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INDEX THEORY OF THE DE RHAM COMPLEX ON MANIFOLDS WITH PERIODIC ENDS

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ABSTRACT. We study the de Rham complex on a smooth manifold with a periodic end modeled on an infinite cyclic cover $\tilde{X} \rightarrow X$. The completion of this complex in exponentially weighted L^2 -norms is Fredholm for all but finitely many exceptional weights determined by the eigenvalues of the covering translation map $H_*(\tilde{X}) \rightarrow H_*(\tilde{X})$. We calculate the index of this weighted de Rham complex for all weights away from the exceptional ones.

1. INTRODUCTION

Let M a smooth closed orientable manifold of dimension n . The de Rham complex of complex valued differential forms on M ,

$$0 \rightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \rightarrow \cdots \rightarrow \Omega^n(M) \rightarrow 0,$$

is known to be Fredholm in a suitable L^2 completion. This means as usual that the images of d_k are closed and the vector spaces $\ker d_k / \text{im } d_{k-1}$ are finite dimensional. The alternating sum of the dimensions of these spaces is called the index of the de Rham complex. Since $\ker d_k / \text{im } d_{k-1}$ is isomorphic to the singular cohomology $H^k(M; \mathbb{C})$ by the de Rham theorem, the above index equals $\chi(M)$, the Euler characteristic of M .

This paper extends these classical results to certain non-compact manifolds, those with a periodic end modeled on an infinite cyclic cover \tilde{X} of a closed manifold X . It builds on the earlier work of Miller [8] and Taubes

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[14] and can be viewed as a continuation of our research in [10] and [11] on the index theory of elliptic operators on such manifolds.

Let M be a Riemannian manifold with periodic end modeled on \tilde{X} . Its de Rham complex can be completed in the L^2 norm using (on the end) a Riemannian measure dx lifted from that on X . This completion is, however, not Fredholm; see Remark 2.3. To rectify this problem, we will use L^2_δ norms, which are the L^2 norms on M with respect to the measure $e^{\delta f(x)} dx$ over the end. Here, δ is a real number and $f : \tilde{X} \rightarrow \mathbb{R}$ is a smooth function such that $f(\tau(x)) = f(x) + 1$ with respect to the covering translation $\tau : \tilde{X} \rightarrow \tilde{X}$. The L^2_δ completion of the de Rham complex on M will be denoted by $\Omega^*_\delta(M)$.

Theorem 1.1. *Let M be a smooth Riemannian manifold with a periodic end modeled on \tilde{X} , and suppose that $H_*(M; \mathbb{C})$ is finite dimensional. Then $\Omega^*_\delta(M)$ is Fredholm for all but finitely many δ of the form $\delta = \ln |\lambda|$, where λ is a root of the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$.*

Conditions on X that guarantee that $H_*(M; \mathbb{C})$ is finite dimensional can be found in Section 2, together with a proof of Theorem 1.1.

Given a manifold M as in the above theorem, the complex $\Omega^*_\delta(M)$ has a well defined index $\text{ind}_\delta(M)$. Miller [8] showed that $\text{ind}_\delta(M)$ is an even or odd function of δ according to whether $\dim M = n$ is even or odd, and that $\text{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. We add to this knowledge the following result.

Theorem 1.2. *Let M be as in Theorem 1.1. Then $\text{ind}_\delta(M)$ is a piecewise constant function of δ whose only jumps occur at $\delta = \ln |\lambda|$, where λ a root of the characteristic polynomial $A_k(t)$ of $\tau_* : H_k(\tilde{X}; \mathbb{C}) \rightarrow H_k(\tilde{X}; \mathbb{C})$ for some $k = 0, \dots, n - 1$. Every such λ contributes $(-1)^{k+1}$ times its multiplicity as the root of $A_k(t)$ to the jump.*

Together with the results of [8] this completes the calculation of the function $\text{ind}_\delta(M)$. Theorem 1.2 is proved in Section 3. The last section of the

paper contains discussion as well as calculations of $\text{ind}_\delta(M)$ for two important classes of examples. The first class consists of manifolds with infinite cylindrical ends studied earlier by Atiyah, Patodi and Singer [1], and the second of manifolds arising in the study of knotted $S^2 \subset S^4$.

