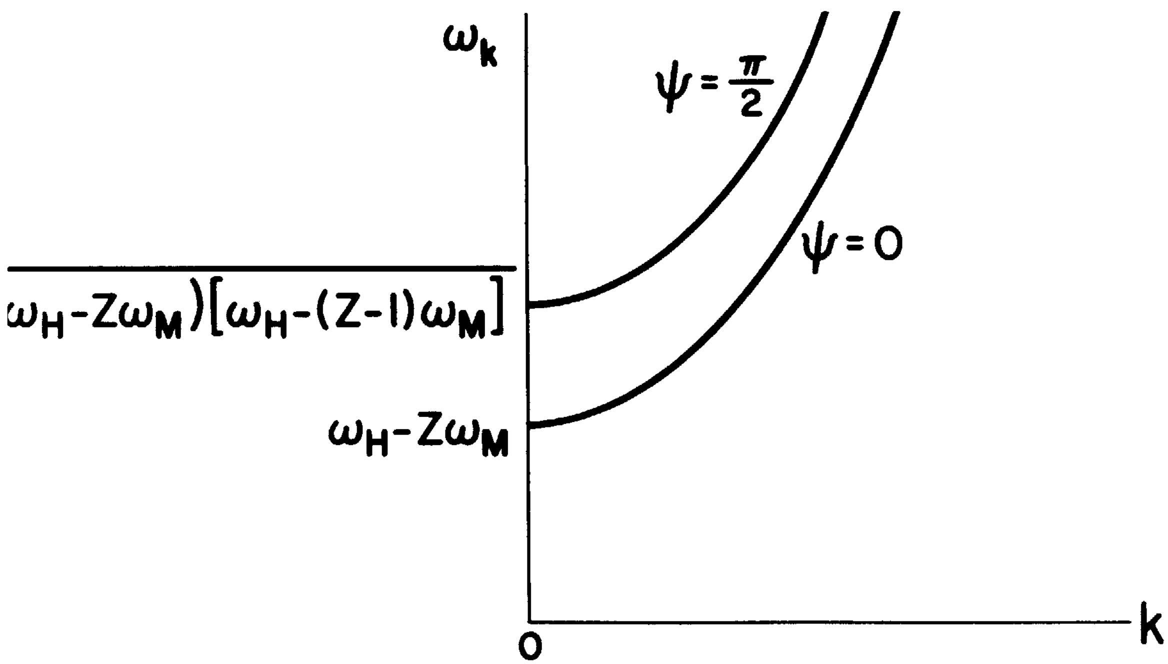


Fig. 2-5

extremes of the instantaneous torque, as is true for the Kittel frequency. The spin wave amplitude is modulated in the same interrelated manner with respect to ϕ_k, as the uniform precession cone angle θ is with respect to ϕ_o. This means that to second order, no spin wave harmonics are created in the transverse plane. The precession path is an ellipse.

Traveling Waves. It was stated earlier that

$\vec{i}'\,\delta M\cos(\phi_k - \vec{k}\cdot\vec{r}) \quad + \quad \vec{j}'\,\delta M\sin(\phi_k - \vec{k}\cdot\vec{r}) \quad + \quad \vec{k}'\,M_z$, does not, in general, describe a progressive wave. This statement may be verified by expanding the transverse components into two standing waves

$\vec{i}'\,\delta M(\cos\phi_k \cos\vec{k}\cdot\vec{r} \;+\; \sin\phi_k \sin\vec{k}\cdot\vec{r}) \;+\; \vec{j}'\,\delta M(\sin\phi_k \cos\vec{k}\cdot\vec{r} - \cos\phi_k \sin\vec{k}$

The cos $\vec{k}\cdot\vec{r}$ terms, already treated, lead to (2-39). It may be shown similarly th the sin $\vec{k}\cdot\vec{r}$ terms lead to (for $\cos\theta = 1$)

$$\dot{\phi}_K = \omega_H - Z\omega_M + \lambda K^2\omega_M + \frac{\sin^2\psi}{2}\omega_M\left[1 - \cos(2\phi_K - 2\xi)\right] \qquad (2\text{-}46)$$

The two equations for $\dot{\phi}_k$ are in obvious contradiction except when $\psi = 0$ so that the assumed spatial variation is incompatible. The difficulty is resolved if the transverse components are initially assumed to be

$$\begin{aligned}(\delta\vec{M})_{transverse} = {} & \vec{i}'\left(\delta M\cos\phi_K \cos\vec{K}\cdot\vec{r} + \delta M'\sin\phi_K' \sin\vec{K}\cdot\vec{r}\right) \\ & + \vec{j}'\left(\delta M\sin\phi_K \cos\vec{K}\cdot\vec{r} - \delta M'\cos\phi_K' \sin\vec{K}\cdot\vec{r}\right)\end{aligned} \qquad (2\text{-}47)$$

where the corresponding primed and unprimed quantities are not equal. Then (2-46), in terms of φ'_k, is no longer inconsistent and may be integrated to give

$$\cos(\varphi_K' - \xi) = \frac{\sqrt{1-\cos\eta_K}\;\cos(\omega_K t + \alpha_K' - \xi)}{\sqrt{1 - \cos\eta_K \cos(2\omega_K t + 2\alpha_K' - 2\xi)}} \tag{2-48}$$

and

$$\sin(\varphi_K' - \xi) = \frac{\sqrt{1+\cos\eta_K}\;\sin(\omega_K t + \alpha_K' - \xi)}{\sqrt{1 - \cos\eta_K \cos(2\omega_K t + 2\alpha_K' - 2\xi)}} \tag{2-49}$$

where α'_k may be set equal to α_k. The equation for $\delta M'$, similar to (2-43) integrates to

$$\delta M' = \delta M'_{rms}\sqrt{1 - \cos\eta_K \cos\left(2\omega_K t + 2\alpha_K' - 2\xi\right)} \tag{2-50}$$

where $\delta M'_{rms} = \delta M_{rms}$, if the effective driving fields for the two components are equal. Without loss of generality, let $\alpha_K - \xi = 0$; the transverse spin wave components are then

$$(\vec{\delta M})_{transverse} = \vec{i}\,'\,\delta M_{rms}\sqrt{1+\cos\eta_K}\;\cos(\omega_K t - \vec{K}\cdot\vec{r}) + \vec{j}\,'\,\delta M_{rms}\sqrt{1-\cos\eta_K}\;\sin(\omega_K t - \vec{K}\cdot\vec{r}) \tag{2-51}$$

and form a traveling wave.

The path of the transverse component of magnetization at any position in space is again an ellipse, except for $\psi = 0$, in which case it is a circle. It is the ellipticity that makes it necessary to construct traveling waves with care.

2.4. Instability Thresholds -- Linear Response. It was shown in Chapter 1, Section 1.4, that under suitable conditions nonlinear as well as linear responses may occur. By this we mean that the magnetization vector can sometimes precess at other than the driving frequency. When including spin wave coupling effects, it is convenient to treat the two classes separately and we start by considering the linear responses first.

Transverse Microwave Excitation. The simplest excitation of a general ferrimagnetic ellipsoid involves a single, positive elliptically-polarized field of frequency ω maintained in the x'y' plane. The degree of ellipticity is assumed to cause the maximum response, and was discussed in detail in Chapter 1. There it was shown that the positive and negative circularly-polarized components h_1 and h_2, respectively, satisfy the relationships

$$\omega_{h1} + \omega_{h2} = \omega_h \sqrt{1 + \cos\eta} \tag{2-52}$$

and

$$\omega_{h1} - \omega_{h2} = \omega_h \sqrt{1 - \cos\eta} \tag{2-53}$$

where

$$\cos\eta = \frac{-\sqrt{(B^2 + W^2)}\,\omega_M}{\sqrt{(B^2 + W^2)\omega_M^2 + \omega^2}} \tag{2-54}$$

This field produces an rms value of the precession cone angle given by

$$\theta_{rms} = \frac{\omega_h}{\left[2(B^2+W^2)\omega_M^2+(\omega_0^2+\omega^2)-2\sqrt{[(B^2+W^2)\omega_M^2+\omega_0^2][(B^2+W^2)\omega_M^2+\omega^2]} + \omega_{10}^2\right]^{1/2}} \tag{2-55}$$

where the resonance frequency ω_o is

$$\omega_o^2 = \left[\omega_H - (Z-A)\omega_M\right]^2 - (B^2+W^2)\,\omega_M^2 \tag{2-56}$$

If only the important coupling terms in the general equations of motion are retained (δM is assumed to be very small), (2-35) through (2-38) become for small θ ,

$$\begin{aligned} \dot{\varphi}_o = & \sqrt{(B^2+W^2)\,\omega_M^2 + \omega^2} + \sqrt{B^2+W^2}\,\omega_M \cos\left(2\varphi_o - \tan^{-1}\frac{W}{B}\right) \\ & - \sqrt{C^2+D^2}\,\omega_M \theta \sin\left(\varphi_o + \tan^{-1}\frac{C}{D}\right) \end{aligned} \tag{2-57}$$

$$\dot{\theta} = \sqrt{B^2+W^2}\,\omega_M \sin\left(2\varphi_o - \tan^{-1}\frac{W}{B}\right)\theta \tag{2-58}$$

$$\dot{\phi}_K = \omega_H - (Z - \lambda K^2)\omega_M + \frac{\sin^2\psi}{2}\,\omega_M\left[1 + \cos 2(\phi_K - \xi)\right]$$
$$- \sqrt{C^2 + D^2}\,\sin\left(\phi_0 + \tan^{-1}\frac{C}{D}\right)\omega_M\,\theta \qquad (2\text{-}59)$$
$$- \frac{\sin 2\psi}{2}\,\omega_M\,\theta\left[\cos(2\phi_K - \phi_0 - \xi) + \cos(\phi_0 - \xi)\right]$$

and

$$\delta\dot{M} = \left[\frac{\sin^2\psi}{2}\sin 2(\phi_K - \xi) - \frac{\theta}{2}\sin 2\psi\,\sin(2\phi_K - \phi_0 - \xi)\right.$$
$$\left. + \frac{\theta^2}{2}(\lambda K^2 + \cos^2\psi - A)\sin 2X - \frac{\Delta H_K}{M}\right]\omega_M\,\delta M \qquad (2\text{-}60)$$

It is apparent from the inspection of (2-60) that the spin wave loss may be overcome by at least two different mechanisms. The first occurs when $\omega_k = \omega/2$, and is a first order coupling in θ;* the second occurs when $\omega_k = \omega$, and involves a second order coupling. It is important only when the former mechanism is not possible, that is, when the frequency $\omega/2$ lies below the spin wave manifold.

<u>1) First Order Coupling.</u> If $\omega > 2(\omega_H - Z\,\omega_M)$, the first

*Suhl has shown that pairs of modes, whose frequencies sum to the uniform precession, may also go unstable via a first order coupling mechanism.[13] The half-frequency waves are merely a special case of the general condition and they do not necessarily have the lowest threshold. Nevertheless, the threshold of the $\psi = \pi/4$ half-frequency wave may be considered typical.

order process will dominate, and all the terms arising from (2-60), which depend on θ^2 or higher, may be ignored. Since the term $-\frac{\sin 2\psi}{2}\theta\sin(2\phi_k - \phi_o - \xi)\,\omega_M \delta M$ is already to first order, it is not necessary to include in (2-57) through (2-59) any of the terms that are proportional to θ. It then follows from the methods previously discussed that

$$
\begin{aligned}
&-\frac{\sin 2\psi}{2}\theta\sin(2\phi_K-\phi_o-\xi)\,\omega_M\,\delta M = \\
&-\frac{\sin 2\psi}{2}\theta_{rms}\Big[\sqrt{1+\cos\eta}\,\cos(\omega t-\beta_1)\sin(2\phi_K-\xi-\beta_1) \\
&-\sqrt{1-\cos\eta}\,\sin(\omega t-\beta_1)\cos(2\phi_K-\xi-\beta_1)\Big]\,\omega_M\,\delta M
\end{aligned}
\tag{2-61}
$$

where

$$\beta_1 = \frac{1}{2}\tan^{-1}\frac{W}{B}$$

The spin wave phase is given by (2-41) and its amplitude by

$$\delta M = \delta M_o\sqrt{1 + \cos\eta_K\cos(2\omega_K t + 2\alpha_K - 2\xi)} \tag{2-62}$$

In many cases, the values of $\cos\eta$ and $\cos\eta_k$ are small compared to unity, and terms involving their squares may be neglected. If this is done, (2-61) simplifies to

$$
\begin{aligned}
&-\frac{\sin 2\psi}{2}\theta\sin(2\phi_K-\phi_o-\xi)\,\omega_M\,\delta M = \\
&-\frac{\sin 2\psi}{2}\theta_{rms}\sin\Big[(2\omega_K-\omega)t+(2\alpha_K-\xi)\Big]\,\omega_M\,\delta M_o
\end{aligned}
\tag{2-63}
$$

The other important term in (2-60) is $\frac{\sin^2\psi}{2}\sin(2\phi_k - 2\xi)\,\omega_M\,\delta M$ and it may be expanded in a similar manner except that now the first order terms in (2-59) must be retained. We use the unperturbed solution for θ and ϕ_o and let $\phi_k = \phi_k^\circ + \delta_1 \cos(\omega t - \beta_1) - \delta_2 \sin(\omega t - \beta_1) + \ldots$ (ϕ_k° is the unperturbed solution). It then follows that

$$\delta_1 = \frac{\omega_M}{\omega}\,\theta_{rms}\sqrt{1-\cos\eta}\left[\sqrt{C^2+D^2}\cos(\beta_1+\beta_2) - \frac{\sin 2\psi}{2}\sin(\beta_1-\xi)\right] \qquad (2\text{-}64)$$

and

$$\delta_2 = \frac{\omega_M}{\omega}\,\theta_{rms}\sqrt{1+\cos\eta}\left[\sqrt{C^2+D^2}\sin(\beta_1+\beta_2) + \frac{\sin 2\psi}{2}\cos(\beta_1-\xi)\right] \qquad (2\text{-}65)$$

where

$$\beta_2 = \tan^{-1}\frac{C}{D}$$

If terms in $(\cos\eta_k)^2$ or higher are again neglected, the result is

$$\begin{aligned}\frac{\sin^2\psi}{2}\sin(2\phi_K - 2\xi)\,\omega_M\,\delta M \;\simeq\;& \frac{\sin^2\psi}{2}\Big\{\delta_1\cos\left[(2\omega_K-\omega)t + (2\alpha_K - 2\xi + \beta_1)\right] \\ &+ \delta_2 \sin\left[(2\omega_K-\omega)t + (2\alpha_K-2\xi+\beta_1)\right]\Big\}\,\omega_M\,\delta M_o\end{aligned} \qquad (2\text{-}66)$$

An instability threshold occurs when $\omega_k = \omega/2$, and the sum of (2-63) and (2-66) just equals the loss term. The critical value of θ_{rms} is given by the equation

$$\theta_{rms}\Big\{ \sin^2\psi \frac{\omega_M}{\omega} \sqrt{C^2+D^2}\Big[\sqrt{1-\cos\eta}\cos(\beta_1+\beta_2)\cos(2\alpha_K+\beta_1-2\xi)$$

$$+\sqrt{1+\cos\eta}\sin(\beta_1+\beta_2)\sin(2\alpha_K+\beta_1-2\xi)\Big]$$

$$+\frac{\sin^2\psi\sin 2\psi}{2}\frac{\omega_M}{\omega}\Big[\sqrt{1+\cos\eta}\cos(\beta_1-\xi)\sin(2\alpha_K+\beta_1-2\xi) \qquad (2\text{-}67)$$

$$-\sqrt{1-\cos\eta}\sin(\beta_1-\xi)\cos(2\alpha_K+\beta_1-2\xi)\Big]$$

$$-\sin 2\psi\sin(2\alpha_K-\xi)\Big\} = \frac{2\Delta H_K}{M}$$

The critical field required to produce this θ_{rms} may be found from (2-55). The values of ψ, α_k, and ξ are chosen to maximize the left-hand side of (2-67), but ψ must correspond to one of the allowed half-frequency spin waves. For example, if half the driving frequency lies very near the bottom of the spin wave spectrum, the angle ψ is constrained to be small.

