Studies on Quasisymmetric Functions

by

Darij Grinberg

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2016

Author . . . .
Department of Mathematics
June 3, 2016

Certified by . . . .
Alexander Postnikov
Professor
Thesis Supervisor

Accepted by . . . .
Alexei Borodin
Chairman, Department Committee on Graduate Theses

Studies on Quasisymmetric Functions

by

Darij Grinberg

Submitted to the Department of Mathematics
on June 3, 2016, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

In 1983, Ira Gessel introduced the ring of quasisymmetric functions (QSym), an extension of the ring of symmetric functions and nowadays one of the standard examples of a combinatorial Hopf algebra. In this thesis, I elucidate three aspects of its theory:

1) Gessel's P-partition enumerators are quasisymmetric functions that generalize, and share many properties of, the Schur functions; their Hopf-algebraic antipode satisfies a simple and explicit formula. Malvenuto and Reutenauer have generalized this formula to quasisymmetric functions "associated to a set of equalities and inequalities". I reformulate their generalization in the handier terminology of double posets, and present a new proof and an even further generalization in which a group acts on the double poset.

2) There is a second bialgebra structure on QSym, with its own "internal" comultiplication. I show how this bialgebra can be constructed using the Aguiar-Bergeron-Sottile universal property of QSym by extending the base ring; the same approach also constructs the so-called "Bernstein homomorphism", which makes any connected graded commutative Hopf algebra into a comodule over this second bialgebra QSym.

3) I prove a recursive formula for the "dual immaculate quasisymmetric functions" (a certain special case of P-partition enumerators) conjectured by Mike Zabrocki. The proof introduces a dendriform algebra structure on QSym.

Two further results appearing in this thesis, but not directly connected to QSym, are:

4) generalizations of Whitney's formula for the chromatic polynomial of a graph in terms of broken circuits. One of these generalizations involves weights assigned to the broken circuits. A formula for the chromatic symmetric function is also obtained.

5) a proof of a conjecture by Bergeron, Ceballos and Labbé on reduced-word graphs in Coxeter groups (joint work with Alexander Postnikov). Given an element of a Coxeter group, we can form a graph whose vertices are the reduced expressions of this element, and whose edges connect two reduced expressions which are "a single braid move apart". The simplest part of the conjecture says that this graph is bipartite; we show finer claims about its cycles.

Thesis Supervisor: Alexander Postnikov
Title: Professor

Acknowledgments

More has gone into this thesis than the bibliography suggests. I owe my initial mathematical upbringing to my parents – Natalia Grinberg and Viktor Palamodov –, who have taught me from my earliest years on and through turbulent times of migration and travel (which would probably have been a lot less turbulent if not for my presence).

I was happy to have two intellectual communities save me from boredom in my middle school years: the German mathematical-olympiad scene on the one hand, held together by a number of (mostly unpaid) teachers, academics and enthusiastic students; on the other, a wide and welcoming global mathematical community on the Internet (various geometry mailing lists, Art of Problem Solving, later MathOverflow). Special thanks are due to my friends from these times, whose intellectual curiosity guarded me from my laziness of mind: Peter Scholze, Daniel Harrer, Christian Sattler, Katharina Jochemko, Christopher Wulff, and many more; to the Euclidean geometers of the wide open web with whom I have made my first stints in research – Floor van Lamoen, Paul Yiu, in particular –; and to the volunteers (too many to mention) of the German and international mathematical contests and the QED and CdE societies.

My road to algebraic combinatorics was indirect and occasionally circuitous; I have had the fortune of a five-year undergraduate program at the LMU Munich (having been admitted in the last year before the LMU switched to the much more condensed and rigid bachelor/master system), which allowed me to roam several places and to pick up the kinds of intellectual souvenirs that tend to unexpectly prove useful in the middle of seemingly unrelated research. My trajectory has been influenced by collisions with Hans-Jürgen Schneider (from whom I learnt Hopf algebras), Katharina Jochemko (who suggested I do so), Damien Calaque (who mentored, if not formally advised, my diploma thesis about PBW theorems), Annette Huber-Klawitter (whose seminar made me learn about lambda-rings), Pavel Etingof (whose lecture notes on representation theory made me believe, for a while, that it would be my destiny), and probably various others.

It is hard to overstate the amount of intellectual enrichment I have received at the MIT since 2011; I merely observe that my frequency of MathOverflow questions has significantly decreased after my admission, since at MIT I almost always have had a real person to ask, a few offices away. I have learnt much from Alexander Postnikov, Pavel Etingof, George Lusztig, Richard Stanley, and I wish I had had the foresight to learn more from Nancy Lynch, Victor Kac, Sigurdur Helgason and Alexei Borodin while I was here. Special thanks are owed to Tom Roby and Keith Conrad for familiarizing me with American academia and life in the US. Miriam Farber, Sam Hopkins and Wuttisak Trongsiriwat have been great officemates (and occasional collaborators), and Alexander Postnikov a great advisor. Travis Scrimshaw, Nicolas Thiéry and Florent Hivert have tirelessly tutored me in the use of software (SageMath, Coq/ssreflect), some of which has yet to bear fruit. Collaborations with Victor Reiner, Tom Roby, Viviane Pons, James Borger, William Kuszmaul, Eric Neyman, and (again) Alex Postnikov have been interesting and rewarding. Thanks to Ira Gessel for joining my thesis committee, and to everyone who attended my defense. Finally, I want to thank the departmental administration (Barbara Peskin in particular) and the ISO for clearing most of the bureaucratic hurdles of immigration and work for me (all while creating none of their own).

Contents

Preface

Mountains will labour: what's
born? A ridiculous mouse!

Horace

This thesis gathers some of the work I have done within the last five years. It is, per se, not an integral whole, but four of its five chapters are united by a common thread (quasisymmetric and symmetric functions). Specifically, the thesis consists of the following:

Chapter 1: *Double posets and the antipode of* QSym. This chapter (which also appears on the arXiv as preprint `arXiv:1509.08355v2`) reproves and generalizes a formula for the antipode of certain quasisymmetric functions due to Malvenuto and Reutenauer. Only basic knowledge of quasisymmetric functions is assumed, and an even more basic understanding of Hopf algebras (reading [GriRei15, §1 and §5] should be more than sufficient).

Chapter 2: *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions.* This chapter (which has also been posted on the arXiv as preprint `arXiv:1410.0079v6` in a slightly modified edition[1]) constructs four new operations on the ring of quasisymmetric functions, two of which make it into a dendriform algebra. The operations are used to prove a conjecture of Mike Zabrocki that gives an

[1] More precisely, the preprint `arXiv:1410.0079v6` comes in two versions: a short ("default") one, and a detailed one (available as an ancillary file). Our Chapter 2 is a mix of the two.

alternative definition of the so-called "dual immaculate quasisymmetric functions" (an analogue of the Schur functions). This paper has been accepted for publication at the Canadian Journal of Mathematics; it is fairly light on prerequisites (again, the reader will find everything she needs in [GriRei15, §1 and §5], except for some further material from [GriRei15, §8] used in the last section).

Chapter 3: *The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality.* In this chapter (which has also been released on the arXiv as preprint `arXiv:1604.02969v2`), I use the Aguiar-Bergeron-Sottile universal property of the ring of quasisymmetric functions to construct (what Hazewinkel calls) the Bernstein homomorphism (actually, a generalization thereof), which generalizes the internal comultiplication on quasisymmetric functions. The reader is, again, expected to have a good understanding of Hopf algebras and quasisymmetric functions ([GriRei15, §1, §5 and §7]); prior familiarity with the internal comultiplication is not required.

Chapter 4: *A note on non-broken-circuit sets and the chromatic polynomial.* This chapter (also appearing on the arXiv as preprint `arXiv:1604.03063v1`) explores several generalizations of Whitney's formula for the chromatic polynomial of a graph in terms of subsets containing no broken circuits. In particular, the graph is replaced by the matroid, the chromatic polynomial by the chromatic symmetric function (although not both at once!), and the subsets containing no broken circuits are replaced by subsets avoiding a certain pre-selected set of broken circuits). This note has an expositive character, even if the generalizations are new (to my knowledge); in particular, I believe it to be readable with no preknowledge whatsoever in algebraic combinatorics.[2]

[2]Remarkably, the main lemma in this chapter (Lemma 4.2.7) is proven using a bijection Φ highly reminiscent of the involution T in the proof of Theorem 1.4.2 in Chapter 1. (Actually, Φ can be extended to an involution, thus making the analogy even more palpable.) This suggests a connection

Chapter 5: *Proof of a conjecture of Bergeron, Ceballos and Labbé* (joint work with Alexander Postnikov). This chapter (which also appears on the arXiv as preprint `arXiv:1603.03138v1`) proves a conjecture about reduced words of elements in Coxeter groups (more precisely, about cycles in reduced word graphs). Unlike the previous four chapters, this one has no direct connection with combinatorial Hopf algebras and symmetric functions; the reader is assumed to be familiar with basic combinatorial properties of Coxeter groups [Lusztig14].

Various other work done during my stay at MIT has not found its way into this thesis, including the two-part paper, joint with Tom Roby, on birational rowmotion [GriRob15]; the study of dual stable Grothendieck polynomials, joint with Pavel Galashin and Gaku Liu [GaGrLi16]; and some minor results that have become exercises in [GriRei15].

between the two results and possibly even a common generalization; I have not, however, been able to take hold of such a generalization so far.

Chapter 1

Double posets and the antipode of QSym

Abstract

We assign a quasisymmetric function to every double poset (that is, every finite set endowed with two partial orders) and any weight function on its ground set. This generalizes well-known objects such as monomial and fundamental quasisymmetric functions, (skew) Schur functions, dual immaculate functions, and quasisymmetric (P, ω)-partition enumerators. We then prove a formula for the antipode of this function that holds under certain conditions (which are satisfied when the second order of the double poset is total, but also in some other cases); this restates (in a way that to us seems more natural) a result by Malvenuto and Reutenauer, but our proof is new and self-contained. We generalize it further to an even more comprehensive setting, where a group acts on the double poset by automorphisms.

1.1 Introduction

Double posets and **E**-partitions (for **E** a double poset) have been introduced by Claudia Malvenuto and Christophe Reutenauer [MalReu09] in order to construct a combinatorial Hopf algebra which harbors a noticeable amount of structure, including an analogue of the Littlewood-Richardson rule and a lift of the internal product operation of the Malvenuto-Reutenauer Hopf algebra of permutations. In this note, we shall employ these same notions to restate in a simpler form, and reprove in a more elementary fashion, a formula for the antipode in the Hopf algebra QSym of quasisymmetric functions due to (the same) Malvenuto and Reutenauer (generalizing an earlier result by Gessel), and extend it further to a case in which a group acts on the double poset.

In the present version of the paper, some (classical and/or straightforward) proofs are missing or sketched. A more detailed version exists, in which at least a few of these proofs are elaborated on more[1].

Acknowledgments

Katharina Jochemko's work [Joch13] provoked this research. I learnt a lot about QSym from Victor Reiner.

1.2 Quasisymmetric functions

Let us first briefly introduce the notations that will be used in the following.

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *composition* means a finite sequence of positive integers. We let Comp be the set of all compositions. For $n \in \mathbb{N}$, a *composition of* n means a composition whose entries sum to n (that is, a composition $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_k = n$).

Let $\mathbf{k}$ be an arbitrary commutative ring. We shall keep $\mathbf{k}$ fixed throughout this paper. We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in infinitely many (commuting) indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. A *monomial* shall always mean a monomial (without coefficients) in the variables $x_1, x_2, x_3, \ldots$. [2]

---

[1] It can be downloaded from `http://web.mit.edu/~darij/www/algebra/dp-abstr-long.pdf`

[2] For the sake of completeness, let us give a detailed definition of monomials and of the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. (This definition also appears in Section 2.2 of this thesis.)

Let $x_1, x_2, x_3, \ldots$ be countably many distinct symbols. We let Mon be the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (written multiplicatively); it consists of elements of the form $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ for finitely supported $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^\infty$ (where "finitely supported" means that all but finitely many positive integers i satisfy $a_i = 0$). A *monomial* will mean an element of Mon. Thus, a monomial is a combinatorial object, independent of $\mathbf{k}$; it does not carry a coefficient.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. By abuse of notation, we shall identify every monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \in \text{Mon}$ with the corresponding element $x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \cdot \cdots$ of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ when

Inside the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a subalgebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ consisting of the *bounded-degree* formal power series; these are the power series f for which there exists a $d \in \mathbb{N}$ such that no monomial of degree $> d$ appears in f [3]. This $\mathbf{k}$-subalgebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ becomes a topological $\mathbf{k}$-algebra, by inheriting the topology from $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent*[4] if they have the forms $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_\ell}^{a_\ell}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_\ell}^{a_\ell}$ for two strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ and $(j_1 < j_2 < \cdots < j_\ell)$ of positive integers and one (common) sequence $(a_1, a_2, \ldots, a_\ell)$ of positive integers.[5] A power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if every two pack-equivalent monomials have equal coefficients in front of them in f. It is easy to see that the quasisymmetric power series form a $\mathbf{k}$-subalgebra of

---

necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of $\mathbf{k}$). (To be very pedantic, this identification is slightly dangerous, because it can happen that two distinct monomials in Mon get identified with two identical elements of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. However, this can only happen when the ring $\mathbf{k}$ is trivial, and even then it is not a real problem unless we infer the equality of monomials from the equality of their counterparts in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, which we are not going to do.)

We furthermore endow the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ with the following topology (as in [GriRei15, Section 2.6]):

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients (indexed by the set Mon). More precisely, we have a $\mathbf{k}$-module isomorphism

$$\prod_{\mathfrak{m} \in \text{Mon}} \mathbf{k} \to \mathbf{k}[[x_1, x_2, x_3, \ldots]], \qquad (\lambda_{\mathfrak{m}})_{\mathfrak{m} \in \text{Mon}} \mapsto \sum_{\mathfrak{m} \in \text{Mon}} \lambda_{\mathfrak{m}} \mathfrak{m}.$$

We use this isomorphism to transport the product topology on $\prod_{\mathfrak{m} \in \text{Mon}} \mathbf{k}$ to $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. The resulting topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ turns $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ into a polynomial $\mathbf{k}$-algebra; this is the topology that we will be using whenever we make statements about convergence in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ or write down infinite sums of power series. A sequence $(a_n)_{n \in \mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

Note that this topological $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is **not** the completion of $\mathbf{k}[x_1, x_2, x_3, \ldots]$ with respect to the standard grading (in which all x_i have degree 1). (They are distinct even as sets.)

[3] The *degree* of a monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ is defined to be the nonnegative integer $a_1 + a_2 + a_3 + \cdots$. A monomial $\mathfrak{m}$ is said to *appear* in a power series $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ if and only if the coefficient of $\mathfrak{m}$ in f is nonzero.

[4] Pack-equivalence and the related notions of packed combinatorial objects that we will encounter below originate in work of Hivert, Novelli and Thibon [NovThi05]. Simple as they are, they are of great help in dealing with quasisymmetric functions.

[5] For instance, $x_2^2 x_3 x_4^2$ is pack-equivalent to $x_1^2 x_4 x_8^2$ but not to $x_2 x_3^2 x_4^2$.

$\mathbf{k}[[x_1, x_2, x_3, \ldots]]$; but usually one is interested in the set of quasisymmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This latter set is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$, and is known as the $\mathbf{k}$-*algebra of quasisymmetric functions over* $\mathbf{k}$. It is denoted by QSym. It is clear that symmetric functions (in the usual sense of this word in combinatorics – so, really, symmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$) form a $\mathbf{k}$-subalgebra of QSym. The quasisymmetric functions have a rich theory which is related to, and often sheds new light on, the classical theory of symmetric functions; expositions can be found in [Stan99, §§7.19, 7.23] and [GriRei15, §§5-6] and other sources.

As a $\mathbf{k}$-module, QSym has a basis $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ indexed by all compositions, where the quasisymmetric function M_α for a given composition α is defined as follows: Writing α as $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we set

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}}} \mathfrak{m}$$

(where the i_k in the first sum are positive integers). This basis $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is known as the *monomial basis* of QSym, and is the simplest to define among many. (We shall briefly encounter another basis in Example 1.3.6.)

The $\mathbf{k}$-algebra QSym can be endowed with a structure of a $\mathbf{k}$-coalgebra which, combined with its $\mathbf{k}$-algebra structure, turns it into a Hopf algebra. We refer to the literature both for the theory of coalgebras and Hopf algebras (see [Montg93], [GriRei15, §1], [Manchon04, §1-§2], [Abe77], [Sweed69], [DNR01] or [Fresse14, Chapter 7]) and for a deeper study of the Hopf algebra QSym (see [Malve93], [HaGuKi10, Chp. 6] or [GriRei15, §5]); in this note we shall need but the very basics of this structure, and so it is only them that we introduce.

We define a $\mathbf{k}$-linear map $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ (here and in the following, all tensor products are over $\mathbf{k}$ by default) by requiring that

$$\Delta\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = \sum_{k=0}^{\ell} M_{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \otimes M_{(\alpha_{k+1},\alpha_{k+2},\ldots,\alpha_\ell)} \qquad \text{for every } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \operatorname{Comp} \tag{1.1}$$

[6]. We further define a $\mathbf{k}$-linear map $\varepsilon : \operatorname{QSym} \to \mathbf{k}$ by requiring that

$$\varepsilon\left(M_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}\right) = \delta_{\ell,0} \qquad \text{for every } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \in \operatorname{Comp}$$

[7]. (Equivalently, ε sends every power series $f \in \operatorname{QSym}$ to the result $f(0,0,0,\ldots)$ of substituting zeroes for the variables $x_1, x_2, x_3, \ldots$ in f. The map Δ can also be described in such terms, but with greater difficulty [GriRei15, (5.3)].) It is well-known that these maps Δ and ε make the three diagrams

$$\begin{CD}
\operatorname{QSym} @>{\Delta}>> \operatorname{QSym} \otimes \operatorname{QSym} \\
@V{\Delta}VV @VV{\Delta \otimes \operatorname{id}}V \\
\operatorname{QSym} \otimes \operatorname{QSym} @>>{\operatorname{id} \otimes \Delta}> \operatorname{QSym} \otimes \operatorname{QSym} \otimes \operatorname{QSym}
\end{CD} ,$$

$$\begin{array}{ccc}
\operatorname{QSym} & \xrightarrow{\Delta} & \operatorname{QSym} \otimes \operatorname{QSym} \\
& \underset{\cong}{\searrow} & \downarrow {\scriptstyle \varepsilon \otimes \operatorname{id}} \\
& & \mathbf{k} \otimes \operatorname{QSym}
\end{array} , \qquad
\begin{array}{ccc}
\operatorname{QSym} & \xrightarrow{\Delta} & \operatorname{QSym} \otimes \operatorname{QSym} \\
& \underset{\cong}{\searrow} & \downarrow {\scriptstyle \operatorname{id} \otimes \varepsilon} \\
& & \operatorname{QSym} \otimes \mathbf{k}
\end{array}$$

(where the $\cong$ arrows are the canonical isomorphisms) commutative, and so $(\operatorname{QSym}, \Delta, \varepsilon)$ is what is commonly called a $\mathbf{k}$-*coalgebra*. Furthermore, Δ and ε are $\mathbf{k}$-algebra homomorphisms, which is what makes this $\mathbf{k}$-coalgebra QSym into a $\mathbf{k}$-*bialgebra*. Finally, let $m : \operatorname{QSym} \otimes \operatorname{QSym} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map sending every pure tensor $a \otimes b$ to ab, and let $u : \mathbf{k} \to \operatorname{QSym}$ be the $\mathbf{k}$-linear map sending $1 \in \mathbf{k}$ to $1 \in \operatorname{QSym}$. Then,

---

[6] This definition relies on the fact that $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym.

[7] Here, $\delta_{u,v}$ is defined to be $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$ whenever u and v are two objects.

there exists a unique $\mathbf{k}$-linear map $S : \mathrm{QSym} \to \mathrm{QSym}$ making the diagram

$$\begin{array}{ccccccc} & & \mathrm{QSym}\otimes\mathrm{QSym} & \xrightarrow{S\otimes\mathrm{id}} & \mathrm{QSym}\otimes\mathrm{QSym} & & \\ & \nearrow^{\Delta} & & & & \searrow^{m} & \\ \mathrm{QSym} & & \xrightarrow{\varepsilon} & \mathbf{k} & \xrightarrow{u} & & \mathrm{QSym} \\ & \searrow_{\Delta} & & & & \nearrow_{m} & \\ & & \mathrm{QSym}\otimes\mathrm{QSym} & \xrightarrow{\mathrm{id}\otimes S} & \mathrm{QSym}\otimes\mathrm{QSym} & & \end{array} \tag{1.2}$$

commutative. This map S is known as the *antipode* of QSym. It is known to be an involution and an algebra automorphism of QSym, and its action on the various quasisymmetric functions defined combinatorially is the main topic of this note. The existence of the antipode S makes QSym into a *Hopf algebra*.

1.3 Double posets

Next, we shall introduce the notion of a double poset, following Malvenuto and Reutenauer [MalReu09].

Definition 1.3.1. (a) We shall encode posets as pairs $(P, <)$, where P is a set and $<$ is a strict partial order relation (i.e., an irreflexive, transitive and antisymmetric binary relation) on the set P; this relation $<$ will be regarded as the smaller relation of the poset. (All binary relations will be written in infix notation: i.e., we write "$a < b$" for "a is related to b by the relation $<$".)

(b) If $<$ is a strict partial order relation on a set P, and if a and b are two elements of P, then we say that a and b are $<$-*comparable* if we have either $a < b$ or $a = b$ or $b < a$. A strict partial order relation $<$ on a set P is said to be a *total order* if and only if every two elements of P are $<$-comparable.

(c) If $<$ is a strict partial order relation on a set P, and if a and b are two elements of P, then we say that a is $<$-*covered by* b if we have $a < b$ and there exists no $c \in P$ satisfying $a < c < b$. (For instance, if $<$ is the standard smaller relation on $\mathbb{Z}$, then each $i \in \mathbb{Z}$ is $<$-covered by $i + 1$.)

(d) A *double poset* is defined as a triple $(E, <_1, <_2)$ where E is a finite set and $<_1$ and $<_2$ are two strict partial order relations on E.

(e) A double poset $(E, <_1, <_2)$ is said to be *special* if the relation $<_2$ is a total order.

(f) A double poset $(E, <_1, <_2)$ is said to be *semispecial* if every two $<_1$-comparable elements of E are $<_2$-comparable.

(g) A double poset $(E, <_1, <_2)$ is said to be *tertispecial* if it satisfies the following condition: If a and b are two elements of E such that a is $<_1$-covered by b, then a and b are $<_2$-comparable.

(h) If $<$ is a binary relation on a set P, then the *opposite relation* of $<$ is defined to be the binary relation $>$ on the set P which is defined as follows: For any $e \in P$ and $f \in P$, we have $e > f$ if and only if $f < e$. Notice that if $<$ is a strict partial order relation, then so is the opposite relation $>$ of $<$.

Clearly, every special double poset is semispecial, and every semispecial double poset is tertispecial.[8]

Definition 1.3.2. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, then an $\mathbf{E}$-*partition* shall mean a map $\phi : E \to \{1, 2, 3, \ldots\}$ such that:

- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$;
- every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

[8]The notions of a double poset and of a special double poset come from [MalReu09]. The notion of a "tertispecial double poset" (Dog Latin for "slightly less special than semispecial") appears to be new and arguably sounds artificial, but is the most suitable setting for some of the results below (and appears in nature, beyond the particular case of special double posets – see Example 1.3.3). We shall not use semispecial double posets in the following; they were only introduced as a middle ground between special and tertispecial double posets with a less daunting definition.

Example 1.3.3. The notion of an **E**-partition (which was inspired by the earlier notions of P-partitions and (P, ω)-partitions as studied by Gessel and Stanley[9]) generalizes various well-known combinatorial concepts. For example:

- If $<_2$ is the same order as $<_1$ (or any extension of this order), then **E**-partitions are weakly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

- If $<_2$ is the opposite relation of $<_1$ (or any extension of this opposite relation), then **E**-partitions are strictly increasing maps from the poset $(E, <_1)$ to the totally ordered set $\{1, 2, 3, \ldots\}$.

For a more interesting example, let $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be two partitions such that $\mu \subseteq \lambda$. (See [GriRei15, §2] for the notations we are using here.) The skew Young diagram $Y(\lambda/\mu)$ is then defined as the set of all $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfying $\mu_i < j \leq \lambda_i$. On this set $Y(\lambda/\mu)$, we define two partial order relations $<_1$ and $<_2$ by

$$(i, j) <_1 (i', j') \iff (i \leq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j'))$$

and

$$(i, j) <_2 (i', j') \iff (i \geq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j')) .$$

The resulting double poset $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$ has the property that the $\mathbf{Y}(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ. (Again, see [GriRei15, §2] for the meaning of these words.)

This double poset $\mathbf{Y}(\lambda/\mu)$ is not special (in general), but it is tertispecial. (Indeed, if a and b are two elements of $Y(\lambda/\mu)$ such that a is $<_1$-covered by b, then a is either the left neighbor of b or the top neighbor of b, and thus we have either $a <_2 b$ (in the former case) or $b <_2 a$ (in the latter case).) Some authors prefer to use a special double poset instead, which is defined as follows: We define a total order $<_h$ on $Y(\lambda/\mu)$ by

$$(i,j) <_h (i',j') \iff (i > i' \text{ or } (i = i' \text{ and } j < j')).$$

Then, $\mathbf{Y}_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$ is a special double poset, and the $\mathbf{Y}_h(\lambda/\mu)$-partitions are precisely the semistandard tableaux of shape λ/μ.

We now assign a certain formal power series to every double poset:

Definition 1.3.4. If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1,2,3,\ldots\}$ is a map, then we define a power series $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}, \qquad \text{where } \mathbf{x}_{\pi,w} = \prod_{e \in E} x_{\pi(e)}^{w(e)}.$$

The following fact is easy to see (but will be reproven below):

Proposition 1.3.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset, and $w : E \to \{1,2,3,\ldots\}$ be a map. Then, $\Gamma(\mathbf{E}, w) \in \operatorname{QSym}$.

Example 1.3.6. The power series $\Gamma(\mathbf{E}, w)$ generalize various well-known quasisymmetric functions.

(a) If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1,2,3,\ldots\}$ is the constant function sending everything to 1, then $\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_\pi$, where $\mathbf{x}_\pi = \prod_{e \in E} x_{\pi(e)}$. We shall denote this power series $\Gamma(\mathbf{E}, w)$ by $\Gamma(\mathbf{E})$; it is exactly what has been called $\Gamma(\mathbf{E})$ in [MalReu09, §2.2]. All results proven below for $\Gamma(\mathbf{E}, w)$ can be applied to $\Gamma(\mathbf{E})$, yielding simpler (but less general) statements.

[9] See [Gessel15] for the history of these notions, and see [Gessel84], [Stan71], [Stan11, §3.15] and [Stan99, §7.19] for some of their theory. Mind that these sources use different and sometimes incompatible notations – e.g., the P-partitions of [Stan11, §3.15] and [Gessel15] differ from those of [Gessel84] by a sign reversal.

(b) If $E = \{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$, if $<_1$ is the usual total order inherited from $\mathbb{Z}$, and if $<_2$ is the opposite relation of $<_1$, then the special double poset $\mathbf{E} = (E, <_1, <_2)$ satisfies $\Gamma(\mathbf{E}, w) = M_\alpha$, where α is the composition $(w(1), w(2), \ldots, w(\ell))$. Thus, the elements of the monomial basis $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ are special cases of the functions $\Gamma(\mathbf{E}, w)$.

(c) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n. Let $D(\alpha)$ be the set $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}$. Let E be the set $\{1, 2, \ldots, n\}$, and let $<_1$ be the total order inherited on E from $\mathbb{Z}$. Let $<_2$ be some partial order on E with the property that

$$i + 1 <_2 i \qquad \text{for every } i \in D(\alpha)$$

and

$$i <_2 i + 1 \qquad \text{for every } i \in \{1, 2, \ldots, n-1\} \setminus D(\alpha).$$

(There are several choices for such an order; in particular, we can find one which is a total order.) Then,

$$\begin{aligned} \Gamma((E, <_1, <_2)) &= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\beta \text{ is a composition of } n;\ D(\beta) \supseteq D(\alpha)} M_\beta. \end{aligned}$$

This power series is known as the α-th *fundamental quasisymmetric function*, usually called F_α (in [BBSSZ13a, §2.4] and Section 2.2 of this thesis) or L_α (in [Stan99, §7.19] or [GriRei15, Def. 5.15]).

(d) Let $\mathbf{E}$ be one of the two double posets $\mathbf{Y}(\lambda/\mu)$ and $\mathbf{Y}_h(\lambda/\mu)$ defined as in Example 1.3.3 for two partitions μ and λ. Then, $\Gamma(\mathbf{E})$ is the skew Schur function $s_{\lambda/\mu}$.

(e) Similarly, *dual immaculate functions* as defined in [BBSSZ13a, §3.7] can be realized as $\Gamma(\mathbf{E})$ for conveniently chosen $\mathbf{E}$ (see Proposition 2.4.4), which helped the author to prove one of their properties (see Chapter 2 of this thesis). (The **E**-partitions here are the so-called *immaculate tableaux*.)

(f) When the relation $<_2$ of a double poset $\mathbf{E} = (E, <_1, <_2)$ is a total order (i.e., when the double poset $\mathbf{E}$ is special), the $\mathbf{E}$-partitions are precisely the reverse (P, ω)-partitions (for $P = (E, <_1)$ and ω being a labelling of P dictated by $<_2$) in the terminology of [Stan99, §7.19], and the power series $\Gamma(\mathbf{E})$ is the $K_{P,\omega}$ of [Stan99, §7.19]. This can also be rephrased using the notations of [GriRei15, §5.2]: When the relation $<_2$ of a double poset $\mathbf{E} = (E, <_1, <_2)$ is a total order, we can relabel the elements of E by the integers $1, 2, \ldots, n$ in such a way that $1 <_2 2 <_2 \cdots <_2 n$; then, the $\mathbf{E}$-partitions are the P-partitions in the terminology of [GriRei15, Def. 5.12], where P is the labelled poset $(E, <_1)$; and furthermore, our $\Gamma(\mathbf{E})$ is the $F_P(\mathbf{x})$ of [GriRei15, Def. 5.12]. Conversely, if P is a labelled poset, then the $F_P(\mathbf{x})$ of [GriRei15, Def. 5.12] is our $\Gamma((P, <_P, <_{\mathbb{Z}}))$.

1.4 The antipode theorem

We now come to the main results of this note. We first state a theorem and a corollary which are not new, but will be reproven in a more self-contained way which allows them to take their (well-deserved) place as fundamental results rather than afterthoughts in the theory of QSym.

Definition 1.4.1. We let S denote the antipode of QSym.

Theorem 1.4.2. Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$, where $>_1$ denotes the opposite relation of $<_1$.

Corollary 1.4.3. Let $(E, <_1, <_2)$ be a tertispecial double poset. Then, $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$, where $>_1$ denotes the opposite relation of $<_1$.

We shall give examples for consequences of these facts shortly (Example 1.4.7), but let us first explain where they have already appeared. Corollary 1.4.3 is equivalent to [GriRei15, Corollary 5.27][10] (a result found by Malvenuto and Reutenauer, as well as by Ehrenborg in an equivalent form). Theorem 1.4.2 is equivalent to Malvenuto's and Reutenauer's [MalReu98, Theorem 3.1][11]. We nevertheless believe that our versions of these facts are more natural and simpler than the ones appearing in existing literature[12], and if not, then at least their proofs below are more in the nature of things.

To these known results, we add another, which seems to be unknown so far (probably because it is far harder to state in the terminologies of (P, ω)-partitions or equality-and-inequality conditions appearing in literature). First, we need to introduce some notation:

[10] It is easiest to derive [GriRei15, Corollary 5.27] from our Corollary 1.4.3, as this only requires setting $\mathbf{E} = (P, <_P, <_{\mathbb{Z}})$ (this is a special double poset, thus in particular a tertispecial one) and noticing that $\Gamma((P, <_P, <_{\mathbb{Z}})) = F_P(\mathbf{x})$ and $\Gamma((P, >_P, <_{\mathbb{Z}})) = F_{P^{\mathrm{opp}}}(\mathbf{x})$, where all unexplained notations are defined in [GriRei15, Chp. 5]. But one can also proceed in the opposite direction.

[11] This equivalence requires a bit of work to set up. To derive [MalReu98, Theorem 3.1] from our Theorem 1.4.2, it is enough to contract all undirected edges in G, denoting the vertex set of the new graph by E, and then define two order relations $<_1$ and $<_2$ on E by

$$(a <_1 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G)$$

and

$$(a <_2 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G').$$

The map w sends every $e \in E$ to the number of vertices of G that became e when the edges were contracted. To show that the resulting double poset $(E, <_1, <_2)$ is tertispecial, we must notice that if a is $<_1$-covered by b, then G had an edge from one of the vertices that became a to one of the vertices that became b. The "x_i's in X satisfying a set of conditions" (in the language of [MalReu98, Section 3]) are then in 1-to-1 correspondence with $(E, <_1, <_2)$-partitions (at least when $X = \{1, 2, 3, \ldots\}$); this is not immediately obvious but not hard to check either (the acyclicity of G and G' is used in the proof). As a result, [MalReu98, Theorem 3.1] follows from Theorem 1.4.2 above. With some harder work, one can conversely derive our Theorem 1.4.2 from [MalReu98, Theorem 3.1].

[12] That said, we would not be surprised if Malvenuto and Reutenauer are aware of them and just have not published them; after all, they have discovered both the original version of Theorem 1.4.2 in [MalReu98] and the notion of double posets in [MalReu09].

Definition 1.4.4. Let G be a group, and let E be a G-set.

(a) Let $<$ be a strict partial order relation on E. We say that G *preserves the relation* $<$ if the following holds: For every $g \in G$, $a \in E$ and $b \in E$ satisfying $a < b$, we have $ga < gb$.

(b) Let $w : E \to \{1, 2, 3, \ldots\}$. We say that G *preserves* w if every $g \in G$ and $e \in E$ satisfy $w(ge) = w(e)$.

(c) Let $g \in G$. Assume that the set E is finite. We say that g is E-*even* if the action of g on E (that is, the permutation of E that sends every $e \in E$ to ge) is an even permutation of E.

(d) If X is any set, then the set X^E of all maps $E \to X$ becomes a G-set in the following way: For any $\pi \in X^E$ and $g \in G$, we define the element $g\pi \in X^E$ to be the map sending each $e \in E$ to $\pi(g^{-1}e)$.

(e) Let F be a further G-set. Assume that the set E is finite. An element $\pi \in F$ is said to be E-*coeven* if every $g \in G$ satisfying $g\pi = \pi$ is E-even. A G-orbit O on F is said to be E-*coeven* if all elements of O are E-coeven.

Before we come to the promised result, let us state a simple fact:

Lemma 1.4.5. Let G be a group. Let F and E be G-sets such that E is finite. Let O be a G-orbit on F. Then, O is E-coeven if and only if at least one element of O is E-coeven.

Theorem 1.4.6. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par}\mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par}\mathbf{E}$ of all $\mathbf{E}$-partitions; namely, $\operatorname{Par}\mathbf{E}$ is a G-subset of the G-set $\{1, 2, 3, \ldots\}^E$ (see Definition 1.4.4 (d) for the definition of the latter). For any G-orbit O on $\operatorname{Par}\mathbf{E}$, we define a monomial $\mathbf{x}_{O,w}$

by

$$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \qquad \text{for some element } \pi \text{ of } O$$

(this does not depend on the choice of π). Let

$$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \mathbf{x}_{O,w}$$

and

$$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \mathbf{x}_{O,w}.$$

Then, $\Gamma(\mathbf{E}, w, G)$ and $\Gamma^+(\mathbf{E}, w, G)$ belong to QSym and satisfy

$$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G).$$

This theorem, which combines Theorem 1.4.2 with the ideas of Pólya enumeration, is inspired by Jochemko's reciprocity result for order polynomials [Joch13, Theorem 2.8], which can be obtained from it by specializations (see Section 1.8 for the details of how Jochemko's result follows from ours).

We shall now review a number of particular cases of Theorem 1.4.2. Details on most of them will be provided in forthcoming work.

Example 1.4.7. (a) Corollary 1.4.3 follows from Theorem 1.4.2 by letting w be the function which is constantly 1.

(b) Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition of a nonnegative integer n, and let $\mathbf{E} = (E, <_1, <_2)$ be the double poset defined in Example 1.3.6 (b). Let $w : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ be the map sending every i to α_i. As Example 1.3.6 (b) shows, we have $\Gamma(\mathbf{E}, w) = M_\alpha$. Thus, applying Theorem 1.4.2

to these $\mathbf{E}$ and w yields

$$S(M_\alpha) = (-1)^\ell \Gamma((E, >_1, <_2), w) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \cdots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$
$$= (-1)^\ell \sum_{i_1 \leq i_2 \leq \cdots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \cdots x_{i_\ell}^{\alpha_1} = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1))}} M_\gamma.$$

This is the formula for $S(M_\alpha)$ given in [Malve93, (4.26)], in [GriRei15, Theorem 5.11], and in [BenSag14, Theorem 4.1] (originally due to Ehrenborg and to Malvenuto and Reutenauer). It also shows that the $\Gamma(\mathbf{E}, w)$ for varying $\mathbf{E}$ and w span the $\mathbf{k}$-module QSym.

(c) Applying Corollary 1.4.3 to the double poset of Example 1.3.6 (c) (where the relation $<_2$ is chosen to be a total order) yields the formula for the antipode of a fundamental quasisymmetric function ([Malve93, (4.27)], [GriRei15, (5.9)], [BenSag14, Theorem 5.1]).

(d) Let us use the notations of Example 1.3.3. For any partition λ, let λ^t denote the conjugate partition of λ. Let μ and λ be two partitions satisfying $\mu \subseteq \lambda$. Then, there is a bijection $\tau : Y(\lambda/\mu) \to Y(\lambda^t/\mu^t)$ sending each $(i,j) \in Y(\lambda/\mu)$ to (j,i). This bijection is an isomorphism of double posets from $(Y(\lambda/\mu), >_1, <_2)$ to $(Y(\lambda^t/\mu^t), >_1, >_2)$. Thus, applying Corollary 1.4.3 to the tertispecial double poset $\mathbf{Y}(\lambda/\mu)$, we obtain

$$S(\Gamma(\mathbf{Y}(\lambda/\mu))) = (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda/\mu), >_1, <_2))$$
$$= (-1)^{|\lambda/\mu|} \Gamma\left(\left(Y\left(\lambda^t/\mu^t\right), >_1, >_2\right)\right). \tag{1.3}$$

But from Example 1.3.6 (d), we know that $\Gamma(\mathbf{Y}(\lambda/\mu)) = s_{\lambda/\mu}$. Moreover, a similar argument using [GriRei15, Remark 2.12] shows that $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$. Hence, (1.3) rewrites as

$$S\left(s_{\lambda/\mu}\right) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}. \tag{1.4}$$

This is a well-known formula, and is usually stated for S being the antipode of the Hopf algebra of symmetric (rather than quasisymmetric) functions, but the latter antipode is a restriction of the antipode of QSym.

It is also possible (but more difficult) to derive (1.4) by using the double poset $\mathbf{Y}_h(\lambda/\mu)$ instead of $\mathbf{Y}(\lambda/\mu)$. (This boils down to what was done in [GriRei15, proof of Corollary 5.29].)

(e) Two results of Benedetti and Sagan [BenSag14, Theorems 8.1–8.2] on the antipodes of immaculate functions can be obtained from Corollary 1.4.3 using dualization.

1.5 Lemmas: packed E-partitions and comultiplications

We shall now prepare for the proofs of our results. To this end, we introduce the notion of a *packed map*.

Definition 1.5.1. (a) An *initial interval* will mean a set of the form $\{1, 2, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$.

(b) If T is a set and $\pi : T \to \{1, 2, 3, \ldots\}$ is a map, then π is said to be *packed* if $\pi(T)$ is an initial interval. Clearly, this initial interval must be $\{1, 2, \ldots, |\pi(T)|\}$.

Proposition 1.5.2. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. For every packed map $\pi : E \to \{1, 2, 3, \ldots\}$, we define $\operatorname{ev}_w \pi$ to be the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where $\ell = |\pi(E)|$ (so that $\pi(E) = \{1, 2, \ldots, \ell\}$, since π is packed), and where each α_i is defined as $\sum_{e \in \pi^{-1}(i)} w(e)$. Then,

$$\Gamma(\mathbf{E}, w) = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi}. \tag{1.5}$$

Proof of Proposition 1.5.2. For every finite subset T of $\{1, 2, 3, \ldots\}$, there exists a unique strictly increasing bijection $\{1, 2, \ldots, |T|\} \to T$. We shall denote this bijection by r_T. For every map $\pi : E \to \{1, 2, 3, \ldots\}$, we define the *packing of* π as the map $r_{\pi(E)}^{-1} \circ \pi : E \to \{1, 2, 3, \ldots\}$; this is a packed map (indeed, its image is $\{1, 2, \ldots, |\pi(E)|\}$), and will be denoted by $\operatorname{pack} \pi$. This map $\operatorname{pack} \pi$ is an $\mathbf{E}$-partition if and only if π is an $\mathbf{E}$-partition[13].

We shall show that for every packed $\mathbf{E}$-partition φ, we have

$$\sum_{\pi \text{ is an } \mathbf{E}\text{-partition; } \operatorname{pack} \pi = \varphi} \mathbf{x}_{\pi,w} = M_{\operatorname{ev}_w \varphi}. \tag{1.6}$$

Once this is proven, it will follow that

$$\begin{aligned}
\Gamma(\mathbf{E}, w) &= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w} = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition; } \operatorname{pack} \pi = \varphi} \mathbf{x}_{\pi,w}}_{\substack{= M_{\operatorname{ev}_w \varphi} \\ \text{(by (1.6))}}} \\
&\qquad (\text{since } \operatorname{pack} \pi \text{ is a packed } \mathbf{E}\text{-partition for every } \mathbf{E}\text{-partition } \pi) \\
&= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\operatorname{ev}_w \varphi},
\end{aligned}$$

and Proposition 1.5.2 will be proven.

So it remains to prove (1.6). Let φ be a packed $\mathbf{E}$-partition. Let $\ell = |\varphi(E)|$; thus $\varphi(E) = \{1, 2, \ldots, \ell\}$ (since φ is packed). Let $\alpha_i = \sum_{e \in \varphi^{-1}(i)} w(e)$ for every $i \in \{1, 2, \ldots, \ell\}$; thus, $\operatorname{ev}_w \varphi = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ (by the definition of $\operatorname{ev}_w \varphi$). Then, the

[13] Indeed, $\operatorname{pack} \pi = r_{\pi(E)}^{-1} \circ \pi$. Since $r_{\pi(E)}$ is strictly increasing, we thus see that, for any given $e \in E$ and $f \in E$, the equivalences

$$((\operatorname{pack} \pi)(e) \leq (\operatorname{pack} \pi)(f)) \Longleftrightarrow (\pi(e) \leq \pi(f))$$

and

$$((\operatorname{pack} \pi)(e) < (\operatorname{pack} \pi)(f)) \Longleftrightarrow (\pi(e) < \pi(f))$$

hold. Hence, $\operatorname{pack} \pi$ is an $\mathbf{E}$-partition if and only if π is an $\mathbf{E}$-partition.

right hand side of (1.6) rewrites as follows:

$$M_{\operatorname{ev}_w \varphi} = \sum_{i_1 < i_2 < \cdots < i_\ell} \underbrace{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}}_{= \prod_{k=1}^{\ell} x_{i_k}^{\alpha_k}} = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\alpha_k}}_{\substack{= x_{i_k}^{\sum_{e \in \varphi^{-1}(k)} w(e)} \\ (\text{since } \alpha_k = \sum_{e \in \varphi^{-1}(k)} w(e))}}$$

$$= \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\sum_{e \in \varphi^{-1}(k)} w(e)}}_{\substack{= \prod_{e \in \varphi^{-1}(k)} x_{i_k}^{w(e)} \\ = \prod_{e \in E;\ \varphi(e) = k} x_{i_k}^{w(e)}}} = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{k=1}^{\ell} \prod_{e \in E;\ \varphi(e) = k} \underbrace{x_{i_k}^{w(e)}}_{\substack{= x_{i_{\varphi(e)}}^{w(e)} \\ (\text{since } k = \varphi(e))}}$$

$$= \sum_{i_1 < i_2 < \cdots < i_\ell} \underbrace{\prod_{k=1}^{\ell} \prod_{e \in E;\ \varphi(e) = k} x_{i_{\varphi(e)}}^{w(e)}}_{= \prod_{e \in E} x_{i_{\varphi(e)}}^{w(e)}} = \sum_{i_1 < i_2 < \cdots < i_\ell} \prod_{e \in E} x_{i_{\varphi(e)}}^{w(e)}$$

$$= \sum_{T \subseteq \{1,2,3,\ldots\};\ |T| = \ell} \underbrace{\prod_{e \in E} x_{r_T(\varphi(e))}^{w(e)}}_{= \prod_{e \in E} x_{(r_T \circ \varphi)(e)}^{w(e)} = \mathbf{x}_{r_T \circ \varphi, w}} = \sum_{T \subseteq \{1,2,3,\ldots\};\ |T| = \ell} \mathbf{x}_{r_T \circ \varphi, w} \tag{1.7}$$

[14].

On the other hand, recall that φ is an **E**-partition. Hence, every map π satisfying $\operatorname{pack} \pi = \varphi$ is an **E**-partition (because, as we know, $\operatorname{pack} \pi$ is an **E**-partition if and only if π is an **E**-partition). Thus, the **E**-partitions π satisfying $\operatorname{pack} \pi = \varphi$ are precisely the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $\operatorname{pack} \pi = \varphi$. Hence,

$$\sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ \operatorname{pack} \pi = \varphi} \mathbf{x}_{\pi, w} = \sum_{\pi : E \to \{1,2,3,\ldots\};\ \operatorname{pack} \pi = \varphi} \mathbf{x}_{\pi, w}$$
$$= \sum_{T \subseteq \{1,2,3,\ldots\};\ |T| = \ell} \sum_{\pi : E \to \{1,2,3,\ldots\};\ \operatorname{pack} \pi = \varphi;\ \pi(E) = T} \mathbf{x}_{\pi, w}$$

(because if $\pi : E \to \{1, 2, 3, \ldots\}$ is a map satisfying $\operatorname{pack} \pi = \varphi$, then $|\pi(E)| = \ell$

[14]In the second-to-last equality, we have used the fact that the strictly increasing sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers are in bijection with the subsets $T \subseteq \{1, 2, 3, \ldots\}$ such that $|T| = \ell$. The bijection sends a sequence $(i_1 < i_2 < \cdots < i_\ell)$ to the set of its entries; its inverse map sends every T to the sequence $(r_T(1), r_T(2), \ldots, r_T(|T|))$.

[15]). But for every ℓ-element subset T of $\{1,2,3,\ldots\}$, there exists exactly one $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$: namely, $\pi = r_T \circ \varphi$ [16]. Therefore, for every ℓ-element subset T of $\{1,2,3,\ldots\}$, we have

$$\sum_{\pi : E \to \{1,2,3,\ldots\};\ \operatorname{pack}\pi = \varphi;\ \pi(E) = T} \mathbf{x}_{\pi,w} = \mathbf{x}_{r_T \circ \varphi, w}.$$

Hence,

$$\begin{aligned}\sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ \operatorname{pack}\pi = \varphi} \mathbf{x}_{\pi,w} &= \sum_{T \subseteq \{1,2,3,\ldots\};\ |T| = \ell} \underbrace{\sum_{\pi : E \to \{1,2,3,\ldots\};\ \operatorname{pack}\pi = \varphi;\ \pi(E) = T} \mathbf{x}_{\pi,w}}_{= \mathbf{x}_{r_T \circ \varphi, w}} \\ &= \sum_{T \subseteq \{1,2,3,\ldots\};\ |T| = \ell} \mathbf{x}_{r_T \circ \varphi, w} = M_{\operatorname{ev}_w \varphi}\end{aligned}$$

---

[15] *Proof.* Let $\pi : E \to \{1,2,3,\ldots\}$ be a map satisfying $\operatorname{pack}\pi = \varphi$. The definition of $\operatorname{pack}\pi$ yields $\operatorname{pack}\pi = r_{\pi(E)}^{-1} \circ \pi$. Hence, $|(\operatorname{pack}\pi)(E)| = \left|\left(r_{\pi(E)}^{-1} \circ \pi\right)(E)\right| = \left|r_{\pi(E)}^{-1}(\pi(E))\right| = |\pi(E)|$ (since $r_{\pi(E)}^{-1}$ is a bijection). Since $\operatorname{pack}\pi = \varphi$, this rewrites as $|\varphi(E)| = |\pi(E)|$. Hence, $|\pi(E)| = |\varphi(E)| = \ell$, qed.

[16] *Proof.* Let T be an ℓ-element subset of $\{1,2,3,\ldots\}$. We need to show that there exists exactly one $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$: namely, $\pi = r_T \circ \varphi$. In other words, we need to prove the following two claims:

Claim 1: The map $r_T \circ \varphi$ is a map $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$.

Claim 2: If $\pi : E \to \{1,2,3,\ldots\}$ is a map satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$, then $\pi = r_T \circ \varphi$.

Proof of Claim 1. We have $(r_T \circ \varphi)(E) = r_T\left(\underbrace{\varphi(E)}_{=\{1,2,\ldots,\ell\}}\right) = r_T\left(\left\{1,2,\ldots,\underbrace{\ell}_{\substack{=|T| \\ (\text{since } T \text{ is } \ell\text{-element})}}\right\}\right) = r_T(\{1,2,\ldots,|T|\}) = T$ (by the definition of r_T). Now, the definition of $\operatorname{pack}(r_T \circ \varphi)$ shows that

$$\begin{aligned}\operatorname{pack}(r_T \circ \varphi) &= r_{(r_T \circ \varphi)(E)}^{-1} \circ (r_T \circ \varphi) = r_T^{-1} \circ (r_T \circ \varphi) \qquad (\text{since } (r_T \circ \varphi)(E) = T) \\ &= \varphi.\end{aligned}$$

Thus, the map $r_T \circ \varphi : E \to \{1,2,3,\ldots\}$ satisfies $\operatorname{pack}(r_T \circ \varphi) = \varphi$ and $(r_T \circ \varphi)(E) = T$. In other words, $r_T \circ \varphi$ is a map $\pi : E \to \{1,2,3,\ldots\}$ satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$. This proves Claim 1.

Proof of Claim 2. Let $\pi : E \to \{1,2,3,\ldots\}$ be a map satisfying $\operatorname{pack}\pi = \varphi$ and $\pi(E) = T$. The definition of $\operatorname{pack}\pi$ shows that $\operatorname{pack}\pi = r_{\pi(E)}^{-1} \circ \pi = r_T^{-1} \circ \pi$ (since $\pi(E) = T$). Hence, $r_T^{-1} \circ \pi = \operatorname{pack}\pi = \varphi$, so that $\pi = r_T \circ \varphi$. This proves Claim 2.

Now, both Claims 1 and 2 are proven; hence, our proof is complete.

(by (1.7)). Thus, (1.6) is proven, and with it Proposition 1.5.2. □

Proof of Proposition 1.3.5. Proposition 1.3.5 follows immediately from Proposition 1.5.2. □

We shall now describe the coproduct of $\Gamma(\mathbf{E}, w)$, essentially giving the proof that is left to the reader in [MalReu09, Theorem 2.2].

Definition 1.5.3. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset.

(a) Then, $\operatorname{Adm} \mathbf{E}$ will mean the set of all pairs (P, Q), where P and Q are subsets of E satisfying $P \cap Q = \varnothing$ and $P \cup Q = E$ and having the property that no $p \in P$ and $q \in Q$ satisfy $q <_1 p$. These pairs (P, Q) are called the *admissible partitions* of $\mathbf{E}$. (In the terminology of [MalReu09], they are the *decompositions* of $(E, <_1)$.)

(b) For any subset T of E, we let $\mathbf{E} \mid_T$ denote the double poset $(T, <_1, <_2)$, where $<_1$ and $<_2$ (by abuse of notation) denote the restrictions of the relations $<_1$ and $<_2$ to T.

Proposition 1.5.4. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta(\Gamma(\mathbf{E}, w)) = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma(\mathbf{E} \mid_Q, w \mid_Q). \qquad (1.8)$$

A particular case of Proposition 1.5.4 (namely, the case when $w(e) = 1$ for each $e \in E$) appears in [Malve93, Théorème 4.16].

We shall now outline a proof of this fact. The proof relies on a simple bijection that an experienced combinatorialist will have no trouble finding (and proving even less); let us just give a brief outline of the argument[17]:

[17] See the detailed version of this note for an (almost) completely written-out proof; I am afraid that the additional level of detail is of no help to the understanding.

Proof of Proposition 1.5.4. Whenever $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ is a composition and $k \in \{0, 1, \dots, \ell\}$, we introduce the notation $\alpha[: k]$ for the composition $(\alpha_1, \alpha_2, \dots, \alpha_k)$, and the notation $\alpha[k :]$ for the composition $(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_\ell)$. Now, the formula (1.1) can be rewritten as follows:

$$\Delta(M_\alpha) = \sum_{k=0}^{\ell} M_{\alpha[:k]} \otimes M_{\alpha[k:]} \tag{1.9}$$

for every $\ell \in \mathbb{N}$ and every composition α with ℓ entries.

Now, applying Δ to the equality (1.5) yields

$$\begin{aligned}\Delta(\Gamma(\mathbf{E}, w)) &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\Delta(M_{\mathrm{ev}_w \varphi})}_{\substack{= \sum_{k=0}^{|\varphi(E)|} M_{(\mathrm{ev}_w \varphi)[:k]} \otimes M_{(\mathrm{ev}_w \varphi)[k:]} \\ \text{(by (1.9))}}} \\ &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \sum_{k=0}^{|\varphi(E)|} M_{(\mathrm{ev}_w \varphi)[:k]} \otimes M_{(\mathrm{ev}_w \varphi)[k:]}. \end{aligned} \tag{1.10}$$

On the other hand, rewriting each of the tensorands on the right hand side of (1.8) using (1.5), we obtain

$$\begin{aligned}&\sum_{(P,Q) \in \mathrm{Adm}\, \mathbf{E}} \Gamma(\mathbf{E}\mid_P, w\mid_P) \otimes \Gamma(\mathbf{E}\mid_Q, w\mid_Q) \\ &= \sum_{(P,Q) \in \mathrm{Adm}\, \mathbf{E}} \left(\sum_{\varphi \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\mathrm{ev}_{w|_P} \varphi} \right) \otimes \left(\sum_{\varphi \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\mathrm{ev}_{w|_Q} \varphi} \right) \\ &= \sum_{(P,Q) \in \mathrm{Adm}\, \mathbf{E}} \left(\sum_{\sigma \text{ is a packed } \mathbf{E}|_P\text{-partition}} M_{\mathrm{ev}_{w|_P} \sigma} \right) \otimes \left(\sum_{\tau \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\mathrm{ev}_{w|_Q} \tau} \right) \\ &= \sum_{(P,Q) \in \mathrm{Adm}\, \mathbf{E}} \sum_{\sigma \text{ is a packed } \mathbf{E}|_P\text{-partition}} \sum_{\tau \text{ is a packed } \mathbf{E}|_Q\text{-partition}} M_{\mathrm{ev}_{w|_P} \sigma} \otimes M_{\mathrm{ev}_{w|_Q} \tau}.\end{aligned}$$

We need to prove that the right hand sides of this equality and of (1.10) are equal (because then, it will follow that so are the left hand sides, and thus Proposition 1.5.4 will be proven). For this, it is clearly enough to exhibit a bijection between

- the pairs (φ, k) consisting of a packed $\mathbf{E}$-partition φ and a $k \in \{0, 1, \ldots, |\varphi(E)|\}$

and

- the triples $((P, Q), \sigma, \tau)$ consisting of a $(P, Q) \in \operatorname{Adm} \mathbf{E}$, a packed $\mathbf{E}\mid_P$-partition σ and a packed $\mathbf{E}\mid_Q$-partition τ

which bijection has the property that whenever it maps (φ, k) to $((P, Q), \sigma, \tau)$, we have the equalities $(\operatorname{ev}_w \varphi)[: k] = \operatorname{ev}_{w|_P} \sigma$ and $(\operatorname{ev}_w \varphi)[k :] = \operatorname{ev}_{w|_Q} \tau$. Such a bijection is easy to construct: Given (φ, k), it sets $P = \varphi^{-1}(\{1, 2, \ldots, k\})$, $Q = \varphi^{-1}(\{k+1, k+2, \ldots, |\varphi(E)|\})$, $\sigma = \varphi\mid_P$ and $\tau = \operatorname{pack}(\varphi\mid_Q)$ [18]. Conversely, given $((P, Q), \sigma, \tau)$, the inverse bijection sets $k = |\sigma(P)|$ and constructs φ as the map $E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$. Proving that this alleged bijection and its alleged inverse bijection are well-defined and actually mutually inverse is straightforward and left to the reader[19]. □

[18] We notice that these P, Q, σ and τ satisfy $\sigma(e) = \varphi(e)$ for every $e \in P$, and $\tau(e) = \varphi(e) - k$ for every $e \in Q$.

[19] The only part of the argument that is a bit trickier is proving the well-definedness of the inverse bijection: We need to show that if $((P, Q), \sigma, \tau)$ is a triple consisting of a $(P, Q) \in \operatorname{Adm} \mathbf{E}$, a packed $\mathbf{E}\mid_P$-partition σ and a packed $\mathbf{E}\mid_Q$-partition τ, and if we set $k = |\sigma(P)|$, then the map $\varphi : E \to \{1, 2, 3, \ldots\}$ which sends every $e \in E$ to $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$ is actually a packed $\mathbf{E}$-partition.

Indeed, it is clear that this map φ is packed. It remains to show that it is an $\mathbf{E}$-partition. To do so, we must prove the following two claims:

Claim 1: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\varphi(e) \leq \varphi(f)$.

Claim 2: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\varphi(e) < \varphi(f)$.

We shall only prove Claim 1 (as the proof of Claim 2 is similar). So let $e \in E$ and $f \in E$ be such that $e <_1 f$. We need to show that $\varphi(e) \leq \varphi(f)$. We are in one of the following four cases:

Case 1: We have $e \in P$ and $f \in P$.

Case 2: We have $e \in P$ and $f \in Q$.

Case 3: We have $e \in Q$ and $f \in P$.

Case 4: We have $e \in Q$ and $f \in Q$.

In Case 1, our claim $\varphi(e) \leq \varphi(f)$ follows from the assumption that σ is an $\mathbf{E}\mid_P$-partition (because in Case 1, we have $\varphi(e) = \sigma(e)$ and $\varphi(f) = \sigma(f)$). In Case 4, it follows from the assumption that τ is an $\mathbf{E}\mid_Q$-partition (since in Case 4, we have $\varphi(e) = \tau(e) + k$ and $\varphi(f) = \tau(f) + k$). In Case 2, it clearly holds (indeed, if $e \in P$, then the definition of φ yields $\varphi(e) = \sigma(e) \leq k$, and if $f \in Q$, then the definition of φ yields $\varphi(f) = \tau(f) + k > k$; therefore, in Case 2, we have $\varphi(e) \leq k < \varphi(f)$). Finally, Case 3 is impossible (because having $e \in Q$ and $f \in P$ and $e <_1 f$ would contradict $(P, Q) \in \operatorname{Adm} \mathbf{E}$). Thus, we have proven the claim in each of the four cases, and consequently Claim 1 is proven. As we have said above, Claim 2 is proven similarly.

We note in passing that there is also a rule for multiplying quasisymmetric functions of the form $\Gamma(\mathbf{E}, w)$. Namely, if $\mathbf{E}$ and $\mathbf{F}$ are two double posets and u and v are corresponding maps, then $\Gamma(\mathbf{E}, u) \Gamma(\mathbf{F}, v) = \Gamma(\mathbf{EF}, w)$ for a map w which is defined to be u on the subset $\mathbf{E}$ of $\mathbf{EF}$, and v on the subset $\mathbf{F}$ of $\mathbf{EF}$. Here, $\mathbf{EF}$ is a double poset defined as in [MalReu09, §2.1]. Combined with Proposition 1.3.5, this fact gives a combinatorial proof for the fact that QSym is a $\mathbf{k}$-algebra, as well as for some standard formulas for multiplications of quasisymmetric functions; similarly, Proposition 1.5.4 can be used to derive the well-known formulas for ΔM_α, ΔL_α, $\Delta s_{\lambda/\mu}$ etc. (although, of course, we have already used the formula for ΔM_α in our proof of Proposition 1.5.4).

1.6 Proof of Theorem 1.4.2

Before we come to the proof of Theorem 1.4.2, let us state three simple lemmas:

Lemma 1.6.1. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let P and Q be subsets of E such that $P \cap Q = \varnothing$ and $P \cup Q = E$. Assume that there exist no $p \in P$ and $q \in Q$ such that q is $<_1$-covered by p. Then, $(P, Q) \in \operatorname{Adm} \mathbf{E}$.

Proof of Lemma 1.6.1. For any $a \in E$ and $b \in E$, we let $[a, b]$ denote the subset $\{e \in E \mid a <_1 e <_1 b\}$ of E. It is clear that if a, b and c are three elements of E satisfying $a <_1 c <_1 b$, then both $[a, c]$ and $[c, b]$ are proper subsets of $[a, b]$, and therefore

$$\text{both numbers } |[a, c]| \text{ and } |[c, b]| \text{ are smaller than } |[a, b]| . \tag{1.11}$$

A pair $(p, q) \in P \times Q$ is said to be a *malposition* if it satisfies $q <_1 p$. Now, let us assume (for the sake of contradiction) that there exists a malposition. Fix a malposition (u, v) for which the value $|[u, v]|$ is minimum. Thus, $u \in P$, $v \in Q$ and $v <_1 u$, but v is not $<_1$-covered by u (since there exist no $p \in P$ and $q \in Q$ such that q is $<_1$-covered by p). Hence, there exists a $w \in E$ such that $v <_1 w <_1 u$ (since $v <_1 u$). Consider this w. Applying (1.11) to $a = v$, $c = w$ and $b = u$, we see that both numbers $|[u,w]|$ and $|[w,v]|$ are smaller than $|[u,v]|$, and therefore neither (u,w) nor (w,v) is a malposition (since we picked (u,v) to be a malposition with minimum $|[u,v]|$). But $w \in E = P \cup Q$, so that either $w \in P$ or $w \in Q$. If $w \in P$, then (w,v) is a malposition; if $w \in Q$, then (u,w) is a malposition. In either case, we obtain a contradiction to the fact that neither (u,w) nor (w,v) is a malposition. This contradiction shows that our assumption was wrong. Hence, there exists no malposition. Consequently, $(P,Q) \in \operatorname{Adm} \mathbf{E}$. □

Lemma 1.6.2. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $(P,Q) \in \operatorname{Adm} \mathbf{E}$. Then, $\mathbf{E} \mid_P$ is a tertispecial double poset.