Finally, we will remark that the de Rham complex is a special case of the more general concept of an elliptic complex. The index theory for elliptic complexes on closed manifolds was developed by Atiyah and Singer whose famous index theorem [2] expresses the index of such a complex in purely topological terms. More generally, Atiyah, Patodi and Singer [1] computed the index for certain elliptic operators (that is, elliptic complexes of length two) on manifolds with cylindrical ends. In our paper [11] we extended their result to general manifolds with periodic ends; our index formula involves a new end-periodic η -invariant, generalizing the η -invariant of Atiyah, Patodi and Singer from the cylindrical setting. It would be interesting to compare the index formula of Theorem 1.2 with that of [11] for the operator $d + d^*$ obtained by wrapping up the de Rham complex.

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2. THE FREDHOLM PROPERTY

In this section, we will prove Theorem 1.1 by reducing it to a statement about twisted cohomology of X .

2.1. The Fourier–Laplace transform. The de Rham complex of M is an elliptic complex on the end-periodic manifold M hence we can use the general theory of such complexes due to Taubes [14]. According to that theory, it is sufficient to check the Fredholm property of the de Rham complex of \tilde{X} completed in the L^2 norm on \tilde{X} with respect to the measure $e^{\delta f(x)} dx$. The latter complex can be studied using the Fourier–Laplace transform, which

is defined by the formula

$$\hat{\omega}_z = \sum z^k \cdot (\tau^*)^k \omega, \quad z \in \mathbb{C}^*,$$

on compactly supported forms ω on \tilde{X} , and is extended by continuity to L^2 forms. The summation in the above formula extends to all integer k , which makes the form $\hat{\omega}_z$ invariant with respect to τ^* . The form $\hat{\omega}_z$ then defines a form on X which is denoted by the same symbol. An application of the Fourier–Laplace transform to the de Rham complex on \tilde{X} results in a family of twisted de Rham complexes

$$\dots \longrightarrow \Omega^k(X) \xrightarrow{d - \ln z \, df} \Omega^{k+1}(X) \longrightarrow \dots \quad (1)$$

parameterized by $z \in \mathbb{C}^*$. The following result is proved in Taubes [14, Lemma 4.3].

Proposition 2.1. *Let M be a smooth Riemannian manifold with a periodic end modeled on \tilde{X} . For any given δ , the complex $\Omega_\delta^*(M)$ is Fredholm if and only if the complexes (1) are exact for all z such that $|z| = e^\delta$.*

The cohomology of complex (1) is of course the twisted de Rham cohomology $H_z^*(X; \mathbb{C})$ with coefficients in the complex line bundle with flat connection $- \ln z \, df$.

Proposition 2.2. *Let M be a smooth Riemannian manifold with a periodic end modeled on \tilde{X} . Then the following three conditions are equivalent:*

- (1) $\Omega_\delta^*(M)$ is Fredholm for all $\delta \in \mathbb{R}$ away from a discrete set;
- (2) $H_z^*(X; \mathbb{C})$ vanishes for all $z \in \mathbb{C}^*$ away from a discrete set;
- (3) $H_z^*(X; \mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$.

Proof. Observe that the complexes (1) form a holomorphic family of elliptic complexes on \mathbb{C}^* ; therefore, exactness of (1) at one point is equivalent to exactness away from a discrete set. The statement now follows from the preceding discussion. \square

Remark 2.3. The usual L^2 completion of the de Rham complex on M , that is, the complex $\Omega_0^*(M)$, is not Fredholm because $H_z^0(X; \mathbb{C})$ is not zero when $z = 1$.

2.2. Finite dimensionality. Fix a finite cell complex structure on X , lift it to \tilde{X} , and consider the chain complex $C_*(\tilde{X}, \mathbb{C})$ and its homology $H_*(\tilde{X}; \mathbb{C})$. The group of integers acts on both by covering translations making them into finitely generated modules over $\mathbb{C}[t, t^{-1}]$. The twisted de Rham theorem tells us that the cohomology $H_z^*(X; \mathbb{C})$ of complex (1) is isomorphic to the cohomology of the complex $\text{Hom}_{\mathbb{C}[t, t^{-1}]}(C_*(\tilde{X}, \mathbb{C}), \mathbb{C}_z)$, where \mathbb{C}_z denotes a copy of \mathbb{C} viewed as a $\mathbb{C}[t, t^{-1}]$ module with $p(t)$ acting via multiplication by $p(z)$.

Proposition 2.4. *Let M be a smooth Riemannian manifold with a periodic end modeled on \tilde{X} . Then the following two conditions are equivalent:*

- (1) $H_*(M; \mathbb{C})$ is a finite dimensional vector space;
- (2) $H_z^*(X; \mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$.

