UNIFORM-INLET THREE-DIMENSIONAL TRANSONIC
BELTRAMI FLOW THROUGH A DUCTED FAN

by

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I. INTRODUCTION

This report is a continuation of a line of analytical treatment of three dimensional flows in compressor or ducted fans. In the earlier theories [1,2,9,10], the blade row is modelled as a set of spinning lifting lines and the induced velocities are treated as acoustic perturbations. While these studies have been useful in advancing our knowledge of the three dimensional nature of the flow, it has been difficult until now to correlate the theory with experiments. The reason, of course, is that the strong overall swirl induced by such blade rows is by no means a small perturbation in practical applications.

It is realized that the high pressure stage of a compressor usually has a large number of blades (~40 to 100). Therefore although the collective effect of all the baldes can be a large disturbance, each blade contributes only a small disturbance. Thus a linearized theory is still possible. To extend the previous acoustic theory to a rotor with large turning we may still represent the blades as super-positions of source and lifting lines, but we have to calculate the exit flow by linearizing about a non-zero (and large) swirl velocity profile. This is the approach taken by the theory proposed by McCune and Hawthorne [3]. They calculated the velocities induced by the trailing vorticity of a nonuniformly loaded rectilinear cascade for incompressible flow. This work was later generalized to the compressible case by Morton [4]. Cheng [5] treated incompressible flow in an annular geometry. In that analysis the blades are represented as lifting lines of nearly constant circulation, and the exit flow is therefore, to

lowest order, of "free vortex" type. Linearizing about the free vortex
flow, the velocity induced by the trailing vortex sheets due to non-
uniform blade bonding are calculated in [5] to order $\delta\Gamma$. It is the
purpose of this report to treat the general compressible case, including
the results of Ref. [5] as the incompressible limit. The result of this
analysis also serves as the Green's function for constructing a lifting
surface theory for transonic rotors with practical loading.

Before going into the details of the theory, let us examine some
simple pictures of its findings. The nonuniformly loaded blades shed
off the excess circulation as wakes. The induced velocity of the wakes
is found to cause a "downwash" at the blade which has the effect of
partially nullifying the nonuniformity in loading. This can easily be
understood by a consideration of the wake system of a blade. Fig. 1
shows a blade with the tip region more heavily loaded than the inboard
stations. We can see that the wake induces a tangential velocity com-
ponent which lowers the angle of attack at the high work (tip) region
and increases that of the low work (hub) region.

Another major development is the mode matching of the upstream
and downstream flow. The presence of the strong swirl makes the
acoustic mode shapes downstream drastically different from those up-
stream. For example, we can have upstream hyperbolic modes in a tran-
sonic rotor while all the downstream modes are elliptic because the
relative velocity is subsonic there. Simple modewise matching is no
longer possible. In the present work, a method of mixed-mode matching
is developed. A result is that a pure tone downstream (upstream) can
excite a whole spectrum of tones upstream (downstream). In particular,

any source downstream can excite the acoustic radiations upstream of a transonic rotor.

II. THE FORMULATION OF THE PROBLEM

For the sake of brevity in the presentation we make the following simplifications.[*] We assume:

(i)   Inviscid, isentropic flow.

(ii)   Isolated rotor in an infinitely long annular duct.

(iii)   Steady flow in the rotor frame.

(iv)   The flow for upstream of the rotor is uniform axial flow with no swirl.

(v)   The number of blades of the rotor is large so that while the overall induced swirl is large, the loading per blade remains small.

(vi)   The axisymmetric part of the velocity downstream has, to a first approximation, a free vortex profile.

(vii)   The three dimensional nature of the flow is due to the presence of discreet vortex sheets trailing from the nonuniformly loaded blades.

The equations of motion for a steady flow in the rotor frame is

$$q' \cdot \nabla q' - \nu^2 \underline{r} + 2\underline{\nu}x q' = - \frac{\nabla p}{\rho} \qquad (2.1)$$

where primes are used to denote flow propertities measured in the relative frame. Thus

$$q' = q - \nu r \hat{\underline{\theta}} \qquad (2.2)$$

_____

[*] The assumptions (i)-(vii) are not mutually independent. Moreover, most of the implied restrictions can be relaxed as the present theory is further refined.

where ν is the angular frequency of the rotor. In the following discussion all quantities are nondimensionalized with respect to a length scale given by the blade tip radius r_T, a time scale of $r_T/W_{-\infty}$, and a mass scale of $\rho_{-\infty} r_T^3$ where $W_{-\infty}$ and $\rho_{-\infty}$ are the inlet axial velocity and density.

The continuity equation is

$$\rho \nabla \cdot q' + q' \cdot \nabla \rho = 0 \qquad (2.3)$$

Taking the dot product of (2.1) with q' and using the isentropic condition

$$dp = a^2 d\rho \qquad (2.4)$$

we obtain

$$a^2 \nabla \cdot q' = 1/2\ q' \cdot \nabla q'^2 - \nu^2 q' \cdot r \qquad (2.5)$$

The model we adopt consists of a set of lifting lines representing the blades. Due to spanwise loading variation there will be a trailing vortex sheet originating from each blade so that the Kutta condition can be satisfied. These vortex sheets are assumed to remain thin[*] and will be shown to first order in the blade loading variation to lie on surfaces governed by the θ-averaged (pitchwise-averaged) flow.

With the assumptions stated at the beginning of this section we have Beltrami flow[3],[5]

$$\underline{\omega} \times q' = 0 \qquad (2.6)$$

[*] In a recent report Kerrebrock [6] noted that if the background flow is not a free vortex (in particular he considered a wheel flow) the wakes will be spread by shear waves even if we assume inviscid flow. Here we only look at flow produced by a rotor of nearly constant spanwise blade loading so that the background flow is very nearly a free vortex and the mechanism for shear wave is absent.

where

$$\underline{\omega} = \nabla \times \underline{q} \qquad (2.7)$$

is the vorticity measured in the absolute frame. To obtain (2.6) we

write Crocco's equation in the rotating frame

$$\frac{\partial \underline{q}'}{\partial t} - T\nabla S + \nabla I - \underline{q}' \times \underline{\omega} = 0 \qquad (2.8)$$

where

$$I \equiv h + 1/2 \, q'^2 - 1/2 (\nu r)^2 \qquad (2.9)$$

The rothalpy I in the rotor frame is the analogue of the total enthalpy

$h°$ in the stationary (absolute) frame. The first two terms in (2.8)

are zero for the assumed steady relative isentropic flow. From (2.9)

the equation of motion for I is

$$\frac{DI}{Dt} = \frac{Dh}{Dt} + \frac{D}{Dt} (1/2 \, q'^2) - \frac{D}{Dt} (1/2 (\nu r)^2)$$

Some manipulation yields

$$\frac{DI}{Dt} = T \frac{DS}{Dt} + \frac{1}{\rho} \frac{\partial p}{\partial t}$$

For isentropic steady relative flow with uniform inlet conditions this

leads to $\nabla I = 0$ and (2.8) reduces to (2.6).

Let the vortex sheets lie on surfaces of

$$\alpha = \alpha(r, \theta, x)$$

From (2.6) $\underline{\omega}$ and $\underline{q}'$ are parallel and the vortex sheets can be re-

presented as

$$\underline{\omega} = \nabla \times \underline{q} = f(x,r) \sum_{i=1}^{B} \delta(\alpha - \alpha_i) \hat{\underline{q}}' \qquad (2.10)$$

$$\alpha_i = 0, \ 2\pi/B, \ 2(2\pi/B), \ 3(2\pi/B), \ldots$$

where B is the number of blades and $\char94$ denotes unit vector. f represents the strength of the vortex sheet, still to be determined.

Equations (2.5) and (2.10) form the basic set of equations for the present analysis. (The functions f and α will be furnished from the averaged flow later in this section.) Essentially (2.5) specifies the divergence of the velocity field and (2.10) specifies the curl. Together with appropriate boundary and upstream-downstream matching conditions, they completely define the velocities.

We shall solve for the velocities by linearizing about a base flow q_o which is a uniform axial flow upstream and a free vortex flow downstream:

$$q = q_o + q_T \qquad (2.11)$$

$$q_o = (u_o, v_o, w_o)$$

$$u_o = 0 \quad \text{both upstream and downstream}$$

$$w_o = C_1, C_2 \quad \text{(constants) upstream and downstream} \qquad (2.12)$$

$$v_o = 0 \quad \text{for upstream} \quad \text{and } \sim 1/r \text{ for downstream}$$

Let $C_\Gamma \Gamma(r)$ be the (nondimensionalized) circulation on each blade with $\Gamma(r)$ representing the nonuniform loading profile.* Then

$$v_o = \frac{BC_\Gamma \Gamma}{2\pi v} \qquad x < 0 \qquad (2.13)$$

We assume that $\Gamma(r)$ is a function of spanwise mean 1 and

$$\frac{1}{B\Gamma} \frac{d\Gamma}{dr} = O(\varepsilon) \qquad \varepsilon \ll 1$$

q_T and $\underline{\omega}$ are thus $O(\varepsilon)$. A study of the flow for stronger loading variations is underway, and will be reported subsequently.

Before we can solve (2.5) and (2.10) it is necessary to find f

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* As pointed out by many authors, $d\Gamma/dr = 0$ at $r = h,1$ to be consistent with the boundary condition $u(h) = u(1) = 0$.

and α. We shall derive expressions for them first by a heuristic argument and then in a more formal way.

Fig. 2a shows a developed stream surface. Since the vortex sheets are going along the stream lines, $\alpha =$ cst is a streamline on this surface. It is easy to see that to the lowest order of approximation

$$\alpha = - \theta - \frac{x}{r} \tan \beta'_2$$

$$= - \theta - \frac{x}{r} \frac{(\nu r - v_o)}{w_o} = - \theta + \frac{x}{r} \frac{v'_o}{w_o} \qquad (2.14)$$

in the downstream flow. It will be shown that f in (2.10) is $O(\epsilon)$. Therefore in the present work we only need the lowest order approximation of α to be used in (2.10).

Let each blade carry a circulation of $C_T\Gamma(r)$ with $C_T\Gamma > 0$ in the direction as shown in Fig. 3a. For isentropic flow Kelvin's theorem applies. Consider the circulation circuit as shown in Fig. 3b. The radial velocity at the trailing edge is

$$2|u| = C \frac{(\Gamma_2 - \Gamma_1)}{\delta r}$$

or

$$u = \mp \frac{1}{2} C_P \frac{d\Gamma}{dr} \qquad (2.15)$$

where the upper (lower) sign refers to the pressure (suction) side of the blade.

Figure 3c shows the form of the wakes on a developed stream surface downstream of the blade. (To O(1) the stream surface remains a cylinder.) The radial velocity jump across the vortex sheet is given by

$$2|u| = \lim_{\epsilon \to 0} \int_{\alpha = -\epsilon}^{\epsilon} (\omega \cos \beta'_2) \, r d\alpha$$

Thus

$$|\underline{\omega}| = \frac{2|u|}{r\cos\beta_2^{\,\prime}}\ \delta(\alpha)$$

This can be generalized to include all the vortex sheets and after taking care of proper signs and also using (2.15) we obtain

$$\underline{\omega} = C_\Gamma \frac{d\Gamma}{dr}\ \frac{\hat{q}_o^{\,\prime}}{r\cos\beta_2^{\,\prime}}\ \sum_{i=1}^{B}\delta(\alpha-\alpha_i)$$

where $\alpha_i = 0,\ 2\pi/B,\ 4\pi/B,\ldots,(B-1)2\pi/B$.

From the velocity triangle

$$\cos\beta_2^{\,\prime} = w_o/q_o^{\,\prime}$$

Thus

$$\underline{\omega} = \frac{1}{r}\ C_\Gamma \frac{\partial}{\partial r}\ \frac{q_o^{\,\prime}}{w_o}\ \sum_{i=1}^{B}\delta(\alpha-\alpha_i) \qquad (2.16)$$

What follows is a formal derivation of (2.16). Let

$$\begin{array}{c}\underline{Q}\\ \underline{Q}^{\,\prime}\\ R\\ \underline{\Omega}\end{array} = \frac{1}{2\pi}\int_0^{2\pi}\begin{array}{c}\underline{a}\\ \underline{q}^{\,\prime}\\ \rho\\ \underline{\omega}\end{array}\,d\theta$$

The nonaxisymetric part of the flow quantities are all $0(\varepsilon)$. From (2.6) we have

$$\underline{\omega} = \lambda\underline{q}^{\,\prime}$$

Since $\underline{\omega} = 0(\varepsilon)$ we can write

$$\underline{\omega} = \lambda\underline{Q}^{\,\prime} + 0(\varepsilon^2) \qquad (2.17)$$

From the identity

$$\nabla\cdot\underline{\omega} = 0 \simeq \nabla\cdot(\frac{\lambda}{R}\,R\underline{Q}^{\,\prime}) = \nabla\,(\frac{\lambda}{R})\,\cdot\,R\underline{Q}^{\,\prime}$$

where continuity is used in the last quality, it can be concluded therefore, that there must be a functional relationship between λ/R and $R\underline{Q}^{\,\prime}$. Furthermore, continuity

$$\nabla\,\cdot\,(R\underline{Q}^{\,\prime}) = 0$$

also enables us to write $R\underline{Q}'$ as the cross product of two gradients

$$RQ' = \nabla\alpha \times \nabla\psi \qquad (2.18)$$

where α and ψ are yet to be defined. Thus λ/R will be a function of α and ψ. We pick ψ to be the Stokes stream function of $\underline{Q}$.

$$U = -\frac{1}{Rr}\frac{\partial\psi}{\partial x} \quad , \quad W = \frac{1}{Rr}\frac{\partial\psi}{\partial r} \qquad (2.19)$$

Without loss of generality we define

$$\alpha(x,r,\theta) = -\theta - F(x,r) \qquad (2.20)$$

Since the trailing vortex sheets remain thin we can write (2.17) as

$$\underline{\omega} = \nabla\times\underline{q} = Rf(x,r)\sum_{i=1}^{B}\delta(\alpha-\alpha_i)\underline{Q}' \qquad (2.21)$$

$$\alpha_i = 0, \ 2\pi/B, \ 4\pi/B,\ldots$$

Averaging over θ we obtain

$$\underline{\Omega} = \nabla\times Q = Rf\frac{B}{2\pi}\underline{Q}' \qquad (2.22)$$

The x and r component of (2.21) are

$$\frac{\partial r v_o}{\partial r} = \frac{B}{2\pi}f\frac{\partial\psi}{\partial r} \qquad \frac{\partial r v_o}{\partial x} = \frac{B}{2\pi}f\frac{\partial\psi}{\partial x} \qquad (2.23)$$

Thus

$$f = \frac{2\pi}{B}\frac{\partial(rv_o)}{\partial\psi} \equiv C_\Gamma\frac{\partial\Gamma}{\partial\psi} \qquad (2.24)$$

To find α we go back to the defination (2.18). The θ component is

$$\frac{vr - v_o}{r} = W\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial r}$$

This can be integrated immediately by the method of characteristics

with boundary condition $F = 0$ at $X = 0$ (the rotor plane). The
solution is

$$F(x,r) = \int_o^x \frac{\nu r(x) - \upsilon_o}{w_o} \frac{x}{r} + O(\varepsilon) \qquad (2.25)$$

with $r = r(x)$ along a streamline of $\psi = $ constant. Noticing that f
given by (2.24) is already $O(\varepsilon)$ we only have to determine α or F in
(2.21) to $O(1)$. To this order the stream surfaces remain parallel to
the axis of the annulus and (2.25) becomes

$$F(x,r) = \frac{\nu r - \upsilon_o}{w_o} \frac{x}{r} + O(\varepsilon) \qquad (2.26)$$

Summarizing, we have

$$\underline{\omega} = R \ C_\Gamma \ \frac{\partial \Gamma}{\partial \psi} \sum_{i=1}^B \delta(\alpha - \alpha_i) \ \underline{Q}' \qquad (2.27)$$

$$\alpha = -\theta - \frac{\nu r - \upsilon_o}{w_o} \frac{x}{r} \qquad (2.28)$$

This is the same as our previous result if we remember

$$\frac{\partial \Gamma}{\partial \psi} = \frac{1}{rRw_o} \frac{\partial \Gamma}{\partial r} + O(\varepsilon) \qquad \qquad \underline{Q}' = \underline{q_o'} + O(\varepsilon)$$

so (2.27) and (2.28) are equivalent to (2.16) and (2.14) with a
difference of $O(\varepsilon^2)$.