Proof of Lemma 1.6.2. We need to show that the double poset $\mathbf{E} \mid_P = (P, <_1, <_2)$ is tertispecial. In other words, we need to show that if a and b are two elements of P such that a is $<_1$-covered by b as element of the set P, then a and b are $<_2$-comparable.[20]

Let a and b be two elements of P such that a is $<_1$-covered by b as element of the set P. Thus, $a <_1 b$, and

$$\text{there exists no } c \in P \text{ satisfying } a <_1 c <_1 b. \tag{1.12}$$

Now, if $c \in E$ is such that $a <_1 c <_1 b$, then c must belong to P [21], which entails a contradiction to (1.12). Thus, there is no $c \in E$ satisfying $a <_1 c <_1 b$. Therefore (and because we have $a <_1 b$), we see that a is $<_1$-covered by b as element of the set E. Since $\mathbf{E}$ is tertispecial, this yields that a and b are $<_2$-comparable.

[20]Here, we are using the following notation: If T is a subset of E, and if u and v are two elements of T, then we say that "u is $<_1$-covered by v as element of the set T" if and only if u is $<_{1,T}$-covered by v, where $<_{1,T}$ denotes the relation $<_1$ on the set T (not the relation $<_1$ on the set E). In general, saying that u is $<_1$-covered by v as element of the set T is **not** equivalent to saying that u is $<_1$-covered by v as element of the set E (because there might be an element w of E satisfying $u <_1 w <_1 v$, but no such element that belongs to T). Rather, u is $<_1$-covered by v as element of the set T if and only if $u <_1 v$ and there exists no $w \in T$ satisfying $u <_1 w <_1 v$.

[21]*Proof.* Assume the contrary. Thus, $c \notin P$. But $(P,Q) \in \operatorname{Adm} \mathbf{E}$. Thus, $P \cap Q = \varnothing$, $P \cup Q = E$, and

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{1.13}$$

From $c \in E$ and $c \notin P$, we obtain $c \in E \setminus P \subseteq Q$ (since $P \cup Q = E$). Applying (1.13) to $p = b$ and $q = c$, we thus conclude that we cannot have $c <_1 b$. This contradicts $c <_1 b$. This contradiction shows that our assumption was false, qed.

Thus, we have shown that if a and b are two elements of P such that a is $<_1$-covered by b as element of the set P, then a and b are $<_2$-comparable. This proves Lemma 1.6.2.

(We could similarly show that $\mathbf{E}\mid_Q$ is a tertispecial double poset; but we will not use this.) $\square$

Lemma 1.6.3. Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map.

(a) If $E = \varnothing$, then $\Gamma(\mathbf{E}, w) = 1$.

(b) If $E \neq \varnothing$, then $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$.

Proof of Lemma 1.6.3. (a) Part (a) is obvious (since there is only one $\mathbf{E}$-partition when $E = \varnothing$).

(b) Observe that $\Gamma(\mathbf{E}, w)$ is a homogeneous power series of degree $\sum\limits_{e \in E} w(e)$. When $E \neq \varnothing$, this degree is > 0, and thus the power series $\Gamma(\mathbf{E}, w)$ is annihilated by ε (since ε annihilates any homogeneous power series in QSym whose degree is > 0). $\square$

Proof of Theorem 1.4.2. We shall prove Theorem 1.4.2 by strong induction over $|E|$. The induction base ($|E| = 0$) is left to the reader; we start with the induction step. Consider a tertispecial double poset $\mathbf{E} = (E, <_1, <_2)$ with $|E| > 0$ and a map $w : E \to \{1, 2, 3, \ldots\}$, and assume that Theorem 1.4.2 is proven for all tertispecial double posets of smaller size.

We have $|E| > 0$ and thus $E \neq \varnothing$. Hence, Lemma 1.6.3 (b) shows that $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$. Thus, $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = u\left(\underbrace{\varepsilon(\Gamma(\mathbf{E}, w))}_{=0}\right) = u(0) = 0$.

The upper commutative pentagon of (1.2) shows that $u \circ \varepsilon = m \circ (S \otimes \mathrm{id}) \circ \Delta$. Applying both sides of this equality to $\Gamma(\mathbf{E}, w)$, we obtain $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) =$

$(m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w))$. Since $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = 0$, this becomes

$$\begin{aligned}
0 &= (m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \mathrm{id})(\Delta(\Gamma(\mathbf{E}, w)))) \\
&= m\left((S \otimes \mathrm{id})\left(\sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} \Gamma(\mathbf{E}\mid_P, w\mid_P) \otimes \Gamma(\mathbf{E}\mid_Q, w\mid_Q)\right)\right) \qquad (\text{by } (1.8)) \\
&= m\left(\sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} S(\Gamma(\mathbf{E}\mid_P, w\mid_P)) \otimes \Gamma(\mathbf{E}\mid_Q, w\mid_Q)\right) \\
&= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} S(\Gamma(\mathbf{E}\mid_P, w\mid_P))\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q) \\
&= S(\Gamma(\mathbf{E}\mid_E, w\mid_E))\,\Gamma(\mathbf{E}\mid_\varnothing, w\mid_\varnothing) + \sum_{\substack{(P,Q)\in \operatorname{Adm}\mathbf{E};\\ |P|<|E|}} S(\Gamma(\mathbf{E}\mid_P, w\mid_P))\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q)
\end{aligned} \tag{1.14}$$

(since the only pair $(P,Q) \in \operatorname{Adm}\mathbf{E}$ satisfying $|P| = |E|$ is $(E, \varnothing)$, whereas all other pairs $(P,Q) \in \operatorname{Adm}\mathbf{E}$ satisfy $|P| < |E|$).

But whenever $(P,Q) \in \operatorname{Adm}\mathbf{E}$ is such that $|P| < |E|$, the double poset $\mathbf{E}\mid_P = (P, <_1, <_2)$ is tertispecial (by Lemma 1.6.2), and therefore we have $S(\Gamma(\mathbf{E}\mid_P, w\mid_P)) = S(\Gamma((P, <_1, <_2), w\mid_P)) = (-1)^{|P|}\,\Gamma((P, >_1, <_2), w\mid_P)$ (by the induction hypothesis). Hence, (1.14) rewrites as

$$\begin{aligned}
0 &= S\left(\Gamma\left(\underbrace{\mathbf{E}\mid_E}_{=\mathbf{E}}, \underbrace{w\mid_E}_{=w}\right)\right) \underbrace{\Gamma(\mathbf{E}\mid_\varnothing, w\mid_\varnothing)}_{\substack{=\Gamma((\varnothing,<_1,<_2),w\mid_\varnothing)=1\\ (\text{by Lemma 1.6.3 (a)})}} \\
&\quad + \sum_{\substack{(P,Q)\in \operatorname{Adm}\mathbf{E};\\ |P|<|E|}} (-1)^{|P|}\,\Gamma((P, >_1, <_2), w\mid_P)\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q) \\
&= S(\Gamma(\mathbf{E}, w)) + \sum_{\substack{(P,Q)\in \operatorname{Adm}\mathbf{E};\\ |P|<|E|}} (-1)^{|P|}\,\Gamma((P, >_1, <_2), w\mid_P)\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q).
\end{aligned}$$

Thus,

$$S\left(\Gamma\left(\mathbf{E},w\right)\right)=-\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ |P|<|E|}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right). \tag{1.15}$$

We shall now prove that

$$0=\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right). \tag{1.16}$$

But first, let us explain how this will complete our proof. In fact, the only pair $(P,Q)\in\operatorname{Adm}\mathbf{E}$ satisfying $|P|=|E|$ is $(E,\varnothing)$, whereas all other pairs $(P,Q)\in\operatorname{Adm}\mathbf{E}$ satisfy $|P|<|E|$. Hence, if (1.16) is proven, then we can conclude that

$$\begin{aligned}
0&=\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)\\
&=\left(-1\right)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),\underbrace{w\mid_E}_{=w}\right)\underbrace{\Gamma\left(\mathbf{E}\mid_\varnothing,w\mid_\varnothing\right)}_{=\Gamma\left(\left(\varnothing,<_1,<_2\right),w\mid_\varnothing\right)=1}\\
&\qquad+\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ |P|<|E|}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)\\
&=\left(-1\right)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right)\\
&\qquad+\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ |P|<|E|}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right),
\end{aligned}$$

so that

$$\begin{aligned}
\left(-1\right)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right)&=-\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ |P|<|E|}}\left(-1\right)^{|P|}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q,w\mid_Q\right)\\
&=S\left(\Gamma\left(\underbrace{\mathbf{E}}_{=(E,<_1,<_2)},w\right)\right)\qquad\text{(by (1.15))}\\
&=S\left(\Gamma\left(\left(E,<_1,<_2\right),w\right)\right),
\end{aligned}$$

and thus $S\left(\Gamma\left(\left(E,<_1,<_2\right)\right),w\right)=(-1)^{|E|}\Gamma\left(\left(E,>_1,<_2\right),w\right)$, which completes the induction step and thus the proof of Theorem 1.4.2. It thus remains to prove (1.16).

For every subset P of E, we have

$$\begin{aligned}\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)&=\sum_{\pi\text{ is a }(P,>_1,<_2)\text{-partition}}\mathbf{x}_{\pi,w|_P}\\&\qquad\left(\text{by the definition of }\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)\right)\\&=\sum_{\sigma\text{ is a }(P,>_1,<_2)\text{-partition}}\mathbf{x}_{\sigma,w|_P}.\end{aligned}\tag{1.17}$$

For every subset Q of E, we have

$$\begin{aligned}\Gamma\left(\underbrace{\mathbf{E}\mid_Q}_{=(Q,<_1,<_2)},w\mid_Q\right)&=\Gamma\left(\left(Q,<_1,<_2\right),w\mid_Q\right)\\&=\sum_{\pi\text{ is a }(Q,<_1,<_2)\text{-partition}}\mathbf{x}_{\pi,w|_Q}\\&\qquad\left(\text{by the definition of }\Gamma\left(\left(Q,<_1,<_2\right),w\mid_Q\right)\right)\\&=\sum_{\tau\text{ is a }(Q,<_1,<_2)\text{-partition}}\mathbf{x}_{\tau,w|_Q}.\end{aligned}\tag{1.18}$$

Now,

$$\sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \underbrace{\Gamma\left(\left(P,>_1,<_2\right),w\mid_P\right)}_{\substack{=\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P} \\ \text{(by (1.17))}}} \underbrace{\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)}_{\substack{=\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q} \\ \text{(by (1.18))}}}$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \left(\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P}\right)\left(\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q}\right)$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \ \sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q}$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\sigma,w|_P}\mathbf{x}_{\tau,w|_Q}$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \underbrace{\mathbf{x}_{\pi|_P,w|_P}\mathbf{x}_{\pi|_Q,w|_Q}}_{=\mathbf{x}_{\pi,w}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } (\pi\mid_P,\pi\mid_Q) \text{ for } (\sigma,\tau) \text{ in the inner sum,}\\ \text{since every pair } (\sigma,\tau) \text{ consisting of a map } \sigma:P\to\{1,2,3,\ldots\}\\ \text{and a map } \tau:Q\to\{1,2,3,\ldots\}\\ \text{can be written as } (\pi\mid_P,\pi\mid_Q) \text{ for a unique } \pi:E\to\{1,2,3,\ldots\}\\ \text{(namely, for the } \pi:E\to\{1,2,3,\ldots\} \text{ that is defined to send every}\\ e\in P \text{ to } \sigma(e) \text{ and to send every } e\in Q \text{ to } \tau(e)\text{)}\end{array}\right)$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi:E\to\{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\pi,w}$$

$$= \sum_{\pi:E\to\{1,2,3,\ldots\}} \left(\sum_{\substack{(P,Q)\in \operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} (-1)^{|P|}\right)\mathbf{x}_{\pi,w}.$$

In order to prove that this sum is 0 (and thus to prove (1.16) and finish our proof of Theorem 1.4.2), it therefore is enough to show that for every map $\pi : E \to$

$\{1, 2, 3, \ldots\}$, we have

$$\sum_{\substack{(P,Q)\in \operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} (-1)^{|P|} = 0. \tag{1.19}$$

Hence, let us fix a map $\pi : E \to \{1, 2, 3, \ldots\}$. Our goal is now to prove (1.19). To do so, we denote by Z the set of all $(P, Q) \in \operatorname{Adm}\mathbf{E}$ such that $\pi\mid_P$ is a $(P, >_1, <_2)$-partition and $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition. We are going to define an involution $T : Z \to Z$ of the set Z having the property that, for any $(P, Q) \in Z$, if we write $T((P, Q))$ in the form (P', Q'), then $(-1)^{|P'|} = -(-1)^{|P|}$. Once such an involution T is found, it will be clear that it matches the addends on the left hand side of (1.19) into pairs of mutually cancelling addends[22], and so (1.19) will follow and we will be done. It thus remains to find T.

The definition of T is simple (although it will take us a while to prove that it is well-defined): Let F be the subset of E consisting of those $e \in E$ which have minimum $\pi(e)$. Then, F is a nonempty subposet of the poset $(E, <_2)$, and hence has a minimal element f (that is, an element f such that no $g \in F$ satisfies $g <_2 f$). Fix such an f. Now, the map T sends a $(P, Q) \in Z$ to $\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases}$.

In order to prove that the map T is well-defined, we need to prove that its output values all belong to Z. In other words, we need to prove that

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \in Z \tag{1.20}$$

for every $(P, Q) \in Z$.

Proof of (1.20)*:* Fix $(P, Q) \in Z$. Thus, (P, Q) is an element of $\operatorname{Adm}\mathbf{E}$ with the property that $\pi\mid_P$ is a $(P, >_1, <_2)$-partition and $\pi\mid_Q$ is a $(Q, <_1, <_2)$-partition.

---

[22]In fact, the $(-1)^{|P'|} = -(-1)^{|P|}$ condition makes it clear that T has no fixed points. Therefore, to each addend on the left hand side of (1.19) corresponds an addend with opposite sign, which cancels it: Namely, for each $(A, B) \in Z$, the addend for $(P, Q) = (A, B)$ is cancelled by the addend for $(P, Q) = T((A, B))$.

From $(P, Q) \in \operatorname{Adm} \mathbf{E}$, we see that $P \cap Q = \varnothing$ and $P \cup Q = E$, and furthermore that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{1.21}$$

We know that f belongs to the set F, which is the subset of E consisting of those $e \in E$ which have minimum $\pi(e)$. Thus,

$$\pi(f) \leq \pi(h) \qquad \text{for every } h \in E. \tag{1.22}$$

Moreover,

$$\pi(f) < \pi(h) \qquad \text{for every } h \in E \text{ satisfying } h <_2 f \tag{1.23}$$

[23].

We need to prove (1.20). We are in one of the following two cases:

Case 1: We have $f \in P$.

Case 2: We have $f \notin P$.

Let us first consider Case 1. In this case, we have $f \in P$.

Recall that $P \cap Q = \varnothing$ and $P \cup Q = E$. From this, we easily obtain $(P \setminus \{f\}) \cap (Q \cup \{f\}) = \varnothing$ and $(P \setminus \{f\}) \cup (Q \cup \{f\}) = E$.

Furthermore, there exist no $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ such that q is $<_1$-covered by p [24]. Hence, Lemma 1.6.1 (applied to $P \setminus \{f\}$ and $Q \cup \{f\}$ instead of P and Q)

[23] *Proof of* (1.23)*:* Let $h \in E$ be such that $h <_2 f$. We must prove (1.23). Indeed, assume the contrary. Thus, $\pi(f) \geq \pi(h)$. Combined with (1.22), this shows that $\pi(f) = \pi(h)$. Our definition of F shows that F is the subset of E consisting of those $e \in E$ satisfying $\pi(e) = \pi(f)$ (since $f \in F$). Therefore, $h \in F$ (since $\pi(h) = \pi(f)$). But f is a minimal element of F. In other words, no $g \in F$ satisfies $g <_2 f$. This contradicts the fact that $h \in F$ satisfies $h <_2 f$. This contradiction proves that our assumption was wrong, qed.

[24] *Proof.* Assume the contrary. Thus, there exist $p \in P \setminus \{f\}$ and $q \in Q \cup \{f\}$ such that q is $<_1$-covered by p. Consider such p and q.

We know that q is $<_1$-covered by p, and thus we have $q <_1 p$. Also, $p \in P \setminus \{f\} \subseteq P$. Hence, if we had $q \in Q$, then we would obtain a contradiction to (1.21). Hence, we cannot have $q \in Q$. Therefore, $q = f$ (since $q \in Q \cup \{f\}$ but not $q \in Q$). Hence, $f = q <_1 p$, so that $p >_1 f$. Therefore, $\pi(p) \leq \pi(f)$ (since $\pi\mid_P$ is a $(P, >_1, <_2)$-partition, and since both f and p belong to P).

Now, recall that q is $<_1$-covered by p. Hence, q and p are $<_2$-comparable (since E is tertispecial). In other words, f and p are $<_2$-comparable (since $q = f$). In other words, either $f <_2 p$ or $f = p$ or $p <_2 f$. But $p <_2 f$ cannot hold (because if we had $p <_2 f$, then (1.23) (applied to $h = p$) would lead to $\pi(f) < \pi(p)$, which would contradict $\pi(p) \leq \pi(f)$), and $f = p$ cannot hold either (since shows that $(P \setminus \{f\}, Q \cup \{f\}) \in \operatorname{Adm} \mathbf{E}$.

Furthermore, $\pi \mid_P$ is a $(P, >_1, <_2)$-partition, and therefore $\pi \mid_{P \setminus \{f\}}$ is a $(P \setminus \{f\}, >_1, <_2)$-partition (since $P \setminus \{f\} \subseteq P$).

Furthermore, $\pi \mid_{Q \cup \{f\}}$ is a $(Q \cup \{f\}, <_1, <_2)$-partition[25].

Altogether, we now know that $(P \setminus \{f\}, Q \cup \{f\}) \in \operatorname{Adm} \mathbf{E}$, that $\pi \mid_{P \setminus \{f\}}$ is a $(P \setminus \{f\}, >_1, <_2)$-partition, and that $\pi \mid_{Q \cup \{f\}}$ is a $(Q \cup \{f\}, <_1, <_2)$-partition. In other words, $(P \setminus \{f\}, Q \cup \{f\}) \in Z$ (by the definition of Z). Thus,

$$\begin{aligned} \begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} &= (P \setminus \{f\}, Q \cup \{f\}) \qquad \text{(since } f \in P\text{)} \\ &\in Z. \end{aligned}$$

Hence, (1.20) is proven in Case 1.

Let us next consider Case 2. In this case, we have $f \notin P$.

Recall that $P \cap Q = \varnothing$ and $P \cup Q = E$. From this, we easily obtain $(P \cup \{f\}) \cap (Q \setminus \{f\}) = \varnothing$ and $(P \cup \{f\}) \cup (Q \setminus \{f\}) = E$.

Furthermore, there exist no $p \in P \cup \{f\}$ and $q \in Q \setminus \{f\}$ such that q is $<_1$-covered by p [26]. Hence, Lemma 1.6.1 (applied to $P \cup \{f\}$ and $Q \setminus \{f\}$ instead of P and Q)

$f <_1 p$). Thus, we must have $f <_2 p$.

Now, $\pi \mid_P$ is a $(P, >_1, <_2)$-partition. Hence, $\pi(p) < \pi(f)$ (since $p >_1 f$ and $f <_2 p$, and since p and f both lie in P). But (1.22) (applied to $h = p$) shows that $\pi(f) \leq \pi(p)$. Hence, $\pi(p) < \pi(f) \leq \pi(p)$, a contradiction. Thus, our assumption was wrong, qed.

[25] *Proof.* In order to prove this, we need to verify the following two claims:

Claim 1: Every $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ satisfying $a <_1 b$ satisfy $\pi(a) \leq \pi(b)$;

Claim 2: Every $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ satisfying $a <_1 b$ and $b <_2 a$ satisfy $\pi(a) < \pi(b)$.

Proof of Claim 1: Let $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ be such that $a <_1 b$. We need to prove that $\pi(a) \leq \pi(b)$. If $a = f$, then this follows immediately from (1.22) (applied to $h = b$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in Q$ (since $a \in Q \cup \{f\}$). Now, if $b \in P$, then $a <_1 b$ contradicts (1.21) (applied to $p = b$ and $q = a$). Hence, we cannot have $b \in P$. Therefore, $b \in E \setminus P = Q$ (since $P \cap Q = \varnothing$ and $P \cup Q = E$). Thus, $\pi(a) \leq \pi(b)$ follows immediately from the fact that $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition (since $a \in Q$ and $b \in Q$). This proves Claim 1.

Proof of Claim 2: Let $a \in Q \cup \{f\}$ and $b \in Q \cup \{f\}$ be such that $a <_1 b$ and $b <_2 a$. We need to prove that $\pi(a) < \pi(b)$. If $a = f$, then this follows immediately from (1.23) (applied to $h = b$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in Q$ (since $a \in Q \cup \{f\}$). Now, if $b \in P$, then $a <_1 b$ contradicts (1.21) (applied to $p = b$ and $q = a$). Hence, we cannot have $b \in P$. Therefore, $b \in E \setminus P = Q$ (since $P \cap Q = \varnothing$ and $P \cup Q = E$). Thus, $\pi(a) < \pi(b)$ follows immediately from the fact that $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition (since $a \in Q$ and $b \in Q$). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven, and we are done.

[26] *Proof.* Assume the contrary. Thus, there exist $p \in P \cup \{f\}$ and $q \in Q \setminus \{f\}$ such that q is

shows that $(P \cup \{f\}, Q \setminus \{f\}) \in \operatorname{Adm} \mathbf{E}$.

Furthermore, $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition, and therefore $\pi \mid_{Q \setminus \{f\}}$ is a $(Q \setminus \{f\}, <_1, <_2)$-partition (since $Q \setminus \{f\} \subseteq Q$).

Furthermore, $\pi \mid_{P \cup \{f\}}$ is a $(P \cup \{f\}, >_1, <_2)$-partition[27].

Altogether, we now know that $(P \cup \{f\}, Q \setminus \{f\}) \in \operatorname{Adm} \mathbf{E}$, that $\pi \mid_{P \cup \{f\}}$ is a $(P \cup \{f\}, >_1, <_2)$-partition, and that $\pi \mid_{Q \setminus \{f\}}$ is a $(Q \setminus \{f\}, <_1, <_2)$-partition. In other words, $(P \cup \{f\}, Q \setminus \{f\}) \in Z$ (by the definition of Z). Thus,

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} = (P \cup \{f\}, Q \setminus \{f\}) \qquad (\text{since } f \notin P)$$
$$\in Z.$$

$<_1$-covered by p. Consider such p and q.

We have $f \notin P$ and thus $f \in E \setminus P = Q$ (since $P \cap Q = \varnothing$ and $P \cup Q = E$).

We know that q is $<_1$-covered by p, and thus we have $q <_1 p$. Also, $q \in Q \setminus \{f\} \subseteq Q$. Hence, if we had $p \in P$, then we would obtain a contradiction to (1.21). Hence, we cannot have $p \in P$. Therefore, $p = f$ (since $p \in P \cup \{f\}$ but not $p \in P$). Hence, $q <_1 p = f$. Therefore, $\pi(q) \leq \pi(f)$ (since $q \in Q$ and $f \in Q$, and since $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition). Thus, we cannot have $q <_2 f$ (because if we had $q <_2 f$, then (1.23) (applied to $h = q$) would show that $\pi(f) < \pi(q)$, which would contradict $\pi(q) \leq \pi(f)$).

Now, recall that q is $<_1$-covered by p. Hence, q and p are $<_2$-comparable (since E is tertispecial). In other words, q and f are $<_2$-comparable (since $p = f$). In other words, either $q <_2 f$ or $q = f$ or $f <_2 q$. But we cannot have $q <_2 f$ (as we have just shown), and we cannot have $q = f$ either (since $q <_1 f$). Thus, we must have $f <_2 q$.

From $q <_1 f$ and $f <_2 q$, we conclude that $\pi(q) < \pi(f)$ (since $\pi \mid_Q$ is a $(Q, <_1, <_2)$-partition, and since $q \in Q$ and $f \in Q$). But (1.22) (applied to $h = q$) shows that $\pi(f) \leq \pi(q)$. Hence, $\pi(q) < \pi(f) \leq \pi(q)$, a contradiction. Thus, our assumption was wrong, qed.

[27] *Proof.* In order to prove this, we need to verify the following two claims:

Claim 1: Every $a \in P \cup \{f\}$ and $b \in P \cup \{f\}$ satisfying $a >_1 b$ satisfy $\pi(a) \leq \pi(b)$;

Claim 2: Every $a \in P \cup \{f\}$ and $b \in P \cup \{f\}$ satisfying $a >_1 b$ and $b <_2 a$ satisfy $\pi(a) < \pi(b)$.

Proof of Claim 1: Let $a \in P \cup \{f\}$ and $b \in P \cup \{f\}$ be such that $a >_1 b$. We need to prove that $\pi(a) \leq \pi(b)$. If $a = f$, then this follows immediately from (1.22) (applied to $h = b$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in P$ (since $a \in P \cup \{f\}$). Now, if $b \in Q$, then $b <_1 a$ contradicts (1.21) (applied to $p = a$ and $q = b$). Hence, we cannot have $b \in Q$. Therefore, $b \in E \setminus Q = P$ (since $P \cap Q = \varnothing$ and $P \cup Q = E$). Thus, $\pi(a) \leq \pi(b)$ follows immediately from the fact that $\pi \mid_P$ is a $(P, >_1, <_2)$-partition (since $a \in P$ and $b \in P$). This proves Claim 1.

Proof of Claim 2: Let $a \in P \cup \{f\}$ and $b \in P \cup \{f\}$ be such that $a >_1 b$ and $b <_2 a$. We need to prove that $\pi(a) < \pi(b)$. If $a = f$, then this follows immediately from (1.23) (applied to $h = b$). Hence, we WLOG assume that $a \neq f$. Thus, $a \in P$ (since $a \in P \cup \{f\}$). Now, if $b \in Q$, then $b <_1 a$ contradicts (1.21) (applied to $p = a$ and $q = b$). Hence, we cannot have $b \in Q$. Therefore, $b \in E \setminus Q = P$ (since $P \cap Q = \varnothing$ and $P \cup Q = E$). Thus, $\pi(a) < \pi(b)$ follows immediately from the fact that $\pi \mid_P$ is a $(P, >_1, <_2)$-partition (since $a \in P$ and $b \in P$). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven, and we are done.

Hence, (1.20) is proven in Case 2.

We have now proven (1.20) in both Cases 1 and 2. Thus, (1.20) always holds. In other words, the map T is well-defined.

What the map T does to a pair $(P, Q) \in Z$ can be described as moving the element f from the set where it resides (either P or Q) to the other set. Clearly, doing this twice gives us the original pair back. Hence, the map T is an involution. Furthermore, for any $(P, Q) \in Z$, if we write $T((P, Q))$ in the form (P', Q'), then $(-1)^{|P'|} = -(-1)^{|P|}$ (because $P' = \begin{cases} P \cup \{f\}, & \text{if } f \notin P; \\ P \setminus \{f\}, & \text{if } f \in P \end{cases}$). As we have already explained, this proves (1.19). And this, in turn, completes the induction step of the proof of Theorem 1.4.2. □

1.7 Proof of Theorem 1.4.6

Before we begin proving Theorem 1.4.6, we state a criterion for **E**-partitions that is less wasteful (in the sense that it requires fewer verifications) than the definition:

Lemma 1.7.1. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\phi : E \to \{1, 2, 3, \ldots\}$ be a map. Assume that the following two conditions hold:

- *Condition 1:* If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $e <_2 f$, then $\phi(e) \leq \phi(f)$.
- *Condition 2:* If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $f <_2 e$, then $\phi(e) < \phi(f)$.

Then, ϕ is an **E**-partition.

Proof of Lemma 1.7.1. For any $a \in E$ and $b \in E$, we define a subset $[a, b]$ of E as in the proof of Lemma 1.6.1.

We need to show that ϕ is an **E**-partition. In other words, we need to prove the following two claims:

Claim 1: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$.

Claim 2: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

Proof of Claim 1: Assume the contrary. Thus, there exists a pair $(e, f) \in E \times E$ satisfying $e <_1 f$ but not $\phi(e) \leq \phi(f)$. Such a pair will be called a *malrelation.* Fix a malrelation (u, v) for which the value $|[u, v]|$ is minimum (such a (u, v) exists, since there exists a malrelation). Thus, $u \in E$ and $v \in E$ and $u <_1 v$ but not $\phi(u) \leq \phi(v)$.

If u was $<_1$-covered by v, then we would obtain $\phi(u) \leq \phi(v)$ [28], which would contradict the assumption that we do not have $\phi(u) \leq \phi(v)$. Hence, u is not $<_1$-covered by v. Consequently, there exists a $w \in E$ such that $u <_1 w <_1 v$ (since $u <_1 v$). Consider this w. Applying (1.11) to $a = u$, $c = w$ and $b = v$, we see that both numbers $|[u, w]|$ and $|[w, v]|$ are smaller than $|[u, v]|$, and therefore neither (u, w) nor (w, v) is a malrelation (since we picked (u, v) to be a malrelation with minimum $|[u, v]|$). Therefore, we have $\phi(u) \leq \phi(w)$ and $\phi(w) \leq \phi(v)$ (since $u <_1 w$ and $w <_1 v$). Combining these two inequalities, we obtain $\phi(u) \leq \phi(v)$. This contradicts the assumption that we do not have $\phi(u) \leq \phi(v)$. This contradiction concludes the proof of Claim 1.

Instead of Claim 2, we shall prove the following stronger claim:

Claim 3: Every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and not $e <_2 f$ satisfy $\phi(e) < \phi(f)$.

Proof of Claim 3: Assume the contrary. Thus, there exists a pair $(e, f) \in E \times E$ satisfying $e <_1 f$ and not $e <_2 f$ but not $\phi(e) < \phi(f)$. Such a pair will be called a *malrelation.* Fix a malrelation (u, v) for which the value $|[u, v]|$ is minimum (such a (u, v) exists, since there exists a malrelation). Thus, $u \in E$ and $v \in E$ and $u <_1 v$ and not $u <_2 v$ but not $\phi(u) < \phi(v)$.

If u was $<_1$-covered by v, then we would obtain $\phi(u) < \phi(v)$ easily[29], which

[28] *Proof.* Assume that u is $<_1$-covered by v. Thus, u and v are $<_2$-comparable (since the poset $\mathbf{E}$ is tertispecial). In other words, we have either $u <_2 v$ or $u = v$ or $v <_2 u$. In the first of these three cases, we obtain $\phi(u) \leq \phi(v)$ by applying Condition 1 to $e = u$ and $f = v$. In the third of these cases, we obtain $\phi(u) < \phi(v)$ (and thus $\phi(u) \leq \phi(v)$) by applying Condition 2 to $e = u$ and $f = v$. The second of these cases cannot happen because $u <_1 v$. Thus, we always have $\phi(u) \leq \phi(v)$, qed.

[29] *Proof.* Assume that u is $<_1$-covered by v. Thus, u and v are $<_2$-comparable (since the poset $\mathbf{E}$ is tertispecial). In other words, we have either $u <_2 v$ or $u = v$ or $v <_2 u$. Since neither $u <_2 v$ nor $u = v$ can hold (indeed, $u <_2 v$ is ruled out by assumption, whereas $u = v$ is ruled out by $u <_1 v$),

would contradict the assumption that we do not have $\phi(u) < \phi(v)$. Hence, u is not $<_1$-covered by v. Consequently, there exists a $w \in E$ such that $u <_1 w <_1 v$ (since $u <_1 v$). Consider this w. Applying (1.11) to $a = u$, $c = w$ and $b = v$, we see that both numbers $|[u, w]|$ and $|[w, v]|$ are smaller than $|[u, v]|$, and therefore neither (u, w) nor (w, v) is a malrelation (since we picked (u, v) to be a malrelation with minimum $|[u, v]|$).

But $\phi(v) \leq \phi(u)$ (since we do not have $\phi(u) < \phi(v)$). On the other hand, $u <_1 w$ and therefore $\phi(u) \leq \phi(w)$ (by Claim 1). Furthermore, $w <_1 v$ and thus $\phi(w) \leq \phi(v)$ (by Claim 1). The chain of inequalities $\phi(v) \leq \phi(u) \leq \phi(w) \leq \phi(v)$ ends with the same term that it begins with; therefore, it must be a chain of equalities. In other words, we have $\phi(v) = \phi(u) = \phi(w) = \phi(v)$.

Now, using $\phi(w) = \phi(v)$, we can see that $w <_2 v$ [30]. The same argument (applied to u and w instead of w and v) shows that $u <_2 w$. Thus, $u <_2 w <_2 v$, which contradicts the fact that we do not have $u <_2 v$. This contradiction proves Claim 3.

Proof of Claim 2: The condition "$f <_2 e$" is stronger than "not $e <_2 f$". Thus, Claim 2 follows from Claim 3.

Claims 1 and 2 are now both proven, and so Lemma 1.7.1 follows. □

Proof of Lemma 1.4.5. Consider the following three logical statements:

Statement 1: The orbit O is E-coeven.

Statement 2: All elements of O are E-coeven.

Statement 3: At least one element of O is E-coeven.

Statements 1 and 2 are equivalent (according to the definition of when an orbit is E-coeven). Our goal is to prove that Statements 1 and 3 are equivalent (because this is precisely what Lemma 1.4.5 says). Thus, it clearly suffices to show that Statements 2 and 3 are equivalent. Since Statement 2 obviously implies Statement 3, we

---

we thus have $v <_2 u$. Therefore, $\phi(u) < \phi(v)$ by Condition 2 (applied to $e = u$ and $f = v$), qed.

[30] *Proof.* Assume the contrary. Thus, we do not have $w <_2 v$. But $\phi(w) = \phi(v)$ shows that we do not have $\phi(w) < \phi(v)$. Hence, (w, v) is a malrelation (since $w <_1 v$ and not $w <_2 v$ but not $\phi(w) < \phi(v)$). This contradicts the fact that (w, v) is not a malrelation. This contradiction completes the proof.

therefore only need to show that Statement 3 implies Statement 2. Thus, assume that Statement 3 holds. We need to prove that Statement 2 holds.

There exists at least one E-coeven $\phi \in O$ (because we assumed that Statement 3 holds). Consider this ϕ. Now, let $\pi \in O$ be arbitrary. We shall show that π is E-coeven.

We know that ϕ is E-coeven. In other words,

$$\text{every } g \in G \text{ satisfying } g\phi = \phi \text{ is } E\text{-even.} \tag{1.24}$$

Now, let $g \in G$ be such that $g\pi = \pi$. Since ϕ belongs to the G-orbit O, we have $O = G\phi$. Now, $\pi \in O = G\phi$. In other words, there exists some $h \in G$ such that $\pi = h\phi$. Consider this h. We have $g\pi = \pi$. Since $\pi = h\phi$, this rewrites as $gh\phi = h\phi$. In other words, $h^{-1}gh\phi = \phi$. Thus, (1.24) (applied to $h^{-1}gh$ instead of g) shows that $h^{-1}gh$ is E-even. In other words,

$$\text{the action of } h^{-1}gh \text{ on } E \text{ is an even permutation of } E. \tag{1.25}$$

Now, let ε be the group homomorphism from G to $\operatorname{Aut} E$ which describes the G-action on E. Then, $\varepsilon\left(h^{-1}gh\right)$ is the action of $h^{-1}gh$ on E, and thus is an even permutation of E (by (1.25)).

But since ε is a group homomorphism, we have $\varepsilon\left(h^{-1}gh\right) = \varepsilon\left(h\right)^{-1}\varepsilon\left(g\right)\varepsilon\left(h\right)$. Thus, the permutations $\varepsilon\left(h^{-1}gh\right)$ and $\varepsilon\left(g\right)$ of E are conjugate. Since the permutation $\varepsilon\left(h^{-1}gh\right)$ is even, this shows that the permutation $\varepsilon\left(g\right)$ is even. In other words, the action of g on E is an even permutation of E. In other words, g is E-even.

Now, let us forget that we fixed g. We thus have shown that every $g \in G$ satisfying $g\pi = \pi$ is E-even. In other words, π is E-coeven.

Let us now forget that we fixed π. Thus, we have proven that every $\pi \in O$ is E-coeven. In other words, Statement 2 holds. We have thus shown that Statement 3 implies Statement 2. Consequently, Statements 2 and 3 are equivalent, and so the proof of Lemma 1.4.5 is complete. $\square$

Next, we will show three simple properties of posets on which groups act.

Proposition 1.7.2. Let E be a set. Let $<_1$ be a strict partial order relation on E. Let G be a finite group which acts on E. Assume that G preserves the relation $<_1$.

Let $g \in G$. Let E^g be the set of all orbits under the action of g on E. Define a binary relation $<_1^g$ on E^g by

$$(u <_1^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then, $<_1^g$ is a strict partial order relation.

Proposition 1.7.2 is precisely [Joch13, Lemma 2.4], but let us outline the proof for the sake of completeness:

Proof of Proposition 1.7.2. Let us first show that the relation $<_1^g$ is irreflexive. Indeed, assume the contrary. Thus, there exists a $u \in E^g$ such that $u <_1^g u$. Consider this u. Since $u <_1^g u$, there exist $a \in u$ and $b \in u$ with $a <_1 b$. Consider these a and b. There exists a $k \in \mathbb{N}$ such that $b = g^k a$ (since a and b both lie in one and the same g-orbit u). Consider this k.

The g-orbit u of a is finite (since g is finite). Thus, there exists a positive integer n such that $g^n a = a$. Consider this n. Notice that $g^{np} a = (g^n)^p a = a$ for every $p \in \mathbb{N}$ (since $g^n a = a$).

Now, $a <_1 b = g^k a$. Since G preserves the relation $<_1$, this shows that $ha <_1 hg^k a$ for every $h \in G$. Thus, $g^{\ell k} a <_1 g^{\ell k} g^k a$ for every $\ell \in \mathbb{N}$. Hence, $g^{\ell k} a <_1 g^{\ell k} g^k a = g^{(\ell+1)k} a$ for every $\ell \in \mathbb{N}$. Consequently, $g^{0k} a <_1 g^{1k} a <_1 g^{2k} a <_1 \cdots <_1 g^{nk} a$. Thus, $g^{0k} a <_1 g^{nk} a = a$ (since $g^{np} a = a$ for every $p \in \mathbb{N}$), which contradicts $g^{0k} a = 1_G a = a$. This contradiction proves that our assumption was wrong. Hence, the relation $<_1^g$ is irreflexive.

Let us next show that the relation $<_1^g$ is transitive. Indeed, let u, v and w be three elements of E^g such that $u <_1^g v$ and $v <_1^g w$. We must prove that $u <_1^g w$.

There exist $a \in u$ and $b \in v$ with $a <_1 b$ (since $u <_1^g v$). Consider these a and b.

There exist $a' \in v$ and $b' \in w$ with $a' <_1 b'$ (since $v <_1^g w$). Consider these a' and b'.

The elements b and a' lie in one and the same g-orbit (namely, in v). Hence, there exists some $k \in \mathbb{N}$ such that $a' = g^k b$. Consider this k. We have $a <_1 b$ and thus $g^k a <_1 g^k b$ (since G preserves the relation $<_1$). Hence, $g^k a <_1 g^k b = a' <_1 b'$. Since $g^k a \in u$ (because $a \in u$) and $b' \in w$, this shows that $u <_1^g w$. We thus have proven that the relation $<_1^g$ is transitive.

Now, we know that the relation $<_1^g$ is irreflexive and transitive, and thus also antisymmetric (since every irreflexive and transitive binary relation is antisymmetric). In other words, $<_1^g$ is a strict partial order relation. This proves Proposition 1.7.2. $\square$

Remark 1.7.3. Proposition 1.7.2 can be generalized: Let E be a set. Let $<_1$ be a strict partial order relation on E. Let G be a finite group which acts on E. Assume that G preserves the relation $<_1$. Let H be a subgroup of G. Let E^H be the set of all orbits under the action of H on E. Define a binary relation $<_1^H$ on E^H by

$$\left(u <_1^H v\right) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then, $<_1^H$ is a strict partial order relation.

This result (whose proof is quite similar to that of Proposition 1.7.2) implicitly appears in [Stan84, p. 30].

Proposition 1.7.4. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$.

Let $g \in G$. Let E^g be the set of all orbits under the action of g on E. Define a binary relation $<_1^g$ on E^g by

$$(u <_1^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Define a binary relation $<_2^g$ on E^g by

$$(u <_2^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b) .$$

Let $\mathbf{E}^g$ be the triple $(E^g, <_1^g, <_2^g)$. Then, $\mathbf{E}^g$ is a tertispecial double poset.

Proof of Proposition 1.7.4. Both relations $<_1$ and $<_2$ are strict partial order relations (since $\mathbf{E}$ is a double poset). Proposition 1.7.2 shows that $<_1^g$ is a strict partial order relation. Proposition 1.7.2 (applied to $<_2$ and $<_2^g$ instead of $<_1$ and $<_1^g$) shows that $<_2^g$ is a strict partial order relation. Thus, $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$ is a double poset. It remains to show that this double poset $\mathbf{E}^g$ is tertispecial.

Let u and v be two elements of E^g such that u is $<_1^g$-covered by v. We shall prove that u and v are $<_2^g$-comparable.

We have $u <_1^g v$ (since u is $<_1^g$-covered by v). In other words, there exist $a \in u$ and $b \in v$ with $a <_1 b$. Consider these a and b.

If there was a $c \in E$ satisfying $a <_1 c <_1 b$, then we would have $u <_1^g w <_1^g v$ with w being the g-orbit of c, and this would contradict the condition that u is $<_1^g$-covered by v. Hence, no such c can exist. In other words, a is $<_1$-covered by b. Thus, a and b are $<_2$-comparable (since the double poset $\mathbf{E}$ is tertispecial). Consequently, u and v are $<_2^g$-comparable.

Now, let us forget that we fixed u and v. We thus have shown that if u and v are two elements of E^g such that u is $<_1^g$-covered by v, then u and v are $<_2^g$-comparable. In other words, the double poset $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$ is tertispecial. This proves Proposition 1.7.4. □

Proposition 1.7.5. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$.

Let $g \in G$. Define the set E^g, the relations $<_1^g$ and $<_2^g$ and the triple $\mathbf{E}^g$ as in Proposition 1.7.4. Thus, $\mathbf{E}^g$ is a tertispecial double poset (by Proposition 1.7.4).

There is a bijection Φ between

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the maps $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Namely, this bijection Φ sends any map $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$ to the map $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$ defined by

$$\overline{\pi}(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u.$$

(The well-definedness of this map $\overline{\pi}$ is easy to see: Indeed, from $g\pi = \pi$, we can conclude that any two elements a_1 and a_2 of a given g-orbit u satisfy $\pi(a_1) = \pi(a_2)$.)

Consider this bijection Φ. Let $\pi : E \to \{1, 2, 3, \ldots\}$ be a map satisfying $g\pi = \pi$.

(a) If π is an $\mathbf{E}$-partition, then $\Phi(\pi)$ is an $\mathbf{E}^g$-partition.

(b) If $\Phi(\pi)$ is an $\mathbf{E}^g$-partition, then π is an $\mathbf{E}$-partition.

(c) Let $w : E \to \{1, 2, 3, \ldots\}$ be map. Define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by

$$w^g(u) = \sum_{a \in u} w(a) \qquad \text{for every } u \in E^g.$$

Then, $\mathbf{x}_{\Phi(\pi), w^g} = \mathbf{x}_{\pi, w}$.

Proof of Proposition 1.7.5 (sketched). The definition of Φ shows that

$$(\Phi(\pi))(u) = \pi(a) \qquad \text{for every } u \in E^g \text{ and } a \in u. \tag{1.26}$$

(a) Assume that π is an $\mathbf{E}$-partition. We want to show that $\Phi(\pi)$ is an $\mathbf{E}^g$-partition. In order to do so, we can use Lemma 1.7.1 (applied to $\mathbf{E}^g$, $(E^g, <_1^g, <_2^g)$ and $\Phi(\pi)$ instead of $\mathbf{E}$, $(E, <_1, <_2)$ and ϕ); we only need to check the following two conditions:

Condition 1: If $e \in E^g$ and $f \in E^g$ are such that e is $<_1^g$-covered by f, and if we have $e <_2^g f$, then $(\Phi(\pi))(e) \leq (\Phi(\pi))(f)$.

Condition 2: If $e \in E^g$ and $f \in E^g$ are such that e is $<_1^g$-covered by f, and if we have $f <_2^g e$, then $(\Phi(\pi))(e) < (\Phi(\pi))(f)$.

Proof of Condition 1: Let $e \in E^g$ and $f \in E^g$ be such that e is $<_1^g$-covered by f. Assume that we have $e <_2^g f$.

We have $e <_1^g f$ (because e is $<_1^g$-covered by f). In other words, there exist $a \in e$ and $b \in f$ satisfying $a <_1 b$. Consider these a and b. Since π is an $\mathbf{E}$-partition, we have $\pi(a) \leq \pi(b)$ (since $a <_1 b$). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(e) = \pi(a)$ (since $a \in e$) and $(\Phi(\pi))(f) = \pi(b)$ (since $b \in f$). Thus, $(\Phi(\pi))(e) = \pi(a) \leq \pi(b) = (\Phi(\pi))(f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E^g$ and $f \in E^g$ be such that e is $<_1^g$-covered by f. Assume that we have $f <_2^g e$.

We have $e <_1^g f$ (because e is $<_1^g$-covered by f). In other words, there exist $a \in e$ and $b \in f$ satisfying $a <_1 b$. Consider these a and b.

If there was a $c \in E$ satisfying $a <_1 c <_1 b$, then the g-orbit w of this c would satisfy $e <_1^g w <_1^g f$, which would contradict the fact that e is $<_1^g$-covered by f. Hence, there exists no such c. In other words, a is $<_1$-covered by b (since $a <_1 b$). Therefore, a and b are $<_2$-comparable (since $\mathbf{E}$ is tertispecial). In other words, we have either $a <_2 b$ or $a = b$ or $b <_2 a$. Since $a <_2 b$ is impossible (because if we had $a <_2 b$, then we would have $e <_2^g f$ (since $a \in e$ and $b \in f$), which would contradict $f <_2^g e$ (since $<_2^g$ is a strict partial order relation)), and since $a = b$ is impossible (because $a <_1 b$), we therefore must have $b <_2 a$. But since π is an $\mathbf{E}$-partition, we have $\pi(a) < \pi(b)$ (since $a <_1 b$ and $b <_2 a$). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(e) = \pi(a)$ (since $a \in e$) and $(\Phi(\pi))(f) = \pi(b)$ (since $b \in f$). Thus, $(\Phi(\pi))(e) = \pi(a) < \pi(b) = (\Phi(\pi))(f)$. Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 1.7.5 (a) is proven.

(b) Assume that $\Phi(\pi)$ is an $\mathbf{E}^g$-partition. We want to show that π is an $\mathbf{E}$-

partition. In order to do so, we can use Lemma 1.7.1 (applied to $\phi = \pi$); we only need to check the following two conditions:

Condition 1: If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $e <_2 f$, then $\pi(e) \leq \pi(f)$.

Condition 2: If $e \in E$ and $f \in E$ are such that e is $<_1$-covered by f, and if we have $f <_2 e$, then $\pi(e) < \pi(f)$.

Proof of Condition 1: Let $e \in E$ and $f \in E$ be such that e is $<_1$-covered by f. Assume that we have $e <_2 f$.

We have $e <_1 f$ (since e is $<_1$-covered by f). Let u and v be the g-orbits of e and f, respectively. Thus, u and v belong to E^g, and satisfy $u <_1^g v$ (since $e <_1 f$). Hence, $(\Phi(\pi))(u) \leq (\Phi(\pi))(v)$ (since $\Phi(\pi)$ is an $\mathbf{E}^g$-partition). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(u) = \pi(e)$ (since $e \in u$) and $(\Phi(\pi))(v) = \pi(f)$ (since $f \in v$). Thus, $\pi(e) = (\Phi(\pi))(u) \leq (\Phi(\pi))(v) = \pi(f)$. Hence, Condition 1 is proven.

Proof of Condition 2: Let $e \in E$ and $f \in E$ be such that e is $<_1$-covered by f. Assume that we have $f <_2 e$.

We have $e <_1 f$ (since e is $<_1$-covered by f). Let u and v be the g-orbits of e and f, respectively. Thus, u and v belong to E^g, and satisfy $u <_1^g v$ (since $e <_1 f$) and $v <_2^g u$ (since $f <_2 e$). Hence, $(\Phi(\pi))(u) < (\Phi(\pi))(v)$ (since $\Phi(\pi)$ is an $\mathbf{E}^g$-partition). But the definition of $\Phi(\pi)$ shows that $(\Phi(\pi))(u) = \pi(e)$ (since $e \in u$) and $(\Phi(\pi))(v) = \pi(f)$ (since $f \in v$). Thus, $\pi(e) = (\Phi(\pi))(u) < (\Phi(\pi))(v) = \pi(f)$. Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 1.7.5 (b) is proven.

(c) The definition of $\mathbf{x}_{\Phi(\pi),w^g}$ shows that

$$\mathbf{x}_{\Phi(\pi),w^g} = \prod_{e\in E^g} x_{(\Phi(\pi))(e)}^{w^g(e)} = \prod_{u\in E^g} \underbrace{x_{(\Phi(\pi))(u)}^{w^g(u)}}_{\substack{=\prod_{a\in u} x_{(\Phi(\pi))(u)}^{w(a)} \\ (\text{since } w^g(u)=\sum_{a\in u} w(a))}} = \prod_{u\in E^g} \prod_{a\in u} \underbrace{x_{(\Phi(\pi))(u)}^{w(a)}}_{\substack{=x_{\pi(a)}^{w(a)} \\ (\text{by (1.26)})}}$$
$$= \underbrace{\prod_{u\in E^g} \prod_{a\in u}}_{=\prod_{a\in E}} x_{\pi(a)}^{w(a)} = \prod_{a\in E} x_{\pi(a)}^{w(a)} = \prod_{e\in E} x_{\pi(e)}^{w(e)} = \mathbf{x}_{\pi,w}$$

(by the definition of $\mathbf{x}_{\pi,w}$). This proves Proposition 1.7.5 (c). □

Our next lemma is a standard argument in Pólya enumeration theory (compare it with the proof of Burnside's lemma):

Lemma 1.7.6. Let G be a finite group. Let F be a finite G-set. Let O be a G-orbit on F, and let $\pi \in O$.

(a) We have
$$\frac{1}{|O|} = \frac{1}{|G|} \sum_{\substack{g\in G; \\ g\pi=\pi}} 1. \tag{1.27}$$

(b) Let E be a further finite G-set. For every $g \in G$, let $\operatorname{sign}_E g$ denote the sign of the permutation of E that sends every $e \in E$ to ge. (Thus, $g \in G$ is E-even if and only if $\operatorname{sign}_E g = 1$.) Then,
$$\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g\in G; \\ g\pi=\pi}} \operatorname{sign}_E g. \tag{1.28}$$

Proof of Lemma 1.7.6. Let $\operatorname{Stab}_G \pi$ denote the stabilizer of π; this is the subgroup $\{g \in G \mid g\pi = \pi\}$ of G. The G-orbit of π is O (since O is a G-orbit on F, and since

$\pi \in O$). In other words, $O = G\pi$. Therefore,

$$|O| = |G\pi| = \frac{|G|}{|\operatorname{Stab}_G \pi|}$$

(by the orbit-stabilizer theorem) and thus

$$\frac{1}{|O|} = \frac{|\operatorname{Stab}_G \pi|}{|G|}. \tag{1.29}$$

(a) We have

$$\sum_{\substack{g\in G;\\ g\pi=\pi}} 1 = \left| \underbrace{\{g \in G \mid g\pi = \pi\}}_{=\operatorname{Stab}_G \pi} \right| = |\operatorname{Stab}_G \pi| .$$

Hence,

$$\frac{1}{|G|} \underbrace{\sum_{\substack{g\in G;\\ g\pi=\pi}} 1}_{=|\operatorname{Stab}_G \pi|} = \frac{1}{|G|} |\operatorname{Stab}_G \pi| = \frac{|\operatorname{Stab}_G \pi|}{|G|} = \frac{1}{|O|}$$

(by (1.29)). This proves Lemma 1.7.6 (a).

(b) We need to prove (1.28). Assume first that O is E-coeven. Thus, π is E-coeven (by the definition of what it means for O to be E-coeven). This means that every $g \in G$ satisfying $g\pi = \pi$ is E-even. Hence, every $g \in G$ satisfying $g\pi = \pi$ satisfies $\operatorname{sign}_E g = 1$. Thus,

$$\begin{aligned} \frac{1}{|G|} \sum_{\substack{g\in G;\\ g\pi=\pi}} \underbrace{\operatorname{sign}_E g}_{=1} &= \frac{1}{|G|} \sum_{\substack{g\in G;\\ g\pi=\pi}} 1 = \frac{1}{|O|} \qquad \text{(by (1.27))} \\ &= \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \qquad \text{(since } O \text{ is } E\text{-coeven)} . \end{aligned}$$

Thus, we have proven (1.28) under the assumption that O is E-coeven. We can therefore WLOG assume the opposite now. Thus, assume that O is not E-coeven. Hence, no element of O is E-coeven (due to the contrapositive of Lemma 1.4.5). In particular, π is not E-coeven. In other words, not every $g \in G$ satisfying $g\pi = \pi$ is

E-even. In other words, not every $g \in \operatorname{Stab}_G \pi$ is E-even (since the elements $g \in G$ satisfying $g\pi = \pi$ are exactly the elements $g \in \operatorname{Stab}_G \pi$). In other words, not every $g \in \operatorname{Stab}_G \pi$ satisfies $\operatorname{sign}_E g = 1$.

Now, the map

$$\operatorname{Stab}_G \pi \to \{1, -1\}, \qquad g \mapsto \operatorname{sign}_E g$$

is a group homomorphism (since the sign of a permutation is multiplicative) and is not the trivial homomorphism (since not every $g \in \operatorname{Stab}_G \pi$ satisfies $\operatorname{sign}_E g = 1$). Hence, it must send exactly half the elements of $\operatorname{Stab}_G \pi$ to 1 and the other half to -1. Therefore, the addends in the sum $\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g$ cancel each other out (one half of them are 1, and the others are -1). Therefore, $\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g = 0$, so that

$$\frac{1}{|G|} \underbrace{\sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g}_{=0} = 0 = \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}$$

(since O is not E-coeven). Thus,

$$\frac{1}{|G|} \underbrace{\sum_{\substack{g \in G; \\ g\pi = \pi}}}_{= \sum_{g \in \operatorname{Stab}_G \pi}} \operatorname{sign}_E g = \frac{1}{|G|} \sum_{g \in \operatorname{Stab}_G \pi} \operatorname{sign}_E g = \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}.$$

This proves (1.28). Lemma 1.7.6 (b) is thus proven. ☐

Proof of Theorem 1.4.6 (sketched). For every $g \in G$, define a tertispecial double poset $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$ as follows:

Let E^g be the set of all orbits under the action of g on E. Define a binary relation $<_1^g$ on E^g by

$$(u <_1^g v) \Longleftrightarrow (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Similarly, define a strict partial order relation $<_2^g$ on E^g by

$$\left(u <_2^g v\right) \iff \left(\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b\right).$$

Finally, set $\mathbf{E}^g = \left(E^g, <_1^g, <_2^g\right)$. Proposition 1.7.4 shows that this $\mathbf{E}^g$ is a tertispecial double poset.

Furthermore, for every $g \in G$, define a map $w^g : E^g \to \{1, 2, 3, \ldots\}$ by $w^g(u) = \sum_{a \in u} w(a)$. (Since G preserves w, the numbers $w(a)$ for all $a \in u$ are equal (for given u), and thus $\sum_{a \in u} w(a)$ can be rewritten as $|u| \cdot w(b)$ for any particular $b \in u$.) Now,

$$S\left(\Gamma\left(\left(E^g, <_1^g, <_2^g\right), w^g\right)\right) = (-1)^{|E^g|} \Gamma\left(\left(E^g, >_1^g, <_2^g\right), w^g\right) \tag{1.30}$$

(by Theorem 1.4.2, applied to $\left(\left(E^g, <_1^g, <_2^g\right), w^g\right)$ instead of $\left(\left(E, <_1, <_2\right), w\right)$).

For every $g \in G$, we have

$$\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w} = \Gamma\left(\mathbf{E}^g, w^g\right) \tag{1.31}$$

[31].

[31] *Proof of* (1.31)*:* Let $g \in G$. In Proposition 1.7.5, we have introduced a bijection Φ between

- the maps $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the maps $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Parts (a) and (b) of Proposition 1.7.5 show that this bijection Φ restricts to a bijection between

- the $\mathbf{E}$-partitions $\pi : E \to \{1, 2, 3, \ldots\}$ satisfying $g\pi = \pi$

and

- the $\mathbf{E}^g$-partitions $\overline{\pi} : E^g \to \{1, 2, 3, \ldots\}$.

Hence,

$$\sum_{\pi \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\pi, w^g} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \underbrace{\mathbf{x}_{\Phi(\pi), w^g}}_{\substack{= \mathbf{x}_{\pi, w} \\ \text{(by Proposition (1.7.5) (c))}}} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w},$$

whence $\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w} = \sum_{\pi \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\pi, w^g} = \Gamma\left(\mathbf{E}^g, w^g\right)$. This proves (1.31).

It is clearly sufficient to prove Theorem 1.4.6 for $\mathbf{k} = \mathbb{Z}$ (since all the power series that we are discussing are defined functorially in $\mathbf{k}$, and thus any identity between these series that holds over $\mathbb{Z}$ must hold over any $\mathbf{k}$). Therefore, it is sufficient to prove Theorem 1.4.6 for $\mathbf{k} = \mathbb{Q}$ (since $\mathrm{QSym}_{\mathbb{Z}}$ embeds into $\mathrm{QSym}_{\mathbb{Q}}$ [32]). Thus, we WLOG assume that $\mathbf{k} = \mathbb{Q}$. This will allow us to divide by positive integers.

Every G-orbit O on $\operatorname{Par}\mathbf{E}$ satisfies

$$\frac{1}{|O|}\sum_{\pi\in O}\underbrace{\mathbf{x}_{\pi,w}}_{\substack{=\mathbf{x}_{O,w} \\ \text{(since } \mathbf{x}_{O,w} \text{ is defined} \\ \text{to be } \mathbf{x}_{\pi,w})}} = \frac{1}{|O|}\underbrace{\sum_{\pi\in O}\mathbf{x}_{O,w}}_{=|O|\mathbf{x}_{O,w}} = \frac{1}{|O|}\,|O|\,\mathbf{x}_{O,w} = \mathbf{x}_{O,w}. \tag{1.32}$$

[32] Here, we are using the notation $\mathrm{QSym}_{\mathbf{k}}$ for the Hopf algebra QSym defined over a base ring $\mathbf{k}$.

Now,

$$\begin{aligned}
\Gamma(\mathbf{E},w,G) &= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \underbrace{\mathbf{x}_{O,w}}_{\substack{=\frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w} \\ \text{(by (1.32))}}} = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \frac{1}{|O|}\sum_{\pi\in O}\mathbf{x}_{\pi,w} \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \sum_{\pi\in O} \underbrace{\frac{1}{|O|}}_{\substack{=\frac{1}{|G|}\sum_{g\in G;\ g\pi=\pi} 1 \\ \text{(by (1.27), applied to } F=\operatorname{Par}\mathbf{E})}} \mathbf{x}_{\pi,w} \\
&= \underbrace{\sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \sum_{\pi\in O}}_{=\sum_{\pi\in\operatorname{Par}\mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}} \left(\frac{1}{|G|}\sum_{\substack{g\in G;\\ g\pi=\pi}} 1\right)\mathbf{x}_{\pi,w} \\
&= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \left(\frac{1}{|G|}\sum_{\substack{g\in G;\\ g\pi=\pi}} 1\right)\mathbf{x}_{\pi,w} = \frac{1}{|G|} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \sum_{\substack{g\in G;\\ g\pi=\pi}}}_{=\sum_{g\in G}\sum_{\pi \text{ is an } \mathbf{E}\text{-partition};\ g\pi=\pi}} \mathbf{x}_{\pi,w} \\
&= \frac{1}{|G|}\sum_{g\in G} \underbrace{\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition};\\ g\pi=\pi}} \mathbf{x}_{\pi,w}}_{\substack{=\Gamma(\mathbf{E}^g,w^g) \\ \text{(by (1.31))}}} \\
&= \frac{1}{|G|}\sum_{g\in G}\Gamma\left(\underbrace{\mathbf{E}^g}_{=\left(E^g,<_1^g,<_2^g\right)}, w^g\right) = \frac{1}{|G|}\sum_{g\in G}\Gamma\left(\left(E^g,<_1^g,<_2^g\right),w^g\right). \qquad (1.33)
\end{aligned}$$

Hence, $\Gamma(\mathbf{E},w,G) \in \operatorname{QSym}$ (by Proposition 1.3.5).

Applying the map S to both sides of the equality (1.33), we obtain

$$\begin{aligned}
S(\Gamma(\mathbf{E},w,G)) &= \frac{1}{|G|}\sum_{g\in G} \underbrace{S\left(\Gamma\left(\left(E^g,<_1^g,<_2^g\right),w^g\right)\right)}_{\substack{=(-1)^{|E^g|}\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right) \\ \text{(by (1.30))}}} \\
&= \frac{1}{|G|}\sum_{g\in G}(-1)^{|E^g|}\Gamma\left(\left(E^g,>_1^g,<_2^g\right),w^g\right). \qquad (1.34)
\end{aligned}$$

On the other hand, for every $g \in G$, let $\operatorname{sign}_E g$ denote the sign of the permutation of E that sends every $e \in E$ to ge. Thus, $g \in G$ is E-even if and only if $\operatorname{sign}_E g = 1$. Now, every G-orbit O on $\operatorname{Par} \mathbf{E}$ and every $\pi \in O$ satisfy

$$\begin{cases} \dfrac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \tag{1.35}$$

(by (1.28), applied to $F = \operatorname{Par} \mathbf{E}$). Furthermore,

$$\operatorname{sign}_E g = (-1)^{|E| - |E^g|} \tag{1.36}$$

for every $g \in G$ [33].