When the sample is a spheroid magnetized along the axis of revolution, B = W = C = D = 0. It follows therefore that

$$\theta_{crit} = \frac{2\Delta H_K}{M\sin 2\psi\left|1-\frac{\sin^2\psi}{2}\frac{\omega_M}{\omega}\right|} \qquad (2\text{-}68)$$

This case, treated by Suhl,[10] has a minimum threshold near $\psi = \pi/4$.

2) Second Order Coupling. If $\omega < 2(\omega_H - Z\,\omega_M)$, the first order coupling is forbidden, and the second order process occurs when $\omega_k = \omega$. The threshold may be obtained by inspection from (2-60) and is

$$\theta_{rms}^{2}\left(\cos^{2}\psi + \lambda k^{2} - A\right) = \frac{2\Delta H_K}{M} \qquad (2\text{-}69)$$

The maximum coupling occurs for $\psi = 0$, and the frequency condition implies that, very nearly, $\lambda k^2 = A$ (exactly true for a spheroid) so that

$$\theta_{crit} \simeq \sqrt{\frac{2\Delta H_K}{M}} \qquad (2\text{-}70)$$

as given by Suhl.[10,11] It should be noted that the internal modulating fields will add correction terms to this threshold, as they did to the first order threshold. They will be important, however, only at reasonably low frequencies, and then the first order process will usually be the important one.

Additional Microwave Excitation. It seems reasonable to suppose that additional microwave fields can produce coupling between the uniform precession and certain spin waves similar to that already described. A great variety of modulation and cross modulation effects appears possible, all governed by (2-35) through (2-38), and certain basic ones are explored here. For simplicity, we assume a spheroidal sample magnetized along its axis of revolution. For this geometry, it is convenient to define $N_x = N_y = N_t$.

1) Transverse Fields. Let us assume that there are two transverse circularly-polarized driving fields: the one, h_1, is driving the linear resonance ($\omega_1 = \omega_0$), the other, h_2, is at some arbitrary frequency ω_2 not equal to ω_0. The motion of the uniform precession is governed by

$$\phi_0 \simeq \omega_1 t - \delta \sin\left[(\omega_2-\omega_1)t + \alpha_2\right] \tag{2-71}$$

and

$$\theta \simeq \theta_0 \left\{ 1 - \delta \cos\left[(\omega_2-\omega_1)t + \alpha_2\right] \right\} \tag{2-72}$$

where

$$\theta_0 = \frac{\omega_{h1}}{\omega_{\ell 0}} \quad \text{and} \quad \delta = \frac{\omega_{h2}}{\theta_0 (\omega_2-\omega_1)}.$$

These two latter quantities are assumed to be small compared to unity.

The second term in (2-60) has an average value of $-\frac{\theta_0}{2} \sin 2\psi\, \delta \sin(\xi + \alpha_2)$ provided that a spin wave of frequency $\omega_k = \omega_2/2$ exists in the manifold. Since δ is assumed to be small, this term will never be important when the first order coupling process $\omega_k = \omega_1/2$ is possible; if it is not possible the term will compete with the second order coupling ($\omega_k = \omega_1$). The new coupling will dominate when $\theta_0^{\,2} < \frac{2\,\omega_{\ell k}}{\omega_M}$ and under these circumstances the threshold for the $\omega_k = \omega_2/2$ wave is

$$\omega_{hz} = \frac{2\,\omega_{\ell k}}{\omega_M \sin 2\psi}\left(\omega_2 - \omega_1\right) \tag{2-73}$$

This threshold is independent of θ_o provided that the latter is large enough to assure that δ is small. All of these conditions may be met simultaneously if $\sin 2\psi \gg \sqrt{\dfrac{2\,\omega_{\ell k}}{\omega_M}}$.

2) Longitudinal Fields. The presence of a longitudinal modulating field h_z of frequency ω_z causes frequency modulation of both ϕ_o and ϕ_k. They are given by

$$\dot{\phi}_o = \omega_1 + \omega_{hz} \sin\left(\omega_z t + \beta_z\right) \tag{2-74}$$

and

$$\dot{\phi}_K = A_K + B_K \cos\left(2\phi_K - 2\xi\right) + \omega_{hz} \sin\left(\omega_z t + \beta_z\right) \tag{2-75}$$

where

$$A_K = \omega_H - \left(Z - \lambda K^2\right)\omega_M + \frac{\sin^2\psi}{2}\,\omega_M$$

$$B_K = \frac{\sin^2\psi}{2}\,\omega_M$$

and β_z is a phase constant. For simplicity, assume that $\omega_k \gg B_k$ so that cross modulation effects are negligible.* Then retaining important terms, one has

*Cross modulation refers to a possible coupling between the longitudinal and transverse modulating processes. This should be taken into account at very low frequencies since it will increase or decrease δ_z depending on the value of β_z.

$$\phi_0 = \omega_1 t - \delta_z \cos(\omega_z t + \beta_z) \tag{2-76}$$

and

$$\phi_K = \omega_K t + \alpha_K - \delta_z \cos(\omega_z t + \beta_z) \tag{2-77}$$

with

$$\delta_z = \frac{\omega_{hz}}{\omega_z}$$

The transverse circularly-polarized field h_1 of frequency ω_1 is still present, but the cone angle is assumed to be less than that required for a second order instability. We need only the first two terms of (2-60), which are

$$\frac{\sin^2\psi}{2} \sin\left[2\omega_K t + 2\alpha_K - 2\xi + 2\delta \cos(\omega_z t + \beta_z)\right]$$

$$- \frac{\theta_0}{2} \sin 2\psi \sin\left[(2\omega_K - \omega_1)t + 2\alpha_K + \delta \cos(\omega_z t + \beta_z) - \xi\right]$$

These may be expanded to give terms that can possess an average value. The threshold equation (neglecting the $\omega_k = \omega_1/2$ term, which has already been considered) is, then

$$\begin{aligned} &\sin^2\psi \sum_{n=1}^{\infty} J_n(2\delta_z) \sin\left[2(\omega_K - \tfrac{n}{2}\omega_z)t + \phi_n\right] \\ &- \theta_0 \sin 2\psi \sum_{n=1}^{\infty} J_n(\delta_z) \sin\left[(n\omega_z \pm 2\omega_K \mp \omega_1)t - \phi_n^{\pm}\right] = \frac{2\Delta H_K}{M} \end{aligned} \tag{2-78}$$

where $J_n(\delta)$ is a Bessel function of order n; and, ϕ_n, ϕ_n^+, or ϕ_n^- may each be set equal to $\pi/2$ for any n, in order to maximize the coupling. The first set of terms in (2-78) is due to direct parametric coupling, independent of θ (the field h_1 could be zero), between the longitudinal modulating field and the spin wave. An average value exists for the discrete set of spin wave frequencies

$$\omega_K = \frac{n}{2}\,\omega_z \qquad\qquad n = 1, 2, 3, \cdots \tag{2-79}$$

provided they exist within the spin wave manifold. The strongest coupling occurs when n = 1, which is analagous to the direct parametric coupling of Chapter 1 that concerns the uniform precession. In the case of spin wave coupling, the transverse volume dipolar field causes frequency mixing, while in the case of the uniform precession the mixing depends on the shape dependent surface dipolar field. This coupling was discovered independently by the author,[1,14] and by Schlömann et al.[15] The latter have also experimentally verified its existence. It is important to note that with the coupling to the uniform precession, a definite phase relation between the transverse and longitudinal fields is necessary to maximize the effect. This is not true in the case of the spin wave coupling since a spin wave is always available with the correct phase.

Let us assume that a small value of n is allowed, and that θ_o is maintained below any of the first or second order thresholds. Then, according to (2-78) the condition

$$\sin^2\psi \; J_n(2\delta_z) = \frac{2\Delta H_K}{M} \tag{2-80}$$

causes an instability. Since n is assumed small, ω_z must normally be large and δ_z small compared to unity. The Bessel function can therefore be approximated as

$$J_n(2\delta_z) \simeq \frac{1}{n!} \, \delta_z^{\,n} \tag{2-81}$$

which when substituted into (2-80) leads to a threshold

$$\omega_{h_z} = \left(\frac{2n! \; \omega_{\ell K}}{\omega_M \sin^2\psi} \right)^{\frac{1}{n}} \omega_z \tag{2-82}$$

The minimum threshold occurs, of course, for n = 1, provided the corresponding spin wave exists, and is

$$\omega_{h_z} = \frac{2\,\omega_{\ell K}}{\omega_M \sin^2\psi} \, \omega_z \tag{2-83}$$

If ω_z is very small, the above mentioned process is not possible because n would have to be very large in order for $(n/2)\omega_z$ to be in the spin wave manifold. Therefore the coupling coefficient would be too small to allow ω_{h_z} to be physically realized. On the other hand, if ω_z and/or ΔH_k were too large, the process might again require a prohibitively large field.

The second set of terms of (2-78) has an average value if

$$\omega_K = \frac{\omega_1 \pm n\,\omega_z}{2} \qquad n = 1, 2, 3, \cdots \tag{2-84}$$

This first order coupling is important only when $\omega_k = \omega_1/2$ is below the spin wave manifold. Under such conditions, the minus sign in (2-84) is ruled out. The remaining possibility competes with the second order coupling and in order to dominate it must satisfy the inequalities

$$\theta_o \sin 2\psi \, J_n(\delta_z) > \frac{2\Delta H_K}{M} \tag{2-85}$$

and

$$\sin 2\psi \, J_n(\delta_z) > \sqrt{\frac{2\Delta H_K}{M}} \tag{2-86}$$

As an example of this process, consider the case where the transverse driving field is exciting the resonance ($\omega_1 = \omega_o$), and the bottom of the spin wave spectrum (for the spheroid) is $\omega_o/2$. This latter condition requires that $\omega_o = 2N_t\omega_M$. Consider the k = 0 extensions of the spin wave manifold,* which become

$$\omega_K^2 = \frac{\omega_o}{4}\left(\omega_o + 2\omega_M \sin^2\psi\right) \tag{2-87}$$

Since $\omega_k = \dfrac{\omega_o + n\omega_z}{2}$, it follows that

$$\sin^2\psi = \frac{n\omega_z(2\omega_o + n\omega_z)}{2\omega_o\omega_M} \tag{2-88}$$

If $n\omega_z$ is small compared to $2\omega_o$ and ω_M, then

---

*Strictly speaking, the exact magnetostatic modes[16] should be considered in this range, however no very serious error appears to be committed by extrapolating the Suhl spin wave spectrum to zero k values.

$$\sin 2\psi \simeq 2\sqrt{\frac{n\,\omega_z}{\omega_M}} \tag{2-89}$$

and (2-86) reduces to

$$2\sqrt{\frac{n\,\omega_z}{\omega_M}}\; J_n\left(\frac{\omega_{h_z}}{\omega_z}\right) > \sqrt{\frac{2\,\Delta H_K}{M}} \tag{2-90}$$

The value of $J_n\left(\frac{\omega h_z}{\omega_z}\right)$ decreases much more rapidly than $\frac{1}{\sqrt{n}}$ so that the minimum threshold occurs for $n = 1$, and is

$$J_1\left(\frac{\omega_{h_z}}{\omega_z}\right) > \frac{1}{2}\sqrt{\frac{2\,\omega_{\ell K}}{\omega_z}} \tag{2-91}$$

Since $J_1(\delta_z)$ has a maximum value of .5819 when $\delta_z = .84$, we see that ω_z must be at least as large as

$$\omega_z \geqslant 1.5\;\omega_{\ell K} \tag{2-92}$$

for the inequality (2-91) to hold. If the minimum condition exists, the first threshold is given by

$$h_z = 1.26\;\Delta H_K \tag{2-93}$$

<u>2.5. Instability Thresholds -- Nonlinear Resonance.</u> Under certain conditions, the principal response of the magnetization may occur at other than the driving frequency. These cases have in common the presence of longitudinal modulating fields, either applied or self-induced, and an applied transverse

field that is rotating at a frequency different from the magnetization. Both fields can couple to certain of the spin waves and may be treated by the methods of the previous section. As an example, consider a sphere that is excited in the transverse plane by a field h_1 of frequency ω_1 and simultaneously by a strong longitudinal field h_2 of a lower frequency ω_2. The resonant frequency and the principal response are assumed to be at the difference frequency $\omega_o = \omega_H = \omega_1 - \omega_2$. Let us further assume that $\omega_1 = 4\omega_M$, $\omega_2 = 3\omega_M$, and $\omega_o = \omega_M$. Under these conditions, $\omega_k = \omega_o/2$ is below the spin wave manifold and there are three possible instabilities. Transverse and longitudinal couplings occur when $\omega_k = \omega_1/2$ or $\omega_k = \omega_2/2$, respectively, and the normal second order coupling occurs when $\omega_k = \omega_1 - \omega_2$. The latter coupling is negligible compared to the former so that the threshold equation is given by

$$\frac{\sin 2\psi}{2}\frac{\omega_{h1}}{\omega_2}\cos\left(\omega_K - \frac{\omega_1}{2}\right)t + \frac{\sin^2\psi}{2}\frac{\omega_{h2}}{\omega_2}\cos 2\left(\omega_K - \frac{\omega_2}{2}\right)t = \frac{\omega_{\ell K}}{\omega_M} \tag{2-94}$$

The threshold for the $\omega_k = \omega_1/2$, $\psi = \pi/4$ spin wave is

$$\omega_{h1} = \frac{2\,\omega_{\ell K}\,\omega_2}{\omega_M} = 6\,\omega_{\ell K} \tag{2-95}$$

and the threshold for the $\omega_k = \omega_2/2$, $\psi = \pi/2$ spin wave is

$$\omega_{h_2} = \frac{2\,\omega_{\ell K}\,\omega_2}{\omega_M} = 6\,\omega_{\ell K} \tag{2-96}$$

These thresholds are independent of θ_o as long as $\theta_o > \frac{\omega_{h_1}}{\omega_2}$. Since $\theta_o = \frac{\omega_{h_1}}{2\omega_{\ell o}} \frac{\omega_{h_2}}{\omega_2}$ for this case of nonlinear resonance (see Chapter 1, Equation (1-78)) it follows that $\omega_{h_2} > 2\,\omega_{\ell o}$ in order that the nonlinear response is dominant. If $\omega_{\ell k} = \omega_{\ell o}$ this latter condition is met by a factor of 3 at the threshold given by (2-96).

