Chain Polytopes and Algebras with Straightening Laws

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Chain Polytopes and Algebras with Straightening Laws

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Dedicated to Professor Ngô Việt Trung on the occasion of his sixtieth birthday

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Abstract It will be shown that the toric ring of the chain polytope of a finite partially ordered set is an algebra with straightening laws on a finite distributive lattice. Thus, in particular, every chain polytope possesses a regular unimodular triangulation arising from a flag complex.

Keywords Algebra with straightening laws · Chain polytope · Partially ordered set

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1 Introduction

In [6], the order polytope $\mathcal{O}(P)$ and the chain polytope $\mathcal{C}(P)$ of a finite poset (partially ordered set) P are studied in detail from a view point of combinatorics. Toric rings of order polytopes are studied in [1]. In particular, it is shown that the toric ring $K[\mathcal{O}(P)]$ of the order polytope $\mathcal{O}(P)$ is an algebra with straightening laws ([2, p. 124]) on a finite distributive lattice. In the present paper, it will be proved that the toric ring $K[\mathcal{C}(P)]$ of the chain polytope $\mathcal{C}(P)$ is also an algebra with straightening laws on a finite distributive lattice. It then follows immediately that $\mathcal{C}(P)$ possesses a regular unimodular triangulation arising from a flag complex.

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2 Toric Rings of Order Polytopes and Chain Polytopes

Let $P = \{x_1, \ldots, x_d\}$ be a finite poset. For each subset $W \subset P$, we associate $\rho(W) = \sum_{i \in W} \mathbf{e}_i \in \mathbb{R}^d$, where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the unit coordinate vectors of $\mathbb{R}^d$. In particular, $\rho(\emptyset)$ is the origin of $\mathbb{R}^d$. A *poset ideal* of P is a subset I of P such that, for all x_i and x_j with $x_i \in I$ and $x_j \leq x_i$, one has $x_j \in I$. An *antichain* of P is a subset A of P such that x_i and x_j belonging to A with $i \neq j$ are incomparable.

Recall that the *order polytope* is the convex polytope $\mathcal{O}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $0 \leq a_i \leq 1$ for every $1 \leq i \leq d$ together with $a_i \geq a_j$ if $x_i \leq x_j$ in P. The vertices of $\mathcal{O}(P)$ are those $\rho(I)$ such that I is a poset ideal of P ([6, Corollary 1.3]). The *chain polytope* is the convex polytope $\mathcal{C}(P) \subset \mathbb{R}^d$ which consists of those $(a_1, \ldots, a_d) \in \mathbb{R}^d$ such that $a_i \geq 0$ for every $1 \leq i \leq d$ together with

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \leq 1$$

for every maximal chain $x_{i_1} < x_{i_2} < \cdots < x_{i_k}$ of P. The vertices of $\mathcal{C}(P)$ are those $\rho(A)$ such that A is an antichain of P ([6, Theorem 2.2]).

Let $S = K[x_1, \ldots, x_d, t]$ denote the polynomial ring over a field K whose variables are the elements of P together with the new variable t. For each subset $W \subset P$, we associate the squarefree monomial $x(W) = \prod_{i \in W} x_i \in S$. In particular, $x(\emptyset) = 1$. The *toric ring* $K[\mathcal{O}(P)]$ of $\mathcal{O}(P)$ is the subring of R generated by those monomials $t \cdot x(I)$ such that I is a poset ideal of P. The toric ring $K[\mathcal{C}(P)]$ of $\mathcal{C}(P)$ is the subring of R generated by those monomials $t \cdot x(A)$ such that A is an antichain of P.

3 Algebras with Straightening Laws

Let $R = \bigoplus_{n \geq 0} R_n$ be a graded algebra over a field $R_0 = K$. Suppose that P is a poset with an injection $\varphi : P \to R$ such that $\varphi(\alpha)$ is a homogeneous element of R with $\deg \varphi(\alpha) \geq 1$ for every $\alpha \in P$. A *standard monomial* of R is a finite product of the form $\varphi(\alpha_1)\varphi(\alpha_2)\cdots$ with $\alpha_1 \leq \alpha_2 \leq \cdots$. Then, we say that $R = \bigoplus_{n \geq 0} R_n$ is an *algebra with straightening laws* on P over K if the following conditions are satisfied:

- The set of standard monomials is a basis of R as a vector space over K.
- If α and β in P are incomparable and if

$$\varphi(\alpha)\varphi(\beta) = \sum_i r_i \varphi(\gamma_{i_1})\varphi(\gamma_{i_2})\cdots, \tag{1}$$

  where $0 \neq r_i \in K$ and $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots$ is the unique expression for $\varphi(\alpha)\varphi(\beta) \in R$ as a linear combination of distinct standard monomials, then $\gamma_{i_1} \leq \alpha$ and $\gamma_{i_1} \leq \beta$ for every i.