[33] *Proof of* (1.36)*:* Let $g \in G$. Recall that $\operatorname{sign}_E g$ is the sign of the permutation of E that sends every $e \in E$ to ge. But if σ is a permutation of a finite set X, then the sign of σ is $(-1)^{|X| - |X^\sigma|}$, where X^σ is the set of all cycles of σ. Applying this to $X = E$, $\sigma =$ (the permutation of E that sends every $e \in E$ to ge) and $X^\sigma = E^g$, we see that the sign of the permutation of E that sends every $e \in E$ to ge is $(-1)^{|E| - |E^g|}$. In other words, $\operatorname{sign}_E g = (-1)^{|E| - |E^g|}$, qed.

Now,

$$\begin{aligned}
&\Gamma^{+}(\mathbf{E}, w, G) \\
&= \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \underbrace{\mathbf{x}_{O,w}}_{\substack{= \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\ \text{(by (1.32))}}} = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\
&\qquad \left(\begin{array}{c} \text{here, we have extended the sum to all } G\text{-orbits} \\ \text{on } \operatorname{Par}\mathbf{E} \text{ (not just the } E\text{-coeven ones); but all new addends are 0} \\ \text{and therefore do not influence the value of the sum} \end{array} \right) \\
&= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \sum_{\pi \in O} \underbrace{\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}}_{\substack{= \frac{1}{|G|} \sum_{g \in G;\ g\pi = \pi} \operatorname{sign}_E g \\ \text{(by (1.35))}}} \mathbf{x}_{\pi,w} \\
&= \underbrace{\sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \sum_{\pi \in O}}_{= \sum_{\pi \in \operatorname{Par}\mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}} \left(\frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \right) \mathbf{x}_{\pi,w} \\
&= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \left(\frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \operatorname{sign}_E g \right) \mathbf{x}_{\pi,w} = \frac{1}{|G|} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \sum_{\substack{g \in G; \\ g\pi = \pi}}}_{= \sum_{g \in G} \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}}} (\operatorname{sign}_E g)\, \mathbf{x}_{\pi,w} \\
&= \frac{1}{|G|} \sum_{g \in G} \underbrace{\operatorname{sign}_E g}_{\substack{= (-1)^{|E| - |E^g|} \\ \text{(by (1.36))}}} \underbrace{\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi,w}}_{\substack{= \Gamma(\mathbf{E}^g, w^g) \\ \text{(by (1.31))}}} \\
&= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|}\, \Gamma(\mathbf{E}^g, w^g) . \qquad (1.37)
\end{aligned}$$

Hence, $\Gamma^{+}(\mathbf{E}, w, G) \in \operatorname{QSym}$ (by Proposition 1.3.5).

The group G preserves the relation $>_1$ (since it preserves the relation $<_1$). Hence, applying (1.37) to $(E, >_1, <_2)$ instead of $\mathbf{E}$, we obtain

$$\Gamma^{+}((E, >_1, <_2), w, G) = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g).$$

Multiplying both sides of this equality by $(-1)^{|E|}$, we transform it into

$$\begin{aligned}
(-1)^{|E|} \Gamma^{+}((E, >_1, <_2), w, G) &= \frac{1}{|G|} \sum_{g \in G} \underbrace{(-1)^{|E|} (-1)^{|E| - |E^g|}}_{=(-1)^{|E^g|}} \Gamma((E^g, >_1^g, <_2^g), w^g) \\
&= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g) \\
&= S(\Gamma(\mathbf{E}, w, G)) \qquad \text{(by (1.34))}.
\end{aligned}$$

This proves Theorem 1.4.6. □

1.8 Application: Jochemko's theorem

We shall now demonstrate an application of Theorem 1.4.6: namely, we will use it to provide an alternative proof of [Joch13, Theorem 2.13]. The way we derive [Joch13, Theorem 2.13] from Theorem 1.4.6 is classical, and in fact was what originally motivated the discovery of Theorem 1.4.6 (although, of course, it cannot be conversely derived from [Joch13, Theorem 2.13], so it is an actual generalization).

An intermediate step between [Joch13, Theorem 2.13] and Theorem 1.4.6 will be the following fact:

Corollary 1.8.1. Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. For every $q \in \mathbb{N}$, let $\operatorname{Par}_q \mathbf{E}$ denote the set of all $\mathbf{E}$-partitions whose image is contained in $\{1, 2, \ldots, q\}$. Then, the group G also acts on $\operatorname{Par}_q \mathbf{E}$; namely, $\operatorname{Par}_q \mathbf{E}$ is a G-subset of the G-set

$\{1, 2, \ldots, q\}^E$ (see Definition 1.4.4 (d) for the definition of the latter).

(a) There exists a unique polynomial $\Omega_{\mathbf{E},G} \in \mathbb{Q}[X]$ such that every $q \in \mathbb{N}$ satisfies

$$\Omega_{\mathbf{E},G}(q) = (\text{the number of all } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}). \tag{1.38}$$

(b) This polynomial satisfies

$$\begin{aligned} &\Omega_{\mathbf{E},G}(-q) \\ &= (-1)^{|E|} (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, >_1, <_2)) \\ &= (-1)^{|E|} (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, <_1, >_2)) \end{aligned} \tag{1.39}$$

for all $q \in \mathbb{N}$.

Proof of Corollary 1.8.1 (sketched). Set $\mathbf{k} = \mathbb{Q}$. For any $f \in \operatorname{QSym}$ and any $q \in \mathbb{N}$, we define an element $\operatorname{ps}^1(f)(q) \in \mathbb{Q}$ by

$$\operatorname{ps}^1(f)(q) = f\left(\underbrace{1, 1, \ldots, 1}_{q \text{ times}}, 0, 0, 0, \ldots\right)$$

(that is, $\operatorname{ps}^1(f)(q)$ is the result of substituting 1 for $x_1, x_2, \ldots, x_q$ and 0 for $x_{q+1}, x_{q+2}, x_{q+3}, \ldots$ in the power series f).

(a) Consider the elements $\Gamma(\mathbf{E}, w, G)$ and $\Gamma^+(\mathbf{E}, w, G)$ of QSym defined in Theorem 1.4.6. Observe that $\operatorname{Par}_q \mathbf{E}$ is a G-subset of $\operatorname{Par} \mathbf{E}$.

Now, [GriRei15, Proposition 7.7 (i)] shows that, for any given $f \in \operatorname{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(f)(q)$. Applying this to $f = \Gamma(\mathbf{E}, w, G)$, we conclude that there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(\Gamma(\mathbf{E}, w, G))(q)$. But since every $q \in \mathbb{N}$

satisfies

$$\operatorname{ps}^1\left(\Gamma\left(\mathbf{E},w,G\right)\right)(q) = \underbrace{\left(\Gamma\left(\mathbf{E},w,G\right)\right)}_{=\sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}}\left(\underbrace{1,1,\ldots,1}_{q \text{ times}},0,0,0,\ldots\right)$$
$$= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \underbrace{\mathbf{x}_{O,w}\left(\underbrace{1,1,\ldots,1}_{q \text{ times}},0,0,0,\ldots\right)}_{=\begin{cases}1, & \text{if } O\subseteq \operatorname{Par}_q\mathbf{E};\\ 0, & \text{if } O\not\subseteq \operatorname{Par}_q\mathbf{E}\end{cases}}$$
$$= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \begin{cases}1, & \text{if } O\subseteq \operatorname{Par}_q\mathbf{E};\\ 0, & \text{if } O\not\subseteq \operatorname{Par}_q\mathbf{E}\end{cases}$$
$$= \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}_q\mathbf{E}} 1 = \left(\text{the number of all } G\text{-orbits on } \operatorname{Par}_q\mathbf{E}\right), \tag{1.40}$$

this rewrites as follows: There exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals (the number of all G-orbits on $\operatorname{Par}_q\mathbf{E}$). This proves Corollary 1.8.1 (a).

(b) [GriRei15, Proposition 7.7 (i)] shows that, for any given $f \in \operatorname{QSym}$, there exists a unique polynomial in $\mathbb{Q}[X]$ whose value on each $q \in \mathbb{N}$ equals $\operatorname{ps}^1(f)(q)$. This polynomial is denoted by $\operatorname{ps}^1(f)$ in [GriRei15, Proposition 7.7]. From our above proof of Corollary 1.8.1 (a), we see that

$$\Omega_{\mathbf{E},G} = \operatorname{ps}^1\left(\Gamma\left(\mathbf{E},w,G\right)\right).$$

But [GriRei15, Proposition 7.7 (iii)] shows that, for any $f \in \operatorname{QSym}$ and $m \in \mathbb{N}$, we have $\operatorname{ps}^1(S(f))(m) = \operatorname{ps}^1(f)(-m)$. Applying this to $f = \Gamma(\mathbf{E},w,G)$, we obtain

$$\operatorname{ps}^1\left(S\left(\Gamma\left(\mathbf{E},w,G\right)\right)\right)(m) = \underbrace{\operatorname{ps}^1\left(\Gamma\left(\mathbf{E},w,G\right)\right)}_{=\Omega_{\mathbf{E},G}}(-m) = \Omega_{\mathbf{E},G}(-m)$$

for any $m \in \mathbb{N}$. Thus, any $m \in \mathbb{N}$ satsfies

$$\begin{aligned}
\Omega_{\mathbf{E},G}(-m) &= \operatorname{ps}^1 \left(\underbrace{S(\Gamma(\mathbf{E}, w, G))}_{\substack{=(-1)^{|E|}\Gamma^+((E,>_1,<_2),w,G) \\ \text{(by Theorem 1.4.6)}}} \right)(m) \\
&= \operatorname{ps}^1 \left((-1)^{|E|} \Gamma^+ ((E, >_1, <_2), w, G) \right)(m) \\
&= (-1)^{|E|} \operatorname{ps}^1 \left(\Gamma^+ ((E, >_1, <_2), w, G) \right)(m) .
\end{aligned}$$

Renaming m as q in this equality, we see that every $q \in \mathbb{N}$ satisfies

$$\Omega_{\mathbf{E},G}(-q) = (-1)^{|E|} \operatorname{ps}^1 \left(\Gamma^+ ((E, >_1, <_2), w, G) \right)(q) . \tag{1.41}$$

But just as we proved (1.40), we can show that every $q \in \mathbb{N}$ satisfies

$$\operatorname{ps}^1 \left(\Gamma^+ (\mathbf{E}, w, G) \right)(q) = (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q \mathbf{E}) .$$

Applying this to $(E, >_1, <_2)$ instead of $\mathbf{E}$, we obtain

$$\begin{aligned}
&\operatorname{ps}^1 \left(\Gamma^+ ((E, >_1, <_2), w, G) \right)(q) \\
&= (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, >_1, <_2)) .
\end{aligned}$$

Now, (1.41) becomes

$$\begin{aligned}
\Omega_{\mathbf{E},G}(-q) &= (-1)^{|E|} \underbrace{\operatorname{ps}^1 \left(\Gamma^+ ((E, >_1, <_2), w, G) \right)(q)}_{=(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q(E,>_1,<_2))} \\
&= (-1)^{|E|} (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, >_1, <_2)) .
\end{aligned}$$

In order to prove Corollary 1.8.1 (b), it thus remains to show that

$$\begin{aligned}
&(\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, >_1, <_2)) \\
&= (\text{the number of all even } G\text{-orbits on } \operatorname{Par}_q (E, <_1, >_2))
\end{aligned} \tag{1.42}$$

for every $q \in \mathbb{N}$.

Proof of (1.42): Let $q \in \mathbb{N}$. Let $w_0 : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, q\}$ be the map sending each $i \in \{1, 2, \ldots, q\}$ to $q + 1 - i$. Then, the map

$$\operatorname{Par}_q (E, >_1, <_2) \to \operatorname{Par}_q (E, <_1, >_2), \qquad \pi \mapsto w_0 \circ \pi$$

is an isomorphism of G-sets (this is easy to check). Thus,
$\operatorname{Par}_q (E, >_1, <_2) \cong \operatorname{Par}_q (E, <_1, >_2)$ as G-sets. From this, (1.42) follows (by functoriality, if one wishes).

The proof of Corollary 1.8.1 (b) is now complete. □

Now, the second formula of [Joch13, Theorem 2.13] follows from our (1.39), applied to $\mathbf{E} = (P, \prec, <_\omega)$ (where $<_\omega$ is the partial order on P given by $(p <_\omega q) \Longleftrightarrow (\omega(p) < \omega(q))$). The first formula of [Joch13, Theorem 2.13] can also be derived from our above arguments. We leave the details to the reader.

Chapter 2

Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions

Abstract

The dual immaculate functions are a basis of the ring QSym of quasisymmetric functions, and form one of the most natural analogues of the Schur functions. The dual immaculate function corresponding to a composition is a weighted generating function for immaculate tableaux in the same way as a Schur function is for semistandard Young tableaux; an "immaculate tableau" is defined similarly to a semistandard Young tableau, but the shape is a composition rather than a partition, and only the first column is required to strictly increase (whereas the other columns can be arbitrary; but each row has to weakly increase). Dual immaculate functions have been introduced by Berg, Bergeron, Saliola, Serrano and Zabrocki in arXiv:1208.5191, and have since been found to possess numerous nontrivial properties.

In this note, we prove a conjecture of Mike Zabrocki which provides an alternative construction for the dual immaculate functions in terms of certain "vertex operators". The proof uses a dendriform structure on the ring QSym; we discuss the relation of this structure to known dendriform structures on the combinatorial Hopf algebras FQSym and WQSym.

2.1 Introduction

The three most well-known combinatorial Hopf algebras that are defined over any commutative ring $\mathbf{k}$ are the Hopf algebra of symmetric functions (denoted Sym), the Hopf algebra of quasisymmetric functions (denoted QSym), and that of noncommutative symmetric functions (denoted NSym). The first of these three has been studied for several decades, while the latter two are newer; we refer to [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5] for expositions of them[1]. All three of these Hopf algebras are known to carry multiple algebraic structures, and have several bases of combinatorial and algebraic significance. The Schur functions – forming a basis of Sym – are probably the most important of these bases; a natural question is thus to seek similar bases for QSym and NSym.

Several answers to this question have been suggested, but the simplest one appears to be given in a 2013 paper by Berg, Bergeron, Saliola, Serrano and Zabrocki [BBSSZ13a]: They define the *immaculate (noncommutative symmetric) functions* (which form a basis of NSym) and the *dual immaculate (quasi-symmetric) functions* (which form a basis of QSym). These two bases are mutually dual and satisfy analogues of various properties of the Schur functions. Among these are a Littlewood-Richardson rule [BBSSZ13b], a Pieri rule [BSOZ13], and a representation-theoretical interpretation [BBSSZ13c]. The immaculate functions can be defined by an analogue of the Jacobi-Trudi identity (see [BBSSZ13a, Remark 3.28] for details), whereas the dual immaculate functions can be defined as generating functions for "immaculate tableaux" in analogy to the Schur functions being generating functions for semistandard tableaux (see Proposition 2.4.4 below).

The original definition of the immaculate functions ([BBSSZ13a, Definition 3.2]) is by applying a sequence of so-called *noncommutative Bernstein operators* to the constant power series $1 \in \text{NSym}$. Around 2013, Mike Zabrocki conjectured that the dual immaculate functions can be obtained by a similar use of "quasi-symmetric

[1] Historically, the origin of the noncommutative symmetric functions is in [GKLLRT95], whereas the quasisymmetric functions have been introduced in [Gessel84]. See also [Stan99, Section 7.19] specifically for the quasisymmetric functions and their enumerative applications (although the Hopf algebra structure does not appear in this source).

Bernstein operators". The purpose of this note is to prove this conjecture (Corollary 2.5.6 below). Along the way, we define certain new binary operations on QSym; two of them give rise to a structure of a dendriform algebra [EbrFar08], which seems to be interesting in its own right.

This note is organized as follows: In Section 2.2, we recall basic properties of quasisymmetric (and symmetric) functions and introduce the notations that we shall use. In Section 2.3, we define two binary operations $\prec$ and Φ on the power series ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and show that they restrict to operations on QSym which interact with the Hopf algebra structure of QSym in a useful way. In Section 2.4, we define the dual immaculate functions, and show that this definition agrees with the one given in [BBSSZ13a, Remark 3.28]; we then give a combinatorial interpretation of dual immaculate functions (which is not new, but has apparently never been explicitly stated). In Section 2.5, we prove Zabrocki's conjecture. In Section 2.6, we discuss how our binary operations can be lifted to noncommutative power series and restrict to operations on WQSym, which are closely related to similar operations that have appeared in the literature. In the final Section 2.7, we ask some further questions.

This Chapter is a modified version of the preprint arXiv:1410.0079v6. It follows partly the default version of the preprint, partly the detailed version (which is available as an ancillary file).

Acknowledgments

Mike Zabrocki kindly shared his conjecture with me during my visit to University of York, Toronto in March 2014. I am also grateful to Nantel Bergeron for his invitation and hospitality. An anonymous referee made numerous helpful remarks.

2.2 Quasisymmetric functions

We assume that the reader is familiar with the basics of the theory of symmetric and quasisymmetric functions (as presented, e.g., in [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5]). However, let us define all the notations that we need (not least because they are not consistent across literature). We shall try to have our notations match those used in [BBSSZ13a, Section 2] as much as possible.

We use $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$.

A *composition* means a finite sequence of positive integers. For instance, $(2, 3)$ and $(1, 5, 1)$ are compositions. The *empty composition* (i.e., the empty sequence $()$) is denoted by $\varnothing$. We denote by Comp the set of all compositions. For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we denote by $|\alpha|$ the *size* of the composition α; this is the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$. If $n \in \mathbb{N}$, then a *composition of n* simply means a composition having size n. A *nonempty composition* means a composition that is not empty (or, equivalently, that has size > 0).

Let $\mathbf{k}$ be a commutative ring (which, for us, means a commutative ring with unity). This $\mathbf{k}$ will stay fixed throughout the paper. We shall define our symmetric and quasisymmetric functions over this commutative ring $\mathbf{k}$. [2] Every tensor sign $\otimes$ without a subscript should be understood to mean $\otimes_{\mathbf{k}}$.

Let $x_1, x_2, x_3, \ldots$ be countably many distinct indeterminates. We let Mon be the free abelian monoid on the set $\{x_1, x_2, x_3, \ldots\}$ (written multiplicatively); it consists of elements of the form $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ for finitely supported $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^\infty$ (where "finitely supported" means that all but finitely many positive integers i satisfy $a_i = 0$). A *monomial* will mean an element of Mon. Thus, monomials are combinatorial objects (without coefficients), independent of $\mathbf{k}$.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. By abuse of notation, we shall identify every monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots \in$ Mon with the corresponding element $x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \cdot \cdots$ of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ when necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of $\mathbf{k}$); however, monomials don't live in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ per se[3].

[2] We do not require anything from $\mathbf{k}$ other than being a commutative ring. Some authors prefer to work only over specific rings $\mathbf{k}$, such as $\mathbb{Z}$ or $\mathbb{Q}$ (for example, [BBSSZ13a] always works over $\mathbb{Q}$). Usually, their results (and often also their proofs) nevertheless are just as valid over arbitrary $\mathbf{k}$. We see no reason to restrict our generality here.

[3] This is a technicality. Indeed, the monomials 1 and x_1 are distinct, but the corresponding elements 1 and x_1 of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ are identical when $\mathbf{k} = 0$. So we could not regard the monomials as lying in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by default.

The $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a topological $\mathbf{k}$-algebra; its topology is the product topology[4]. The polynomial ring $\mathbf{k}[x_1, x_2, x_3, \ldots]$ is a dense subset of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ with respect to this topology. This allows to prove certain identities in the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (such as the associativity of multiplication, just to give a stupid example) by first proving them in $\mathbf{k}[x_1, x_2, x_3, \ldots]$ (that is, for polynomials), and then arguing that they follow by density in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

If $\mathfrak{m}$ is a monomial, then $\operatorname{Supp} \mathfrak{m}$ will denote the subset

$$\{i \in \{1, 2, 3, \ldots\} \mid \text{the exponent with which } x_i \text{ occurs in } \mathfrak{m} \text{ is } > 0\}$$

of $\{1, 2, 3, \ldots\}$; this subset is finite. The *degree* $\deg \mathfrak{m}$ of a monomial $\mathfrak{m} = x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ is defined to be $a_1 + a_2 + a_3 + \cdots \in \mathbb{N}$.

A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if there exists an $N \in \mathbb{N}$ such that every monomial of degree $> N$ appears with coefficient 0 in P. Let $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ denote the $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ formed by the bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

The $\mathbf{k}$-*algebra of symmetric functions* over $\mathbf{k}$ is defined as the $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ consisting of all bounded-degree power series which are invariant under any permutation of the indeterminates. This $\mathbf{k}$-subalgebra is denoted by Sym. (Notice that Sym is denoted Λ in [GriRei15].) As a $\mathbf{k}$-module, Sym is known to have several bases, such as the basis of complete homogeneous symmetric functions (h_λ) and that of the Schur functions (s_λ), both indexed by the integer partitions.

[4]More precisely, this topology is defined as follows (see also [GriRei15, Section 2.6]):

We endow the ring $\mathbf{k}$ with the discrete topology. To define a topology on the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, we (temporarily) regard every power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as the family of its coefficients. Thus, $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ becomes a product of infinitely many copies of $\mathbf{k}$ (one for each monomial). This allows us to define a product topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This product topology is the topology that we will be using whenever we make statements about convergence in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ or write down infinite sums of power series. A sequence $(a_n)_{n \in \mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

Note that this is **not** the topology obtained by taking the completion of $\mathbf{k}[x_1, x_2, x_3, \ldots]$ with respect to the standard grading (in which all x_i have degree 1). Indeed, this completion is not even the whole $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are said to be *pack-equivalent* if they have the form $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ and $\mathfrak{n} = x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_\ell}^{\alpha_\ell}$ for some $\ell \in \mathbb{N}$, some positive integers α_1, α_2, ..., α_ℓ, some positive integers i_1, i_2, ..., i_ℓ satisfying $i_1 < i_2 < \cdots < i_\ell$, and some positive integers j_1, j_2, ..., j_ℓ satisfying $j_1 < j_2 < \cdots < j_\ell$ [5]. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if any two pack-equivalent monomials have equal coefficients in P. The **k**-*algebra of quasisymmetric functions* over **k** is defined as the **k**-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$ consisting of all bounded-degree power series which are quasisymmetric. It is clear that $\mathrm{Sym} \subseteq \mathrm{QSym}$.

For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, the *monomial quasisymmetric function* M_α is defined by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}.$$

One easily sees that $M_\alpha \in \mathrm{QSym}$ for every $\alpha \in \mathrm{Comp}$. It is well-known that $(M_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the **k**-module QSym; this is the so-called *monomial basis* of QSym. Other bases of QSym exist as well, some of which we are going to encounter below.

It is well-known that the **k**-algebras Sym and QSym can be canonically endowed with Hopf algebra structures such that Sym is a Hopf subalgebra of QSym. We refer to [HaGuKi10, Chapters 4 and 6] and [GriRei15, Chapters 2 and 5] for the definitions of these structures (and for a definition of the notion of a Hopf algebra); at this point, let us merely state a few properties. The comultiplication $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ of QSym satisfies

$$\Delta(M_\alpha) = \sum_{i=0}^{\ell} M_{(\alpha_1, \alpha_2, \ldots, \alpha_i)} \otimes M_{(\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_\ell)}$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \mathrm{Comp}$. The counit $\varepsilon : \mathrm{QSym} \to \mathbf{k}$ of QSym satisfies $\varepsilon(M_\alpha) = \begin{cases} 1, & \text{if } \alpha = \varnothing; \\ 0, & \text{if } \alpha \neq \varnothing \end{cases}$ for every $\alpha \in \mathrm{Comp}$.

[5] For instance, the monomial $x_1^4 x_2^2 x_3 x_7^6$ is pack-equivalent to $x_2^4 x_4^2 x_4 x_5^6$, but not to $x_2^2 x_1^4 x_3 x_7^6$.

We shall always use the notation Δ for the comultiplication of a Hopf algebra, the notation ε for the counit of a Hopf algebra, and the notation S for the antipode of a Hopf algebra. Occasionally we shall use *Sweedler's notation* for working with coproducts of elements of a Hopf algebra[6].

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of an $n \in \mathbb{N}$, then we define a subset $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ by

$$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\}.$$

This subset $D(\alpha)$ is called the *set of partial sums* of the composition α; see [GriRei15, Definition 5.10] for its further properties. Most importantly, a composition α of size n can be uniquely reconstructed from n and $D(\alpha)$.

If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is a composition of an $n \in \mathbb{N}$, then the *fundamental quasisymmetric function* $F_\alpha \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\text{bdd}}$ can be defined by

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{2.1}$$

(This is only one of several possible definitions of F_α. In [GriRei15, Definition 5.15], the power series F_α is denoted by L_α and defined differently; but [GriRei15, Proposition 5.17] proves the equivalence of this definition with ours.[7]) One can easily see that $F_\alpha \in \mathrm{QSym}$ for every $\alpha \in \mathrm{Comp}$. The family $(F_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym as well; it is called the *fundamental basis* of QSym.

[6] In a nutshell, Sweedler's notation (or, more precisely, the special case of Sweedler's notation that we will use) consists in writing $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ for the tensor $\Delta(c) \in C \otimes C$, where c is an element of a $\mathbf{k}$-coalgebra C. The sum $\sum_{(c)} c_{(1)} \otimes c_{(2)}$ symbolizes a representation of the tensor $\Delta(c)$ as a sum $\sum_{i=1}^{N} c_{1,i} \otimes c_{2,i}$ of pure tensors; it allows us to manipulate $\Delta(c)$ without having to explicitly introduce the N and the $c_{1,i}$ and the $c_{2,i}$. For instance, if $f : C \to \mathbf{k}$ is a $\mathbf{k}$-linear map, then we can write $\sum_{(c)} f(c_{(1)}) c_{(2)}$ for $\sum_{i=1}^{N} f(c_{1,i}) c_{2,i}$. Of course, we need to be careful not to use Sweedler's notation for terms which do depend on the specific choice of the N and the $c_{1,i}$ and the $c_{2,i}$; for instance, we must not write $\sum_{(c)} c_{(1)}^2 c_{(2)}$.

[7] In fact, [GriRei15, (5.5)] is exactly our equality (2.1).

2.3 Restricted-product operations

We shall now define two binary operations on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Definition 2.3.1. We define a binary operation $\prec : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation[8]) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases} \tag{2.2}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Some clarifications are in order. First, we are using $\prec$ as an operation symbol (rather than as a relation symbol as it is commonly used)[9]. Second, we consider $\min \varnothing$ to be ∞, and this symbol ∞ is understood to be greater than every integer[10]. Hence, $\mathfrak{m} \prec 1 = \mathfrak{m}$ for every nonconstant monomial $\mathfrak{m}$, and $1 \prec \mathfrak{m} = 0$ for every monomial $\mathfrak{m}$.

Let us first see why the operation $\prec$ in Definition 2.3.1 is well-defined. Recall that the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is the product topology. Hence, if $\prec$ is to be $\mathbf{k}$-bilinear and continuous with respect to it, we must have

$$\left(\sum_{\mathfrak{m} \in \operatorname{Mon}} \lambda_{\mathfrak{m}} \mathfrak{m} \right) \prec \left(\sum_{\mathfrak{n} \in \operatorname{Mon}} \mu_{\mathfrak{n}} \mathfrak{n} \right) = \sum_{\mathfrak{m} \in \operatorname{Mon}} \sum_{\mathfrak{n} \in \operatorname{Mon}} \lambda_{\mathfrak{m}} \mu_{\mathfrak{n}} \mathfrak{m} \prec \mathfrak{n}$$

for any families $(\lambda_{\mathfrak{m}})_{\mathfrak{m} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}}$ and $(\mu_{\mathfrak{n}})_{\mathfrak{n} \in \operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}}$ of scalars. Combined with (2.2), this uniquely determines $\prec$. Therefore, the binary operation $\prec$ satisfying the conditions of Definition 2.3.1 is unique (if it exists). But it also exists, because if we

[8] By this we mean that we write $a \prec b$ instead of $\prec(a, b)$.

[9] Of course, the symbol has been chosen because it is reminiscent of the smaller symbol in "$\min(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n})$".

[10] but not greater than itself

define a binary operation $\prec$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by the explicit formula

$$\left(\sum_{\mathfrak{m}\in\operatorname{Mon}} \lambda_{\mathfrak{m}}\mathfrak{m}\right) \prec \left(\sum_{\mathfrak{n}\in\operatorname{Mon}} \mu_{\mathfrak{n}}\mathfrak{n}\right) = \sum_{\substack{(\mathfrak{m},\mathfrak{n})\in\operatorname{Mon}\times\operatorname{Mon}\,; \\ \min(\operatorname{Supp}\mathfrak{m})<\min(\operatorname{Supp}\mathfrak{n})}} \lambda_{\mathfrak{m}}\mu_{\mathfrak{n}}\mathfrak{m}\mathfrak{n}$$

$$\text{for all } (\lambda_{\mathfrak{m}})_{\mathfrak{m}\in\operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}} \text{ and } (\mu_{\mathfrak{n}})_{\mathfrak{n}\in\operatorname{Mon}} \in \mathbf{k}^{\operatorname{Mon}},$$

then it clearly satisfies the conditions of Definition 2.3.1 (and is well-defined).

The operation $\prec$ is not associative; however, it is part of what is called a *dendriform algebra* structure on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (and on QSym, as we shall see below). The following remark (which will not be used until Section 2.6, and thus can be skipped by a reader not familiar with dendriform algebras) provides some details:

Remark 2.3.2. Let us define another binary operation $\succeq$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ similarly to $\prec$ except that we set

$$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m}\cdot\mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n})\,; \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}) \end{cases}.$$

Then, the structure $(\mathbf{k}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a dendriform algebra augmented to satisfy [EbrFar08, (15)]. In particular, any three elements a, b and c of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ satisfy

$$\begin{aligned} a \prec b + a \succeq b &= ab; \\ (a \prec b) \prec c &= a \prec (bc)\,; \\ (a \succeq b) \prec c &= a \succeq (b \prec c)\,; \\ a \succeq (b \succeq c) &= (ab) \succeq c. \end{aligned}$$

Now, we introduce another binary operation.

Definition 2.3.3. We define a binary operation $\Phi : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \Phi \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.

Here, $\max \varnothing$ is understood as 0. The welldefinedness of the operation Φ in Definition 2.3.3 is proven in the same way as that of the operation $\prec$.

Let us make a simple observation which will not be used until Section 2.6, but provides some context:

Proposition 2.3.4. The binary operation Φ is associative. It is also unital (with 1 serving as the unity).

Proof of Proposition 2.3.4. Let us first show that Φ is associative.

In order to show this, we must prove that

$$(a \Phi b) \Phi c = a \Phi (b \Phi c) \tag{2.3}$$

for any three elements a, b and c of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

But if $\mathfrak{m}$, $\mathfrak{n}$ and $\mathfrak{p}$ are three monomials, then the definition of Φ readily shows that

$$(\mathfrak{m} \Phi \mathfrak{n}) \Phi \mathfrak{p} = \begin{cases} \mathfrak{m}\mathfrak{n}\mathfrak{p}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}) \\ & \quad \text{and } \max(\operatorname{Supp}(\mathfrak{m}\mathfrak{n})) \leq \min(\operatorname{Supp} \mathfrak{p}); \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mathfrak{m} \Phi (\mathfrak{n} \Phi \mathfrak{p}) = \begin{cases} \mathfrak{m}\mathfrak{n}\mathfrak{p}, & \text{if } \max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}) \\ & \qquad \text{and } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}(\mathfrak{n}\mathfrak{p}))\,; \\ 0, & \text{otherwise} \end{cases} ;$$

thus, $(\mathfrak{m} \Phi \mathfrak{n}) \Phi \mathfrak{p} = \mathfrak{m} \Phi (\mathfrak{n} \Phi \mathfrak{p})$ (since it is straightforward to check that the condition $(\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n})$ and $\max(\operatorname{Supp}(\mathfrak{m}\mathfrak{n})) \leq \min(\operatorname{Supp}\mathfrak{p}))$ is equivalent to the condition
$(\max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p})$ and $\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}(\mathfrak{n}\mathfrak{p})))$ [11]). In other words, the equality (2.3) holds when a, b and c are monomials. Thus, this equality also holds whenever a, b and c are polynomials (since it is $\mathbf{k}$-linear in a, b and c), and consequently also holds whenever a, b and c are power series (since it is continuous in a, b and c). This proves that Φ is associative.

The proof of the fact that Φ is unital (with unity 1) is similar and left to the reader. Proposition 2.3.4 is thus shown. □

Here is another property of Φ that will not be used until Section 2.6:

Proposition 2.3.5. Every $a \in \operatorname{QSym}$ and $b \in \operatorname{QSym}$ satisfy $a \prec b \in \operatorname{QSym}$ and $a \Phi b \in \operatorname{QSym}$.

For example, we can explicitly describe the operation Φ on the monomial basis $(M_\gamma)_{\gamma \in \operatorname{Comp}}$ of QSym. Namely, any two nonempty compositions α and β satisfy $M_\alpha \Phi M_\beta = M_{[\alpha,\beta]} + M_{\alpha \odot \beta}$, where $[\alpha, \beta]$ and $\alpha \odot \beta$ are two compositions defined by

$$[(\alpha_1, \alpha_2, \ldots, \alpha_\ell), (\beta_1, \beta_2, \ldots, \beta_m)] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)\,;$$
$$(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \odot (\beta_1, \beta_2, \ldots, \beta_m) = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)\,.$$

[12] If one of α and β is empty, then $M_\alpha \Phi M_\beta = M_{[\alpha,\beta]}$.

[11] Indeed, both conditions are equivalent to
$(\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n})$ and $\max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{p})$ and $\max(\operatorname{Supp}\mathfrak{n}) \leq \min(\operatorname{Supp}\mathfrak{p}))$.

[12] What we call $[\alpha, \beta]$ is denoted by $\alpha \cdot \beta$ in [GriRei15, before Proposition 5.7].

Proposition 2.3.5 can reasonably be called obvious; the below proof owes its length mainly to the difficulty of formalizing the intuition.

Proof of Proposition 2.3.5. We shall first introduce a few more notations.

If $\mathfrak{m}$ is a monomial, then the *Parikh composition* of $\mathfrak{m}$ is defined as follows: Write $\mathfrak{m}$ in the form $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for some $\ell \in \mathbb{N}$, some positive integers α_1, α_2, $\ldots$, α_ℓ, and some positive integers i_1, i_2, $\ldots$, i_ℓ satisfying $i_1 < i_2 < \cdots < i_\ell$. Notice that this way of writing $\mathfrak{m}$ is unique. Then, the Parikh composition of $\mathfrak{m}$ is defined to be the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$.

We denote by $\operatorname{Parikh} \mathfrak{m}$ the Parikh composition of a monomial $\mathfrak{m}$. Now, it is easy to see that the definition of a monomial quasisymmetric function M_α can be rewritten as follows: For every $\alpha \in \operatorname{Comp}$, we have

$$M_\alpha = \sum_{\substack{\mathfrak{m} \in \operatorname{Mon}; \\ \operatorname{Parikh} \mathfrak{m} = \alpha}} \mathfrak{m}. \tag{2.4}$$

(Indeed, for any given composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, the monomials $\mathfrak{m}$ satisfying $\operatorname{Parikh} \mathfrak{m} = \alpha$ are precisely the monomials of the form $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ with i_1, i_2, $\ldots$, i_ℓ being positive integers satisfying $i_1 < i_2 < \cdots < i_\ell$.)

Now, pack-equivalent monomials can be characterized as follows: Two monomials $\mathfrak{m}$ and $\mathfrak{n}$ are pack-equivalent if and only if they have the same Parikh composition.

Now, we come to the proof of Proposition 2.3.5.

Let us first fix two compositions α and β. We shall prove that $M_\alpha \prec M_\beta \in \operatorname{QSym}$.

Write the compositions α and β as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $(\beta_1, \beta_2, \ldots, \beta_m)$. Let $\mathcal{S}_0$ denote the ℓ-element set $\{0\} \times \{1, 2, \ldots, \ell\}$. Let $\mathcal{S}_1$ denote the m-element set $\{1\} \times \{1, 2, \ldots, m\}$. Let $\mathcal{S}$ denote the $(\ell + m)$-element set $\mathcal{S}_0 \cup \mathcal{S}_1$. Let $\operatorname{inc}_0 : \{1, 2, \ldots, \ell\} \to \mathcal{S}$ be the map which sends every $p \in \{1, 2, \ldots, \ell\}$ to $(0, p) \in \mathcal{S}_0 \subseteq \mathcal{S}$. Let $\operatorname{inc}_1 : \{1, 2, \ldots, m\} \to \mathcal{S}$ be the map which sends every $q \in \{1, 2, \ldots, m\}$ to $(1, q) \in \mathcal{S}_1 \subseteq \mathcal{S}$.

Define a map $\rho : \mathcal{S} \to \{1, 2, 3, \ldots\}$ by setting

$$\rho(0, p) = \alpha_p \qquad \text{for all } p \in \{1, 2, \ldots, \ell\};$$
$$\rho(1, q) = \beta_q \qquad \text{for all } q \in \{1, 2, \ldots, m\}.$$

For every composition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we define a *$\gamma$-smap* to be a map $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ satisfying the following three properties:

- The maps $f \circ \operatorname{inc}_0$ and $f \circ \operatorname{inc}_1$ are strictly increasing.

- We have[13] $\min(f(\mathcal{S}_0)) < \min(f(\mathcal{S}_1))$.

- Every $u \in \{1, 2, \ldots, n\}$ satisfies

$$\sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u.$$

These three properties will be called the three *defining properties* of a γ-smap.

Now, we make the following claim:

Claim 1: Let $\mathfrak{q}$ be any monomial. Let γ be the Parikh composition of $\mathfrak{q}$. The coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all γ-smaps.

Proof of Claim 1: Write the composition γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$. Write the monomial $\mathfrak{q}$ in the form $\mathfrak{q} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n}$ for some positive integers k_1, k_2, $\ldots$, k_n satisfying $k_1 < k_2 < \cdots < k_n$. (This is possible because $(\gamma_1, \gamma_2, \ldots, \gamma_n) = \gamma$ is the Parikh composition of $\mathfrak{q}$.) Then, $\operatorname{Supp} \mathfrak{q} = \{k_1, k_2, \ldots, k_n\}$.

[13] Keep in mind that we set $\min \varnothing = \infty$.

From (2.4), we get $M_\alpha = \sum_{\substack{\mathfrak{m}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{m}=\alpha}} \mathfrak{m}$. Similarly, $M_\beta = \sum_{\substack{\mathfrak{n}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{n}=\beta}} \mathfrak{n}$. Hence,

$$\begin{aligned}
& M_\alpha \prec M_\beta \\
&= \left(\sum_{\substack{\mathfrak{m}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{m}=\alpha}} \mathfrak{m} \right) \prec \left(\sum_{\substack{\mathfrak{n}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{n}=\beta}} \mathfrak{n} \right) \\
&= \sum_{\substack{\mathfrak{m}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{m}=\alpha}} \sum_{\substack{\mathfrak{n}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{n}=\beta}} \underbrace{\mathfrak{m} \prec \mathfrak{n}}_{\substack{= \begin{cases} \mathfrak{m}\mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}) \end{cases} \\ \text{(by the definition of } \prec \text{ on monomials)}}} \\
&\qquad \text{(since the operation } \prec \text{ is } \mathbf{k}\text{-bilinear and continuous)} \\
&= \sum_{\substack{\mathfrak{m}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{m}=\alpha}} \sum_{\substack{\mathfrak{n}\in \operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{n}=\beta}} \begin{cases} \mathfrak{m}\mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}) \end{cases} \\
&= \sum_{\substack{(\mathfrak{m},\mathfrak{n})\in \operatorname{Mon}\times\operatorname{Mon};\\ \operatorname{Parikh}\mathfrak{m}=\alpha;\\ \operatorname{Parikh}\mathfrak{n}=\beta;\\ \min(\operatorname{Supp}\mathfrak{m})<\min(\operatorname{Supp}\mathfrak{n})}} \mathfrak{m}\mathfrak{n}.
\end{aligned}$$

Thus, the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all pairs $(\mathfrak{m}, \mathfrak{n}) \in \operatorname{Mon}\times\operatorname{Mon}$ such that $\operatorname{Parikh}\mathfrak{m} = \alpha$, $\operatorname{Parikh}\mathfrak{n} = \beta$, $\min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n})$ and $\mathfrak{m}\mathfrak{n} = \mathfrak{q}$. These pairs shall be called *spairs*. (The concept of a spair depends on $\mathfrak{q}$; we nevertheless omit $\mathfrak{q}$ from the notation, since we regard $\mathfrak{q}$ as fixed.)

Now, we shall construct a bijection between the γ-smaps and the spairs.

Indeed, we first define a map Φ from the set of γ-smaps to the set of spairs as follows: Let $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ be a γ-smap. Then, $\Phi(f)$ is defined to be the spair

$$\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p}, \prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right).$$

[14]

[14] This is a well-defined spair, for the following reasons:

- The first defining property of a γ-smap can be rewritten as "$f(0,1) < f(0,2) < \cdots <$

Conversely, we define a map Ψ from the set of spairs to the set of γ-smaps as follows: Let $(\mathfrak{m}, \mathfrak{n})$ be a spair. Then, we write the monomial $\mathfrak{m}$ in the form $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ for some positive integers $i_1, i_2, \ldots, i_\ell$ satisfying $i_1 < i_2 < \cdots < i_\ell$ (this is possible since $\operatorname{Parikh} \mathfrak{m} = \alpha$), and we write the monomial $\mathfrak{n}$ in the form $\mathfrak{n} = x_{j_1}^{\beta_1} x_{j_2}^{\beta_2} \cdots x_{j_m}^{\beta_m}$ for some positive integers $j_1, j_2, \ldots, j_m$ satisfying $j_1 < j_2 < \cdots < j_m$ (this is possible since $\operatorname{Parikh} \mathfrak{n} = \beta$). Of course, $\operatorname{Supp} \mathfrak{m} = \{i_1, i_2, \ldots, i_\ell\}$ and $\operatorname{Supp} \mathfrak{n} = \{j_1, j_2, \ldots, j_m\}$, so that $\min \{i_1, i_2, \ldots, i_\ell\} < \min \{j_1, j_2, \ldots, j_m\}$ (since

---

$f(0, \ell)$ and $f(1,1) < f(1,2) < \cdots < f(1,m)$". Combined with $k_1 < k_2 < \cdots < k_n$, this shows that $k_{f(0,1)} < k_{f(0,2)} < \cdots < k_{f(0,\ell)}$ and $k_{f(1,1)} < k_{f(1,2)} < \cdots < k_{f(1,m)}$. Hence, $\operatorname{Parikh} \left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) = \alpha$ and $\operatorname{Parikh} \left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) = \beta$.

- The second defining property of a γ-smap shows that $\min (f(\mathcal{S}_0)) < \min (f(\mathcal{S}_1))$, so that $k_{\min(f(\mathcal{S}_0))} < k_{\min(f(\mathcal{S}_1))}$ (since $k_1 < k_2 < \cdots < k_n$). But $\operatorname{Supp} \left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) = \left\{ k_{f(s)} \mid s \in \mathcal{S}_0 \right\}$ and thus $\min \left(\operatorname{Supp} \left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \right) = \min \left\{ k_{f(s)} \mid s \in \mathcal{S}_0 \right\} = k_{\min(f(\mathcal{S}_0))}$ (since $k_1 < k_2 < \cdots < k_n$). Similarly, $\min \left(\operatorname{Supp} \left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) \right) = k_{\min(f(\mathcal{S}_1))}$. Hence,

$$\min \left(\operatorname{Supp} \left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \right) = k_{\min(f(\mathcal{S}_0))} < k_{\min(f(\mathcal{S}_1))} = \min \left(\operatorname{Supp} \left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) \right).$$

- The third defining property of a γ-smap shows that $\sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u$ for every $u \in \{1, 2, \ldots, n\}$. Now, every $p \in \{1, 2, \ldots, \ell\}$ satisfies $\alpha_p = \rho(0, p)$. Hence, $\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} = \prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\rho(0,p)} = \prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}$. Similarly, $\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} = \prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}$. Multiplying these two identities, we obtain

$$\begin{aligned}
&\left(\prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\alpha_p} \right) \left(\prod_{q=1}^{m} x_{k_{f(1,q)}}^{\beta_q} \right) \\
&= \left(\prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)} \right) \left(\prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)} \right) = \prod_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} = \prod_{u=1}^{n} \prod_{s \in f^{-1}(u)} \underbrace{x_{k_{f(s)}}^{\rho(s)}}_{\substack{= x_{k_u}^{\rho(s)} \\ (\text{since } f(s) = u)}} \\
&= \prod_{u=1}^{n} \underbrace{\prod_{s \in f^{-1}(u)} x_{k_u}^{\rho(s)}}_{\substack{= x_{k_u}^{\gamma_u} \\ (\text{since } \sum_{s \in f^{-1}(u)} \rho(s) = \gamma_u)}} = \prod_{u=1}^{n} x_{k_u}^{\gamma_u} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n} = \mathfrak{q}.
\end{aligned}$$

$\min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n})$).

Now, we define a map $f : \mathcal{S} \to \{1, 2, \ldots, n\}$ as follows:

- For every $p \in \{1, 2, \ldots, \ell\}$, we let $f(0, p)$ be the unique $r \in \{1, 2, \ldots, n\}$ such that $i_p = k_r$. [15]

- For every $q \in \{1, 2, \ldots, m\}$, we let $f(1, q)$ be the unique $r \in \{1, 2, \ldots, n\}$ such that $j_q = k_r$. [16]

It is now straightforward to show that f is a γ-smap.[17] We define $\Psi(\mathfrak{m}, \mathfrak{n})$ to be this γ-smap f.

---

[15] To prove that this is well-defined, we need to show that this r exists and is unique. The uniqueness of r is obvious (since $k_1 < k_2 < \cdots < k_n$). To prove its existence, we notice that $i_p \in \operatorname{Supp}\mathfrak{m}$ (since $\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$ and $\alpha_p > 0$) and thus $i_p \in \operatorname{Supp}\mathfrak{m} \subseteq \operatorname{Supp}\underbrace{(\mathfrak{mn})}_{=\mathfrak{q}} = \operatorname{Supp}\mathfrak{q} = \{k_1, k_2, \ldots, k_n\}$.

[16] This is again well-defined, for similar reasons as the r in the definition of $f(0, p)$.

[17] Indeed:

- The first defining property of a γ-smap holds. (*Proof:* Let us show that $f \circ \operatorname{inc}_0$ is strictly increasing (the proof for $f \circ \operatorname{inc}_1$ is similar). Assume it is not. Then there exist some $p, p' \in \{1, 2, \ldots, \ell\}$ satisfying $p < p'$ and $(f \circ \operatorname{inc}_0)(p) \geq (f \circ \operatorname{inc}_0)(p')$. Consider these p, p'. We have $p < p'$, and therefore $i_p < i_{p'}$ (since $i_1 < i_2 < \cdots < i_\ell$). But $(f \circ \operatorname{inc}_0)(p) \geq (f \circ \operatorname{inc}_0)(p')$, and thus $k_{(f \circ \operatorname{inc}_0)(p)} \geq k_{(f \circ \operatorname{inc}_0)(p')}$ (since $k_1 < k_2 < \cdots < k_n$). Since $k_{(f \circ \operatorname{inc}_0)(p)} = k_{f(0,p)} = i_p$ (by the definition of $f(0, p)$) and similarly $k_{(f \circ \operatorname{inc}_0)(p')} = i_{p'}$, this rewrites as $i_p \geq i_{p'}$. This contradicts $i_p < i_{p'}$. This contradiction completes the proof.)

- The second defining property of a γ-smap holds. (*Proof:* We WLOG assume that ℓ and m are positive, since the other case is straightforward. We have $i_1 < i_2 < \cdots < i_\ell$. In other words, $k_{f(0,1)} < k_{f(0,2)} < \cdots < k_{f(0,\ell)}$ (since $k_{f(0,p)} = i_p$ for every $p \in \{1, 2, \ldots, \ell\}$). Hence, $f(0, 1) < f(0, 2) < \cdots < f(0, \ell)$ (since $k_1 < k_2 < \cdots < k_n$). Hence, $\min(f(\mathcal{S}_0)) = f(0, 1)$. Similarly, $\min(f(\mathcal{S}_1)) = f(1, 1)$. But from $i_1 < i_2 < \cdots < i_\ell$, we obtain $i_1 = \min\{i_1, i_2, \ldots, i_\ell\}$; similarly, $j_1 = \min\{j_1, j_2, \ldots, j_m\}$. Hence, $k_{f(0,1)} = i_1 = \min\{i_1, i_2, \ldots, i_\ell\} < \min\{j_1, j_2, \ldots, j_m\} = j_1 = k_{f(1,1)}$, so that $f(0, 1) < f(1, 1)$ (since $k_1 < k_2 < \cdots < k_n$). Hence, $\min(f(\mathcal{S}_0)) = f(0, 1) < f(1, 1) = \min(f(\mathcal{S}_1))$, qed.)

- The third defining property of a γ-smap holds. (*Proof:* We have

$$\mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \prod_{p=1}^{\ell} \underbrace{x_{i_p}^{\alpha_p}}_{\substack{= x_{k_{f(0,p)}}^{\rho(0,p)} \\ (\text{since } \alpha_p = \rho(0,p) \\ \text{and } i_p = k_{f(0,p)})}} = \prod_{p=1}^{\ell} x_{k_{f(0,p)}}^{\rho(0,p)} = \prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}$$

and similarly $\mathfrak{n} = \prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}$. Hence,

$$\mathfrak{mn} = \left(\prod_{s \in \mathcal{S}_0} x_{k_{f(s)}}^{\rho(s)}\right) \left(\prod_{s \in \mathcal{S}_1} x_{k_{f(s)}}^{\rho(s)}\right) = \prod_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} \qquad (\text{since } \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \text{ and } \mathcal{S}_0 \cap \mathcal{S}_1 = \varnothing).$$

We thus have defined a map Φ from the set of γ-smaps to the set of spairs, and a map Ψ from the set of spairs to the set of γ-smaps. It is straightforward to see that these two maps Φ and Ψ are mutually inverse, and thus Φ is a bijection. We thus have found a bijection between the set of γ-smaps and the set of spairs. Consequently, the number of all γ-smaps equals the number of all spairs.

Now, recall that the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all spairs. Hence, the coefficient of $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ equals the number of all γ-smaps (since the number of all γ-smaps equals the number of all spairs). In other words, Claim 1 is proven.

Claim 1 shows that the coefficient of a monomial $\mathfrak{q}$ in $M_\alpha \prec M_\beta$ depends not on $\mathfrak{q}$ but only on the Parikh composition of $\mathfrak{q}$. Thus, any two pack-equivalent monomials have equal coefficients in $M_\alpha \prec M_\beta$ (since any two pack-equivalent monomials have the same Parikh composition). In other words, the power series $M_\alpha \prec M_\beta$ is quasisymmetric. Since $M_\alpha \prec M_\beta \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]_{\text{bdd}}$, this yields that $M_\alpha \prec M_\beta \in \operatorname{QSym}$.

[At this point, let us remark that we can give an explicit formula for $M_\alpha \prec M_\beta$: Namely,

$$M_\alpha \prec M_\beta = \sum_{\gamma \in \operatorname{Comp}} \mathfrak{s}^\gamma_{\alpha,\beta} M_\gamma, \tag{2.5}$$

where $\mathfrak{s}^\gamma_{\alpha,\beta}$ is the number of all γ-smaps. Indeed, for every monomial $\mathfrak{q}$, the coefficient of $\mathfrak{q}$ on the left-hand side of (2.5) equals $\mathfrak{s}^\gamma_{\alpha,\beta}$ where γ is the Parikh composition of $\mathfrak{q}$ (because of Claim 1), whereas the coefficient of $\mathfrak{q}$ on the right-hand side of (2.5) also equals $\mathfrak{s}^\gamma_{\alpha,\beta}$ (for obvious reasons). Hence, every monomial has equal coefficients on the two sides of (2.5), and so (2.5) holds. Of course, (2.5) again proves that $M_\alpha \prec M_\beta \in \operatorname{QSym}$, since the sum $\sum\limits_{\gamma \in \operatorname{Comp}} \mathfrak{s}^\gamma_{\alpha,\beta} M_\gamma$ has only finitely many nonzero addends (indeed, γ-smaps can only exist if $|\gamma| \leq |\alpha| + |\beta|$).]

Now, let us forget that we fixed α and β. We thus have shown that every two

Thus, $\prod\limits_{s \in \mathcal{S}} x_{k_{f(s)}}^{\rho(s)} = \mathfrak{m}\mathfrak{n} = \mathfrak{q} = x_{k_1}^{\gamma_1} x_{k_2}^{\gamma_2} \cdots x_{k_n}^{\gamma_n}$. Now, for any $u \in \{1, 2, \ldots, n\}$, the exponent of x_{k_u} on the left hand side of this equality is $\sum\limits_{s \in f^{-1}(u)} \rho(s)$ (since $k_1 < k_2 < \cdots < k_n$), whereas the exponent of x_{k_u} on the right hand side is γ_u. Comparing these coefficients, we find $\sum\limits_{s \in f^{-1}(u)} \rho(s) = \gamma_u$.)

compositions α and β satisfy $M_\alpha \prec M_\beta \in \mathrm{QSym}$.

Now, let $a \in \mathrm{QSym}$ and $b \in \mathrm{QSym}$. We shall only prove that $a \prec b \in \mathrm{QSym}$ (since the proof of $a \Phi b \in \mathrm{QSym}$ is very similar[18]).

The statement that we need to prove ($a \prec b \in \mathrm{QSym}$) is $\mathbf{k}$-linear in each of a and b. Hence, we can WLOG assume that both a and b are elements of the monomial basis of QSym. Assume this. Thus, $a = M_\alpha$ and $b = M_\beta$ for some compositions α and β. Consider these α and β. Now, as we know, $M_\alpha \prec M_\beta \in \mathrm{QSym}$, so that $\underbrace{a}_{=M_\alpha} \prec \underbrace{b}_{=M_\beta} = M_\alpha \prec M_\beta \in \mathrm{QSym}$. This completes our proof of Proposition 2.3.5. □

Remark 2.3.6. The proof of Proposition 2.3.5 given above actually yields a combinatorial formula for $M_\alpha \prec M_\beta$ whenever α and β are two compositions. Namely, let α and β be two compositions. Then,

$$M_\alpha \prec M_\beta = \sum_{\gamma \in \mathrm{Comp}} \mathfrak{s}^\gamma_{\alpha,\beta} M_\gamma, \tag{2.6}$$

where $\mathfrak{s}^\gamma_{\alpha,\beta}$ is the number of all smaps $(\alpha, \beta) \to \gamma$. Here a *smap* $(\alpha, \beta) \to \gamma$ means what was called a γ-smap in the above proof of Proposition 2.3.5.

This is similar to the well-known formula for $M_\alpha M_\beta$ (see, for example, [GriRei15, Proposition 5.3]) which (translated into our language) states that

$$M_\alpha M_\beta = \sum_{\gamma \in \mathrm{Comp}} \mathfrak{t}^\gamma_{\alpha,\beta} M_\gamma, \tag{2.7}$$

where $\mathfrak{t}^\gamma_{\alpha,\beta}$ is the number of all overlapping shuffles $(\alpha, \beta) \to \gamma$. Here, the *overlapping shuffles* $(\alpha, \beta) \to \gamma$ are defined in the same way as the γ-smaps, with the only difference that the second of the three properties that define a γ-smap (namely, the property $\min(f(\mathcal{S}_0)) < \min(f(\mathcal{S}_1))$) is omitted. Needless to say, (2.7) can be proven similarly to our proof of (2.6) above.

Here is a somewhat nontrivial property of Φ and $\prec$:

[18] Alternatively, of course, $a \Phi b \in \mathrm{QSym}$ can be checked using the formula $M_\alpha \Phi M_\beta = M_{[\alpha,\beta]} + M_{\alpha \odot \beta}$ (which is easily proven). However, there is no such simple proof for $a \prec b \in \mathrm{QSym}$.

Theorem 2.3.7. Let S denote the antipode of the Hopf algebra QSym. Let us use Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$, where b is any element of QSym. Then,

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \Phi\, a\right) b_{(2)} = a \prec b$$

for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \mathrm{QSym}$.

Proof of Theorem 2.3.7. Let $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. We can WLOG assume that a is a monomial (because all operations in sight are $\mathbf{k}$-linear and continuous). So assume this. That is, $a = \mathfrak{n}$ for some monomial $\mathfrak{n}$. Consider this $\mathfrak{n}$. Let $k = \min(\operatorname{Supp} \mathfrak{n})$. Notice that $k \in \{1, 2, 3, \ldots\} \cup \{\infty\}$.

(Some remarks about ∞ are in order. We use ∞ as an object which is greater than every integer. We will use summation signs like $\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}$ and $\sum_{k < i_1 < i_2 < \cdots < i_\ell}$ in the following. Both of these summation signs range over $(i_1, i_2, \ldots, i_\ell) \in \{1, 2, 3, \ldots\}^\ell$ satisfying certain conditions ($1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$ in the first case, and $k < i_1 < i_2 < \cdots < i_\ell$ in the second case). In particular, none of the $i_1, i_2, \ldots, i_\ell$ is allowed to be ∞ (unlike k). So the summation $\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}$ is identical to $\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell}$ when $k = \infty$, whereas the summation $\sum_{k < i_1 < i_2 < \cdots < i_\ell}$ is empty when $k = \infty$ unless $\ell = 0$. (If $\ell = 0$, then the summation $\sum_{k < i_1 < i_2 < \cdots < i_\ell}$ ranges over the empty 0-tuple, no matter what k is.)

We shall also use an additional symbol $\infty + 1$, which is understood to be greater than every element of $\{1, 2, 3, \ldots\} \cup \{\infty\}$.)

Every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$a \prec M_\alpha = \left(\sum_{k < i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a \tag{2.8}$$

[19].

[19] *Proof of (2.8):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

Let us define a map $\mathfrak{B}_k : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\mathfrak{B}_k(p) = p(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \qquad \text{for every } p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$$

(where $p(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots)$ has to be understood as $p(x_1, x_2, x_3, \ldots) = p$ when $k = \infty$). Then, $\mathfrak{B}_k$ is an evaluation map (in an appropriate sense) and thus a continuous $\mathbf{k}$-algebra homomorphism. Any monomial $\mathfrak{m}$ satisfies

$$\mathfrak{B}_k(\mathfrak{m}) = \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases} \tag{2.9}$$

$\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Combined with $a = \mathfrak{n}$, this yields

$$\begin{aligned} a \prec M_\alpha &= \mathfrak{n} \prec \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\mathfrak{n} \prec \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right)}_{\substack{= \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min(\operatorname{Supp}\mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}; \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{n}) \geq \min\{i_1, i_2, \ldots, i_\ell\} \end{cases} \\ \text{(by the definition of } \prec \text{ on monomials)}}} \\ &\qquad \text{(since } \prec \text{ is } \mathbf{k}\text{-bilinear and continuous)} \\ &= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \begin{cases} \mathfrak{n} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \min(\operatorname{Supp}\mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}; \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{n}) \geq \min\{i_1, i_2, \ldots, i_\ell\} \end{cases} \\ &= \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell; \\ \min(\operatorname{Supp}\mathfrak{n}) < \min\{i_1, i_2, \ldots, i_\ell\}}}}_{\substack{= \sum_{\min(\operatorname{Supp}\mathfrak{n}) < i_1 < i_2 < \cdots < i_\ell} \\ = \sum_{k < i_1 < i_2 < \cdots < i_\ell} \\ \text{(since } \min(\operatorname{Supp}\mathfrak{n}) = k)}} \underbrace{\mathfrak{n}}_{=a} \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{k < i_1 < i_2 < \cdots < i_\ell} a \cdot x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \\ &= \left(\sum_{k < i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \cdot a. \end{aligned}$$

This proves (2.8).

[20]. Any $p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ satisfies

$$p \mathbin{\Phi} a = a \cdot \mathfrak{B}_k(p) \tag{2.10}$$

[21]. Also, every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$\mathfrak{B}_k(M_\alpha) = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \tag{2.12}$$

[22].

We shall use one further obvious observation: If $i_1, i_2, \ldots, i_\ell$ are some positive

---

[20] *Proof.* Let $\mathfrak{m}$ be a monomial. Then,

$$\begin{aligned}
\mathfrak{B}_k(\mathfrak{m}) &= \mathfrak{m}(x_1, x_2, \ldots, x_k, 0, 0, 0, \ldots) \qquad \text{(by the definition of } \mathfrak{B}_k\text{)} \\
&= (\text{the result of replacing the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ by } 0 \text{ in } \mathfrak{m}) \\
&= \begin{cases} \mathfrak{m}, & \text{if none of the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ appears in } \mathfrak{m}; \\ 0, & \text{if some of the indeterminates } x_{k+1}, x_{k+2}, x_{k+3}, \ldots \text{ appear in } \mathfrak{m} \end{cases} \\
&= \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases}
\end{aligned}$$

(because none of the indeterminates $x_{k+1}, x_{k+2}, x_{k+3}, \ldots$ appears in $\mathfrak{m}$ if and only if $\max(\operatorname{Supp}\mathfrak{m}) \leq k$). This proves (2.9).

[21] *Proof of (2.10):* Fix $p \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. Since the equality (2.10) is $\mathbf{k}$-linear and continuous in p, we can WLOG assume that p is a monomial. Assume this. Hence, $p = \mathfrak{m}$ for some monomial $\mathfrak{m}$. Consider this $\mathfrak{m}$. We have

$$\mathfrak{B}_k\left(\underbrace{p}_{=\mathfrak{m}}\right) = \mathfrak{B}_k(\mathfrak{m}) = \begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases} \tag{2.11}$$

(by (2.9)). Now,

$$\begin{aligned}
\underbrace{p}_{=\mathfrak{m}} \mathbin{\Phi} \underbrace{a}_{=\mathfrak{n}} = \mathfrak{m} \mathbin{\Phi} \mathfrak{n} &= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > \min(\operatorname{Supp}\mathfrak{n}) \end{cases} \\
&\qquad \text{(by the definition of } \Phi\text{)} \\
&= \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases} \qquad (\text{since } \min(\operatorname{Supp}\mathfrak{n}) = k) \\
&= \underbrace{\mathfrak{n}}_{=a} \cdot \underbrace{\begin{cases} \mathfrak{m}, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) \leq k; \\ 0, & \text{if } \max(\operatorname{Supp}\mathfrak{m}) > k \end{cases}}_{\substack{=\mathfrak{B}_k(p) \\ \text{(by (2.11))}}} \\
&= a \cdot \mathfrak{B}_k(p).
\end{aligned}$$

This proves (2.10).