Subharmonic Resonance. The example just given assumes that the longitudinal modulating field is externally applied. In general, however, there can be longitudinal fields, which are induced by the precessing magnetization, as in the case of subharmonic resonance. In Chapter 1, Section 1.4, it was shown (neglecting spin waves) that for an ellipsoid not magnetized along a principal axis, excited transversely by a circularly-polarized field h of frequency ω_1 and biased to the half-frequency, a resonance can occur at $\omega_1/2$ given by

$$\theta = \frac{h_{crit}}{\Delta H_o} \sqrt{\frac{1}{2}\left(1 - \frac{h_{crit}}{h}\right)} \tag{2-97}$$

provided that h exceeds h_{crit}. For a thin disc ($N_z = 1$), magnetized at an angle α with respect to its plane, the threshold is given by

$$h_{crit} = \frac{2\omega_1}{\omega_M \sin 2\alpha} \Delta H_o \tag{2-98}$$

It is interesting to observe the spin wave behavior as h slowly increases from zero. Below h_{crit}, the subharmonic resonance is impossible, and the frequency ω_1 . The cone angle is given by

$$\theta_o \simeq \frac{\omega_h}{\sqrt{(\omega_o-\omega_1)^2+\omega_{\ell o}^2}} \simeq \frac{2\,\omega_h}{\omega_1} \tag{2-99}$$

and since the sample is not resonant, it is quite small. Now half-frequency spin waves ($\omega_1/2$) are surely present in the spin wave manifold since the disc is biased to the half-frequency, therefore first order spin wave coupling must be considered. For the geometry mentioned above, the resonance frequency is

$$\omega_o^2 = \left(\omega_H - \omega_M \cos^2\alpha\right)\left(\omega_H - \omega_M \cos^2\alpha + \omega_M \sin^2\alpha\right) \tag{2-100}$$

and the spin wave spectrum is

$$\omega_k^2 = \left(\omega_H - \omega_M \cos^2\alpha + \lambda k^2 \omega_M\right)\left(\omega_H - \omega_M \cos^2\alpha + \lambda k^2 \omega_M + \omega_M \sin^2\psi\right) \tag{2-101}$$

If $\omega_k = \omega_o$, it follows that $\psi_{max} = \alpha$ when k = 0 so that the threshold value of θ for the usual first order coupling process is

$$\theta \simeq \frac{2\,\omega_{\ell k}}{\omega_M \sin 2\psi_{max}} = \frac{2\,\omega_{\ell k}}{\omega_M \sin 2\alpha} \tag{2-102}$$

which implies that

$$h_{crit,\,first\ order} = \frac{\omega_1}{\omega_M \sin 2\alpha}\,\Delta H_o \tag{2-103}$$

This field is just one - half of that needed to cause the subharmonic resonance. If h is increased further, the cone angle remains "stuck" at its threshold value. At this point it is worthwhile to pause and reflect upon the obvious relationship between (2-98) and (2-103) and the meaning of the factor of 2. In the first place, the subharmonic resonance can certainly be considered as a half-frequency spin wave in the limit as $k \rightarrow 0$. The entire spin wave theory presented in this thesis is not quantitatively valid in such cases and it is tempting to reconcile the factor of 2 on that account. A detailed analysis of this point indicates however that the spin wave analysis is correct in this instance and that the correct interpretation of the numerical discrepancy arises from the fact that the two thresholds in question do not refer to the same physical event. Equation (2-103) indicates the amplitude of the field needed to cause the first departure from the linear response. As this threshold is overreached, the half-frequency mode will build up to larger amplitude and there will be coexistence of both responses. Equation (2-98) indicates the point at which the half-frequency mode suddenly becomes resonant -- its amplitude grows rapidly according to (2-97). It might be argued that since the response would then be primarily at the half-frequency, the first order instability process would have to shift to the quarter frequency ($\omega_1/4 = \omega_0/2$), and consequently would have a higher threshold than (2-102) -- (infinite if $\omega_0/2$ were below the spin wave band). Although this reasoning is correct it may be shown that there will always be some residual linear response (a modulation product of the subharmonic resonance) sufficient to overcome the loss of $\omega_k = \omega_1/2$ spin waves. It appears, therefore, that the subharmonic response will be limited by a modified first order coupling process to approximately the same value as the saturated linear response.

2.6. The Measurement of ΔH_k. It is clear from the preceding discussions that a knowledge of how ΔH_k depends on the spin wave propagation vector $\vec{k}$ is necessary in order to properly evaluate any of the various instability thresholds. Most of the recent experimental investigations, which aim to uncover such information, have involved measurements of the power required to cause either a first or second order instability. In the latter case, it is the degenerate $\psi = 0$ spin waves that are involved ($\lambda k^2 = A$); whereas, in the former case, it is usually disturbances of small k value that are important. The majority of experimental results cluster therefore in one of the two regions.[17,18] Since the spin wave spectrum is a function of the saturation magnetization, some efforts have been made to alter the particular range of spin waves -- either by changing the temperature of the sample, or by carrying out the experiments on a series of samples that differ in chemical composition. Both methods give various values of the magnetization but unfortunately do not leave other parameters (such as ΔH_k) invariant. Correlations of measurements taken at different frequencies suffer from the obvious disadvantages.

We will now describe a method by which it appears possible to measure, at will, the line width of _any_ degenerate spin wave, so long as the normal first order coupling process is impossible. (This can always be arranged if the operating frequency is high enough.) The key to this method is in the realization that additional microwave fields, in conjunction with the normal transverse positively-polarized excitation, can cause a more flexible parametric excitation. The various couplings outlined in Section 2.4 of this chapter suggest the use of a spheroid that is acted upon simultaneously by two transverse circularly-polarized fields of amplitudes h_1 and h_2, having frequencies of $\omega_1 = \omega_o$ and ω_2 respectively, plus a longitudinal field of amplitude h_3 having a frequency ω_3. Parametric couplings to spin waves of frequencies ω_1, $\omega_2/2$, and $\omega_3/2$, respectively, exist for such fields and normally the process with the lowest threshold dominates. If, however, $\omega_2 = \omega_3 = 2\omega_1$ all the couplings involve $\omega_k = \omega_1$ spin waves, and the remaining questions concern the values ψ and ξ of the spin wave with minimum threshold. This threshold is

$$\sin^2\psi\left(\frac{\omega_{h3}}{2\omega_1}\right)\sin(\beta_3+2\xi-2\alpha_K) - \sin 2\psi\left(\frac{\omega_{h2}}{\omega_1}\right)\sin(\xi+\alpha_2) + \frac{\omega_{h1}^2}{\omega_{\ell o}^2}\left(\cos^2\psi+\lambda K^2-N_t\right)\sin 2\alpha_K = \frac{2\Delta H_K(\psi,\xi)}{M} \tag{2-104}$$

where since $\omega_k = \omega_1 = \omega_o$, the relationship between λk^2 and ψ is known. For simplicity consider $\sin^2\psi\,\omega_M << \omega_H - N_z\omega_M + \lambda k^2\omega_M$ so that the spin wave frequency may be approximated as

$$\omega_K \simeq \omega_H - N_z\omega_M + \lambda K^2\omega_M + \frac{1}{2}\sin^2\psi\ \omega_M \tag{2-105}$$

Because

$$\omega_o = \omega_H - N_z\omega_M + N_t\omega_M \tag{2-106}$$

it follows that

$$\lambda K^2 \simeq N_t - \frac{1}{2}\sin^2\psi \qquad (2\text{-}107)$$

and the quantity $(\cos^2\psi + \lambda k^2 - N_t)$ in (2-104) may be written as $(\cos^2\psi - 1/2\,\sin^2\psi)$. It is apparent from (2-104) that by the appropriate setting of h_1, h_2, and h_3, the net coupling may be maximized for any given ψ. The inverse problem of determining $\Delta H_k(\psi, \xi)$ from a series of measurements, is more difficult since although α_2 and β_3 are controllable phases, ξ and α_K are not. If $\Delta H_k(\psi, \xi)$ were a known function, we could predict exactly which spin wave would go unstable first for any given setting of h_1, h_2, h_3, α_2, and β_3 and thus verify the assumption. However if the loss surface is reasonably well-behaved, it should be possible to arrive at it experimentally by an iterative procedure. For example, the loss surface might be considered plane, and the values of ψ and ξ predicted for a series of experiments. The measured points would be ascribed to these values, which in turn could be used to predict for the next set of experiments etc.

One simple and rather amusing experiment could determine the minimum value of ΔH_k among the degenerate spin waves, although it initially would yield no information as to the appropriate value of ψ. If (2-105) is approximately valid and $h_2 = 0$, the threshold equation becomes

$$\sin^2\psi\,\frac{\omega_{h3}}{2\omega_1}\sin\left(\beta_3 + 2\xi - \frac{\pi}{2}\right) + \left(\cos^2\psi - \frac{1}{2}\sin^2\psi\right)\frac{\omega_{h1}^2}{\omega_{10}^2} = \frac{2\Delta H_K(\psi,\xi)}{M} \qquad (2\text{-}108)$$

where α_k will equal to $\pi/4$ to maximize the coupling. Now let $\frac{\omega h_3}{2\omega_1} = 3/2 \frac{\omega h_1^2}{\omega_{10}^2}$, and let both driving fields be increased until the threshold is reached. If β_3 is experimentally varied until the threshold is minimized, it is certain that $\sin(\beta_3 + 2\xi - \pi/2)$ equals unity. The important feature is that now the left-hand side of (2-108) is independent of ψ so that all degenerate spin waves are being driven equally; the first wave to go unstable obviously has the minimum value of ΔH_k.

The degenerate spin waves are not the only ones that can be measured by such techniques, for many different field combinations may be used. As a single example, an additional field h_2 of frequency ω_2 (greater than ω_1) can produce instabilities among the $\omega_2/2$ spin waves, provided that the frequency $\omega_1/2$ is below the spin wave manifold. The unstable modes presumably have the largest possible value of $\sin 2\psi$ and could have frequencies greater or smaller than the resonant frequency ω_1.

CHAPTER 3

THE TRANSIENT MAGNETIZATION

In the preceding chapters, there were derived for an arbitrarily magnetized ellipsoid, the general equations of motion for the spatially-uniform component of magnetization, and a typical, small-amplitude, spin wave. The results obtained assumed for the most part steady-state conditions. It is of interest, however, to inquire into non steady-state aspects of the magnetization and in this chapter we consider the transient behavior of the uniform precession and the spin wave spectrum together with their mutual interaction. Since the general problems are very difficult, if not impossible, the discussion must be limited in scope to some of the more manageable topics. Those considered fall into two general categories. The first assumes a steady-state magnetizing field and considers the transient response of the magnetization when a circularly-polarized driving field is suddenly turned on or off. The second assumes that the magnetizing field itself has undergone a sudden change (such as 180° reversal) and considers the spin wave interactions that are present until the magnetization reorients itself with the field.

3.1. The Build Up of the Uniform Precession.

The Transient Response to a Transverse Circularly-Polarized Driving Field. As a prelude to the general transient problems involving pulsed excitation, let us review a very much simpler problem -- that of the build up of the uniform precession in a sphere (neglecting spin wave interaction). We assume that the sample is magnetized by a field H_o and that a transverse circularly-polarized driving field h of frequency ω is suddenly applied at time $t = 0$. For $t \geqslant 0$, the equations of motion are

$$\dot{\phi}_o = \omega_o - \omega_h \cot\theta \cos(\omega t - \phi_o + \alpha) \tag{3-1}$$

and

$$\dot{\theta} = \omega_h \sin(\omega t - \phi_o + \alpha) - \omega_{\ell o} \sin\theta \tag{3-2}$$

where

$$\omega_o = \omega_H$$

The precession rate quickly equals the driving frequency, and the cone angle starts to build up. Compared to this build up, the ϕ variation is normally very rapid and we may therefore assume that the phase angle α adjusts instantaneously to its appropriate values during the θ transient. These are given by (3-1), which reduces when $\phi_o = \omega t$ to

$$\cos\alpha \cot\theta = \frac{\omega_o - \omega}{\omega_h} \tag{3-3}$$

Equation (3-2) becomes

$$\dot{\theta} = \omega_h \sin\alpha - \omega_{\ell o} \sin\theta \tag{3-4}$$

The most important case is that of resonance ($\omega = \omega_o$) for which (3-3) simplifies to

$$\cos\alpha \cot\theta = 0 \tag{3-5}$$

Initially $\cos\alpha = 0$ and it remains at that value as long as $\theta < \pi/2$. It follows therefore that

$$\dot{\theta} = \omega_h - \omega_{\ell o} \sin\theta \tag{3-6}$$

which may be integrated to yield

$$\tan\frac{\theta}{2} = \frac{\omega_h\left(e^{\sqrt{\omega_{\ell o}^2 - \omega_h^2}\, t} - 1\right)}{\left(\omega_{\ell o} + \sqrt{\omega_{\ell o}^2 - \omega_h^2}\right) e^{\sqrt{\omega_{\ell o}^2 - \omega_h^2}\, t} + \left(\sqrt{\omega_{\ell o}^2 - \omega_h^2} - \omega_{\ell o}\right)} \tag{3-7}$$

when $\omega_{\ell o} > \omega_h$;

$$\tan\frac{\theta}{2} = \frac{\omega_{\ell o} t}{\omega_{\ell o} t + 2} \tag{3-8}$$

when $\omega_{\ell o} = \omega_h$;

and

$$\tan\frac{\theta}{2} = \frac{\omega_{\ell o}}{\omega_h} + \frac{\sqrt{\omega_h^2 - \omega_{\ell o}^2}}{\omega_h} \tan\left[\frac{\sqrt{\omega_h^2 - \omega_{\ell o}^2}}{2} t - \tan^{-1}\left(\frac{\omega_{\ell o}}{\sqrt{\omega_h^2 - \omega_{\ell o}^2}}\right)\right] \tag{3-9}$$

when $\omega_{\ell o} < \omega_h$ (for all $t \geqslant 0$ such that $\theta < \pi/2$). In the latter case, as soon as $\theta = \pi/2$, (3-5) is satisfied independent of α , and $\sin\alpha$ changes from unity to $\frac{\omega_{\ell o}}{\omega_h}$. The cone angle then remains at its saturation value of $\pi/2$.