Proof. It is immediate from the Mayer–Vietoris principle that $H^*(M; \mathbb{C})$ is finite dimensional if and only if $H^*(\tilde{X}; \mathbb{C})$ is finite dimensional. Since $H_*(\tilde{X}; \mathbb{C})$ is a finitely generated module over the principal ideal domain $\mathbb{C}[t, t^{-1}]$, it admits a primary decomposition

$$H_*(\tilde{X}; \mathbb{C}) = \mathbb{C}[t, t^{-1}]^\ell \oplus \mathbb{C}[t, t^{-1}]/(p_1) \oplus \dots \oplus \mathbb{C}[t, t^{-1}]/(p_m), \quad (2)$$

therefore, $H^*(\tilde{X}; \mathbb{C})$ is a finite dimensional vector space if and only if $\ell = 0$ in this decomposition. According to the universal coefficient theorem,

$$H_z^*(X; \mathbb{C}) = \text{Hom}_{\mathbb{C}[t, t^{-1}]}(H_*(\tilde{X}; \mathbb{C}), \mathbb{C}_z) \oplus \text{Ext}_{\mathbb{C}[t, t^{-1}]}(H_*(\tilde{X}; \mathbb{C}), \mathbb{C}_z),$$

hence vanishing of $H_z^*(X; \mathbb{C})$ for at least one z implies that $\ell = 0$. On the other hand, an easy calculation shows that

$$\text{Hom}_{\mathbb{C}[t, t^{-1}]}(V, \mathbb{C}_z) = \text{Ext}_{\mathbb{C}[t, t^{-1}]}(V, \mathbb{C}_z) = 0$$

for any module $V = \mathbb{C}[t, t^{-1}]/(p)$ such that $p(z) \neq 0$. Therefore, $\ell = 0$ implies that $H_z^*(X; \mathbb{C})$ must vanish for all z away from the roots of the polynomials p_1, \dots, p_m . \square

2.3. Proof of Theorem 1.1. It follows from Propositions 2.2 and 2.4 that, if $H^*(\tilde{X}; \mathbb{C})$ is finite dimensional, the complex $\Omega_\delta^*(M)$ is Fredholm for all δ away from a discrete set. To finish the proof of Theorem 1.1 we just need to identify this discrete set. According to Proposition 2.1, it consists of $\delta = \ln |z|$, where $z \in \mathbb{C}^*$ are the complex numbers for which $H_z^*(X; \mathbb{C})$ fails to be zero. To find them, note that the free part in the prime decomposition (2) vanishes making $H_*(\tilde{X}; \mathbb{C})$ into a torsion module,

$$H_*(\tilde{X}; \mathbb{C}) = \mathbb{C}[t, t^{-1}]/(p_1) \oplus \dots \oplus \mathbb{C}[t, t^{-1}]/(p_m). \quad (3)$$

According to Milnor [9, Assertion 4], the order ideal $(p_1 \cdots p_m)$ of this module is spanned by the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$. The calculation with the universal coefficient theorem as in the proof of Proposition 2.4 now completes the proof.

2.4. A sufficient condition. Let M be a smooth orientable manifold with a periodic end modeled on \tilde{X} . Vanishing of $\chi(X)$ is obviously a necessary condition for the vector space $H_*(M; \mathbb{C})$ to be finite dimensional. To come up with a sufficient condition, observe that the derivative df defines a closed 1-form on X , and let $\xi = [df] \in H^1(X; \mathbb{C})$ be its cohomology class. The cup product with ξ gives rise to the chain complex

$$H^0(X; \mathbb{C}) \xrightarrow{\cup \xi} H^1(X; \mathbb{C}) \xrightarrow{\cup \xi} \dots \xrightarrow{\cup \xi} H^n(X; \mathbb{C}). \quad (4)$$

Proposition 2.5. *Suppose the chain complex (4) is exact. Then $H_*(M; \mathbb{C})$ is a finite dimensional vector space for any smooth orientable manifold with periodic end modeled on \tilde{X} .*

This proposition can be derived as a special case of Taubes [14, Theorem 3.1]. That proof is analytic in nature; here is another proof which is purely topological.