[22] *Proof of (2.12):* Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. The definition of M_α yields $M_\alpha =$

integers satisfying $i_1 < i_2 < \cdots < i_\ell$, then

$$\text{there exists exactly one } j \in \{0, 1, \ldots, \ell\} \text{ satisfying } i_j \leq k < i_{j+1} \tag{2.13}$$

(where i_0 is to be understood as 1, and $i_{\ell+1}$ as $\infty + 1$).

Let us now notice that every $f \in \operatorname{QSym}$ satisfies

$$af = \sum_{(f)} \mathfrak{B}_k \left(f_{(1)} \right) \left(a \prec f_{(2)} \right). \tag{2.14}$$

Proof of (2.14): Both sides of the equality (2.14) are $\mathbf{k}$-linear in f. Hence, it is enough to check (2.14) on the basis $(M_\gamma)_{\gamma \in \operatorname{Comp}}$ of QSym, that is, to prove that (2.14) holds whenever $f = M_\gamma$ for some $\gamma \in \operatorname{Comp}$. In other words, it is enough to show

$\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$. Applying the map $\mathfrak{B}_k$ to both sides of this equality, we obtain

$$\begin{aligned}
\mathfrak{B}_k (M_\alpha) &= \mathfrak{B}_k \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\mathfrak{B}_k \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right)}_{\substack{= \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max \left(\operatorname{Supp} \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \right) \leq k; \\ 0, & \text{if } \max \left(\operatorname{Supp} \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \right) > k \end{cases} \\ \text{(by (2.9), applied to } \mathfrak{m} = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \text{)}}} \\
&\qquad \text{(since } \mathfrak{B}_k \text{ is } \mathbf{k}\text{-linear and continuous)} \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max \left(\operatorname{Supp} \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \right) \leq k; \\ 0, & \text{if } \max \left(\operatorname{Supp} \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) \right) > k \end{cases}}_{\substack{= \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max \{ i_1, i_2, \ldots, i_\ell \} \leq k; \\ 0, & \text{if } \max \{ i_1, i_2, \ldots, i_\ell \} > k \end{cases} \\ \text{(since } \operatorname{Supp} \left(x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} \right) = \{ i_1, i_2, \ldots, i_\ell \} \text{)}}} \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \begin{cases} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}, & \text{if } \max \{ i_1, i_2, \ldots, i_\ell \} \leq k; \\ 0, & \text{if } \max \{ i_1, i_2, \ldots, i_\ell \} > k \end{cases} \\
&= \underbrace{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_\ell; \\ \max \{ i_1, i_2, \ldots, i_\ell \} \leq k}}}_{= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k}} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell \leq k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}.
\end{aligned}$$

This proves (2.12).

that

$$aM_\gamma = \sum_{(M_\gamma)} \mathfrak{B}_k \left((M_\gamma)_{(1)} \right) \cdot \left(a \prec (M_\gamma)_{(2)} \right) \qquad \text{for every } \gamma \in \operatorname{Comp}.$$

But this is easily done: Let $\gamma \in \operatorname{Comp}$. Write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$. Then,

$$\begin{aligned}
&\sum_{(M_\gamma)} \mathfrak{B}_k \left((M_\gamma)_{(1)} \right) \cdot \left(a \prec (M_\gamma)_{(2)} \right) \\
&= \sum_{j=0}^{\ell} \underbrace{\mathfrak{B}_k \left(M_{(\gamma_1, \gamma_2, \ldots, \gamma_j)} \right)}_{\substack{= \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \\ \text{(by (2.12))}}} \cdot \underbrace{\left(a \prec M_{(\gamma_{j+1}, \gamma_{j+2}, \ldots, \gamma_\ell)} \right)}_{\substack{= \left(\sum_{k < i_1 < i_2 < \cdots < i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell} \right) \cdot a \\ \text{(by (2.8))}}} \\
&\qquad \left(\text{since } \sum_{(M_\gamma)} (M_\gamma)_{(1)} \otimes (M_\gamma)_{(2)} = \Delta (M_\gamma) = \sum_{j=0}^{\ell} M_{(\gamma_1, \gamma_2, \ldots, \gamma_j)} \otimes M_{(\gamma_{j+1}, \gamma_{j+2}, \ldots, \gamma_\ell)} \right) \\
&= \sum_{j=0}^{\ell} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \underbrace{\left(\sum_{k < i_1 < i_2 < \cdots < i_{\ell-j}} x_{i_1}^{\gamma_{j+1}} x_{i_2}^{\gamma_{j+2}} \cdots x_{i_{\ell-j}}^{\gamma_\ell} \right)}_{\substack{= \sum_{k < i_{j+1} < i_{j+2} < \cdots < i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \\ \text{(here, we have renamed the summation index} \\ \left(i_1, i_2, \ldots, i_{\ell-j} \right) \text{ as } (i_{j+1}, i_{j+2}, \ldots, i_\ell))}} \cdot a \\
&= \sum_{j=0}^{\ell} \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \left(\sum_{k < i_{j+1} < i_{j+2} < \cdots < i_\ell} x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \right) \cdot a \\
&= \underbrace{\sum_{j=0}^{\ell} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \sum_{k < i_{j+1} < i_{j+2} < \cdots < i_\ell}}_{\substack{= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \sum_{j \in \{0,1,\ldots,\ell\};\ i_j \leq k < i_{j+1}} \\ \text{(where } i_0 \text{ is to be understood as 1, and } i_{\ell+1} \text{ as } \infty+1)}} \underbrace{\left(x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_j}^{\gamma_j} \right) \left(x_{i_{j+1}}^{\gamma_{j+1}} x_{i_{j+2}}^{\gamma_{j+2}} \cdots x_{i_\ell}^{\gamma_\ell} \right)}_{= x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}} \cdot a \\
&= \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} \underbrace{\sum_{\substack{j \in \{0,1,\ldots,\ell\}; \\ i_j \leq k < i_{j+1}}} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{\substack{\text{this sum has precisely one addend,} \\ \text{(because of (2.13)),} \\ \text{and thus equals } x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}} \cdot a = \underbrace{\sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\gamma_1} x_{i_2}^{\gamma_2} \cdots x_{i_\ell}^{\gamma_\ell}}_{= M_\gamma} \cdot a \\
&= M_\gamma \cdot a = aM_\gamma,
\end{aligned}$$

qed. Thus, (2.14) is proven.

Now, every $b \in \mathrm{QSym}$ satisfies

$$\sum_{(b)} \underbrace{\left(S\left(b_{(1)}\right) \Phi\ a\right)}_{\substack{=a\cdot\mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \\ \text{(by (2.10), applied to } p=S\left(b_{(1)}\right))}} b_{(2)}$$

$$= \sum_{(b)} a \cdot \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) b_{(2)} = \sum_{(b)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \cdot \underbrace{ab_{(2)}}_{\substack{=\sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a \prec \left(b_{(2)}\right)_{(2)}\right) \\ \text{(by (2.14), applied to } f=b_{(2)})}}$$

$$= \sum_{(b)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \left(\sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a \prec \left(b_{(2)}\right)_{(2)}\right)\right)$$

$$= \sum_{(b)} \sum_{\left(b_{(2)}\right)} \mathfrak{B}_k\left(S\left(b_{(1)}\right)\right) \mathfrak{B}_k\left(\left(b_{(2)}\right)_{(1)}\right)\left(a \prec \left(b_{(2)}\right)_{(2)}\right)$$

$$= \sum_{(b)} \underbrace{\sum_{\left(b_{(1)}\right)} \mathfrak{B}_k\left(S\left(\left(b_{(1)}\right)_{(1)}\right)\right) \mathfrak{B}_k\left(\left(b_{(1)}\right)_{(2)}\right)}_{\substack{=\mathfrak{B}_k\left(\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\cdot\left(b_{(1)}\right)_{(2)}\right) \\ \text{(since } \mathfrak{B}_k \text{ is a } \mathbf{k}\text{-algebra homomorphism)}}} \left(a \prec b_{(2)}\right)$$

$$\left(\begin{array}{c}\text{since the coassociativity of } \Delta \text{ yields} \\ \sum_{(b)} \sum_{\left(b_{(2)}\right)} b_{(1)} \otimes \left(b_{(2)}\right)_{(1)} \otimes \left(b_{(2)}\right)_{(2)} = \sum_{(b)} \sum_{\left(b_{(1)}\right)} \left(b_{(1)}\right)_{(1)} \otimes \left(b_{(1)}\right)_{(2)} \otimes b_{(2)}\end{array}\right)$$

$$= \sum_{(b)} \mathfrak{B}_k\left(\underbrace{\sum_{\left(b_{(1)}\right)} S\left(\left(b_{(1)}\right)_{(1)}\right)\left(b_{(1)}\right)_{(2)}}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(by one of the defining equations of the antipode)}}}\right)\left(a \prec b_{(2)}\right)$$

$$= \sum_{(b)} \underbrace{\mathfrak{B}_k\left(\varepsilon\left(b_{(1)}\right)\right)}_{\substack{=\varepsilon\left(b_{(1)}\right) \\ \text{(since } \mathfrak{B}_k \text{ is a } \mathbf{k}\text{-algebra} \\ \text{homomorphism, and} \\ \varepsilon\left(b_{(1)}\right)\in\mathbf{k} \text{ is a scalar)}}} \left(a \prec b_{(2)}\right) = \sum_{(b)} \varepsilon\left(b_{(1)}\right) \cdot \left(a \prec b_{(2)}\right)$$

$$= \sum_{(b)} a \prec \left(\varepsilon\left(b_{(1)}\right) b_{(2)}\right) = a \prec \underbrace{\left(\sum_{(b)} \varepsilon\left(b_{(1)}\right) b_{(2)}\right)}_{=b} = a \prec b.$$

This proves Theorem 2.3.7. ☐

Let us connect the Φ operation with the fundamental basis of QSym:

Proposition 2.3.8. For any two compositions α and β, define a composition $\alpha \odot \beta$ as follows:

– If α is empty, then set $\alpha \odot \beta = \beta$.

– Otherwise, if β is empty, then set $\alpha \odot \beta = \alpha$.

– Otherwise, define $\alpha \odot \beta$ as $(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$, where α is written as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and where β is written as $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$.

Then, any two compositions α and β satisfy

$$F_\alpha \Phi F_\beta = F_{\alpha \odot \beta}.$$

Our proof of this proposition will rely on the following lemma:

Lemma 2.3.9. If G is a set of integers and r is an integer, then we let $G+r$ denote the set $\{g + r \mid g \in G\}$ of integers.

Let $p \in \mathbb{N}$ and $q \in \mathbb{N}$. Let α be a composition of p. Let β be a composition of q. Consider the composition $\alpha \odot \beta$ defined in Proposition 2.3.8.

(a) Then, $\alpha \odot \beta$ is a composition of $p + q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

(b) Also, define a composition $[\alpha, \beta]$ by $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, where α and β are written in the forms $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Assume that $p > 0$ and $q > 0$. Then, $[\alpha, \beta]$ is a composition of $p + q$ satisfying $D([\alpha, \beta]) = D(\alpha) \cup \{p\} \cup (D(\beta) + p)$.

(Actually, part **(b)** of this lemma will not be used until much later, but part **(a)** will be used soon.)

Proof of Lemma 2.3.9. Write α in the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Thus, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$, so that $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell = |\alpha| = p$ (since α is a composition of p).

Write β in the form $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Thus, $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_m$, so that $\beta_1 + \beta_2 + \cdots + \beta_m = |\beta| = q$ (since β is a composition of q).

We have $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$, and thus

$$\begin{aligned} D(\beta) &= \{\beta_1, \beta_1 + \beta_2, \beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \cdots + \beta_{m-1}\} \\ &\qquad \text{(by the definition of } D(\beta)) \\ &= \{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\}. \end{aligned}$$

Also, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, and thus

$$\begin{aligned} D(\alpha) &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}\} \\ &\qquad \text{(by the definition of } D(\alpha)) \\ &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell-1\}\}. \end{aligned}$$

(a) If α or β is empty, then Lemma 2.3.9 **(a)** holds for obvious reasons (because of the definition of $\alpha \odot \beta$ in this case). Thus, we WLOG assume that neither α nor β is empty.

We have $\alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$ (by the definition of $\alpha \odot \beta$) and thus

$$\begin{aligned} |\alpha \odot \beta| &= \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_m \\ &= \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{=p} + \underbrace{(\beta_1 + \beta_2 + \cdots + \beta_m)}_{=q} = p + q. \end{aligned}$$

Thus, $\alpha \odot \beta$ is a composition of $p + q$. Hence, it remains to show that $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

Now, $\alpha \odot \beta = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m)$, so that

$$\begin{aligned}
& D(\alpha \odot \beta) \\
& = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}, \\
& \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1), \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2, \\
& \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3, \ldots, \\
& \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_{m-1}\} \\
& \qquad \text{(by the definition of } D(\alpha \odot \beta)) \\
& = \underbrace{\{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}}_{= D(\alpha)} \\
& \qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + (\alpha_\ell + \beta_1) + \beta_2 + \beta_3 + \cdots + \beta_j}_{\substack{= (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell) + (\beta_1 + \beta_2 + \cdots + \beta_j) \\ = (\beta_1 + \beta_2 + \cdots + \beta_j) + (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}} \right. \\
& \qquad\qquad \left. \mid j \in \{1, 2, \ldots, m-1\} \right\} \\
& = D(\alpha) \cup \left\{ (\beta_1 + \beta_2 + \cdots + \beta_j) + \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{= p} \mid j \in \{1, 2, \ldots, m-1\} \right\} \\
& = D(\alpha) \cup \underbrace{\{(\beta_1 + \beta_2 + \cdots + \beta_j) + p \mid j \in \{1, 2, \ldots, m-1\}\}}_{= \{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\} + p} \\
& = D(\alpha) \cup \left(\underbrace{\{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\}}_{= D(\beta)} + p \right) \\
& = D(\alpha) \cup (D(\beta) + p).
\end{aligned}$$

This completes the proof of Lemma 2.3.9 **(a)**.

(b) We have $p > 0$. Thus, the composition α is nonempty (since α is a composition of p). In other words, the composition $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is nonempty (since $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$). Hence, $\ell > 0$.

We have $q > 0$. Thus, the composition β is nonempty (since β is a composition of q). In other words, the composition $(\beta_1, \beta_2, \ldots, \beta_m)$ is nonempty (since $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$). Hence, $m > 0$.

We have $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$ (by the definition of $[\alpha, \beta]$) and thus

$$\begin{aligned} |\alpha \odot \beta| &= \alpha_1 + \alpha_2 + \cdots + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_m \\ &= \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{=p} + \underbrace{(\beta_1 + \beta_2 + \cdots + \beta_m)}_{=q} = p + q. \end{aligned}$$

Thus, $[\alpha, \beta]$ is a composition of $p + q$. Hence, it remains to show that $D([\alpha, \beta]) = D(\alpha) \cup (D(\beta) + p)$.

Now, $[\alpha, \beta] = (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, so that

$$\begin{aligned}
& D([\alpha, \beta]) \\
&= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1}, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2, \ldots, \\
&\qquad \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_{m-1}\} \\
&\qquad \text{(by the definition of } D([\alpha, \beta])\text{)} \\
&= \underbrace{\{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid i \in \{1, 2, \ldots, \ell - 1\}\}}_{= D(\alpha)} \\
&\qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_\ell}_{= p} \right\} \\
&\qquad \cup \left\{ \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell + \beta_1 + \beta_2 + \cdots + \beta_j}_{\substack{= (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell) + (\beta_1 + \beta_2 + \cdots + \beta_j) \\ = (\beta_1 + \beta_2 + \cdots + \beta_j) + (\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}} \right. \\
&\qquad\qquad\qquad \left. \mid j \in \{1, 2, \ldots, m - 1\} \right\} \\
&= D(\alpha) \cup \{p\} \cup \left\{ (\beta_1 + \beta_2 + \cdots + \beta_j) + \underbrace{(\alpha_1 + \alpha_2 + \cdots + \alpha_\ell)}_{= p} \right. \\
&\qquad\qquad\qquad \left. \mid j \in \{1, 2, \ldots, m - 1\} \right\} \\
&= D(\alpha) \cup \{p\} \cup \underbrace{\{(\beta_1 + \beta_2 + \cdots + \beta_j) + p \mid j \in \{1, 2, \ldots, m - 1\}\}}_{= \{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m-1\}\} + p} \\
&= D(\alpha) \cup \{p\} \cup \left(\underbrace{\{\beta_1 + \beta_2 + \cdots + \beta_j \mid j \in \{1, 2, \ldots, m - 1\}\}}_{= D(\beta)} + p \right) \\
&= D(\alpha) \cup \{p\} \cup (D(\beta) + p).
\end{aligned}$$

This completes the proof of Lemma 2.3.9 **(b)**. □

Proof of Proposition 2.3.8. If either α or β is empty, then this is obvious (since Φ is unital with 1 as its unity, and since $F_{\varnothing} = 1$). So let us WLOG assume that neither is. Write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, and write β as $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. Thus, ℓ and m are positive (since α and β are nonempty).

Let $p = |\alpha|$ and $q = |\beta|$. Thus, p and q are positive (since α and β are nonempty). Recall that we use the notation $D(\alpha)$ for the set of partial sums of a composition α. If G is a set of integers and r is an integer, then we let $G + r$ denote the set $\{g + r \mid g \in G\}$ of integers.

Lemma 2.3.9 **(a)** shows that $\alpha \odot \beta$ is a composition of $p+q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta) + p)$.

Applying (2.1) to p instead of n, we obtain

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_p}. \tag{2.15}$$

Applying (2.1) to q and β instead of n and α, we obtain

$$F_\beta = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_q; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)}} x_{i_1} x_{i_2} \cdots x_{i_q} = \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}$$

(here, we renamed the summation index $(i_1, i_2, \ldots, i_q)$ as $(i_{p+1}, i_{p+2}, \ldots, i_{p+q})$). This, together with (2.15), yields

$$
\begin{aligned}
& F_\alpha \mathbin{\Phi} F_\beta \\
&= \left(\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_p} \right) \mathbin{\Phi} \left(\sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}} \right) \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} \underbrace{\left(x_{i_1} x_{i_2} \cdots x_{i_p} \right) \mathbin{\Phi} \left(x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}} \right)}_{\substack{= \begin{cases} x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}, & \text{if } i_p \leq i_{p+1}; \\ 0, & \text{if } i_p > i_{p+1} \end{cases} \\ \text{(by the definition of } \Phi \text{ on monomials)}}} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha)}} \sum_{\substack{i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p}} \begin{cases} x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}, & \text{if } i_p \leq i_{p+1}; \\ 0, & \text{if } i_p > i_{p+1} \end{cases} \\
&= \underbrace{\sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_p; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha); \\ i_{p+1} \leq i_{p+2} \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\beta)+p; \\ i_p \leq i_{p+1}}}}_{= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha) \cup (D(\beta)+p)}}} \underbrace{x_{i_1} x_{i_2} \cdots x_{i_p} x_{i_{p+1}} x_{i_{p+2}} \cdots x_{i_{p+q}}}_{= x_{i_1} x_{i_2} \cdots x_{i_{p+q}}} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha) \cup (D(\beta)+p)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}}. \qquad (2.16)
\end{aligned}
$$

On the other hand, $\alpha \odot \beta$ is a composition of $p+q$ satisfying $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta)+p)$. Thus, (2.1) (applied to $\alpha \odot \beta$ and $p+q$ instead of α and n) yields

$$F_{\alpha \odot \beta} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha \odot \beta)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}} = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_{p+q}; \\ i_j < i_{j+1} \text{ if } j \in D(\alpha) \cup (D(\beta)+p)}} x_{i_1} x_{i_2} \cdots x_{i_{p+q}}$$

(since $D(\alpha \odot \beta) = D(\alpha) \cup (D(\beta)+p)$). Compared with (2.16), this yields $F_\alpha \mathbin{\Phi} F_\beta = F_{\alpha \odot \beta}$. This proves Proposition 2.3.8. □

For our goals, we need a certain particular case of Proposition 2.3.8. Namely, let us recall that for every $m \in \mathbb{N}$, the *m-th complete homogeneous symmetric function*

h_m is defined as the element $\sum\limits_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$ of Sym. It is easy to see that $h_m = F_{(m)}$ for every positive integer m. From this, we obtain:

Corollary 2.3.10. For any two compositions α and β, define a composition $\alpha \odot \beta$ as in Proposition 2.3.8. Then, every composition α and every positive integer m satisfy

$$F_{\alpha \odot (m)} = F_\alpha \Phi h_m. \qquad (2.17)$$

Proof of Corollary 2.3.10. Let α be a composition. Let m be a positive integer. Recall that $h_m = F_{(m)}$. Proposition 2.3.8 yields $F_\alpha \Phi F_{(m)} = F_{\alpha \odot (m)}$. Hence, $F_{\alpha \odot (m)} = F_\alpha \Phi \underbrace{F_{(m)}}_{= h_m} = F_\alpha \Phi h_m$. This proves Corollary 2.3.10. □

Remark 2.3.11. We can also define a binary operation $\divideontimes : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \times \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and that it satisfy

$$\mathfrak{m} \divideontimes \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$

for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$. (Recall that $\max \varnothing = 0$ and $\min \varnothing = \infty$.)

This operation $\divideontimes$ shares some of the properties of Φ (in particular, it is associative and has neutral element 1); an analogue of Theorem 2.3.7 says that

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \divideontimes a \right) b_{(2)} = a \preceq b$$

for any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, where $a \preceq b$ stands for $b \succeq a$. (Of course, we could also define $\preceq$ by changing the "$<$" into a "$\leq$" and the "$\geq$" into a "$>$" in the definition of $\prec$.)

2.4 Dual immaculate functions and the operation $\prec$

We will now study the dual immaculate functions defined in [BBSSZ13a]. However, instead of defining them as was done in [BBSSZ13a, Section 3.7], we shall give a different (but equivalent) definition. First, we introduce immaculate tableaux (which we define as in [BBSSZ13a, Definition 3.9]), which are an analogue of the well-known semistandard Young tableaux (also known as "column-strict tableaux")[23]:

Definition 2.4.1. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition.

(a) The *Young diagram* of α will mean the subset $\{(i,j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \ell;\ 1 \leq j \leq \alpha_i\}$ of $\mathbb{Z}^2$. It is denoted by $Y(\alpha)$.

(b) An *immaculate tableau of shape* α will mean a map $T : Y(\alpha) \to \{1, 2, 3, \ldots\}$ which satisfies the following two axioms:

1. We have $T(i,1) < T(j,1)$ for any integers i and j satisfying $1 \leq i < j \leq \ell$.
2. We have $T(i,u) \leq T(i,v)$ for any integers i, u and v satisfying $1 \leq i \leq \ell$ and $1 \leq u < v \leq \alpha_i$.

The *entries* of an immaculate tableau T mean the images of elements of $Y(\alpha)$ under T.

We will use the same graphical representation of immaculate tableaux (analogous to the "English notation" for semistandard Young tableaux) that was used in [BBSSZ13a]: An immaculate tableau T of shape $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is represented as a table whose rows are left-aligned (but can have different lengths), and whose i-th row (counted from top) has α_i boxes, which are respectively filled with the entries $T(i,1)$, $T(i,2)$, $\ldots$, $T(i,\alpha_i)$ (from left to right). For example, an immaculate tableau T of shape $(3,1,2)$ is represented by the picture

$$\begin{array}{|c|c|c|} \hline a_{1,1} & a_{1,2} & a_{1,3} \\ \hline a_{2,1} & & \\ \cline{1-2} a_{3,1} & a_{3,2} & \\ \cline{1-2} \end{array},$$

[23]See, e.g., [Stan99, Chapter 7] for a study of semistandard Young tableaux. We will not use them in this note; however, our terminology for immaculate tableaux will imitate some of the classical terminology defined for semistandard Young tableaux.

where $a_{i,j} = T(i,j)$ for every $(i,j) \in Y((3,1,2))$. Thus, the first of the above two axioms for an immaculate tableau T says that the entries of T are strictly increasing down the first column of $Y(\alpha)$, whereas the second of the above two axioms says that the entries of T are weakly increasing along each row of $Y(\alpha)$.

(c) Let $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ be a composition of $|\alpha|$. An immaculate tableau T of shape α is said to have *content* β if every $j \in \{1,2,3,\ldots\}$ satisfies

$$\left|T^{-1}(j)\right| = \begin{cases} \beta_j, & \text{if } j \leq k; \\ 0, & \text{if } j > k \end{cases}.$$

Notice that not every immaculate tableau has a content (with this definition), because we only allow compositions as contents. More precisely, if T is an immaculate tableau of shape α, then there exists a composition β such that T has content β if and only if there exists a $k \in \mathbb{N}$ such that $T(Y(\alpha)) = \{1,2,\ldots,k\}$.

(d) Let β be a composition of $|\alpha|$. Then, $K_{\alpha,\beta}$ denotes the number of immaculate tableaux of shape α and content β.

For future reference, let us notice that if α is a nonempty composition and if T is an immaculate tableau of shape α, then

$$\text{the smallest entry of } T \text{ is } T(1,1) \tag{2.18}$$

(because every $(i,j) \in Y(\alpha)$ satisfies $T(1,1) \leq T(i,1) \leq T(i,j)$). Moreover, if α is a composition, if T is an immaculate tableau of shape α, and if $(i,j) \in Y(\alpha)$ is such that $i > 1$, then

$$T(1,1) < T(i,1) \leq T(i,j). \tag{2.19}$$

Definition 2.4.2. Let α be a composition. The *dual immaculate function* $\mathfrak{S}^*_\alpha$ corresponding to α is defined as the quasisymmetric function

$$\sum_{\beta \models |\alpha|} K_{\alpha,\beta} M_\beta.$$

This definition is not identical to the definition of $\mathfrak{S}^*_\alpha$ used in [BBSSZ13a], but it is equivalent to it, as the following proposition shows.

Proposition 2.4.3. Definition 2.4.2 is equivalent to the definition of $\mathfrak{S}^*_\alpha$ used in [BBSSZ13a].

Proof of Proposition 2.4.3. Let $\leq_\ell$ denote the lexicographic order on compositions.

Let α be a composition. Then, [BBSSZ13a, Proposition 3.36] yields the following:

$$\left(\text{the dual immaculate function } \mathfrak{S}^*_\alpha \text{ as defined in [BBSSZ13a]}\right) = \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta.$$

Compared with

$$\begin{aligned}
&\left(\text{the dual immaculate function } \mathfrak{S}^*_\alpha \text{ as defined in Definition 2.4.2}\right)\\
&= \sum_{\beta \models |\alpha|} K_{\alpha,\beta} M_\beta = \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta + \sum_{\substack{\beta \models |\alpha|;\\ \text{not } \beta \leq_\ell \alpha}} \underbrace{K_{\alpha,\beta}}_{\substack{=0 \\ \text{(by [BBSSZ13a, Proposition 3.15 (2)])}}} M_\beta\\
&= \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta + \underbrace{\sum_{\substack{\beta \models |\alpha|;\\ \text{not } \beta \leq_\ell \alpha}} 0 M_\beta}_{=0} = \sum_{\substack{\beta \models |\alpha|;\\ \beta \leq_\ell \alpha}} K_{\alpha,\beta} M_\beta,
\end{aligned}$$

this yields

$$\begin{aligned}
&\left(\text{the dual immaculate function } \mathfrak{S}^*_\alpha \text{ as defined in [BBSSZ13a]}\right)\\
&= \left(\text{the dual immaculate function } \mathfrak{S}^*_\alpha \text{ as defined in Definition 2.4.2}\right).
\end{aligned}$$

Hence, Definition 2.4.2 is equivalent to the definition of $\mathfrak{S}^*_\alpha$ used in [BBSSZ13a]. This proves Proposition 2.4.3. $\square$

It is helpful to think of dual immaculate functions as analogues of Schur functions obtained by replacing semistandard Young tableaux by immaculate tableaux. Definition 2.4.2 is the analogue of the well-known formula $s_\lambda = \sum_{\mu \vdash |\lambda|} k_{\lambda,\mu} m_\mu$ for any partition λ, where s_λ denotes the Schur function corresponding to λ, where m_μ denotes the monomial symmetric function corresponding to the partition μ, and where $k_{\lambda,\mu}$ is the (λ,μ)-th Kostka number (i.e., the number of semistandard Young tableaux of shape λ and content μ). The following formula for the $\mathfrak{S}_\alpha^*$ (known to the authors of [BBSSZ13a] but not explicitly stated in their work) should not come as a surprise:

Proposition 2.4.4. Let α be a composition. Then,

$$\mathfrak{S}_\alpha^* = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T.$$

Here, $\mathbf{x}_T$ is defined as $\prod_{(i,j)\in Y(\alpha)} x_{T(i,j)}$ when T is an immaculate tableau of shape α.

Before we prove this proposition, let us state a fundamental and simple lemma:

Lemma 2.4.5. **(a)** If I is a finite subset of $\{1,2,3,\ldots\}$, then there exists a unique strictly increasing bijection $\{1,2,\ldots,|I|\} \to I$. Let us denote this bijection by r_I. Its inverse r_I^{-1} is obviously again a strictly increasing bijection.

Now, let α be a composition.

(b) If T is an immaculate tableau of shape α, then $r_{T(Y(\alpha))}^{-1} \circ T$ (remember that immaculate tableaux are maps from $Y(\alpha)$ to $\{1,2,3,\ldots\}$) is an immaculate tableau of shape α as well, and has the additional property that there exists a unique composition β of $|\alpha|$ such that $r_{T(Y(\alpha))}^{-1} \circ T$ has content β.

(c) Let Q be an immaculate tableau of shape α. Let β be a composition of $|\alpha|$ such that Q has content β. Then,

$$M_\beta = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r_{T(Y(\alpha))}^{-1} \circ T = Q}} \mathbf{x}_T. \tag{2.20}$$

Proof of Lemma 2.4.5. **(a)** Lemma 2.4.5 **(a)** is obvious.

(b) Let T be an immaculate tableau of shape α. Then, $r_{T(Y(\alpha))}^{-1} \circ T$ is an immaculate tableau of shape α as well[24]. Let $R = r_{T(Y(\alpha))}^{-1} \circ T : Y(\alpha) \to \{1, 2, \ldots, |T(Y(\alpha))|\}$. Then,

$$\underbrace{R}_{= r_{T(Y(\alpha))}^{-1} \circ T} (Y(\alpha)) = \left(r_{T(Y(\alpha))}^{-1} \circ T \right) (Y(\alpha))$$
$$= r_{T(Y(\alpha))}^{-1} (T(Y(\alpha))) = \{1, 2, \ldots, |T(Y(\alpha))|\}.$$

Hence, $(|R^{-1}(1)|, |R^{-1}(2)|, \ldots, |R^{-1}(|T(Y(\alpha))|)|)$ is a composition. Therefore, there exists a unique composition β of $|\alpha|$ such that R has content β (namely, $\beta = (|R^{-1}(1)|, |R^{-1}(2)|, \ldots, |R^{-1}(|T(Y(\alpha))|)|)$). In other words, there exists a unique composition β of $|\alpha|$ such that $r_{T(Y(\alpha))}^{-1} \circ T$ has content β (since $R = r_{T(Y(\alpha))}^{-1} \circ T$). This completes the proof of Lemma 2.4.5 **(b)**.

(c) If T is a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$, then T is automatically an immaculate tableau of shape α [25]. Hence, the summation sign "$\sum\limits_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r_{T(Y(\alpha))}^{-1} \circ T = Q}}$" on the right hand side of (2.20) can be replaced by "$\sum\limits_{\substack{T : Y(\alpha) \to \{1,2,3,\ldots\}; \\ r_{T(Y(\alpha))}^{-1} \circ T = Q}}$".
Hence,

$$\sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha; \\ r_{T(Y(\alpha))}^{-1} \circ T = Q}} \mathbf{x}_T = \sum_{\substack{T : Y(\alpha) \to \{1,2,3,\ldots\}; \\ r_{T(Y(\alpha))}^{-1} \circ T = Q}} \mathbf{x}_T.$$

Now, let us write the composition β in the form $(\beta_1, \beta_2, \ldots, \beta_\ell)$. Then, we have

$$\left| Q^{-1}(k) \right| = \begin{cases} \beta_k, & \text{if } k \leq \ell; \\ 0, & \text{if } k > \ell \end{cases} \qquad \text{for every positive integer } k \tag{2.21}$$

(since Q has content β). Hence, $Q(Y(\alpha)) = \{1, 2, \ldots, \ell\}$. As a consequence, the maps

[24]This is because the map $r_{T(Y(\alpha))}^{-1}$ is strictly increasing, and the inequality conditions which decide whether a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ is an immaculate tableau of shape α are preserved under composition with a strictly increasing map.

[25]*Proof.* Let T be a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$. Thus, $T = r_{T(Y(\alpha))} \circ Q$. Since Q is an immaculate tableau of shape α, this shows that T is an immaculate tableau of shape α (since the map $r_{T(Y(\alpha))}$ is strictly increasing, and the inequality conditions which decide whether a map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ is an immaculate tableau of shape α are preserved under composition with a strictly increasing map).

$T : Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$ are in 1-to-1 correspondence with the ℓ-element subsets of $\{1, 2, 3, \ldots\}$ (the correspondence sends a map T to the ℓ-element subset $T(Y(\alpha))$, and the inverse correspondence sends an ℓ-element subset I to the map $r_I \circ Q$). But these latter subsets, in turn, are in 1-to-1 correspondence with the strictly increasing length-ℓ sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers (the correspondence sends a subset G to the sequence $(r_G(1), r_G(2), \ldots, r_G(\ell))$; of course, this latter sequence is just the list of all elements of G in increasing order). Composing these two 1-to-1 correspondences, we conclude that the maps $T : Y(\alpha) \to \{1, 2, 3, \ldots\}$ satisfying $r_{T(Y(\alpha))}^{-1} \circ T = Q$ are in 1-to-1 correspondence with the strictly increasing length-ℓ sequences $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers (the correspondence sends a map T to the sequence $\left(r_{T(Y(\alpha))}(1), r_{T(Y(\alpha))}(2), \ldots, r_{T(Y(\alpha))}(\ell)\right)$), and this correspondence has the property that $\mathbf{x}_T = x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$ whenever some map T gets sent to some sequence $(i_1 < i_2 < \cdots < i_\ell)$ (because if some map T gets sent to some sequence $(i_1 < i_2 < \cdots < i_\ell)$, then $(i_1, i_2, \ldots, i_\ell) = \left(r_{T(Y(\alpha))}(1), r_{T(Y(\alpha))}(2), \ldots, r_{T(Y(\alpha))}(\ell)\right)$, so that every $k \in \{1, 2, \ldots, \ell\}$ satisfies $i_k = r_{T(Y(\alpha))}(k)$, and now we have

$$\mathbf{x}_T = \prod_{(i,j) \in Y(\alpha)} x_{T(i,j)} = \prod_{k=1}^{\ell} \prod_{\substack{(i,j) \in Y(\alpha); \\ Q(i,j) = k}} \underbrace{x_{T(i,j)}}_{\substack{= x_{r_{T(Y(\alpha))}(Q(i,j))} \\ \text{(since } T(i,j) = r_{T(Y(\alpha))}(Q(i,j)) \\ \text{(because } r_{T(Y(\alpha))}^{-1} \circ T = Q \\ \text{and thus } T = r_{T(Y(\alpha))} \circ Q))}}$$

$$\text{(since } Q(Y(\alpha)) = \{1, 2, \ldots, \ell\})$$

$$= \prod_{k=1}^{\ell} \underbrace{\prod_{\substack{(i,j) \in Y(\alpha); \\ Q(i,j) = k}}}_{= \prod_{(i,j) \in Q^{-1}(k)}} \underbrace{x_{r_{T(Y(\alpha))}(Q(i,j))}}_{\substack{= x_{r_{T(Y(\alpha))}(k)} \\ \text{(since } Q(i,j) = k)}}$$

$$= \prod_{k=1}^{\ell} \underbrace{\prod_{(i,j) \in Q^{-1}(k)} x_{r_{T(Y(\alpha))}(k)}}_{\substack{= x_{r_{T(Y(\alpha))}(k)}^{\left|Q^{-1}(k)\right|} = x_{i_k}^{\left|Q^{-1}(k)\right|} \\ \text{(since } r_{T(Y(\alpha))}(k) = i_k)}} = \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\left|Q^{-1}(k)\right|}}_{\substack{= x_{i_k}^{\beta_k} \\ \text{(since } \left|Q^{-1}(k)\right| = \beta_k \\ \text{(by (2.21)))}}} = \prod_{k=1}^{\ell} x_{i_k}^{\beta_k} = x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell}$$

). Hence,

$$\sum_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T = \sum_{1\leq i_1<i_2<\cdots<i_\ell} x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \cdots x_{i_\ell}^{\beta_\ell} = M_\beta$$

(by the definition of M_β). Altogether, we thus have

$$\sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha;\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T = \sum_{\substack{T:Y(\alpha)\to\{1,2,3,\ldots\};\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T = M_\beta.$$

This proves Lemma 2.4.5 **(c)**. ☐

Proof of Proposition 2.4.4. For every finite subset I of $\{1,2,3,\ldots\}$, we shall use the notation r_I introduced in Lemma 2.4.5 **(a)**. Recall Lemma 2.4.5 **(b)**; it says that if T is an immaculate tableau of shape α, then $r_{T(Y(\alpha))}^{-1} \circ T$ is an immaculate tableau of shape α as well, and has the additional property that there exists a unique composition β of $|\alpha|$ such that $r_{T(Y(\alpha))}^{-1} \circ T$ has content β.

Now,

$$\mathfrak{S}_\alpha^* = \sum_{\beta\models|\alpha|} \underbrace{K_{\alpha,\beta} M_\beta}_{\substack{= \sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} M_\beta \\ \text{(by the definition of } K_{\alpha,\beta})}} = \sum_{\beta\models|\alpha|} \sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} M_\beta. \tag{2.22}$$

But (2.20) shows that every composition β of $|\alpha|$ satisfies

$$\sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} M_\beta = \sum_{\substack{Q \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{and content } \beta}} \sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha;\\ r_{T(Y(\alpha))}^{-1}\circ T=Q}} \mathbf{x}_T = \sum_{\substack{T \text{ is an immaculate}\\ \text{tableau of shape } \alpha\\ \text{such that } r_{T(Y(\alpha))}^{-1}\circ T\\ \text{has content } \beta}} \mathbf{x}_T$$

(because for every immaculate tableau T of shape α, the map $r_{T(Y(\alpha))}^{-1} \circ T$ is an immaculate tableau of shape α as well). Substituting this into (2.22), we obtain

$$\mathfrak{S}^*_\alpha = \sum_{\beta \models |\alpha|} \underbrace{\sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{and content } \beta}} M_\beta}_{= \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{such that } r^{-1}_{T(Y(\alpha))} \circ T \\ \text{has content } \beta}} \mathbf{x}_T} = \sum_{\beta \models |\alpha|} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha \\ \text{such that } r^{-1}_{T(Y(\alpha))} \circ T \\ \text{has content } \beta}} \mathbf{x}_T = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T$$

(because for every immaculate tableau T of shape α, there exists a unique composition β of $|\alpha|$ such that $r^{-1}_{T(Y(\alpha))} \circ T$ has content β), whence Proposition 2.4.4 follows. □

Corollary 2.4.6. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition with $\ell > 0$. Let $\overline{\alpha}$ denote the composition $(\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ of $|\alpha| - \alpha_1$. Then,

$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}.$$

Here, h_n denotes the n-th complete homogeneous symmetric function for every $n \in \mathbb{N}$.

Proof of Corollary 2.4.6. Proposition 2.4.4 shows that

$$\mathfrak{S}^*_\alpha = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T = \sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q \tag{2.23}$$

(here, we have renamed the summation index T as Q).

Let $n = \alpha_1$. If $i_1, i_2, \ldots, i_n$ are positive integers satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$, and if T is an immaculate tableau of shape $\overline{\alpha}$, then

$$\begin{aligned}
&(x_{i_1}x_{i_2}\cdots x_{i_n}) \prec \mathbf{x}_T \\
&= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } \min(\operatorname{Supp}(x_{i_1}x_{i_2}\cdots x_{i_n})) < \min(\operatorname{Supp}(\mathbf{x}_T)); \\ 0, & \text{if } \min(\operatorname{Supp}(x_{i_1}x_{i_2}\cdots x_{i_n})) \geq \min(\operatorname{Supp}(\mathbf{x}_T)) \end{cases} \\
&\qquad \text{(by the definition of } \prec \text{ on monomials)} \\
&= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min(T(Y(\overline{\alpha}))); \\ 0, & \text{if } i_1 \geq \min(T(Y(\overline{\alpha}))) \end{cases} \qquad (2.24) \\
&\qquad \text{(since } \min(\operatorname{Supp}(x_{i_1}x_{i_2}\cdots x_{i_n})) = i_1 \text{ and } \operatorname{Supp}(\mathbf{x}_T) = T(Y(\overline{\alpha}))).
\end{aligned}$$

But from $n = \alpha_1$, we obtain $h_n = h_{\alpha_1}$, so that $h_{\alpha_1} = h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n}$ and $\mathfrak{S}^*_{\overline{\alpha}} = \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \mathbf{x}_T$ (by Proposition 2.4.4). Hence,

$$\begin{aligned}
&h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}} \\
&= \left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2}\cdots x_{i_n} \right) \prec \left(\sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \mathbf{x}_T \right) \\
&= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \underbrace{(x_{i_1}x_{i_2}\cdots x_{i_n}) \prec \mathbf{x}_T}_{\substack{= \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min(T(Y(\overline{\alpha}))); \\ 0, & \text{if } i_1 \geq \min(T(Y(\overline{\alpha}))) \end{cases} \\ \text{(by (2.24))}}} \\
&= \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} \sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}}} \begin{cases} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T, & \text{if } i_1 < \min(T(Y(\overline{\alpha}))); \\ 0, & \text{if } i_1 \geq \min(T(Y(\overline{\alpha}))) \end{cases} \\
&= \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}; \\ i_1 < \min(T(Y(\overline{\alpha})))}} x_{i_1}x_{i_2}\cdots x_{i_n}\mathbf{x}_T. \qquad (2.25)
\end{aligned}$$

We need to check that this equals $\mathfrak{S}^*_{\alpha} = \sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q$.

Now, let us define a map Φ from:

- the set of all pairs $((i_1, i_2, \ldots, i_n), T)$, where i_1, i_2, $\ldots$, i_n are positive integers satisfying $i_1 \leq i_2 \leq \cdots \leq i_n$, and where T is an immaculate tableau of shape $\overline{\alpha}$ satisfying $i_1 < \min(T(Y(\overline{\alpha})))$

to:

- the set of all immaculate tableaux of shape α.

Namely, we define the image of a pair $((i_1, i_2, \ldots, i_n), T)$ under Φ to be the immaculate tableau obtained by adding a new row, filled with the entries $i_1, i_2, \ldots, i_n$ (from left to right), to the top[26] of the tableau T [27].

This map Φ is a bijection[28], and has the property that if Q denotes the image of a pair $((i_1, i_2, \ldots, i_n), T)$ under the bijection Φ, then $\mathbf{x}_Q = x_{i_1} x_{i_2} \cdots x_{i_n} \mathbf{x}_T$. Hence,

$$\sum_{\substack{Q \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_Q = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ T \text{ is an immaculate} \\ \text{tableau of shape } \overline{\alpha}; \\ i_1 < \min(T(Y(\overline{\alpha})))}} x_{i_1} x_{i_2} \cdots x_{i_n} \mathbf{x}_T.$$

In light of (2.23) and (2.25), this rewrites as $\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}$. So Corollary 2.4.6 is proven. □

Corollary 2.4.7. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition. Then,

$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)).$$

Proof of Corollary 2.4.7. We prove Corollary 2.4.7 by induction over ℓ:

[26] Here, we are using the graphical representation of immaculate tableaux introduced in Definition 2.4.1.

[27] Formally speaking, this means that the image of $((i_1, i_2, \ldots, i_n), T)$ is the map $Y(\alpha) \to \{1, 2, 3, \ldots\}$ which sends every $(u, v) \in Y(\alpha)$ to $\begin{cases} i_v, & \text{if } u = 1; \\ T(u-1, v), & \text{if } u \neq 1 \end{cases}$. Proving that this map is an immaculate tableau is easy.

[28] *Proof.* The injectivity of the map Φ is obvious. Its surjectivity follows from the observation that if Q is an immaculate tableau of shape α, then the first entry of its top row is smaller than the smallest entry of the immaculate tableau formed by all other rows of Q. (This is a consequence of (2.19), applied to Q instead of T.)

Induction base: If $\ell = 0$, then $\alpha = \varnothing$ and thus $\mathfrak{S}^*_\alpha = \mathfrak{S}^*_\varnothing = 1$. But if $\ell = 0$, then we also have $h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)) = 1$. Hence, if $\ell = 0$, then $\mathfrak{S}^*_\alpha = 1 = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$. Thus, Corollary 2.4.7 is proven when $\ell = 0$. The induction base is complete.

Induction step: Let L be a positive integer. Assume that Corollary 2.4.7 holds for $\ell = L - 1$. We now need to prove that Corollary 2.4.7 by holds for $\ell = L$.

So let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a composition with $\ell = L$. Then, $\ell = L > 0$. Now, let $\overline{\alpha}$ denote the composition $(\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ of $|\alpha| - \alpha_1$. Then, Corollary 2.4.6 yields $\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \mathfrak{S}^*_{\overline{\alpha}}$. But by our induction hypothesis, we can apply Corollary 2.4.7 to $\overline{\alpha} = (\alpha_2, \alpha_3, \ldots, \alpha_\ell)$ instead of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ (since $\ell - 1 = L - 1$). As a result, we obtain $\mathfrak{S}^*_{\overline{\alpha}} = h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$. Hence,

$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec \underbrace{\mathfrak{S}^*_{\overline{\alpha}}}_{= h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))} = h_{\alpha_1} \prec (h_{\alpha_2} \prec (h_{\alpha_3} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)))$$
$$= h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots)).$$

Now, let us forget that we fixed α. We thus have shown that $\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1) \cdots))$ for every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ which satisfies $\ell = L$. In other words, Corollary 2.4.7 holds for $\ell = L$. This completes the induction step. The induction proof of Corollary 2.4.7 is thus complete. □

2.5 An alternative description of $h_m \prec$

In this section, we shall also use the Hopf algebra of *noncommutative symmetric functions*. This Hopf algebra (a noncommutative one, for a change) is denoted by NSym and has been discussed in [GriRei15, Section 5] and [HaGuKi10, Chapter 6]; all we need to know about it are the following properties:

- There is a nondegenerate pairing between NSym and QSym, that is, a nondegenerate $\mathbf{k}$-bilinear form $\mathrm{NSym} \times \mathrm{QSym} \to \mathbf{k}$. We shall denote this bilinear form by $(\cdot,\cdot)$. This **k**-bilinear form is a Hopf algebra pairing, i.e., it satisfies

$$(ab, c) = \sum_{(c)} \left(a, c_{(1)}\right) \left(b, c_{(2)}\right) \qquad (2.26)$$

$$\text{for all } a \in \operatorname{NSym}, \ b \in \operatorname{NSym} \ \text{ and } c \in \operatorname{QSym};$$

$$(1, c) = \varepsilon(c) \qquad \text{for all } c \in \operatorname{QSym};$$

$$\sum_{(a)} \left(a_{(1)}, b\right) \left(a_{(2)}, c\right) = (a, bc)$$

$$\text{for all } a \in \operatorname{NSym}, \ b \in \operatorname{QSym} \ \text{ and } c \in \operatorname{QSym};$$

$$(a, 1) = \varepsilon(a) \qquad \text{for all } a \in \operatorname{NSym};$$

$$(S(a), b) = (a, S(b)) \qquad \text{for all } a \in \operatorname{NSym} \ \text{ and } b \in \operatorname{QSym}$$

(where we use Sweedler's notation).

- There is a basis of the **k**-module NSym which is dual to the fundamental basis $(F_\alpha)_{\alpha\in\mathrm{Comp}}$ of QSym with respect to the bilinear form $(\cdot,\cdot)$. This basis is called the *ribbon basis* and will be denoted by $(R_\alpha)_{\alpha\in\mathrm{Comp}}$.

Both of these properties are immediate consequences of the definitions of NSym and of $(R_\alpha)_{\alpha\in\mathrm{Comp}}$ given in [GriRei15, Section 5] (although other sources define these objects differently, and then the properties no longer are immediate). The notations we are using here are the same as the ones used in [GriRei15, Section 5] (except that [GriRei15, Section 5] calls L_α what we denote by F_α), and only slightly differ from those in [BBSSZ13a] (namely, [BBSSZ13a] denotes the pairing $(\cdot,\cdot)$ by $\langle\cdot,\cdot\rangle$ instead).

We need some more definitions. For any $g \in \operatorname{NSym}$, let $\mathrm{L}_g : \operatorname{NSym} \to \operatorname{NSym}$ denote the left multiplication by g on NSym (that is, the **k**-linear map $\operatorname{NSym} \to \operatorname{NSym}, \ f \mapsto gf$). For any $g \in \operatorname{NSym}$, let $g^\perp : \operatorname{QSym} \to \operatorname{QSym}$ be the **k**-linear map adjoint to $\mathrm{L}_g : \operatorname{NSym} \to \operatorname{NSym}$ with respect to the pairing $(\cdot,\cdot)$ between NSym and

QSym. Thus, for any $g \in \operatorname{NSym}$, $a \in \operatorname{NSym}$ and $c \in \operatorname{QSym}$, we have

$$\left(a, g^{\perp} c\right) = \left(\underbrace{\mathrm{L}_g \, a}_{= ga}, c \right) = (ga, c) . \tag{2.27}$$

The following fact is well-known (and also is an easy formal consequence of the definition of $g^{\perp}$ and of (2.26)):

Lemma 2.5.1. Every $g \in \operatorname{NSym}$ and $f \in \operatorname{QSym}$ satisfy

$$g^{\perp} f = \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)}. \tag{2.28}$$

Proof of Lemma 2.5.1. Let $g \in \operatorname{NSym}$ and $f \in \operatorname{QSym}$. For every $a \in \operatorname{NSym}$, we have

$$\begin{aligned}
\left(a, g^{\perp} f\right) &= \left(\underbrace{\mathrm{L}_g \, a}_{\substack{= ga \\ \text{(by the definition of } \mathrm{L}_g \text{)}}}, f \right) && \left(\begin{array}{c} \text{since the map } g^{\perp} \text{ is adjoint to } \mathrm{L}_g \\ \text{with respect to the pairing } (\cdot, \cdot) \end{array} \right) \\
&= (ga, f) = \sum_{(f)} \left(g, f_{(1)}\right) \left(a, f_{(2)}\right) && \left(\begin{array}{c} \text{by (2.26), applied to } g,\ a \text{ and } f \\ \text{instead of } a,\ b \text{ and } c \end{array} \right) \\
&= \left(a, \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)} \right) && \left(\text{since the pairing } (\cdot, \cdot) \text{ is } \mathbf{k}\text{-bilinear}\right).
\end{aligned}$$

Since the pairing $(\cdot, \cdot)$ is nondegenerate, this entails that $g^{\perp} f = \sum_{(f)} \left(g, f_{(1)}\right) f_{(2)}$. This proves Lemma 2.5.1. □

For any composition α, we define a composition $\omega(\alpha)$ as follows: Let $n = |\alpha|$, and write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let $\operatorname{rev} \alpha$ denote the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ of n. Then, $\omega(\alpha)$ shall be the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev} \alpha)$. (This definition is identical with that in [GriRei15, Definition 5.22]. Some authors denote $\omega(\alpha)$ by α' instead.) We notice that $\omega(\omega(\alpha)) = \alpha$ for any composition α.

Here is a simple property of the composition $\omega(\alpha)$ that will later be used:

Proposition 2.5.2. **(a)** We have $\omega([\alpha,\beta]) = \omega(\beta) \odot \omega(\alpha)$ for any two compositions α and β.

(b) We have $\omega(\alpha \odot \beta) = [\omega(\beta), \omega(\alpha)]$ for any two compositions α and β.

(c) We have $\omega(\omega(\gamma)) = \gamma$ for every composition γ.

Proof of Proposition 2.5.2. For any composition α, we define a composition $\operatorname{rev}\alpha$ as follows: Let $n = |\alpha|$, and write α as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Let $\operatorname{rev}\alpha$ denote the composition $(\alpha_\ell, \alpha_{\ell-1}, \ldots, \alpha_1)$ of n. (This definition of $\operatorname{rev}\alpha$ is the same as the one we gave above during the definition of $\omega(\alpha)$.) Clearly,

$$|\operatorname{rev}\gamma| = |\gamma| \qquad \text{for any composition } \gamma. \tag{2.29}$$

It is easy to see that

$$\operatorname{rev}([\alpha,\beta]) = [\operatorname{rev}\beta, \operatorname{rev}\alpha] \qquad \text{and} \tag{2.30}$$

$$\operatorname{rev}(\alpha \odot \beta) = (\operatorname{rev}\beta) \odot (\operatorname{rev}\alpha) \tag{2.31}$$

for any two compositions α and β.

Recall that a composition γ of a nonnegative integer n is uniquely determined by the set $D(\gamma)$ and the number n. Thus, if γ_1 and γ_2 are two compositions of one and the same nonnegative integer n satisfying $D(\gamma_1) = D(\gamma_2)$, then

$$\gamma_1 = \gamma_2. \tag{2.32}$$

For every composition γ, we define a composition $\rho(\gamma)$ as follows: Let $n = |\gamma|$. Let $\rho(\gamma)$ be the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$. (This is well-defined, because for every subset T of $\{1, 2, \ldots, n-1\}$, there exists a unique composition τ of n which satisfies $D(\tau) = T$.) Notice that

$$|\rho(\gamma)| = |\gamma| \qquad \text{for any composition } \gamma. \tag{2.33}$$

Also, if $n \in \mathbb{N}$, and if γ is a composition of n, then

$$D(\rho(\gamma)) = \{1, 2, \ldots, n-1\} \setminus D(\gamma) \qquad (2.34)$$

(by the definition of $\rho(\gamma)$).

Notice also that

$$\omega(\alpha) = \rho(\operatorname{rev}\alpha) \qquad \text{for any composition } \alpha \qquad (2.35)$$

[29].

Now, we shall prove that

$$\rho([\alpha, \beta]) = \rho(\alpha) \odot \rho(\beta) \qquad (2.36)$$

for any two compositions α and β.

Proof of (2.36): Let α and β be two compositions. Let $p = |\alpha|$ and $q = |\beta|$; thus, α and β are compositions of p and q, respectively. We WLOG assume that both compositions α and β are nonempty (since otherwise, (2.36) is fairly obvious). The composition α is a composition of p. Thus, $p > 0$ (since α is nonempty). Similarly, $q > 0$.

Hence, $[\alpha, \beta]$ is a composition of $p + q$ satisfying $D([\alpha, \beta]) = D(\alpha) \cup \{p\} \cup$

[29] *Proof of (2.35):* Let α be a composition. Let $n = |\alpha|$. Thus, α is a composition of n. Hence, $\omega(\alpha)$ is a composition of n as well. Also, $\operatorname{rev}\alpha$ is a composition of n. Now, the definition of $\rho(\operatorname{rev}\alpha)$ shows that $\rho(\operatorname{rev}\alpha)$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha)$. Hence, $\rho(\operatorname{rev}\alpha)$ is a composition of n and satisfies $D(\rho(\operatorname{rev}\alpha)) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha)$.

On the other hand, $\omega(\alpha)$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha)$ (by the definition of $\omega(\alpha)$). Thus, $\omega(\alpha)$ is a composition of n and satisfies $D(\omega(\alpha)) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha)$.

Hence,

$$D(\rho(\operatorname{rev}\alpha)) = \{1, 2, \ldots, n-1\} \setminus D(\operatorname{rev}\alpha) = D(\omega(\alpha)).$$

Applying (2.32) to $\gamma_1 = \rho(\operatorname{rev}\alpha)$ and $\gamma_2 = \omega(\alpha)$, we therefore obtain $\rho(\operatorname{rev}\alpha) = \omega(\alpha)$. Qed.

$(D(\beta)+p)$ (by Lemma 2.3.9 **(b)**). The definition of $\rho([\alpha,\beta])$ thus yields

$$D(\rho([\alpha,\beta])) = \{1,2,\ldots,p+q-1\} \setminus \underbrace{D([\alpha,\beta])}_{=D(\alpha)\cup\{p\}\cup(D(\beta)+p)}$$
$$= \{1,2,\ldots,p+q-1\} \setminus (\{p\} \cup D(\alpha) \cup (D(\beta)+p)). \qquad (2.37)$$

Applying (2.33) to $\gamma = \alpha$, we obtain $|\rho(\alpha)| = |\alpha| = p$. Thus, $\rho(\alpha)$ is a composition of p. Similarly, $\rho(\beta)$ is a composition of q. Thus, Lemma 2.3.9 **(a)** (applied to $\rho(\alpha)$ and $\rho(\beta)$ instead of α and β) shows that $\rho(\alpha) \odot \rho(\beta)$ is a composition of $p+q$ satisfying $D(\rho(\alpha) \odot \rho(\beta)) = D(\rho(\alpha)) \cup (D(\rho(\beta))+p)$. Also, applying (2.33) to $\gamma = [\alpha,\beta]$, we obtain $|\rho([\alpha,\beta])| = |[\alpha,\beta]| = p+q$ (since $[\alpha,\beta]$ is a composition of $p+q$). In other words, $\rho([\alpha,\beta])$ is a composition of $p+q$.

But the definition of $\rho(\alpha)$ shows that $D(\rho(\alpha)) = \{1,2,\ldots,p-1\} \setminus D(\alpha)$. Also, the definition of $\rho(\beta)$ shows that $D(\rho(\beta)) = \{1,2,\ldots,q-1\} \setminus D(\beta)$. Hence,

$$\underbrace{D(\rho(\beta))}_{=\{1,2,\ldots,q-1\}\setminus D(\beta)} + p = (\{1,2,\ldots,q-1\} \setminus D(\beta)) + p$$
$$= \underbrace{(\{1,2,\ldots,q-1\}+p)}_{=\{p+1,p+2,\ldots,p+q-1\}} \setminus (D(\beta)+p)$$
$$= \{p+1,p+2,\ldots,p+q-1\} \setminus (D(\beta)+p).$$

Also, $D(\beta) \subseteq \{1,2,\ldots,q-1\}$, so that
$D(\beta)+p \subseteq \{1,2,\ldots,q-1\}+p = \{p+1,p+2,\ldots,p+q-1\}$.

Now, it is well-known that if X, Y, X' and Y' are four sets such that $X' \subseteq X$, $Y' \subseteq Y$ and $X \cap Y = \varnothing$, then

$$(X \setminus X') \cup (Y \setminus Y') = (X \cup Y) \setminus (X' \cup Y'). \qquad (2.38)$$

Now,

$$\begin{aligned}
& D\left(\rho\left(\alpha\right)\odot\rho\left(\beta\right)\right) \\
&= \underbrace{D\left(\rho\left(\alpha\right)\right)}_{=\{1,2,\ldots,p-1\}\setminus D(\alpha)} \cup \underbrace{\left(D\left(\rho\left(\beta\right)\right)+p\right)}_{=\{p+1,p+2,\ldots,p+q-1\}\setminus(D(\beta)+p)} \\
&= \left(\{1,2,\ldots,p-1\}\setminus D\left(\alpha\right)\right)\cup\left(\{p+1,p+2,\ldots,p+q-1\}\setminus\left(D\left(\beta\right)+p\right)\right) \\
&= \underbrace{\left(\{1,2,\ldots,p-1\}\cup\{p+1,p+2,\ldots,p+q-1\}\right)}_{=\{1,2,\ldots,p+q-1\}\setminus\{p\}} \\
&\qquad \setminus\left(D\left(\alpha\right)\cup\left(D\left(\beta\right)+p\right)\right) \\
&\qquad \left(\begin{array}{c}\text{by (2.38), applied to } X=\{1,2,\ldots,p-1\}, \\ Y=\{p+1,p+2,\ldots,p+q-1\},\ X'=D\left(\alpha\right) \text{ and } Y'=D\left(\beta\right)+p\end{array}\right) \\
&= \left(\{1,2,\ldots,p+q-1\}\setminus\{p\}\right)\setminus\left(D\left(\alpha\right)\cup\left(D\left(\beta\right)+p\right)\right) \\
&= \{1,2,\ldots,p+q-1\}\setminus\underbrace{\left(\{p\}\cup D\left(\alpha\right)\cup\left(D\left(\beta\right)+p\right)\right)}_{=D(\alpha)\cup\{p\}\cup(D(\beta)+p)} \\
&= \{1,2,\ldots,p+q-1\}\setminus\left(\{p\}\cup D\left(\alpha\right)\cup\left(D\left(\beta\right)+p\right)\right) \\
&= D\left(\rho\left([\alpha,\beta]\right)\right) \qquad \left(\text{by (2.37)}\right).
\end{aligned}$$

Thus, (2.32) (applied to $n=p+q$, $\gamma_1=\rho\left(\alpha\right)\odot\rho\left(\beta\right)$ and $\gamma_2=\rho\left([\alpha,\beta]\right)$) shows that $\rho\left(\alpha\right)\odot\rho\left(\beta\right)=\rho\left([\alpha,\beta]\right)$. This proves (2.36).

(a) Let α and β be two compositions. Then, (2.35) yields $\omega\left(\alpha\right)=\rho\left(\operatorname{rev}\alpha\right)$. Also, (2.35) (applied to β instead of α) yields $\omega\left(\beta\right)=\rho\left(\operatorname{rev}\beta\right)$.