If the sphere is not in resonance, (3-4) becomes

$$\dot{\theta} = \sqrt{\omega_h^2 - (\omega_0 - \omega)^2 \tan^2\theta} - \omega_{\ell 0} \sin\theta \tag{3-10}$$

When $\theta = 0$, the detuning term in (3-10) is zero, and the cone angle starts to build up as in the resonant case. As θ grows this term increases also (α is shifting away from $\pi/2$), and the effective driving field decreases. If the detuning is appreciable so that $|\omega_0 - \omega| >> \omega_{\ell 0}$ (or if ω_h is small), the cone angle will not build up to any large value. Then $\tan\theta$ and $\sin\theta$ may be replaced by their arguments and (3-10) approximated as

$$\dot{\theta} = \sqrt{\omega_h^2 - (\omega_0 - \omega)^2 \theta^2} - \omega_{\ell 0}\, \theta \tag{3-11}$$

This may be integrated and gives

$$\frac{\omega_0 - \omega}{\omega_{\ell 0}} \sin^{-1}\left(\frac{\omega_0 - \omega}{\omega_h}\, \theta\right) + \ln\left(\frac{\omega_h}{\sqrt{\omega_h^2 - (\omega_0 - \omega)^2 \theta^2} - \omega_{\ell 0}\theta}\right) = \left[\frac{(\omega_0 - \omega)^2 + \omega_{\ell 0}^2}{\omega_{\ell 0}}\right] t \tag{3-12}$$

which is more conveniently solved for t than for θ. If $\omega_0 = \omega$, (3-12) reduces to

$$\theta \simeq \frac{\omega_h}{\omega_{\ell 0}}\left(1 - e^{-\omega_{\ell 0} t}\right) \tag{3-13}$$

For small values of θ , (3-7) agrees with (3-13).

When the sample shape is not spherical the resonance frequency is a function of θ and therefore of the incident power. If the sample is biased to be resonant when $\theta = 0$, it shifts out of resonance as θ increases. We may compensate for this by initially detuning to the other side of the resonance so that an increasing θ tends to restore resonance. As we shall see, this procedure is limited. Assume that the sample is a spheroid, and that θ is small. Then

$$\begin{aligned} \omega_o &= \omega_H - (N_z - N_t)\,\omega_M \cos\theta \\ &\simeq \omega_o' + W\theta^2 \end{aligned} \tag{3-14}$$

where $\omega_o' = \omega_H - (N_Z - N_t)\,\omega_M$ and $W = \frac{N_Z - N_t}{2}\,\omega_M$. Equation (3-11) may be written as

$$\dot{\theta} = \sqrt{\omega_h^2 - D} - \omega_{lo}\theta \tag{3-15}$$

where D, the detuning term, is given by

$$D = \left[(\omega_o' - \omega) + W\theta^2\right]^2 \theta^2 \tag{3-16}$$

If $\omega = \omega_o'$, D is zero when $\theta = 0$ and builds up monotonically as θ^6 for increasing cone angle. Assume now that $\omega_o' < \omega$ if W is positive ($N_Z > N_t$), or $\omega_o' > \omega$ if W is negative ($N_Z < N_t$). Under these conditions D becomes

$$D = W^2\left(\theta^2 - \theta_o^2\right)^2 \theta^2 \tag{3-17}$$

where $\theta_o^{\ 2} = \frac{\omega - \omega_o'}{W}$, as is shown in Fig. 3-1. The build up transient is now very interesting, for if the cone angle can attain a value $\theta_o/\sqrt{3}$, the peak in D is surmounted and (provided ω_h is great enough) θ will increase to θ_o.

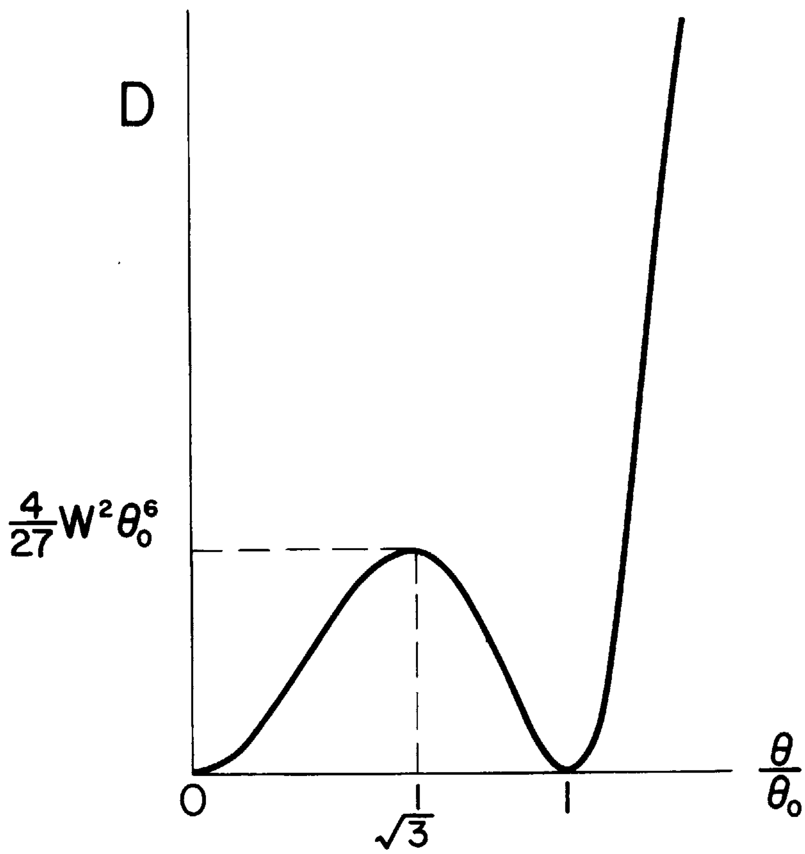


Fig. 3-1

Since the height of the barrier is proportional to the sixth power of θ_o, we would expect that it can be exceeded easily for small values but that at some critical point it cannot -- without additional excitation. The condition for scaling the top of the barrier is

$$\omega_h^2 - \frac{4}{27} W^2 \theta_o^6 \geqslant \omega_{\ell o}^2 \frac{\theta_o^2}{3} \tag{3-18}$$

Any further increase in the cone angle will bring about a reduction in the detuning term. If this decrease is sufficient to offset the increase in the loss term, a "runaway" condition will result. Anderson and Suhl have analyzed such behavior, on the basis of steady-state theory, and have determined the minimum instability threshold.[19] We are interested here, however, in determining under what conditions θ can increase until D = 0. A necessary condition is that the driving field be given by

$$\omega_h = \omega_{\ell o} \theta_o \tag{3-19}$$

so that (3-18) becomes

$$\frac{9}{2} \omega_{\ell o}^2 \geqslant W^2 \theta_o^4 \tag{3-20}$$

The initial detuning, necessary to make D equal zero when $\theta = \theta_o$, is evidently

$$|\omega - \omega_o'| = |W| \theta_o^2 \leq \frac{3}{\sqrt{2}} \omega_{\ell o} \tag{3-21}$$

Substituting the value of W into (3-21) gives

$$\theta_o^2 \leq \frac{3\sqrt{2}\ \omega_{\ell o}}{|N_z - N_t|\ \omega_M} = \theta_{crit}^2 \tag{3-22}$$

which is minimized for a thin disc ($N_z = 1$). If θ_o is slightly greater than θ_{crit}, the cone angle can no longer reach θ_o but it will arrive at a value just under $\theta_o/\sqrt{3}$. It should be noted that θ_{crit} is normally greater than the threshold cone angle, for second order spin wave instabilities, which is

$$\theta^2 = \frac{2\,\omega_{\ell K}}{\omega_M} \tag{3-23}$$

<u>Premature Saturation.</u> The simple theory of the transient build up just given does not prevent the cone angle from attaining large values (when h is comparable to ΔH) unless the resonant frequency is a function of θ. In practice, the latter never approaches $\pi/2$ in the steady-state, even for a sphere, because, as we have previously discussed, Suhl has shown that certain spin waves become unstable at power levels much smaller than those implied by $\omega_h = \omega_{\ell o}$.* These modes feed upon the uniform precession and cause a premature saturation of θ. The initial transient state is now very much complicated. As long as the cone angle is below the minimum spin wave instability threshold, the equations previously given apply. As θ becomes greater than this threshold, spin waves start to build up, and θ is reduced as the loss term increases. There is a time lag before this starts to make any appreciable difference, however, and so θ continues to increase. More

*The power required to keep the cone of precession open to some angle θ is equal to the damping torque $\times\, \omega_{precession}$. Therefore, Power/volume = $\mu_o M\, \Delta H_o \sin^2\theta\, \omega$.

and more spin waves become unstable as their thresholds are reached, and the initial instabilities grow at rapidly increasing rates. The amplitude of maximally-coupled spin waves is approximately given by

$$\delta\dot{M} \simeq \left(\frac{\sin\theta}{2} - \frac{\Delta H_K}{M} \right) \omega_M \delta M \tag{3-24}$$

for first order coupling ($\omega_k = \omega/2$), and

$$\delta\dot{M} \simeq \left(\frac{\tan\theta \sin\theta}{2} - \frac{\Delta H_K}{M} \right) \omega_M \delta M \tag{3-25}$$

for second order coupling ($\omega_k = \omega$). (Note that if $\theta > \pi/4$ the latter is the dominant process and near $\theta = \pi/2$, the build-up rate is enormous.) The increasing loss finally limits the growth of θ and makes it decrease to the critical value corresponding to the instability of minimum threshold. The detailed dynamics of this turbulent system is beyond hope of solving. Even if only one spin wave were involved, there would result four coupled nonlinear differential equations. Many such waves are involved, however, and the number depends on by how much θ tries to exceed θ_{crit}.

Practical considerations of these transients make a solution desirable. If spin wave instabilities, of one form or another, are used to limit pulsed r-f power incident upon a ferrite, the time lag associated with their build up allows an initial "spike" to remain on the transmitted pulse. This is often undesirable.

3.2. The Decay of the Uniform Precession. Another interesting transient condition is found when the incident pulse shuts off. We might expect, a priori, that θ , which is at its critical value, (the pulse is assumed to be of long enough duration to insure this) would merely decay (along with the excited spin waves) when deprived of the driving field. It has been found, however, that this is not always the case; the cone angle sometimes grows before finally subsiding. Evidentally in these cases, the spin waves give back to the uniform precession some of the energy they previously had coupled from it. This implies that the parametric coupling coefficient changes sign in an interval, short compared to the relaxation time. Although a solution to this problem is also very difficult, a number of qualitative statements can be made about the equations. First of all, to assume that the driving field suddenly steps to zero will probably not yield a correct model of the usual laboratory experiment, although it is of interest in its own right. An example may clarify this remark. Consider that a spheroid is biased to some resonant frequency ω_0 and that a transverse driving field of frequency ω , not equal to ω_0 , is present. Also, assume that the incident power level is sufficient to create an excited spin wave. Now if the driving field steps to zero the frequency of the uniform precession suddenly shifts from ω to ω_0, which in general bears no relation to the frequency of the excited spin wave. The two modes are decoupled therefore and the parametric coupling averages to zero over a time interval that is small compared to the subsequent decay time. If, however, the driving field does not suddenly go to zero, the precession rate does not immediately change, and it is possible for the parametric coupling to maintain an average value over the time interval of interest. It appears, therefore, that a correct model of the driving field decay is important except perhaps when $\omega = \omega_o$; then no large change in the precession rate occurs in either case. A qualitative picture of the interrelationship between the amplitude and phases of the two modes may be obtained from an inspection of the differential equations. Let us consider those appropriate for the first order coupled modes ($\omega_k = \omega/2$) , assuming a spheroidal sample shape. If higher order terms are neglected, (2-35) through (2-38), Chapter 2, reduce to

$$\dot{\phi}_o = \omega_o - \frac{\omega_h}{\theta}\cos(\omega t - \phi_o + \alpha_o) + \frac{\delta M^2}{8M^2}\omega_M\left(3\cos^2\psi - 1 - \frac{\sin 2\psi}{\theta}\cos y\right) \qquad (3\text{-}26)$$

$$\dot{\theta} = \omega_h \sin(\omega t - \phi_o + \alpha_o) - \omega_{ro}\theta + \frac{\delta M^2}{8M^2}\omega_M \sin 2\psi \sin y \qquad (3\text{-}27)$$

$$\dot{\phi}_K = \omega_H - N_z\omega_M + \lambda K^2\omega_M + \sin^2\psi\,\omega_M\cos^2(\phi_K - \xi) - \frac{\sin 2\psi}{2}\omega_M\theta\left[\cos(\phi_o - \xi) + \cos y\right] \qquad (3\text{-}28)$$

and

$$\delta\dot{M} = \left[\frac{\sin^2\psi}{2}\sin 2(\phi_K - \xi) - \frac{\sin 2\psi}{2}\theta\sin y - \frac{\Delta H_K}{M}\right]\omega_M\,\delta M \qquad (3\text{-}29)$$

where

$$y = 2\phi_K - \phi_o - \xi$$

We assume that the driving field h (large enough to have caused saturation of θ)

steps to zero at t = 0 so that the initial conditions are

$$\theta = \theta^c = \frac{2\Delta H_K}{M \sin 2\psi\left(1 - \frac{\sin^2\psi}{2}\frac{\omega_M}{\omega}\right)} = \frac{\omega_h{}^c}{\sqrt{(\omega_o - \omega)^2 + \omega_{\ell o}^2}} \qquad (3\text{-}30)$$

where sin 2ψ (consistent with $\omega_k = \omega/2$) is chosen to minimize θ^c, and sin y $\simeq$ -1. The phase of the uniform precession, α_o , for t $\leq$ 0 is given by

$$n\left(\cos\alpha_o - \frac{3\cos^2\psi - 1}{\sin 2\psi}\theta^c \sin\alpha_o\right) = \cos\alpha^c - \frac{3\cos^2\psi - 1}{\sin 2\psi}\theta^c \sin\alpha^c \qquad (3\text{-}31)$$

where

$$n = \frac{\omega_h}{\omega_h{}^c} \qquad \text{and} \qquad \cot\alpha^c = \frac{\omega_o - \omega}{\omega_{\ell o}}$$