According to Proposition 2.4, it is sufficient to prove that the twisted cohomology $H_z^*(X; \mathbb{C})$ vanishes for at least one $z \in \mathbb{C}^*$. We will show that it does so for all $z \neq 1$ in a sufficiently small neighborhood of 1. For such z , there is a spectral sequence (E_r^*, d_r) which starts at $E_1^* = H^*(X; \mathbb{C})$, converges to $H_z^*(X; \mathbb{C})$, and whose differentials are given by the Massey products with the class ξ ; see Farber [5, Section 10.9]. The convergence of this spectral sequence to zero is therefore a necessary and sufficient condition for the finite dimensionality of $H_*(M; \mathbb{C})$. The chain complex (4) is the term (E_1^*, d_1) of that spectral sequence, hence its exactness is sufficient for the finite dimensionality of $H^*(X; \mathbb{C})$.

Note that the vanishing of the Euler characteristic of X is not a sufficient condition for $H_*(M; \mathbb{C})$ to be finite dimensional. An example is provided by the connected sum of $S^1 \times S^{n-1}$ with any manifold that is not a rational homology sphere but has Euler characteristic 2.

3. THE INDEX CALCULATION

Let M be a smooth Riemannian manifold with periodic end modeled on the infinite cyclic cover \tilde{X} , and assume that $H_*(M; \mathbb{C})$ is finite dimensional. Let $\tau : \tilde{X} \rightarrow \tilde{X}$ be a covering translation, and denote by $A_k(t)$ the characteristic polynomial of $\tau_* : H_k(\tilde{X}; \mathbb{C}) \rightarrow H_k(\tilde{X}; \mathbb{C})$ (the polynomial $A_1(t)$ is traditionally referred to as the Alexander polynomial of the fundamental group of X). Denote by Δ the set of all δ of the form $\delta = \ln |\lambda|$, where λ is a root of the product polynomial $A_0(t) \cdot \dots \cdot A_{n-1}(t)$.

According to Theorem 1.1, the complex $\Omega_\delta^*(M)$ is Fredholm for all δ away from Δ . Its index $\text{ind}_\delta(M)$ is a piecewise constant function away from Δ , where it may jump. We wish to calculate the size of these jumps.

3.1. Excision principle. Let δ_1 and δ_2 be two weights in $\mathbb{R} - \Delta$ and complete the de Rham complex of \tilde{X} in the L^2 norm with respect to the measure $e^{\delta_1 f(x)} dx$ on the negative end of \tilde{X} , and $e^{\delta_2 f(x)} dx$ on the positive end. This

complex will be denoted by $\Omega_{\delta_1\delta_2}^*(\tilde{X})$. This is a Fredholm complex whose index will be denoted by $\text{ind}_{\delta_1\delta_2}(\tilde{X})$.

Proposition 3.1. $\text{ind}_{\delta_2}(M) - \text{ind}_{\delta_1}(M) = \text{ind}_{\delta_1\delta_2}(\tilde{X})$.

Proof. Let $c \in \mathbb{R}$ be a regular value of $f : \tilde{X} \rightarrow \mathbb{R}$ then $Y = f^{-1}(c)$ is a submanifold of \tilde{X} separating it as $\tilde{X} = \tilde{X}_- \cup \tilde{X}_+$. Write $M = Z \cup \tilde{X}_+$ for some smooth compact manifold Z with boundary Y . An application of the excision principle to these two splittings yields

$$\text{ind}_{\delta_1}(M) + \text{ind}_{\delta_1\delta_2}(\tilde{X}) = \text{ind}_{\delta_2}(M) + \text{ind}_{\delta_1\delta_1}(\tilde{X}).$$

Note that the complex $\Omega_{\delta_1\delta_1}^*(\tilde{X})$ is exact because the complexes (1) obtained from it by the Fourier–Laplace transform are exact for all z with $|z| = e^{\delta_1}$. Therefore, $\text{ind}_{\delta_1\delta_1}(\tilde{X}) = 0$ and the proof is complete. \square

3.2. Computing cohomology of $\Omega_{\delta_1\delta_2}^*(\tilde{X})$. We will proceed by several reductions, the first being from weighted forms to weighted cellular cochains. To be precise, fix a finite cell complex structure on X and lift it to \tilde{X} . Also, introduce the Hilbert space $\ell_{\delta_1\delta_2}^2$ of the sequences $\{x_k \mid k \in \mathbb{Z}\}$ of complex numbers such that

$$\sum_{k < 0} e^{2\delta_1 k} |x_k|^2 < \infty \quad \text{and} \quad \sum_{k > 0} e^{2\delta_2 k} |x_k|^2 < \infty.$$

Theorem 2.17 of Miller [8], which is a weighted version of the L^2 de Rham theorem, see [4, 7], establishes an isomorphism between the cohomology of $\Omega_{\delta_1\delta_2}^*(\tilde{X})$ and the cellular cohomology of \tilde{X} with $\ell_{\delta_1\delta_2}^2$ coefficients. Miller actually uses weighted *simplicial* cohomology, but this is readily seen to be isomorphic to the more standard and convenient cellular version.