From (2.35) (applied to $[\alpha,\beta]$ instead of α), we obtain

$$\begin{aligned}
\omega\left([\alpha,\beta]\right) &= \rho\left(\underbrace{\operatorname{rev}\left([\alpha,\beta]\right)}_{\substack{=[\operatorname{rev}\beta,\operatorname{rev}\alpha] \\ \text{(by (2.30))}}}\right) = \rho\left([\operatorname{rev}\beta,\operatorname{rev}\alpha]\right) \\
&= \underbrace{\rho\left(\operatorname{rev}\beta\right)}_{=\omega(\beta)}\odot\underbrace{\rho\left(\operatorname{rev}\alpha\right)}_{=\omega(\alpha)} \qquad \left(\begin{array}{c}\text{by (2.36), applied to } \operatorname{rev}\beta \\ \text{and } \operatorname{rev}\alpha \text{ instead of } \alpha \text{ and } \beta\end{array}\right) \\
&= \omega\left(\beta\right)\odot\omega\left(\alpha\right).
\end{aligned}$$

This proves Proposition 2.5.2 **(a)**.

(c) First of all, it is clear that

$$\operatorname{rev}(\operatorname{rev}\gamma) = \gamma \qquad \text{for every composition } \gamma. \tag{2.39}$$

Furthermore,

$$\rho(\rho(\gamma)) = \gamma \qquad \text{for every composition } \gamma \tag{2.40}$$

[30].

On the other hand, if G is a set of integers and r is an integer, then we let $r - G$ denote the set $\{r - g \mid g \in G\}$ of integers. Then, for any $n \in \mathbb{N}$ and any composition γ of n, we have

$$D(\operatorname{rev}\gamma) = n - D(\gamma) \tag{2.41}$$

[31].

---

[30] *Proof of (2.40):* Let γ be a composition. Let $n = |\gamma|$. Thus, γ is a composition of n. The definition of $\rho(\gamma)$ shows that $\rho(\gamma)$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$. Thus, $\rho(\gamma)$ is a composition of n and satisfies $D(\rho(\gamma)) = \{1, 2, \ldots, n-1\} \setminus D(\gamma)$.

Therefore, the definition of $\rho(\rho(\gamma))$ shows that $\rho(\rho(\gamma))$ is the unique composition β of n which satisfies $D(\beta) = \{1, 2, \ldots, n-1\} \setminus D(\rho(\gamma))$. Thus, $\rho(\rho(\gamma))$ is a composition of n and satisfies $D(\rho(\rho(\gamma))) = \{1, 2, \ldots, n-1\} \setminus D(\rho(\gamma))$. Hence,

$$\begin{aligned} D(\rho(\rho(\gamma))) &= \{1, 2, \ldots, n-1\} \setminus \underbrace{D(\rho(\gamma))}_{=\{1,2,\ldots,n-1\}\setminus D(\gamma)} \\ &= \{1, 2, \ldots, n-1\} \setminus (\{1, 2, \ldots, n-1\} \setminus D(\gamma)) \\ &= D(\gamma) \qquad (\text{since } D(\gamma) \subseteq \{1, 2, \ldots, n-1\}). \end{aligned}$$

Hence, (2.32) (applied to $\gamma_1 = \rho(\rho(\gamma))$ and $\gamma_2 = \gamma$) shows that $\rho(\rho(\gamma)) = \gamma$. This proves (2.40).

[31] *Proof of (2.41):* Let $n \in \mathbb{N}$. Let γ be a composition of n. Thus, γ is a composition satisfying $|\gamma| = n$.

Write γ in the form $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$. Then, $\operatorname{rev}\gamma = (\gamma_\ell, \gamma_{\ell-1}, \ldots, \gamma_1)$ (by the definition of $\operatorname{rev}\gamma$). Also, from $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$, we obtain $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_\ell$, whence $\gamma_1 + \gamma_2 + \cdots + \gamma_\ell = |\gamma| = n$. Hence, every $i \in \{1, 2, \ldots, \ell - 1\}$ satisfies

$$\begin{aligned} n = \gamma_1 + \gamma_2 + \cdots + \gamma_\ell &= (\gamma_1 + \gamma_2 + \cdots + \gamma_i) + \underbrace{(\gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_\ell)}_{=\gamma_\ell+\gamma_{\ell-1}+\cdots+\gamma_{i+1}} \\ &= (\gamma_1 + \gamma_2 + \cdots + \gamma_i) + (\gamma_\ell + \gamma_{\ell-1} + \cdots + \gamma_{i+1}). \end{aligned} \tag{2.42}$$

Now,

$$\rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma)) \qquad \text{for every composition } \gamma \tag{2.44}$$

[32].

Now, let γ be a composition. Then, (2.35) (applied to $\alpha = \gamma$) yields $\omega(\gamma) =$

Also, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell)$, so that the definition of $D(\gamma)$ yields

$$\begin{aligned} D(\gamma) &= \{\gamma_1, \gamma_1+\gamma_2, \gamma_1+\gamma_2+\gamma_3, \ldots, \gamma_1+\gamma_2+\cdots+\gamma_{\ell-1}\} \\ &= \{\gamma_1+\gamma_2+\cdots+\gamma_i \mid i \in \{1,2,\ldots,\ell-1\}\}. \end{aligned} \tag{2.43}$$

But $\operatorname{rev}\gamma = (\gamma_\ell, \gamma_{\ell-1}, \ldots, \gamma_1)$. Hence, the definition of $D(\operatorname{rev}\gamma)$ yields

$$\begin{aligned} D(\operatorname{rev}\gamma) &= \{\gamma_\ell, \gamma_\ell+\gamma_{\ell-1}, \gamma_\ell+\gamma_{\ell-1}+\gamma_{\ell-2}, \ldots, \gamma_\ell+\gamma_{\ell-1}+\gamma_{\ell-2}+\cdots+\gamma_2\} \\ &= \left\{ \underbrace{\gamma_\ell+\gamma_{\ell-1}+\cdots+\gamma_{i+1}}_{\substack{=n-(\gamma_1+\gamma_2+\cdots+\gamma_i) \\ (\text{by } (2.42))}} \mid i \in \{1,2,\ldots,\ell-1\} \right\} \\ &= \{n-(\gamma_1+\gamma_2+\cdots+\gamma_i) \mid i \in \{1,2,\ldots,\ell-1\}\} \\ &= n - \underbrace{\{\gamma_1+\gamma_2+\cdots+\gamma_i \mid i \in \{1,2,\ldots,\ell-1\}\}}_{\substack{=D(\gamma) \\ (\text{by } (2.43))}} \\ &= n - D(\gamma). \end{aligned}$$

This proves (2.41).

[32] *Proof of (2.44):* Let γ be a composition. Let $n = |\gamma|$. Thus, γ is a composition of n.

Now, (2.33) (applied to $\operatorname{rev}\gamma$ instead of γ) yields $|\rho(\operatorname{rev}\gamma)| = |\operatorname{rev}\gamma| = |\gamma|$ (by (2.29)). Also, (2.29) (applied to $\rho(\gamma)$ instead of γ) yields $|\operatorname{rev}(\rho(\gamma))| = |\rho(\gamma)| = |\gamma|$ (by (2.33)). Now, $|\rho(\operatorname{rev}\gamma)| = |\gamma| = n$, $|\operatorname{rev}\gamma| = |\gamma| = n$, $|\rho(\gamma)| = |\gamma| = n$ and $|\operatorname{rev}(\rho(\gamma))| = |\gamma| = n$. Hence, all of $\rho(\operatorname{rev}\gamma)$, $\operatorname{rev}\gamma$, $\rho(\gamma)$ and $\operatorname{rev}(\rho(\gamma))$ are compositions of n.

Applying (2.34) to $\operatorname{rev}\gamma$ instead of γ, we obtain

$$\begin{aligned} D(\rho(\operatorname{rev}\gamma)) &= \underbrace{\{1,2,\ldots,n-1\}}_{=n-\{1,2,\ldots,n-1\}} \setminus \underbrace{D(\operatorname{rev}\gamma)}_{\substack{=n-D(\gamma) \\ (\text{by } (2.41))}} \\ &= (n - \{1,2,\ldots,n-1\}) \setminus (n - D(\gamma)) \\ &= n - \underbrace{(\{1,2,\ldots,n-1\} \setminus D(\gamma))}_{\substack{=D(\rho(\gamma)) \\ (\text{by } (2.34))}} \\ &= n - D(\rho(\gamma)). \end{aligned}$$

Comparing this with

$$D(\operatorname{rev}(\rho(\gamma))) = n - D(\rho(\gamma)) \qquad (\text{by } (2.41), \text{ applied to } \rho(\gamma) \text{ instead of } \gamma),$$

we obtain $D(\rho(\operatorname{rev}\gamma)) = D(\operatorname{rev}(\rho(\gamma)))$. Hence, (2.32) (applied to $\gamma_1 = \rho(\operatorname{rev}\gamma)$ and $\gamma_2 = \operatorname{rev}(\rho(\gamma))$) yields $\rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma))$. This proves (2.44).

$\rho(\operatorname{rev}\gamma) = \operatorname{rev}(\rho(\gamma))$ (by (2.44)). But (2.35) (applied to $\alpha = \omega(\gamma)$) yields

$$\omega(\omega(\gamma)) = \rho\left(\operatorname{rev}\left(\underbrace{\omega(\gamma)}_{=\operatorname{rev}(\rho(\gamma))}\right)\right) = \rho\left(\underbrace{\operatorname{rev}(\operatorname{rev}(\rho(\gamma)))}_{\substack{=\rho(\gamma) \\ \text{(by (2.39), applied to} \\ \rho(\gamma) \text{ instead of } \gamma)}}\right)$$
$$= \rho(\rho(\gamma)) = \gamma \qquad \text{(by (2.40))}.$$

This proves Proposition 2.5.2 **(c)**.

(b) Let α and β be two compositions. Then, Proposition 2.5.2 **(a)** (applied to $\omega(\beta)$ and $\omega(\alpha)$ instead of α and β) yields

$$\omega([\omega(\beta), \omega(\alpha)]) = \underbrace{\omega(\omega(\alpha))}_{\substack{=\alpha \\ \text{(by Proposition 2.5.2 (c),} \\ \text{applied to } \gamma=\alpha)}} \odot \underbrace{\omega(\omega(\beta))}_{\substack{=\beta \\ \text{(by Proposition 2.5.2 (c),} \\ \text{applied to } \gamma=\beta)}}$$
$$= \alpha \odot \beta.$$

Hence, $\alpha \odot \beta = \omega([\omega(\beta), \omega(\alpha)])$. Applying the map ω to both sides of this equality, we conclude that

$$\omega(\alpha \odot \beta) = \omega(\omega([\omega(\beta), \omega(\alpha)])) = [\omega(\beta), \omega(\alpha)]$$

(by Proposition 2.5.2 **(c)**, applied to $\gamma = [\omega(\beta), \omega(\alpha)]$). This proves Proposition 2.5.2 **(b)**. $\square$

The notion of $\omega(\alpha)$ gives rise to a simple formula for the antipode S of the Hopf algebra QSym in terms of its fundamental basis:

Proposition 2.5.3. Let α be a composition. Then, $S(F_\alpha) = (-1)^{|\alpha|} F_{\omega(\alpha)}$.

This is proven in [GriRei15, Proposition 5.23].

We now state the main result of this note:

Theorem 2.5.4. Let $f \in \operatorname{QSym}$ and let m be a positive integer. For any two compositions α and β, define a composition $\alpha \odot \beta$ as in Proposition 2.3.8. Then,

$$h_m \prec f = \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_{\alpha \odot (m)} R_{\omega(\alpha)}^{\perp} f.$$

(Here, the sum on the right hand side converges, because all but finitely many compositions α satisfy $R_{\omega(\alpha)}^{\perp} f = 0$ for degree reasons.)

The proof is based on the following simple lemma:

Lemma 2.5.5. Let $a \in \operatorname{QSym}$ and $f \in \operatorname{QSym}$. Then,

$$\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R_{\omega(\alpha)}^{\perp} f = a \prec f.$$

Proof of Lemma 2.5.5. The basis $(F_\alpha)_{\alpha \in \operatorname{Comp}}$ of QSym and the basis $(R_\alpha)_{\alpha \in \operatorname{Comp}}$ of NSym are dual bases. Thus,

$$\sum_{\alpha \in \operatorname{Comp}} F_\alpha (R_\alpha, g) = g \qquad \text{for every } g \in \operatorname{QSym}. \tag{2.45}$$

Let us use Sweedler's notation. The map Comp $\to$ Comp, $\alpha \mapsto \omega(\alpha)$ is a bijection (since $\omega(\omega(\alpha)) = \alpha$ for any composition α). Hence, we can substitute $\omega(\alpha)$ for α in the sum $\sum\limits_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R^\perp_{\omega(\alpha)} f$. We thus obtain

$$\begin{aligned}
&\sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} a) R^\perp_{\omega(\alpha)} f \\
&= \sum_{\alpha\in\mathrm{Comp}} \underbrace{(-1)^{|\omega(\alpha)|}}_{\substack{=(-1)^{|\alpha|} \\ (\text{since } |\omega(\alpha)|=|\alpha|)}} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \underbrace{R^\perp_{\omega(\omega(\alpha))}}_{\substack{=R^\perp_\alpha \\ (\text{since } \omega(\omega(\alpha))=\alpha)}} f \\
&= \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \underbrace{R^\perp_\alpha f}_{\substack{=\sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\ (\text{by } (2.28))}} \\
&= \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} \left(F_{\omega(\alpha)} \mathbin{\Phi} a\right) \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \left(\left(\sum_{\alpha\in\mathrm{Comp}} \underbrace{(-1)^{|\alpha|} F_{\omega(\alpha)}}_{\substack{=S(F_\alpha) \\ (\text{by Proposition 2.5.3})}} \left(R_\alpha, f_{(1)}\right) \right) \mathbin{\Phi} a \right) f_{(2)} \\
&= \sum_{(f)} \left(\left(\sum_{\alpha\in\mathrm{Comp}} S\left(F_\alpha\right) \left(R_\alpha, f_{(1)}\right) \right) \mathbin{\Phi} a \right) f_{(2)} \\
&= \sum_{(f)} \left(S \left(\underbrace{\sum_{\alpha\in\mathrm{Comp}} F_\alpha \left(R_\alpha, f_{(1)}\right)}_{\substack{=f_{(1)} \\ (\text{by } (2.45), \text{ applied to } g=f_{(1)})}} \right) \mathbin{\Phi} a \right) f_{(2)} = \sum_{(f)} \left(S\left(f_{(1)}\right) \mathbin{\Phi} a\right) f_{(2)} = a \prec f
\end{aligned}$$

(by Theorem 2.3.7, applied to $b = f$). This proves Lemma 2.5.5. □

Proof of Theorem 2.5.4. We have

$$\begin{aligned}
&\sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} \underbrace{F_{\alpha\odot(m)}}_{\substack{=F_\alpha \mathbin{\Phi} h_m \\ (\text{by } (2.17))}} R^\perp_{\omega(\alpha)} f \\
&= \sum_{\alpha\in\mathrm{Comp}} (-1)^{|\alpha|} (F_\alpha \mathbin{\Phi} h_m) R^\perp_{\omega(\alpha)} f = h_m \prec f
\end{aligned}$$

(by Lemma 2.5.5, applied to $a = h_m$). This proves Theorem 2.5.4. $\square$

As a consequence, we obtain the following result, conjectured by Mike Zabrocki (private correspondence):

Corollary 2.5.6. For every positive integer m, define a $\mathbf{k}$-linear operator $\mathbf{W}_m$: QSym → QSym by

$$\mathbf{W}_m = \sum_{\alpha \in \text{Comp}} (-1)^{|\alpha|} F_{\alpha \odot (m)} R^{\perp}_{\omega(\alpha)}$$

(where $F_{\alpha \odot (m)}$ means left multiplication by $F_{\alpha \odot (m)}$). Then, every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ satisfies

$$\mathfrak{S}^*_\alpha = (\mathbf{W}_{\alpha_1} \circ \mathbf{W}_{\alpha_2} \circ \cdots \circ \mathbf{W}_{\alpha_\ell})(1).$$

Proof of Corollary 2.5.6. For every positive integer m and every $f \in$ QSym, we have

$$\mathbf{W}_m f = \sum_{\alpha \in \text{Comp}} (-1)^{|\alpha|} F_{\alpha \odot (m)} R^{\perp}_{\omega(\alpha)} f = h_m \prec f \qquad \text{(by Theorem 2.5.4)}.$$

Hence, by induction, for every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we have

$$\mathbf{W}_{\alpha_1}(\mathbf{W}_{\alpha_2}(\cdots(\mathbf{W}_{\alpha_\ell}(1))\cdots)) = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_\ell} \prec 1)\cdots)) = \mathfrak{S}^*_\alpha$$

(by Corollary 2.4.7). In other words,

$$\mathfrak{S}^*_\alpha = \mathbf{W}_{\alpha_1}(\mathbf{W}_{\alpha_2}(\cdots(\mathbf{W}_{\alpha_\ell}(1))\cdots)) = (\mathbf{W}_{\alpha_1} \circ \mathbf{W}_{\alpha_2} \circ \cdots \circ \mathbf{W}_{\alpha_\ell})(1).$$

This proves Corollary 2.5.6. $\square$

Let us finish this section with two curiosities: two analogues of Theorem 2.5.4, one of which can be viewed as an "$m = 0$ version" and the other as a "negative m version". We begin with the "$m = 0$ one", as it is the easier one to state:

Proposition 2.5.7. Let $f \in \operatorname{QSym}$. Then,

$$\varepsilon(f) = \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} f.$$

Proof of Proposition 2.5.7. Let us use Sweedler's notation. The map

$$\operatorname{Comp} \to \operatorname{Comp}, \ \alpha \mapsto \omega(\alpha)$$

is a bijection (since $\omega(\omega(\alpha)) = \alpha$ for any composition α). Hence, we can substitute $\omega(\alpha)$ for α in the sum $\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} f$. We thus obtain

$$\begin{aligned}
&\sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} f \\
&= \sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\omega(\alpha)|}}_{\substack{=(-1)^{|\alpha|} \\ (\text{since } |\omega(\alpha)|=|\alpha|)}} F_{\omega(\alpha)} \underbrace{R^\perp_{\omega(\omega(\alpha))}}_{\substack{=R^\perp_\alpha \\ (\text{since } \omega(\omega(\alpha))=\alpha)}} f = \sum_{\alpha \in \operatorname{Comp}} \underbrace{(-1)^{|\alpha|} F_{\omega(\alpha)}}_{\substack{=S(F_\alpha) \\ (\text{by Proposition 2.5.3})}} \underbrace{R^\perp_\alpha f}_{\substack{=\sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} \\ (\text{by (2.28)})}} \\
&= \sum_{\alpha \in \operatorname{Comp}} S(F_\alpha) \sum_{(f)} \left(R_\alpha, f_{(1)}\right) f_{(2)} = \sum_{\alpha \in \operatorname{Comp}} \sum_{(f)} S(F_\alpha) \left(R_\alpha, f_{(1)}\right) f_{(2)} \\
&= \sum_{(f)} \left(\sum_{\alpha \in \operatorname{Comp}} S(F_\alpha) \left(R_\alpha, f_{(1)}\right) \right) f_{(2)} = \sum_{(f)} S \left(\underbrace{\sum_{\alpha \in \operatorname{Comp}} F_\alpha \left(R_\alpha, f_{(1)}\right)}_{\substack{=f_{(1)} \\ (\text{by (2.45), applied to } g=f_{(1)})}} \right) f_{(2)} \\
&= \sum_{(f)} S\left(f_{(1)}\right) f_{(2)} = \varepsilon(f)
\end{aligned}$$

(by one of the defining properties of the antipode). This proves Proposition 2.5.7. □

The "negative m" analogue is less obvious:[33]

[33] Proposition 2.5.8 does not literally involve a negative m, but it involves an element $F_\alpha^{\backslash m}$ which can be viewed as "something like $F_{(\alpha)\odot(-m)}$".

Proposition 2.5.8. Let $f \in \operatorname{QSym}$ and let m be a positive integer. For any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we define an element $F_\alpha^{\backslash m}$ of QSym as follows:

- If $\ell = 0$ or $\alpha_\ell < m$, then $F_\alpha^{\backslash m} = 0$.
- If $\alpha_\ell = m$, then $F_\alpha^{\backslash m} = F_{(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1})}$.
- If $\alpha_\ell > m$, then $F_\alpha^{\backslash m} = F_{(\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell - m)}$.

(Here, any equality or inequality in which α_ℓ is mentioned is understood to include the statement that $\ell > 0$.)

Then,

$$(-1)^m \sum_{\alpha \in \operatorname{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R_{\omega(\alpha)}^\perp f = \varepsilon \left(R_{(1^m)}^\perp f \right).$$

Here, (1^m) denotes the composition $\left(\underbrace{1, 1, \ldots, 1}_{m \text{ times}} \right)$.

Proof of Proposition 2.5.8. Let us first make some auxiliary observations.

Any two elements a and b of NSym satisfy

$$(ab)^\perp = b^\perp \circ a^\perp \tag{2.46}$$

[34].

For every two compositions α and β, we define a composition $[\alpha, \beta]$ by $[\alpha, \beta] =$

[34] *Proof of (2.46):* Let a and b be two elements of NSym. Let $c \in \operatorname{QSym}$. Then,

$$(ab)^\perp c = \sum_{(c)} \underbrace{\left(ab, c_{(1)}\right)}_{\substack{= \sum_{\left(c_{(1)}\right)} \left(a, \left(c_{(1)}\right)_{(1)}\right) \left(b, \left(c_{(1)}\right)_{(2)}\right) \\ \text{(by (2.26), applied to } c_{(1)} \text{ instead of } c)}} c_{(2)} \qquad \text{(by (2.28), applied to } g = ab \text{ and } f = c)$$

$$= \sum_{(c)} \sum_{\left(c_{(1)}\right)} \left(a, \left(c_{(1)}\right)_{(1)}\right) \left(b, \left(c_{(1)}\right)_{(2)}\right) c_{(2)} = \sum_{(c)} \sum_{\left(c_{(2)}\right)} \left(a, c_{(1)}\right) \left(b, \left(c_{(2)}\right)_{(1)}\right) \left(c_{(2)}\right)_{(2)}$$

$$\left(\begin{array}{c} \text{since the coassociativity of } \Delta \text{ yields} \\ \sum_{(c)} \sum_{\left(c_{(1)}\right)} \left(c_{(1)}\right)_{(1)} \otimes \left(c_{(1)}\right)_{(2)} \otimes c_{(2)} = \sum_{(c)} \sum_{\left(c_{(2)}\right)} c_{(1)} \otimes \left(c_{(2)}\right)_{(1)} \otimes \left(c_{(2)}\right)_{(2)} \end{array} \right)$$

$$= \sum_{(c)} \left(a, c_{(1)}\right) \sum_{\left(c_{(2)}\right)} \left(b, \left(c_{(2)}\right)_{(1)}\right) \left(c_{(2)}\right)_{(2)}.$$

$(\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m)$, where α and β are written as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$. We further define a composition $\alpha \odot \beta$ as in Proposition 2.3.8. Then, every two nonempty compositions α and β satisfy

$$R_\alpha R_\beta = R_{[\alpha,\beta]} + R_{\alpha\odot\beta}. \tag{2.47}$$

(This is part of [GriRei15, Theorem 5.42(c)].) Now it is easy to see that

$$R_{\omega([\alpha,(m)])} + R_{\omega(\alpha\odot(m))} = R_{(1^m)} R_{\omega(\alpha)} \tag{2.48}$$

for every nonempty composition α [35]. Hence, for every nonempty composition α,

---

Compared with

$$\left(b^\perp \circ a^\perp\right)(c) = b^\perp \left(\underbrace{a^\perp c}_{\substack{=\sum_{(c)} \left(a, c_{(1)}\right) c_{(2)} \\ \text{(by (2.28), applied to } g=a \text{ and } f=c)}} \right) = b^\perp \left(\sum_{(c)} \left(a, c_{(1)}\right) c_{(2)} \right)$$
$$= \sum_{(c)} \left(a, c_{(1)}\right) \underbrace{b^\perp \left(c_{(2)}\right)}_{\substack{= \sum_{\left(c_{(2)}\right)} \left(b, \left(c_{(2)}\right)_{(1)}\right) \left(c_{(2)}\right)_{(2)} \\ \text{(by (2.28), applied to } g=b \text{ and } f=c_{(2)})}} \qquad \left(\text{since the map } b^\perp \text{ is } \mathbf{k}\text{-linear}\right)$$
$$= \sum_{(c)} \left(a, c_{(1)}\right) \sum_{\left(c_{(2)}\right)} \left(b, \left(c_{(2)}\right)_{(1)}\right) \left(c_{(2)}\right)_{(2)},$$

this yields $(ab)^\perp c = \left(b^\perp \circ a^\perp\right)(c)$.

Now, let us forget that we fixed c. We thus have shown that $(ab)^\perp c = \left(b^\perp \circ a^\perp\right)(c)$ for every $c \in \operatorname{QSym}$. In other words, $(ab)^\perp = b^\perp \circ a^\perp$. This proves (2.46).

[35] *Proof of (2.48):* Let α be a nonempty composition. Proposition 2.5.2 **(a)** shows that $\omega([\alpha, \beta]) = \omega(\beta) \odot \omega(\alpha)$ for every nonempty composition β. Applying this to $\beta = (m)$, we obtain $\omega([\alpha,(m)]) = \underbrace{\omega((m))}_{=(1^m)} \odot \omega(\alpha) = (1^m) \odot \omega(\alpha)$. But Proposition 2.5.2**(b)** shows that $\omega(\alpha \odot \beta) = [\omega(\beta), \omega(\alpha)]$ for every nonempty composition β. Applying this to $\beta = (m)$, we obtain $\omega(\alpha \odot (m)) = \left[\underbrace{\omega((m))}_{=(1^m)}, \omega(\alpha)\right] = [(1^m), \omega(\alpha)]$. Now,

$$R_{\omega([\alpha,(m)])} + R_{\omega(\alpha\odot(m))} = R_{\omega(\alpha\odot(m))} + R_{\omega([\alpha,(m)])} = R_{[(1^m),\omega(\alpha)]} + R_{(1^m)\odot\omega(\alpha)}$$
$$\left(\text{since } \omega(\alpha \odot (m)) = [(1^m), \omega(\alpha)] \text{ and } \omega([\alpha,(m)]) = (1^m) \odot \omega(\alpha)\right)$$
$$= R_{(1^m)} R_{\omega(\alpha)}$$

(since (2.47) (applied to (1^m) and $\omega(\alpha)$ instead of α and β) shows that $R_{(1^m)} R_{\omega(\alpha)} = R_{[(1^m),\omega(\alpha)]} +$

we have

$$\left(\underbrace{R_{\omega([\alpha,(m)])}+R_{\omega(\alpha\odot(m))}}_{=R_{(1^m)}R_{\omega(\alpha)}}\right)^{\perp}=\left(R_{(1^m)}R_{\omega(\alpha)}\right)^{\perp}=R_{\omega(\alpha)}^{\perp}\circ R_{(1^m)}^{\perp} \tag{2.49}$$

(by (2.46), applied to $a=R_{(1^m)}$ and $b=R_{\omega(\alpha)}$).

We furthermore notice that $\omega(\varnothing)=\varnothing$ and thus $R_{\omega(\varnothing)}^{\perp}=R_{\varnothing}^{\perp}=\operatorname{id}$ (since $R_{\varnothing}=1$).

Now,

$$\begin{aligned}
&\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\text{Comp};\\ \alpha_\ell=m}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}}_{\substack{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)|}\\ \text{(since } \alpha_\ell=m)}} \underbrace{F^{\backslash m}_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}}_{\substack{=F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})}\\ \text{(since } \alpha_\ell=m)}} \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}}_{\substack{=R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m))}\\ \text{(since } \alpha_\ell=m)}} f \\
&= \underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\text{Comp};\\ \alpha_\ell=m}}}_{=\sum_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})\in\text{Comp}}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)|}}_{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})|+m}} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})} \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m))}}_{\substack{=R^{\perp}_{\omega([(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])}\\ \text{(since } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},m)\\ =[(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])}} f \\
&= \sum_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})\in\text{Comp}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})|+m} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})} R^{\perp}_{\omega([(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}),(m)])} f \\
&= \sum_{\alpha\in\text{Comp}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega([\alpha,(m)])} f \\
&\qquad \left(\text{here, we have substituted } \alpha \text{ for } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1}) \text{ in the sum}\right) \\
&= (-1)^{|\varnothing|+m} F_\varnothing R^{\perp}_{\omega([\varnothing,(m)])} f + \sum_{\substack{\alpha\in\text{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega([\alpha,(m)])} f
\end{aligned} \tag{2.50}$$

$R_{(1^m)\odot\omega(\alpha)})$. This proves (2.48).

(here, we have split off the addend for $\alpha = \varnothing$ from the sum). On the other hand,

$$\begin{aligned}
&\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \alpha_\ell>m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} \underbrace{F^{\backslash m}_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}}_{\substack{=F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m)}\\ (\text{since } \alpha_\ell>m)}} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))} f \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \alpha_\ell>m}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|} F_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m)} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))} f \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \ell>0}} \underbrace{(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m)|}}_{=(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|+m}} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} \\
&\qquad \underbrace{R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m))}}_{\substack{=R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))}\\ (\text{since } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+m)=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))}} f \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } (\alpha_1,\alpha_2,\ldots,\alpha_\ell) \\ \text{for } (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell-m) \text{ in the sum}\end{array}\right) \\
&= \sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \ell>0}} (-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|+m} F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} R^{\perp}_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(m))} f \\
&= \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^{\perp}_{\omega(\alpha\odot(m))} f \qquad (2.51)
\end{aligned}$$

(here, we have substituted α for $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ in the sum).

But

$$\sum_{\alpha\in\operatorname{Comp}}(-1)^{|\alpha|}F_{\alpha}^{\backslash m}R_{\omega(\alpha)}^{\perp}f$$
$$=\sum_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp}}(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m}R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp}f$$
$$\left(\text{here, we have renamed the summation index } \alpha \text{ as } (\alpha_1,\alpha_2,\ldots,\alpha_\ell)\right)$$
$$=\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \ell=0\text{ or }\alpha_\ell<m}}(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}\underbrace{F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m}}_{\substack{=0\\ (\text{since }\ell=0\text{ or }\alpha_\ell<m)}}R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp}f$$
$$+\underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \alpha_\ell=m}}(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m}R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp}f}_{\substack{=(-1)^{|\varnothing|+m}F_{\varnothing}R_{\omega([\varnothing,(m)])}^{\perp}f+\sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha\text{ is nonempty}}}(-1)^{|\alpha|+m}F_{\alpha}R_{\omega([\alpha,(m)])}^{\perp}f\\ (\text{by (2.50)})}}$$
$$+\underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \alpha_\ell>m}}(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}F_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}^{\backslash m}R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp}f}_{\substack{=\sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha\text{ is nonempty}}}(-1)^{|\alpha|+m}F_{\alpha}R_{\omega(\alpha\odot(m))}^{\perp}f\\ (\text{by (2.51)})}}$$
$$=\underbrace{\sum_{\substack{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\in\operatorname{Comp};\\ \ell=0\text{ or }\alpha_\ell<m}}(-1)^{|(\alpha_1,\alpha_2,\ldots,\alpha_\ell)|}0R_{\omega((\alpha_1,\alpha_2,\ldots,\alpha_\ell))}^{\perp}f}_{=0}$$
$$+(-1)^{|\varnothing|+m}F_{\varnothing}R_{\omega([\varnothing,(m)])}^{\perp}f+\sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha\text{ is nonempty}}}(-1)^{|\alpha|+m}F_{\alpha}R_{\omega([\alpha,(m)])}^{\perp}f$$
$$+\sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha\text{ is nonempty}}}(-1)^{|\alpha|+m}F_{\alpha}R_{\omega(\alpha\odot(m))}^{\perp}f$$

$$= (-1)^{|\varnothing|+m} F_\varnothing R^\perp_{\omega([\varnothing,(m)])} f + \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega([\alpha,(m)])} f$$

$$+ \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega(\alpha\odot(m))} f$$

$$= (-1)^{|\varnothing|+m} F_\varnothing \underbrace{R^\perp_{\omega([\varnothing,(m)])}}_{\substack{=R^\perp_{(1^m)} \\ (\text{since } \omega([\varnothing,(m)])=\omega((m))=(1^m))}} f$$

$$+ \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha \underbrace{\left(R_{\omega([\alpha,(m)])} + R_{\omega(\alpha\odot(m))}\right)^\perp}_{\substack{=R^\perp_{\omega(\alpha)}\circ R^\perp_{(1^m)} \\ (\text{by } (2.49))}} f$$

$$= (-1)^{|\varnothing|+m} F_\varnothing \underbrace{R^\perp_{(1^m)} f}_{\substack{=R^\perp_{\omega(\varnothing)}\left(R^\perp_{(1^m)} f\right) \\ (\text{since } R^\perp_{\omega(\varnothing)}=\operatorname{id} \text{ and thus} \\ R^\perp_{\omega(\varnothing)}\left(R^\perp_{(1^m)} f\right)=R^\perp_{(1^m)} f)}} + \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha \underbrace{\left(R^\perp_{\omega(\alpha)} \circ R^\perp_{(1^m)}\right) f}_{=R^\perp_{\omega(\alpha)}\left(R^\perp_{(1^m)} f\right)}$$

$$= (-1)^{|\varnothing|+m} F_\varnothing R^\perp_{\omega(\varnothing)} \left(R^\perp_{(1^m)} f\right) + \sum_{\substack{\alpha\in\operatorname{Comp};\\ \alpha \text{ is nonempty}}} (-1)^{|\alpha|+m} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right)$$

$$= \sum_{\alpha\in\operatorname{Comp}} \underbrace{(-1)^{|\alpha|+m}}_{=(-1)^m(-1)^{|\alpha|}} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right) \qquad \left(\begin{array}{c}\text{here, we have incorporated the} \\ \alpha=\varnothing \text{ addend into the sum}\end{array}\right)$$

$$= (-1)^m \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right).$$

Multiplying both sides of this equality with $(-1)^m$, we obtain

$$(-1)^m \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R^\perp_{\omega(\alpha)} f = \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right).$$

Comparing this with

$$\varepsilon\left(R^\perp_{(1^m)} f\right) = \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha R^\perp_{\omega(\alpha)} \left(R^\perp_{(1^m)} f\right)$$

(by Proposition 2.5.7, applied to $R^\perp_{(1^m)} f$ instead of f), we obtain

$$(-1)^m \sum_{\alpha\in\operatorname{Comp}} (-1)^{|\alpha|} F_\alpha^{\backslash m} R^\perp_{\omega(\alpha)} f = \varepsilon\left(R^\perp_{(1^m)} f\right).$$

This proves Proposition 2.5.8. □

2.6 Lifts to WQSym and FQSym

We have so far been studying the Hopf algebras Sym, QSym and NSym. These are merely the tip of an iceberg; dozens of combinatorial Hopf algebras are currently known, many of which are extensions of these. In this final section, we shall discuss how (and whether) our operations $\prec$ and Φ as well as some similar operations can be lifted to the bigger Hopf algebras WQSym and FQSym. We shall give no proofs, as these are not difficult and the whole discussion is tangential to this note.

Let us first define these two Hopf algebras (which are discussed, for example, in [FoiMal14]).

We start with WQSym. (Our definition of WQSym follows the papers of the Marne-la-Vallée school, such as [AFNT13, Section 5.1][36]; it will differ from that in [FoiMal14], but we will explain why it is equivalent.)

Let $X_1, X_2, X_3, \ldots$ be countably many distinct symbols. These symbols will be called *letters*. We define a *word* to be an ℓ-tuple of elements of $\{X_1, X_2, X_3, \ldots\}$ for some $\ell \in \mathbb{N}$. Thus, for example, (X_3, X_5, X_2) and (X_6) are words. We denote the empty word $()$ by 1, and we often identify the one-letter word (X_i) with the symbol X_i for every $i > 0$. For any two words $u = (X_{i_1}, X_{i_2}, \ldots, X_{i_n})$ and $v = (X_{j_1}, X_{j_2}, \ldots, X_{j_m})$, we define the concatenation uv as the word $(X_{i_1}, X_{i_2}, \ldots, X_{i_n}, X_{j_1}, X_{j_2}, \ldots, X_{j_m})$. Concatenation is an associative operation and the empty word 1 is a neutral element for it; thus, the words form a monoid. We let Wrd denote this monoid. This monoid is the free monoid on the set $\{X_1, X_2, X_3, \ldots\}$. Concatenation allows us to rewrite any word $(X_{i_1}, X_{i_2}, \ldots, X_{i_n})$ in the shorter form $X_{i_1} X_{i_2} \cdots X_{i_n}$.

Notice that Mon (the set of all monomials) is also a monoid under multiplication. We can thus define a monoid homomorphism $\pi : \mathrm{Wrd} \to \mathrm{Mon}$ by $\pi(X_i) = x_i$ for all $i \in \{1, 2, 3, \ldots\}$. This homomorphism π is surjective.

[36] where WQSym is denoted by **WQSym**

We define $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ to be the $\mathbf{k}$-module $\mathbf{k}^{\mathrm{Wrd}}$; its elements are all families $(\lambda_w)_{w\in\mathrm{Wrd}} \in \mathbf{k}^{\mathrm{Wrd}}$. We define a multiplication on $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ by

$$(\lambda_w)_{w\in\mathrm{Wrd}} \cdot (\mu_w)_{w\in\mathrm{Wrd}} = \left(\sum_{(u,v)\in\mathrm{Wrd}^2;\ uv=w} \lambda_u \mu_v\right)_{w\in\mathrm{Wrd}}. \qquad (2.52)$$

This makes $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ into a $\mathbf{k}$-algebra, with unity $(\delta_{w,1})_{w\in\mathrm{Wrd}}$. This $\mathbf{k}$-algebra is called the $\mathbf{k}$-*algebra of noncommutative power series in* $X_1, X_2, X_3, \ldots$. For every $u \in \mathrm{Wrd}$, we identify the word u with the element $(\delta_{w,u})_{w\in\mathrm{Wrd}}$ of $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ [37]. The $\mathbf{k}$-algebra $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ becomes a topological $\mathbf{k}$-algebra via the product topology (recalling that $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle = \mathbf{k}^{\mathrm{Wrd}}$ as sets). Thus, every element $(\lambda_w)_{w\in\mathrm{Wrd}}$ of $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ can be rewritten in the form $\sum_{w\in\mathrm{Wrd}} \lambda_w w$. This turns the equality (2.52) into a distributive law (for infinite sums), and explains why we refer to elements of $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ as "noncommutative power series". We think of words as noncommutative analogues of monomials.

The *degree* of a word w will mean its length (i.e., the integer n for which w is an n-tuple). Let $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle_{\mathrm{bdd}}$ denote the $\mathbf{k}$-subalgebra of $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ formed by the *bounded-degree noncommutative power series*[38] in $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$. The surjective monoid homomorphism $\pi : \mathrm{Wrd} \to \mathrm{Mon}$ canonically gives rise to surjective $\mathbf{k}$-algebra homomorphisms $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle_{\mathrm{bdd}} \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]_{\mathrm{bdd}}$, which we also denote by π. Notice that the $\mathbf{k}$-algebra $\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle_{\mathrm{bdd}}$ is denoted $R\langle\mathbf{X}\rangle$ in [GriRei15, Section 8.1].

If w is a word, then we denote by $\operatorname{Supp} w$ the subset

$$\{i \in \{1, 2, 3, \ldots\} \mid \text{the symbol } X_i \text{ is an entry of } w\}$$

of $\{1, 2, 3, \ldots\}$. Notice that $\operatorname{Supp} w = \operatorname{Supp}(\pi(w))$ is a finite set.

A word w is said to be *packed* if there exists an $\ell \in \mathbb{N}$ such that $\operatorname{Supp} w = \{1, 2, \ldots, \ell\}$.

[37] This identification is harmless, since the map $\mathrm{Wrd} \to \mathbf{k}\langle\langle\mathbf{X}\rangle\rangle,\ u \mapsto (\delta_{w,u})_{w\in\mathrm{Wrd}}$ is a monoid homomorphism from Wrd to $(\mathbf{k}\langle\langle\mathbf{X}\rangle\rangle, \cdot)$. (However, it fails to be injective if $\mathbf{k} = 0$.)

[38] A noncommutative power series $(\lambda_w)_{w\in\mathrm{Wrd}} \in \mathbf{k}\langle\langle\mathbf{X}\rangle\rangle$ is said to be *bounded-degree* if there is an $N \in \mathbb{N}$ such that every word w of length $> N$ satisfies $\lambda_w = 0$.

For each word w, we define a packed word $\operatorname{pack} w$ as follows: Replace the smallest letter[39] that appears in w by X_1, the second-smallest letter by X_2, etc..[40] This word $\operatorname{pack} w$ is called the *packing* of w. For example, $\operatorname{pack}(X_3X_1X_6X_1) = X_2X_1X_3X_1$.

For every packed word u, we define an element $\mathbf{M}_u$ of $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle_{\mathrm{bdd}}$ by

$$\mathbf{M}_u = \sum_{\substack{w \in \mathrm{Wrd}; \\ \operatorname{pack} w = u}} w.$$

(This element $\mathbf{M}_u$ is denoted P_u in [AFNT13, Section 5.1].) We denote by WQSym the $\mathbf{k}$-submodule of $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle_{\mathrm{bdd}}$ spanned by the $\mathbf{M}_u$ for all packed words u. It is known that WQSym is a $\mathbf{k}$-subalgebra of $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle_{\mathrm{bdd}}$ which can furthermore be endowed with a Hopf algebra structure (the so-called *Hopf algebra of word quasisymmetric functions*) such that π restricts to a Hopf algebra surjection WQSym $\to$ QSym. Notice that $\pi(\mathbf{M}_u) = M_{\operatorname{Parikh}(\pi(u))}$ for every packed word u, where the Parikh composition $\operatorname{Parikh} \mathfrak{m}$ of any monomial $\mathfrak{m}$ is defined as in the proof of Proposition 2.3.5.

The elements $\mathbf{M}_u$ with u ranging over all packed words form a basis of the $\mathbf{k}$-module WQSym, which is usually called the *monomial basis*[41]. Furthermore, the product of two such elements can be computed by the well-known formula[42]

$$\mathbf{M}_u\mathbf{M}_v = \sum_{\substack{w \text{ is a packed word}; \\ \operatorname{pack}(w[:\ell]) = u;\ \operatorname{pack}(w[\ell:]) = v}} \mathbf{M}_w, \tag{2.53}$$

where ℓ is the length of u, and where we use the notation $w[:\ell]$ for the word formed by the first ℓ letters of w and we use the notation $w[\ell:]$ for the word formed by the remaining letters of w. This equality (which should be considered a noncommutative analogue of (2.7), and can be proven similarly) makes it possible to give

[39]We use the total ordering on the set $\{X_1, X_2, X_3, \ldots\}$ given by $X_1 < X_2 < X_3 < \cdots$.

[40]Here is a more pedantic way to restate this definition: Write w as $(X_{i_1}, X_{i_2}, \ldots, X_{i_\ell})$, and let $I = \operatorname{Supp} w$ (so that $I = \{i_1, i_2, \ldots, i_\ell\}$). Let r_I be the unique increasing bijection $\{1, 2, \ldots, |I|\} \to I$. Then, $\operatorname{pack} w$ denotes the word $\left(X_{r_I^{-1}(i_1)}, X_{r_I^{-1}(i_2)}, \ldots, X_{r_I^{-1}(i_\ell)}\right)$.

[41]Sometimes it is parametrized not by packed words but instead by set compositions (i.e., ordered set partitions) of sets of the form $\{1, 2, \ldots, n\}$ with $n \in \mathbb{N}$. But the packed words of length n are in a 1-to-1 correspondence with set compositions of $\{1, 2, \ldots, n\}$, so this is merely a matter of relabelling.

[42]This formula appears in [MeNoTh11, Proposition 4.1].

an alternative definition of WQSym, by defining WQSym as the free $\mathbf{k}$-module with basis $(\mathbf{M}_u)_{u \text{ is a packed word}}$ and defining multiplication using (2.53). This is precisely the approach taken in [FoiMal14, Section 1.1].

The Hopf algebra WQSym has also appeared under the name NCQSym ("quasisymmetric functions in noncommuting variables") in [BerZab05, Section 5.2] and other sources.

We now define five binary operations $\prec$, $\circ$, $\succ$, Φ, and $\between$ on $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$.

Definition 2.6.1. **(a)** We define a binary operation $\prec : \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ and that it satisfy

$$u \prec v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) < \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \geq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(b) We define a binary operation $\circ : \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ and that it satisfy

$$u \circ v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) = \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \neq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(c) We define a binary operation $\succ : \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ (written in infix notation) by the requirements that it be $\mathbf{k}$-bilinear and continuous with respect to the topology on $\mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ and that it satisfy

$$u \succ v = \begin{cases} uv, & \text{if } \min(\operatorname{Supp} u) > \min(\operatorname{Supp} v); \\ 0, & \text{if } \min(\operatorname{Supp} u) \leq \min(\operatorname{Supp} v) \end{cases}$$

for any two words u and v.

(d) We define a binary operation $\Phi : \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \times \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle \to \mathbf{k}\langle\langle \mathbf{X}\rangle\rangle$ (written in infix notation) by the requirements that it be **k**-bilinear and continuous with respect to the topology on $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ and that it satisfy

$$u \mathbin{\Phi} v = \begin{cases} uv, & \text{if } \max\left(\operatorname{Supp} u\right) \leq \min\left(\operatorname{Supp} v\right); \\ 0, & \text{if } \max\left(\operatorname{Supp} u\right) > \min\left(\operatorname{Supp} v\right) \end{cases}$$

for any two words u and v.

(e) We define a binary operation $\divideontimes : \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle \times \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle \to \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ (written in infix notation) by the requirements that it be **k**-bilinear and continuous with respect to the topology on $\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ and that it satisfy

$$u \divideontimes v = \begin{cases} uv, & \text{if } \max\left(\operatorname{Supp} u\right) < \min\left(\operatorname{Supp} v\right); \\ 0, & \text{if } \max\left(\operatorname{Supp} u\right) \geq \min\left(\operatorname{Supp} v\right) \end{cases}$$

for any two words u and v.

The first three of these five operations are closely related to those defined by Novelli and Thibon in [NovThi05a]; the main difference is the use of minima instead of maxima in our definitions.

The operations $\prec$, Φ and $\divideontimes$ on WQSym lift the operations $\prec$, Φ and $\divideontimes$ on QSym. More precisely, any $a \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ and $b \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ satisfy

$$\pi\left(a\right) \prec \pi\left(b\right) = \pi\left(a \prec b\right) = \pi\left(b \succ a\right);$$
$$\pi\left(a\right) \mathbin{\Phi} \pi\left(b\right) = \pi\left(a \mathbin{\Phi} b\right);$$
$$\pi\left(a\right) \divideontimes \pi\left(b\right) = \pi\left(a \divideontimes b\right)$$

(and similar formulas would hold for $\circ$ and $\succ$ had we bothered to define such operations on QSym). Also, using the operation $\succeq$ defined in Remark 2.3.2, we have

$$\pi\left(a\right) \succeq \pi\left(b\right) = \pi\left(a \succ b + a \circ b\right) \qquad \text{for any } a \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle \text{ and } b \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle.$$

We now have the following analogue of Proposition 2.3.5:

Proposition 2.6.2. Every $a \in \operatorname{WQSym}$ and $b \in \operatorname{WQSym}$ satisfy $a \prec b \in \operatorname{WQSym}$, $a \circ b \in \operatorname{WQSym}$, $a \succ b \in \operatorname{WQSym}$, $a \mathbin{\Phi} b \in \operatorname{WQSym}$ and $a \divideontimes b \in \operatorname{WQSym}$.

The proof of Proposition 2.6.2 is easier than that of Proposition 2.3.5; we omit it here. In analogy to Remark 2.3.6 and to (2.53), let us give explicit formulas for these five operations on the basis $(\mathbf{M}_u)_{u \text{ is a packed word}}$ of WQSym:

Remark 2.6.3. Let u and v be two packed words. Let ℓ be the length of u. Then:

(a) We have

$$\mathbf{M}_u \prec \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))<\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(b) We have

$$\mathbf{M}_u \circ \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))=\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(c) We have

$$\mathbf{M}_u \succ \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \min(\operatorname{Supp}(w[:\ell]))>\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

(d) We have

$$\mathbf{M}_u \mathbin{\Phi} \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \max(\operatorname{Supp}(w[:\ell]))\leq\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

The sum on the right hand side consists of two addends (unless u or v is empty), namely $\mathbf{M}_{uv^{+h-1}}$ and $\mathbf{M}_{uv^{+h}}$, where $h = \max(\operatorname{Supp} u)$, and where v^{+j} denotes the word obtained by replacing every letter X_k in v by X_{k+j}.

(e) We have

$$\mathbf{M}_u \ast \mathbf{M}_v = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{pack}(w[:\ell])=u;\ \operatorname{pack}(w[\ell:])=v; \\ \max(\operatorname{Supp}(w[:\ell]))<\min(\operatorname{Supp}(w[\ell:]))}} \mathbf{M}_w.$$

The sum on the right hand side consists of one addend only, namely $\mathbf{M}_{uv^{+h}}$.

Let us now move on to the combinatorial Hopf algebra FQSym, which is known as the *Malvenuto-Reutenauer Hopf algebra* or the *Hopf algebra of free quasi-symmetric functions*. We shall define it as a Hopf subalgebra of WQSym. This is not identical to the definition in [GriRei15, Section 8.1], but equivalent to it.

For every $n \in \mathbb{N}$, we let $\mathfrak{S}_n$ be the symmetric group on the set $\{1, 2, \ldots, n\}$. (This notation is identical with that in [GriRei15]. It has nothing to do with the $\mathfrak{S}_\alpha$ from [BBSSZ13a].) We let $\mathfrak{S}$ denote the disjoint union $\bigsqcup_{n\in\mathbb{N}} \mathfrak{S}_n$. We identify permutations in $\mathfrak{S}$ with certain words – namely, every permutation $\pi \in \mathfrak{S}$ is identified with the word $\left(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)}\right)$, where n is such that $\pi \in \mathfrak{S}_n$. The words thus identified with permutations in $\mathfrak{S}$ are precisely the packed words which do not have repeated elements.

For every word w, we define a word $\operatorname{std} w \in \mathfrak{S}$ as follows: Write w in the form $(X_{i_1}, X_{i_2}, \ldots, X_{i_n})$. Then, $\operatorname{std} w$ shall be the unique permutation $\pi \in \mathfrak{S}_n$ such that, whenever u and v are two elements of $\{1, 2, \ldots, n\}$ satisfying $u < v$, we have $(\pi(u) < \pi(v)$ if and only if $i_u \leq i_v)$. Equivalently (and less formally), $\operatorname{std} w$ is the word which is obtained by

- replacing the leftmost smallest letter of w by X_1, and marking it as "processed";
- then replacing the leftmost smallest letter of w that is not yet processed by X_2, and marking it as "processed";
- then replacing the leftmost smallest letter of w that is not yet processed by X_3, and marking it as "processed";
- etc., until all letters of w are processed.

For instance, $\operatorname{std}(X_3X_5X_2X_3X_2X_3) = X_3X_6X_1X_4X_2X_5$ (which, regarded as permutation, is the permutation written in one-line notation as $(3, 6, 1, 4, 2, 5)$).

We call $\operatorname{std} w$ the *standardization* of w.

Now, for every $\sigma \in \mathfrak{S}$, we define an element $\mathbf{G}_\sigma \in \mathrm{WQSym}$ by

$$\mathbf{G}_\sigma = \sum_{\substack{w \text{ is a packed word;} \\ \operatorname{std} w = \sigma}} \mathbf{M}_w = \sum_{\substack{w \in \mathrm{Wrd}; \\ \operatorname{std} w = \sigma}} w.$$

(The second equality sign can easily be checked.) Then, the $\mathbf{k}$-submodule of WQSym spanned by $(\mathbf{G}_\sigma)_{\sigma \in \mathfrak{S}}$ turns out to be a Hopf subalgebra, with basis $(\mathbf{G}_\sigma)_{\sigma \in \mathfrak{S}}$. This Hopf subalgebra is denoted by FQSym. This definition is not identical with the one given in [GriRei15, Section 8.1]; however, it gives an isomorphic Hopf algebra, as our $\mathbf{G}_\sigma$ correspond to the images of the G_σ introduced in [GriRei15, Section 8.1] under the embedding $\mathrm{FQSym} \to R\left\langle \{X_i\}_{i \in I} \right\rangle$ also defined therein.

Only two of the five operations $\prec$, $\circ$, $\succ$, Φ, and Ж defined in Definition 2.6.1 can be restricted to binary operations on FQSym:

Proposition 2.6.4. Every $a \in \mathrm{FQSym}$ and $b \in \mathrm{FQSym}$ satisfy $a \succ b \in \mathrm{FQSym}$ and $a \Phi b \in \mathrm{FQSym}$.

Moreover, we have the following explicit formulas on the basis $(\mathbf{G}_\sigma)_{\sigma \in \mathfrak{S}}$:

Remark 2.6.5. Let $\sigma \in \mathfrak{S}$ and $\tau \in \mathfrak{S}$. Let ℓ be the length of σ (so that $\sigma \in \mathfrak{S}_\ell$).

(a) We have

$$\mathbf{G}_\sigma \succ \mathbf{G}_\tau = \sum_{\substack{\pi \in \mathfrak{S}; \\ \operatorname{std}(\pi[:\ell]) = \sigma; \ \operatorname{std}(\pi[\ell:]) = \tau; \\ \min(\operatorname{Supp}(\pi[:\ell])) > \min(\operatorname{Supp}(\pi[\ell:]))}} \mathbf{G}_\pi.$$

(b) We have

$$\mathbf{G}_\sigma \Phi \mathbf{G}_\tau = \sum_{\substack{\pi \in \mathfrak{S}; \\ \operatorname{std}(\pi[:\ell]) = \sigma; \ \operatorname{std}(\pi[\ell:]) = \tau; \\ \max(\operatorname{Supp}(\pi[:\ell])) \leq \min(\operatorname{Supp}(\pi[\ell:]))}} \mathbf{G}_\pi.$$

$\blacksquare$ The sum on the right hand side consists of one addend only, namely $\mathbf{G}_{\sigma\tau^{+\ell}}$.

The statements of Remark 2.6.5 can be easily derived from Remark 2.6.3. The proof for **(a)** rests on the following simple observations:

- Every word w satisfies $\operatorname{std}(\operatorname{pack} w) = \operatorname{std} w$.
- Every $n \in \mathbb{N}$, every word w of length n and every $\ell \in \{0, 1, \ldots, n\}$ satisfy

  $$\operatorname{std}((\operatorname{std} w)[:\ell]) = \operatorname{std}(w[:\ell]) \qquad \text{and} \qquad \operatorname{std}((\operatorname{std} w)[\ell:]) = \operatorname{std}(w[\ell:]).$$

- Every $n \in \mathbb{N}$, every word w of length n and every $\ell \in \{0, 1, \ldots, n\}$ satisfy the equivalence

  $$\begin{aligned} &(\min(\operatorname{Supp}(w[:\ell])) > \min(\operatorname{Supp}(w[\ell:]))) \\ \iff &(\min(\operatorname{Supp}((\operatorname{std} w)[:\ell])) > \min(\operatorname{Supp}((\operatorname{std} w)[\ell:]))). \end{aligned}$$

The third of these three observations would fail if the greater sign were to be replaced by a smaller sign; this is essentially why $\operatorname{FQSym} \subseteq \operatorname{WQSym}$ is not closed under $\prec$.

The operation $\succ$ on FQSym defined above is closely related to the operation $\succ$ on FQSym introduced by Foissy in [Foissy07, Section 4.2]. Indeed, the latter differs from the former in the use of max instead of min.

2.7 Epilogue

We have introduced five binary operations $\prec$, $\circ$, $\succ$, Φ, and $\divideontimes$ on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and their restrictions to QSym; we have further introduced five analogous operations on $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$ and their restrictions to WQSym (as well as the restrictions of two of them to FQSym). We have used these operations (specifically, $\prec$ and Φ) to prove a formula (Corollary 2.5.6) for the dual immaculate functions $\mathfrak{S}^*_\alpha$. Along the way, we have found that the $\mathfrak{S}^*_\alpha$ can be obtained by repeated application of the operation $\prec$

(Corollary 2.4.7). A similar (but much more obvious) result can be obtained for the fundamental quasisymmetric functions: For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \in \operatorname{Comp}$, we have

$$F_\alpha = h_{\alpha_1} \mathbin{\divideontimes} h_{\alpha_2} \mathbin{\divideontimes} \cdots \mathbin{\divideontimes} h_{\alpha_\ell} \mathbin{\divideontimes} 1$$

(we do not use parentheses here, since $\divideontimes$ is associative). This shows that the $\mathbf{k}$-algebra $(\operatorname{QSym}, \divideontimes)$ is free. Moreover,

$$F_{\omega(\alpha)} = e_{\alpha_\ell} \mathbin{\Phi} e_{\alpha_{\ell-1}} \mathbin{\Phi} \cdots \mathbin{\Phi} e_{\alpha_1} \mathbin{\Phi} 1,$$

where e_m stands for the m-th elementary symmetric function; thus, the $\mathbf{k}$-algebra $(\operatorname{QSym}, \Phi)$ is also free.[43] (Incidentally, this shows that $S(a \divideontimes b) = S(b) \mathbin{\Phi} S(a)$ for any $a, b \in \operatorname{QSym}$. But this does not hold for $a, b \in \operatorname{WQSym}$.)

One might wonder what "functions" can be similarly constructed using the operations $\prec$, $\circ$, $\succ$, Φ, and $\divideontimes$ in WQSym, using the noncommutative analogues $H_m = \sum_{i_1 \leq i_2 \leq \cdots \leq i_m} X_{i_1} X_{i_2} \cdots X_{i_m} = \mathbf{G}_{(1,2,\ldots,m)}$ and $E_m = \sum_{i_1 > i_2 > \cdots > i_m} X_{i_1} X_{i_2} \cdots X_{i_m} = \mathbf{G}_{(m,m-1,\ldots,1)}$ of h_m and e_m. (These analogues actually live in NSym, where NSym is embedded into FQSym as in [GriRei15, Corollary 8.14]; but the operations do not preserve NSym, and only two of them preserve FQSym.) However, it seems somewhat tricky to ask the right questions here; for instance, the $\mathbf{k}$-linear span of the $\succ$-closure of $\{H_m \mid m \geq 0\}$ is not a $\mathbf{k}$-subalgebra of FQSym (since $H_2 H_1$ is not a $\mathbf{k}$-linear combination of H_3, $H_1 \succ (H_1 \succ H_1)$, $(H_1 \succ H_1) \succ H_1$, $H_1 \succ H_2$ and $H_2 \succ H_1$).

On the other hand, one might also try to write down the set of identities satisfied by the operations $\cdot$, $\prec$, $\circ$, $\succeq$, Φ and $\divideontimes$ on the various spaces ($\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, QSym, $\mathbf{k}\langle\langle \mathbf{X} \rangle\rangle$, WQSym and FQSym), or by subsets of these operations; these identities could then be used to define new operads, i.e., algebraic structures comprising a $\mathbf{k}$-module and some operations on it that imitate (some of) the operations $\cdot$, $\prec$, $\circ$, $\succeq$, Φ and $\divideontimes$. For instance, apart from being associative, the operations Φ and $\divideontimes$ on

[43] We owe these two observations to the referee.

$\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$ satisfy the identity

$$(a \mathbin{\Phi} b) \divideontimes c + (a \divideontimes b) \mathbin{\Phi} c = a \mathbin{\Phi} (b \divideontimes c) + a \divideontimes (b \mathbin{\Phi} c) \tag{2.54}$$

for all $a, b, c \in \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle$. This follows from the (easily verified) identities

$$\begin{aligned}(a \mathbin{\Phi} b) \divideontimes c - a \mathbin{\Phi} (b \divideontimes c) &= \varepsilon(b)(a \divideontimes c - a \mathbin{\Phi} c);\\ (a \divideontimes b) \mathbin{\Phi} c - a \divideontimes (b \mathbin{\Phi} c) &= \varepsilon(b)(a \mathbin{\Phi} c - a \divideontimes c),\end{aligned}$$

where $\varepsilon : \mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle \to \mathbf{k}$ is the map which sends every noncommutative power series to its constant term. The equality (2.54) (along with the associativity of $\mathbin{\Phi}$ and $\divideontimes$) makes $\left(\mathbf{k}\left\langle\left\langle \mathbf{X}\right\rangle\right\rangle, \mathbin{\Phi}, \divideontimes\right)$ into what is called an $As^{\langle 2\rangle}$-*algebra* (see [Zinbie10, p. 39]). Is QSym or WQSym a free $As^{\langle 2\rangle}$-algebra? What if we add the existence of a common neutral element for the operations $\mathbin{\Phi}$ and $\divideontimes$ to the axioms of this operad?

Chapter 3

The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality

Abstract

If H is a commutative connected graded Hopf algebra over a commutative ring $\mathbf{k}$, then a certain canonical $\mathbf{k}$-algebra homomorphism $H \to H \otimes \operatorname{QSym}_{\mathbf{k}}$ is defined, where $\operatorname{QSym}_{\mathbf{k}}$ denotes the Hopf algebra of quasisymmetric functions. This homomorphism generalizes the "internal comultiplication" on $\operatorname{QSym}_{\mathbf{k}}$, and extends what Hazewinkel (in §18.24 of his "Witt vectors") calls the Bernstein homomorphism.

We construct this homomorphism with the help of the universal property of $\operatorname{QSym}_{\mathbf{k}}$ as a combinatorial Hopf algebra (a well-known result by Aguiar, Bergeron and Sottile) and extension of scalars (the commutativity of H allows us to consider, for example, $H \otimes \operatorname{QSym}_{\mathbf{k}}$ as an H-Hopf algebra, and this change of viewpoint significantly extends the reach of the universal property).

***

One of the most important aspects of QSym (the Hopf algebra of quasisymmetric functions) is a universal property discovered by Aguiar, Bergeron and Sottile in 2003 [ABS03]; among other applications, it gives a unifying framework for various quasisymmetric and symmetric functions constructed from combinatorial objects (e.g., the chromatic symmetric function of a graph).