We refer the reader to [2, Chapter XIII] for fundamental material on algebras with straightening laws. The relations (1) are called the *straightening relations* of R.

Let P be an arbitrary finite poset and $\mathcal{J}(P)$ the finite distributive lattice ([7, p. 252]), consisting of all poset ideals of P ordered by inclusion. The toric ring $K[\mathcal{O}(P)]$ of the order polytope $\mathcal{O}(P)$ is a graded ring with $\deg(t \cdot x(I)) = 1$ for every $I \in \mathcal{J}(P)$. We then define the injection $\varphi : \mathcal{J}(P) \to K[\mathcal{O}(P)]$ by setting $\varphi(I) = t \cdot x(I)$ for every $I \in \mathcal{J}(P)$. One of the fundamental results obtained in [1] is that $K[\mathcal{O}(P)]$ is an algebra with straightening laws on $\mathcal{J}(P)$. Its straightening relations are

$$\varphi(I)\varphi(J) = \varphi(I \cap J)\varphi(I \cup J), \tag{2}$$

where I and J are poset ideals of P which are incomparable in $\mathcal{J}(P)$.

Theorem 3.1 *The toric ring of the chain polytope of a finite poset is an algebra with straightening laws on a finite distributive lattice.*

Proof Let P be an arbitrary finite poset and $\mathcal{C}(P)$ its chain polytope. The toric ring $K[\mathcal{C}(P)]$ is a graded ring with $\deg(t \cdot x(A)) = 1$ for every antichain A of P.

For a subset $Z \subset P$, we write $\max(Z)$ for the set of maximal elements of Z. In particular, $\max(Z)$ is an antichain of P. The poset ideal of P generated by a subset $Y \subset P$ is the smallest poset ideal of P which contains Y.

Now, we define the injection $\psi : \mathcal{J}(P) \to K[\mathcal{C}(P)]$ by setting $\psi(I) = t \cdot x(\max(I))$ for all poset ideals I of P. If I and J are poset ideals of P, then

$$\psi(I)\psi(J) = \psi(I \cup J)\psi(I * J), \tag{3}$$

where $I * J$ is the poset ideal of P generated by $\max(I \cap J) \cap (\max(I) \cup \max(J))$. Since $I * J \subset I$ and $I * J \subset J$, the relations (3) satisfy the condition of the straightening relations.

It remains to prove that the set of standard monomials of $K[\mathcal{C}(P)]$ is a K-basis of $K[\mathcal{C}(P)]$. It follows from [6, Theorem 4.1] that the Hilbert function ([2, p. 33]) of the Ehrhart ring ([2, p. 97]) of $\mathcal{O}(P)$ coincides with that of $\mathcal{C}(P)$. Since $\mathcal{O}(P)$ and $\mathcal{C}(P)$ possess the integer decomposition property ([4, Lemma 2.1]), the Ehrhart ring of $\mathcal{O}(P)$ coincides with $K[\mathcal{O}(P)]$ and the Ehrhart ring of $\mathcal{C}(P)$ coincides with $K[\mathcal{C}(P)]$. Hence, the Hilbert function of $K[\mathcal{O}(P)]$ is equal to that of $K[\mathcal{C}(P)]$. Thus, the set of standard monomials of $K[\mathcal{C}(P)]$ is the K-basis of $K[\mathcal{C}(P)]$ as desired. □

4 Flag and Unimodular Triangulations

The fact that $K[\mathcal{C}(P)]$ is an algebra with straightening laws guarantees that the toric ideal of $\mathcal{C}(P)$ possesses an initial ideal generated by squarefree quadratic monomials. We refer the reader to [3] and [5, Appendix] for the background of the existence of squarefree quadratic initial ideals of toric ideals. By virtue of [8, Theorem 8.3], it follows that

Corollary 4.1 *Every chain polytope possesses a regular unimodular triangulation arising from a flag complex.*

5 Further Questions

Let, as before, P be a finite poset and $\mathcal{J}(P)$ the finite distributive lattice consisting of all poset ideals of P ordered by inclusion. Let $S = K[x_1, \ldots, x_n, t]$ denote the polynomial ring and $\Omega = \{w_I\}_{I \in \mathcal{J}(P)}$ a set of monomials in $x_1, \ldots, x_n$ indexed by $\mathcal{J}(P)$. We write $K[\Omega]$ for the subring of S generated by those monomials $w_I \cdot t$ with $I \in \mathcal{J}(P)$ and define the injection $\varphi : \mathcal{J}(P) \to K[\Omega]$ by setting $\varphi(I) = w_I \cdot t$ for every $I \in \mathcal{J}(P)$.