III. THE MEAN FLOW

The basic equations we wish to solve are (2.5) and (2.16) which are repeated here for convenience:

$$a^2 \nabla \cdot \underline{q}' = \frac{1}{2} \underline{q}' \cdot \nabla \underline{q}'^2 - \nu^2 \underline{q}' \cdot \underline{r} \tag{3.1}$$

$$\underline{\omega} = \frac{c_\Gamma}{w_0 r} \frac{\partial \Gamma}{\partial r} \underline{q}'_0 \sum_{i=1}^{B} \delta(\alpha - \alpha_i) \tag{3.2}$$

$$\alpha_i = 0, \ 2\pi/B, \ \ldots, \ \frac{(B-1)2\pi}{B}$$

$$\alpha = -\theta - \frac{x(\nu r - v_0)}{r w_0} \tag{3.3}$$

In the following, (3.1) is conveniently linearized with respect to the free vortex profile (2.12), although

$$\underline{q} = \underline{q}_0 + \underline{q}_1 \quad \text{and} \quad \underline{q}' = \underline{q}'_0 + \underline{q}'_1$$

where $q_1 = 0(\varepsilon)$. Since the difference between the relative and absolute velocity is a zeroth-order quantity $(\nu r \hat{\underline{\theta}})$, $\underline{q}_1$ and $\underline{q}'_1$ are the same and we shall drop all primes on the perturbation quantities. Taking the θ – average of (3.2) we obtain

$$\nabla \times \bar{\underline{q}}_1 = \frac{B c_\Gamma}{2\pi w_0} \frac{\partial \Gamma}{\partial r} \frac{\underline{q}'_0}{r} \tag{3.4}$$

Writing $a = a_0 + a_1$, $a_1 = 0(\varepsilon)$, we obtain after some algebra

$$a_0^2 \ (\frac{\partial u_1}{\partial r} + \frac{u_1}{r} + \frac{1}{r} \frac{\partial v_1}{\partial \theta} + \frac{\partial w_1}{\partial x}) =$$

$$= -\frac{v_0^2}{r} u_1 + (w_0 \frac{\partial}{\partial x} + \frac{v_0'}{r} \frac{\partial}{\partial \theta}) (w_0 w_1 + v_0' v_1) \tag{3.5}$$

The θ - averaged equation is

$$\frac{\partial \bar{u}_1}{\partial r} + \frac{\bar{u}_1}{r} (1 + \frac{v_0^2}{a_0^2}) \frac{\bar{u}_1}{r} + (1 - \frac{w_0^2}{a_0^2}) \frac{\partial \bar{w}_1}{\partial x} = 0$$

or

$$\frac{\partial \bar{u}_1}{\partial r} + \frac{\bar{u}_1}{r} (1 + M_\theta^2) + (1 - M_x^2) \frac{\partial \bar{w}_1}{\partial x} = 0 \qquad (3.6)$$

The θ component of (3.4) is

$$\frac{\partial \bar{u}_1}{\partial x} - \frac{\partial \bar{w}_1}{\partial r} = \frac{BC_\Gamma}{2\pi w_0 r} \frac{\partial \Gamma}{\partial r} v_0' \qquad (3.7)$$

(3.6) and (3.7) form two equations for the two unknowns $\bar{u}_1$ and $\bar{w}_1$. Note that different $M_x(r)$ and $M_\theta(r)$ have to be used for the upstream and downstream flow and the RHS of (3.7) is zero for upstream. The difference of the base flow for upstream and downstream means that mode-by-mode matching across the rotor plane is no longer possible. We can only match the sum of all the modes. The matching problem will be dealt with in a later section.

To solve the equations (downstream) let

$$\bar{q}_1 = \nabla \Phi + \underline{E} \qquad (3.8)$$

so that $\bar{u}_1 = \frac{\partial \Phi}{\partial r} + E_r$, $\bar{w}_1 = \frac{\partial \Phi}{\partial x} + E_x$ Pick $E_\theta = E_r = 0$ and

$$E_x = - \frac{BC_\Gamma}{2\pi w_0} \int_h^r dr \frac{1}{r} \frac{\partial \Gamma}{\partial r} v_0' + C \qquad (3.9)$$

Then (3.7) will be identically satisfied. The constant of integration C will be determined in the matching process to conserve mass flux.

(3.6) now becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \Phi}{\partial r} + (1 - M_x^2) \frac{\partial^2 \Phi}{\partial x^2} = 0 \qquad (3.10)$$

The boundary conditions are

$$\bar{u}_1 = \frac{\partial \Phi}{\partial r} = 0 \text{ at } r = h_1, 1 \qquad (3.11)$$

and $|\nabla \Phi|$ vanishes far downstream. The matching conditions across the rotor plane are that there should be no change of radial momentum across the matching plame and axial mass flux is conserved. The details will be dealt with in section 8.

For the incompressible case (3.6) reduces to $\nabla \cdot \bar{q}_1 = 0$. Analytic solution can easily be worked out. It is more convenient in this case to use ψ where

$$\bar{u}_1 = -\frac{1}{r} \frac{\partial \psi}{\partial x} \text{ and } \bar{w}_1 = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

whence continuity is automatically satisfied. (3.7) then becomes

$$-\frac{1}{r}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r}\right) = \begin{cases} 0 & x < 0 \\ \dfrac{BC_\Gamma v_0'}{v\pi w_0 r} \dfrac{\partial \Gamma}{\partial r} & x > 0 \end{cases} \qquad (3.12)$$

The boundary conditions are

$$\frac{\partial \psi}{\partial x} = 0 \text{ at } r = h_1, 1$$

and velocities remain bounded far from the rotor. The solution of (3.12)

is

$$\psi = \begin{cases} \dfrac{1}{2} \sum_{\ell=1}^{\infty} A_\ell \, e^{\lambda_{1\ell} x} \, r \, Z_{1\ell}(r) \\[2ex] \sum_{\ell=1}^{\infty} A_\ell \left(1 - \dfrac{1}{2} e^{-\lambda_{1\ell} x}\right) r \, Z_{1\ell}(r) \end{cases} \qquad (3.13)$$

where

$$A_1 \equiv \frac{1}{\lambda_{1\ell}^2} \int_h^1 \left(\frac{BC_\Gamma v_0}{2\pi r w_0} \frac{\partial \Gamma}{\partial r}\right) Z_{1\ell}(r)$$

$$Z_{1\ell}(r) \equiv \frac{(J_1(\lambda_{1\ell} r) + b_\ell Y_1(\lambda_{1\ell} r))}{\left(\int_h^1 r(J_1(\lambda_{1\ell} r) + b_\ell Y_1(\lambda_{1\ell} r))^2 dr\right)^{1/2}}$$

The eigenvalue $\lambda_{1\ell}$ and phase-factors b_ℓ are chosen to satisfy the boundary conditions

$$Z_{1\ell}(h) = Z_{1\ell}(1) = 0$$

For further details, see Ref.9.

IV. THE θ VARYING FLOW

In section III we have effectively written

$$\underline{q}(x,r,\theta) = \underline{q}_0(r) + \overline{\underline{q}}_1(x,r) + \tilde{\underline{q}}_1(x,r,\theta) \tag{4.1}$$

where $\tilde{\underline{q}}_1$ averaged to zero over θ. We found $\overline{\underline{q}}_1$ from the θ averaged equations.[*] We shall find $\tilde{\underline{q}}_1$ here. The governing equation may be obtained from subtracting from (3.5) its θ-averaged part. We then obtain

$$(a_0^2 + v_0^2) \frac{\tilde{u}_1}{r} + a_0^2 \frac{\partial \tilde{u}_1}{\partial r} + \frac{(a_0^2 - v_0'^2)}{r} \frac{\partial \tilde{v}_1}{\partial \theta} -$$

$$- v_0' \frac{\partial \tilde{v}_1}{\partial x} + (a_0^2 - w_0^2) \frac{\partial \tilde{w}_1}{\partial x} - \frac{v_0'}{r} \frac{\partial \tilde{w}_1}{\partial \theta} = 0 \tag{4.2}$$

or

$$(1 + M_\theta^2) \frac{\tilde{u}_1}{r} + \frac{\partial \tilde{u}_1}{\partial r} + \frac{(1 - M_\theta'^2)}{r} \frac{\partial \tilde{v}_1}{\partial \theta} -$$

$$- M_x M_\theta' (\frac{\partial v_1}{\partial x} + \frac{1}{r} \frac{\partial \tilde{w}_1}{\partial \theta}) + (1 - M_x^2) \frac{\partial \tilde{w}_1}{\partial x} = 0$$

M_θ and M_θ' are the tangential Mach number of the base flow in the absolute and relative frames respectively. From (3.2)

$$\nabla \times \tilde{\underline{q}}_1 = \frac{C_\Gamma}{w_0 r} \frac{\partial \Gamma}{\partial r} \underline{q}_0' \sum_{i=1}^{B} [\delta(\alpha - \alpha_i) - \frac{1}{2\pi}] \tag{4.3}$$

Using $\underline{q}_0' = w_0 \hat{x} + v_0'\hat{\theta}$ and (3.3) we can write

$$\underline{q}_0' = w_0 r \nabla\alpha \times \hat{r} + O(\varepsilon) \tag{4.4}$$

---

[*]Recall that $\overline{\underline{q}}_1$ represents simply the extent to which the θ-averaged flow deviates from a pure free vortex type associated with $\underline{q}_0$. $\underline{q}_0 + \underline{q}_1$, therefore, is the present approximation to the mean flow corresponding to the reference flows in Refs. 3 and 5.

as can be directly verified. Then (4.3) becomes

$$\nabla \times \tilde{\underline{q}}_1 = C_\Gamma \frac{\partial \Gamma}{\partial r} \sum_{i=1}^{B} [\delta(\alpha - \alpha_i) - \frac{1}{2\pi}] \nabla\alpha \times \hat{r} + O(\epsilon^2) \qquad (4.5)$$

It is convenient to write

$$\sum_{i=1}^{B} [\delta(\alpha - \alpha_i) - \frac{1}{2\pi}] = H'(\alpha) - (\frac{B\alpha}{2\pi} - \frac{1}{2})'$$
$$\equiv S'(\alpha)$$

Where primes here denote differentiation with respect to α. H and S are the step and sawtooth functions illustrated in Figure 4a and Figure 4b. Thus

$$\nabla \times \tilde{\underline{q}}_1 = C_\Gamma \frac{\partial \Gamma}{\partial r} S'(\alpha) \nabla\alpha \times \hat{r} = C_\Gamma \nabla S \times \nabla\Gamma \qquad (4.6)$$

Let

$$\tilde{\underline{q}}_1 = \nabla\phi + \underline{A} \qquad (4.7)$$

Then

$$\nabla \times \underline{A} = C_\Gamma \nabla S \times \nabla\Gamma$$

If we chose

$$\underline{A} = C_\Gamma S \nabla\Gamma \qquad (4.8)$$

(4.6) will be satisfied identically. (The other choice

$$\underline{A} = -C_\Gamma \Gamma \nabla S$$

only differs from our choice by a curl free vector which can be absorbed in the $\nabla\phi$ part of (4.7).) Now ϕ has to satisfy (4.2). Substituting (4.7) and (4.8) into (4.2) we obtain an equation for ϕ

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \phi}{\partial r} + \frac{(1 - M_\theta'^2)}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + (1 - M_x^2) \frac{\partial^2 \phi}{\partial x^2} -$$

$$- 2 \frac{M_x M_\theta'}{r} \frac{\partial^2 \phi}{\partial x \partial \bar{\bar{\theta}}} \equiv (\frac{1 + M_\theta^2}{r} + \frac{\partial}{\partial r}) \ (-C_\Gamma S \frac{\partial \Gamma}{r}) \qquad (4.9)$$

where

$$M_\theta' = v_0'/a_0 = (v_0 - \nu r)/a_0$$

The type of partial differential to which equation (4.9) belongs depends on the coefficient matrix

$$
\begin{array}{c}
\quad \begin{array}{ccc} x & r & \theta \end{array} \\
\begin{array}{c} x \\ \\ r \\ \\ \theta \end{array}
\begin{bmatrix}
1 - M_x^2 & 0 & -M_x M_\theta'/r \\
0 & 1 & 0 \\
-M_x M_\theta'/r & 0 & (1 - M_\theta'^2)/r^2
\end{bmatrix}
\equiv
\begin{bmatrix}
a & 0 & -c \\
0 & 1 & 0 \\
-c & 0 & b
\end{bmatrix}
\end{array}
$$

The eigenvalues are

$$\lambda = 1 \text{ and } \frac{1}{2}[(a + b) \pm \sqrt{(a - b)^2 + 4c^2}]$$

The equation will be hyperbolic, parabolic or elliptic according to whether

$$(a + b) \underset{>}{\overset{<}{=}} (a - b)^2 + 4c^2$$

This relation may be simplified to the requirement

$$M_x^2 + M_\theta'^2 \quad \begin{array}{l} > \quad \text{hyperbolic} \\ = \quad 1 \quad \text{for} \quad \text{parabolic} \quad \text{equation type.} \\ < \quad \text{elliptic} \end{array}$$

The LHS is just the square of the relative mach number, in analogy with the less general result of Ref. 10. Thus we only have radiation if the

relative mach number (including finite swirl) is greater than one. For many practical transonic compressors $(M_x^2 + M_\theta'^2)$ can be less than 1 downstream and greater than 1 for part of the radius upstream.

(4.9) reduces to McCune's acoustic equation [10] if we let $v_0 \to 0$. The homogeneous equation becomes, in that limit,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}(1 - \frac{v^2 r^2}{a_0^2})\frac{\partial^2 \phi}{\partial \theta^2} + (1 - M_x^2)\frac{\partial^2 \phi}{\partial x^2} + 2M_x\frac{vr}{a_0}\frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial x} = 0$$

(θ here is $-\theta$ in [10])

V. THE WAKE FUNCTION

We just seek a particular solution to (4.9). It will be called the wake function. The sawtooth function may be expanded in Forrier series as

$$S(\alpha) = \sum_{n = 1}^{\infty} - \frac{ie^{inB\alpha}}{n\pi} \tag{5.1}$$

Let

$$\phi = \sum_{n = 1}^{\infty} i\phi_n(r) e^{inB\alpha} \tag{5.2}$$

(4.9) becomes

$$\sum_{n = 1}^{\infty} [\frac{\partial^2 \phi_n}{\partial r^2} + 2inB \frac{\partial \phi_n}{\partial r}\frac{\partial \alpha}{\partial r} - n^2 B^2 \phi_n (\frac{\partial \alpha}{\partial r})^2 + inB\phi_n \frac{\partial^2 \alpha}{\partial r^2}]$$

$$+ \frac{(1 + M_\theta^2)}{r}[\frac{\partial \phi_n}{\partial r} + inB\phi_n \frac{\partial \alpha}{\partial r}] \tag{5.3}$$

$$+ \frac{(1 - M_\theta'^2)}{r^2}\phi_n[\frac{\partial^2 \alpha}{\partial \theta^2} inB - n^2 B^2 (\frac{\partial \alpha}{\partial \theta})^2]$$

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$$+ (1 - M_x^2)\phi_n I \frac{\partial^2 \alpha}{\partial x^2} inB - n^2 B^2 (\frac{\partial \alpha}{\partial x})^2]$$

$$- 2 \frac{M_x M_\theta'}{r} \phi_n I \frac{\partial^2 \alpha}{\partial x \partial \theta} inB - n^2 B^2 (\frac{\partial \alpha}{\partial x} \frac{\partial \alpha}{\partial \theta})] e^{inB\alpha}$$

$$= (\frac{1 + M_\theta^2}{r} + \frac{\partial}{\partial r}) I - C_\Gamma \frac{\partial \Gamma}{\partial r} \cdot \sum_{n=1}^{\infty} \frac{-ie^{inB\alpha}}{n\pi}]$$

with the expression for α in (2.14) the first terms in the last three square brackets on the LHS are zero.

As we are only interested in the solution at the vicinity of the blade, it is only neccessary to get a solution for small x. However care must be taken to apply the boundary condition at $x \to \infty$. Under the assumption that the wakes follow the free vortex flow we have rolled up vortex sheets at large x. Whether the wakes still follow the primary flow after they roll up is irrelevant. We expect that the rolled up wakes at large x do not induce any significant varying velocity disturbances. Then the solution obtained by combining a particular solution to (5.3) for small x and a solution to the homogeneous part of (4.9) with boundary condition of vanishing ϕ at $x \to \infty$ (or radiation condition if the equation is hyperbolic) will be valid for small x.