On the other hand, let $\Lambda_{\mathbf{k}}$ be the Hopf algebra of symmetric functions over a commutative ring $\mathbf{k}$. If H is any commutative cocommutative connected graded $\mathbf{k}$-Hopf algebra, then a certain $\mathbf{k}$-algebra homomorphism $H \to H \otimes \Lambda_{\mathbf{k}}$ (not a Hopf algebra homomorphism!) was defined by Joseph N. Bernstein, and used by Zelevinsky in [Zelevi81, §5.2] to classify PSH-algebras. In [Haz08, §18.24], Hazewinkel observed that this homomorphism generalizes the second comultiplication of $\Lambda_{\mathbf{k}}$, and asked for "more study" and a better understanding of this homomorphism.

In this note, I shall define an extended version of this homomorphism: a $\mathbf{k}$-algebra homomorphism $H \to H \otimes \mathrm{QSym}_{\mathbf{k}}$ for any commutative (but not necessarily cocommutative) connected graded $\mathbf{k}$-Hopf algebra H. This homomorphism, which I will call the *Bernstein homomorphism*, will generalize the second comultiplication of $\mathrm{QSym}_{\mathbf{k}}$, or rather its variant with the two tensorands flipped. When H is cocommutative, this homomorphism has its image contained in $H \otimes \Lambda_{\mathbf{k}}$ and thus becomes Bernstein's original homomorphism.

The Bernstein homomorphism $H \to H \otimes \mathrm{QSym}_{\mathbf{k}}$ is not fully new (although I have not seen it appear explicitly in the literature). Its dual version is a coalgebra homomorphism $H' \otimes \mathrm{NSym}_{\mathbf{k}} \to H'$, where H' is a cocommutative connected graded Hopf algebra; i.e., it is an action of $\mathrm{NSym}_{\mathbf{k}}$ on any such H'. This action is implicit in the work of Patras and Reutenauer on descent algebras, and a variant of it for Hopf monoids instead of Hopf algebras appears in [Aguiar13, Propositions 84 and 88, and especially the Remark after Proposition 88]. What I believe to be new in this note is the way I will construct the Bernstein homomorphism: as a consequence of the Aguiar-Bergeron-Sottile universal property of QSym, but applied not to the $\mathbf{k}$-Hopf algebra $\mathrm{QSym}_{\mathbf{k}}$ but to the H-Hopf algebra QSym_{H}. The commutativity of H is being used here to deploy H as the base ring.

Acknowledgments

Thanks to Marcelo Aguiar for enlightening discussions.

3.1 Definitions and conventions

For the rest of this note, we fix a commutative ring[1] $\mathbf{k}$. All tensor signs ($\otimes$) without a subscript will mean $\otimes_{\mathbf{k}}$. We shall use the notions of $\mathbf{k}$-algebras, $\mathbf{k}$-coalgebras and $\mathbf{k}$-Hopf algebras as defined (e.g.) in [GriRei15, Chapter 1]. We shall also use the notions of graded $\mathbf{k}$-algebras, graded $\mathbf{k}$-coalgebras and graded $\mathbf{k}$-Hopf algebras as defined in [GriRei15, Chapter 1]; in particular, we shall not use the topologists' sign conventions[2]. The comultiplication and the counit of a $\mathbf{k}$-coalgebra C will be denoted by Δ_C and ε_C, respectively; when the C is unambiguously clear from the context, we will omit it from the notation (so we will just write Δ and ε).

If V and W are two $\mathbf{k}$-modules, then we let $\tau_{V,W}$ be the $\mathbf{k}$-linear map $V \otimes W \to W \otimes V$, $v \otimes w \mapsto w \otimes v$. This $\mathbf{k}$-linear map $\tau_{V,W}$ is called the *twist map*, and is a $\mathbf{k}$-module isomorphism.

The next two definitions are taken from [GriRei15, §1.4][3]:

Definition 3.1.1. Let A be a $\mathbf{k}$-algebra. Let m_A denote the $\mathbf{k}$-linear map $A \otimes A \to A$, $a \otimes b \mapsto ab$. Let u_A denote the $\mathbf{k}$-linear map $\mathbf{k} \to A$, $\lambda \mapsto \lambda \cdot 1_A$. (The maps m_A and u_A are often denoted by m and u when A is unambiguously clear from the context.) For any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear map $m^{(k-1)} : A^{\otimes k} \to A$ recursively as follows: We set $m^{(-1)} = u_A$, $m^{(0)} = \mathrm{id}_A$ and

$$m^{(k)} = m \circ \left(\mathrm{id}_A \otimes m^{(k-1)}\right) \qquad \text{for every } k \geq 1.$$

The maps $m^{(k-1)} : A^{\otimes k} \to A$ are called the *iterated multiplication maps* of A.

Notice that for every $\mathbf{k} \in \mathbb{N}$, the map $m^{(k-1)}$ is the $\mathbf{k}$-linear map $A^{\otimes k} \to A$ which sends every $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A^{\otimes k}$ to $a_1 a_2 \cdots a_k$.

[1] The word "ring" always means "associative ring with 1" in this note. Furthermore, a $\mathbf{k}$-algebra (when $\mathbf{k}$ is a commutative ring) means a $\mathbf{k}$-module A equipped with a ring structure such that the multiplication map $A \times A \to A$ is $\mathbf{k}$-bilinear.

[2] Thus, the twist map $V \otimes V \to V \otimes V$ for a graded $\mathbf{k}$-module V sends $v \otimes w \mapsto w \otimes v$, even if v and w are homogeneous of odd degree.

[3] The objects we are defining are classical and standard; however, the notation we are using for them is not. For example, what we call $\Delta^{(k-1)}$ in Definition 3.1.2 is denoted by Δ_{k-1} in [Sweed69], and is called $\Delta^{(k)}$ in [Fresse14, §7.1].

Definition 3.1.2. Let C be a $\mathbf{k}$-coalgebra. For any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear map $\Delta^{(k-1)} : C \to C^{\otimes k}$ recursively as follows: We set $\Delta^{(-1)} = \varepsilon_C$, $\Delta^{(0)} = \mathrm{id}_C$ and

$$\Delta^{(k)} = \left(\mathrm{id}_C \otimes \Delta^{(k-1)}\right) \circ \Delta \qquad \text{for every } k \geq 1.$$

The maps $\Delta^{(k-1)} : C \to C^{\otimes k}$ are called the *iterated comultiplication maps* of C.

A *composition* shall mean a finite sequence of positive integers. The *size* of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is defined to be the nonnegative integer $\alpha_1 + \alpha_2 + \cdots + \alpha_k$, and is denoted by $|\alpha|$. Let Comp denote the set of all compositions.

Let $\mathbb{N}$ denote the set $\{0, 1, 2, \ldots\}$.

Definition 3.1.3. Let H be a graded $\mathbf{k}$-module. For every $n \in \mathbb{N}$, we let $\pi_n : H \to H$ be the canonical projection of H onto the n-th graded component H_n of H. We shall always regard π_n as a map from H to H, not as a map from H to H_n, even though its image is H_n.

For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, we let $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ be the tensor product $\pi_{a_1} \otimes \pi_{a_2} \otimes \cdots \otimes \pi_{a_k}$ of the canonical projections $\pi_{a_i} : H \to H$. Thus, the image of π_α can be identified with $H_{a_1} \otimes H_{a_2} \otimes \cdots \otimes H_{a_k}$.

Let $\mathrm{QSym}_{\mathbf{k}}$ denote the $\mathbf{k}$-Hopf algebra of quasisymmetric functions defined over $\mathbf{k}$. (This is defined and denoted by $\mathcal{QS}ym$ in [ABS03, §3]; it is also defined and denoted by QSym in [GriRei15, Chapter 5].) We shall follow the notations and conventions of [GriRei15, §5.1] as far as $\mathrm{QSym}_{\mathbf{k}}$ is concerned; in particular, we regard $\mathrm{QSym}_{\mathbf{k}}$ as a subring of the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$.

Let ε_P denote the $\mathbf{k}$-linear map $\mathrm{QSym}_{\mathbf{k}} \to \mathbf{k}$ sending every $f \in \mathrm{QSym}_{\mathbf{k}}$ to $f(1, 0, 0, 0, \ldots) \in \mathbf{k}$. (This map ε_P is denoted by $\zeta_{\mathcal{Q}}$ in [ABS03, §4] and by ζ_Q in [GriRei15, Example 7.2].) Notice that ε_P is a $\mathbf{k}$-algebra homomorphism.

Definition 3.1.4. For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we define a power series $M_\alpha \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

(where the sum is over all strictly increasing ℓ-tuples $(i_1 < i_2 < \cdots < i_\ell)$ of positive integers). It is well-known (and easy to check) that this M_α belongs to $\operatorname{QSym}_{\mathbf{k}}$. The power series M_α is called the *monomial quasisymmetric function* corresponding to α. The family $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module $\operatorname{QSym}_{\mathbf{k}}$; this is the so-called *monomial basis* of $\operatorname{QSym}_{\mathbf{k}}$. (See [ABS03, §3] and [GriRei15, §5.1] for more about this basis.)

It is well-known that every $(b_1, b_2, \ldots, b_\ell) \in \operatorname{Comp}$ satisfies

$$\Delta\left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \sum_{i=0}^{\ell} M_{(b_1, b_2, \ldots, b_i)} \otimes M_{(b_{i+1}, b_{i+2}, \ldots, b_\ell)} \tag{3.1}$$

and

$$\varepsilon\left(M_{(b_1, b_2, \ldots, b_\ell)}\right) = \begin{cases} 1, & \text{if } \ell = 0; \\ 0, & \text{if } \ell \neq 0 \end{cases}.$$

These two equalities can be used as a definition of the $\mathbf{k}$-coalgebra structure on $\operatorname{QSym}_{\mathbf{k}}$ (because $(M_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module $\operatorname{QSym}_{\mathbf{k}}$, and thus the $\mathbf{k}$-linear maps Δ and ε are uniquely determined by their values on the M_α).

3.2 The Aguiar-Bergeron-Sottile theorem

The cornerstone of the Aguiar-Bergeron-Sottile paper [ABS03] is the following result:

Theorem 3.2.1. Let $\mathbf{k}$ be a commutative ring. Let H be a connected graded $\mathbf{k}$-Hopf algebra. Let $\zeta : H \to \mathbf{k}$ be a $\mathbf{k}$-algebra homomorphism.

(a) Then, there exists a unique graded $\mathbf{k}$-coalgebra homomorphism $\Psi : H \to$

$\operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Psi} & \operatorname{QSym} \\ & {\scriptstyle\zeta}\searrow \quad \swarrow{\scriptstyle\varepsilon_P} & \\ & \mathbf{k} & \end{array}$$

is commutative.

(b) This unique $\mathbf{k}$-coalgebra homomorphism $\Psi : H \to \operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-Hopf algebra homomorphism.

(c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\zeta_\alpha : H \to \mathbf{k}$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\zeta^{\otimes k}} \mathbf{k}^{\otimes k} \xrightarrow{\cong} \mathbf{k} .$$

(Here, the map $\mathbf{k}^{\otimes k} \xrightarrow{\cong} \mathbf{k}$ is the canonical $\mathbf{k}$-algebra isomorphism from $\mathbf{k}^{\otimes k}$ to $\mathbf{k}$. Recall also that $\Delta^{(k-1)} : H \to H^{\otimes k}$ is the "iterated comultiplication map"; see [GriRei15, §1.4] for its definition. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 3.1.3.)

Then, the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 3.2.1 **(a)** is given by the formula

$$\Psi(h) = \sum_{\substack{\alpha \in \operatorname{Comp}; \\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha \qquad \text{whenever } n \in \mathbb{N} \text{ and } h \in H_n.$$

(Recall that H_n denotes the n-th graded component of H.)

(d) The unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 3.2.1 **(a)** is also given by

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \zeta_\alpha(h) \cdot M_\alpha \qquad \text{for every } h \in H$$

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

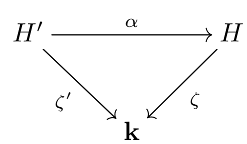
(e) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, the unique $\mathbf{k}$-

coalgebra homomorphism Ψ of Theorem 3.2.1 **(a)** satisfies $\Psi(H) \subseteq \Lambda_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$. (See [GriRei15, §2] for the definition of $\Lambda_{\mathbf{k}}$. We regard $\Lambda_{\mathbf{k}}$ as a $\mathbf{k}$-subalgebra of $\operatorname{QSym}_{\mathbf{k}}$ in the usual way.)

Parts **(a)**, **(b)** and **(c)** of Theorem 3.2.1 are proven in [ABS03, proof of Theorem 4.1] and [GriRei15, proof of Theorem 7.3] (although we are using different notations here[4], and avoiding the standing assumptions of [ABS03] which needlessly require $\mathbf{k}$ to be a field and H to be of finite type). Theorem 3.2.1 **(d)** easily follows from Theorem 3.2.1 **(c)**[5]. Theorem 3.2.1 **(e)** appears in [GriRei15, Remark 7.4] (and something very close is proven in [ABS03, Theorem 4.3]). For the sake of completeness, let me give some details on the proof of Theorem 3.2.1 **(e)**:

*Proof of Theorem 3.2.1 **(e)**.* Let $\varepsilon_p : \Lambda_{\mathbf{k}} \to \mathbf{k}$ be the restriction of the $\mathbf{k}$-algebra homomorphism $\varepsilon_P : \operatorname{QSym}_{\mathbf{k}} \to \mathbf{k}$ to $\Lambda_{\mathbf{k}}$. From [ABS03, Theorem 4.3], we know that

[4]The paper [ABS03] defines a *combinatorial coalgebra* to be a pair (H, ζ) consisting of a connected graded $\mathbf{k}$-coalgebra H (where "connected" means that $\varepsilon\mid_{H_0} : H_0 \to \mathbf{k}$ is a $\mathbf{k}$-module isomorphism) and a $\mathbf{k}$-linear map $\zeta : H \to \mathbf{k}$ satisfying $\zeta\mid_{H_0} = \varepsilon\mid_{H_0}$. Furthermore, it defines a *morphism* from a combinatorial coalgebra (H', ζ') to a combinatorial coalgebra (H, ζ) to be a homomorphism $\alpha : H' \to H$ of graded $\mathbf{k}$-coalgebras for which the diagram



is commutative. Theorem 3.2.1 **(a)** translates into this language as follows: There exists a unique morphism from the combinatorial coalgebra (H, ζ) to the combinatorial coalgebra $(\operatorname{QSym}_{\mathbf{k}}, \varepsilon_P)$. (Apart from this, [ABS03] is also using the notations $\Bbbk$, $\mathcal{H}$, $\mathcal{QSym}$ and $\zeta_{\mathcal{Q}}$ for what we call $\mathbf{k}$, H, $\operatorname{QSym}_{\mathbf{k}}$ and ε_P.)

[5]*Proof.* Let Ψ be the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 3.2.1 **(a)**. It is easy to see that every $n \in \mathbb{N}$, every composition α with $|\alpha| \neq n$ and every $h \in H_n$ satisfy $\zeta_\alpha(h) = 0$ (because $\pi_\alpha\left(\Delta^{(k-1)}\left(\underbrace{h}_{\in H_n}\right)\right) \in \pi_\alpha\left(\Delta^{(k-1)}(H_n)\right) = 0$ (for reasons of gradedness)). Hence, for every $n \in \mathbb{N}$ and every $h \in H_n$, we have

$$\sum_{\alpha \in \operatorname{Comp}} \zeta_\alpha(h) \cdot M_\alpha = \sum_{\substack{\alpha \in \operatorname{Comp};\\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha + \sum_{\substack{\alpha \in \operatorname{Comp};\\ |\alpha| \neq n}} \underbrace{\zeta_\alpha(h)}_{=0} \cdot M_\alpha$$
$$= \sum_{\substack{\alpha \in \operatorname{Comp};\\ |\alpha| = n}} \zeta_\alpha(h) \cdot M_\alpha = \Psi(h) \qquad \text{(by Theorem 3.2.1 } \textbf{(c)}\text{)}.$$

Both sides of this equality are $\mathbf{k}$-linear in h; thus, it also holds for every $h \in H$ (even if h is not homogeneous). This proves Theorem 3.2.1 **(d)**.

there exists a unique graded $\mathbf{k}$-coalgebra homomorphism $\Psi' : H \to \Lambda_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Psi'} & \Lambda_{\mathbf{k}} \\ & {\scriptstyle \zeta} \searrow \quad \swarrow {\scriptstyle \varepsilon_p} & \\ & \mathbf{k} & \end{array} \tag{3.2}$$

is commutative. Consider this Ψ'. Let $\iota : \Lambda_{\mathbf{k}} \to \operatorname{QSym}_{\mathbf{k}}$ be the canonical inclusion map; this is a $\mathbf{k}$-Hopf algebra homomorphism. Also, $\varepsilon_p = \varepsilon_P \circ \iota$ (by the definition of ε_p). The commutative diagram (3.2) yields $\zeta = \underbrace{\varepsilon_p}_{= \varepsilon_P \circ \iota} \circ \Psi' = \varepsilon_P \circ \iota \circ \Psi'$.

Now, consider the unique $\mathbf{k}$-coalgebra homomorphism Ψ of Theorem 3.2.1 **(a)**. Due to its uniqueness, it has the following property: If $\widetilde{\Psi}$ is any $\mathbf{k}$-coalgebra homomorphism $H \to \operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\widetilde{\Psi}} & \operatorname{QSym} \\ & {\scriptstyle \zeta} \searrow \quad \swarrow {\scriptstyle \varepsilon_P} & \\ & \mathbf{k} & \end{array} \tag{3.3}$$

is commutative, then $\widetilde{\Psi} = \Psi$. Applying this to $\widetilde{\Psi} = \iota \circ \Psi'$, we obtain $\iota \circ \Psi' = \Psi$ (since the diagram (3.3) is commutative for $\widetilde{\Psi} = \iota \circ \Psi'$ (because $\zeta = \varepsilon_P \circ \iota \circ \Psi'$)). Hence, $\underbrace{\Psi}_{= \iota \circ \Psi'} (H) = (\iota \circ \Psi') (H) = \iota \left(\underbrace{\Psi' (H)}_{\subseteq \Lambda_{\mathbf{k}}} \right) \subseteq \iota (\Lambda_{\mathbf{k}}) = \Lambda_{\mathbf{k}}$. This proves Theorem 3.2.1 **(e)**. □

Remark 3.2.2. Let $\mathbf{k}$, H and ζ be as in Theorem 3.2.1. Then, the $\mathbf{k}$-module $\operatorname{Hom}(H, \mathbf{k})$ of all $\mathbf{k}$-linear maps from H to $\mathbf{k}$ has a canonical structure of a $\mathbf{k}$-algebra; its unity is the map $\varepsilon \in \operatorname{Hom}(H, \mathbf{k})$, and its multiplication is the binary operation $\bigstar$ defined by

$$f \bigstar g = m_{\mathbf{k}} \circ (f \otimes g) \circ \Delta_H : H \to \mathbf{k} \qquad \text{for every } f, g \in \operatorname{Hom}(H, \mathbf{k})$$

(where $m_{\mathbf{k}}$ is the canonical isomorphism $\mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$). This $\mathbf{k}$-algebra is called the *convolution algebra* of H and $\mathbf{k}$; it is a particular case of the construction in [GriRei15, Definition 1.27]. Using this convolution algebra, we can express the map

ζ_α in Theorem 3.2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\zeta_\alpha : H \to \mathbf{k}$ is given by

$$\zeta_\alpha = (\zeta \circ \pi_{a_1}) \bigstar (\zeta \circ \pi_{a_2}) \bigstar \cdots \bigstar (\zeta \circ \pi_{a_k}) .$$

(This follows from [GriRei15, Exercise 1.43].)

3.3 Extension of scalars and $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms

Various applications of Theorem 3.2.1 can be found in [ABS03] and [GriRei15, Chapter 7]. We are going to present another application, which we will obtain by "leveraging" Theorem 3.2.1 through an extension-of-scalars argument[6]. Let us first introduce some more notations.

Definition 3.3.1. Let H be a $\mathbf{k}$-algebra (possibly with additional structure, such as a grading or a Hopf algebra structure). Then, $\underline{H}$ will mean the $\mathbf{k}$-algebra H without any additional structure (for instance, the $\mathbf{k}$-coalgebra structure on H is forgotten if H was a $\mathbf{k}$-bialgebra, and the grading is forgotten if H was graded). Sometimes we will use the notation $\underline{H}$ even when H has no additional structure beyond being a H-algebra; in this case, it means the same as H, just stressing the fact that it is a plain $\mathbf{k}$-algebra with nothing up its sleeves.

In other words, $\underline{H}$ will denote the image of H under the forgetful functor from whatever category H belongs to to the category of $\mathbf{k}$-algebras. We shall often use $\underline{H}$ and H interchangeably, whenever H is merely a $\mathbf{k}$-algebra or the other structures on H cannot cause confusion.

[6] I have learned this extension-of-scalars trick from Petracci's [Petra02, proof of Lemma 2.1.1]; similar ideas appear in various other algebraic arguments.

Definition 3.3.2. Let A be a commutative $\mathbf{k}$-algebra.

(a) If H is a $\mathbf{k}$-module, then $\underline{A} \otimes H$ will be understood to mean the A-module $A \otimes H$, on which A acts by the rule

$$a(b \otimes h) = ab \otimes h \qquad \text{for all } a \in A,\ b \in A \text{ and } h \in H.$$

This A-module $\underline{A} \otimes H$ is called the *$\mathbf{k}$-module H with scalars extended to $\underline{A}$*.

We can define a functor $\operatorname{Mod}_{\mathbf{k}} \to \operatorname{Mod}_A$ (where Mod_B denotes the category of B-modules) which sends every object $H \in \operatorname{Mod}_{\mathbf{k}}$ to $\underline{A} \otimes H$ and every morphism $f \in \operatorname{Mod}_{\mathbf{k}}(H_1, H_2)$ to $\operatorname{id} \otimes f \in \operatorname{Mod}_A(\underline{A} \otimes H_1, \underline{A} \otimes H_2)$; this functor is called *extension of scalars* (from $\mathbf{k}$ to A).

(b) If H is a graded $\mathbf{k}$-module, then the A-module $\underline{A} \otimes H$ canonically becomes a graded $\underline{A}$-module (namely, its n-th graded component is $\underline{A} \otimes H_n$, where H_n is the n-th graded component of H). Notice that even if A is graded, we disregard its grading when defining the grading on $\underline{A} \otimes H$; this is why we are calling it $\underline{A} \otimes H$ and not $A \otimes H$.

As before, we can define a functor from the category of graded $\mathbf{k}$-modules to the category of graded A-modules (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars*.

(c) If H is a $\mathbf{k}$-algebra, then the A-module $\underline{A} \otimes H$ becomes an A-algebra according to the rule

$$(a \otimes h)(b \otimes g) = ab \otimes hg \qquad \text{for all } a \in A,\ b \in A,\ h \in H \text{ and } g \in H.$$

(This is, of course, the same rule as used in the standard definition of the tensor product $A \otimes H$; but notice that we are regarding $\underline{A} \otimes H$ as an A-algebra, not just as a $\mathbf{k}$-algebra.) This A-algebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-algebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-algebras to the category of A-algebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars.*

(d) If H is a $\mathbf{k}$-coalgebra, then the A-module $\underline{A} \otimes H$ becomes an A-coalgebra. Namely, its comultiplication is defined to be

$$\mathrm{id}_A \otimes \Delta_H : A \otimes H \to A \otimes (H \otimes H) \cong (A \otimes H) \otimes_A (A \otimes H),$$

and its counit is defined to be

$$\mathrm{id}_A \otimes \varepsilon_H : A \otimes H \to A \otimes \mathbf{k} \cong A$$

(recalling that Δ_H and ε_H are the comultiplication and the counit of H, respectively). Note that both the comultiplication and the counit are A-linear, so this A-coalgebra $\underline{A} \otimes H$ is well-defined. This A-coalgebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-coalgebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-coalgebras to the category of A-coalgebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars.*

Notice that $\underline{A} \otimes H$ is an A-coalgebra, not a $\mathbf{k}$-coalgebra. If A has a pre-existing $\mathbf{k}$-coalgebra structure, then the A-coalgebra structure on $\underline{A} \otimes H$ usually has nothing to do with the $\mathbf{k}$-coalgebra structure on $A \otimes H$ obtained by tensoring the $\mathbf{k}$-coalgebras A and H.

(e) If H is a $\mathbf{k}$-bialgebra, then the A-module $\underline{A} \otimes H$ becomes an A-bialgebra. (Namely, the A-algebra structure and the A-coalgebra structure previously defined on $\underline{A} \otimes H$, combined, form an A-bialgebra structure.) This A-bialgebra $\underline{A} \otimes H$ is called the *$\mathbf{k}$-bialgebra H with scalars extended to $\underline{A}$*.

As before, we can define a functor from the category of $\mathbf{k}$-bialgebras to the category of A-bialgebras (which functor sends every object H to $\underline{A} \otimes H$), which is called *extension of scalars.*

(f) Similarly, extension of scalars is defined for $\mathbf{k}$-Hopf algebras, graded $\mathbf{k}$-bialgebras, etc.. Again, all structures on A that go beyond the $\mathbf{k}$-algebra structure are irrelevant and can be forgotten.

Definition 3.3.3. Let A be a commutative $\mathbf{k}$-algebra.

(a) Let H be a $\mathbf{k}$-module, and let G be an A-module. For any $\mathbf{k}$-linear map $f : H \to G$, we let $f^{\sharp}$ denote the A-linear map

$$\underline{A} \otimes H \to G, \qquad a \otimes h \mapsto af(h).$$

(It is easy to see that this latter map is indeed well-defined and A-linear.) For any A-linear map $g : \underline{A} \otimes H \to G$, we let $g^{\flat}$ denote the $\mathbf{k}$-linear map

$$H \to G, \qquad h \mapsto g(1 \otimes h).$$

Sometimes we will use the notations $f^{\sharp(A,\mathbf{k})}$ and $g^{\flat(A,\mathbf{k})}$ instead of $f^{\sharp}$ and $g^{\flat}$ when the A and the $\mathbf{k}$ are not clear from the context.

It is easy to see that $\left(f^{\sharp}\right)^{\flat} = f$ for any $\mathbf{k}$-linear map $f : H \to G$, and that $\left(g^{\flat}\right)^{\sharp} = g$ for any A-linear map $g : \underline{A} \otimes H \to G$. Thus, the maps

$$\begin{aligned} \{\mathbf{k}\text{-linear maps } H \to G\} &\to \{A\text{-linear maps } \underline{A} \otimes H \to G\}, \\ f &\mapsto f^{\sharp} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \{A\text{-linear maps } \underline{A} \otimes H \to G\} &\to \{\mathbf{k}\text{-linear maps } H \to G\}, \\ g &\mapsto g^{\flat} \end{aligned} \tag{3.5}$$

are mutually inverse.

This is a particular case of an adjunction between functors (namely, the Hom-tensor adjunction, with a slight simplification, also known as the induction-restriction adjunction); this is also the reason why we are using the $\sharp$ and $\flat$ notations. The maps (3.4) and (3.5) are natural in H and G.

(b) Let H be a $\mathbf{k}$-coalgebra, and let G be an A-coalgebra. A $\mathbf{k}$-linear map $f : H \to G$ is said to be a $(\mathbf{k}, \underline{A})$*-coalgebra homomorphism* if the A-linear map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism.

Proposition 3.3.4. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-algebra. Let G be an A-algebra. Let $f : H \to G$ be a $\mathbf{k}$-linear map. Then, f is a $\mathbf{k}$-algebra homomorphism if and only if $f^\sharp$ is an A-algebra homomorphism.

Proof of Proposition 3.3.4. Straightforward and left to the reader. (The main step is to observe that $f^\sharp$ is an A-algebra homomorphism if and only if every $a, b \in A$ and $h, g \in H$ satisfy $f^\sharp\left((a \otimes h)(b \otimes g)\right) = f^\sharp(a \otimes h)\, f^\sharp(b \otimes g)$. This is because the tensor product $\underline{A} \otimes H$ is spanned by pure tensors.) □

Proposition 3.3.5. Let A be a commutative $\mathbf{k}$-algebra. Let H be a graded $\mathbf{k}$-module. Let G be an A-module. Let $f : H \to G$ be a $\mathbf{k}$-linear map. Then, the $\mathbf{k}$-linear map f is graded if and only if the $\mathbf{k}$-linear map $f^\sharp$ is graded.

Proof of Proposition 3.3.5. Again, straightforward and therefore omitted. □

Let us first prove some easily-checked properties of $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms.

Proposition 3.3.6. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G and I be two A-coalgebras. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $g : G \to I$ be an A-coalgebra homomorphism. Then, $g \circ f$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.3.6. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Now, straightforward elementwise computation (using the fact that the map f is $\mathbf{k}$-linear, and the map g is A-linear) shows that

$$(g \circ f)^\sharp = g \circ f^\sharp. \tag{3.6}$$

Thus, $(g \circ f)^\sharp$ is an A-coalgebra homomorphism (since g and $f^\sharp$ are A-coalgebra homomorphisms). In other words, $g \circ f$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. This proves Proposition 3.3.6. □

Proposition 3.3.7. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let F and H be two $\mathbf{k}$-coalgebras. Let G be an A-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $e : F \to H$ be a $\mathbf{k}$-coalgebra homomorphism. Then, $f \circ e$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.3.7. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. The map $\mathrm{id}_A \otimes e : \underline{A} \otimes F \to \underline{A} \otimes H$ is an A-coalgebra homomorphism (since $e : F \to H$ is a $\mathbf{k}$-coalgebra homomorphism). Now, straightforward computation shows that $(f \circ e)^\sharp = f^\sharp \circ (\mathrm{id}_A \otimes e)$. Hence, $(f \circ e)^\sharp$ is an A-coalgebra homomorphism (since $f^\sharp$ and $\mathrm{id}_A \otimes e$ are A-coalgebra homomorphisms). In other words, $f \circ e$ is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. This proves Proposition 3.3.7. □

Proposition 3.3.8. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let B be a commutative A-algebra. Let $p : A \to B$ be an A-algebra homomorphism. (Actually, p is uniquely determined by the A-algebra structure on B.) Let $p_G : G \to B \otimes_A G$ be the canonical A-module homomorphism defined as the composition

$$G \xrightarrow{\cong} A \otimes_A G \xrightarrow{p \otimes_A \mathrm{id}} B \otimes_A G.$$

Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $p_G \circ f : H \to \underline{B} \otimes_A G$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.3.8. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp = f^{\sharp(A,\mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes_A f^\sharp : \underline{B} \otimes_A (\underline{A} \otimes H) \to \underline{B} \otimes_A G$ is a B-coalgebra homomorphism.

Let $\kappa : \underline{B} \otimes H \to \underline{B} \otimes_A (\underline{A} \otimes H)$ be the canonical B-module isomorphism (sending each $b \otimes h \in \underline{B} \otimes H$ to $b \otimes_A (1 \otimes h)$). It is well-known that κ is a B-coalgebra isomorphism[7]. Thus, $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ is a B-coalgebra homomorphism (since both $\mathrm{id}_B \otimes_A f^\sharp$ and κ are B-coalgebra homomorphisms).

The definition of p_G yields that

$$p_G(u) = 1 \otimes_A u \tag{3.7}$$

for every $u \in G$.

The map $p_G \circ f : H \to \underline{B} \otimes_A G$ gives rise to a map $(p_G \circ f)^{\sharp(B,\mathbf{k})} : \underline{B} \otimes H \to \underline{B} \otimes_A G$. But easy computations show that $(p_G \circ f)^{\sharp(B,\mathbf{k})} = \left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ (because the map $(p_G \circ f)^{\sharp(B,\mathbf{k})}$ sends a pure tensor $b \otimes h \in \underline{B} \otimes H$ to $b \underbrace{(p_G \circ f)(h)}_{\substack{= p_G(f(h)) = 1 \otimes_A f(h) \\ \text{(by (3.7))}}} = b(1 \otimes_A f(h)) = b \otimes_A f(h)$, whereas the map $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ sends a pure tensor $b \otimes h \in \underline{B} \otimes H$ to

$$\begin{aligned} \left(\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa\right)(b \otimes h) &= \left(\mathrm{id}_B \otimes_A f^\sharp\right)\left(\underbrace{\kappa(b \otimes h)}_{= b \otimes_A (1 \otimes h)}\right) = \left(\mathrm{id}_B \otimes_A f^\sharp\right)(b \otimes_A (1 \otimes h)) \\ &= b \otimes_A \underbrace{f^\sharp(1 \otimes h)}_{= 1 f(h) = f(h)} = b \otimes_A f(h) \end{aligned}$$

as well). Thus, $(p_G \circ f)^{\sharp(B,\mathbf{k})}$ is a B-coalgebra homomorphism (since $\left(\mathrm{id}_B \otimes_A f^\sharp\right) \circ \kappa$ is a B-coalgebra homomorphism). In other words, $p_G \circ f$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism. This proves Proposition 3.3.8. ☐

Proposition 3.3.9. Let $\mathbf{k}$ be a commutative ring. Let A and B be two commutative $\mathbf{k}$-algebras. Let H and G be two $\mathbf{k}$-coalgebras. Let $f : H \to \underline{A} \otimes G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $p : A \to B$ be a $\mathbf{k}$-algebra homomorphism. Then, $(p \otimes \mathrm{id}) \circ f : H \to \underline{B} \otimes G$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.3.9. Consider B as an A-algebra by means of the $\mathbf{k}$-algebra homomorphism $p : A \to B$. Thus, p becomes an A-algebra homomorphism $A \to B$. Now, $\underline{A} \otimes G$ is an A-coalgebra. Let $p_{\underline{A} \otimes G} : \underline{A} \otimes G \to B \otimes_A (\underline{A} \otimes G)$ be the canonical

[7] In fact, it is part of the natural isomorphism $\mathrm{Ind}_A^B \circ \mathrm{Ind}_{\mathbf{k}}^A \cong \mathrm{Ind}_{\mathbf{k}}^B$, where Ind_P^Q means extension of scalars from P to Q (as a functor from the category of P-coalgebras to the category of Q-coalgebras).

A-module homomorphism defined as the composition

$$\underline{A} \otimes G \xrightarrow{\cong} A \otimes_A (\underline{A} \otimes G) \xrightarrow{p \otimes_A \mathrm{id}} B \otimes_A (\underline{A} \otimes G).$$

Proposition 3.3.8 (applied to $\underline{A} \otimes G$ and $p_{\underline{A} \otimes G}$ instead of G and p_G) shows that $p_{\underline{A} \otimes G} \circ f : H \to \underline{B} \otimes_A (\underline{A} \otimes G)$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

But let ϕ be the canonical B-module isomorphism $\underline{B} \otimes_A (\underline{A} \otimes G) \to \underbrace{(\underline{B} \otimes_A \underline{A})}_{\cong \underline{B}} \otimes G \to \underline{B} \otimes G$. Then, ϕ is a B-coalgebra homomorphism, and has the property that $p \otimes \mathrm{id} = \phi \circ p_{\underline{A} \otimes G}$ as maps $\underline{A} \otimes G \to \underline{B} \otimes G$ (this can be checked by direct computation). Now,

$$\underbrace{(p \otimes \mathrm{id})}_{= \phi \circ p_{\underline{A} \otimes G}} \circ f = \phi \circ p_{\underline{A} \otimes G} \circ f = \phi \circ (p_{\underline{A} \otimes G} \circ f)$$

must be a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism (by Proposition 3.3.6, since $p_{\underline{A} \otimes G} \circ f$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism and since ϕ is a B-coalgebra homomorphism). This proves Proposition 3.3.9. □

Proposition 3.3.10. Let $\mathbf{k}$ be a commutative ring. Let A and B be two commutative $\mathbf{k}$-algebras. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ is a $(\underline{B}, \underline{B} \otimes \underline{A})$-coalgebra homomorphism.

Proof of Proposition 3.3.10. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^\sharp = f^{\sharp(A,\mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes f^\sharp : \underline{B} \otimes (\underline{A} \otimes H) \to \underline{B} \otimes G$ is a B-coalgebra homomorphism.

But the $\underline{B}$-linear map $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ gives rise to a $\underline{B} \otimes \underline{A}$-linear map $(\mathrm{id} \otimes f)^{\sharp(\underline{B} \otimes \underline{A}, \underline{B})} : (\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H) \to \underline{B} \otimes G$.

Now, let γ be the canonical B-module isomorphism $(\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H) \to \underline{B} \otimes (\underline{A} \otimes H)$ (sending each $(b \otimes a) \otimes_B (b' \otimes h) \in (\underline{B} \otimes \underline{A}) \otimes_B (\underline{B} \otimes H)$ to $bb' \otimes (a \otimes h)$). Then, γ is a B-coalgebra isomorphism (this is easy to check). Hence, $\left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ is a B-coalgebra isomorphism (since $\mathrm{id}_B \otimes f^\sharp$ and γ are B-coalgebra isomorphisms).

Now, it is straightforward to see that $(\mathrm{id} \otimes f)^{\sharp(\underline{B \otimes A}, \underline{B})} = \left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ [8]. Hence, the map $(\mathrm{id} \otimes f)^{\sharp(\underline{B \otimes A}, \underline{B})}$ is a B-coalgebra homomorphism (since $\left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma$ is a B-coalgebra homomorphism). In other words, $\mathrm{id} \otimes f : \underline{B} \otimes H \to \underline{B} \otimes G$ is a $(\underline{B}, \underline{B} \otimes \underline{A})$-coalgebra homomorphism. This proves Proposition 3.3.10. □

Proposition 3.3.11. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let B be a commutative A-algebra. Let H be a $\mathbf{k}$-coalgebra. Let G be an A-coalgebra. Let I be a B-coalgebra. Let $f : H \to G$ be a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Let $g : G \to I$ be an $(\underline{A}, \underline{B})$-coalgebra homomorphism. Then, $g \circ f : H \to I$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism.

Proof of Proposition 3.3.11. Since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism, the map $f^{\sharp(A,\mathbf{k})} : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism. Thus, the map $\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})} : \underline{B} \otimes_A (\underline{A} \otimes H) \to \underline{B} \otimes_A G$ is a B-coalgebra homomorphism.

Since $g : G \to I$ is an $(\underline{A}, \underline{B})$-coalgebra homomorphism, the map $g^{\sharp(\underline{B},\underline{A})} : \underline{B} \otimes_A G \to I$ is a B-coalgebra homomorphism.

Let $\delta : \underline{B} \otimes H \to \underline{B} \otimes_A (\underline{A} \otimes H)$ be the canonical B-module isomorphism (sending each $b \otimes h$ to $b \otimes_A (1 \otimes h)$). Then, δ is a B-coalgebra isomorphism. Straightforward elementwise computation shows that $(g \circ f)^{\sharp(\underline{B},\mathbf{k})} = g^{\sharp(\underline{B},\underline{A})} \circ \left(\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})}\right) \circ \delta$. Hence, $(g \circ f)^{\sharp(\underline{B},\mathbf{k})}$ is a B-coalgebra homomorphism (since $g^{\sharp(\underline{B},\underline{A})}$, $\mathrm{id}_B \otimes_A f^{\sharp(A,\mathbf{k})}$ and δ are B-coalgebra homomorphisms). In other words, $g \circ f : H \to I$ is a $(\mathbf{k}, \underline{B})$-coalgebra homomorphism. This proves Proposition 3.3.11. □

With these basics in place, we can now "escalate" Theorem 3.2.1 to the following setting:

Corollary 3.3.12. Let $\mathbf{k}$ be a commutative ring. Let H be a connected graded $\mathbf{k}$-Hopf algebra. Let A be a commutative $\mathbf{k}$-algebra. Let $\xi : H \to \underline{A}$ be a $\mathbf{k}$-algebra homomorphism.

[8] Indeed, it suffices to check it on pure tensors, i.e., to prove that

$$(\mathrm{id} \otimes f)^{\sharp(\underline{B \otimes A}, \underline{B})} ((b \otimes a) \otimes_B (b' \otimes h)) = \left(\left(\mathrm{id}_B \otimes f^\sharp\right) \circ \gamma\right) ((b \otimes a) \otimes_B (b' \otimes h))$$

for each $b \in B$, $a \in A$, $b' \in B$ and $h \in H$. But this is easy (both sides turn out to be $bb' \otimes_B af(h)$).

(a) Then, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad \Xi \quad} & \underline{A} \otimes \mathrm{QSym} \\ & \searrow^{\xi} \quad \swarrow_{\mathrm{id}_A \otimes \varepsilon_P} & \\ & \underline{A} & \end{array} \tag{3.8}$$

is commutative (where we regard $\mathrm{id}_A \otimes \varepsilon_P : \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k}$ as a map from $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ to $\underline{A}$, by canonically identifying $\underline{A} \otimes \mathbf{k}$ with $\underline{A}$).

(b) This unique $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-algebra homomorphism.

(c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to A$ (not to $\mathbf{k}$!) as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\xi^{\otimes k}} A^{\otimes k} \xrightarrow{m^{(k-1)}} A .$$

(Recall that $\Delta^{(k-1)} : H \to H^{\otimes k}$ and $m^{(k-1)} : A^{\otimes k} \to A$ are the "iterated comultiplication and multiplication maps"; see [GriRei15, §1.4] for their definitions. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 3.1.3.)

Then, the unique $(\mathbf{k}, \underline{A})$-coalgebra homomorphism Ξ of Corollary 3.3.12 **(a)** is given by

$$\Xi(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H$$

(in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

(d) If the $\mathbf{k}$-coalgebra H is cocommutative, then $\Xi(H)$ is a subset of the subring $\underline{A} \otimes \Lambda_{\mathbf{k}}$ of $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

Proof of Corollary 3.3.12. We have $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \cong \mathrm{QSym}_{\underline{A}}$ as A-bialgebras canonically (since $\mathrm{QSym}_{\mathbf{k}}$ is defined functorially in $\mathbf{k}$, with a basis that is independent of $\mathbf{k}$).

Recall that we have defined a $\mathbf{k}$-algebra homomorphism $\varepsilon_P : \mathrm{QSym}_{\mathbf{k}} \to \mathbf{k}$. We shall now denote this ε_P by $\varepsilon_{P,\mathbf{k}}$ in order to stress that it depends on $\mathbf{k}$. Similarly, an

$\mathbf{m}$-algebra homomorphism $\varepsilon_{P,\mathbf{m}} : \operatorname{QSym}_{\mathbf{m}} \to \mathbf{m}$ is defined for any commutative ring $\mathbf{m}$. In particular, an $\underline{A}$-algebra homomorphism $\varepsilon_{P,\underline{A}} : \operatorname{QSym}_{\underline{A}} \to \underline{A}$ is defined. The definitions of $\varepsilon_{P,\mathbf{m}}$ for all $\mathbf{m}$ are essentially identical; thus, the map $\varepsilon_{P,\underline{A}} : \operatorname{QSym}_{\underline{A}} \to \underline{A}$ can be identified with the map $\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}} : \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k}$ (if we identify $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ with $\operatorname{QSym}_{\underline{A}}$ and identify $\underline{A} \otimes \mathbf{k}$ with $\underline{A}$). We shall use this identification below.

The $\mathbf{k}$-linear map $\xi : H \to \underline{A}$ induces an A-linear map $\xi^\sharp : \underline{A} \otimes H \to \underline{A}$ (defined by $\xi^\sharp (a \otimes h) = a\xi (h)$ for all $a \in A$ and $h \in H$). Proposition 3.3.4 (applied to $G = \underline{A}$ and $f = \xi$) shows that $\xi^\sharp$ is an A-algebra homomorphism (since ξ is a $\mathbf{k}$-algebra homomorphism).

Theorem 3.2.1 **(a)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^\sharp$ instead of $\mathbf{k}$, H and ζ) shows that there exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ for which the diagram

$$\begin{array}{ccc} \underline{A} \otimes H & \xrightarrow{\Psi} & \operatorname{QSym}_{\underline{A}} \\ & {\scriptstyle \xi^\sharp} \searrow \quad \swarrow {\scriptstyle \varepsilon_{P,\underline{A}}} & \\ & \underline{A} & \end{array} \tag{3.9}$$

is commutative. Since we are identifying the map $\varepsilon_{P,\underline{A}} : \operatorname{QSym}_{\underline{A}} \to \underline{A}$ with the map $\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}} : \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}} \to \underline{A} \otimes \mathbf{k} = \underline{A}$, we can rewrite this as follows: There exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} \underline{A} \otimes H & \xrightarrow{\Psi} & \underline{A} \otimes \operatorname{QSym} \\ & {\scriptstyle \xi^\sharp} \searrow \quad \swarrow {\scriptstyle \operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}} & \\ & \underline{A} & \end{array}$$

is commutative. In other words, there exists a unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Psi = \xi^\sharp$. Let us refer to this observation as the *intermediate universal property*.

The $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms $H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ are in a 1-to-1 correspondence with the A-coalgebra homomorphisms $\underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$, which is the same as the A-coalgebra homomorphisms $\underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ (since $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}} \cong \operatorname{QSym}_{\underline{A}}$). The correspondence is given by sending a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism

$\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ to the A-coalgebra homomorphism $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$. Moreover, this correspondence has the property that Ξ is graded if and only if $\Xi^{\sharp}$ is (according to Proposition 3.3.5). Thus, this correspondence restricts to a correspondence between the graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphisms $H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ and the graded A-coalgebra homomorphisms $\underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$. Using this correspondence, we can rewrite the intermediate universal property as follows: There exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi^{\sharp} = \xi^{\sharp}$. In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $((\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi)^{\sharp} = \xi^{\sharp}$ (since (3.6) shows that $((\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi)^{\sharp} = (\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi^{\sharp}$). In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ such that $(\operatorname{id}_A \otimes \varepsilon_{P,\mathbf{k}}) \circ \Xi = \xi$ (since the map (3.4) is a bijection). In other words, there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram (3.8) is commutative. This proves Corollary 3.3.12 **(a)**.

By tracing back the above argument, we see that it yields an explicit construction of the unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ for which the diagram (3.8) is commutative: Namely, it is defined by $\Xi^{\sharp} = \Psi$, where Ψ is the unique graded $\underline{A}$-coalgebra homomorphism $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ for which the diagram (3.9) is commutative. Consider these Ξ and Ψ.

Theorem 3.2.1 **(b)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$ instead of $\mathbf{k}$, H and ζ) shows that $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$ is an $\underline{A}$-Hopf algebra homomorphism, thus an $\underline{A}$-algebra homomorphism. In other words, $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is an $\underline{A}$-algebra homomorphism (since $\Xi^{\sharp} : \underline{A} \otimes H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is the same as $\Psi : \underline{A} \otimes H \to \operatorname{QSym}_{\underline{A}}$, up to our identifications). Hence, $\Xi : H \to \underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ is a $\mathbf{k}$-algebra homomorphism as well (by Proposition 3.3.4, applied to $\underline{A}$, $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$ and Ξ instead of A, G and f). This proves Corollary 3.3.12 **(b)**.

(c) Theorem 3.2.1 **(d)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^{\sharp}$ instead of $\mathbf{k}$, H and ζ) shows that Ψ is given by

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \left(\xi^{\sharp}\right)_{\alpha}(h) \cdot M_{\alpha} \qquad \text{for every } h \in \underline{A} \otimes H, \tag{3.10}$$

where the map $\left(\xi^\sharp\right)_\alpha : \underline{A} \otimes H \to \underline{A}$ is defined in the same way as the map $\zeta_\alpha : H \to \mathbf{k}$ was defined in Theorem 3.2.1 **(d)** (but for $\underline{A}$, $\underline{A} \otimes H$ and $\xi^\sharp$ instead of $\mathbf{k}$, H and ζ). Notice that (3.10) is an equality inside $\operatorname{QSym}_{\underline{A}}$. Recalling that we are identifying $\operatorname{QSym}_{\underline{A}}$ with $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$, we can rewrite it as an equality in $\underline{A} \otimes \operatorname{QSym}_{\mathbf{k}}$; it then takes the form

$$\Psi(h) = \sum_{\alpha \in \operatorname{Comp}} \left(\xi^\sharp\right)_\alpha (h) \otimes M_\alpha \qquad \text{for every } h \in \underline{A} \otimes H. \tag{3.11}$$

Let ι_H be the $\mathbf{k}$-module homomorphism

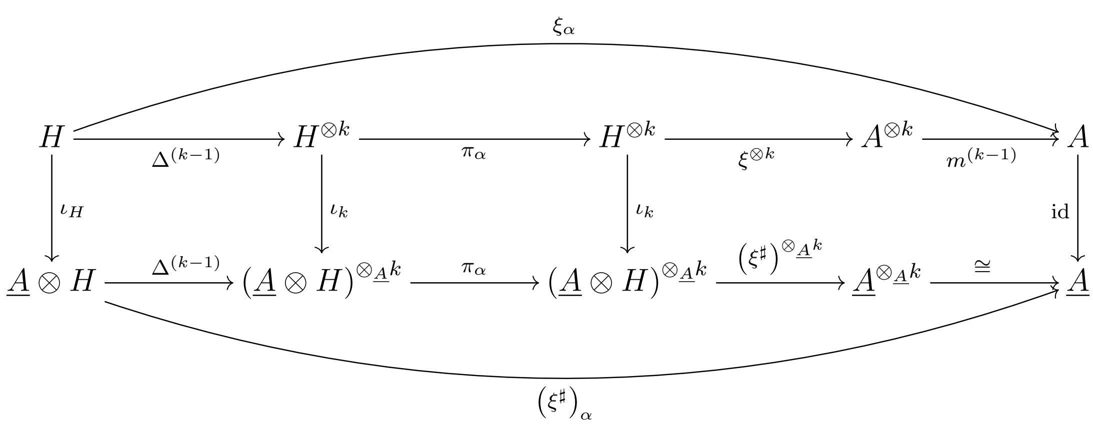
$$H \to \underline{A} \otimes H, \qquad h \mapsto 1 \otimes h.$$

Also, for every $k \in \mathbb{N}$, we let ι_k be the $\mathbf{k}$-module homomorphism

$$H^{\otimes k} \to (\underline{A} \otimes H)^{\otimes_{\underline{A}} k}, \qquad g \mapsto 1 \otimes g \in \underline{A} \otimes H^{\otimes k} \cong (\underline{A} \otimes H)^{\otimes_{\underline{A}} k}$$

(where $U^{\otimes_{\underline{A}} k}$ denotes the k-th tensor power of an $\underline{A}$-module U); this homomorphism sends every $h_1 \otimes h_2 \otimes \cdots \otimes h_k \in H^{\otimes k}$ to $(1 \otimes h_1) \otimes_{\underline{A}} (1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} (1 \otimes h_k)$.

On the other hand, fix some $\alpha \in \operatorname{Comp}$. Write the composition α in the form $\alpha = (a_1, a_2, \ldots, a_k)$. The diagram



is commutative[9]. Therefore, $\left(\xi^\sharp\right)_\alpha \circ \iota_H = \operatorname{id} \circ \xi_\alpha = \xi_\alpha$.

[9] *Proof.* In fact:

Now, forget that we fixed α. We thus have shown that

$$\left(\xi^{\sharp}\right)_{\alpha} \circ \iota_H = \xi_{\alpha} \qquad \text{for every } \alpha \in \operatorname{Comp}. \tag{3.12}$$

- Its upper pentagon is commutative (by the definition of ξ_α).

- Its lower pentagon is commutative (by the definition of $\left(\xi^{\sharp}\right)_{\alpha}$).

- Its left square is commutative (since the operation $\Delta^{(k-1)}$ on a $\mathbf{k}$-coalgebra is functorial with respect to the base ring, i.e., commutes with extension of scalars).

- Its middle square is commutative (since the operation π_α on a graded $\mathbf{k}$-module is functorial with respect to the base ring, i.e., commutes with extension of scalars).

- Its right rectangle is commutative. (Indeed, every $h_1, h_2, \ldots, h_k \in H$ satisfy

$$\begin{aligned}
&\left(\operatorname{id} \circ m^{(k-1)} \circ \xi^{\otimes k}\right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k) \\
&= m^{(k-1)}\left(\underbrace{\xi^{\otimes k}(h_1 \otimes h_2 \otimes \cdots \otimes h_k)}_{=\xi(h_1)\otimes\xi(h_2)\otimes\cdots\otimes\xi(h_k)}\right) = m^{(k-1)}\left(\xi(h_1) \otimes \xi(h_2) \otimes \cdots \otimes \xi(h_k)\right) \\
&= \xi(h_1)\,\xi(h_2)\cdots\xi(h_k)
\end{aligned}$$

and thus

$$\begin{aligned}
&\left(\left(\xi^{\sharp}\right)^{\otimes_{\underline{A}} k} \circ \iota_k\right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k) \\
&= \left(\xi^{\sharp}\right)^{\otimes_{\underline{A}} k} \underbrace{\left(\iota_k(h_1 \otimes h_2 \otimes \cdots \otimes h_k)\right)}_{=(1\otimes h_1)\otimes_{\underline{A}}(1\otimes h_2)\otimes_{\underline{A}}\cdots\otimes_{\underline{A}}(1\otimes h_k)} \\
&= \left(\xi^{\sharp}\right)^{\otimes_{\underline{A}} k}\left((1 \otimes h_1) \otimes_{\underline{A}} (1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} (1 \otimes h_k)\right) \\
&= \xi^{\sharp}(1 \otimes h_1) \otimes_{\underline{A}} \xi^{\sharp}(1 \otimes h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \xi^{\sharp}(1 \otimes h_k) \\
&= \xi(h_1) \otimes_{\underline{A}} \xi(h_2) \otimes_{\underline{A}} \cdots \otimes_{\underline{A}} \xi(h_k) \qquad \left(\text{since } \xi^{\sharp}(1 \otimes y) = \xi(y) \text{ for every } y \in H\right) \\
&= \xi(h_1)\,\xi(h_2)\cdots\xi(h_k) \qquad \left(\text{since } \underline{A}^{\otimes_{\underline{A}} k} \cong \underline{A}\right) \\
&= \left(\operatorname{id} \circ m^{(k-1)} \circ \xi^{\otimes k}\right)(h_1 \otimes h_2 \otimes \cdots \otimes h_k).
\end{aligned}$$

Hence, $\left(\xi^{\sharp}\right)^{\otimes_{\underline{A}} k} \circ \iota_k = \operatorname{id} \circ m^{(k-1)} \circ \xi^{\otimes k}$. In other words, the right rectangle is commutative.)

Now, every $h \in H$ satisfies

$$\begin{aligned}
\Xi(h) &= \underbrace{\Xi^\sharp}_{=\Psi}(1 \otimes h) = \Psi(1 \otimes h) \\
&= \sum_{\alpha \in \mathrm{Comp}} \left(\xi^\sharp\right)_\alpha \underbrace{(1 \otimes h)}_{=\iota_H(h)} \otimes M_\alpha \qquad \text{(by (3.11), applied to } 1 \otimes h \text{ instead of } h) \\
&= \sum_{\alpha \in \mathrm{Comp}} \underbrace{\left(\xi^\sharp\right)_\alpha (\iota_H(h))}_{=\left(\left(\xi^\sharp\right)_\alpha \circ \iota_H\right)(h)} \otimes M_\alpha = \sum_{\alpha \in \mathrm{Comp}} \underbrace{\left(\left(\xi^\sharp\right)_\alpha \circ \iota_H\right)}_{\substack{=\xi_\alpha \\ \text{(by (3.12))}}}(h) \otimes M_\alpha \\
&= \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha.
\end{aligned}$$

This proves Corollary 3.3.12 **(c)**.

(d) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, the A-coalgebra $\underline{A} \otimes H$ is cocommutative as well.

Let us first see why $\underline{A} \otimes \Lambda_{\mathbf{k}}$ is a subring of $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$. Indeed, recall that we are using the standard A-Hopf algebra isomorphism $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\underline{A}}$ to identify $\mathrm{QSym}_{\underline{A}}$ with $\underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$. Similarly, let us use the standard A-Hopf algebra isomorphism $\underline{A} \otimes \Lambda_{\mathbf{k}} \to \Lambda_{\underline{A}}$ to identify $\Lambda_{\underline{A}}$ with $\underline{A} \otimes \Lambda_{\mathbf{k}}$. Now, $\underline{A} \otimes \Lambda_{\mathbf{k}} = \Lambda_{\underline{A}} \subseteq \mathrm{QSym}_{\underline{A}} = \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$.

Theorem 3.2.1 **(e)** (applied to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^\sharp$ instead of $\mathbf{k}$, H and ζ) shows that $\Psi(\underline{A} \otimes H) \subseteq \Lambda_{\underline{A}} = \underline{A} \otimes \Lambda_{\mathbf{k}}$. Since $\Psi = \Xi^\sharp$, this rewrites as $\Xi^\sharp(\underline{A} \otimes H) \subseteq \underline{A} \otimes \Lambda_{\mathbf{k}}$. But $\Xi(H) \subseteq \Xi^\sharp(\underline{A} \otimes H)$ (since every $h \in H$ satisfies $\Xi(h) = \Xi^\sharp(1 \otimes h) \in \Xi^\sharp(\underline{A} \otimes H)$). Hence, $\Xi(H) \subseteq \Xi^\sharp(\underline{A} \otimes H) \subseteq \underline{A} \otimes \Lambda_{\mathbf{k}}$. This proves Corollary 3.3.12 **(d)**. □

Remark 3.3.13. Let $\mathbf{k}$, H, A and ξ be as in Corollary 3.3.12. Then, the $\mathbf{k}$-module $\mathrm{Hom}(H, A)$ of all $\mathbf{k}$-linear maps from H to A has a canonical structure of a $\mathbf{k}$-algebra; its unity is the map $u_A \circ \varepsilon_H \in \mathrm{Hom}(H, A)$ (where $u_A : \mathbf{k} \to A$ is the $\mathbf{k}$-linear map sending 1 to 1), and its multiplication is the binary operation $\bigstar$ defined by

$$f \bigstar g = m_A \circ (f \otimes g) \circ \Delta_H : H \to A \qquad \text{for every } f, g \in \mathrm{Hom}(H, A)$$

(where m_A is the $\mathbf{k}$-linear map $A \otimes A \to A,\ a \otimes b \mapsto ab$). This $\mathbf{k}$-algebra is called the *convolution algebra* of H and A; it is precisely the $\mathbf{k}$-algebra defined in [GriRei15, Definition 1.27]. Using this $\mathbf{k}$-algebra, we can express the map ξ_α in Theorem 3.2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\xi_\alpha : H \to A$ is given by

$$\xi_\alpha = (\xi \circ \pi_{a_1}) \bigstar (\xi \circ \pi_{a_2}) \bigstar \cdots \bigstar (\xi \circ \pi_{a_k}).$$

(This follows easily from [GriRei15, Exercise 1.43].)

3.4 The second comultiplication on $\operatorname{QSym}_{\mathbf{k}}$

Convention 3.4.1. In the following, we do **not** identify compositions with infinite sequences, as several authors do. As a consequence, the composition $(1, 3)$ does not equal the vector $(1, 3, 0)$ or the infinite sequence $(1, 3, 0, 0, 0, \ldots)$.

We now recall the definition of the *second comultiplication* (a.k.a. *internal comultiplication*) of $\operatorname{QSym}_{\mathbf{k}}$. Several definitions of this operation appear in the literature; we shall use the one in [Haz08, §11.39]:[10]

Definition 3.4.2. (a) Given a $u \times v$-matrix $A = (a_{i,j})_{1 \le i \le u,\ 1 \le j \le v} \in \mathbb{N}^{u \times v}$ (where $u, v \in \mathbb{N}$) with nonnegative entries, we define three tuples of nonnegative integers:

- The v-tuple $\operatorname{column} A \in \mathbb{N}^v$ is the v-tuple whose j-th entry is $\sum_{i=1}^{u} a_{i,j}$ (that is, the sum of all entries in the j-th column of A) for each j. (In other words, $\operatorname{column} A$ is the sum of all rows of A, regarded as vectors.)
- The u-tuple $\operatorname{row} A \in \mathbb{N}^u$ is the u-tuple whose i-th entry is $\sum_{j=1}^{v} a_{i,j}$ (that is, the sum of all entries in the i-th row of A) for each i. (In other words, $\operatorname{row} A$ is the sum of all columns of A, regarded as vectors.)

[10] The second comultiplication seems to be as old as $\operatorname{QSym}_{\mathbf{k}}$; it first appeared in Gessel's [Gessel84, §4] (the same article where $\operatorname{QSym}_{\mathbf{k}}$ was first defined).

- The uv-tuple read $A \in \mathbb{N}^{uv}$ is the uv-tuple whose $(v(i-1)+j)$-th entry is $a_{i,j}$ for all $i \in \{1, 2, \ldots, u\}$ and $j \in \{1, 2, \ldots, v\}$. In other words,

$$\begin{aligned}&\operatorname{read} A \\ &= (a_{1,1}, a_{1,2}, \ldots, a_{1,v}, a_{2,1}, a_{2,2}, \ldots, a_{2,v}, \ldots, a_{u,1}, a_{u,2}, \ldots, a_{u,v}).\end{aligned}$$

We say that the matrix A is *column-reduced* if column A is a composition (i.e., contains no zero entries). Equivalently, A is column-reduced if and only if no column of A is the 0 vector.

We say that the matrix A is *row-reduced* if row A is a composition (i.e., contains no zero entries). Equivalently, A is row-reduced if and only if no row of A is the 0 vector.

We say that the matrix A is *reduced* if A is both column-reduced and row-reduced.

(b) If $w \in \mathbb{N}^k$ is a k-tuple of nonnegative integers (for some $k \in \mathbb{N}$), then w^{red} shall mean the composition obtained from w by removing each entry that equals 0. For instance, $(3, 1, 0, 1, 0, 0, 2)^{\mathrm{red}} = (3, 1, 1, 2)$.

(c) Let $\mathbb{N}_{\mathrm{red}}^{\bullet,\bullet}$ denote the set of all reduced matrices in $\mathbb{N}^{u\times v}$, where u and v both range over $\mathbb{N}$. In other words, we set

$$\mathbb{N}_{\mathrm{red}}^{\bullet,\bullet} = \bigcup_{(u,v)\in\mathbb{N}^2} \left\{A \in \mathbb{N}^{u\times v} \mid A \text{ is reduced}\right\}.$$

(d) Let $\Delta_P : \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$ be the $\mathbf{k}$-linear map defined by setting

$$\Delta_P(M_\alpha) = \sum_{\substack{A\in\mathbb{N}_{\mathrm{red}}^{\bullet,\bullet};\\ (\operatorname{read} A)^{\mathrm{red}}=\alpha}} M_{\operatorname{row} A} \otimes M_{\operatorname{column} A} \qquad \text{for each } \alpha \in \mathrm{Comp}.$$

This map Δ_P is called the *second comultiplication* (or *internal comultiplication*) of $\mathrm{QSym}_{\mathbf{k}}$.