Suppose that $K[\Omega]$ is an algebra with straightening laws on $\mathcal{J}(P)$ over K. We say that $K[\Omega]$ is *compatible* if each of its straightening relations is of the form $\varphi(I)\varphi(I') = \varphi(J)\varphi(J')$ such that $J \leq I \wedge I'$ and $J' \geq I \vee I'$, where I and I' are poset ideals of P which are incomparable in $\mathcal{J}(P)$.

Let $K[\Omega]$ and $K[\Omega']$ be compatible algebras with straightening laws on $\mathcal{J}(P)$ over K. Then, we identify $K[\Omega]$ with $K[\Omega']$ if the straightening relations of $K[\Omega]$ coincide with those of $K[\Omega']$.

Let P^* be the *dual poset* ([7, p. 247]) of a poset P. The toric ring $K[\mathcal{C}(P^*)]$ of $\mathcal{C}(P^*)$ can be regarded as an algebra with straightening laws on $\mathcal{J}(P)$ over K in the obvious way. Clearly, each of the toric rings $K[\mathcal{O}(P)]$, $K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$ is a compatible algebra with straightening laws on $\mathcal{J}(P)$ over K.

Question 5.1 (a) Given a finite poset P, find all possible compatible algebras with straightening laws on $\mathcal{J}(P)$ over K.

(b) In particular, for which posets P, does there exist a unique compatible algebra with straightening laws on $\mathcal{J}(P)$ over K ?

Example 5.2 (a) Let P be the poset of Fig. 1. Then, $K[\mathcal{O}(P)] = K[\mathcal{C}(P)]$, and there exists a unique compatible algebra with straightening laws on $\mathcal{J}(P)$ over K. In fact, $\mathcal{J}(P)$ for P is the poset in Fig. 2. In the corresponding algebra, we must have $be = af$. Then, we have either $bc = ad$ or $bc = af$. However, since $bc \neq be$, it follows that $bc = ad$. Similarly, we have $de = cf$. Hence, the ASL relations are unique.

(b) Let P be the poset of Fig. 3. Then, there exist three compatible algebras with straightening laws on $\mathcal{J}(P)$ over K. They are $K[\mathcal{O}(P)]$, $K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$.

(c) Let P be the poset of Fig. 4. Then, there exist nine compatible algebras with straightening laws on $\mathcal{J}(P)$ over K.

Fig. 1 Example 5.2 (a)

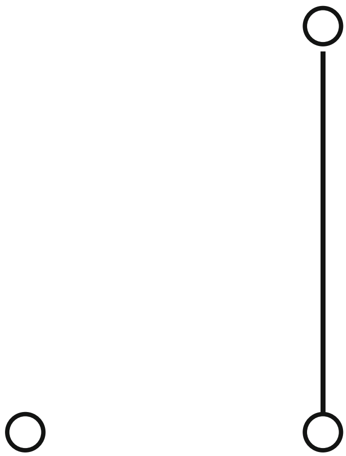


Fig. 2 Example 5.2 (b)

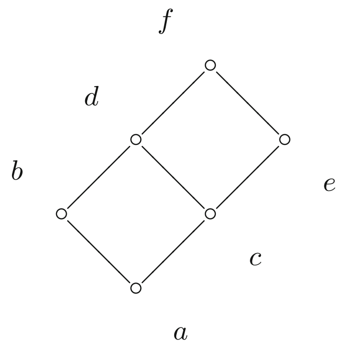


Fig. 3 Example 5.2 (c)

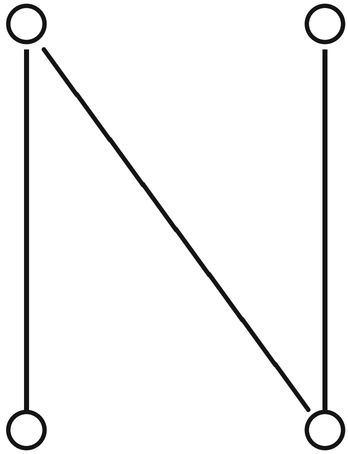
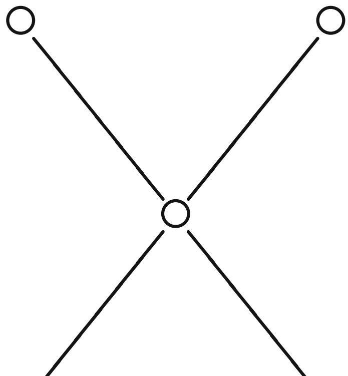


Fig. 4 $\mathcal{J}(P)$ in Example 5.2 (a)



Conjecture 5.3 *If P is a disjoint union of chains, then the compatible algebras with straightening laws on $\mathcal{J}(P)$ over K are $K[\mathcal{O}(P)]$, $K[\mathcal{C}(P)]$, and $K[\mathcal{C}(P^*)]$.*

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