Ordering $x = O(\varepsilon)$ and $\phi_n = O(\varepsilon)$ and neglecting terms of $O(\varepsilon^2)$ (5.3) becomes

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \phi_n}{\partial r} - n^2 B^2 \phi_n \frac{1 - M_\theta'^2}{r^2} + (1 - M_x^2) \frac{v_0'^2}{r^2} + 2 \frac{M_x M_\theta'}{r} \frac{v_0'}{r} = g_n(r)$$

$$(5.4a)$$

where

$$g_n(r) \equiv \frac{1}{n\pi} \left(\frac{1 + M_\theta^2}{r} + \frac{\partial}{\partial r} \right) \left(C_\Gamma \frac{\partial \Gamma}{\partial r} \right) \tag{5.5}$$

(4.13a) may be simplified as

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \phi_n}{\partial r} - n^2 B^2 \phi_n \frac{1}{r^2} + \frac{v_0'^2}{r^2} = g_n(r) \tag{5.4b}$$

The effect of compressibility only brings in the term

$$\frac{M_\theta^2}{r} \left(\frac{\partial}{\partial r} \phi_n - \frac{1}{n\pi} C_\Gamma \frac{\partial \Gamma}{\partial r} \right)$$

The incompressible limit is

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_n}{\partial r} - n^2 B^2 \phi_n \frac{1}{r^2} + \frac{v_0'^2}{r^2} = \frac{1}{n\pi} \left(\frac{1}{r} + \frac{\partial}{\partial r} \right) \left(C_\Gamma \frac{\partial \Gamma}{\partial r} \right) \tag{5.4c}$$

We can get a glimpse at how the flow is set up from the RHS (the driving term) of (5.4c). It was mentioned that [(2.15)]

$$u \sim C_\Gamma \frac{\partial \Gamma}{\partial r}$$

Thus

$$\text{RHS of (5.4c)} \sim \frac{1}{r} \frac{\partial}{\partial r} (ru)$$

which is the r derivative component of the continuity equation. What has happened is the nonuniform spanwise loading of the blade induces a radial velocity. The rest of the flow field then tries to readjust so that continuity can be satisfied.

We shall first give an analytic solution for (5.4) in the limit of $v_0 \to 0$ and then describe a very efficient numerical algorithm for the general case.

For $v_0 \to 0$, (5.4b) reduces to

$$\frac{\partial^2 \phi_n}{\partial \alpha^2} + \frac{1}{r}\frac{\partial \phi_n}{\partial r} - n^2 B^2 (v^2 + \frac{1}{r^2}) \phi_n = g_n(r) \qquad (5.6)$$

with $g_n(r)$ now given by

$$g_n(r) = \frac{1}{n\pi} (\frac{1}{r} + \frac{\partial}{\partial r})(C_\Gamma \frac{\partial \Gamma}{\partial r}) \qquad (5.7)$$

The boundary condition is $\dfrac{\partial \phi_n}{\partial r} = 0$ at $r = h, 1$. The solution is given in terms of the modified Beasel functions.

$$\phi_n(r) = I(r) \int_r^1 \xi d\xi\, g_n(\xi)\, K(\xi) + K(r) \int_h^r \xi d\xi\, g_n(\xi)\, I(\xi)$$

$$+ \frac{K'(h)K'(1)S_I - I'(h)K'(1)S_K}{Z} I(r) + \frac{I'(h)I'(1)S_K - I'(h)K'(1)S_I}{Z} K(r)$$

where

$$S_I = \int_h^1 \xi d\xi\, g_n(\xi)I(\xi) \qquad S_K = \int_L^1 \xi d\xi\, g_n(\xi)K(\xi)$$

$$Z = I'(h)K'(1) - K'(h)I'(1)$$

The I and K are the modified Beasel functions of first and second kind

$$I(r) = I_{nB} (nBvr) \qquad\qquad K(r) = K_{nB} (nBvr)$$

$$I'(r) = \frac{\partial}{\partial r} I_{nB} (nBvr) \qquad\qquad K'(r) = \frac{\partial}{\partial r} K_{nB} (nBvr)$$

This is identical to the wake function used by McCune and Okurounmu [11] since $v_0 \to 0$ reduces (4.9) to the acoustic equation.

A very efficient numerical method to solve (5.4) for the general case is the Alternate Gradient Implicit scheme. (See e.g. Richtmeyer

and Morton [12].) Let us write (5.4b) as (dropping the subscript n's)

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1 + M_\theta^2}{r} \frac{\partial \phi}{\partial r} - f\phi - g = 0$$

where

$$f \equiv n^2 B^2 (\frac{1}{r^2} + \frac{v_0'^2}{r^2})$$

Let k denote the iteration index and j denote the r station number.
Then to second order in δr

$$\frac{\phi_j^{k+1} - \phi_j^k}{\delta t} = \frac{(\phi_{j+1}^{k+1} + \phi_{j-1}^{k+1} - 2\phi_j^{k+1})}{\delta r^2} +$$

$$+ \frac{(1 + M_{\theta j}^2)}{r_j} \frac{(\phi_{j+1}^{k+1} - \phi_{j-1}^{k+1})}{2\delta r} - f_j \phi_j^{k+1} - g_j$$

δt is a tuning parameter normally set to 1. Rearranging we have

$$- A_j \phi_{j+1}^{k+1} + B_j \phi_j^{k+1} - C_j \phi_j^{k+1} = D_j$$

where

$$A_j = (1 + \frac{1 + M_{\theta j}^2}{r_j} \frac{\delta r}{2}) \, \delta t$$

$$B_j = \delta r^2 + (2 + f_j \delta r^2) \, \delta t$$

$$C_j = (1 - \frac{(1 + M_{\theta j}^2)}{2r_j} \delta r) \, \delta t$$

$$D_j = (\phi_j^k - g_j \delta t) \, \delta r^2$$

Then we can solve for ϕ_j^{k+1} in two sweeps as

$$\phi_j^{k+1} = E_j \phi_{j+1}^{k+1} + F_j \quad \text{(backward sweep)}$$

where

$$E_j = \frac{A_j}{B_j - C_j E_{j-1}} \qquad F_j = \frac{D_j + C_j F_{j-1}}{B_j - C_j E_{j-1}} \quad \text{(forward sweep)}$$

To start the sweep the boundary condition $\frac{\partial \phi}{\partial r} = 0$ at $r = h$ and 1 has to be applied. A second order accurate finite difference approximation of the boundary condition is

$$E_1 = \frac{1}{1 + f_1 \delta r^2/2} \qquad F_1 = - E_1 g_1 \frac{\delta r^2}{2} \qquad \begin{array}{l}\text{starter} \\ \text{forward sweep}\end{array}$$

$$\phi_n = \frac{F_{n-1}(1 + f_{n-1}\delta r^2/2) + g_{n-1}\delta r^2/2}{1 - E_{n-1}(1 + f_{n-1}\delta r^2/2)} \qquad \begin{array}{l}\text{starter} \\ \text{backward sweep}\end{array}$$

The scheme is unconditionally stable for $f > 0$ and typically converges in 3 to 4 iterations.

VI. THE ACOUSTIC MODES

Let us now investigate the homogeneous part of (4.9), which is repeated here

$$\frac{\partial^2 \phi}{\partial r^2} + (1 + M_\theta^2)\frac{1}{r}\frac{\partial \phi}{\partial r} + (1 - M_\theta'^2)\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} +$$

$$+ (1 - M_x^2)\frac{\partial^2 \phi}{\partial x^2} - 2M_x M_\theta' \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta \partial x} = 0 \tag{6.1}$$

We have to remember that the M_x, M_θ and M_θ' are different for upstream and downstream.[*] In particular $M_\theta = 0$ upstream in the present study. Often relative Mach number $M_x^2 + M_\theta'^2$ is less than 1 downstream but may well be greater than 1 over part of the radius upstream. Let

$$\phi = \sum_n \phi_n \, e^{inB\theta + ik_n x} \qquad (6.2)$$

real part implied. Then for each ϕ_n

$$\frac{\partial^2 \phi}{\partial r^2} + (1 + M_\theta^2)\frac{1}{r}\frac{\partial \phi}{\partial r} + \phi \left[-(1 - M_x^2)k^2 + 2\frac{M_\theta' M_x}{r}nBk - \frac{n^2 B^2(1 - M_\theta'^2)}{r^2} \right] = 0 \qquad (6.3)$$

where the subscript n has been dropped temporarily for simplicity. The boundary conditions are

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{at} \quad r = h, \, 1. \qquad (6.4)$$

Defining

$$\lambda \equiv \left[k - \frac{M_\theta' M_x nB}{r(1 - M_x^2)} \right] \qquad (6.5)$$

(6.3) becomes

$$\frac{\partial^2 \phi}{\partial r^2} + (1 + M_\theta^2)\frac{1}{r}\frac{\partial \phi}{\partial r} + \phi \left[- \lambda^2(1 - M_x^2) + \frac{n^2 B^2}{r^2}\left[\frac{M_\theta'^2 + M_x^2 - 1}{1 - M_x^2}\right] \right] = 0 \qquad (6.6)$$

(6.6) may be put into self-adjoint form by introducing

$$p = \exp \left(\int_h^r dr(1 + M_\theta^2)/r \right) \qquad (6.7)$$

---

[*] M_x is always assumed to be less than unity in the present work.

Then

$$(p\phi')' + \phi \quad p[- \lambda^2(1 - M_x^2) - \frac{n^2B^2}{r^2} (\frac{1 - M_\theta'^2 - M_x^2}{1 - M_x^2})] = 0 \tag{6.8}$$

Here primes denote differentiation with respect to r except for M_θ' which as before denotes the relative tangential Mach number. In order for expansion (6.2) to be valid we must have an infinite number of eigenvalues k_n and corresponding eigenfunctions ϕ_n. By the Oscillation Theorem [13] this is only possible if the expression in the curly braces is greater than zero and goes to infinity for some value of k (which may be infinite).

Let ϕ_1 and ϕ_2 be two solutions with eigenvalues k_1 and k_2 (with the corresponding λ_1 and λ_2).

$$(p\phi_1')' + p\phi_1 \quad - (1 - M_x^2)\lambda_1^2 - \frac{n^2B^2}{r^2} (\frac{1 - M_\theta'^2 - M_x^2}{1 - M_x^2}) = 0 \tag{6.9a}$$

$$(p\phi_2')' + p\phi_2 \quad - (1 - M_x^2)\lambda_2^2 - \frac{n^2B^2}{r^2} (\frac{1 - M_\theta'^2 - M_x^2}{1 - M_x^2}) = 0 \tag{6.9b}$$

Subtracting ϕ_2 (6.9a) from ϕ_1 (6.9b) and integrating the result from r = h to 1 and making use of the boundary condition (6.4) we obtain

$$\int_h^1 p\phi_1\phi_2 (1 - M_x^2) (\lambda_2 - \lambda_1) (\lambda_2 + \lambda_1) dr = 0 \tag{6.10a}$$

or, using (6.5)

$$\int_h^1 p\phi_1\phi_2 (1 - M_x^2) (k_2 - k_1) (k_2 + k_1 - \frac{2M_\theta'M_x nB}{r(1 - M_x^2)}) dr = 0 \tag{6.10b}$$

All the coefficients in the differential equations (6.3) are real, therefore, if k is an eigenvalue and ϕ the corresponding eigenfunction

so are k^* and ϕ^*. Let $\phi_1 = \phi$, $\phi_2 = \phi^*$, $k_1 = k$, $k_2 = k^*$. Then

$$\int_h^1 p|\phi|^2 \ (1 - M_x^2) \ (2k_i) \ (2k_r - \frac{2M_\theta' M_x nB}{r(1 - M_x^2)}) \ dr = 0 \qquad (6.11)$$

The Incompressible Case

In the limit of all Mach numbers $\to 0$ (6.6) and (6.11) reduce to

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \phi \ [-k^2 - \frac{n^2 B^2}{r^2}] = 0 \qquad (6.12)$$

$$\int_h^1 p|\phi|^2 \ k_i \ k_r \ dr = 0 \qquad (6.13)$$

For solution to exist it is obvious that k must be purely imaginary,

$$k = \mp i\alpha$$

where the upper (lower) sign refers to the upstream (downstream) flow.

This represents an elliptic field vanishing at $|x| \to \infty$.

The solution can be written down explicitly in this case:

$$\phi = \begin{cases} \sum_{np}\sum A_{np}^u \phi_{np} \ e^{inB\theta} \ e^{\gamma_{np}x} & x < 0 \\ \\ \sum_{np}\sum A_{np}^d \phi_{np} \ e^{inB\theta} \ e^{-\gamma_{np}x} & x > 0 \end{cases}$$

$$\phi_{np} = (J_{nB}(\gamma_{np}r) + b_{np} \ Y_{nB}(\gamma_{np}r))/\sqrt{N_{np}}$$

$$N_{np} = \int_u^1 \xi d\xi \ (J_{nB}(\gamma_{np}\xi) + b_{np}Y_{nB}(\gamma_{np}\xi))^2$$

$$b_{np} = - J_{nB}'(\gamma_{np}h)/Y_{nB}'(\gamma_{np}h)$$

The γ_{np}'s are the solutions to the algebric equation

$$J'_{nB} (\gamma_{np}) \; Y'_{nB} (\gamma_{np}h) - Y'_{nB} (\gamma_{np}) \; J'_{nB} (\gamma_{np}h) = 0$$

The induced velocity decays exponentially with no oscillation away from the rotor. Details on obtaining the γ_{np} and b_{np} values are available in Reference 9.

The Case of $V_0 \to 0$

This includes the acoustic theory and the upstream part of the present theory. There is no difference between the two for $x < 0$. Now the relative Mach number may be greater than 1 locally and radiation of sound is possible. This can easily be seen from (6.8). For if $(M_x^2 + M_\theta'^2 - 1) > 0$, $-\lambda^2$ can be negative and the expression inside the curly braces is still positive. For a given n there will only be a finite number of real k's (λ's) because the first term in the curly braces will dominate for large λ^2. (cf. Ref.1)

For the complex eigenvalues, we can find their real parts by (6.11). When $k_i \neq 0$, (6.11) gives

$$k_r = \frac{\int_h^1 p|\phi|^2 (\frac{M_\theta'}{r}) M_x nB \, dr}{\int_h^1 p|\phi|^2 (1 - M_x^2) dr} \qquad (6.14)$$

As there is no radial pressure gradient in the equilibrium upstream flow

$M_x = Cst$ and $M'_\theta \sim \nu r$ [*]

$$k_r = - \frac{\nu M_x^2 nB}{1 - M_x^2} \qquad (6.15)$$

Fig. 5 shows the locations of the eigenvalues in the complex k plane.

(6.3) is quadratic in k. Therefore if $k_i \neq 0$ both k and k^* are eigenroots. We keep only the one with $k_i < 0$ so that (6.2) will give a vanishing solution for $x \to -\infty$. If k is purely real, roots will occur in pairs. We keep only the one which gives the correct Doppler effect. These roots are illustrated in Fig. 5. It con easily be seen ((6.3), (6.5) and (6.15)) that the line defined by (6.15) is midway between a pair of real k's. In this case of $V_0 \to 0$ the solution can be written down explicitly. (6.3) becomes

$$\phi'' + \frac{1}{r} \phi' + \phi [\ell^2 - \frac{n^2 R^2}{r^2}] = 0 \qquad (6.16)$$

where

$$\ell^2 = n^2 B^2 \nu^2 M_x^2 - k^2 (1 - M_x^2) - 2 M_x \nu nBk \qquad (6.17)$$

The solution is

$$\phi_n = Z_{nB} (\ell r)$$

where

$$Z_{nB} (\ell r) = J_{nB} (\ell r) + 2Y_{nB} (\ell r)$$

and ℓ and α are the eigenvalues and phase factors required to satisfy the two point boundary condition

$$Z'_{nB} (\ell r) = 0 \quad \text{at} \quad r = h \text{ and } 1 \qquad (6.18)$$

---

[*] Recall that, in the present notation, M_x, M_θ, M'_θ, etc. are mean value (θ-averaged) quantities.

The eigenvalues ℓ are all real and greater than 0. However the k obtained from (6.17) may be complex:

$$k = \frac{M_x \, nB \pm \sqrt{(M_x \, nB)^2 - [\ell^2 - (nB \, M_x^2)](1 - M_x^2)}}{(1 - M_x^2)} \qquad (6.19)$$

For sufficiently large ℓ, k is always complex. This result is identical with that of McCune [9]. Transonic resonance is also possible. For k = 0 (6.16) becomes

$$\phi'' + \frac{1}{\gamma} \phi' + \phi \left(\frac{n^2 B^2 \nu^2}{a_0^2} - \frac{n^2 B^2}{r^2} \right) = 0$$

which has nontrivial solution $\phi = Z_{nB} \left(\frac{nB\nu r}{a_0} \right)$ if $\frac{nB\nu}{a_0}$ happens to be one of the eigenvalues ℓ defined in (6.18). McCune [10] overcame the mathematical singularity by putting in a viscous dissipation term. There is no need to repeat this here. By contrast, non-linear effects were used in landing this singularity in Refs. 1 and 11.

The Downstream Flow

This is typified by a large V_0. As a result, for a compressor, the relative Mach number is often subsonic. From (6.6) it can be concluded that λ (and k) must be complex so that the boundary condition (6.4) can be satisfied. Since $k_i \neq 0$ we can estimate k_r by (6.14)

$$k_r = \frac{\int_h^1 \rho |\phi|^2 \frac{M_\theta'}{r} M_x \, nB \, dr}{\int_h^1 \rho |\phi|^2 (1 - M_x^2) dr} \sim \left. \frac{M_\theta' M_x \, nB}{r(1 - M_x^2)} \right|_{r = \xi} \qquad (6.20)$$

with ξ at some point along the span (h,1). For many operating
compressors $M'_\theta < 0$ with our definition. In that case, $k_r < 0$. Instead
of a monotonic exponential decay in the incompressible case, ϕ has an
oscillatory decay from the rotor. Comparing the estimates (6.20) and
(6.15) the main effect of the swirl is to decrease k_r. Thus we have
longer waves downstream.