(e) Let τ denote the twist map $\tau_{\mathrm{QSym}_{\mathbf{k}},\mathrm{QSym}_{\mathbf{k}}} : \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$. Let $\Delta'_P = \tau \circ \Delta_P : \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$.

Example 3.4.3. The matrix $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 0 & 5 \\ 0 & 0 & 3 & 1 \end{pmatrix} \in \mathbb{N}^{3\times 4}$ is row-reduced but not column-reduced (and thus not reduced). If we denote it by A, then $\operatorname{row} A = (3, 7, 4)$ and $\operatorname{column} A = (3, 0, 5, 6)$ and $\operatorname{read} A = (1, 0, 2, 0, 2, 0, 0, 5, 0, 0, 3, 1)$.

Proposition 3.4.4. The $\mathbf{k}$-algebra $\mathrm{QSym}_{\mathbf{k}}$, equipped with comultiplication Δ_P and counit ε_P, is a $\mathbf{k}$-bialgebra (albeit not a connected graded one, and not a Hopf algebra).

Proposition 3.4.4 is a well-known fact (appearing, for example, in [MalReu95, first paragraph of §3]), but we shall actually derive it further below using our results.

3.5 The (generalized) Bernstein homomorphism

Let us now define the Bernstein homomorphism of a commutative connected graded $\mathbf{k}$-Hopf algebra, generalizing [Haz08, §18.24]:

Definition 3.5.1. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to H$ (not to $\mathbf{k}$!) as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{m^{(k-1)}} H .$$

(Recall that $\Delta^{(k-1)} : H \to H^{\otimes k}$ and $m^{(k-1)} : H^{\otimes k} \to H$ are the "iterated comultiplication and multiplication maps"; see [GriRei15, §1.4] for their definitions. The map $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ is the one defined in Definition 3.1.3.) Define a map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ by

$$\beta_H(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H. \tag{3.13}$$

It is easy to see that this map β_H is well-defined (i.e., the sum on the right hand side of (3.13) has only finitely many nonzero addends[11]) and $\mathbf{k}$-linear.

Remark 3.5.2. Let $\mathbf{k}$ and H be as in Definition 3.5.1. Then, the $\mathbf{k}$-module $\mathrm{Hom}(H, H)$ of all $\mathbf{k}$-linear maps from H to H has a canonical structure of a $\mathbf{k}$-algebra, defined as in Remark 3.3.13 (for $A = H$). Using this $\mathbf{k}$-algebra, we can express the map ξ_α in Theorem 3.2.1 **(c)** as follows: For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, the map $\xi_\alpha : H \to A$ is given by

$$\xi_\alpha = \pi_{a_1} \bigstar \pi_{a_2} \bigstar \cdots \bigstar \pi_{a_k}.$$

(This follows easily from [GriRei15, Exercise 1.43].)

The graded $\mathbf{k}$-Hopf algebra $\mathrm{QSym}_{\mathbf{k}}$ is commutative and connected; thus, Definition 3.5.1 (applied to $H = \mathrm{QSym}_{\mathbf{k}}$) constructs a $\mathbf{k}$-linear map $\beta_{\mathrm{QSym}_{\mathbf{k}}} : \mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes$

[11] *Proof.* Let $h \in H$. Then, there exists some $N \in \mathbb{N}$ such that $h \in H_0 + H_1 + \cdots + H_{N-1}$ (since $h \in H = \bigoplus_{i \in \mathbb{N}} H_i$). Consider this N. Now, it is easy to see that every composition $\alpha = (a_1, a_2, \ldots, a_k)$ of size $\geq N$ satisfies $\left(\pi_\alpha \circ \Delta^{(k-1)}\right)(h) = 0$ (because $\Delta^{(k-1)}(h)$ is concentrated in the first N homogeneous components of the graded $\mathbf{k}$-module $H^{\otimes k}$, and all of these components are annihilated by π_α) and therefore $\xi_\alpha(h) = 0$. Thus, the sum on the right hand side of (3.13) has only finitely many nonzero addends (namely, all its addends with $|\alpha| \geq N$ are 0).

$\mathrm{QSym}_{\mathbf{k}}$. We shall now prove that this map is identical with the Δ'_P from Definition 3.4.2 **(e)**:

Proposition 3.5.3. We have $\beta_{\mathrm{QSym}_{\mathbf{k}}} = \Delta'_P$.

Before we prove this, let us recall a basic formula for multiplication of monomial quasisymmetric functions:

Proposition 3.5.4. Let $k \in \mathbb{N}$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be k compositions. Let $\mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ denote the set of all column-reduced matrices in $\mathbb{N}^{k \times v}$ with v ranging over $\mathbb{N}$. In other words, let

$$\mathbb{N}^{k,\bullet}_{\mathrm{Cred}} = \bigcup_{v \in \mathbb{N}} \left\{ A \in \mathbb{N}^{k \times v} \mid A \text{ is column-reduced} \right\}.$$

Then,

$$M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} = \sum_{\substack{A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}; \\ (A_{g,\bullet})^{\mathrm{red}} = \alpha_g \text{ for each } g}} M_{\mathrm{column}\, A}.$$

Here, $A_{i,\bullet}$ denotes the i-th row of A (regarded as a list of nonnegative integers).

Notice that the $k = 2$ case of Proposition 3.5.4 is a restatement of the standard formula for the multiplication of monomial quasisymmetric functions (e.g., [GriRei15, Proposition 5.3][12] or [Haz08, §11.26]). The general case is still classical, but since an explicit proof is hard to locate in the literature, let me sketch it here.

Proof of Proposition 3.5.4. We begin by introducing notations:

- Let $\mathbb{N}^{k,\infty}$ denote the set of all matrices with k rows (labelled $1, 2, \ldots, k$) and countably many columns (labelled $1, 2, 3, \ldots$) whose entries all belong to $\mathbb{N}$.
- Let $\mathbb{N}^{k,\infty}_{\mathrm{fin}}$ denote the set of all matrices in $\mathbb{N}^{k,\infty}$ which have only finitely many nonzero entries.

[12] Actually, [GriRei15, Proposition 5.3] is slightly more general (the $k = 2$ case of Proposition 3.5.4 is obtained from [GriRei15, Proposition 5.3] by setting $I = \{1, 2, 3, \ldots\}$). That said, our proof can easily be extended to work in this greater generality.

- Let $\mathbb{N}^\infty$ denote the set of all infinite sequences $(a_1, a_2, a_3, \ldots)$ of elements of $\mathbb{N}$.

- Let $\mathbb{N}^\infty_{\text{fin}}$ denote the set of all sequences in $\mathbb{N}^\infty$ which have only finitely many nonzero entries.

- For every $B \in \mathbb{N}^{k,\infty}_{\text{fin}}$ and $i \in \{1, 2, \ldots, k\}$, we let $B_{i,\bullet} \in \mathbb{N}^\infty_{\text{fin}}$ be the i-th row of B.

- For every $B = (b_{i,j})_{1 \leq i \leq k,\ 1 \leq j} \in \mathbb{N}^{k,\infty}_{\text{fin}}$, we let $\operatorname{column} B \in \mathbb{N}^\infty_{\text{fin}}$ be the sequence whose j-th entry is $\sum_{i=1}^{k} a_{i,j}$ (that is, the sum of all entries in the j-th column of B) for each j. (In other words, $\operatorname{column} B$ is the sum of all rows of B, regarded as vectors.)

- We extend Definition 3.4.2 **(b)** to the case when $w \in \mathbb{N}^\infty_{\text{fin}}$: If $w \in \mathbb{N}^\infty_{\text{fin}}$, then w^{red} shall mean the composition obtained from w by removing each entry that equals 0 [13].

- For every $\beta = (b_1, b_2, b_3, \ldots) \in \mathbb{N}^\infty_{\text{fin}}$, we define a monomial $\mathbf{x}^\beta$ in the indeterminates $x_1, x_2, x_3, \ldots$ by
$$\mathbf{x}^\beta = x_1^{b_1} x_2^{b_2} x_3^{b_3} \cdots .$$

Then, it is easy to see that

$$M_\alpha = \sum_{\substack{\beta \in \mathbb{N}^\infty_{\text{fin}}; \\ \beta^{\text{red}} = \alpha}} \mathbf{x}^\beta \qquad \text{for every composition } \alpha. \tag{3.14}$$

[13] Here is a more rigorous definition of w^{red}: Let $w = (w_1, w_2, w_3, \ldots)$. Let $\mathcal{J}$ be the set of all positive integers j such that $w_j \neq 0$. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements of $\mathcal{J}$, in increasing order. Then, w^{red} is defined to be the composition $(w_{j_1}, w_{j_2}, \ldots, w_{j_h})$.

This rigorous definition of w^{red} has the additional advantage of making sense in greater generality than "remove each entry that equals 0"; namely, it still works when $w \in \mathbb{N}^I_{\text{fin}}$ for some totally ordered set I.

Now,

$$\begin{aligned}
& M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} \\
&= \prod_{g=1}^{k} \underbrace{M_{\alpha_g}}_{\substack{= \sum\limits_{\substack{\beta \in \mathbb{N}_{\text{fin}}^{\infty}; \\ \beta^{\text{red}} = \alpha_g}} \mathbf{x}^{\beta} \\ \text{(by (3.14))}}} = \prod_{g=1}^{k} \sum_{\substack{\beta \in \mathbb{N}_{\text{fin}}^{\infty}; \\ \beta^{\text{red}} = \alpha_g}} \mathbf{x}^{\beta} \\
&= \sum_{\substack{(\beta_1, \beta_2, \ldots, \beta_k) \in \left(\mathbb{N}_{\text{fin}}^{\infty}\right)^k; \\ (\beta_g)^{\text{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\beta_1} \mathbf{x}^{\beta_2} \cdots \mathbf{x}^{\beta_k} = \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \underbrace{\mathbf{x}^{B_{1,\bullet}} \mathbf{x}^{B_{2,\bullet}} \cdots \mathbf{x}^{B_{k,\bullet}}}_{\substack{= \mathbf{x}^{\text{column } B} \\ \text{(since column } B \text{ is the sum of the} \\ \text{rows of } B \text{ (as vectors))}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } (B_{1,\bullet}, B_{2,\bullet}, \ldots, B_{k,\bullet}) \text{ for} \\ (\beta_1, \beta_2, \ldots, \beta_k) \text{ in the sum, since the map} \\ \mathbb{N}_{\text{fin}}^{k,\infty} \to \left(\mathbb{N}_{\text{fin}}^{\infty}\right)^k, \ B \mapsto (B_{1,\bullet}, B_{2,\bullet}, \ldots, B_{k,\bullet}) \text{ is a bijection} \end{array} \right) \\
&= \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\text{column } B}. \qquad (3.15)
\end{aligned}$$

Now, let us introduce one more notation: For every matrix $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$, let B^{Cred} be the matrix obtained from B by removing all zero columns (i.e., all columns containing only zeroes)[14]. It is easy to see that $B^{\text{Cred}} \in \mathbb{N}_{\text{Cred}}^{k,\bullet}$ for every $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$. Moreover, every $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$ satisfies the following fact: If $A = B^{\text{Cred}}$, then

$$(B_{g,\bullet})^{\text{red}} = (A_{g,\bullet})^{\text{red}} \text{ for each } g \qquad (3.16)$$

(indeed, $A_{g,\bullet}$ is obtained from $B_{g,\bullet}$ by removing some zero entries).

[14] Again, we can define B^{Cred} more rigorously as follows: Let $\mathcal{J}$ be the set of all positive integers j such that the j-th column of B is nonzero. Let $(j_1 < j_2 < \cdots < j_h)$ be the list of all elements of $\mathcal{J}$, in increasing order. Then, B^{Cred} is defined to be the $k \times h$-matrix whose columns (from left to right) are the j_1-th column of B, the j_2-nd column of B, ..., the j_h-th column of B.

Now, (3.15) becomes

$$\begin{aligned}
M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} &= \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \mathbf{x}^{\text{column } B} \\
&= \sum_{A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}} \underbrace{\sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g; \\ B^{\text{Cred}} = A}}}_{\substack{= \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A; \\ (B_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} = \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \\ \text{(because if } B^{\text{Cred}} = A \text{, then } (B_{g,\bullet})^{\text{red}} = (A_{g,\bullet})^{\text{red}} \\ \text{for each } g \text{ (because of (3.16)))}}} \mathbf{x}^{\text{column } B} \\
&\qquad \left(\text{since } B^{\text{Cred}} \in \mathbb{N}_{\text{Cred}}^{k,\bullet} \text{ for each } B \in \mathbb{N}_{\text{fin}}^{k,\infty}\right) \\
&= \underbrace{\sum_{A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}} \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}}}_{= \sum_{\substack{A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A}}} \mathbf{x}^{\text{column } B} \\
&= \sum_{\substack{A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}; \\ (A_{g,\bullet})^{\text{red}} = \alpha_g \text{ for each } g}} \sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A}} \mathbf{x}^{\text{column } B}. \qquad (3.17)
\end{aligned}$$

But for every matrix $A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}$, we have

$$\sum_{\substack{B \in \mathbb{N}_{\text{fin}}^{k,\infty}; \\ B^{\text{Cred}} = A}} \mathbf{x}^{\text{column } B} = M_{\text{column } A}. \qquad (3.18)$$

Proof of (3.18): Let $A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}$. We need to prove (3.18).

For every $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$, we have $(\text{column } B)^{\text{red}} = \text{column}\left(B^{\text{Cred}}\right)$ (because first taking the sum of each column of B and then removing the zeroes among these sums results in the same list as first removing the zero columns of B and then taking the sum of each remaining column). Thus, for every $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$ satisfying $B^{\text{Cred}} = A$, we have column $B \in \mathbb{N}_{\text{fin}}^{\infty}$ and $(\operatorname{column} B)^{\text{red}} = \operatorname{column} \underbrace{\left(B^{\text{Cred}}\right)}_{=A} = \operatorname{column} A$. Hence, the map

$$\left\{B \in \mathbb{N}_{\text{fin}}^{k,\infty} \mid B^{\text{Cred}} = A\right\} \to \left\{\beta \in \mathbb{N}_{\text{fin}}^{\infty} \mid \beta^{\text{red}} = \operatorname{column} A\right\},$$
$$B \mapsto \operatorname{column} B \qquad (3.19)$$

is well-defined.

On the other hand, if $\beta \in \mathbb{N}_{\text{fin}}^{\infty}$ satisfies $\beta^{\text{red}} = \operatorname{column} A$, then there exists a unique $B \in \mathbb{N}_{\text{fin}}^{k,\infty}$ satisfying $B^{\text{Cred}} = A$ and $\operatorname{column} B = \beta$ [15]. In other words, the map (3.19)

[15] Namely, this B can be computed as follows: Write the sequence β in the form $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$. Let $(i_1 < i_2 < \cdots < i_h)$ be the list of all c satisfying $\beta_c \neq 0$, written in increasing order. Then, B shall be the matrix whose i_1-st, i_2-nd, ..., i_h-th columns are the columns of A (from left to right), whereas all its other columns are 0.

Let us briefly sketch a proof of the fact that this B is indeed an element of $\mathbb{N}_{\text{fin}}^{k,\bullet}$ satisfying $B^{\text{Cred}} = A$ and $\operatorname{column} B = \beta$:

Indeed, it is clear that $B \in \mathbb{N}_{\text{fin}}^{k,\bullet}$.

We shall now show that

$$(\text{the } j\text{-th entry of } \operatorname{column} B) = \beta_j \qquad (3.20)$$

for every $j \in \{1, 2, 3, \ldots\}$.

Proof of (3.20): Let $j \in \{1, 2, 3, \ldots\}$. We must prove (3.20). We are in one of the following two cases:

Case 1: We have $j \in \{i_1, i_2, \ldots, i_h\}$.

Case 2: We have $j \notin \{i_1, i_2, \ldots, i_h\}$.

Let us first consider Case 1. In this case, we have $j \in \{i_1, i_2, \ldots, i_h\}$. Hence, there exists a $g \in \{1, 2, \ldots, h\}$ such that $j = i_g$. Consider this g. Now,

$$\begin{aligned}
&(\text{the } j\text{-th entry of } \operatorname{column} B) \\
&= \left(\text{the sum of the entries of the } \underbrace{j}_{=i_g}\text{-th column of } B\right) \\
&= \left(\text{the sum of the entries of } \underbrace{\text{the } i_g\text{-th column of } B}_{\substack{=(\text{the } g\text{-th column of } A) \\ (\text{by the definition of } B)}}\right) \\
&= (\text{the sum of the entries of the } g\text{-th column of } A) \\
&= \left(\text{the } g\text{-th entry of } \underbrace{\operatorname{column} A}_{\substack{=\beta^{\text{red}} = \left(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_h}\right) \\ (\text{by the definition of } \beta^{\text{red}})}}\right) \\
&= \left(\text{the } g\text{-th entry of } \left(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_h}\right)\right) = \beta_{i_g} = \beta_j \qquad (\text{since } i_g = j).
\end{aligned}$$

Thus, (3.20) is proven in Case 1.

Let us now consider Case 2. In this case, we have $j \notin \{i_1, i_2, \ldots, i_h\}$. Hence, j does not belong is bijective. Thus, we can substitute β for column B in the sum $\sum_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B}$, and obtain

$$\sum_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B} = \sum_{\substack{\beta\in\mathbb{N}_{\text{fin}}^{\infty};\\ \beta^{\text{red}}=\operatorname{column} A}} \mathbf{x}^{\beta} = M_{\operatorname{column} A}$$

(by (3.14), applied to $\alpha = \operatorname{column} A$). This proves (3.18).

Now, (3.17) becomes

$$M_{\alpha_1} M_{\alpha_2} \cdots M_{\alpha_k} = \sum_{\substack{A\in\mathbb{N}_{\text{Cred}}^{k,\bullet};\\ (A_{g,\bullet})^{\text{red}}=\alpha_g \text{ for each } g}} \underbrace{\sum_{\substack{B\in\mathbb{N}_{\text{fin}}^{k,\infty};\\ B^{\text{Cred}}=A}} \mathbf{x}^{\operatorname{column} B}}_{\substack{=M_{\operatorname{column} A}\\ \text{(by (3.18))}}} = \sum_{\substack{A\in\mathbb{N}_{\text{Cred}}^{k,\bullet};\\ (A_{g,\bullet})^{\text{red}}=\alpha_g \text{ for each } g}} M_{\operatorname{column} A}.$$

---

to the list $(i_1 < i_2 < \cdots < i_h)$. In other words, j does not belong to the list of all $c \in \{1, 2, 3, \ldots\}$ satisfying $\beta_c \neq 0$ (since this list is $(i_1 < i_2 < \cdots < i_h)$). Hence, $\beta_j = 0$.

Recall that $j \notin \{i_1, i_2, \ldots, i_h\}$. Hence, the j-th column of B is the 0 vector (by the definition of B). Now,

$$\begin{aligned}
&(\text{the } j\text{-th entry of } \operatorname{column} B)\\
&= \left(\text{the sum of the entries of } \underbrace{\text{the } j\text{-th column of } B}_{=(\text{the } 0 \text{ vector})}\right)\\
&= (\text{the sum of the entries of the } 0 \text{ vector})\\
&= 0 = \beta_j.
\end{aligned}$$

Thus, (3.20) is proven in Case 2.

Hence, (3.20) is proven in both Cases 1 and 2. Thus, the proof of (3.20) is complete.

Now, from (3.20), we immediately obtain $\operatorname{column} B = (\beta_1, \beta_2, \beta_3, \ldots) = \beta$.

It remains to prove that $B^{\text{Cred}} = A$. This can be done as follows: We have $A \in \mathbb{N}_{\text{Cred}}^{k,\bullet}$; thus, the matrix A is column-reduced. Hence, no column of A is the zero vector. Therefore, none of the i_1-st, i_2-nd, $\ldots$, i_h-th columns of B is the zero vector (since these columns are the columns of A). On the other hand, each of the remaining columns of B is the zero vector (due to the definition of B). Thus, the set of all positive integers j such that the j-th column of B is nonzero is precisely $\{i_1, i_2, \ldots, i_h\}$. The list of all elements of this set, in increasing order, is $(i_1 < i_2 < \cdots < i_h)$. Hence, the definition of B^{Cred} shows that B^{Cred} is the $k \times h$-matrix whose columns (from left to right) are the i_1-th column of B, the i_2-nd column of B, $\ldots$, the i_h-th column of B. Since these columns are precisely the columns of A, this entails that B^{Cred} is the matrix A. In other words, $B^{\text{Cred}} = A$.

Thus, we have proven that B is an element of $\mathbb{N}_{\text{fin}}^{k,\bullet}$ satisfying $B^{\text{Cred}} = A$ and $\operatorname{column} B = \beta$. It is fairly easy to see that it is the only such element (because the condition $\operatorname{column} B = \beta$ determines which columns of B are nonzero, whereas the condition $B^{\text{Cred}} = A$ determines the precise values of these columns).

This proves Proposition 3.5.4. $\square$

We need one more piece of notation:

Definition 3.5.5. We define a (multiplicative) monoid structure on the set Comp as follows: If $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$ are two compositions, then we set $\alpha\beta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$. Thus, Comp becomes a monoid with neutral element $\varnothing = ()$ (the empty composition). (This monoid is actually the free monoid on the set $\{1, 2, 3, \ldots\}$.)

Proposition 3.5.6. Let $\gamma \in \mathrm{Comp}$ and $k \in \mathbb{N}$. Then,

$$\Delta^{(k-1)} M_\gamma = \sum_{\substack{(\gamma_1, \gamma_2, \ldots, \gamma_k) \in \mathrm{Comp}^k; \\ \gamma_1 \gamma_2 \cdots \gamma_k = \gamma}} M_{\gamma_1} \otimes M_{\gamma_2} \otimes \cdots \otimes M_{\gamma_k}.$$

Proof of Proposition 3.5.6 (sketched). We can rewrite (3.1) as follows:

$$\Delta M_\beta = \sum_{\substack{(\sigma, \tau) \in \mathrm{Comp} \times \mathrm{Comp}; \\ \sigma\tau = \beta}} M_\sigma \otimes M_\tau \qquad \text{for every } \beta \in \mathrm{Comp}. \tag{3.21}$$

Proposition 3.5.6 can easily be proven by induction using (3.21). $\square$

Proof of Proposition 3.5.3. Fix $\alpha \in \mathrm{Comp}$ and $\gamma \in \mathrm{Comp}$. Write α in the form $\alpha = (a_1, a_2, \ldots, a_k)$; thus, a $\mathbf{k}$-linear map $\xi_\alpha : \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}}$ is defined (as in Definition 3.5.1, applied to $H = \mathrm{QSym}_{\mathbf{k}}$).

We shall prove that

$$\xi_\alpha (M_\gamma) = \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\mathrm{red}}; \\ (\mathrm{read}\, A)^{\mathrm{red}} = \gamma; \\ \mathrm{row}\, A = \alpha}} M_{\mathrm{column}\, A}. \tag{3.22}$$

Proof of (3.22): The definition of ξ_α yields $\xi_\alpha = m^{(k-1)} \circ \pi_\alpha \circ \Delta^{(k-1)}$. Thus,

$$\begin{aligned}
\xi_\alpha (M_\gamma) &= \left(m^{(k-1)} \circ \pi_\alpha \circ \Delta^{(k-1)}\right)(M_\gamma) \\
&= \left(m^{(k-1)} \circ \pi_\alpha\right) \underbrace{\left(\Delta^{(k-1)}(M_\gamma)\right)}_{\substack{= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} M_{\gamma_1}\otimes M_{\gamma_2}\otimes\cdots\otimes M_{\gamma_k} \\ \text{(by Proposition 3.5.6)}}} \\
&= \left(m^{(k-1)} \circ \pi_\alpha\right) \left(\sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} M_{\gamma_1}\otimes M_{\gamma_2}\otimes\cdots\otimes M_{\gamma_k} \right) \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} m^{(k-1)} \underbrace{\left(\pi_\alpha\left(M_{\gamma_1}\otimes M_{\gamma_2}\otimes\cdots\otimes M_{\gamma_k}\right)\right)}_{\substack{=\pi_{a_1}(M_{\gamma_1})\otimes\pi_{a_2}(M_{\gamma_2})\otimes\cdots\otimes\pi_{a_k}(M_{\gamma_k}) \\ \text{(since } \pi_\alpha=\pi_{a_1}\otimes\pi_{a_2}\otimes\cdots\otimes\pi_{a_k})}} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \underbrace{m^{(k-1)}\left(\pi_{a_1}(M_{\gamma_1})\otimes\pi_{a_2}(M_{\gamma_2})\otimes\cdots\otimes\pi_{a_k}(M_{\gamma_k})\right)}_{=\pi_{a_1}(M_{\gamma_1})\cdot\pi_{a_2}(M_{\gamma_2})\cdot\cdots\cdot\pi_{a_k}(M_{\gamma_k}) = \prod_{g=1}^k \pi_{a_g}(M_{\gamma_g})} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \prod_{g=1}^k \underbrace{\pi_{a_g}(M_{\gamma_g})}_{\substack{= \begin{cases} M_{\gamma_g}, & \text{if } |\gamma_g| = a_g; \\ 0, & \text{if } |\gamma_g| \neq a_g \end{cases} \\ \text{(since the power series } M_{\gamma_g} \text{ is} \\ \text{homogeneous of degree } |\gamma_g|)}} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \underbrace{\prod_{g=1}^k \begin{cases} M_{\gamma_g}, & \text{if } |\gamma_g| = a_g; \\ 0, & \text{if } |\gamma_g| \neq a_g \end{cases}}_{= \begin{cases} \prod_{g=1}^k M_{\gamma_g}, & \text{if } |\gamma_g| = a_g \text{ for all } g; \\ 0, & \text{otherwise} \end{cases}} \\
&= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in \operatorname{Comp}^k; \\ \gamma_1\gamma_2\cdots\gamma_k=\gamma}} \begin{cases} \prod_{g=1}^k M_{\gamma_g}, & \text{if } |\gamma_g| = a_g \text{ for all } g; \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

$$= \sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}} \underbrace{\prod_{g=1}^{k} M_{\gamma_g}}_{\substack{=M_{\gamma_1}M_{\gamma_2}\cdots M_{\gamma_k} \\ = \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g}} M_{\text{column } A} \\ \text{(by Proposition 3.5.4, applied to} \\ \alpha_i=\gamma_i)}} \tag{3.23}$$

$$\text{(here, we have filtered out the zero addends)} \tag{3.24}$$

$$= \underbrace{\sum_{\substack{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}} \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g}}}_{= \sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k} \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}}} M_{\text{column } A}$$

$$= \sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k} \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g;\\ \gamma_1\gamma_2\cdots\gamma_k=\gamma;\\ |\gamma_g|=a_g \text{ for all } g}}}_{\substack{= \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g;\\ (A_{1,\bullet})^{\text{red}}(A_{2,\bullet})^{\text{red}}\cdots\left(A_{k,\bullet}\right)^{\text{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\text{red}}\right|=a_g \text{ for all } g}} \\ \text{(here, we have replaced every } \gamma_g \\ \text{in the conditions } (\gamma_1\gamma_2\cdots\gamma_k=\gamma) \\ \text{and } (|\gamma_g|=a_g \text{ for all } g) \text{ by the} \\ \text{corresponding } (A_{g,\bullet})^{\text{red}}\text{, because} \\ \text{of the condition that } (A_{g,\bullet})^{\text{red}}=\gamma_g)}} M_{\text{column } A}$$

$$= \underbrace{\sum_{(\gamma_1,\gamma_2,\ldots,\gamma_k)\in\operatorname{Comp}^k} \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{g,\bullet})^{\text{red}}=\gamma_g \text{ for each } g;\\ (A_{1,\bullet})^{\text{red}}(A_{2,\bullet})^{\text{red}}\cdots\left(A_{k,\bullet}\right)^{\text{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\text{red}}\right|=a_g \text{ for all } g}}}_{= \sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{1,\bullet})^{\text{red}}(A_{2,\bullet})^{\text{red}}\cdots\left(A_{k,\bullet}\right)^{\text{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\text{red}}\right|=a_g \text{ for all } g}}} M_{\text{column } A}$$

$$= \sum_{\substack{A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}; \\ (A_{1,\bullet})^{\mathrm{red}} (A_{2,\bullet})^{\mathrm{red}} \cdots \left(A_{k,\bullet}\right)^{\mathrm{red}} = \gamma; \\ \left|(A_{g,\bullet})^{\mathrm{red}}\right| = a_g \text{ for all } g}} M_{\mathrm{column}\, A}. \tag{3.25}$$

Now, we observe that every $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$ satisfies

$$(A_{1,\bullet})^{\mathrm{red}} (A_{2,\bullet})^{\mathrm{red}} \cdots (A_{k,\bullet})^{\mathrm{red}} = (\mathrm{read}\, A)^{\mathrm{red}} \tag{3.26}$$

[16].

Also, for every every $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$, we have the logical equivalence

$$\left(\left|(A_{g,\bullet})^{\mathrm{red}}\right| = a_g \text{ for all } g\right) \iff (\mathrm{row}\, A = \alpha) \tag{3.27}$$

[17].

[16] *Proof of (3.26):* Let $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$.

Let $\mathbb{N}^\bullet$ be the set of all finite lists of nonnegative integers. Then, $\mathrm{Comp} \subseteq \mathbb{N}^\bullet$. In Definition 3.5.5, we have defined a monoid structure on the set Comp. We can extend this monoid structure to the set $\mathbb{N}^\bullet$ (by the same rule: namely, if $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_m)$, then $\alpha\beta = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$). (Of course, this monoid $\mathbb{N}^\bullet$ is just the free monoid on the set $\mathbb{N}$.) Using the latter structure, we can rewrite the definition of $\mathrm{read}\, A$ as follows:

$$\mathrm{read}\, A = A_{1,\bullet} A_{2,\bullet} \cdots A_{k,\bullet}.$$

Clearly, the map $\mathbb{N}^\bullet \to \mathrm{Comp},\ \beta \mapsto \beta^{\mathrm{red}}$ is a monoid homomorphism. Thus,

$$(A_{1,\bullet})^{\mathrm{red}} (A_{2,\bullet})^{\mathrm{red}} \cdots (A_{k,\bullet})^{\mathrm{red}} = \left(\underbrace{A_{1,\bullet} A_{2,\bullet} \cdots A_{k,\bullet}}_{=\mathrm{read}\, A}\right)^{\mathrm{red}} = (\mathrm{read}\, A)^{\mathrm{red}}.$$

This proves (3.26).

[17] *Proof of (3.27):* Let $A \in \mathbb{N}^{k,\bullet}_{\mathrm{Cred}}$. Then, every $g \in \{1, 2, \ldots, k\}$ satisfies

$$\begin{aligned} \left|(A_{g,\bullet})^{\mathrm{red}}\right| &= \left(\text{sum of all entries of } (A_{g,\bullet})^{\mathrm{red}}\right) = (\text{sum of all nonzero entries in } A_{g,\bullet}) \\ &\qquad \left(\text{by the definition of } (A_{g,\bullet})^{\mathrm{red}}\right) \\ &= (\text{sum of all nonzero entries in the } g\text{-th row of } A) \\ &= (\text{sum of all entries in the } g\text{-th row of } A) = (\text{the } g\text{-th entry of } \mathrm{row}\, A). \end{aligned}$$

Also, every $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$ satisfying $\operatorname{row} A = \alpha$ belongs to $\mathbb{N}^{\bullet,\bullet}_{\text{red}}$ [18]. Conversely, every $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$ satisfying $\operatorname{row} A = \alpha$ belongs to $\mathbb{N}^{k,\bullet}_{\text{Cred}}$ [19]. Combining these two observations, we see that

$$\left(\begin{array}{c}\text{the matrices } A \in \mathbb{N}^{k,\bullet}_{\text{Cred}} \text{ satisfying } \operatorname{row} A = \alpha \\ \text{are precisely the matrices } A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}} \text{ satisfying } \operatorname{row} A = \alpha\end{array}\right). \tag{3.28}$$

Now, (3.25) becomes

$$\begin{aligned}\xi_\alpha (M_\gamma) &= \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (A_{1,\bullet})^{\text{red}}(A_{2,\bullet})^{\text{red}}\cdots\left(A_{k,\bullet}\right)^{\text{red}}=\gamma;\\ \left|(A_{g,\bullet})^{\text{red}}\right|=a_g \text{ for all } g}}}_{\substack{=\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \operatorname{row} A=\alpha}}\\ \text{(by (3.26) and (3.27))}}} M_{\operatorname{column} A} = \underbrace{\sum_{\substack{A\in\mathbb{N}^{k,\bullet}_{\text{Cred}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \operatorname{row} A=\alpha}}}_{\substack{=\sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \operatorname{row} A=\alpha}}\\ \text{(by (3.28))}}} M_{\operatorname{column} A} \\ &= \sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\text{red}};\\ (\text{read } A)^{\text{red}}=\gamma;\\ \operatorname{row} A=\alpha}} M_{\operatorname{column} A}.\end{aligned}$$

This proves (3.22).

Hence, we have the following chain of equivalences:

$$\begin{aligned}&\Bigl(\underbrace{\left|(A_{g,\bullet})^{\text{red}}\right|}_{=(\text{the } g\text{-th entry of } \operatorname{row} A)} = a_g \text{ for all } g\Bigr) \\ \iff &\left((\text{the } g\text{-th entry of } \operatorname{row} A) = a_g \text{ for all } g\right) \\ \iff &\Bigl(\operatorname{row} A = \underbrace{(a_1, a_2, \ldots, a_k)}_{=\alpha}\Bigr) = (\operatorname{row} A = \alpha).\end{aligned}$$

This proves (3.27).

[18] *Proof.* Let $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$ be such that $\operatorname{row} A = \alpha$. We must show that $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$. The sequence $\operatorname{row} A = \alpha$ is a composition; hence, A is row-reduced. Since A is also column-reduced (because $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$), this shows that A is reduced. Hence, $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$, qed.

[19] *Proof.* Let $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$ be such that $\operatorname{row} A = \alpha$. We must show that $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$. The number of rows of A is clearly the length of the vector $\operatorname{row} A$ (where the "length" of a vector just means its number of entries). But this length is k (since $\operatorname{row} A = \alpha = (a_1, a_2, \ldots, a_k)$). Therefore, the number of rows of A is k. Also, A is reduced (since $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$) and therefore column-reduced. Hence, $A \in \mathbb{N}^{k,\bullet}_{\text{Cred}}$ (since A is column-reduced and the number of rows of A is k), qed.

Now, forget that we fixed α and γ. For every $\gamma \in \operatorname{Comp}$, we have

$$\begin{aligned}
\beta_{\operatorname{QSym}_{\mathbf{k}}}(M_\gamma) &= \sum_{\alpha \in \operatorname{Comp}} \underbrace{\xi_\alpha(M_\gamma)}_{\substack{= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A = \alpha}} M_{\operatorname{column} A} \\ \text{(by (3.22))}}} \otimes M_\alpha \qquad \left(\text{by the definition of } \beta_{\operatorname{QSym}_{\mathbf{k}}}\right) \\
&= \sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A = \alpha}} M_{\operatorname{column} A} \otimes \underbrace{M_\alpha}_{\substack{= M_{\operatorname{row} A} \\ (\text{since } \operatorname{row} A = \alpha)}} \\
&= \underbrace{\sum_{\alpha \in \operatorname{Comp}} \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A = \alpha}}}_{= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A \in \operatorname{Comp}}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A} \\
&= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A \in \operatorname{Comp}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A}. \qquad (3.29)
\end{aligned}$$

But every $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$ satisfies $\operatorname{row} A \in \operatorname{Comp}$ [20]. Hence, the summation sign $\sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A \in \operatorname{Comp}}}$ on the right hand side of (3.29) can be replaced by $\sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma}}$. Thus, (3.29) becomes

$$\begin{aligned}
\beta_{\operatorname{QSym}_{\mathbf{k}}}(M_\gamma) &= \underbrace{\sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma; \\ \operatorname{row} A \in \operatorname{Comp}}}}_{= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma}}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A} \\
&= \sum_{\substack{A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}; \\ (\operatorname{read} A)^{\text{red}} = \gamma}} M_{\operatorname{column} A} \otimes M_{\operatorname{row} A}. \qquad (3.30)
\end{aligned}$$

[20] *Proof.* Let $A \in \mathbb{N}^{\bullet,\bullet}_{\text{red}}$. Then, the matrix A is reduced, and therefore row-reduced. In other words, $\operatorname{row} A$ is a composition. In other words, $\operatorname{row} A \in \operatorname{Comp}$, qed.

On the other hand, every $\gamma \in \mathrm{Comp}$ satisfies

$$\underbrace{\Delta'_P}_{=\tau\circ\Delta_P}(M_\gamma) = (\tau \circ \Delta_P)(M_\gamma) = \tau\left(\underbrace{\Delta_P(M_\gamma)}_{\substack{=\sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\mathrm{red}};\\ (\mathrm{read}\,A)^{\mathrm{red}}=\gamma}} M_{\mathrm{row}\,A}\otimes M_{\mathrm{column}\,A} \\ \text{(by the definition of } \Delta_P)}}\right)$$

$$= \tau\left(\sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\mathrm{red}};\\ (\mathrm{read}\,A)^{\mathrm{red}}=\gamma}} M_{\mathrm{row}\,A}\otimes M_{\mathrm{column}\,A}\right) = \sum_{\substack{A\in\mathbb{N}^{\bullet,\bullet}_{\mathrm{red}};\\ (\mathrm{read}\,A)^{\mathrm{red}}=\gamma}} M_{\mathrm{column}\,A}\otimes M_{\mathrm{row}\,A}$$

$$\text{(by the definition of } \tau)$$

$$= \beta_{\mathrm{QSym}_{\mathbf{k}}}(M_\gamma) \qquad \text{(by (3.30))}.$$

Since both maps Δ'_P and $\beta_{\mathrm{QSym}_{\mathbf{k}}}$ are $\mathbf{k}$-linear, this yields $\Delta'_P = \beta_{\mathrm{QSym}_{\mathbf{k}}}$ (since $(M_\gamma)_{\gamma\in\mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$). This proves Proposition 3.5.3. □

The next theorem is an analogue for QSym of the Bernstein homomorphism ([Haz08, §18.24]) for the symmetric functions:

Theorem 3.5.7. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. For every composition α, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to H$ as in Definition 3.5.1. Define a map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ as in Definition 3.5.1.

(a) The map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism.

(b) We have $(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H = \mathrm{id}$, where we regard $\mathrm{id} \otimes \varepsilon_P : \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{H} \otimes \mathbf{k}$ as a map from $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ to $\underline{H}$ (by identifying $\underline{H} \otimes \mathbf{k}$ with $\underline{H}$).

(c) Define a map $\Delta'_P : \mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$ as in Definition 3.4.2 **(e)**. The diagram

$$\begin{array}{ccc} H & \xrightarrow{\beta_H} & \underline{H} \otimes \mathrm{QSym} \\ \downarrow{\scriptstyle \beta_H} & & \downarrow{\scriptstyle \beta_H\otimes\mathrm{id}} \\ \underline{H} \otimes \mathrm{QSym} & \xrightarrow[\mathrm{id}\otimes\Delta'_P]{} & \underline{H} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \end{array}$$

is commutative.

(d) If the $\mathbf{k}$-coalgebra H is cocommutative, then $\beta_H(H)$ is a subset of the subring $\underline{H} \otimes \Lambda_{\mathbf{k}}$ of $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$, where $\Lambda_{\mathbf{k}}$ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

Parts **(b)** and **(c)** of Theorem 3.5.7 can be combined into "β_H makes H into a QSym_2-comodule, where QSym_2 is the coalgebra $(\mathrm{QSym}, \Delta'_P, \varepsilon_P)$" (the fact that this QSym_2 is actually a coalgebra follows from Proposition 3.4.4).

What Hazewinkel actually calls the Bernstein homomorphism in [Haz08, §18.24] is the $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \Lambda_{\mathbf{k}}$ obtained from our map $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ by restricting the codomain when H is both commutative and cocommutative[21]. His observation that the second comultiplication of $\Lambda_{\mathbf{k}}$ is a particular case of the Bernstein homomorphism is what gave the original motivation for the present note; its analogue for $\mathrm{QSym}_{\mathbf{k}}$ is our Proposition 3.5.3.

Proof of Theorem 3.5.7. Set $A = \underline{H}$ and $\xi = \mathrm{id}$. Then, the map ξ_α defined in Corollary 3.3.12 **(c)** is precisely the map ξ_α defined in Definition 3.5.1 (because $\xi^{\otimes k} = \mathrm{id}^{\otimes k} = \mathrm{id}$). Thus, we can afford calling both maps ξ_α without getting confused.

(a) Corollary 3.3.12 **(a)** shows that there exists a unique graded $(\mathbf{k}, \underline{A})$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram (3.8) is commutative. Since $A = \underline{H}$ and $\xi = \mathrm{id}$, we can rewrite this as follows: There exists a unique graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism $\Xi : H \to \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad\Xi\quad} & \underline{H} \otimes \mathrm{QSym} \\ & {\scriptstyle \mathrm{id}} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_H \otimes \varepsilon_P} & \\ & \underline{H} & \end{array} \tag{3.31}$$

is commutative. Consider this Ξ. Corollary 3.3.12 **(c)** shows that this homomorphism

[21] Hazewinkel neglects to require the cocommutativity of H in [Haz08, §18.24], but he uses it nevertheless.

Ξ is given by

$$\Xi(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H.$$

Comparing this equality with (3.13), we obtain $\Xi(h) = \beta_H(h)$ for every $h \in H$. In other words, $\Xi = \beta_H$. Thus, β_H is a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism (since Ξ is a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism).

Corollary 3.3.12 **(b)** shows that Ξ is a $\mathbf{k}$-algebra homomorphism. In other words, β_H is a $\mathbf{k}$-algebra homomorphism (since $\Xi = \beta_H$). This completes the proof of Theorem 3.5.7 **(a)**.

(b) Consider the map Ξ defined in our above proof of Theorem 3.5.7 **(a)**. We have shown that $\Xi = \beta_H$.

The commutative diagram (3.31) shows that $(\mathrm{id} \otimes \varepsilon_P) \circ \Xi = \mathrm{id}$. In other words, $(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H = \mathrm{id}$ (since $\Xi = \beta_H$). This proves Theorem 3.5.7 **(b)**.

(c) Theorem 3.5.7 **(a)** shows that the map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \mathrm{QSym}_\mathbf{k}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism. Theorem 3.5.7 **(a)** (applied to $\mathrm{QSym}_\mathbf{k}$ instead of H) shows that the map $\beta_{\mathrm{QSym}_\mathbf{k}}$ is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_\mathbf{k} \to \underline{\mathrm{QSym}_\mathbf{k}} \otimes \mathrm{QSym}_\mathbf{k}$ and a graded $\left(\mathbf{k}, \underline{\mathrm{QSym}_\mathbf{k}}\right)$-coalgebra homomorphism. Since $\Delta'_P = \beta_{\mathrm{QSym}_\mathbf{k}}$ (by Proposition 3.5.3), this rewrites as follows: The map Δ'_P is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_\mathbf{k} \to \underline{\mathrm{QSym}_\mathbf{k}} \otimes \mathrm{QSym}_\mathbf{k}$ and a graded $\left(\mathbf{k}, \underline{\mathrm{QSym}_\mathbf{k}}\right)$-coalgebra homomorphism.

Applying Corollary 3.3.12 **(a)** to $\underline{H} \otimes \underline{\mathrm{QSym}_\mathbf{k}}$ and β_H instead of A and ξ, we see that there exists a unique graded $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_\mathbf{k}}\right)$-coalgebra homomorphism $\Xi : H \to \underline{H} \otimes \underline{\mathrm{QSym}_\mathbf{k}} \otimes \mathrm{QSym}_\mathbf{k}$ for which the diagram

$$\begin{array}{ccc} H & \xrightarrow{\quad \Xi \quad} & \underline{H} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \\ & {\scriptstyle \beta_H} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}}} \otimes \varepsilon_P} & \\ & \underline{H} \otimes \underline{\mathrm{QSym}} & \end{array} \tag{3.32}$$

is commutative. Thus, if we have two graded $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_\mathbf{k}}\right)$-coalgebra homomorphisms $\Xi : H \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ for which the diagram (3.32) is commutative, then these two homomorphisms must be identical. We will now show that the two homomorphisms $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ both fit the bill; this will then yield that $(\beta_H \otimes \mathrm{id}) \circ \beta_H = (\mathrm{id} \otimes \Delta'_P) \circ \beta_H$, and thus Theorem 3.5.7 **(c)** will follow.

Recall that β_H and Δ'_P are graded maps. Thus, so are $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$. Moreover, β_H is a $(\mathbf{k}, \underline{H})$-coalgebra homomorphism, and Δ'_P is a $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. From this, it is easy to see that $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ and $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ are $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphisms[22].

Now, we shall show that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{(\beta_H \otimes \mathrm{id}) \circ \beta_H} & \underline{H} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \\ & \searrow^{\beta_H} \quad \swarrow_{\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}}} \otimes \varepsilon_P} & \\ & \underline{H} \otimes \underline{\mathrm{QSym}} & \end{array} \qquad (3.33)$$

and

$$\begin{array}{ccc} H & \xrightarrow{\left(\mathrm{id} \otimes \Delta'_P\right) \circ \beta_H} & \underline{H} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \\ & \searrow^{\beta_H} \quad \swarrow_{\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}}} \otimes \varepsilon_P} & \\ & \underline{H} \otimes \underline{\mathrm{QSym}} & \end{array} \qquad (3.34)$$

are commutative. This follows from the computations

$$\underbrace{\left(\mathrm{id}_{\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}} \otimes \varepsilon_P\right) \circ (\beta_H \otimes \mathrm{id})}_{= \beta_H \otimes \varepsilon_P = \beta_H \circ (\mathrm{id} \otimes \varepsilon_P)} \circ \beta_H = \beta_H \circ \underbrace{(\mathrm{id} \otimes \varepsilon_P) \circ \beta_H}_{\substack{= \mathrm{id} \\ \text{(by Theorem 3.5.7 (b))}}} = \beta_H$$

[22] *Proof.* Proposition 3.3.9 (applied to H, $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}$, H, $\mathrm{QSym}_{\mathbf{k}}$, β_H and β_H instead of A, B, H, G, f and p) shows that $(\beta_H \otimes \mathrm{id}) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. It remains to show that $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism.

Recall that Δ'_P is a $\left(\mathbf{k}, \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$. Hence, Proposition 3.3.10 (applied to $\underline{\mathrm{QSym}_{\mathbf{k}}}$, $\underline{H}$, $\mathrm{QSym}_{\mathbf{k}}$, $\underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ and Δ'_P instead of A, B, H, G and f) shows that $\mathrm{id} \otimes \Delta'_P : \underline{H} \otimes \mathrm{QSym}_{\mathbf{k}} \to \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ is an $\left(\underline{H}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. Therefore, Proposition 3.3.11 (applied to $\underline{H}$, $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}$, H, $\underline{H} \otimes \mathrm{QSym}_{\mathbf{k}}$, $\underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$, β_H and $\mathrm{id} \otimes \Delta'_P$ instead of A, B, H, G, I, f and g) shows that $(\mathrm{id} \otimes \Delta'_P) \circ \beta_H$ is a $\left(\mathbf{k}, \underline{H} \otimes \underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. This completes the proof.

and

$$\left(\underbrace{\mathrm{id}_{\underline{H}\otimes\underline{\mathrm{QSym}_{\mathbf{k}}}}}_{=\mathrm{id}_{\underline{H}}\otimes\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}}}\otimes\varepsilon_P\right)\circ\left(\underbrace{\mathrm{id}}_{=\mathrm{id}_{\underline{H}}}\otimes\underbrace{\Delta'_P}_{=\beta_{\mathrm{QSym}_{\mathbf{k}}}}\right)\circ\beta_H$$

$$=\underbrace{\left(\mathrm{id}_{\underline{H}}\otimes\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}}\otimes\varepsilon_P\right)\circ\left(\mathrm{id}_{\underline{H}}\otimes\beta_{\mathrm{QSym}_{\mathbf{k}}}\right)}_{=\mathrm{id}_{\underline{H}}\otimes\left(\left(\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}}\otimes\varepsilon_P\right)\circ\beta_{\mathrm{QSym}_{\mathbf{k}}}\right)}\circ\beta_H$$

$$=\left(\mathrm{id}_{\underline{H}}\otimes\underbrace{\left(\left(\mathrm{id}_{\underline{\mathrm{QSym}_{\mathbf{k}}}}\otimes\varepsilon_P\right)\circ\beta_{\mathrm{QSym}_{\mathbf{k}}}\right)}_{\substack{=\mathrm{id}\\ \text{(by Theorem 3.5.7 (b),}\\ \text{applied to }\mathrm{QSym}_{\mathbf{k}}\text{ instead of }H)}}\right)\circ\beta_H$$

$$=\underbrace{\left(\mathrm{id}_{\underline{H}}\otimes\mathrm{id}\right)}_{=\mathrm{id}}\circ\beta_H=\beta_H.$$

Thus, we know that $(\beta_H\otimes\mathrm{id})\circ\beta_H$ and $(\mathrm{id}\otimes\Delta'_P)\circ\beta_H$ are two graded $\left(\mathbf{k},\underline{H}\otimes\underline{\mathrm{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphisms $\Xi : H\to\underline{H}\otimes\underline{\mathrm{QSym}_{\mathbf{k}}}\otimes\mathrm{QSym}_{\mathbf{k}}$ for which the diagram (3.32) is commutative (since the diagrams (3.33) and (3.34) are commutative). But we have shown before that any two such homomorphisms must be identical. Thus, we conclude that $(\beta_H\otimes\mathrm{id})\circ\beta_H=(\mathrm{id}\otimes\Delta'_P)\circ\beta_H$. This completes the proof of Theorem 3.5.7 **(c)**.

(d) Consider the map Ξ defined in our above proof of Theorem 3.5.7 **(a)**. We have shown that $\Xi=\beta_H$.

Assume that H is cocommutative. Corollary 3.3.12 **(d)** then shows that $\Xi(H)$ is a subset of the subring $\underline{A}\otimes\Lambda_{\mathbf{k}}$ of $\underline{A}\otimes\mathrm{QSym}_{\mathbf{k}}$. In other words, $\beta_H(H)$ is a subset of the subring $\underline{H}\otimes\Lambda_{\mathbf{k}}$ of $\underline{H}\otimes\mathrm{QSym}_{\mathbf{k}}$ (since $\Xi=\beta_H$ and $A=H$). This proves Theorem 3.5.7 **(d)**.

(Alternatively, we could prove **(d)** by checking that for any element h of a commutative cocommutative Hopf algebra H, the element $\xi_\alpha(h)$ of H depends only on the result of sorting α, rather than on the composition α itself.) □

Proof of Proposition 3.4.4. Let τ be the twist map $\tau_{\mathrm{QSym}_{\mathbf{k}},\mathrm{QSym}_{\mathbf{k}}}:\mathrm{QSym}_{\mathbf{k}}\otimes\mathrm{QSym}_{\mathbf{k}}\to$

$\mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$. This twist map clearly satisfies $\tau \circ \tau = \mathrm{id}$. Hence, $\tau \circ \underbrace{\Delta'_P}_{=\tau \circ \Delta_P} = \underbrace{\tau \circ \tau}_{=\mathrm{id}} \circ \Delta_P = \Delta_P$.

Theorem 3.5.7 **(c)** (applied to $H = \mathrm{QSym}_{\mathbf{k}}$) shows that the diagram

$$\begin{array}{ccc} \mathrm{QSym} & \xrightarrow{\beta_{\mathrm{QSym}}} & \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \\ {\scriptstyle \beta_{\mathrm{QSym}}} \downarrow & & \downarrow {\scriptstyle \beta_{\mathrm{QSym}} \otimes \mathrm{id}} \\ \underline{\mathrm{QSym}} \otimes \mathrm{QSym} & \xrightarrow[\mathrm{id} \otimes \Delta'_P]{} & \underline{\mathrm{QSym}} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \end{array}$$

is commutative. In other words, $(\mathrm{id} \otimes \Delta'_P) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}} = \left(\beta_{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{id}\right) \circ \beta_{\mathrm{QSym}_{\mathbf{k}}}$. Since $\beta_{\mathrm{QSym}_{\mathbf{k}}} = \Delta'_P$ (by Proposition 3.5.3), this rewrites as $(\mathrm{id} \otimes \Delta'_P) \circ \Delta'_P = (\Delta'_P \otimes \mathrm{id}) \circ \Delta'_P$. Thus, the operation Δ'_P is coassociative. Therefore, the operation $\Delta_P = \tau \circ \Delta'_P$ is also coassociative (because the coassociativity of a map $H \to H \otimes H$ does not change if we compose this map with the twist map $\tau_{H,H} : H \otimes H \to H \otimes H$). It is furthermore easy to see that the operation ε_P is counital with respect to the operation Δ_P (see, for example, [Haz08, §11.45]). Hence, the $\mathbf{k}$-module $\mathrm{QSym}_{\mathbf{k}}$, equipped with the comultiplication Δ_P and the counit ε_P, is a $\mathbf{k}$-coalgebra. Our goal is to prove that it is a $\mathbf{k}$-bialgebra. Hence, it remains to show that Δ_P and ε_P are $\mathbf{k}$-algebra homomorphisms. For ε_P, this is again obvious (indeed, ε_P sends any $f \in \mathrm{QSym}_{\mathbf{k}}$ to $f(1, 0, 0, 0, \ldots)$). It remains to prove that Δ_P is a $\mathbf{k}$-algebra homomorphism.

The map $\beta_{\mathrm{QSym}_{\mathbf{k}}}$ is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \underline{\mathrm{QSym}_{\mathbf{k}}} \otimes \mathrm{QSym}_{\mathbf{k}}$ (by Theorem 3.5.7 **(a)**, applied to $H = \mathrm{QSym}_{\mathbf{k}}$). In other words, the map Δ'_P is a $\mathbf{k}$-algebra homomorphism $\mathrm{QSym}_{\mathbf{k}} \to \mathrm{QSym}_{\mathbf{k}} \otimes \mathrm{QSym}_{\mathbf{k}}$ (since $\beta_{\mathrm{QSym}_{\mathbf{k}}} = \Delta'_P$, and since $\underline{\mathrm{QSym}_{\mathbf{k}}} = \mathrm{QSym}_{\mathbf{k}}$ as $\mathbf{k}$-algebras). Thus, $\Delta_P = \tau \circ \Delta'_P$ is also a $\mathbf{k}$-algebra homomorphism (since both τ and Δ'_P are $\mathbf{k}$-algebra homomorphisms). This completes the proof of Proposition 3.4.4. $\square$

3.6 Remark on antipodes

We have hitherto not really used the antipode of a Hopf algebra; thus, we could just as well have replaced the words "Hopf algebra" by "bialgebra" throughout the entire preceding text[23]. Let us now connect the preceding results with antipodes.

The antipode of any Hopf algebra H will be denoted by S_H.

Proposition 3.6.1. Let $\mathbf{k}$ be a commutative ring. Let A be a commutative $\mathbf{k}$-algebra. Let H be a $\mathbf{k}$-Hopf algebra. Let G be an A-Hopf algebra. Then, every $\mathbf{k}$-algebra homomorphism $f : H \to G$ which is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism must also satisfy $f \circ S_H = S_G \circ f$.

Proof of Proposition 3.6.1. We know that H is a $\mathbf{k}$-Hopf algebra. Thus, $\underline{A} \otimes H$ is an A-Hopf algebra. Its definition by extending scalars yields that its antipode is given by $S_{\underline{A} \otimes H} = \mathrm{id}_A \otimes S_H$.

Let $f : H \to G$ be a $\mathbf{k}$-algebra homomorphism which is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism. Then, $f^\sharp : \underline{A} \otimes H \to G$ is an A-coalgebra homomorphism (since f is a $(\mathbf{k}, \underline{A})$-coalgebra homomorphism) and an A-algebra homomorphism (by Proposition 3.3.4). Hence, $f^\sharp$ is an A-bialgebra homomorphism, thus an A-Hopf algebra homomorphism (since every A-bialgebra homomorphism between two A-Hopf algebras is an A-Hopf algebra homomorphism). Thus, $f^\sharp$ commutes with the antipodes, i.e., satisfies $f^\sharp \circ S_{\underline{A} \otimes H} = S_G \circ f^\sharp$.

Now, let ι be the canonical $\mathbf{k}$-module homomorphism $H \to \underline{A} \otimes H$, $h \mapsto 1 \otimes h$. Then, $(\mathrm{id}_A \otimes S_H) \circ \iota = \iota \circ S_H$. On the other hand, $f^\sharp \circ \iota = f$ (this is easy to check). Thus,

$$\begin{aligned} \underbrace{f}_{= f^\sharp \circ \iota} \circ S_H &= f^\sharp \circ \underbrace{\iota \circ S_H}_{= (\mathrm{id}_A \otimes S_H) \circ \iota} = f^\sharp \circ \underbrace{(\mathrm{id}_A \otimes S_H)}_{= S_{\underline{A} \otimes H}} \circ \iota = \underbrace{f^\sharp \circ S_{\underline{A} \otimes H}}_{= S_G \circ f^\sharp} \circ \iota \\ &= S_G \circ \underbrace{f^\sharp \circ \iota}_{= f} = S_G \circ f. \end{aligned}$$

This proves Proposition 3.6.1. □

[23] That said, we would not have gained anything this way, because any connected graded $\mathbf{k}$-bialgebra is a $\mathbf{k}$-Hopf algebra (see [GriRei15, Proposition 1.36]).

Corollary 3.6.2. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. Define a map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ as in Definition 3.5.1. Then,

$$\beta_H \circ S_H = \left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \beta_H.$$

Proof of Corollary 3.6.2. Theorem 3.5.7 **(a)** shows that the map β_H is a $\mathbf{k}$-algebra homomorphism $H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ and a graded $(\mathbf{k}, \underline{H})$-coalgebra homomorphism. Thus, Proposition 3.6.1 (applied to $A = H$, $G = \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ and $f = \beta_H$) shows that $\beta_H \circ S_H = S_{\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}} \circ \beta_H$.

But the H-Hopf algebra $\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ is defined by extension of scalars; thus, its antipode is given by $S_{\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}} = \operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}$. Hence,

$$\beta_H \circ S_H = \underbrace{S_{\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}}}_{= \operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}} \circ \beta_H = \left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \beta_H.$$

This proves Corollary 3.6.2. □

Corollary 3.6.3. Let $\mathbf{k}$ be a commutative ring. Let H be a commutative connected graded $\mathbf{k}$-Hopf algebra. Define a map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ as in Definition 3.5.1. Then,

$$S_H = \left(\operatorname{id}_H \otimes \left(\varepsilon_P \circ S_{\operatorname{QSym}_{\mathbf{k}}}\right)\right) \circ \beta_H.$$

Proof of Corollary 3.6.3. We have

$$\begin{aligned} \underbrace{\left(\operatorname{id}_H \otimes \left(\varepsilon_P \circ S_{\operatorname{QSym}_{\mathbf{k}}}\right)\right)}_{= (\operatorname{id}_H \otimes \varepsilon_P) \circ \left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right)} \circ \beta_H &= (\operatorname{id}_H \otimes \varepsilon_P) \circ \underbrace{\left(\operatorname{id}_H \otimes S_{\operatorname{QSym}_{\mathbf{k}}}\right) \circ \beta_H}_{\substack{= \beta_H \circ S_H \\ \text{(by Corollary 3.6.2)}}} \\ &= \underbrace{(\operatorname{id}_H \otimes \varepsilon_P) \circ \beta_H}_{\substack{= \operatorname{id} \\ \text{(by Theorem 3.5.7 (b))}}} \circ S_H = S_H, \end{aligned}$$

and thus Corollary 3.6.3 is proven. □

Remark 3.6.4. What I find remarkable about Corollary 3.6.3 is that it provides a formula for the antipode S_H of H in terms of β_H and $\operatorname{QSym}_{\mathbf{k}}$. Thus, in order to understand the antipode of H, it suffices to study the map β_H and the antipode of $\operatorname{QSym}_{\mathbf{k}}$ well enough.

Similar claims can be made about other endomorphisms of H, such as the Dynkin idempotent or the Eulerian idempotent (when $\mathbf{k}$ is a $\mathbb{Q}$-algebra). Better yet, we can regard the map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ as an "embedding" of the $\mathbf{k}$-Hopf algebra H into the H-Hopf algebra $\underline{H} \otimes \operatorname{QSym}_{\mathbf{k}} \cong \operatorname{QSym}_H$. Here, I am using the word "embedding" in scare quotes, since this map is not a Hopf algebra homomorphism (its domain and its target are Hopf algebras over different base rings); nevertheless, the map β_H is injective (by Theorem 3.5.7 **(b)**), and the corresponding map $(\beta_H)^\sharp : \underline{H} \otimes H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ (sending every $a \otimes h$ to $a\beta_H(h)$) is a graded H-Hopf algebra homomorphism (because it is graded, an H-algebra homomorphism and an H-coalgebra homomorphism); this shows that β_H commutes with various maps defined canonically in terms of a commutative connected graded Hopf algebra. It appears possible to use this for proving identities in commutative connected graded Hopf algebra.

3.7 Questions

Let me finish with some open-ended questions, which probably are not particularly insightful, but (in my opinion) rather natural.

Question 3.7.1. It is well-known (see, e.g., [GriRei15, §5.3]) that the graded Hopf-algebraic dual of the graded Hopf algebra QSym is a graded Hopf algebra NSym. The second comultiplication Δ_P and the second counit ε_P on QSym dualize to a second multiplication m_P and a second unit u_P on NSym, albeit u_P does not really live inside NSym (in fact, it lives in the completion of NSym with respect to its grading). We denote the "almost-$\mathbf{k}$-bialgebra" $(\operatorname{NSym}, m_P, u_P, \Delta, \varepsilon)$ ("almost" because $u_P \notin \operatorname{NSym}$) by NSym_2.

We can more or less dualize Theorem 3.5.7. As a result, instead of a QSym_2-comodule structure on every commutative graded connected Hopf algebra H, we obtain an NSym_2-module structure on every cocommutative graded connected Hopf algebra H. This structure is rather well-known: (I believe) it has $H_\alpha \in \operatorname{NSym}_2$ act as the convolution product $\pi_{a_1} \bigstar \pi_{a_2} \bigstar \cdots \bigstar \pi_{a_k} \in \operatorname{End} H$ for every composition $\alpha = (a_1, a_2, \ldots, a_k)$ (where $\bigstar$ denotes the convolution product in $\operatorname{End} H$). This should be somewhere in the papers by Patras and Reutenauer on descent algebras; it is essentially the way to transfer information from **the** descent algebra NSym_2 to **a** descent algebra $(\operatorname{End}_{\text{graded}} H, \circ)$ of a cocommutative graded connected Hopf algebra H.