If $\dot{\theta}$ is to be positive at any time during the ensuing transient, either sin y must change algebraic sign before θ does or vice versa. A change in the sign of sin y implies a phase shift between the uniform precession and the coupled spin wave, which would then act as a source instead of as a sink. If this could happen immediately, θ would initially increase from its value of θ^c before finally decaying. If, on the other hand, the phase did not shift at all, θ would decrease very rapidly because of the normal damping and the spin wave term. In fact, it would decrease to zero and then become negative (provided δM was still non-zero). A negative value of θ effectively means that the phase of the uniform precession has changed, but only after θ has gone to zero first. If δM is large enough, $|\theta|$ will then increase rapidly and may attain a value larger than θ^c before finally decaying together with δM.*

In general, the relative phase between the coupled modes is not independent of their amplitudes, as is clear from inspection of the equations. There is a special case, however, in which a remarkable simplification occurs. Consider that $\omega = \omega_o$ and $\cos^2\psi = 1/3$, then $\cos \alpha_o = 0$ and (3-26) and (3-28) reduce, respectively to $\dot{\phi}_o = \omega_o$ and $\overline{\dot{\phi}}_k = \omega_k$. Since $\sin y = -1$ for the duration of the transient, (3-27) and (3-29) become

$$\dot{\theta} = -\omega_{\ell o}\theta - \left(\frac{\delta M}{\delta M_o}\right)^2 (n-1)\,\omega_{\ell o}\theta^c \tag{3-32}$$

and

$$\delta\dot{M} = \left(\frac{\theta}{\theta^c} - 1\right)\omega_{\ell k}\,\delta M \tag{3-33}$$

where $\theta(0) = \theta^c$, and $\delta M(0) = \delta M_o$. Although there are now only two coupled equations instead of four, an exact solution is impossible and we must be content with what appears to be a reasonable approximation. This is derived in the Appendix and results in

$$\frac{\theta}{\theta^c} = e^{-\omega_{\ell o}t} + \left(\frac{n-1}{\beta-1}\right)\left[e^{\beta(1-\omega_{\ell o}t - e^{-\omega_{\ell o}t})}P(t) - e^{-\omega_{\ell o}t}P(0)\right] \tag{3-34}$$

for $n \geqslant 1$ and $t \geqslant 0$ where

$$P(t) = 1 + e^{-\omega_{\ell o}t} + \frac{\beta}{\beta+1}e^{-2\omega_{\ell o}t} + \frac{\beta^2}{(\beta+1)(\beta+2)}e^{-3\omega_{\ell o}t} + \cdots \tag{3-35}$$

---

*Note that if θ rises above θ^c other spin waves will go unstable. Because of the time lag in their growth, they probably can be ignored in most cases.

and $\beta = \dfrac{2n\,\omega_{\ell K}}{\omega_{\ell o}}$. Equation (3-34) is plotted in Fig. 3-2 for the case n = 5, $\omega_{\ell o} = \omega_{\ell K}$. The type of behavior that was predicted is clearly evident, although it should be mentioned that the approximate solution always underestimates the peak values.

Qualitatively, the discussion above applies also to second order coupled spin waves. For second order coupling, however, (2-35) through (2-38) reduce instead to

$$\dot{\varphi}_o = \omega_o - \frac{\omega_h}{\theta}\cos(\omega t - \varphi_o + \alpha_o) + \frac{\delta M^2}{2M^2}\omega_M \cos^2 X \tag{3-36}$$

$$\dot{\theta} = \omega_h \sin(\omega t - \varphi_o + \alpha_o) - \omega_{\ell o}\theta + \frac{\delta M^2}{4M^2}\omega_M \theta^2 \sin 2X \tag{3-37}$$

$$\dot{\varphi}_K = \omega_o - N_t\omega_M + \lambda K^2\omega_M + \omega_M\theta^2\cos^2 X \tag{3-38}$$

and

$$\delta\dot{M} = -\omega_{\ell K}\delta M + \frac{\theta^2}{2}\omega_M \sin 2X\,(1 + \lambda K^2 - N_t)\,\delta M \tag{3-39}$$

where ψ has been set equal to zero.

<u>3.3. The Transient Spin Wave Spectrum.</u> The previous discussions concerned a typical resonance experiment involving a dc biasing magnetic field and pulsed microwave excitation. The transient behavior of the magnetization,

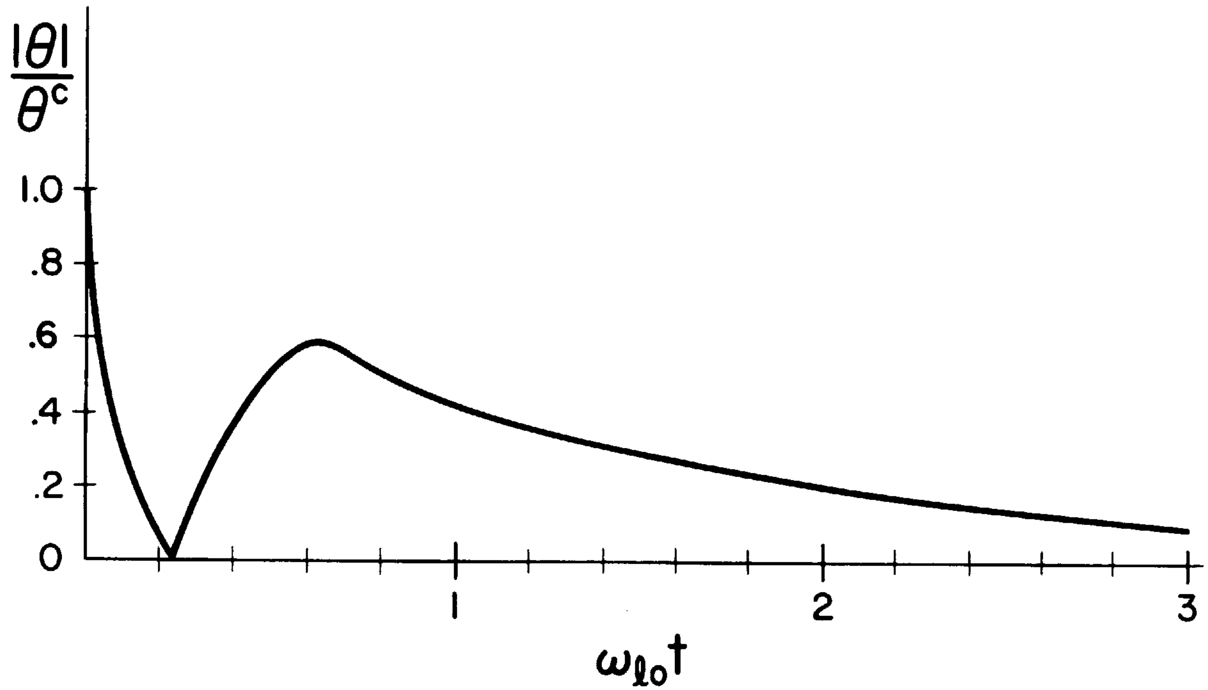


Fig. 3-2

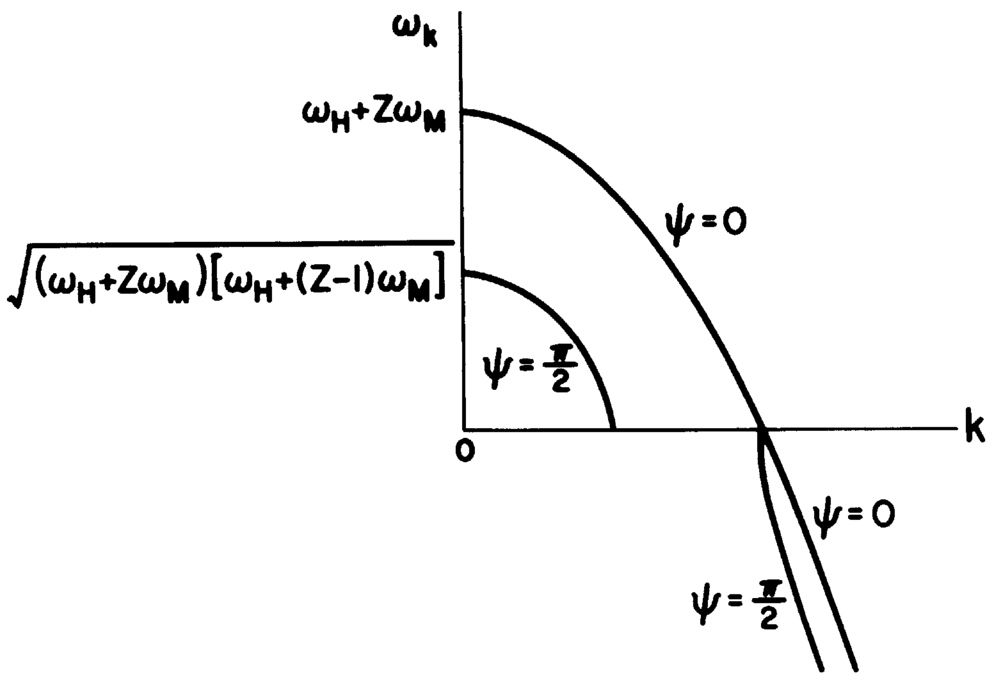
when the biasing magnetic field varies in amplitude and direction, is also of considerable interest for a number of reasons. For one, a knowledge of the dynamics of magnetization reversal in ferrite computer elements is necessary in order to predict the limits on switching times. For another, the possibility of coupling magnetic energy from transient fields and converting it into pulsed microwave energy has been theoretically analyzed by the author,[20,21] and experimentally demonstrated by Stiglitz and Morgenthaler[22] as well as by Elliot, Shaw, and Schaug-Petterson.[23] The behavior of the spin wave spectrum under transient conditions is naturally of great interest. It is particularly important to know what relaxation processes would occur if the magnetization were momentarily inverted ($\theta = \pi$). This problem was treated by the author,[1, 14, 24] and more recently identical results were obtained independently by Schaug-Petterson.[25] The magnetization will change rapidly and it is important to follow the spin wave spectrum and the strength of the various couplings for succeeding stages of the transient. Before we attempt this, it will be helpful to review the $\theta = \pi$ case.

<u>The Inverted Spectrum.</u> The spin wave spectrum, which is valid when negligible transverse components of the magnetization exist ($\sin\theta \simeq 0$), was derived in Chapter 2 and is given by

$$\omega_K^2 = \left(\omega_H - Z\omega_M\cos\theta + \lambda K^2\omega_M\cos\theta\right) \times \left(\omega_H - Z\omega_M\cos\theta + \lambda K^2\omega_M\cos\theta + \sin^2\psi\,\omega_M\cos\theta\right) \tag{3-40}$$

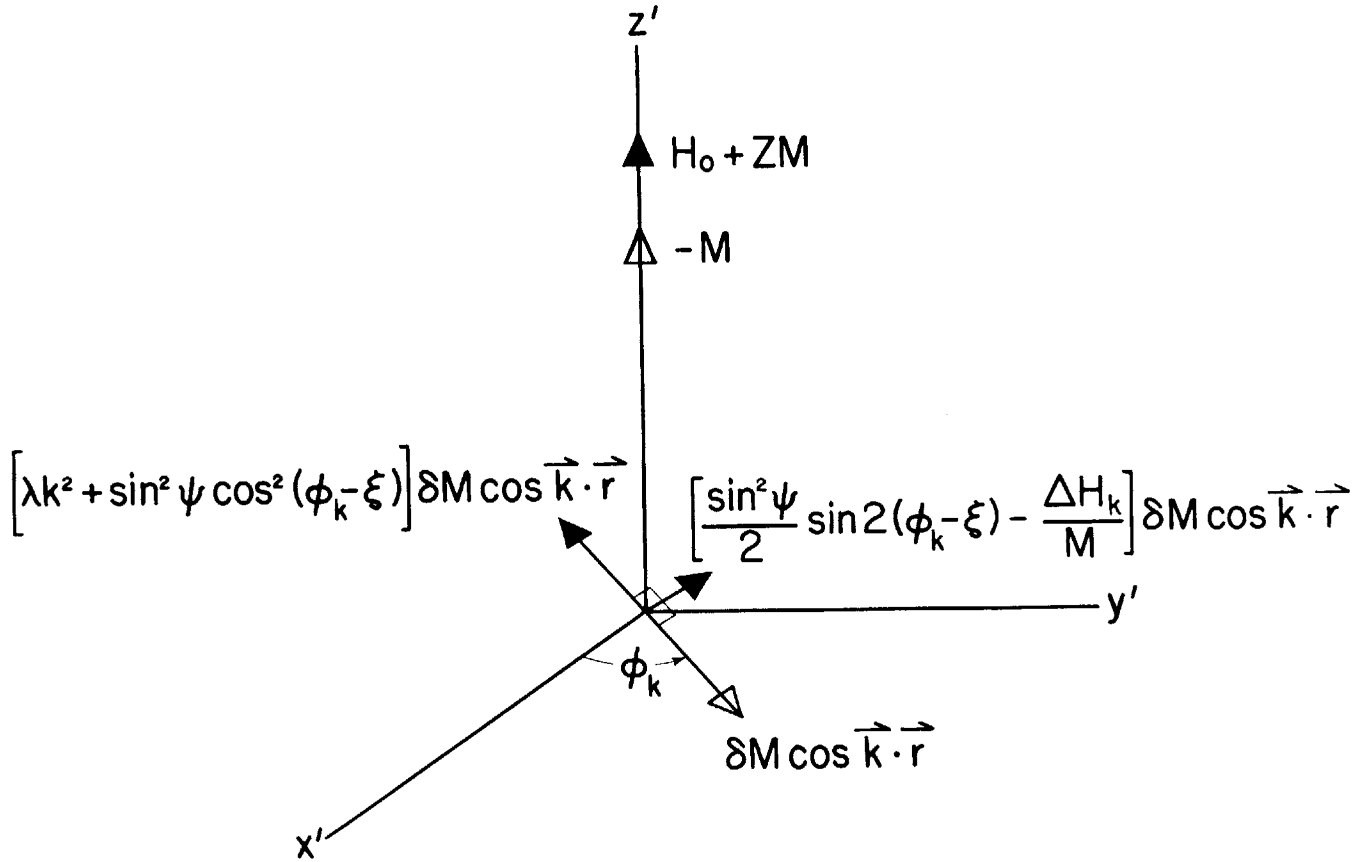
For $\theta = \pi$, the value of cos θ is minus unity and a region in which spin wave propagation is cut off may be shown to exist. This region is largest for $\psi = \pi/2$ directed spin waves (recall that ψ is the angle that the spin wave propagation vector makes with the direction of the internal magnetizing field) and decreases to zero when $\psi = 0$. The spectrum is plotted in Fig. 3-3 for the case of a sphere (Z = 1/3) with $\omega_H = \omega_M$.