The solution to (6.3) in general cannot be obtained in closed form.
In the next sections we shall discuss a numerical way of generating the
approximate eigenfunctions and eigenvalues together with the matching
of the solutions upstream and downstream.

VII. SOLUTION BY GALERKIN'S METHOD

In this section we shall solve the equations (3.10) (mean flow) and
(6.3) (θ varying flow) by a numerical expansion procedure. Since the
coefficients of both equations take different forms for upstream and
downstream, it is desirable to expand both the upstream and downstream
flows in terms of the same set of functions to facilitate the matching
procedure. Galerkin's method, frequently used in problem in elasticity
[14], suits our purpose. Basically the solution modes (by this we mean
the solutions to (3.10) and (6.3) with definite axial wave number k) are
expanded in terms of a complete set of functions satisfying the same set
of boundary conditions. The coefficients of the expansion are found by
making the residual orthogonal to all the functions in the complete set.
The task of solving a differential equation for eigenfunctions is thus
reduced to solving for the eigenvectors of a matrix. We also obtain the
axial wave number (which may be complex) as the eigenvalues of the matrix.

The equations we are solving take the general form of

$$L(k,\phi) = 0 \tag{7.1}$$

with boundary conditions

$$\frac{\partial}{\partial r} \phi \ (h) = \frac{\partial}{\partial r} \phi \ (1) = 0 \tag{7.2}$$

For the mean flow the operator L takes the form

$$L_M(K,\Phi) = \frac{\partial^2 \Phi}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \Phi}{\partial r} + (1 - M_x^2)K^2\Phi \tag{7.3}$$

For the θ varying flow it takes the form

$$L_T(k, \phi_n) = \frac{\partial^2 \phi_n}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \phi_n}{\partial r} +$$

$$+ \phi_n \{ -k^2(1 - M_x^2) + \frac{2M_\theta' M_x}{r} nBk - \frac{n^2 B^2 (1 - M_\theta'^2)}{r^2} \} \tag{7.4}$$

with the understanding that M_θ, M_x and M_θ' take different form for up-
stream and downstream flow. We have also assumed that in the mean flow

$$\Phi(x,r) \sim \Phi(r)e^{Kx} \tag{7.5}$$

and in the θ varying flow

$$\phi(x,r,\theta) \sim \phi_n(r)e^{inB\theta} e^{ik_n x} \tag{7.6}$$

We shall solve (7.1) by Galerkin's method and apply the result to
(7.3) and (7.4). Let

$$\phi(r) = \sum_{\ell=1}^{N} a_\ell \psi_\ell(r) \tag{7.7}$$

where $\{\psi_\ell\}_1^\infty$, is a complete set of independent functions (they need not

be orthogonal) satisfying

$$\frac{\partial}{\partial r}\,\psi_\ell(h) = \frac{\partial}{\partial r}\,\psi_\ell(1) = 0 \tag{7.8}$$

Then the solution to (7.1) is given by (for every eigenvalue k)

$$\phi = \lim_{N \to \infty} \sum_1^N a_\ell(k)\psi_\ell(r) \tag{7.9}$$

with the eigenvalues k and eigenvectors $[a_\ell]$ determined by making the residue $L(k, \sum_{\ell=1}^N a_\ell\psi_\ell)$ orthogonal to the ψ_j's:

$$\int_h^1 drw(r)L(k, \sum_{\ell=1}^N a_\ell\psi_\ell)\psi_j = 0 \qquad j=1,2,\ldots,n \tag{7.10}$$

where $w(r)$ is an appropriate weighing function. In general (7.10) is a matrix relation of the form

$$\begin{bmatrix} M_{ij}(k) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = 0 \tag{7.11}$$

The k's are determine to make $[M_{ij}]$ singular so that $[a_\ell]$ is not identically zero. It can be shown that (7.9) under certain mathematical boundedness and smoothness conditions converges (in the mean). The convergence goes as

$$\sim \int_h^1 |L(k, \sum_{\ell=1}^N a_\ell\psi_\ell)|^2 dr/N$$

which is very fast if a suitable set $\{\psi\}$ is chosen.

The Mean Flow

For the mean flow let

$$\Phi = \sum_{\ell=1}^{n} a_\ell \psi_\ell \qquad (7.12)$$

(To facilitate matching, the same set of ψ's should be used for both

$x > 0$ and $x < 0$.) Then the M_{ij}'s in (7.11) become

$$M_{ij}(k) = \int_h^1 wdr \; \Psi_i \frac{\partial^2 \Psi_j}{\partial r^2} +$$

$$+ \int_h^1 wdr \; \Psi_i \frac{(1 + M_\theta^2)}{r} \frac{\partial \Psi_j}{\partial r} + \qquad (7.13)$$

$$+ K^2 \int_h^1 wdr(1 - M_x^2)\Psi_i \Psi_j \equiv B_{ij} + K^2 C_{ij}$$

where

$$B_{ij} = \int_h^1 wdr \; \Psi_i \left(\frac{\partial^2 \Psi_j}{\partial r^2} + \frac{1 + M_\theta^2}{r}\frac{\partial \Psi_j}{\partial r}\right) \qquad (7.14a)$$

$$C_{ij} = \int_h^1 wdr(1 - M_x^2)\Psi_i \Psi_j \qquad (7.14b)$$

Let $A^T = (a_1, \ldots a_N)$. Then (7.11) may be written as $C^{-1} BA = -K^2 A$

and, hence,

$$K = \pm\sqrt{-\text{ (eigenvalue of } C^{-1} B)} \qquad (7.15a)$$

$$A = \text{eigenvector of } C^{-1} B \qquad (7.15b)$$

The upper (lower) sign in K are for the upstream (downstream) solution.

(It will turn out that the eigenvalues of C^{-1} B are negative and K is

real.) Suitable normalization of a will be carried out in the upstream-

downstream matching process (Chapter 8).

The θ Varying Flow

The procedure is identical to the above except now not only k^2 but k is involved in the dispersion relation (7.11). We shall use m_{ij} instead of M_{ij} to distinguish the present from the mean flow treatment. Let

$$\phi_n = \sum_{\ell=1}^{N} a_{n\ell}\, \psi_{n\ell}$$

(the index n denotes the nth azimuthal mode)

Then

$$m_{ij}^n = \int_h^1 wdr\{\frac{\partial^2 \psi_{ij}}{\partial r^2} + \frac{(1 + M_\theta^2)}{r} \frac{\partial \psi_{nj}}{\partial r} -$$

$$- \frac{n^2 B^2}{r^2} (1 - M_\theta'^2)\psi_{nj}\}\psi_{ni} +$$

$$(7.16a)$$

$$+ k \int_h^1 wdr\, \frac{2M_\theta' M_x}{r} nB\, \psi_{nj}\psi_{ni} -$$

$$- k^2 \int_h^1 wdr(1 - M_x^2)\psi_{nj}\psi_{ni}$$

or

$$m_{ij}^n \equiv - k^2 D_{ij}^n + k E_{ij}^n + F_{ij}^n \qquad (7.16b)$$

where the D_{ij}^n, E_{ij}^n and F_{ij}^n matrices are easily identifiable from (7.16a). The eigenvalue problem is

$$\sum_j (-k^2 D_{ij}^n + k E_{ij}^n + F_{ij}^n)A_j^n = 0 \qquad i = 1,..,n$$

when $A_j^n = a_{nj}$. Dropping the index n temporarily for simplicity we have in matrix notation

$$(-k^2 D + k E + F)A = 0 \qquad (7.17)$$

The matrices involved in (7.17) are N x N and k appears quadratically.
Therefore we would have 2N eigenvalues for k in general. However only N
of them are retained by requiring either vanishing solutions at
$|x| \rightarrow \infty$ on the appropriate radiation condition.

To solve (7.17) for k, it is noted that if a suitable set of ψ's
is chosen, D is going to be diagonal dominant; also D^{-1} exists and is
well-behaved. Multiplying (7.17) by D^{-1}

$$(-k^2 I + k D^{-1} E + D^{-1} F)A = 0 \tag{7.18}$$

Let

$$G \equiv D^{-1} E, \quad H \equiv D^{-1} F$$
$$A_1 \equiv A \quad A_2 \equiv k A_1 \tag{7.19}$$

Then the N x N matrix equation (7.18) may be written as a 2N x 2N matrix
equation

$$\begin{bmatrix} 0 & I \\ H & G \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = k \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{7.20}$$

Thus

$$k = \text{eigenvalues of } \begin{bmatrix} 0 & I \\ H & G \end{bmatrix} \tag{7.21}$$

After finding k, the eigenvector A can be solved easily. Normalization
of A will be carried out in the matching of the upstream and downstream
flow in (Chapter 8).

VIII. MATCHING OF UPSTREAM AND DOWNSTREAM FLOW

The matching conditions are

$$[u]_{0-}^{0+} = 0, \text{ no radial force} \tag{8.1}$$

$$[\rho w]_{0-}^{0+} = 0, \text{ mass flux conservation} \tag{8.2}$$

The difficulty of the matching lies in the fact that, because of the strong swirl, the downstream duct modes are different from the upstream modes. Mode by mode matching is no longer possible. For each radial position, all the modes are added up to satisfy (8.1) and (8.2). Say if we limit ourselves to N θ-modes and P radial modes, then (8.1) and (8.2) furnish 2 x N x P simultaneous algebric relations between the 2 x N x P amplitudes. The details of the mixed mode matching go as follows. Let

$$\rho = \rho_0 + \rho_1$$

where ρ_0 is the density corresponding to the base flow (2.12) which is different far upstream and downstream, then (8.2) becomes

$$[(\rho_0 + \rho_1)(w_0 + w_1)]_u^d = 0 \tag{8.3}$$

(We shall use 0-, 0+ and u, d interchangeably.) In our model of the base flow we have assumed a jump of tangential velocity across the x = 0 plane (actuator disk). There must be a corresponding jump in radial pressure gradient to supply the neccessary centrifugal force. Hawthorne and Ringrose [8] pointed out that this pressure variation will further yield a modification of the radial density profile and therefore a redistribution of mass flux. Hence radial flow will occur even if the far downstream flow is of free vortex type. Since the observed radial flow is small we can assume the difference of the two 0(1) quantities to be

$$(\rho_0 w_0)_d - (\rho_0 w_0)_u = 0(\varepsilon)$$

Then, to $0(\varepsilon)$, (8.3) can be written as

$$[\rho_0 w_1 + \rho_1 w_0 + \rho_0 w_0]_u^d = 0 \qquad (8.4)$$

Taking the θ-average of (8.4) yields

$$[\rho_0 \overline{w}_1 + \overline{\rho}_1 w_0 + \rho_0 w_0]_u^d = 0 \qquad (8.5)$$

and the difference of (8.4) and (8.5) gives the corresponding matching condition far the perturbation flow:

$$[\rho_0 \tilde{w}_1 + \tilde{\rho}_1 w_0]_u^d = 0 \qquad (8.6)$$

Since we have so far only calculated the velocities, the presence of the density perturbations $\overline{\rho}_1$ and $\tilde{\rho}_1$ in (8.5) and (8.6) are not very convenient. However they can be eliminated via the energy equation (isentropic flow has been assumed throughout)

$$C_p T + \frac{1}{2} q^2 = C_p T_t \qquad \text{(along stream line)}$$

or

$$\rho = \rho_t (1 - \frac{\gamma - 1}{2} \frac{q^2}{a_t^2})^{1/\gamma - 1}$$

where the subscript t denotes total (stagnation) quantities. Taking the logarithmic derivative and remembering that T_t is independent of r for uniform axial flow and/or for free vortex flow, we have

$$\frac{d\rho}{\rho} = \frac{- q dq}{a_t^2 - (\gamma - 1)q/2}$$

Replacing $d\rho$ by ρ_1, ρ by ρ_0, q by q_0 and dq by q_1

$$\rho_1 = \frac{\rho_0 (-q_0 \cdot q_1)}{a_x^2 - \frac{\gamma - 1}{2} q_0^2} = - \frac{\rho_0 q_0 \cdot q_1}{a^2} \simeq - \rho_0 \frac{q_0 \cdot q_1}{w_0^2} M_x^2 \qquad (8.7)$$

θ averaging (8.7) we obtain

$$\bar{\rho}_1 \cong - M_x^2 \, \rho_0 \, \frac{\bar{w}_1}{w_0} \qquad (8.8)$$

The mass flux matching condition for the mean flow is therefore

$$[(1 - M_x^2) \, \rho_0 \bar{w}_1 + \rho_0 w_0]_u^d = 0 \qquad (8.9)$$

The θ varying part of (8.7) is

$$\tilde{\rho}_1 = - M_x^2 \, \rho_0 (w_0 \tilde{w}_1 + v_0 \tilde{v}_1)/w_0^2 \qquad (8.10)$$

Hence the mass flux matching condition for the θ-varying flow is

$$[(1 - M_x^2) \, \rho_0 \, \tilde{w}_1/w_0 - M_x M_\theta \, \rho_0 \, \tilde{v}_1/w_0]_u^d = 0 \qquad (8.11)$$

The Mean Flow

The θ-averaged velocity perturbation is given by

$$\overline{\underline{q}_1} = \nabla \Phi + \underline{E} \qquad (8.12)$$

where

$$\underline{E} = \begin{array}{ll} 0 & ; \; x < 0 \\[2mm] (E_0 + C)\hat{x} & ; \; x > 0 \end{array} \qquad (8.13)$$

$$E_0 \equiv - \frac{B C_\Gamma}{2\pi w_0} \int_h^r \frac{v_0'}{r} \frac{\partial \Gamma}{\partial r} \, dr \qquad (8.14)$$

and C is the constant of integration in (3.9). According to the Galerkin procedure

$$\Phi = \begin{array}{cc} \sum_{p=1}^{N} \sum_{l=1}^{N} a^u_{ep} \Psi_l(r) e^{K^u_p x} = \sum_{p=1}^{N} \sum_{l=1}^{N} N^u_p \alpha^d_{lp} \Psi_l(r) e^{K^u_p x} & x < 0 \\[4mm] \sum_{p=1}^{N} \sum_{l=1}^{N} a^u_{ep} \Psi_l(r) e^{K^d_p x} = \sum_{p=1}^{N} \sum_{l}^{N} N^d_p \alpha^d_{lp} \Psi_l(r) e^{K^d_p x} & x > 0 \end{array} \qquad (8.15)$$

where the same Ψ_l's are used for both upstream and downstream. The coefficients of expansion are expressed as $N_p \alpha_{lp}$ where the α_{lp}'s make up the unnormalized eigenvector discussed in the last section; the N_p's provided their normalization. (p indexes the eigenvectors and l the components of each.)

We shall pick the normalized $Z_0(\lambda_l r)$ (which satisfy the boundary conditions) as our expansion functions $\psi_l(r)$:

$$Z_0(\lambda_l r) \equiv [J_0(\lambda_l r) + b_l Y_0(\lambda_l r)]/\sqrt{N_1}$$

$$N_l = \int_h^1 r[J_0(\lambda_l r) + b_l Y_0(\lambda_l r)]^2 dr$$

$$\frac{\partial}{\partial r}(Z_0(\lambda_l r)) = 0 \qquad \text{at } r = h, 1$$

This choice turns out to satisfy exactly the differential equation for Φ upstream [see also (3.13)] and hence saves the labour of solving the matrix eigenvalue problem upstream.

However the set Z_0 is not complete unless we include a constant. This manisfests itself as the constant of integration in (3.9) and re-presents a net mass flux.

From (8.1) we have

$$\sum_{\ell p} N^u_{\ell p} \, \alpha^u_{\ell p} = \sum_{\ell p} N^d_{\ell p} \, \alpha^d_{\ell p} \tag{8.16}$$

Substituting the various expansions into (8.9) we obtain

$$I(1 - M^2_x)\rho_0 \sum_{\ell p} N_p \alpha_{\ell p} \, \Psi_\ell \, K_p]^d_u = - [\rho_0 w_0]^d_u - ((1 - M^2_x)\rho_0)_d (E_0 + C) \tag{8.17}$$

By using the same Ψ's for the upstream and downstream solutions we have been able to carry out the matching of the radial velocity (8.16) without difficulty. However in the matching of the mass flux (8.17) we get on each side of the equality a product of $\Psi_\ell(r)$ and $\rho_0(r)$ with different radial dependence upstream and downstream. Therefore we have to reexpand both sides of (8.16) in terms of a common set of complete functions $\boxed{-}_m(r)$ before we can compare the coefficients. It is convenient to use the same set of Ψ's for the $\boxed{-}$'s so that orthogonal relations can be used wherever possible. Upon substituting (8.16) into (8.17) the problem can be cast in the form of the following matrix equation for $[N^d_p]$

$$([a_{mp}]]b_{mp}] + \frac{1}{f} [C_m d_p]) \, [N^d_p] = [g_m] - \frac{e}{f} [C_m] \tag{8.18}$$

where [] represents an N x N matrix or an N vector depending on the number of subscripts its element carries. We have assumed that the re-expansion is also carried out to N terms so that (8.18) remains an N x N system and thus does not become over or under determined.