Proposition 3.2. *View $\ell_{\delta_1\delta_2}^2$ as a $\mathbb{C}[t, t^{-1}]$ module with t acting as the right shift operator, $t(x_k) = x_{k+1}$. Then for all but finitely many δ_1 and δ_2 , the cohomology of \tilde{X} with $\ell_{\delta_1\delta_2}^2$ coefficients equals the homology of the complex*

$$\text{Hom}_{\mathbb{C}[t, t^{-1}]}(C_*(\tilde{X}, \mathbb{C}), \ell_{\delta_1\delta_2}^2). \quad (5)$$

Proof. This will follow as soon as we show that the images of the boundary operators ∂ in complex (5) are closed. These boundary operators

$$\partial : (\ell_{\delta_1\delta_2}^2)^k \rightarrow (\ell_{\delta_1\delta_2}^2)^\ell$$

are matrices whose entries are Laurent polynomials in t . Since $\mathbb{C}[t, t^{-1}]$ is a principal ideal domain, each ∂ will have a diagonal matrix in properly chosen bases. The statement now follows from the fact that the operator $t - \lambda : \ell_{\delta_1\delta_2}^2 \rightarrow \ell_{\delta_1\delta_2}^2$ is Fredholm for all λ with $|\lambda|$ different from e^{δ_1} and e^{δ_2} ; see for instance Conway [3, Proposition 27.7 (c)]. \square

The universal coefficient theorem now tells us that the $\ell_{\delta_1\delta_2}^2$ cohomology of \tilde{X} is isomorphic to

$$\mathrm{Hom}_{\mathbb{C}[t, t^{-1}]} (H_*(\tilde{X}; \mathbb{C}), \ell_{\delta_1\delta_2}^2) \oplus \mathrm{Ext}_{\mathbb{C}[t, t^{-1}]} (H_*(\tilde{X}; \mathbb{C}), \ell_{\delta_1\delta_2}^2). \quad (6)$$

Recall from the proof of Theorem 1.1 that $H_*(\tilde{X}; \mathbb{C})$ is a torsion module (3) whose order ideal is spanned by the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$. Therefore, our next step will be to compute (6), one cyclic module at a time.

Lemma 3.3. *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2} . Then, for any cyclic module $V = \mathbb{C}[t, t^{-1}]/(t - \lambda)^m$, we have*

$$\mathrm{Ext}_{\mathbb{C}[t, t^{-1}]} (V, \ell_{\delta_1\delta_2}^2) = 0.$$

Proof. We already know from the proof of Proposition 3.2 that the operator $t - \lambda : \ell_{\delta_1\delta_2}^2 \rightarrow \ell_{\delta_1\delta_2}^2$ is Fredholm. In addition, one can easily check that all finite sequences belong to its image. Since such sequences are dense in $\ell_{\delta_1\delta_2}^2$, the operator $t - \lambda$ is surjective, and so are the operators $(t - \lambda)^m$ for all m . The result is now immediate from the definition of Ext . \square

Lemma 3.4. *Let λ be a complex number such that $|\lambda|$ is different from e^{δ_1} and e^{δ_2} . Assume that $\delta_2 < \delta_1$. Then, for any cyclic module $V = \mathbb{C}[t, t^{-1}]/(t - \lambda)^m$, the dimension of $\mathrm{Hom}_{\mathbb{C}[t, t^{-1}]} (V, \ell_{\delta_1\delta_2}^2)$ is m if $e^{\delta_2} < |\lambda| < e^{\delta_1}$, and zero otherwise.*

Proof. For such a module, $\text{Hom}_{\mathbb{C}[t,t^{-1}]}(V, \ell_{\delta_1\delta_2}^2)$ equals the kernel of the operator $(t - \lambda)^m : \ell_{\delta_1\delta_2}^2 \rightarrow \ell_{\delta_1\delta_2}^2$. Computing this kernel is a straightforward exercise with infinite series. \square