Mathematically, the cut-off condition comes about when the two factors in (3-40) have opposite algebraic signs, which lead to imaginary values of ω_k. The physics can best be understood by studying a graphical representation of the field vectors. This is given in Fig. 3-4. In the first place, because of the inversion of $\vec{M}$, the "demagnetizing field" ZM is adding to H_o while the exchange field is effectively subtracting from it. There is a similar reversal in the effect of the volume dipolar field so that $\psi = 0$ spin waves now have the highest frequencies, and $\psi = \pi/2$ waves the lowest. Consider for a moment these latter waves and assume that they are lossless and propagate when k = 0. As k increases the exchange field reduces the frequency more and more until finally all torques balance and ω_k becomes zero. At this point, $\sin 2(\phi_k - \zeta) = 0$ so that the component of the volume dipolar field, which is perpendicular to the spin wave (see Fig. 3-4), is also zero and the equilibrium position and magnitude of δM is maintained. If k increases slightly, one should expect the exchange field to overbalance the applied field, and the precession to proceed in the opposite direction (negative ω_k). Actually, the phase of ϕ_k



ω_k
$\omega_H + Z\omega_M$
$\sqrt{(\omega_H + Z\omega_M)[\omega_H + (Z-1)\omega_M]}$
$\psi = 0$
$\psi = \frac{\pi}{2}$
0
k
$\psi = 0$
$\psi = \frac{\pi}{2}$

Fig. 3-3



z'
$H_o + ZM$
$-M$
$\left[\lambda k^2 + \sin^2\psi \cos^2(\phi_k - \xi)\right] \delta M \cos \vec{k}\cdot\vec{r}$
$\left[\frac{\sin^2\psi}{2} \sin 2(\phi_k - \xi) - \frac{\Delta H_k}{M}\right] \delta M \cos \vec{k}\cdot\vec{r}$
y'
ϕ_k
$\delta M \cos \vec{k}\cdot\vec{r}$
x'

Fig. 3-4

changes but $\dot{\phi}_k$ remains zero. Since the $\sin 2(\phi_k - \xi) \neq 0$, the perpendicular component of the volume dipolar field becomes non-zero and $\dot{\delta M}$ is either increased or decreased (both phases of ϕ_k are permitted). If it is positive an unstable situation exists. As k increases still more, the phase of ϕ_k changes ($\dot{\phi}_k$ still zero) until the $\sin 2(\phi_k - \xi)$ reaches a maximum and then it starts to decrease. When k increases to the point where $\sin 2(\phi_k - \xi) = 0$ again, the end of the cut-off region has been reached. Any further increase in k now unbalances the precession torques and ω_k becomes negative. The situation is much the same for any other direction of ψ, except that a greater k value is needed to reach the beginning of the cut-off band. It is interesting to note that the end of the latter occurs for the same value of k, regardless of ψ. As this angle approaches 0, the band shrinks until it becomes only a single point. Since the transverse volume dipolar field is proportional to $\sin^2 \psi$, the parametric coupling decreases to zero in the latter case.

Further mathematical insight comes from the consideration of the lossless equations of motion for $\vec{\delta M}$, which are

$$\dot{\phi}_K = \omega_H + Z\omega_M - \lambda K^2 \omega_M - \sin^2\psi \, \omega_M \cos^2(\phi_K - \xi) \tag{3-41}$$

and

$$\dot{\delta M} = - \sin^2\psi \, \sin(\phi_K - \xi) \cos(\phi_K - \xi) \, \omega_M \, \delta M \tag{3-42}$$

In the cut off region, the precessional motion of the spin wave has stopped ($\dot{\phi}_k = 0$). Therefore

$$\omega_M \sin^2\psi \cos^2(\phi_k - \xi) = \omega_H + z\omega_M - \lambda K^2 \omega_M \qquad (3\text{-}43)$$

Solving this equation for $\cos(\phi_k - \xi)$ and $\sin(\phi_k - \xi)$ and substituting these expressions into (3-42) yields

$$\delta\dot{M} = \pm i\, \omega_K\, \delta M \qquad (3\text{-}44)$$

where ω_k is given by (3-40). The plus or minus sign means that both growing and decaying solutions are possible; the former being the only one of interest. Since ω_k is imaginary, let $\omega_k = i\Omega_k$. Then

$$\delta M = \delta M_o\, e^{\Omega_K t} \qquad (3\text{-}45)$$

The maximum value of Ω_k for arbitrary ψ occurs when $\phi_k - \xi = \pi/4$, since we know from (3-42) that

$$\Omega_K = \frac{\sin^2\psi}{2}\,\omega_M \sin 2(\phi_K - \xi) \qquad (3\text{-}46)$$

It is also seen that $\psi = \pi/2$ leads to its largest possible value in agreement with the previous discussion; then

$$\delta M = \delta M_o\, e^{\frac{\omega_M}{2} t} \qquad (3\text{-}47)$$

The growth of this individual spin wave is therefore extremely rapid unless the saturation magnetization has a small amplitude. The total effect of the unstable spin waves is very much greater since a whole band of them exist. The k value of the fastest growing spin wave in any paritcular direction is easily found from (3-43) with $\cos^2(\phi_k - \xi) = 1/2$ and is given by

$$K^2 = \frac{H + (z - \frac{\sin^2\psi}{2})M}{\lambda M} \qquad (3\text{-}48)$$

provided that this value of k^2 is positive.

The unstable spin wave growth, together with all other relaxation processes, will certainly result in the rapid return of the magnetization vector from its inverted to its normal position. The spectrum just described will, therefore, continuously change until it finally reaches the steady-state Suhl spectrum. The actual transient behavior will depend to a large extent on how much uniform precession is initially present when the spin waves first go unstable. The case $\theta = \pi$ is a point of unstable equilibrium for $\vec{M}$ and implies no such initial component. It is possible, therefore, to assume that the unstable spin waves could act to increase M_z from its initial value of $-M$ to $+M$ without there ever being formed a spatially coherent transverse component of the magnetization. Many of these spin waves would grow to large amplitude and each would in turn effect the frequencies of the others. The determination of such interactions, even using quasi-static approximations, is too difficult to carry out.

<u>The General Spectrum.</u> Let us consider instead the more general case when θ is not exactly π. We assume that the transverse component of the uniform precession is very much larger than any initial spin wave amplitude.

The initial spectrum is essentially the same as that for $\theta = \pi$ but the unstable spin waves will constitute an additional loss as far as the uniform precession is concerned (now not in unstable equilibrium). This tends to make θ decrease much faster than usual and causes an initial increase in $\sin\theta$; hence, the uniform component. If this latter component is large compared to the spin wave amplitudes, the spectrum will be governed by the uniform precession and not by the spin waves. If in addition the quasi-static approximation (that $\dot{\theta} \ll \dot{\phi}$) is used, the problem becomes identical to the one of determining a succession of spin wave spectrums assuming no appreciable spin wave amplitudes, but some arbitrary value of θ .

Again, for simplicity, we consider the sample as a spheroid magnetized along its major axis (z). We assume also that all r-f driving fields are zero. Equations (2-35) through (2-38) are then

$$\dot{\phi}_0 = \omega_H - (N_z - N_t)\,\omega_M \cos\theta \tag{3-49}$$

$$\dot{\theta} = -\,\omega_{\lambda 0} \sin\theta \tag{3-50}$$

$$\begin{aligned}\dot{\phi}_K = \omega_H - (N_z - \lambda K^2)\omega_M \cos\theta + \sin^2\psi\, \omega_M \cos\theta \cos^2(\phi_K - \xi) \\ - \sin 2\psi\, \omega_M \sin\theta \cos\chi \cos(\phi_K - \xi) \\ + (\lambda K^2 + \cos^2\psi - N_t)\,\omega_M \sin\theta \tan\theta \cos^2\chi\end{aligned} \tag{3-51}$$

and

$$\delta\dot{M} = \left\{ \frac{\sin^2\psi}{2}\cos\theta\sin 2(\varphi_K - \xi) - \frac{\sin 2\psi}{2}\sin\theta\sin(2\varphi_K - \varphi_o - \xi) - \frac{\Delta H_K}{M}\cos\theta + (\lambda K^2 + \cos^2\psi - N_t)\frac{\sin\theta\tan\theta}{2}\sin 2\chi \right\}\omega_M \delta M \qquad (3\text{-}52)$$

In general ω_k will not be equal to either ω_o or $\omega_o/2$ so that the sin 2ψ term in (3-51) has no average value, whereas, $\cos^2 x$ averages to 1/2. (If $\omega_k = \omega_o$ or $\omega_o/2$, the equation for $\dot{\varphi}_k$ is in general modified. It is easily shown, however, that those spin waves that are maximally-coupled to the uniform precession, and that therefore meet these frequency conditions, are not significantly modified. Spin waves that are modified will be discussed in a later section.) It then follows that

$$\dot{\varphi}_K \simeq \omega_H - (N_z - \lambda K^2)\omega_M\cos\theta + \frac{\sin^2\psi}{2}\omega_M\cos\theta\left[1 + \cos 2(\varphi_K - \xi)\right] + (\lambda K^2 + \cos^2\psi - N_t)\frac{\sin\theta\tan\theta}{2}\omega_M \qquad (3\text{-}53)$$

Since θ is assumed to change negligibly during a precession period, it may be treated as a constant and the $\dot{\varphi}_k$ equation integrated directly as before. The result of the integration is to determine ω_k as

$$\omega_K^2 = (a + b\cos^2\psi + c\sin^2\psi + d\lambda K^2)(a + b\cos^2\psi + d\lambda K^2) \qquad (3\text{-}54)$$

where

$$a = \omega_H - N_z\omega_M\cos\theta - \tfrac{1}{2}N_t\omega_M\sin\theta\tan\theta$$

$$b = \tfrac{1}{2}\omega_M\sin\theta\tan\theta$$

$$c = \omega_M\cos\theta$$

and

$$d = \omega_M\left(\cos\theta + \tfrac{1}{2}\sin\theta\tan\theta\right)$$

Notice that the coefficient d, which determines the slope of the spectrum, is essentially independent of θ except near $\theta = \pi/2$. (This point $\pi/2$ is excluded from consideration since, as derived in Chapter 2, the spin wave equations are not valid unless $\delta M \ll M\cos\theta$ or $\frac{M\cos^2\theta}{\sin\theta}$.) The dependence of d, on θ, is shown in Fig. 3-5. The spectrum is plotted in Figs. 3-6 and 3-7 for various values of θ for a spherical sample. It is assumed that $\omega_H = \omega_M$, and only the $\psi = 0$ and $\psi = \pi/2$ spin waves are shown in the figures. An interesting feature is that these $\psi = 0$ and $\psi = \pi/2$ waves alternate in being lower in frequency as θ varies from 0 to π, and that neither of these directions constitute the upper limit to the spectrum in certain ranges of θ. This may be seen by examining the extrapolated limits of the spectrum for $k = 0$. If ω_k does not equal either $\omega_o/2$ or ω_o, they are given by