The definition of the various terms in (8.18) are:

$$a_{mp} = (1 - (M_x^u)^2)\rho_0^u \, K_m^u \, \alpha_{mp}^d$$

$$b_{mp} = \sum_{\ell \equiv 1}^{n} q_{m\ell} \, K_p^d \, \alpha_{\ell p}^d$$

$$q_{m\ell} = \int_h^1 ((1 - M_x^2)\rho_0 \Psi_\ell)_d \, r\Psi_m \, dr$$

$$c_m = \int_h^1 ((1 - M_x^2)\rho_0)_d \, r\Psi_m \, dr$$

$$d_p = \sum_{\ell \equiv 1}^{n} \alpha_{\ell p}^d \, B_\ell^d \, K_p^d$$

$$B_\ell = \int_h^1 ((1 - M_x^2)\rho_0 \Psi_\ell)_d \, r \, dr$$

$$g_m = \int_h^1 \{((1 - M_x^2)\rho_0 E_0)_d + [\rho_0 w_0]_u^d \} \, r\Psi_m \, dr$$

$$e = \int_h^1 \{((1 - M_x^2)\rho_0 E_0)_d + [\rho_0 w_0]_u^d \} \, r \, dr$$

$$f = \int_h^1 ((1 - M_x^2)\rho_0)_d \, r \, dr$$

Finally the constant of integration in (3.9) is

$$C = - \frac{1}{f} (e + \sum_\ell \sum_p N_p^d \, \alpha_{\ell p}^d \, K_p^d \, B_\ell)$$

The θ-varying Flow

The procedure is exactly the same as the previous one except we have to carry another index n to denote the θ modes. From Chapter 4

$$\tilde{q}_1 = \begin{cases} \nabla\phi & x < 0 \\ \nabla\phi + C_\Gamma S \nabla\Gamma & x > 0 \end{cases} \tag{8.19}$$

$$\phi = \phi_w + \phi_h$$

$$\phi_w = \sum_{n=1}^{\infty} \, \begin{matrix} 0 \\ i\,Rn(r)\,e^{inB\alpha} \end{matrix} \quad \begin{matrix} x<0 \\ \\ x>0 \end{matrix} \tag{8.20}$$

The Rn's are the particular solutions (wake functions) to (5.4). The method by which they are calculated numerically is described in Chapter 5. In addition,

$$\phi_h = \begin{matrix} \sum_{n=1}^{\infty} \sum_{\ell=1}^{N} \sum_{p=1}^{N} a_{n\ell p}^{u}\, \psi_{n\ell}\, e^{inB\theta}\, e^{ik_{np}^{u}x} & x<0 \\[2ex] \sum_{m=1}^{\infty} \sum_{\ell=1}^{N} \sum_{p=1}^{N} a_{n\ell p}^{u}\, \psi_{n\ell}\, e^{inB\theta}\, e^{ik_{np}^{d}x} & x>0 \end{matrix} \tag{8.21}$$

where the Galerkin expansion in Chapter 7 has been carried to N terms. Applying the matching condition (8.1) we have

$$\sum_{n\ell p} a_{n\ell p}^{u}\, \psi_{n\ell}' = \sum_{n\ell p} a_{n\ell p}^{d}\, \psi_{n\ell}' + \sum_{n=1} i\left(R_n' - \frac{C_\Gamma}{n\pi}\frac{\partial\Gamma}{\partial r}\right) \tag{8.22}$$

$$\left(Rn - \frac{C_\Gamma}{n\pi}\Gamma\right) = \sum_{\ell=1}^{N} b_{n\ell}\psi_{n\ell} \tag{8.23}$$

Then

$$\sum_{p=1}^{N} \left(a_{n\ell p}^{u} - a_{n\ell p}^{d}\right) = i\, b_{n\ell} \quad \begin{matrix} n=1,\dots,\infty \\ \ell=1,\dots,N \end{matrix} \tag{8.24}$$

The mass-flux matching (8.11) becomes

$$[(1 - M_x^2)\rho_0/W_0]^u \; {\textstyle\sum_{n\ell p}} \; a_{n\ell p}^u \; \psi_{n\ell} \; i \; k_{np}^u =$$

$$= [(1 - M_x^2)\rho_0/W_0]^d \; {\textstyle\sum_{n\ell p}} \; a_{n\ell p}^d \; \psi_{n\ell} \; i \; k_{np}^d + {\textstyle\sum_{n=1}^{\infty}} \; iR_n(r) \frac{\partial\alpha}{\partial x} \; (inB) \; -$$

$$-[\rho_0 M_x M_\theta/w_0]^d \; {\textstyle\sum_{n\ell p}} \; a_{n\ell p}^d \; \psi_{n\ell} \; inB + {\textstyle\sum_{n=1}^{\infty}} \; i \; R_n(r)(\frac{\partial\alpha}{\partial\theta})(inB)$$

$$(8.25)$$

This can be written as

$$\sum_{n\ell p} a_{n\ell p}^u f_{n\ell p}^u(r) = \sum_{n\ell p} a_{n\ell p}^d f_{n1p}^d(r) + \sum_n i \; g_n(r) \qquad (8.26)$$

where

$$f_{n\ell p}^u(r) = [(1 - M_x^2)\rho_0 k_{np}/w_0]^u \psi_{n\ell}(r)$$

$$f_{n\ell p}^d(r) = [((1 - M_x^2)\rho_0 k_{np} - \rho_0 M_x M_\theta nB)/w_0]^d \psi_{n\ell}(r)$$

$$g_n(r) = [\rho_0(\frac{\partial\alpha}{\partial x}(1 - M_x^2) - M_x M_\theta(\frac{\partial\alpha}{\partial\theta}))/w_0]^d nBR_n(r)$$

Reexpanding these functions in terms of a complete set $\chi_m(r)$ to N terms we have

$$f_{n\ell p}^{u,d} = \sum_{n=1}^{N} b_{n\ell pm}^{u,d} \; \chi_m(r) \qquad (8.27a)$$

$$g_n = \sum_{m=1}^{N} c_{nm} \; \chi_m(r) \qquad (8.27b)$$

(8.26) becomes

$$\sum_{n\ell pm} a_{n\ell p}^u b_{n\ell pm}^u \chi_m(r) = \sum_{n\ell pm} a_{n\ell p}^d b_{n\ell pm}^d \chi_m(r) + \sum_{n,m} ic_{nm}\chi_m(r) \qquad (8.28)$$

For each m and fixed n we equate teh two sides of (8.28) and get N relations. They are

$$\sum_{\ell,p} a^n_{n\ell p} \, b^u_{n\ell pm} = \sum_{\ell p} a^d_{n\ell p} \, b^d_{n\ell pm} + i \, c_{nm} \qquad m=1,2,\ldots,N \qquad (8.29)$$

For each n, (8.24) and (8.29) provide 2N normalization conditions for the 2N eigenvectors $[a^u_{n\ell p}]$ and $[a^d_{n\ell p}]$. Let $N^{u,d}_{np}$ be the normalization constant amd $[\alpha^{u,d}_{n\ell p}]$ be the unnormalized eigenvectors for the $[m_{ij}]$ matrix in section 7.

$$a^{u,d}_{n\ell p} = N^{u,d}_{np} \, \alpha^{u,d}_{n\ell p} \qquad \begin{array}{l} n=1,\ldots,\infty \\ \ell=1,\ldots,N \\ p=1,\ldots,N \end{array} \qquad (8.30)$$

(8.24) and (8.29) become

$$\sum_p N^u_{np} \, \alpha^u_{n\ell p} = \sum_p N^d_{np} \, \alpha^d_{n\ell p} + i \, b_{n\ell} \qquad \ell=1,\ldots,N \qquad (8.31)$$

$$\sum_{\ell,p} N^u_{np} \, \alpha^u_{n\ell p} \, b^u_{n\ell pm} = \sum_{\ell p} N^d_{np} \, \alpha^u_{n\ell p} \, b^d_{n\ell pm} + i \, c_{nm} \qquad m=1,\ldots,N \qquad (8.32)$$

In matrix notation we have, for each n

$$\begin{array}{ll} [\alpha^u_{n\ell p}] & -[\alpha^d_{n\ell p}] \\ [\beta^u_{nmp}] & -[\beta^d_{nmp}] \end{array} \quad \begin{array}{l} [N^u_{np}] \\ [N^d_{np}] \end{array} = \begin{array}{l} [ib_{n\ell}] \\ [ic_{nm}] \end{array} \qquad (8.33)$$

where

$$\beta_{nmp} = \sum_\ell \alpha_{n\ell p} \, b_{n\ell pm}$$

From (8.33) we can solve for $N^{u,d}_{np}$ easily and (8.30) furnishes the coefficients for (8.21).

IX. NUMERICAL CALCULATION

The following gives some of the details of the calculation procedure. Results are presented at the end of this section.

(i) Parameters of the Calculation

The inputs to the computation are: the number of blades B, the hub to tip ratio h, the axial Mach number far upstream $M_{x-\infty}$, the rotor speed ν, the average loading per blade C_Γ and the loading profile $\Gamma(r)$. All quantities are nondimensionalized as discribed in section 2. The loading profile we choose is

$$\Gamma(r) = 1 - \varepsilon \cos \left(\frac{r - h}{1 - h} \pi\right)$$

where ε is a small parameter of about $0.1 \sim 0.2$. Note that Γ has zero derivatives at the tip and hub radii.

Instead of ν and C_Γ it is sometimes more convenient to specify the blade tip Mach number M_T and the total pressure ratio P_{t_2}/P_{t_1}. They are related by

$$\frac{P_{t_2}}{P_{t_1}} = (1 + M_{x_{-\infty}}^2 (\gamma - 1) \frac{\nu B C_\Gamma}{2\pi})^{\gamma/\gamma - 1}$$

$$M_T^2 = (\,^2 + 1)M_{x_{-\infty}}^2$$

(ii) The Base Flow

The base flow from which perturbation is made consists of a uniform

axial flow upstream of the actuator dish and a free vortex flow with

constant axial velocity downstream. Each can be shown to be a self con-

sistent flow field in the applicable region.

Let the upstream and downstream quantities carry subscripts 1 and

2. For a given density ρ_1 and velocity w_1 upstream, the downstream

axial velocity is

$$w_2 = \frac{\rho_1 w_1}{<\rho_2>} \qquad (9.1)$$

where

$$<\rho_2> = \frac{\int_h^1 \rho_2 r dr}{\int_h^1 r dr} \qquad (9.2)$$

The value of $<\rho_2>$ is obtained through the Euler turbine equation and the

isentropic relation.

$$(\frac{a_2}{a_1})^2 = 1 + \frac{\gamma - 1}{2a_1^2} (1 - w_2^2 - v_2^2) + \frac{\gamma - 1}{a_1^2} rv_2 \qquad (9.3)$$

$$\frac{\rho_2}{\rho_1} = (\frac{a_2}{a_1})^{1/\gamma - 1} \qquad (9.4)$$

where a is the velocity of sound and v the θ velocity. $<\rho_2>$ is obtained

by substituting (9.1) into (9.3) and solving the system (9.2), (9.3)

and (9.4) by iteration. The result of some cases under study is shown

in Figure 6.

(iii) The Mean and θ varying Flow

The precedures outlined in the previous sections are straightfor-

ward but tedious to carry out. The wake function calculated by the

mentioned implicit alternate gradient method converges very rapidly.
In the Galerkin expansions normalized Bessel functions are used with
weight equal to r. These Bessel functions are chosen to have vanishing
derivatives at hub and tip and to satisfy the differential equations for
the upstream flow. (This is the reason why they are used.) Orthogonal-
ity is used whenever possible to reduce the labor of computation.

The only difficulty comes in the upstream and downstream matching
of the θ varying part of the perturbation flow. With our model, velo-
cities become singular at the lifting line. The singularity manifests
itself as a failure of convergence of the Fourier expansion of velocities.
For example with velocity potential

$$\phi = \sum_{np} a_{np} \, \phi_{np} \, e^{inB\theta} \, e^{ik_{np}x}$$

The θ velocity component at the lifting line is

$$v_1(x = 0, r, \theta = 0) = \sum_{np} a_{np} \, \frac{inB}{r} \, \phi np$$

In general the above sum over n diverges.

Theoratically it is still possible to apply the matching condition
to the divergent series and obtain meaningful values for the a_n's. This
was done for the small swirl case in [11] and for the incompressible
rectilinear cascade in [3] by requiring the correct singular behavior
at the lifting line. In those work, however, such a step is possible
analytically because the mode shapes are the same for both upstream
and downstream. In the strong swirl case studied here, the mode shapes
are different for upstream and downstream. The mixed mode matching of
the singular velocities is not so straightforward. Direct application

of the procedure outlined in section 8 yields series that do not converge (at least numerically). McCune[*] pointed out that the matching condition (8.2) is an indirect specification of the correct singular behavior of the velocities at the matching plane. The following replacement may be more appropriate

$$[v_1]_{0-}^{0+} = \frac{c_\Gamma}{r} \left(\sum_{i=1}^{B} \delta(\theta - \theta_i) - \frac{B}{2\pi} \right) \tag{9.5}$$

$$\theta_i = 0, \frac{2\pi}{B}, \frac{4\pi}{B}, \ldots$$

(9.5) and (8.2) are shown to be identical in the incompressible rectilinear cascade calculation [3].

For the time being we avoid this problem[†] by addressing ourselves only to the calculation the total deflection angle. In this case the acoustic modes have all decayed away at the Trefftz plane. The perturbation potential there is then given by ϕ_w alone in (8.20). The wake induced velocities which modify the deflection angle are given by $\nabla \phi_w$ at $\alpha = 0$. (We have neglected here the effect of the upstream acoustic radiation, if any, on the incoming flow direction.)

(iv) <u>Numerical Results</u>

Figures 7 to 9 show the total deflection angles for different

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[*] J. E. McCune, Private communication 1976.

[†] Study of the method of determining the correct Fourier coefficients of the divergent series is underway. These coefficients are important for determining the upstream acoustic radiation spectrum in the presence of strong downstream swirl.

blades and loadings. The mean solution takes into account the
θ-averaged contribution of the wakes (associated with $\varepsilon \neq 0$) and the 3D
solution includes also the contribution from ϕ_w. As expected the effect
of the wake generally is to try to "bend" the loading profile towards
a free vortix profile.

Fig. 10 shows the axial decay rate of the θ-averaged flow induced
by the wake. That the magnitudes of the downstream decay rates are
smaller than those of the corresponding upstream radial modes is due to
the Prandtl Glauert effect. Fig. 11 shows the axial complex wave
number of the three dimensional perturbation velocities. The downstream
values are all complex as the flow is subsonic there in the examples
taken. To each azimuthal mode m, there correspond a finite number of
upstream purleyreal wave numbers. These are the acoustic radiating
modes (modes above cut-off).

Fig. 12 shows the deflection correction ΔD due to the three
dimensional part of the perturbation velocities for a rotor of fixed
total pressure ratio as a function of rotor speed. The decrease in ΔD
here is artificial because as we run up the rotor speed ν the loading
per blade decreases $\sim 1/\nu$ if the pressure ratio is to remain constant.
The strength of the shed vorticity also goes down by $1/\nu$. Fig. 14
shows ΔD verses rotor speed for a rotor of constant work per blade so
that the pressure ratio increases as the rotor speed increases. It is
found that ΔD still decreases as speed increases. This is because the
angle the wake makes with the axial direction increases as ν increases.

REFERENCES

1. Okurounmu, O. and McCune, J. E., "Lifting Surface Theory of Axial Compressor Blade Rows," AIAA J., 12, 10, Oct. 1974, pp. 1363-1380.

2. McCune, J. E. and Dharwadkar, S. P., "Lifting Line Theory for Subsonic Axial Compressor Rotors," M.I.T. Gas Turbine Lab. Rep. No. 110, July 1972.

3. McCune, J. E. and Hawthorne, W. R., "The Effects of Trailing Vorticity on the Flow Through Highly-Loaded Cascades," J. Fluid Mech., 74, 4, 1976, pp. 721-740.

4. Morton, Jr., K. B., "Three Dimensional, Compressible Flow Through Highly-Loaded, Rectilinear Cascade," S.M. Thesis, Dept. Aero. & Astro., M.I.T., 1974.

5. Cheng, W. K., "A Three Dimensional Theory for the Velocity Induced by a Heavily Loaded Annular Cascade of Blades," S.M. Thesis, Dept. of Aero. & Astro., M.I.T., June 1975.