3.3. Proof of Theorem 1.2. Let λ be a root of the product polynomial $A_0(t) \cdots A_{n-1}(t)$ of multiplicity $m = m_0 + \dots + m_{n-1}$, where m_k is the multiplicity of λ as a root of $A_k(t)$. Choose generic δ_1 and δ_2 so that $e^{\delta_2} < |\lambda| < e^{\delta_1}$ and there are no other roots of $A_0(t) \cdots A_{n-1}(t)$ whose absolute values fit in this interval. It follows from Proposition 3.1 and the cohomology calculation in the previous section that

$$\text{ind}_{\delta_1}(M) = \text{ind}_{\delta_2}(M) - \sum (-1)^k m_k,$$

which is exactly the formula claimed in Theorem 1.2.

4. DISCUSSION AND EXAMPLES

Let M be a smooth Riemannian manifold of dimension n with a periodic end modeled on \tilde{X} and suppose that $H^*(\tilde{X}; \mathbb{C})$ is finite dimensional. Then, for any $\delta \in \mathbb{R} - \Delta$, the de Rham complex $\Omega_\delta^*(M)$ is Fredholm and its index is given by the formula

$$\text{ind}_\delta(M) = (-1)^n \chi(M) + \sum (-1)^k \#\{\lambda \mid A_k(\lambda) = 0, |\lambda| > e^\delta\},$$

where the roots λ of $A_k(t)$ are counted with their multiplicities. This formula is obtained by combining Theorem 1.2 with Miller's theorem [8] that $\text{ind}_\delta(M) = (-1)^n \chi(M)$ for sufficiently large $\delta > 0$. Miller [8] also shows that the function $\text{ind}_\delta(M)$ is even or odd depending on whether n is even or odd. This is consistent with the above formula because of Blanchfield duality which says that $A_k(\lambda) = 0$ if and only if $A_{n-k-1}(1/\lambda) = 0$ with matching multiplicities.

Example 4.1. A manifold with product end is a smooth Riemannian manifold whose end is modeled on $\tilde{X} = \mathbb{R} \times Y$, where Y is a closed Riemannian manifold. The metric on $\mathbb{R} \times Y$ is presumed to be the product metric.

The index theory on such manifolds has been studied by Atiyah, Patodi and Singer [1]. The covering translation induces an identity map τ_* on the homology of $\mathbb{R} \times Y$. Since $\lambda = 1$ is the only root of the characteristic polynomial of τ_* the complex $\Omega_\delta^*(M)$ is Fredholm for all $\delta \neq 0$. Its index $\text{ind}_\delta(M)$ equals $\chi(M)$ if the dimension of M is even, and $\text{sign}(-\delta) \cdot \chi(M)$ if the dimension of M is odd. Note that the same is true for any manifold whose periodic end is modeled on \tilde{X} such that the characteristic polynomial of $\tau_* : H_*(\tilde{X}; \mathbb{C}) \rightarrow H_*(\tilde{X}; \mathbb{C})$ only has unitary roots.

Example 4.2. This example originates in Fox's 'Quick Trip' [6, Example 11]. Fox constructs a 2-knot in the 4-sphere with with the property that the infinite cyclic cover of its exterior has first homology isomorphic to the additive group of dyadic rationals. A nice plumbing construction of this knot described in Rolfsen's book [13, Section 7.F] shows that it has a Seifert surface diffeomorphic to $S^1 \times S^2 - D^3$. Perform a surgery on the knot so that the Seifert surface is capped off by the core 3-disk of the surgery. The resulting manifold X has integral homology of $S^1 \times S^3$. It follows from a calculation in [13, Section 7.F] that the characteristic polynomials $A_k(t)$ of the covering translation $\tau_* : H_k(\tilde{X}; \mathbb{C}) \rightarrow H_k(\tilde{X}; \mathbb{C})$ are as follows: $A_0(t) = t - 1$, $A_1(t) = t - 2$, $A_2(t) = t - 1/2$, and $A_3(t) = t - 1$. Cut \tilde{X} along a copy of $S^1 \times S^2$ and fill it in by $D^2 \times S^2$ to obtain an end-periodic manifold M . A straightforward calculation shows that $\chi(M) = 2$. The complex $\Omega_\delta^*(M)$ is Fredholm away from $\delta = 0$ and $\delta = \pm \ln 2$. Its index is equal to 1 if $0 < |\delta| < \ln 2$, and is equal to 2 otherwise.

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