$$\omega_K^2(o) = (a + b\cos^2\psi)(a + b\cos^2\psi + c\sin^2\psi) \tag{3-55}$$

The maximum and minimum values of $\omega_k(0)$ must satisfy the condition

$$\sin\psi\cos\psi\left[2b(b-c)\cos^2\psi + a(2b-c) + bc\right] = 0 \tag{3-56}$$

provided $\omega_k \neq 0$. Three possibilities exist: 1) $\psi = 0$ for which

$$\omega_K(o) \equiv \omega_1 = a + b \tag{3-57}$$

2) $\psi = \pi/2$ for which

$$\omega_K(o) \equiv \omega_2 = \sqrt{a(a+c)} \tag{3-58}$$

and 3) $\cos^2\psi = \frac{a(2b - c) + bc}{2b(c - b)}$ (if ψ is real) for which

$$\omega_K(o) \equiv \omega_3 = \frac{c}{2\sqrt{b(c-b)}}(a+b) \tag{3-59}$$

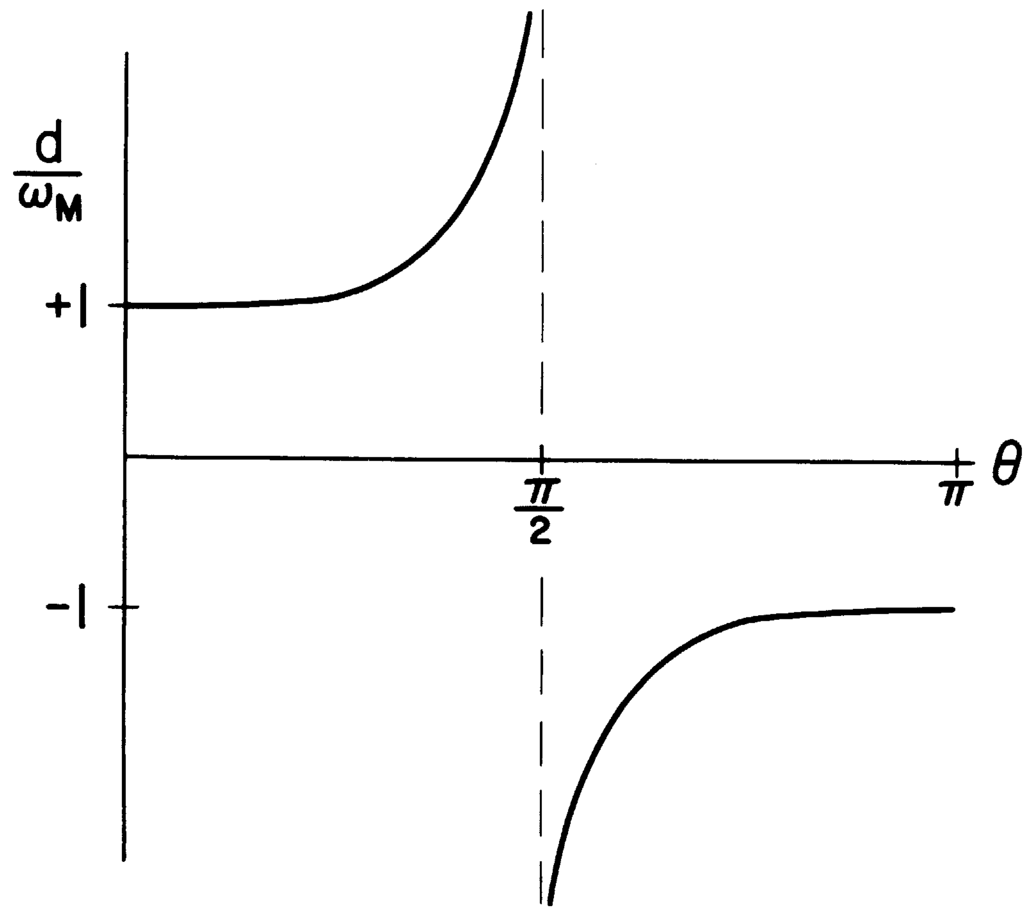


Fig. 3-5

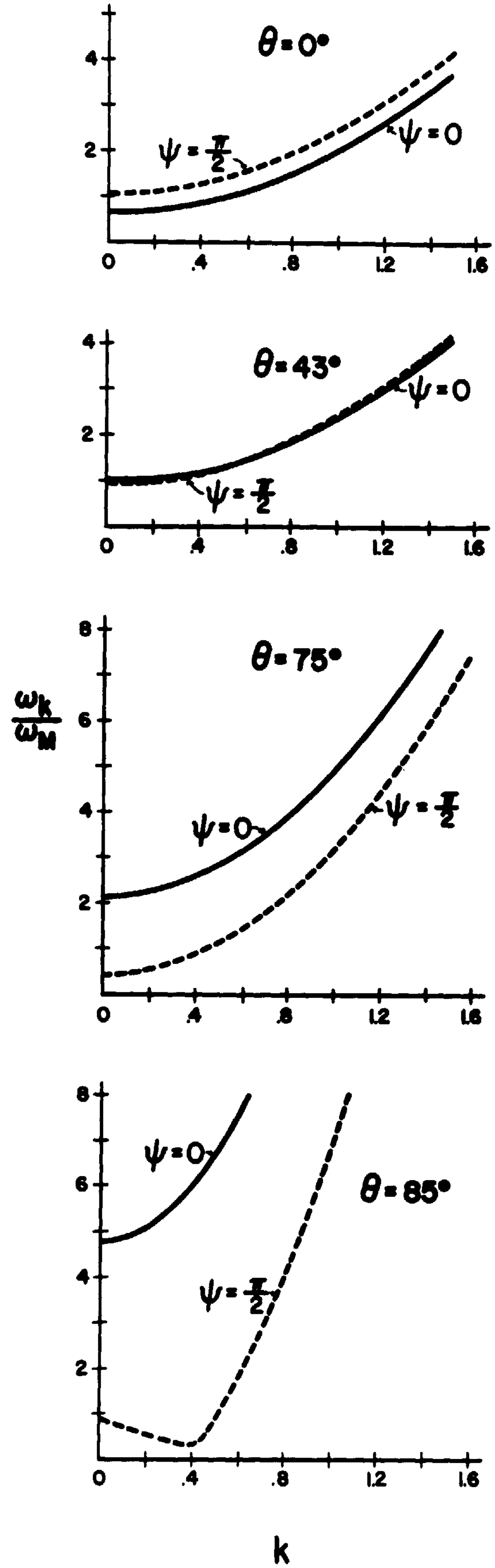


Fig. 3-6

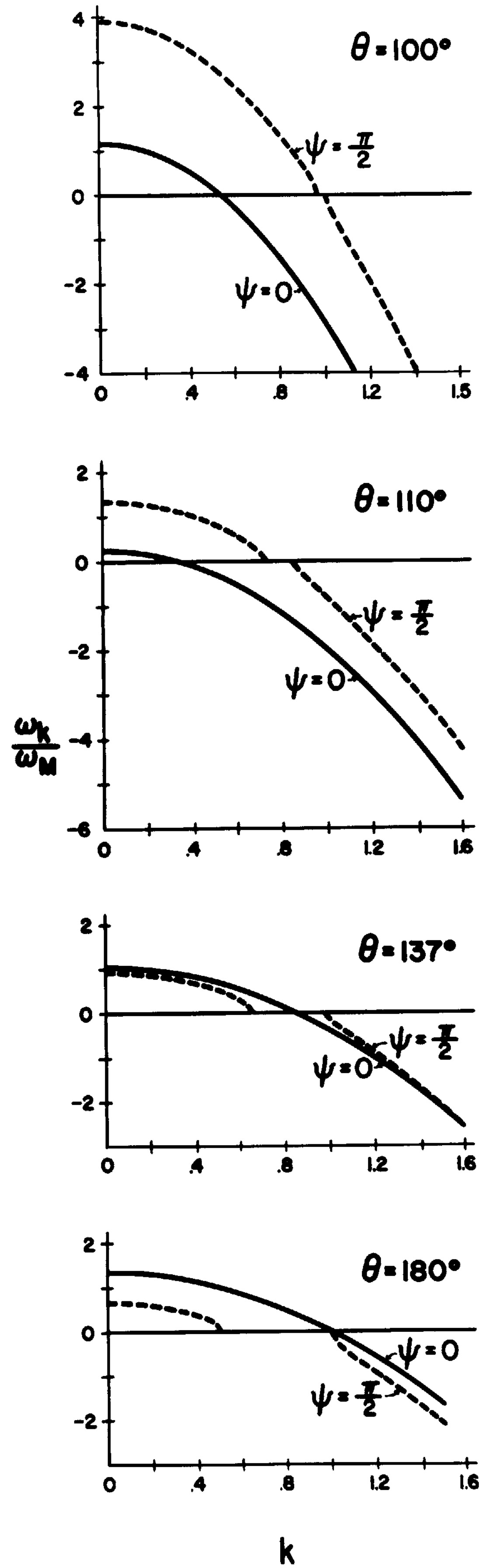


Fig. 3-7

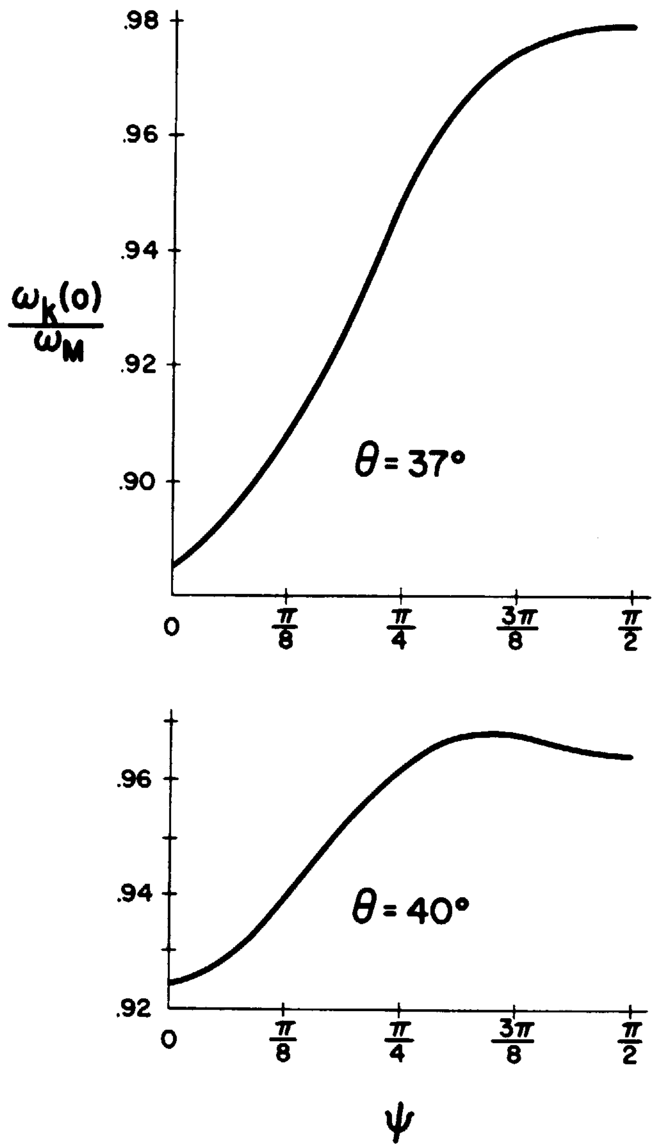
If $\theta < \pi/4$ $(c > 2b)$, it may be shown that ω_3 is the absolute maximum provided $\frac{bc}{c - 2b} > a > -b$. In Figs. 3-8 and 3-9 the $k = 0$ spin wave frequencies are plotted as a function of ψ for several increasing values of θ. The latter are chosen to indicate the manner in which the spectrum turns over. Again, the case $Z = 1/3$, $\omega_H = \omega_M$ was chosen.

3.4. The Position of the Uniform Precession in the Manifold. The position of the uniform precession in the spin wave manifold as a function of the cone angle is important, since its relative position determines which spin waves will be strongly coupled at any given time. For example, Schlömann has shown that the discrete nature of the spin wave modes together with their frequency dependence on power can account for certain fine structure, which has been observed in high power resonance experiments.[26] Moreover, it appears that under certain conditions, the uniform precession can drop below the main spin wave band, although certain other spin waves, with precise phase relationships, will still be degenerate with the uniform precession. For example, assume that $\psi = 0$ spin waves lie lowest in the band and that $\theta < \pi/4$. Then (for a spheroid)

$$\omega_K(\psi=0) = \omega_H - N_z\omega_M\cos\theta + \lambda K^2\omega_M\cos\theta + (1+\lambda K^2 - N_t)\omega_M\sin\theta\tan\theta\cos^2\chi \quad (3\text{-}60)$$

and

$$\omega_0 = \omega_H - N_z\omega_M\cos\theta + N_t\omega_M\cos\theta \quad (3\text{-}61)$$



.98
.96
.94
$\frac{\omega_k(0)}{\omega_M}$
.92
.90
$\theta = 37°$
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
.96
.94
$\theta = 40°$
.92
0
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{3\pi}{8}$
$\frac{\pi}{2}$
ψ

Fig. 3-8

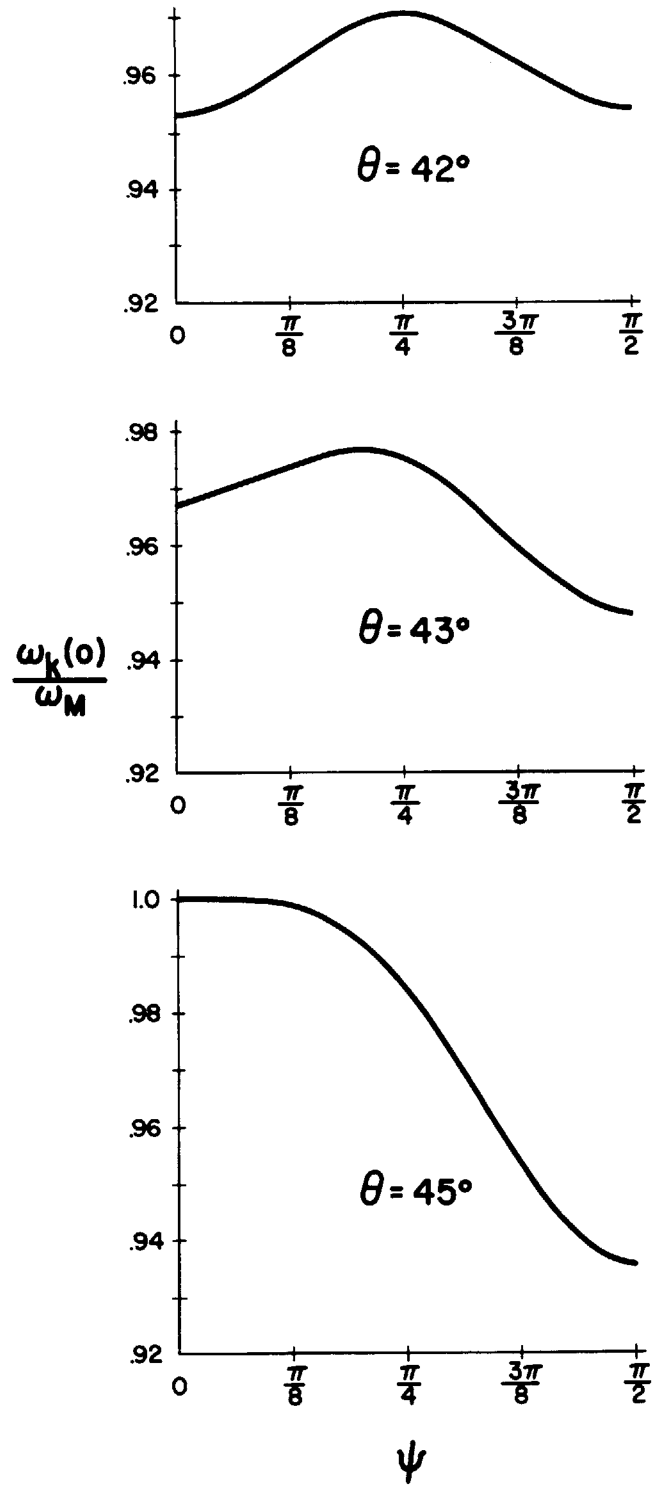


Fig. 3-9

where $x = \phi_k - \phi_o$, as before, and all spin waves are assumed to have infinitesimal amplitudes. If ω_o is to drop below the spin wave band, we must have

$$N_t < \lambda K^2 + (1 + \lambda K^2 - N_t) \tan^2\theta \, \overline{\cos^2 x} \tag{3-62}$$

for all real k. The most stringent requirement is for $k = 0$ and then

$$N_t < (1 - N_t) \overline{\cos^2 x} \tan\theta \tag{3-63}$$

In general, $\dot{\phi}_k \neq \dot{\phi}_o$ so that $\overline{\cos^2 x} = 1/2$, and

$$\tan^2\theta > \frac{2N_t}{1 - N_t} \tag{3-64}$$

gives the critical value of θ. Notice that for a sphere ($N_t = 1/3$), $\theta = \pi/4$ appears to fulfill the condition. However, when $\theta = \pi/4$, the $\psi = 0$ waves are actually at the top rather than at the bottom of the spectrum, and ω_o has not escaped from the spectrum. Further increase of θ will not alter the situation. If N_t is very small (flat disc magnetized normal to its plane) the critical value of θ decreases. This is reasonable since for small N_t and $\theta = 0$, the uniform precession begins close to the bottom of the spin wave manifold and moves only a short distance before escaping. In fact, for $N_t = 0$, the uniform precession is initially at the very bottom of the manifold and any non-zero θ will cause it to drop below. In practice, of course, there is always some non-zero N_t.