6. Kerrebrock, J. L., "Small Disturbances in Turbomachine Annuli with Swirl," M.I.T. Gas Turbine Lab. Rep. No. 125, Oct. 1975.

7. Hawthorne, W. R. and Horlock, J. H., "Actuator Disc Theory of the Incompressible Flow in Axial Compressors," Proc. Inst. Mech. Eng., 176, 1962, pp. 789-814.

8. Hawthorne, W. R. and Ringrose, J., "Actuator Disc Theory of the Compressible Flow in Free-Vortex Turbomachinary," Proc. Inst. Mech. Eng., 178, 3 I, 1964, 1-3.

9. McCune, J. E., "A Three Dimensional Theory of Axial Compressor Blade Rows - Application in Subsonic and Supersonic Flows," J. Aero. Space Sci., 25, 1958, pp. 544-560.

10. McCune, J. E., "The Transonic Flow Field of an Axial Compressor Blade Row," J. Aero. Space Sci., 25, 1958, pp. 616-626.

11. Okurounmu, O. and McCune, J. E., "Three-Dimensional Vortex Theory of Axial Compressor Blade Rows at Subsonic and Transonic Speeds," AIAA J., 8, 7, July 1970, pp. 1275-1283.

12. Richtmeyer and Morton, Difference Methods for Initial Value Problems, John Wiley & Sones, 1967, p. 200.

Figure 1

$$\alpha = \frac{2\pi}{B}$$

$$-\frac{2\pi r}{B}$$

Lines of constant α

$$\alpha = 0$$

$$x \tan \beta_2'$$

$$0$$

$$x$$

$$\alpha = -\frac{2\pi}{B}$$

$$\beta_2'$$

$$+\frac{2\pi r}{B}$$

$$\beta_1'$$

$$r\theta$$
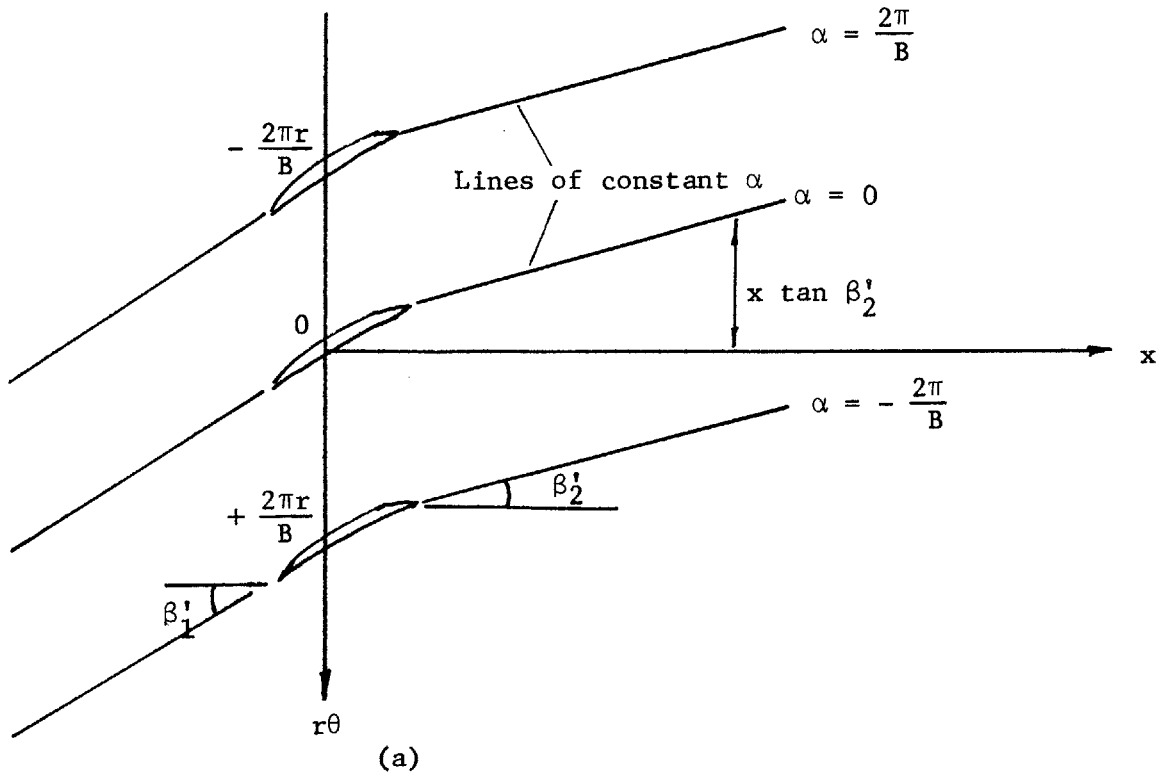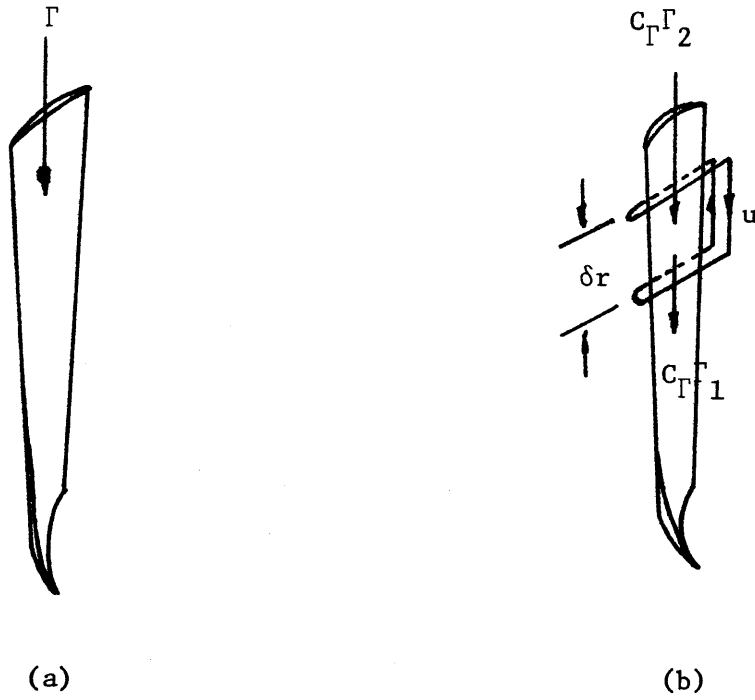
(a)

$$\upsilon_o$$

$$\upsilon r$$

$$\beta_1'$$

$$\upsilon_o'$$

$$\beta_2'$$

$$w_o$$

(b)
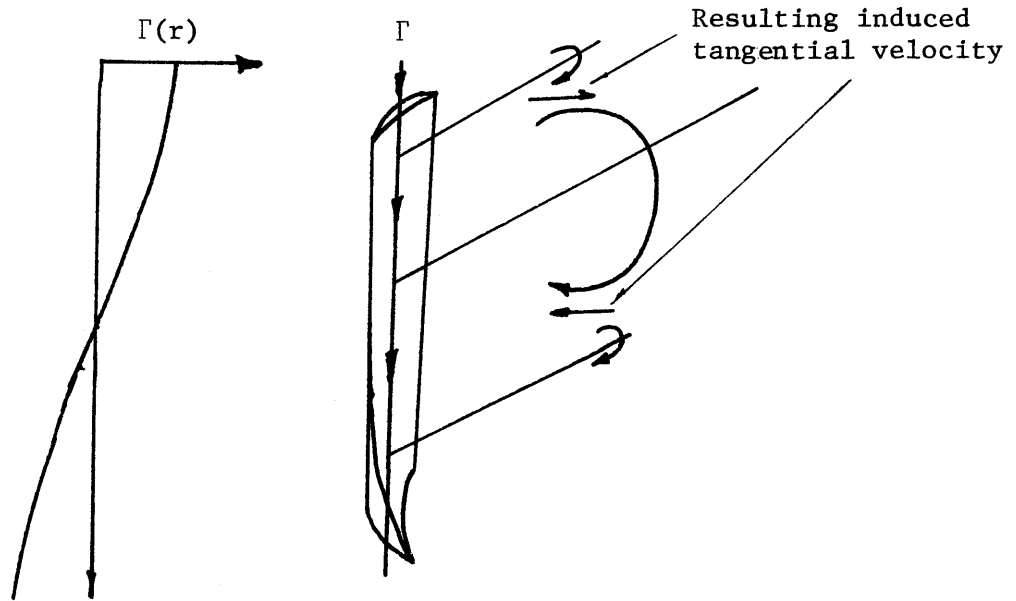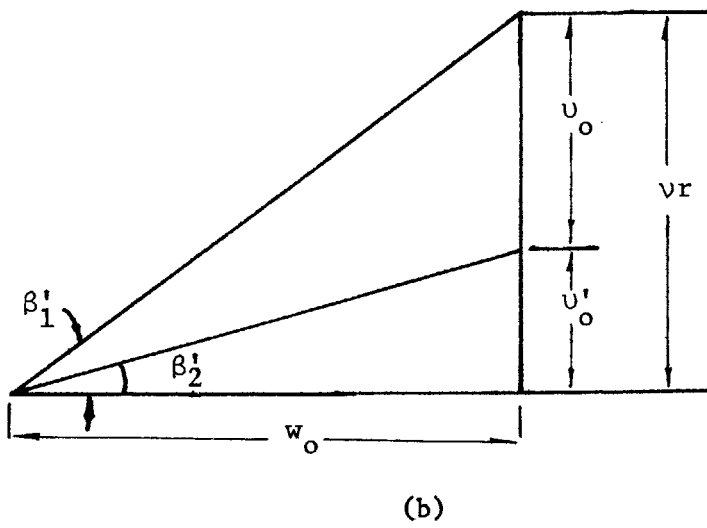
Figure 2

(a)

(b)

(c)

Figure 3

$Y = H(\alpha)$

$Y = \alpha - \frac{1}{2}$

$S(\alpha) = H(\alpha) - (\alpha - \frac{1}{2})$
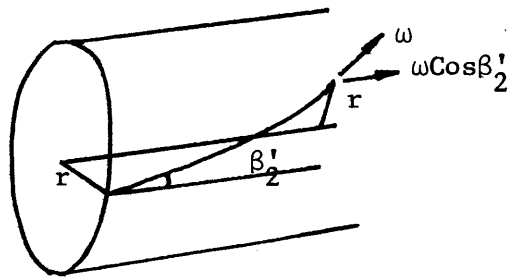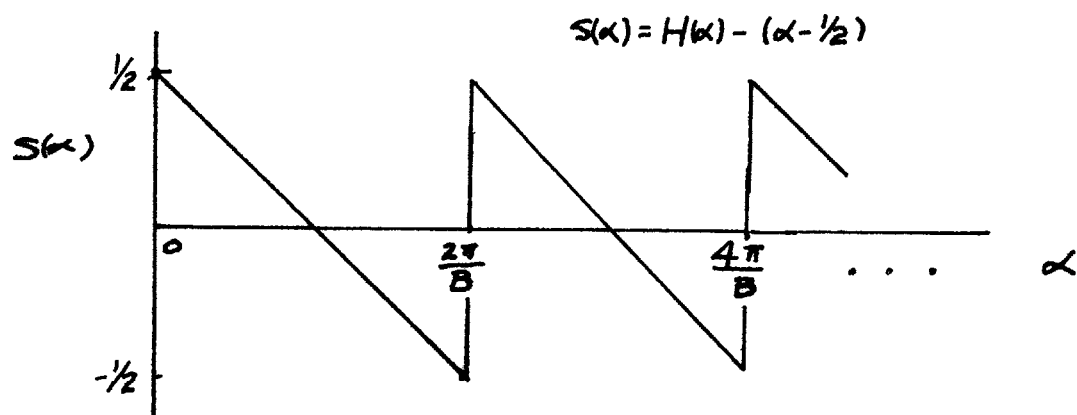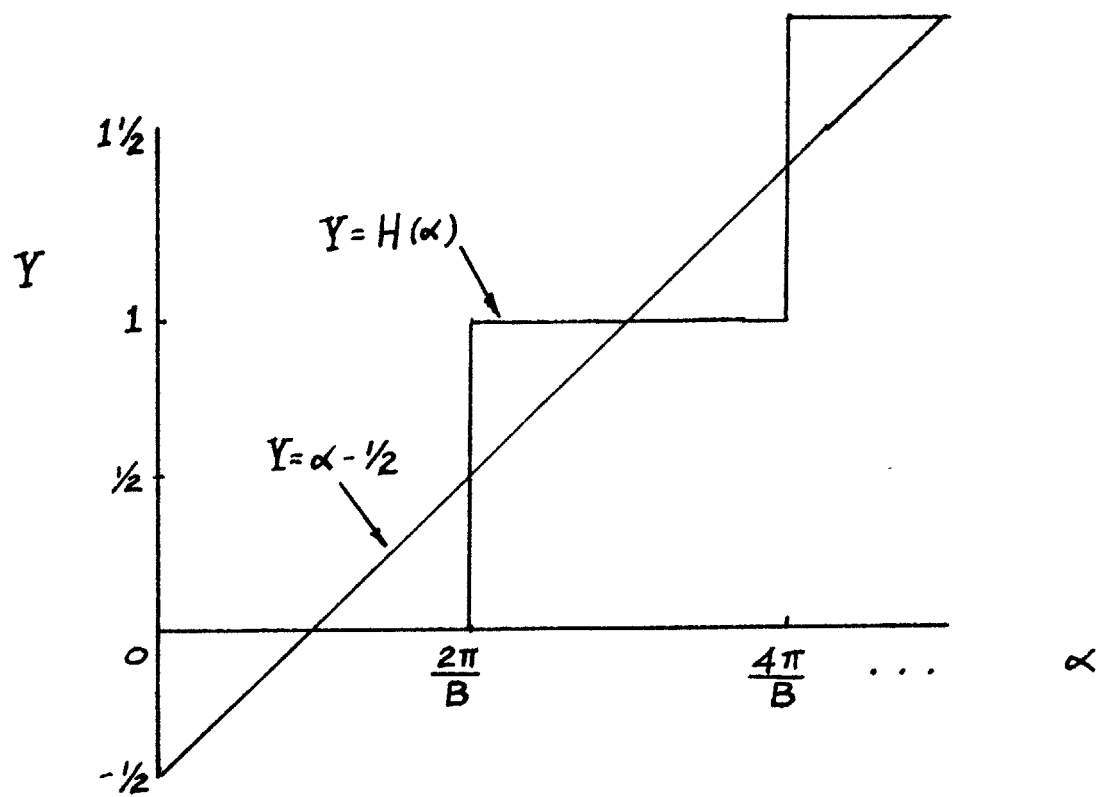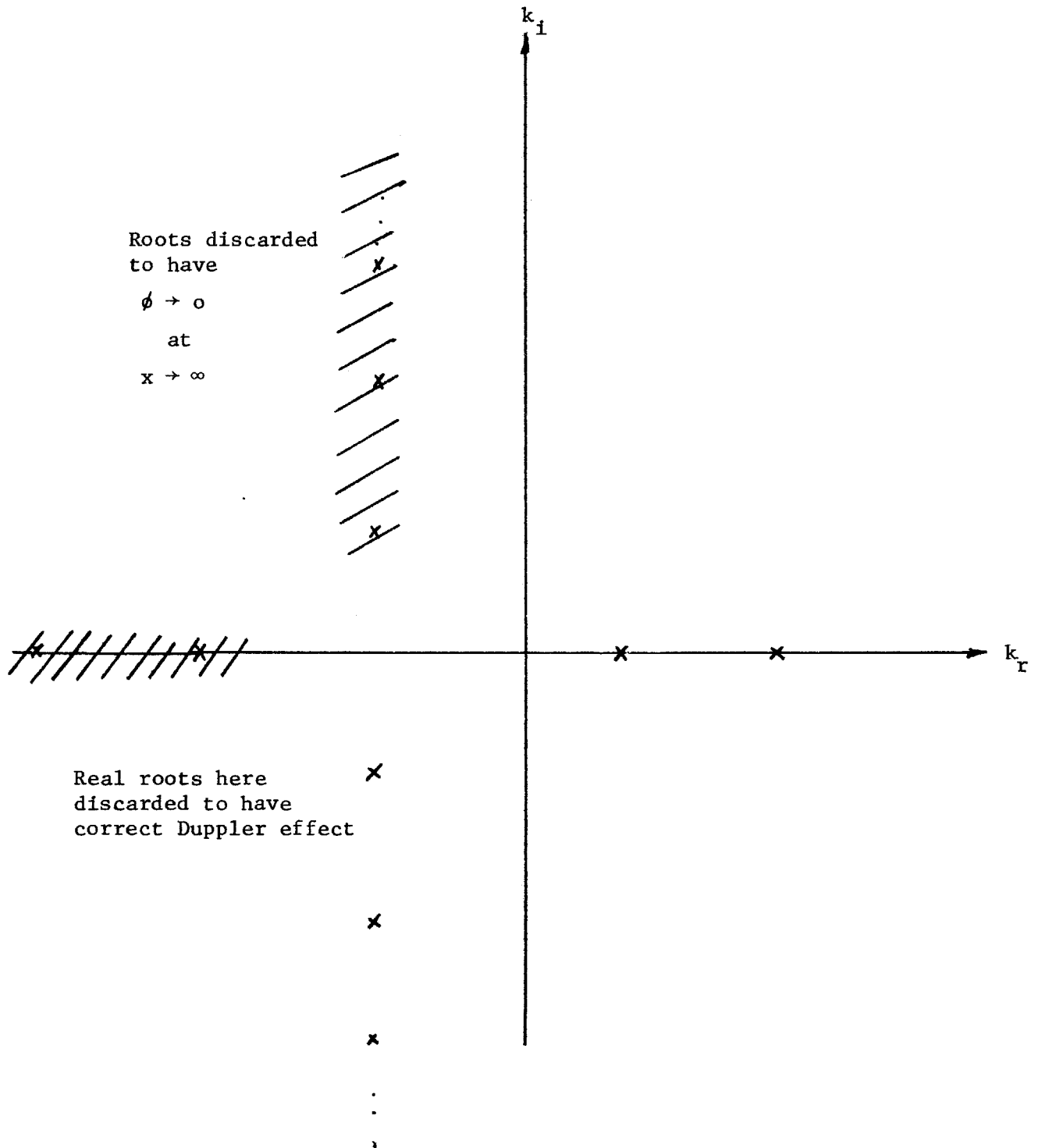
Figure 4

Sketch of upstream eigenvalue map for the transonic ducted fan

Figure 5

The annulus averaged downstream density

$B = 40 \quad h = 0.7$

$$<\bar{\rho}_2> = \frac{2}{1-h^2} \int_h^1 r\rho_2 dr$$

$M_x(-\infty)=0.65$
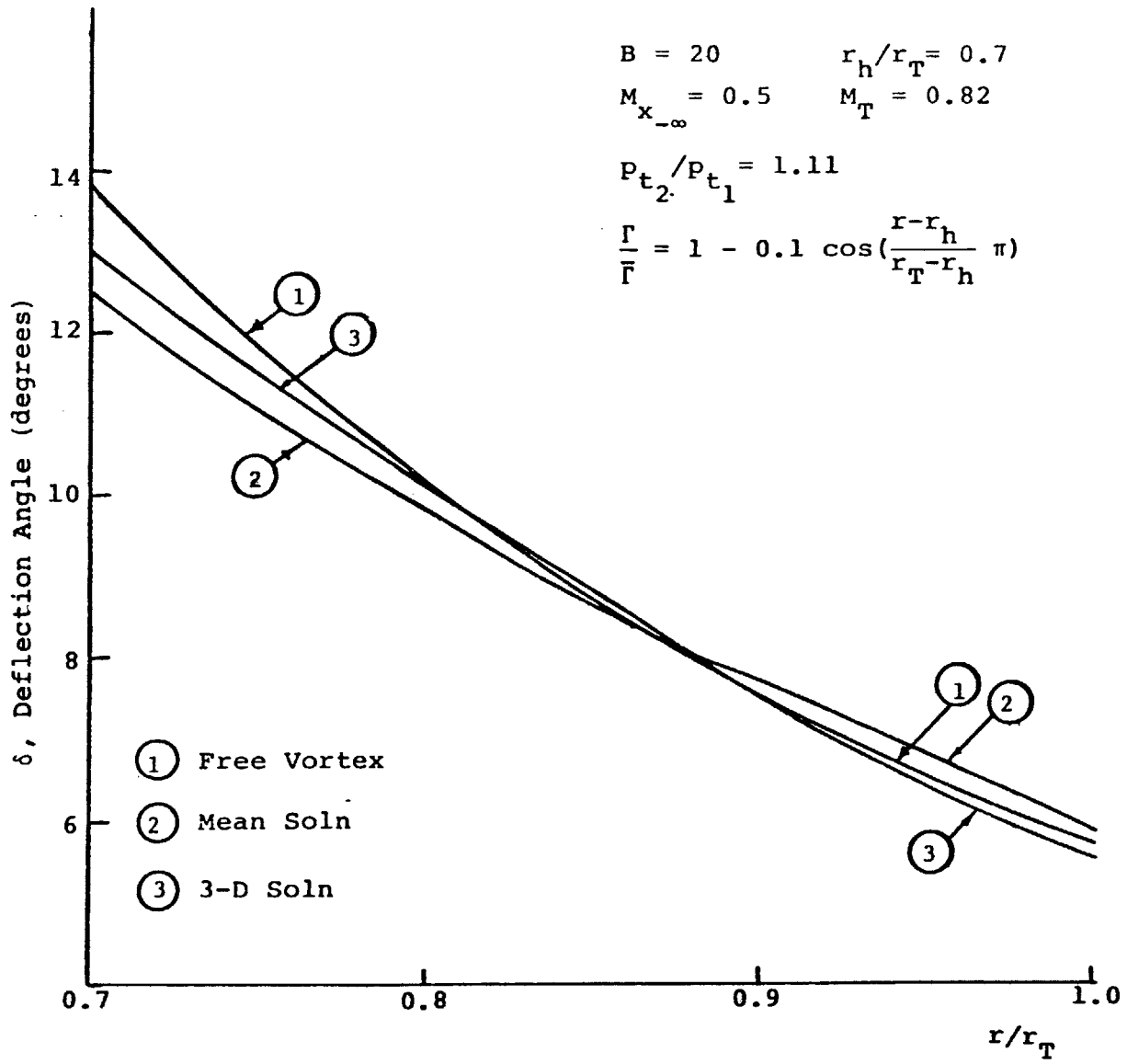$P_{t_2}/P_{t_1}=2.0$

$M_x(-\infty)=0.65$
$P_{t_2}/P_{t_1}=1.8$

$M_x(-\infty)=0.65$
$P_{t_2}/P_{t_1}=1.7$

$M_x(-\infty)=0.5$
$P_{t_2}/P_{t_1}=1.5$
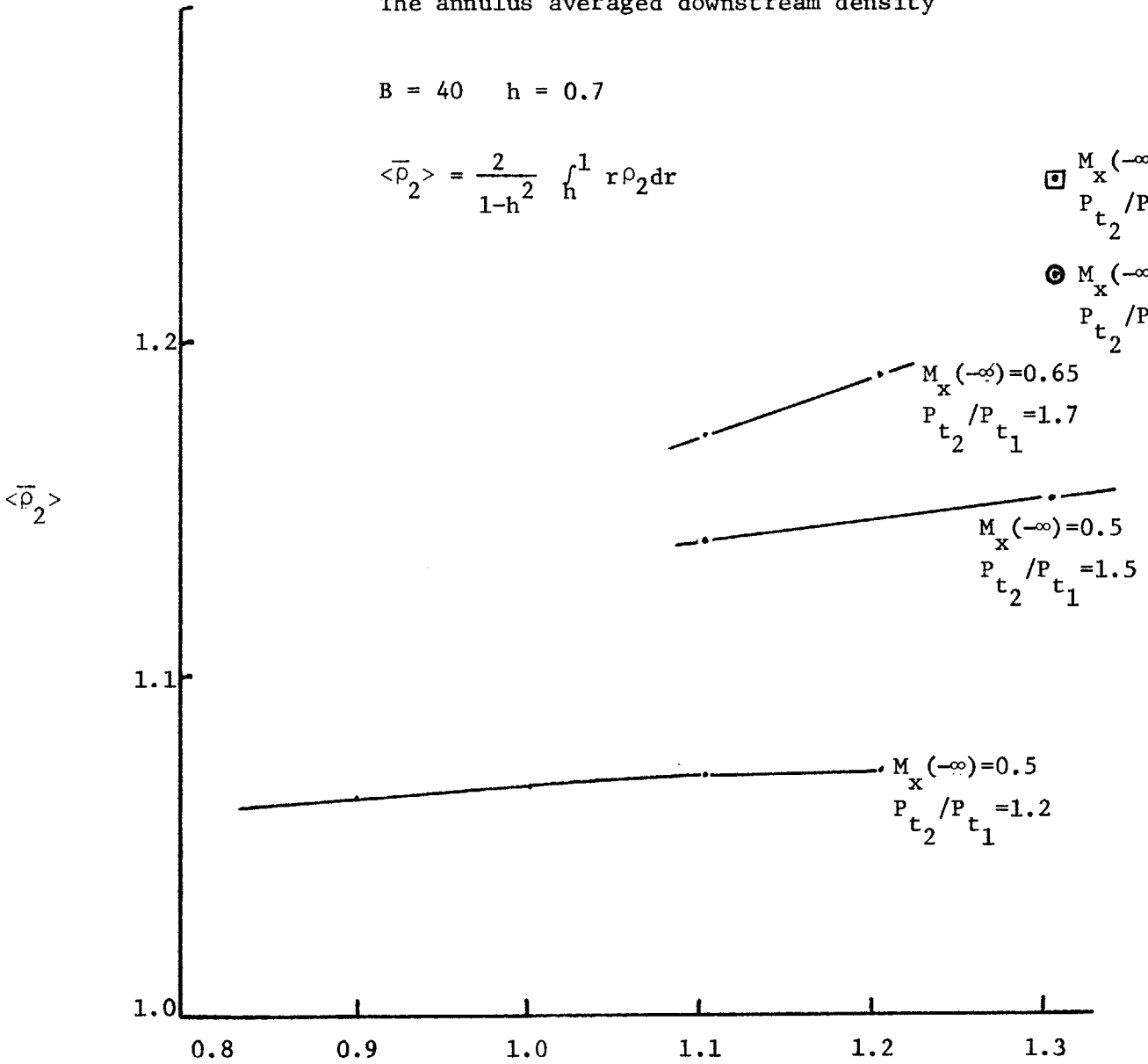
$M_x(-\infty)=0.5$
$P_{t_2}/P_{t_1}=1.2$

Figure 6

Figure 7

Figure 8

$B = 40$ $r_h/r_T = 0.8$
$M_{x_{-\infty}} = 0.65$ $M_T = 1.3$
$p_{t_2}/p_{t_1} = 1.8$

$\dfrac{\Gamma}{\bar{\Gamma}} = 1 - 0.2 \cos(\dfrac{r - r_h}{r_t - r_h} \pi)$

① Free Vortex

② Mean Soln

③ 3-D Soln

δ, Deflection Angle (degrees)

r/r_T
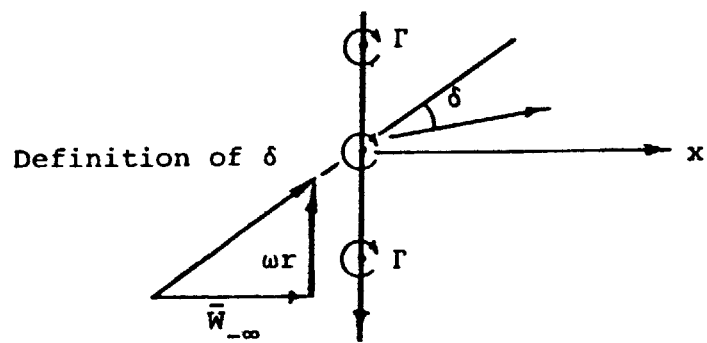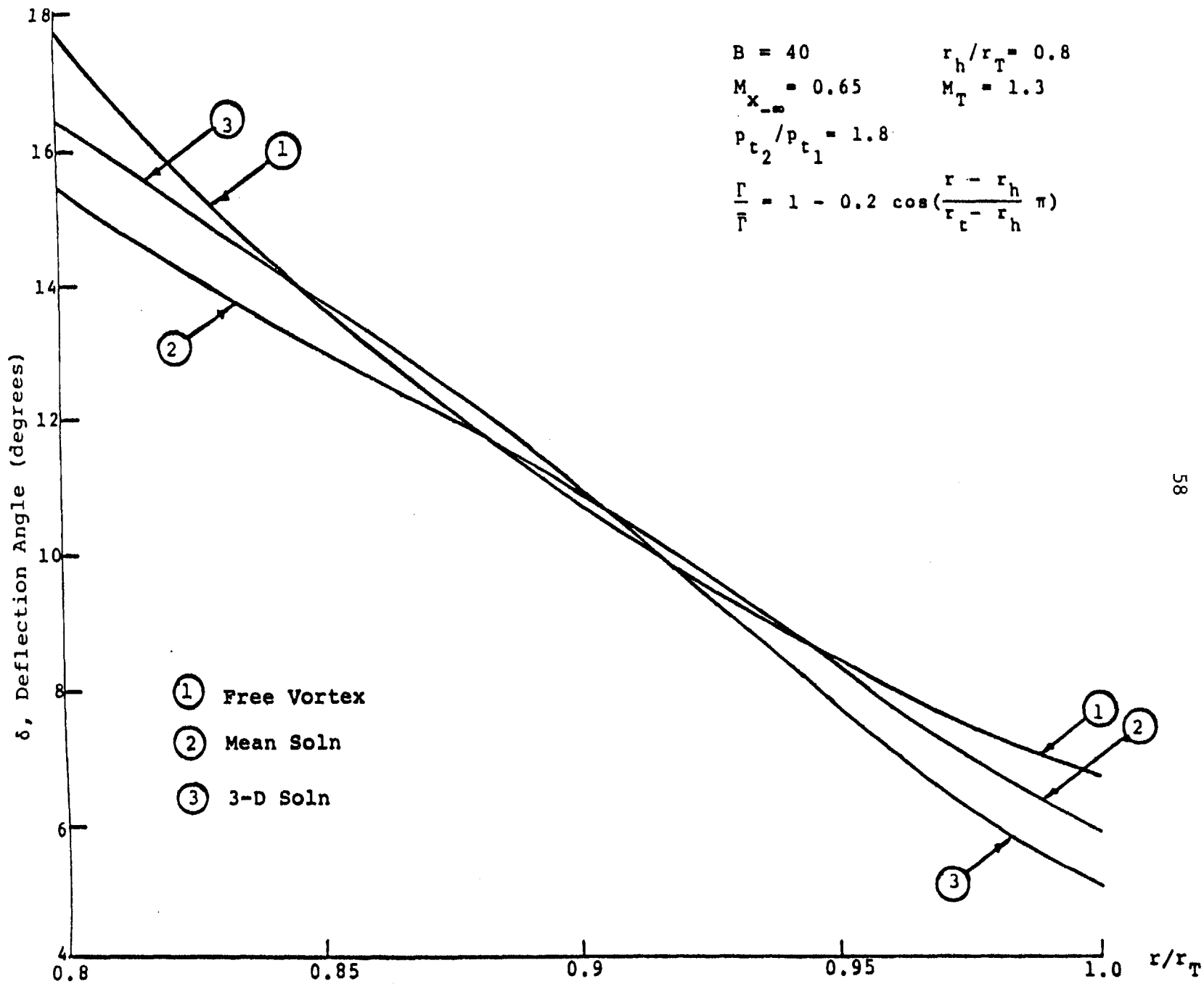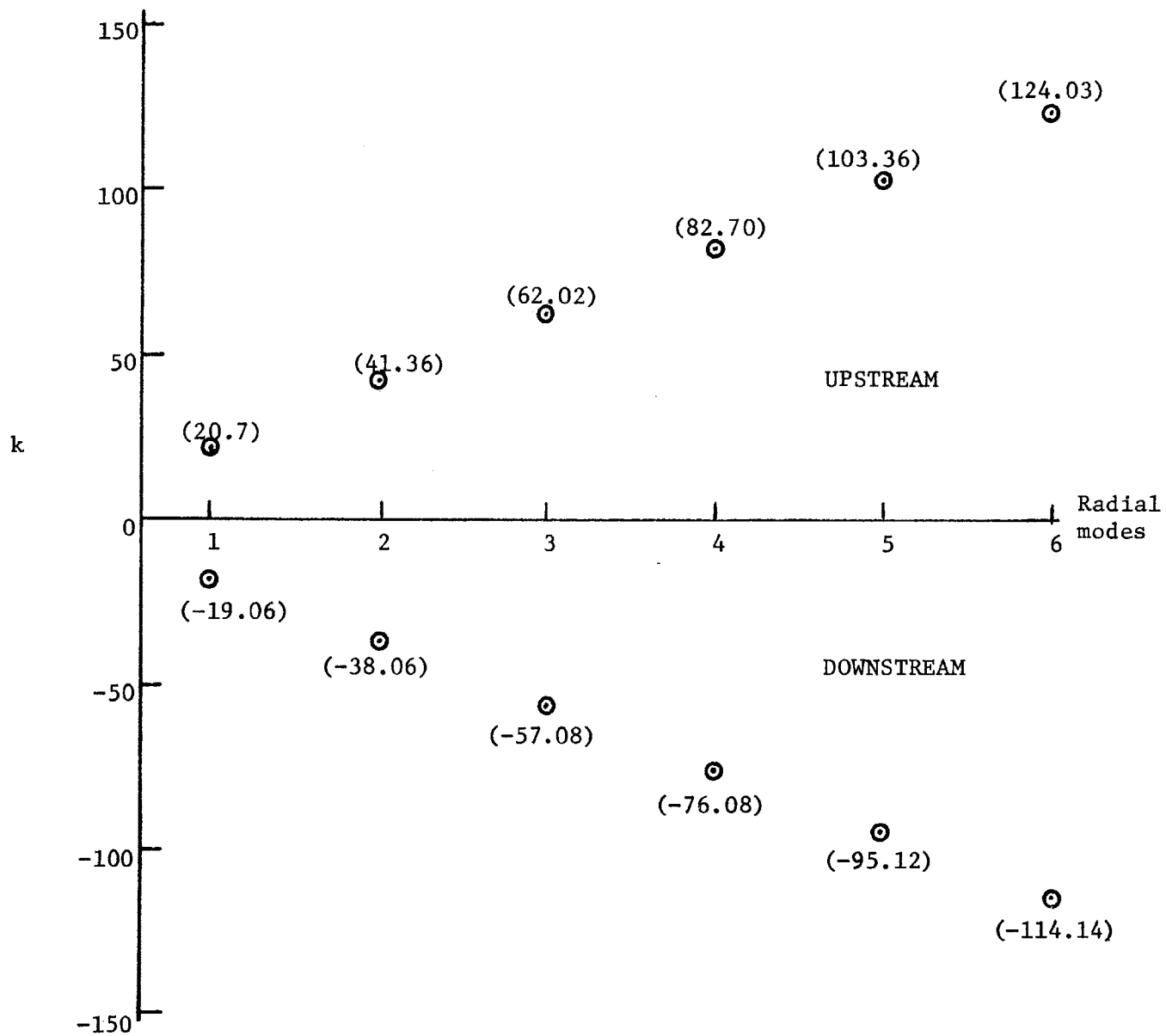
58
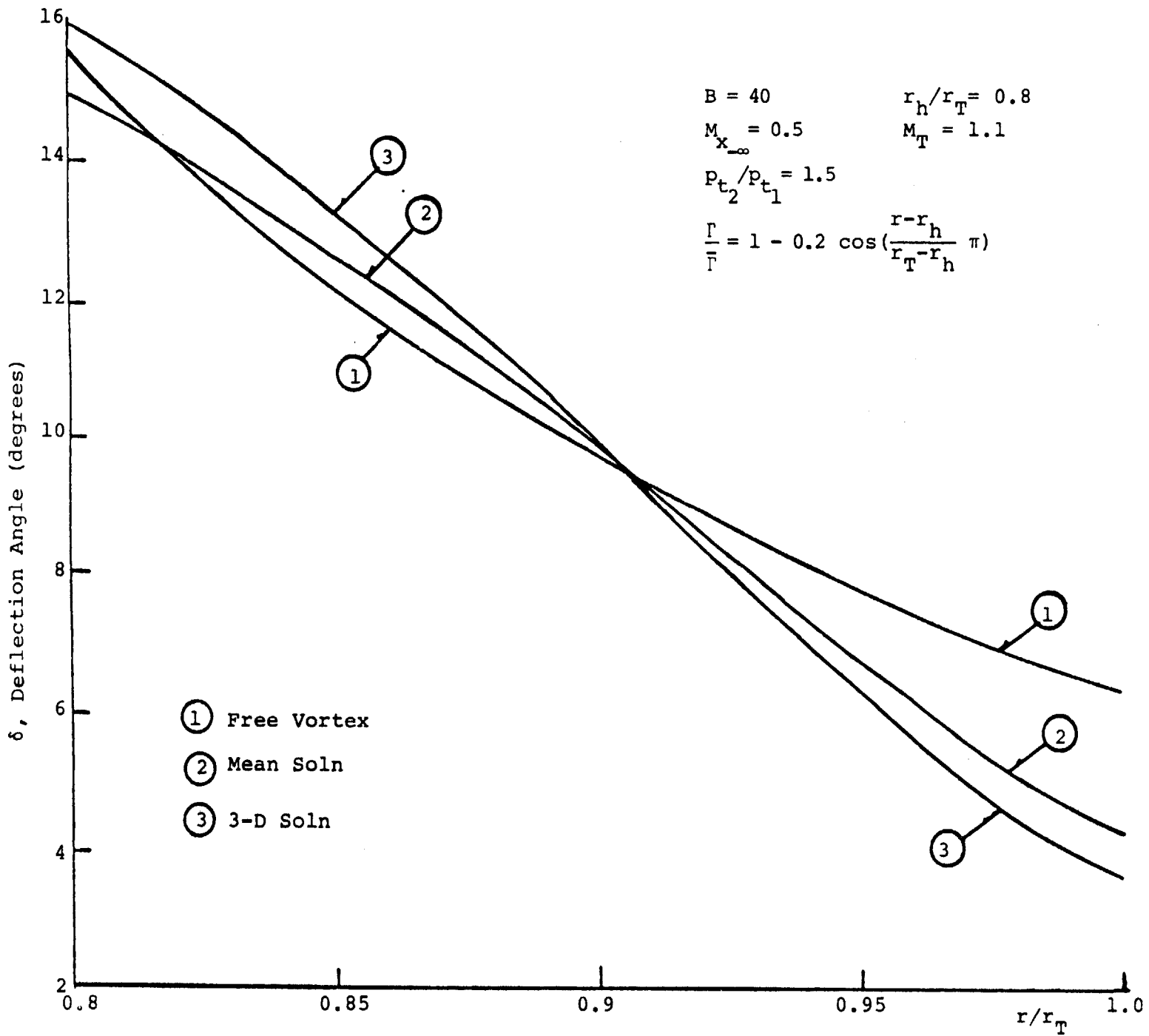
Figure 9