If $\dot{\phi}_k = \dot{\phi}_o$ (assume that $\dot{\phi}_k$ cannot equal $\dot{\phi}_o/2$), it follows that (for $\psi = 0$)

$$N_t = \lambda K^2 + (1 + \lambda K^2 - N_t) \tan^2\theta \, \overline{\cos^2 x} \tag{3-65}$$

Now $\overline{\cos^2 x}$ need not be 1/2 so that their exists a range of degenerate $\psi = 0$ spin waves (with precise phase relationships) even though the inequality (3-64) is valid. They exist from $k = 0$ for which $\overline{\cos^2 x} = \frac{N_t}{(1 - N_t)\tan^2\theta} < 1/2$ to $k = \sqrt{\frac{N_t}{\lambda}}$ for which $\overline{\cos^2 x} = 0$. Notice that the degenerate band has exactly the same width (in k) as when $\theta = 0$, and that the degenerate range of spin waves is not restricted to $\psi = 0$. Equation (3-65) is generalized to

$$\left[\omega_H - N_z\omega_M\cos\theta + \lambda K^2\omega_M\cos\theta + (\cos^2\psi + \lambda K^2 - N_t)\omega_M\sin\theta\tan\theta\cos^2 x\right]$$

$$\times\sqrt{1 + \frac{\omega_M\cos\theta\sin^2\psi}{\omega_H - N_z\omega_M\cos\theta + \lambda K^2\omega_M\cos\theta + (\cos^2\psi + \lambda K^2 - N_t)\omega_M\sin\theta\tan\theta\cos^2 x}} \tag{3-66}$$

$$= \omega_H - N_z\omega_M\cos\theta + N_t\omega_M\cos\theta$$

If $\frac{\omega_M \sin^2\psi}{\omega_H - N_z\omega_M} << 1$, (3-66) simplifies to

$$N_t - \frac{\sin^2\psi}{2} \simeq \lambda K^2 + (\cos^2\psi + \lambda K^2 - N_t)\tan^2\theta\cos^2 X \tag{3-67}$$

No degenerate spin waves exist (for small θ) if $\sin^2\psi > 2N_t$.

In Chapter 2, the critical value of θ leading to instability for $\psi = 0$ spin waves (for which $\omega_k = \omega_o$), was given and may be written in the form

$$\tan^2\theta = \frac{2\Delta H_K}{(1 + \lambda K^2 - N_t)\,M\,\overline{\sin 2X}} \tag{3-68}$$

Since $\cos^2 x$ must be less than 1/2, if the inequality (3-64) holds, the instability thresholds are raised accordingly. Solving (3-67) and (3-68) together yields

$$\tan^2\theta = \frac{(N_t - \lambda K^2)^2 + \left(\frac{\Delta H_K}{M}\right)^2}{(1+\lambda K^2 - N_t)(N_t - \lambda K^2)} \tag{3-69}$$

The minimum value of $\tan\theta$ occurs when

$$\lambda K^2 = N_t + \left(\frac{\Delta H_K}{M}\right)^2 - \frac{\Delta H_K}{M}\sqrt{1+\left(\frac{\Delta H_K}{M}\right)^2} \tag{3-70}$$

and if $\frac{\Delta H_k}{M} \ll 1$, as is usual, (3-70) may be approximated as

$$\lambda K^2 \simeq N_t - \left(\frac{\Delta H_K}{M}\right) \tag{3-71}$$

Substituting this last result into (3-69) yields

$$\tan^2\theta \simeq \frac{2\Delta H_K}{M} \tag{3-72}$$

When $N_t < \frac{\Delta H_k}{M}$ this solution does not exist for any real k and the minimum threshold occurs for $k = 0$. Then (3-69) becomes

$$\tan^2\theta = \frac{N_t^2 + \left(\frac{\Delta H_K}{M}\right)^2}{(1-N_t)N_t} \tag{3-73}$$

If the uniform precession is to drop below the main spin wave band before this threshold is reached, the following inequalities must hold

$$\frac{2N_t}{1-N_t} < \tan^2\theta < \frac{N_t^2 + \left(\frac{\Delta H_K}{M}\right)^2}{(1-N_t)N_t} \tag{3-74}$$

This is possible since by assumption $N_t < \frac{\Delta H_k}{M}$. The smaller the value of N_t, the wider the range of θ, which satisfies the above conditions. Although the spin wave instability threshold is raised by using an extremely thin disc, instabilities of the type discussed by Anderson and Suhl,[19] which depend on the resonant frequency being a function of θ, will remain with unchanged thresholds. These exist, however, only for certain values of the biasing magnetic field.

The spin wave spectrum is shown in Fig. 3-10 for conditions under which the uniform precession has dropped below it. One may inquire why the uniform precession is apparently privileged in this respect. The answer lies in the assumption that only the uniform precession has a significant amplitude. The transverse precessional component interacts with all other spin waves or magnetostatic modes* and under certain conditions tends to raise their frequencies. As long as the other modes possess negligible amplitudes they cannot in turn effect the uniform precession frequency. If some other mode were excited to a large amplitude, the spectrum shift could be quite different.

*Actually this has only been proved for the spin wave spectrum, extrapolated to zero k values. For $\theta = 0$ no magnetostatic modes are below this extrapolated limit and it is reasonable to suppose that this holds generally, at least for small θ.

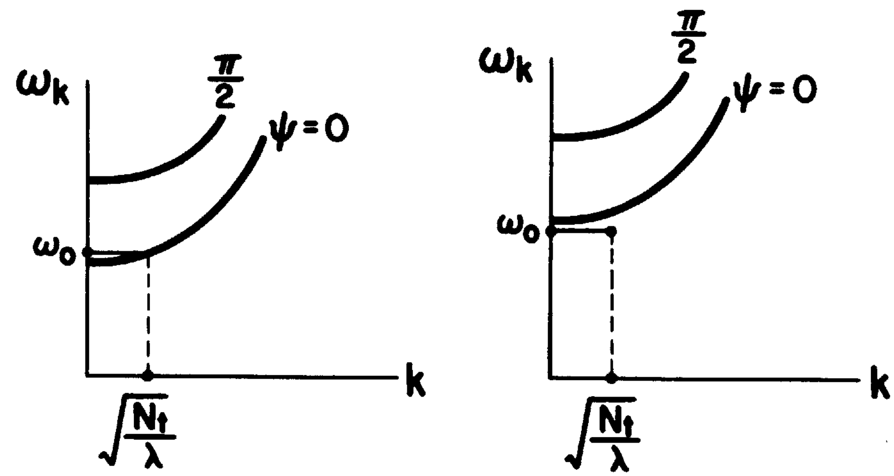


Fig. 3-10

APPENDIX

We establish (3-34) by an iterative procedure. Notice that $\dot{\delta M}(0) = 0$ and that $\dot{\delta M}$ does not change appreciably (compared to $\dot{\theta}$) until θ has gone negative. As a first approximation, therefore, let $\delta M = \delta M_o$ be a constant. Then

$$\dot{\theta} \simeq -\omega_{\ell o}\theta - (n-1)\,\omega_{\ell o}\theta^c \tag{A-1}$$

which may be integrated immediately to give

$$\left(\frac{\theta}{\theta^c} - 1\right) = -n\left(1 - e^{-\omega_{\ell o}t}\right) \tag{A-2}$$

Now use this to get a better approximation of $\dot{\delta M}$ namely

$$\dot{\delta M} \simeq -n\omega_{\ell K}\left(1 - e^{-\omega_{\ell o}t}\right)\delta M \tag{A-3}$$

This may be integrated to the form

$$\delta M = \delta M_o\, e^{-n\omega_{\ell K}\left[t - \frac{1}{\omega_{\ell o}}\left(1 - e^{-\omega_{\ell o}t}\right)\right]} \tag{A-4}$$

which has the correct behavior at $t = 0$ and yet leads to the eventual decay of δM which is necessary.

We use this expression for δM to improve on (A-1). This is a first order equation so that the solution is given by

$$\theta = A(t)\, e^{-\omega_{\ell o}t} \tag{A-5}$$

where

$$A = -(n-1)\omega_{\ell o}\theta^c e^{\beta}\int e^{-(\beta-1)\omega_{\ell o}t}\, e^{-\beta e^{-\omega_{\ell o}t}}\, dt \quad + \; A_o \tag{A-6}$$

and

$$\beta = \frac{2n\,\omega_{\ell K}}{\omega_{\ell o}} \tag{A-7}$$

The integral in (A-6), which must be evaluated, is of the form

$$I = \int e^{-\alpha t}\, e^{-\beta e^{-\gamma t}}\, dt \tag{A-8}$$

with $\gamma = \omega_{\ell o}$ and $\alpha = \gamma(\beta - 1)$. Successive integration by parts gives rise to the series solution

$$I = -\frac{1}{\alpha}\, e^{-\alpha t}\, e^{-\beta e^{-\gamma t}} \left[1 + \frac{\beta\gamma}{\alpha+\gamma}\, e^{-\gamma t} + \frac{\beta^2\gamma^2}{(\alpha+\gamma)(\alpha+2\gamma)}\, e^{-2\gamma t} + \frac{\beta^3\gamma^3}{(\alpha+\gamma)(\alpha+2\gamma)(\alpha+3\gamma)}\, e^{-3\gamma t} + \cdots \right] \tag{A-9}$$

Then

$$A = \left(\frac{n-1}{\beta-1}\right)\theta^c e^{\beta}\, e^{-(\beta-1)\omega_{\ell o}t}\, e^{-\beta e^{-\omega_{\ell o}t}}\, P(t) \; + \; A_o \tag{A-10}$$

where

$$P(t) = 1 + e^{-\omega_{\ell o}t} + \frac{\beta}{\beta+1}\, e^{-2\omega_{\ell o}t} + \frac{\beta^2}{(\beta+1)(\beta+2)}\, e^{-3\omega_{\ell o}t} + \cdots \tag{A-11}$$

The constant A_o is evaluated so that $\theta(0) = \theta^c$. Thus

$$A_o = \theta^c - \left(\frac{n-1}{\beta-1}\right)\theta^c\, P(0) \tag{A-12}$$

and (3-34) is established.

CONCLUSION

In this thesis, there were formulated, for a small ferrimagnetic ellipsoid, the general equations of motion governing the uniform precession and a typical small-amplitude spin wave. The internal field used in setting up the equations consists of the applied fields together with those arising from the demagnetizing, volume dipolar, and exchange energies. Schlömann has shown that fields due to inhomogeneities are important in some cases (particularly with porous polycrystalline samples) since they can cause a gradual decline in the susceptibility for increasing power levels.[27] However, for simplicity, such terms, as well as those due to anistropy, were neglected; their inclusion in the general equations poses no serious problem.

The equations of motion, neglecting spin wave terms, were analyzed with respect to both linear and nonlinear effects. The generalized ferromagnetic resonance frequency was derived and the intermodulation between the precession cone angle and phase was discussed. The general linear external permeability tensor was obtained and proved to be complicated due to the elliptical precession path of the magnetization. Harmonic generation occurs in the longitudinal direction because of this ellipticity but it was also pointed out that longitudinal components of time-varying demagnetizing fields cause harmonic generation in the transverse plane. These field components may also cause subharmonic resonance when an applied transverse driving field is above a certain threshold, provided that the ellipsoid is biased to the subharmonic frequency. The half-frequency subharmonic is the only case of importance and was analyzed in detail. The relationship between this resonance and a general class of nonlinear responses was pointed out.

Next, spin wave interactions with the uniform precession were taken into account. Solutions of the equations led to a determination of the generalized spin wave spectrum and a physical picture of the interrelationship between the spin wave amplitude and phase. This was found to be similar to the case of the uniform precession but due to volume rather than surface effects. The resulting elliptical precession path is of major importance in providing parametric coupling between certain spin waves and the uniform precession and was found to be responsible for the first order instability threshold in a spheroid, as discovered by Suhl. The generalized first order threshold was obtained for an ellipsoid and it was found that both transverse and longitudinal demagnetizing fields are important. The second order instability threshold for a spheroid, also discovered by Suhl, was found to be modified in an ellipsoid because of these fields, but normally only to a minor extent. These above-mentioned thresholds involve a single transverse driving field but the effects of additional applied microwave fields, of various frequencies, were sought. It was found that direct parametric coupling, between a longitudinal pumping field and both the uniform precession and spin waves, can exist under certain frequency conditions, and that transverse pumping fields of the proper frequency can couple directly to spin waves. An experiment utilizing a combination of driving fields was described, which should make possible the measurement of ΔH_k for selected spin waves. Much of the material discussed has direct and obvious application to the field of ferrite parametric amplifiers and oscillators.

The transient build up and decay of the uniform precession in response to a pulsed microwave driving field was discussed, as was the dynamic behavior of spin wave interaction. In particular, it was shown that if the magnetization of a spheroid could suddenly be inverted with respect to the magnetizing field, certain cut-off spin waves would grow very rapidly at the expense of the uniform precession. The spin wave spectrums, appropriate to succeeding stages of the ensuing transient, were derived using a quasi-static approximation and they give at least a qualitative picture of the highly nonlinear loss mechanism. Finally, an analysis of the position of the uniform precession relative to the spin wave manifold indicates that under certain conditions the uniform mode might drop below the main spin wave band. Under such conditions it appears that the second order instability threshold in a very thin disc can be made to rise above its usual value.

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BIOGRAPHY

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