Axial decay rate of mean flow solution for the first 6 radial modes

Figure 10
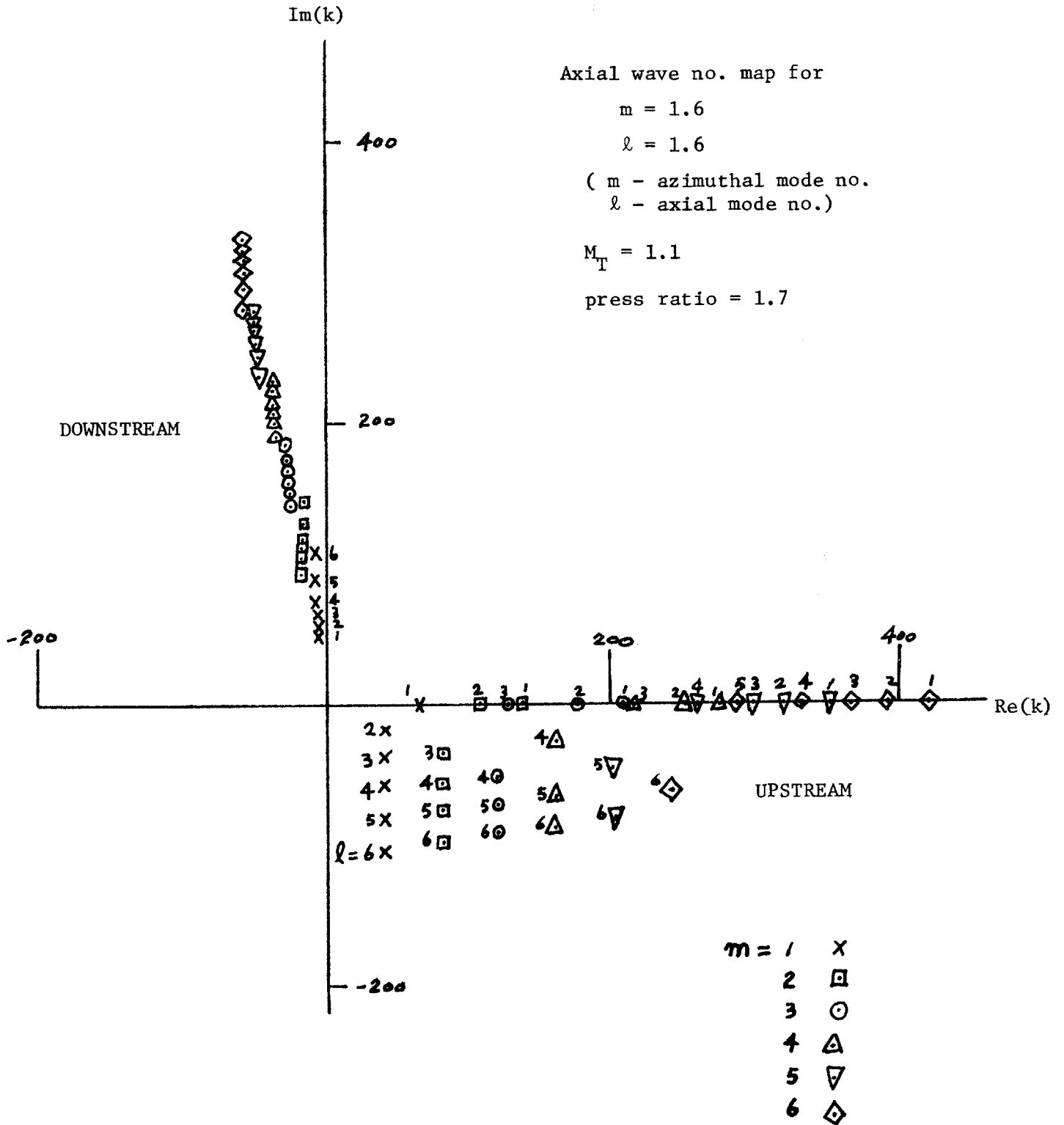
Figure 11

30 Deflection angle correction V_s M_T

$$\Gamma = 1 - 0.2 \cos \left(\frac{r-h}{1-h}\pi\right)$$

$B=40$ $h=0.8$ $P_{t_2}/P_{t_1} = 1.2$ $M_x(-\infty)=0.5$

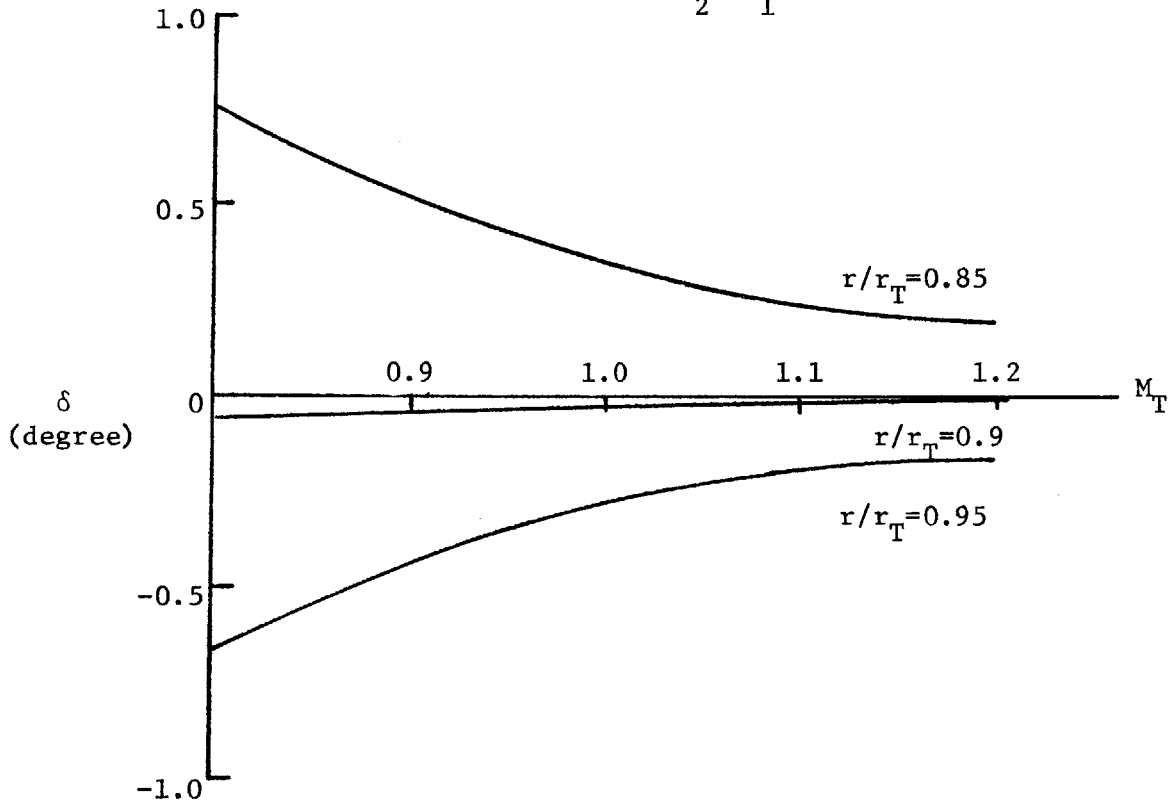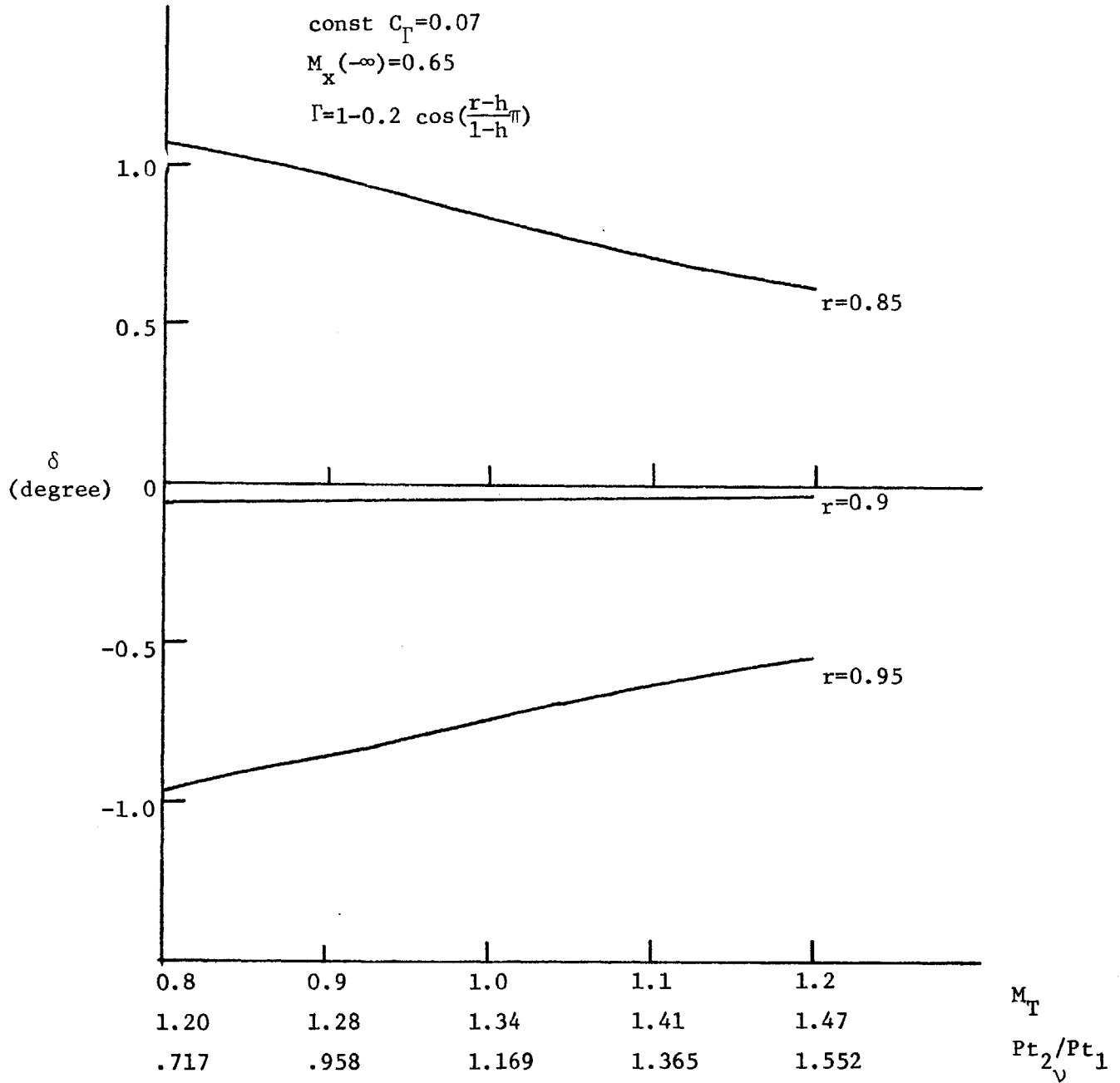

Figure 12

3D deflection angle correction Vs M_T

const $C_\Gamma = 0.07$

$M_x(-\infty) = 0.65$

$\Gamma = 1 - 0.2 \cos(\frac{r-h}{1-h}\pi)$

Figure 13