Is it possible to prove that this works using universal properties like I have done above for Theorem 3.5.7? (Just saying "dualize Theorem 3.5.7" is not enough, because dualization over arbitrary commutative rings is a heuristic, not a proof strategy; there does not seem to be a general theorem stating that "the dual of a correct result is correct", at least when the result has assumptions about gradedness and similar things.)

If the answer is positive, can we use this to give a slick proof of Solomon's Mackey formula? (I am not saying that there is need for slick proofs of this formula – not after those by Gessel and Bidigare –, but it would be interesting to have a new one. I am thinking of letting both NSym_2 and the symmetric groups act on the tensor algebra $T(V)$ of an infinite-dimensional free $\mathbf{k}$-module V; one then only needs to check that the actions match.)

Note that if u and v are two elements of NSym_2, then the action of the NSym-product uv (not the NSym_2-product!) on H is the convolution of the actions of u and v. So the action map $\operatorname{NSym}_2 \to \operatorname{End} H$ takes the multiplication of NSym_2 to composition, and the multiplication of NSym to convolution.

Question 3.7.2. In Question 3.7.1, we found a $\mathbf{k}$-algebra homomorphism $\operatorname{NSym}_2 \to (\operatorname{End} H, \circ)$ for every cocommutative connected graded Hopf algebra H. This is functorial in H, and so is really a map from the constant functor NSym_2

to the functor

$$\{\text{cocommutative connected graded Hopf algebras}\} \to \{\mathbf{k}\text{-modules}\}, \qquad H \mapsto \operatorname{End} H.$$

Does the image of this action span (up to topology) the whole functor? I guess I am badly abusing categorical language here, so let me restate the question in simpler terms: If a natural endomorphism of the $\mathbf{k}$-module H is given for every cocommutative connected graded Hopf algebra H, and this endomorphism is known to annihilate all homogeneous components H_m for sufficiently high m (this is what I mean by "up to topology"), then must there be an element v of NSym_2 such that this endomorphism is the action of v ?

If the answer is "No", then does it change if we require the endomorphism of H to be graded? If we require $\mathbf{k}$ to be a field of characteristic 0 ?

What if we restrict ourselves to commutative cocommutative connected graded Hopf algebras? At least then, if $\mathbf{k}$ is a finite field $\mathbb{F}_q$, there are more natural endomorphisms of H, such as the Frobenius morphism $x \mapsto x^q$ and its powers. One can then ask for the graded endomorphisms of H, but actually it is also interesting to see how the full $\mathbf{k}$-algebra of natural endomorphisms looks like (how do the endomorphisms coming from NSym_2 interact with the Frobenii?). And what about characteristic 0 here?

Question 3.7.3. What are the natural endomorphisms of connected graded Hopf algebras, without any cocommutativity or commutativity assumption? I suspect that they will form a connected graded Hopf algebra, with two multiplications (one for composition and the other for convolution), but now with a basis indexed by "mopiscotions" (i.e., pairs (α, σ) of a composition α and a permutation $\sigma \in \mathfrak{S}_{\ell(\alpha)}$). Is this a known combinatorial Hopf algebra?

Question 3.7.4. Can we extend the map $\beta_H : H \to \underline{H} \otimes \operatorname{QSym}_{\mathbf{k}}$ to a map $H \to \underline{H} \otimes U$ for some combinatorial Hopf algebra U bigger than $\operatorname{QSym}_{\mathbf{k}}$? What if we require some additional (say, dendriform?) structure on H ? Can we achieve $U = \operatorname{NCQSym}_{\mathbf{k}}$ or $U = \operatorname{DoublePosets}_{\mathbf{k}}$ (the combinatorial Hopf algebra of double posets, which is defined for $\mathbf{k} = \mathbb{Z}$ and denoted by $\mathbb{Z}\mathbf{D}$ in [MalReu09], and can be similarly defined over any $\mathbf{k}$) ? (I am singling out these two Hopf algebras because they have fairly nice internal comultiplications. Actually, the internal comultiplication of $\operatorname{NCQSym}_{\mathbf{k}}$ is the key to Bidigare's proof of Solomon's Mackey formula [Schock04, §2], and I feel it will tell us more if we listen to it.)

Aguiar suggests that the map $H \to \underline{H} \otimes \operatorname{NCQSym}_{\mathbf{k}}$ I am looking for is the dual of his action of the Tits algebra on Hopf monoids [Aguiar13, Proposition 88].

Question 3.7.5. Do we gain anything from applying Corollary 3.6.2 to $H = \operatorname{QSym}_{\mathbf{k}}$ (thus getting a statement about Δ'_P) ? Probably not much for Δ'_P that the Marne-la-Vallée people haven't already found using virtual alphabets (the dual version is the statement that $S(a * b) = a * S(b)$ for all $a, b \in \operatorname{NSym}_{\mathbf{k}}$, where $*$ is the internal product).

Question 3.7.6. From Theorem 3.5.7 **(a)** and Proposition 3.5.3, we can conclude that Δ'_P is a $\left(\mathbf{k}, \underline{\operatorname{QSym}_{\mathbf{k}}}\right)$-coalgebra homomorphism. If I am not mistaken, this can be rewritten as the equality

$$(AB) * G = \sum_{(G)} \left(A * G_{(1)}\right) \left(B * G_{(2)}\right) \qquad \text{(using Sweedler's notation)}$$

for any three elements A, B and G of NSym. This is the famous splitting formula.

Now, it is known from [DHNT08, §7] that the same splitting formula holds when A and B are elements of FQSym (into which NSym is known to inject), as long as G is still an element of NSym (actually, it can be an element of the bigger Patras-Reutenauer algebra, but let us settle for NSym so far). Can this be proven in a similar vein? How much of the Marne-la-Vallée theory follows from Theorem 3.2.1?

Chapter 4

A note on non-broken-circuit sets and the chromatic polynomial

Abstract

We demonstrate several generalizations of a classical formula for the chromatic polynomial of a graph – namely, of Whitney's theorem. One generalization allows the exclusion of only some broken circuits, whereas another weighs these broken circuits with weight monomials instead of excluding them; yet another extends the theorem to the chromatic symmetric functions, and yet another replaces the graph by a matroid. Most of these generalizations can be combined (albeit not all of them: matroids do not seem to have chromatic symmetric functions).

***

The purpose of this note is to demonstrate several generalizations of Whitney's theorem [BlaSag86] – a classical formula for the chromatic polynomial of a graph. The directions in which we generalize this formula are the following:

- Instead of summing over the sets which contain no broken circuits, we can sum over the sets which are "$\mathfrak{K}$-free" (i.e., contain no element of $\mathfrak{K}$ as a subset), where $\mathfrak{K}$ is some fixed set of broken circuits (in particular, $\mathfrak{K}$ can be $\varnothing$, yielding another well-known formula for the chromatic polynomial).

- Even more generally, instead of summing over $\mathfrak{K}$-free subsets, we can make a weighted sum over all subsets, where the weight depends on the broken circuits contained in the subset.

- Analogous (and more general) results hold for chromatic symmetric functions.
- Analogous (and more general) results hold for matroids instead of graphs.

Note that, to my knowledge, the last two generalizations cannot be combined: Unlike graphs, matroids do not seem to have a well-defined notion of a chromatic symmetric function.

We shall explore these generalizations in the note below. We shall also use them to prove an apparently new formula for the chromatic polynomial of a graph obtained from a transitive digraph by forgetting the orientations of the edges (Proposition 4.4.2). This latter formula was suggested to me as a conjecture by Alexander Postnikov, during a discussion on hyperplane arrangements on a space with a bilinear form; it is this formula which gave rise to this whole note. The subject of hyperplane arrangements, however, will not be breached here.

Acknowledgments

I thank Alexander Postnikov and Richard Stanley for discussions on hyperplane arrangements that led to the results in this note.

4.1 Definitions and a main result

4.1.1 Graphs and colorings

This note will be concerned with finite graphs. While some results of this note can be generalized to matroids, we shall not discuss this generalization here. Let us start with the definition of a graph that we shall be using:

Definition 4.1.1. **(a)** If V is any set, then $\binom{V}{2}$ will denote the set of all 2-element subsets of V. In other words, if V is any set, then we set

$$\binom{V}{2} = \{S \in \mathcal{P}(V) \mid |S| = 2\}$$
$$= \{\{s,t\} \mid s \in V,\ t \in V,\ s \neq t\}$$

(where $\mathcal{P}(V)$ denotes the powerset of V).

(b) A *graph* means a pair (V, E), where V is a set, and where E is a subset of $\binom{V}{2}$. A graph (V, E) is said to be *finite* if the set V is finite. If $G = (V, E)$ is a graph, then the elements of V are called the *vertices* of the graph G, while the elements of E are called the *edges* of the graph G. If e is an edge of a graph G, then the two elements of e are called the *endpoints* of the edge e. If $e = \{s,t\}$ is an edge of a graph G, then we say that the edge e *connects the vertices s and t* of G.

Comparing our definition of a graph with some of the other definitions used in the literature, we thus observe that our graphs are undirected (i.e., their edges are sets, not pairs), loopless (i.e., the two endpoints of an edge must always be distinct), edge-unlabelled (i.e., their edges are just 2-element sets of vertices, rather than objects with "their own identity"), and do not have multiple edges (or, more precisely, there is no notion of several edges connecting two vertices, since the edges form a set, nor a multiset, and do not have labels).

Definition 4.1.2. Let $G = (V, E)$ be a graph. Let X be a set.

(a) An *X-coloring* of G is defined to mean a map $V \to X$.

(b) An X-coloring f of G is said to be *proper* if every edge $\{s,t\} \in E$ satisfies $f(s) \neq f(t)$.

4.1.2 Symmetric functions

We shall now briefly introduce the notion of symmetric functions. We shall not use any nontrivial results about symmetric functions; we will merely need some notations.[1]

In the following, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$. Also, $\mathbb{N}_+$ shall mean the set $\{1, 2, 3, \ldots\}$.

A *partition* will mean a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathbb{N}^\infty$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and such that all sufficiently high integers $i \geq 1$ satisfy $\lambda_i = 0$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition, and if a positive integer n is such that all integers $i \geq n$ satisfy $\lambda_i = 0$, then we shall identify the partition λ with the finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. Thus, for example, the sequences $(3, 1)$ and $(3, 1, 0)$ and the partition $(3, 1, 0, 0, 0, \ldots)$ are all identified. Every weakly decreasing finite list of positive integers thus is identified with a unique partition.

Let $\mathbf{k}$ be a commutative ring with unity. We shall keep $\mathbf{k}$ fixed throughout the paper. The reader will not be missing out on anything if she assumes that $\mathbf{k} = \mathbb{Z}$.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. It is a topological $\mathbf{k}$-algebra[2]. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if there exists an $N \in \mathbb{N}$ such that every monomial of degree $> N$ appears with coefficient 0 in P. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *symmetric* if and only if P is invariant under any permutation of the indeterminates. We let Λ be the subset of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ consisting of all symmetric bounded-degree power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This subset Λ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and is called the $\mathbf{k}$-*algebra of symmetric functions* over $\mathbf{k}$.

We shall now define the few families of symmetric functions that we will be con-

[1] For an introduction to symmetric functions, see any of [Stan99, Chapter 7], [Martin15, Chapter 9] and [GriRei15, Chapter 2] (and a variety of other texts).

[2] See [GriRei15, Section 2.6] or Section 1.2 of this thesis for the definition of its topology. This topology makes sure that a sequence $(P_n)_{n\in\mathbb{N}}$ of power series converges to some power series P if and only if, for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the } \mathfrak{m}\text{-coefficient of } P_n) = (\text{the } \mathfrak{m}\text{-coefficient of } P)$$

(where the meaning of "sufficiently high" can depend on the $\mathfrak{m}$).

cerned with in this note. The first are the *power-sum symmetric functions*:

Definition 4.1.3. Let n be a positive integer. We define a power series $p_n \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_n = x_1^n + x_2^n + x_3^n + \cdots = \sum_{j \geq 1} x_j^n. \tag{4.1}$$

This power series p_n lies in Λ, and is called the *n-th power-sum symmetric function*.

We also set $p_0 = 1 \in \Lambda$. Thus, p_n is defined not only for all positive integers n, but also for all $n \in \mathbb{N}$.

Definition 4.1.4. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a partition. We define a power series $p_\lambda \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_\lambda = \prod_{i \geq 1} p_{\lambda_i}.$$

This is well-defined, because the infinite product $\prod_{i \geq 1} p_{\lambda_i}$ converges (indeed, all but finitely many of its factors are 1 (because every sufficiently high integer i satisfies $\lambda_i = 0$ and thus $p_{\lambda_i} = p_0 = 1$)).

We notice that every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (written as a finite list of nonnegative integers) satisfies

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}. \tag{4.2}$$

4.1.3 Chromatic symmetric functions

The next symmetric functions we introduce are the actual subject of this note; they are the *chromatic symmetric functions* and originate in [Stanle95, Definition 2.1]:

Definition 4.1.5. Let $G = (V, E)$ be a finite graph. For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$, we let $\mathbf{x}_f$ denote the monomial $\prod_{v \in V} x_{f(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$. We define a power series $X_G \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_G = \sum_{\substack{f:V \to \mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f .$$

This power series X_G is called the *chromatic symmetric function* of G.

We have $X_G \in \Lambda$ for every finite graph $G = (V, E)$; this will follow from Theorem 4.1.8 further below (but is also rather obvious).

We notice that X_G is denoted by $\Psi[G]$ in [GriRei15, §7.3.3].

4.1.4 Connected components

We shall now briefly recall the notion of connected components of a graph.

Definition 4.1.6. Let $G = (V, E)$ be a finite graph. Let u and v be two elements of V (that is, two vertices of G). A *walk* from u to v in G will mean a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = u$ and $w_k = v$ and

$$(\{w_i, w_{i+1}\} \in E \qquad \text{for every } i \in \{0, 1, \ldots, k-1\}) .$$

We say that u and v are *connected (in G)* if there exists a walk from u to v in G.

Definition 4.1.7. Let $G = (V, E)$ be a graph.

(a) We define a binary relation $\sim_G$ (written infix) on the set V as follows: Given $u \in V$ and $v \in V$, we set $u \sim_G v$ if and only if u and v are connected (in G). It is well-known that this relation $\sim_G$ is an equivalence relation. The $\sim_G$-equivalence classes are called the *connected components* of G.

(b) Assume that the graph G is finite. We let $\lambda(G)$ denote the list of the sizes of all connected components of G, in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view $\lambda(G)$ as a partition (since $\lambda(G)$ is a weakly decreasing finite list of positive integers).

Now, we can state a formula for chromatic symmetric functions:

Theorem 4.1.8. Let $G = (V, E)$ be a finite graph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 4.1.7 **(b)**.)

This theorem is not new; it appears, e.g., in [Stanle95, Theorem 2.5]. We shall show a far-reaching generalization of it (Theorem 4.1.11) soon.

4.1.5 Circuits and broken circuits

Let us now define the notions of cycles and circuits of a graph:

Definition 4.1.9. Let $G = (V, E)$ be a graph. A *cycle* of G denotes a list $(v_1, v_2, \ldots, v_{m+1})$ of elements of V with the following properties:

- We have $m > 1$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

If $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G, then the set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ is called a *circuit* of G.

Definition 4.1.10. Let $G = (V, E)$ be a graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every edge e of G, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C). Of course, the notion of a broken circuit of G depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

Thus, if G is a graph with a labeling function, then any circuit C of G gives rise to a broken circuit provided that among the edges in C, only one attains the maximum label. (If more than one of the edges of C attains the maximum label, then C does not give rise to a broken circuit.) Notice that two different circuits may give rise to one and the same broken circuit.

Theorem 4.1.11. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 4.1.7 **(b)**.)

Before we come to the proof of this result, let us explore some of its particular cases. First, a definition is in order:

Definition 4.1.12. Let E be a set. Let $\mathfrak{K}$ be a subset of the powerset of E (that is, a set of subsets of E). A subset F of E is said to be $\mathfrak{K}$-*free* if F contains no $K \in \mathfrak{K}$ as a subset. (For instance, if $\mathfrak{K} = \varnothing$, then every subset F of E is $\mathfrak{K}$-free.)

Corollary 4.1.13. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 4.1.14. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

Corollary 4.1.14 appears in [Stanle95, Theorem 2.9], at least in the particular case in which ℓ is supposed to be injective.

Let us now see how Theorem 4.1.8, Corollary 4.1.13 and Corollary 4.1.14 can be derived from Theorem 4.1.11:

Proof of Corollary 4.1.13 using Theorem 4.1.11. For every subset F of E, we have

$$\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free}; \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{4.3}$$

(because if F is $\mathfrak{K}$-free, then the product $\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0$ is empty and thus equals 1; otherwise, the product $\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0$ contains at least one factor and thus equals 0). Now, Theorem

4.1.11 (applied to 0 instead of a_K) yields

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 \right)}_{\substack{= \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free}; \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \\ \text{(by (4.3))}}} p_{\lambda(V,F)}$$

$$= \sum_{F \subseteq E} (-1)^{|F|} \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free}; \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} p_{\lambda(V,F)} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}.$$

This proves Corollary 4.1.13. □

Proof of Corollary 4.1.14 using Corollary 4.1.13. Corollary 4.1.14 follows from Corollary 4.1.13 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of G. □

Proof of Theorem 4.1.8 using Theorem 4.1.11. Let X be the totally ordered set $\{1\}$, and let $\ell : E \to X$ be the only possible map. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. For every $F \subseteq E$, the product $\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0$ is empty (since K is the empty set), and thus equals 1. Now, Theorem 4.1.11 (applied to 0 instead of a_K) yields

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 \right)}_{=1} p_{\lambda(V,F)} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

This proves Theorem 4.1.8. □

4.2 Proof of Theorem 4.1.11

We shall now prepare for the proof of Theorem 4.1.11 with some notations and some lemmas. Our proof will imitate [BlaSag86, proof of Whitney's theorem].

4.2.1 $\operatorname{Eqs} f$ and basic lemmas

Definition 4.2.1. Let V and X be two sets. Let $f : V \to X$ be a map. We let $\operatorname{Eqs} f$ denote the subset

$$\left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\}$$

of $\dbinom{V}{2}$. (This is well-defined, because any two elements s and t of V satisfying $s \neq t$ clearly satisfy $\{s,t\} \in \dbinom{V}{2}$.)

We shall now state some first properties of this notion:

Lemma 4.2.2. Let $G = (V, E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Then, the X-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$.

Proof of Lemma 4.2.2. The set $E \cap \operatorname{Eqs} f$ is precisely the set of edges $\{s,t\}$ of G satisfying $f(s) = f(t)$; meanwhile, the X-coloring f is called proper if and only if no such edges exist. Thus, Lemma 4.2.2 becomes obvious. □

Lemma 4.2.3. Let $G = (V, E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Let C be a circuit of G. Let $e \in C$ be such that $C \setminus \{e\} \subseteq \operatorname{Eqs} f$. Then, $e \in E \cap \operatorname{Eqs} f$.

Proof of Lemma 4.2.3. The set C is a circuit of G. Hence, we can write C in the form

$$C = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$$

for some cycle $(v_1, v_2, \ldots, v_{m+1})$ of G. Consider this cycle $(v_1, v_2, \ldots, v_{m+1})$. According to the definition of a "cycle", the cycle $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V having the following properties:

- We have $m > 1$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

Recall that $e \in C$. We can thus WLOG assume that $e = \{v_m, v_{m+1}\}$ (since otherwise, we can simply relabel the vertices along the cycle $(v_1, v_2, \ldots, v_{m+1})$). Assume this. Since $\{v_m, v_{m+1}\} = e$, we have

$$C \setminus \{e\} = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}\}$$

(since $v_1, v_2, \ldots, v_m$ are distinct, and since $m > 1$ and $v_{m+1} = v_1$). For every $i \in \{1, 2, \ldots, m-1\}$, we have $f(v_i) = f(v_{i+1})$ (since

$$\{v_i, v_{i+1}\} \subseteq \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}\} = C \setminus \{e\} \subseteq \operatorname{Eqs} f$$

). Hence, $f(v_1) = f(v_2) = \cdots = f(v_m)$, so that $f(v_m) = f\left(\underbrace{v_1}_{=v_{m+1}}\right) = f(v_{m+1})$. Thus, $\{v_m, v_{m+1}\} \in \operatorname{Eqs} f$. Thus, $e = \{v_m, v_{m+1}\} \in \operatorname{Eqs} f$. Combined with $e \in E$, this yields $e \in E \cap \operatorname{Eqs} f$. This proves Lemma 4.2.3. □

Lemma 4.2.4. Let (V, B) be a finite graph. Then,

$$\sum_{\substack{f: V \to \mathbb{N}_+; \\ B \subseteq \operatorname{Eqs} f}} \mathbf{x}_f = p_{\lambda(V,B)}$$

(Here, $\mathbf{x}_f$ is defined as in Definition 4.1.5, and the expression $\lambda(V, B)$ is understood according to Definition 4.1.7 **(b)**.)

Proof of Lemma 4.2.4. Let $(C_1, C_2, \ldots, C_k)$ be a list of all connected components of (V, B), ordered such that $|C_1| \geq |C_2| \geq \cdots \geq |C_k|$. [3] Then, $\lambda(V, B) =$

[3] Every connected component of (V, B) should appear exactly once in this list.

$(|C_1|, |C_2|, \ldots, |C_k|)$ (by the definition of $\lambda(V, B)$). Hence, (4.2) (applied to $\lambda(V, B)$ and $|C_i|$ instead of λ and λ_i) shows that

$$p_{\lambda(V,B)} = p_{|C_1|} p_{|C_2|} \cdots p_{|C_k|} = \prod_{i=1}^{k} p_{|C_i|}. \tag{4.4}$$

But for every $i \in \{1, 2, \ldots, k\}$, we have $p_{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}$ (by the definition of $p_{|C_i|}$). Hence, (4.4) becomes

$$p_{\lambda(V,B)} = \prod_{i=1}^{k} \underbrace{p_{|C_i|}}_{= \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}} = \prod_{i=1}^{k} \sum_{s \in \mathbb{N}_+} x_s^{|C_i|} = \sum_{(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|} \tag{4.5}$$

(by the product rule).

The list $(C_1, C_2, \ldots, C_k)$ contains all connected components of (V, B), each exactly once. Thus, $V = \bigsqcup_{i=1}^{k} C_i$.

We now define a map

$$\Phi : (\mathbb{N}_+)^k \to \{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}$$

as follows: Given any $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$, we let $\Phi(s_1, s_2, \ldots, s_k)$ be the map $V \to \mathbb{N}_+$ which sends every $v \in V$ to s_i, where $i \in \{1, 2, \ldots, k\}$ is such that $v \in C_i$. (This is well-defined, because for every $v \in V$, there exists a unique $i \in \{1, 2, \ldots, k\}$ such that $v \in C_i$; this follows from $V = \bigsqcup_{i=1}^{k} C_i$.) This map Φ is well-defined, because for every $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$, the map $\Phi(s_1, s_2, \ldots, s_k)$ actually belongs to $\{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}$ [4].

A moment's thought reveals that the map Φ is injective[5]. Let us now show that the map Φ is surjective.

[4] *Proof.* We just need to check that $B \subseteq \operatorname{Eqs}(\Phi(s_1, s_2, \ldots, s_k))$. But this is easy: For every $(u, v) \in B$, the vertices u and v of (V, B) lie in one and the same connected component C_i of the graph (V, B), and thus (by the definition of $\Phi(s_1, s_2, \ldots, s_k)$) the map $\Phi(s_1, s_2, \ldots, s_k)$ sends both of them to s_i; but this shows that $(u, v) \in \operatorname{Eqs}(\Phi(s_1, s_2, \ldots, s_k))$.

[5] In fact, we can reconstruct $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ from its image $\Phi(s_1, s_2, \ldots, s_k)$, because each s_i is the image of any element of C_i under $\Phi(s_1, s_2, \ldots, s_k)$ (and this allows us to compute s_i, since C_i is nonempty).

In order to show this, we must prove that every map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$ has the form $\Phi(s_1, s_2, \ldots, s_k)$ for some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$. So let us fix a map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$. We must find some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ such that $f = \Phi(s_1, s_2, \ldots, s_k)$.

We have $B \subseteq \operatorname{Eqs} f$. Thus, for every $\{s, t\} \in B$, we have $\{s, t\} \in B \subseteq \operatorname{Eqs} f$ and thus

$$f(s) = f(t). \tag{4.6}$$

Now, if x and y are two elements of V lying in the same connected component of (V, B), then

$$f(x) = f(y) \tag{4.7}$$

[6]. In other words, the map f is constant on each connected component of (V, B). Thus, the map f is constant on C_i for each $i \in \{1, 2, \ldots, k\}$ (since C_i is a connected component of (V, B)). Hence, for each $i \in \{1, 2, \ldots, k\}$, we can define a positive integer $s_i \in \mathbb{N}_+$ to be the image of any element of C_i under f (this is well-defined, because f is constant on C_i and thus the choice of the element does not matter). Define $s_i \in \mathbb{N}_+$ for each $i \in \{1, 2, \ldots, k\}$ this way. Thus, we have defined a k-tuple $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$. Now, $f = \Phi(s_1, s_2, \ldots, s_k)$ (this follows immediately by recalling the definitions of Φ and s_i).

Let us now forget that we fixed f. We thus have shown that for every map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$, there exists some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ such that $f = \Phi(s_1, s_2, \ldots, s_k)$. In other words, the map Φ is surjective. Since Φ is both injective and surjective, we conclude that Φ is a bijection.

---

[6] *Proof of (4.7):* Let x and y be two elements of V lying in the same connected component of (V, B). Then, the vertices x and y are connected by a walk in the graph (V, B) (by the definition of a "connected component"). Let $(v_0, v_1, \ldots, v_j)$ be this walk (regarded as a sequence of vertices); thus, $v_0 = x$ and $v_j = y$. For every $i \in \{0, 1, \ldots, j-1\}$, we have $\{v_i, v_{i+1}\} \in B$ (since $(v_0, v_1, \ldots, v_j)$ is a walk in the graph (V, B)) and thus $f(v_i) = f(v_{i+1})$ (by (4.6), applied to $(s, t) = (v_i, v_{i+1})$). In other words, $f(v_0) = f(v_1) = \cdots = f(v_j)$. Hence, $f(v_0) = f(v_j)$, so that $f\left(\underbrace{x}_{=v_0}\right) = f(v_0) = f\left(\underbrace{v_j}_{=y}\right) = f(y)$, qed.

Moreover, it is straightforward to see that every map $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ satisfies

$$\mathbf{x}_{\Phi(s_1,s_2,\ldots,s_k)} = \prod_{i=1}^{k} x_{s_i}^{|C_i|} \tag{4.8}$$

(by the definitions of $\mathbf{x}_{\Phi(s_1,s_2,\ldots,s_k)}$ and of Φ). Now,

$$\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f = \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \underbrace{\mathbf{x}_{\Phi(s_1,s_2,\ldots,s_k)}}_{\substack{=\prod_{i=1}^{k} x_{s_i}^{|C_i|} \\ \text{(by (4.8))}}}$$

$$\left(\begin{array}{c}\text{here, we have substituted } \Phi(s_1,s_2,\ldots,s_k) \text{ for } f \text{ in the sum,}\\ \text{since the map } \Phi : (\mathbb{N}_+)^k \to \{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}\\ \text{is a bijection}\end{array}\right)$$

$$= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|} = p_{\lambda(V,B)} \qquad \text{(by (4.5))}.$$

This proves Lemma 4.2.4. ∎

Lemma 4.2.5. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let K be a broken circuit of G. Then, $K \neq \varnothing$.

Proof of Lemma 4.2.5. The set K is a broken circuit of G, and thus is a circuit of G with an edge removed (by the definition of a broken circuit). Thus, the set K contains at least 1 edge (since every circuit of G contains at least 2 edges). This proves Lemma 4.2.5. ∎

4.2.2 Alternating sums

We shall now come to less simple lemmas.

Definition 4.2.6. We shall use the so-called *Iverson bracket notation*: If $\mathcal{S}$ is any logical statement, then $[\mathcal{S}]$ shall mean the integer $\begin{cases} 1, & \text{if } \mathcal{S} \text{ is true;} \\ 0, & \text{if } \mathcal{S} \text{ is false} \end{cases}$.

The following lemma is probably the most crucial one in this note:

Lemma 4.2.7. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let Y be any set. Let $f : V \to Y$ be any map. Then,

$$\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = [E \cap \operatorname{Eqs} f = \varnothing] .$$

Proof of Lemma 4.2.7. We WLOG assume that $E \cap \operatorname{Eqs} f \neq \varnothing$ (since otherwise, the claim is obvious[7]). Thus, $[E \cap \operatorname{Eqs} f = \varnothing] = 0$.

Pick any $d \in E \cap \operatorname{Eqs} f$ with maximum $\ell(d)$ (among all $d \in E \cap \operatorname{Eqs} f$). (This is clearly possible, since $E \cap \operatorname{Eqs} f \neq \varnothing$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ as follows:

$$\mathcal{U} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \notin F\} ;$$
$$\mathcal{V} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \in F\} .$$

Thus, we have $\mathcal{P}(E \cap \operatorname{Eqs} f) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}) .$$

[7] In (slightly) more detail: If $E \cap \operatorname{Eqs} f = \varnothing$, then the sum $\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K$ has only one addend (namely, the addend for $B = \varnothing$), and thus simplifies to

$$\underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \underbrace{\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \varnothing}}}_{=\prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}}} a_K = \prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}} a_K = (\text{empty product}) \qquad \left(\begin{array}{c}\text{since no } K \in \mathfrak{K} \text{ satisfies } K = \varnothing \\ \text{(by Lemma 4.2.5)}\end{array}\right)$$
$$= 1 = [E \cap \operatorname{Eqs} f = \varnothing] .$$

This map Φ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ [8]) and a bijection[9]. Moreover, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|} \tag{4.9}$$

[10].

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{4.10}$$

Proof of (4.10): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We must prove the equivalence (4.10). The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$. Hence, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Therefore, the forward implication of the equivalence (4.10) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if $K \subseteq \Phi(B)$, then $K \subseteq B$. So let us assume that $K \subseteq \Phi(B)$.

We want to prove that $K \subseteq B$. Assume the contrary. Thus, $K \not\subseteq B$. We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of G (since $\mathfrak{K}$ is a set of broken circuits of G). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) (because this is how a broken circuit is defined). Consider these C and e. Thus, $K = C \setminus \{e\}$.

The element e is the unique edge in C having maximum label (among the edges in C). Thus, if e' is any edge in C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{4.11}$$

[8]This follows from the fact that $d \in E \cap \operatorname{Eqs} f$.

[9]Its inverse is the map $\Psi : \mathcal{V} \to \mathcal{U}$ defined by ($\Psi(B) = B \setminus \{d\}$ for every $B \in \mathcal{V}$).

[10]*Proof.* Let $B \in \mathcal{U}$. Thus, $d \notin B$ (by the definition of $\mathcal{U}$). Now, $\left|\underbrace{\Phi(B)}_{=B\cup\{d\}}\right| = |B \cup \{d\}| = |B| + 1$ (since $d \notin B$), so that $(-1)^{|\Phi(B)|} = -(-1)^{|B|}$, qed.

But $\underbrace{K}_{\subseteq\Phi(B)=B\cup\{d\}} \setminus \{d\} \subseteq (B\cup\{d\}) \setminus \{d\} \subseteq B$.

If we had $d \notin K$, then we would have $K\setminus\{d\} = K$ and therefore $K = K\setminus\{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq E \cap \operatorname{Eqs} f$ (since $\Phi(B) \in \mathcal{P}(E \cap \operatorname{Eqs} f)$), so that $C \setminus \{e\} \subseteq E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f$. Hence, Lemma 4.2.3 (applied to Y instead of X) shows that $e \in E \cap \operatorname{Eqs} f$. Thus, $\ell(d) \geq \ell(e)$ (since d was defined to be an element of $E \cap \operatorname{Eqs} f$ with maximum $\ell(d)$ among all $d \in E \cap \operatorname{Eqs} f$).

Also, $d \in C$. Since $\ell(d) \geq \ell(e)$, we can therefore apply (4.11) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven. Thus, we have proven the backward implication of the equivalence (4.10); this completes the proof of (4.10).

Now, recall that we have $\mathcal{P}(E \cap \operatorname{Eqs} f) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Hence, the sum $\sum\limits_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K$ can be split into two sums as follows:

$$\begin{aligned}
&\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \\
&= \sum_{B \in \mathcal{U}} \underbrace{(-1)^{|B|}}_{\substack{= -(-1)^{|\Phi(B)|} \\ \text{(by (4.9))}}} \underbrace{\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}}}_{\substack{= \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} \\ \text{(because of the equivalence (4.10))}}} a_K + \underbrace{\sum_{B \in \mathcal{V}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K}_{\substack{= \sum\limits_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\ \text{(here, we have substituted } \Phi(B) \text{ for } B \text{ in the sum,} \\ \text{since the map } \Phi: \mathcal{U} \to \mathcal{V} \text{ is a bijection)}}} \\
&= \sum_{B \in \mathcal{U}} \left(-(-1)^{|\Phi(B)|}\right) \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K + \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\
&= -\sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K + \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\
&= 0 = [E \cap \operatorname{Eqs} f = \varnothing] \qquad (\text{since } [E \cap \operatorname{Eqs} f = \varnothing] = 0). \qquad (4.12)
\end{aligned}$$

This proves Lemma 4.2.7. □

We now finally proceed to the proof of Theorem 4.1.11:

Proof of Theorem 4.1.11. The definition of X_G shows that

$$X_G = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f$$

$$= \sum_{f:V\to\mathbb{N}_+} \left[\underbrace{f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ \iff (E\cap \operatorname{Eqs} f = \varnothing) \\ (\text{by Lemma 4.2.2, applied to } \mathbb{N}_+ \text{ instead of } X)}} \right] \mathbf{x}_f$$

$$= \sum_{f:V\to\mathbb{N}_+} \underbrace{[E\cap \operatorname{Eqs} f = \varnothing]}_{\substack{= \sum_{B\subseteq E\cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{K\in\mathfrak{K};\ K\subseteq B} a_K \\ (\text{by Lemma 4.2.7, applied to } Y=\mathbb{N}_+)}} \mathbf{x}_f$$

$$= \sum_{f:V\to\mathbb{N}_+} \underbrace{\sum_{B\subseteq E\cap \operatorname{Eqs} f}}_{= \sum_{B\subseteq E;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f = \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{= \sum_{B\subseteq E} \sum_{f:V\to\mathbb{N}_+;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f$$

$$= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f = \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f}_{\substack{= p_{\lambda(V,B)} \\ (\text{by Lemma 4.2.4})}}$$

$$= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) p_{\lambda(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) p_{\lambda(V,F)}$$

(here, we have renamed the summation index B as F). This proves Theorem 4.1.11. $\square$

Thus, Theorem 4.1.11 is proven; as we know, this entails the correctness of Theorem 4.1.8, Corollary 4.1.13 and Corollary 4.1.14.

4.3 The chromatic polynomial

4.3.1 Definition

We have so far studied the chromatic symmetric function. We shall now apply the above results to the chromatic polynomial. The definition of the chromatic polynomial rests upon the following fact:

Theorem 4.3.1. Let $G = (V, E)$ be a finite graph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

Definition 4.3.2. Let $G = (V, E)$ be a finite graph. Theorem 4.3.1 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)$. This polynomial P is called the *chromatic polynomial* of G, and will be denoted by χ_G.

We shall later prove Theorem 4.3.1 (as a consequence of something stronger that we show). First, we shall state some formulas for the chromatic polynomial which are analogues of results proven before for the chromatic symmetric function.

4.3.2 Formulas for χ_G

Before we state several formulas for χ_G, we need to introduce one more notation:

Definition 4.3.3. Let G be a finite graph. We let $\operatorname{conn} G$ denote the number of connected components of G.

The following results are analogues of Theorem 4.1.8, Theorem 4.1.11, Corollary 4.1.13 and Corollary 4.1.14, respectively:

Theorem 4.3.4. Let $G = (V, E)$ be a finite graph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 4.3.3.)

Theorem 4.3.5. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 4.3.3.)

Corollary 4.3.6. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

Corollary 4.3.7. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

4.3.3 Proofs

There are two approaches to these results: One is to derive them similarly to how we derived the analogous results about X_G; the other is to derive them from the latter. We shall take the first approach, since it yields a proof of the classical Theorem 4.3.1 "for free". We begin with an analogue of Lemma 4.2.4:

Lemma 4.3.8. Let (V, B) be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} 1 = q^{\operatorname{conn}(V,B)}.$$

(Here, the expression $\operatorname{conn}(V, B)$ is understood according to Definition 4.1.7 **(b)**.)

One way to prove Lemma 4.3.8 is to evaluate the equality given by Lemma 4.2.4 at $x_k = \begin{cases} 1, & \text{if } k \leq q \\ 0, & \text{if } k > q \end{cases}$. Another proof can be obtained by mimicking our proof of Lemma 4.2.4:

Proof of Lemma 4.3.8. Define $(C_1, C_2, \ldots, C_k)$ as in the proof of Lemma 4.2.4. Thus, $\operatorname{conn}(V, B) = k$. Define a map Φ as in the proof of Lemma 4.2.4, but with $\mathbb{N}_+$ replaced by $\{1, 2, \ldots, q\}$. Then,

$$\Phi : \{1, 2, \ldots, q\}^k \to \{f : V \to \{1, 2, \ldots, q\} \mid B \subseteq \operatorname{Eqs} f\}$$

is a bijection[11]. Now,

$$\begin{aligned}
&\sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq\operatorname{Eqs}f}} 1 \\
&= \sum_{(s_1,s_2,\ldots,s_k)\in\{1,2,\ldots,q\}^k} 1 \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \Phi(s_1,s_2,\ldots,s_k) \text{ for } f \text{ in the sum,}\\ \text{since the map } \Phi:\{1,2,\ldots,q\}^k\to\{f:V\to\{1,2,\ldots,q\}\ \mid\ B\subseteq\operatorname{Eqs}f\}\\ \text{is a bijection}\end{array}\right) \\
&= \left(\text{the number of all } (s_1,s_2,\ldots,s_k)\in\{1,2,\ldots,q\}^k\right) \\
&= q^k = q^{\operatorname{conn}(V,B)} \qquad (\text{since } k=\operatorname{conn}(V,B)).
\end{aligned}$$

This proves Lemma 4.3.8. □

We shall now show a weaker version of Theorem 4.3.5 (as a stepping stone to the actual theorem):

Lemma 4.3.9. Let $G=(V,E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E\to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K\in\mathfrak{K}$. Let $q\in\mathbb{N}$. Then,

$$\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K\right) q^{\operatorname{conn}(V,F)}.
\end{aligned}$$

(Here, of course, the pair (V,F) is regarded as a graph, and the expression $\operatorname{conn}(V,F)$ is understood according to Definition 4.3.3.)

[11] This can be shown in the same way as for the map Φ in the proof of Lemma 4.2.4; we just have to replace every $\mathbb{N}_+$ by $\{1,2,\ldots,q\}$.

Proof of Lemma 4.3.9. We have[12]

$$
\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \left[\underbrace{f \text{ is a proper } \{1,2,\ldots,q\}\text{-coloring of } G}_{\substack{\iff (\text{the } \{1,2,\ldots,q\}\text{-coloring } f \text{ of } G \text{ is proper}) \\ \iff (E\cap \operatorname{Eqs} f=\varnothing) \\ (\text{by Lemma 4.2.2, applied to } \{1,2,\ldots,q\} \text{ instead of } X)}} \right] \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \underbrace{[E\cap \operatorname{Eqs} f=\varnothing]}_{\substack{=\sum_{B\subseteq E\cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{K\in\mathfrak{K};\ K\subseteq B} a_K \\ (\text{by Lemma 4.2.7, applied to } Y=\mathbb{N}_+)}} \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \underbrace{\sum_{B\subseteq E\cap \operatorname{Eqs} f}}_{=\sum_{B\subseteq E;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) = \underbrace{\sum_{f:V\to\{1,2,\ldots,q\}} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{B\subseteq E} \sum_{f:V\to\mathbb{N}_+;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) = \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} 1}_{\substack{=q^{\operatorname{conn}(V,B)} \\ (\text{by Lemma 4.3.8})}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) q^{\operatorname{conn}(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)}
\end{aligned}
$$

(here, we have renamed the summation index B as F). This proves Theorem 4.1.11.

□

From Lemma 4.3.9, we obtain the following consequence:

[12] We are again using the Iverson bracket notation, as defined in Definition 4.2.6.

Lemma 4.3.10. Let $G = (V, E)$ be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\begin{aligned}&(\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)\\ &= \sum_{F \subseteq E} (-1)^{|F|} q^{\operatorname{conn}(V,F)}.\end{aligned}$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 4.3.3.)

Proof of Lemma 4.3.10. This is derived from Lemma 4.3.9 in the same way as Theorem 4.1.8 was derived from Theorem 4.3.5. □

Next, we recall a classical fact about a polynomials over fields: Namely, if a polynomial (in one variable) over a field has infinitely many roots, then this polynomial is 0. Let us state this more formally:

Proposition 4.3.11. Let K be a field. Let $P \in K[x]$ be a polynomial over K. Assume that there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$. Then, $P = 0$.

We shall use the following consequence of this proposition:

Corollary 4.3.12. Let R be an integral domain. Assume that the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective. Let $P \in R[x]$ be a polynomial over R. Assume that $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$ (where 1_R denotes the unity of R). Then, $P = 0$.

Proof of Corollary 4.3.12. Let K denote the fraction field of the integral domain R. We regard R and $R[x]$ as subrings of K and $K[x]$, respectively. By assumption, we have $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$. But the elements $q \cdot 1_R$ of R for $q \in \mathbb{N}$ are pairwise distinct (since the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective). Hence, there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$ (namely, $\lambda = q \cdot 1_R$ for all $q \in \mathbb{N}$). Thus, Proposition 4.3.11 shows that $P = 0$. This proves Corollary 4.3.12. □

We can now prove the classical Theorem 4.3.1:

Proof of Theorem 4.3.1. We need to show that there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

To see that such a polynomial exists, we notice that $P = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}$ is such a polynomial (by Lemma 4.3.10). It remains to prove that such a polynomial is unique. This follows from the fact that if two polynomials $P_1 \in \mathbb{Z}[x]$ and $P_2 \in \mathbb{Z}[x]$ satisfy

$$P_1(q) = P_2(q) \qquad \text{for all } q \in \mathbb{N},$$

then $P_1 = P_2$ [13]. Theorem 4.3.1 is therefore proven. □

Next, it is the turn of Theorem 4.3.5:

Proof of Theorem 4.3.5. Let R be the polynomial ring $\mathbb{Z}[y_K \mid K \in \mathfrak{K}]$, where y_K is a new indeterminate for each $K \in \mathfrak{K}$.

The claim of Theorem 4.3.5 is a polynomial identity in the elements a_K of $\mathbf{k}$. Hence, we can WLOG assume that $\mathbf{k} = R$ and $a_K = y_K$ for each $K \in \mathfrak{K}$. Assume this. Thus, $\mathbf{k}$ is an integral domain, and the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring $\mathbf{k}$ is injective.

For every $q \in \mathbb{N}$, we have

$$\begin{aligned}\chi_G(q) &= (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \\ &\qquad (\text{by the definition of the chromatic polynomial } \chi_G) \\ &= \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)} \qquad (4.13)\end{aligned}$$

[13] This fact follows from Corollary 4.3.12 (applied to $R = \mathbb{Z}$ and $P = P_1 - P_2$).

(by Lemma 4.3.9). Define a polynomial $P \in \mathbf{k}[x]$ by

$$P = \chi_G - \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}. \tag{4.14}$$

Then, for every $q \in \mathbb{N}$, we have

$$\begin{aligned} P\left(\underbrace{q \cdot 1_{\mathbf{k}}}_{=q} \right) &= P(q) = \chi_G(q) - \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)} \qquad \text{(by (4.14))} \\ &= 0 \qquad \text{(by (4.13))}. \end{aligned}$$

Thus, Corollary 4.3.12 (applied to $R = \mathbf{k}$) shows that $P = 0$. In light of (4.14), this rewrites as follows:

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.$$

This proves Theorem 4.3.5. □

Now that Theorem 4.3.5 is proven, we could derive Theorem 4.3.4, Corollary 4.3.6 and Corollary 4.3.7 from it in the same way as we have derived Theorem 4.1.8, Corollary 4.1.13 and Corollary 4.1.14 from Theorem 4.1.11. We leave the details to the reader.

4.3.4 Special case: Whitney's Broken-Circuit Theorem

Corollary 4.3.7 is commonly stated in the following simplified (if less general) form:

Corollary 4.3.13. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{|V|-|F|}.$$

Corollary 4.3.13 is known as *Whitney's Broken-Circuit theorem* (see, e.g., [BlaSag86]).

Notice that ℓ is required to be injective in Corollary 4.3.13; the purpose of this requirement is to ensure that every circuit of G has a unique edge e with maximum $\ell(e)$, and thus induces a broken circuit of G. The proof of Corollary 4.3.13 relies on the following standard result:

Lemma 4.3.14. Let (V, F) be a finite graph. Assume that (V, F) has no circuits. Then, $\operatorname{conn}(V, F) = |V| - |F|$.

(A graph which has no circuits is commonly known as a *forest*.)

Lemma 4.3.14 is both extremely elementary and well-known; for example, it appears in [Bona11, Proposition 10.6] and in [Bollob79, §I.2, Corollary 6]. Let us now see how it entails Corollary 4.3.13:

Proof of Corollary 4.3.13. Corollary 4.3.13 follows from Corollary 4.3.7. Indeed, the injectivity of ℓ shows that every circuit of G has a unique edge e with maximum $\ell(e)$, and thus contains a broken circuit of G. Therefore, if a subset F of E contains no broken circuit of G as a subset, then F contains no circuit of G either, and therefore the graph (V, F) contains no circuits; but this entails that $\operatorname{conn}(V, F) = |V| - |F|$ (by Lemma 4.3.14). Hence, Corollary 4.3.7 immediately yields Corollary 4.3.13. □

4.4 Application to transitive directed graphs

We shall now see an application of Corollary 4.3.6 to graphs which are obtained from certain directed graphs by "forgetting the directions of the edges". Let us first introduce the notations involved:

Definition 4.4.1. **(a)** A *digraph* means a pair (V, A), where V is a set, and where A is a subset of V^2. Digraphs are also called *directed graphs.* A digraph (V, A) is said to be *finite* if the set V is finite. If $D = (V, A)$ is a digraph, then the elements of V are called the *vertices* of the digraph D, while the elements of A are called the *arcs* (or the *directed edges*) of the digraph D. If $a = (v, w)$ is an arc of a digraph D, then v is called the *source* of a, whereas w is called the *target* of a.

(b) A digraph (V, A) is said to be *loopless* if every $v \in V$ satisfies $(v, v) \notin A$. (In other words, a digraph is loopless if and only if it has no arc whose source and target are identical.)

(c) A digraph (V, A) is said to be *transitive* if it has the following property: For any $u \in V$, $v \in V$ and $w \in V$ satisfying $(u, v) \in A$ and $(v, w) \in A$, we have $(u, w) \in A$.

(d) A digraph (V, A) is said to be 2-*path-free* if there exist no three elements u, v and w of V satisfying $(u, v) \in A$ and $(v, w) \in A$.

(e) Let $D = (V, A)$ be a loopless digraph. Define a map $\operatorname{set} : A \to \binom{V}{2}$ by setting

$$(\operatorname{set}(v, w) = \{v, w\} \qquad \text{for every } (v, w) \in A).$$

(It is easy to see that set is well-defined, because (V, A) is loopless.) The graph $(V, \operatorname{set} A)$ will be denoted by $\underline{D}$.

We can now state our application of Corollary 4.3.6, answering a question suggested by Alexander Postnikov:

Proposition 4.4.2. Let $D = (V, A)$ be a finite transitive loopless digraph. Then,

$$\chi_{\underline{D}} = \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V,F) \text{ is 2-path-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}.$$

Proof of Proposition 4.4.2. Let $E = \operatorname{set} A$. Then, the definition of $\underline{D}$ yields $\underline{D} =$

$$\left(V, \underbrace{\operatorname{set} A}_{=E}\right) = (V, E).$$

The map set $: A \to \binom{V}{2}$ (which sends every arc $(v, w) \in A$ to $\{v, w\} \in \binom{V}{2}$) restricts to a surjection $A \to E$ (since $E = \operatorname{set} A$). Let us denote this surjection by π. Thus, π is a map from A to E sending each arc $(v, w) \in A$ to $\{v, w\} \in E$. We shall soon see that π is a bijection.

We define a partial order on the set V as follows: For $i \in V$ and $j \in V$, we set $i < j$ if and only if $(i, j) \in A$ (that is, if and only if there is an arc from i to j in D). This is a well-defined partial order[14]. Thus, V becomes a poset. For every $i \in V$ and $j \in V$ satisfying $i \leq j$, we let $[i, j]$ denote the interval $\{k \in V \mid i \leq k \leq j\}$ of the poset V.

There exist no $i, j \in V$ such that both (i, j) and (j, i) belong to A (because if such i and j would exist, then they would satisfy $i < j$ and $j < i$, but this would contradict the fact that V is a poset). Hence, the projection $\pi : A \to E$ is injective, and thus bijective (since we already know that π is surjective). Hence, its inverse map $\pi^{-1} : E \to A$ is well-defined. For every subset F of E, we have

$$\begin{aligned} F &= \pi\left(\pi^{-1}(F)\right) && \text{(since } \pi \text{ is bijective)} \\ &= \operatorname{set}\left(\pi^{-1}(F)\right) && (4.15) \end{aligned}$$

(since π is a restriction of the map set).

For any $(u, v) \in A$ and any subset F of E, we have the following logical equivalence:

$$(\{u, v\} \in F) \iff \left((u, v) \in \pi^{-1}(F)\right) \qquad (4.16)$$

[15].

[14] Indeed, the relation $<$ that we have just defined is transitive (since the digraph (V, A) is transitive) and antisymmetric (since the digraph (V, A) is loopless).

[15] *Proof of (4.16):* Let $(u, v) \in A$, and let F be a subset of E. We need to prove the equivalence (4.16).

From $(u, v) \in A$, we see that $\pi(u, v)$ is well-defined. The definition of π shows that $\pi(u, v) =$

Define a function $\ell' : A \to \mathbb{N}$ by

$$\ell'(i,j) = |[i,j]| \qquad \text{for all } (i,j) \in A.$$

Define a function $\ell : E \to \mathbb{N}$ by $\ell = \ell' \circ \pi^{-1}$. Thus, $\ell \circ \pi = \ell'$. Therefore,

$$\ell \left(\underbrace{\{i,j\}}_{=\pi(i,j)} \right) = \underbrace{(\ell \circ \pi)}_{=\ell'} (i,j) = \ell'(i,j) = |[i,j]| \tag{4.17}$$

for all $(i,j) \in A$.

Let $\mathfrak{K}$ be the set

$$\{\{\{i,k\}, \{k,j\}\} \mid (i,k) \in A \text{ and } (k,j) \in A\}.$$

Each $K \in \mathfrak{K}$ is a broken circuit of $\underline{D}$ [16]. Thus, $\mathfrak{K}$ is a set of broken circuits of $\underline{D}$.

$\{u,v\}$. Hence, we have the following chain of equivalences:

$$\left(\underbrace{\{u,v\}}_{=\pi(u,v)} \in F \right) \iff (\pi(u,v) \in F) \iff \left((u,v) \in \pi^{-1}(F)\right).$$

This proves (4.16).

[16] *Proof.* Let $K \in \mathfrak{K}$. Then, $K = \{\{i,k\}, \{k,j\}\}$ for some $(i,k) \in A$ and $(k,j) \in A$ (by the definition of $\mathfrak{K}$). Consider these (i,k) and (k,j). Since (V,A) is transitive, we have $(i,j) \in A$. Thus, $\{i,k\}$, $\{k,j\}$ and $\{i,j\}$ are edges of $\underline{D}$. These edges form a circuit of $\underline{D}$. In particular, i, j and k are pairwise distinct.

Applications of (4.17) yield $\ell(\{i,j\}) = |[i,j]|$, $\ell(\{i,k\}) = |[i,k]|$ and $\ell(\{k,j\}) = |[k,j]|$.

But we have $i < k$ (since $(i,k) \in A$) and $k < j$ (since $(k,j) \in A$). Hence, $[i,k]$ is a proper subset of $[i,j]$. (It is proper because it does not contain j, whereas $[i,j]$ does.) Hence, $|[i,k]| < |[i,j]|$. Thus, $\ell(\{i,j\}) = |[i,j]| > |[i,k]| = \ell(\{i,k\})$. Similarly, $\ell(\{i,j\}) > \ell(\{k,j\})$. The last two inequalities show that $\{i,j\}$ is the unique edge of the circuit $\{\{i,k\}, \{k,j\}, \{i,j\}\}$ having maximum label. Hence, $\{\{i,k\}, \{k,j\}, \{i,j\}\} \setminus \{\{i,j\}\}$ is a broken circuit of $\underline{D}$. Since

$$\begin{aligned} \{\{i,k\}, \{k,j\}, \{i,j\}\} \setminus \{\{i,j\}\} &= \{\{i,k\}, \{k,j\}\} \qquad \text{(since } i, j \text{ and } k \text{ are pairwise distinct)} \\ &= K, \end{aligned}$$

this shows that K is a broken circuit of $\underline{D}$, qed.

A subset F of E is $\mathfrak{K}$-free if and only if the digraph $(V, \pi^{-1}(F))$ is 2-path-free[17]. Now, Corollary 4.3.6 (applied to $X = \mathbb{N}$ and $G = \underline{D}$) shows that

$$\chi_{\underline{D}} = \underbrace{\sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}}}_{\substack{= \sum_{\substack{F \subseteq E; \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-path-free}}} \\ \text{(since we have just shown that} \\ \text{a subset } F \text{ of } E \text{ is } \mathfrak{K}\text{-free if and only if} \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-path-free)}}} \underbrace{(-1)^{|F|}}_{\substack{= (-1)^{\left|\pi^{-1}(F)\right|} \\ \text{(since } \pi \text{ is bijective)}}} \underbrace{x^{\operatorname{conn}(V,F)}}_{\substack{= x^{\operatorname{conn}\left(V, \operatorname{set}\left(\pi^{-1}(F)\right)\right)} \\ \text{(by (4.15))}}}$$

$$= \sum_{\substack{F \subseteq E; \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-path-free}}} (-1)^{\left|\pi^{-1}(F)\right|} x^{\operatorname{conn}\left(V, \operatorname{set}\left(\pi^{-1}(F)\right)\right)}$$

$$= \sum_{\substack{B \subseteq A; \\ \text{the digraph } (V,B) \text{ is 2-path-free}}} (-1)^{|B|} x^{\operatorname{conn}(V, \operatorname{set} B)}$$

$$\left(\begin{array}{c} \text{here, we have substituted } B \text{ for } \pi^{-1}(F) \text{ in the sum,} \\ \text{since the map } \pi : A \to E \text{ is bijective and thus induces} \\ \text{a bijection from the subsets of } E \text{ to the subsets of } A \\ \text{sending each } F \subseteq E \text{ to } \pi^{-1}(F) \end{array} \right)$$

$$= \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V,F) \text{ is 2-path-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}$$

---

[17] *Proof.* Let F be a subset of E. Then, we have the following equivalence of statements:

$$\begin{aligned}
& (F \text{ is } \mathfrak{K}\text{-free}) \\
\iff & (\{\{i,k\}, \{k,j\}\} \not\subseteq F \text{ whenever } (i,k) \in A \text{ and } (k,j) \in A) \\
& \quad \text{(by the definition of } \mathfrak{K}) \\
\iff & (\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } \{\{i,k\}, \{k,j\}\} \subseteq F) \\
\iff & (\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } \{i,k\} \in F \text{ and } \{k,j\} \in F) \\
\iff & \left(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } (i,k) \in \pi^{-1}(F) \text{ and } \{k,j\} \in F\right) \\
& \quad \left(\begin{array}{c} \text{because for } (i,k) \in A \text{, we have } \{i,k\} \in F \text{ if and only if } (i,k) \in \pi^{-1}(F) \\ \text{(by (4.16), applied to } u = i \text{ and } v = k) \end{array} \right) \\
\iff & \left(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } (i,k) \in \pi^{-1}(F) \text{ and } (k,j) \in \pi^{-1}(F)\right) \\
& \quad \left(\begin{array}{c} \text{because for } (k,j) \in A \text{, we have } \{k,j\} \in F \text{ if and only if } (k,j) \in \pi^{-1}(F) \\ \text{(by (4.16), applied to } u = k \text{ and } v = j) \end{array} \right) \\
\iff & \left(\text{the digraph } \left(V, \pi^{-1}(F)\right) \text{ is 2-path-free}\right) \qquad \text{(by the definition of "2-path-free")},
\end{aligned}$$

qed.

(here, we have renamed the summation index B as F). This proves Proposition 4.4.2. □

4.5 A matroidal generalization

4.5.1 An introduction to matroids

We shall now present a result that can be considered as a generalization of Theorem 4.3.5 in a different direction than Theorem 4.1.11: namely, a formula for the characteristic polynomial of a matroid. Let us first recall the basic notions from the theory of matroids that will be needed to state it.

First, we introduce some basic poset-related terminology:

Definition 4.5.1. Let P be a poset.

(a) An element v of P is said to be *maximal* (with respect to P) if and only if every $w \in P$ satisfying $w \geq v$ must satisfy $w = v$.

(b) An element v of P is said to be *minimal* (with respect to P) if and only if every $w \in P$ satisfying $w \leq v$ must satisfy $w = v$.

Definition 4.5.2. For any set E, we shall regard the powerset $\mathcal{P}(E)$ as a poset (with respect to inclusion). Thus, any subset $\mathcal{S}$ of $\mathcal{P}(E)$ also becomes a poset, and therefore the notions of "minimal" and "maximal" elements in $\mathcal{S}$ make sense. Beware that these notions are not related to size; i.e., a maximal element of $\mathcal{S}$ might not be a maximum-size element of $\mathcal{S}$.

Now, let us define the notion of "matroid" that we will use:

Definition 4.5.3. **(a)** A *matroid* means a pair $(E, \mathcal{I})$ consisting of a finite set E and a set $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- *Matroid axiom 1:* We have $\varnothing \in \mathcal{I}$.
- *Matroid axiom 2:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{P}(E)$ are such that $Z \subseteq Y$, then $Z \in \mathcal{I}$.

- *Matroid axiom 3:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are such that $|Y| < |Z|$, then there exists some $x \in Z \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

(b) Let $(E, \mathcal{I})$ be a matroid. A subset S of E is said to be *independent* (for this matroid) if and only if $S \in \mathcal{I}$. The set E is called the *ground set* of the matroid $(E, \mathcal{I})$.

Different texts give different definitions of a matroid; these definitions are (mostly) equivalent, but not always in the obvious way[18]. Definition 4.5.3 is how a matroid is defined in [Schrij13, §10.1] and in [Martin15, Definition 3.15] (where it is called a "(matroid) independence system"). There exist other definitions of a matroid, which turn out to be equivalent. The definition of a matroid given in Stanley's [Stanley06, Definition 3.8] is directly equivalent to Definition 4.5.3, with the only differences that

- Stanley replaces Matroid axiom 1 by the requirement that $\mathcal{I} \neq \varnothing$ (which is, of course, equivalent to Matroid axiom 1 as long as Matroid axiom 2 is assumed), and

- Stanley replaces Matroid axiom 3 by the requirement that for every $T \in \mathcal{P}(E)$, all maximal elements of $\mathcal{I} \cap \mathcal{P}(T)$ have the same cardinality[19] (this requirement is equivalent to Matroid axiom 3 as long as Matroid axiom 2 is assumed).

We now introduce some terminology related to matroids:

Definition 4.5.4. Let $M = (E, \mathcal{I})$ be a matroid.

(a) We define a function $r_M : \mathcal{P}(E) \to \mathbb{N}$ by setting

$$r_M(S) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq S\} \qquad \text{for every } S \subseteq E. \tag{4.18}$$

[18] I.e., it sometimes happens that two different texts both define a matroid as a pair (E, U) of a finite set E and a subset $U \subseteq \mathcal{P}(E)$, but they require these pairs (E, U) to satisfy non-equivalent axioms, and the equivalence between their definitions is more complicated than just "a pair (E, U) is a matroid for one definition if and only if it is a matroid for the other".

[19] Here, as we have already explained, we regard $\mathcal{I} \cap \mathcal{P}(T)$ as a poset with respect to inclusion. Thus, an element Y of this poset is maximal if and only if there exists no $Z \in \mathcal{I} \cap \mathcal{P}(T)$ such that Y is a proper subset of Z.

(Note that the right hand side of (4.18) is well-defined, because there exists at least one $Z \in \mathcal{I}$ satisfying $Z \subseteq S$ (namely, $Z = \varnothing$).) If S is a subset of E, then the nonnegative integer $r_M(S)$ is called the *rank* of S (with respect to M). It is clear that r_M is a weakly increasing function from the poset $\mathcal{P}(E)$ to $\mathbb{N}$.

(b) If $k \in \mathbb{N}$, then a *k-flat* of M means a subset of E which has rank k and is maximal among all such subsets (i.e., it is not a proper subset of any other subset having rank k). (Beware: Not all k-flats have the same size.) A *flat* of M is a subset of E which is a k-flat for some $k \in \mathbb{N}$. We let $\operatorname{Flats} M$ denote the set of all flats of M; thus, $\operatorname{Flats} M$ is a subposet of $\mathcal{P}(E)$.

(c) A *circuit* of M means a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$. (That is, a circuit of M means a subset of E which is not independent (for M) and which is minimal among such subsets.)

(d) An element e of E is said to be a *loop* (of M) if $\{e\} \notin \mathcal{I}$. The matroid M is said to be *loopless* if no loops (of M) exist.

Notice that the function that we called r_M in Definition 4.5.4 **(a)** is denoted by rk in Stanley's [Stanley06, Lecture 3].

One of the most classical examples of a matroid is the *graphical matroid* of a graph:

Example 4.5.5. Let $G = (V, E)$ be a finite graph. Define a subset $\mathcal{I}$ of $\mathcal{P}(E)$ by

$$\mathcal{I} = \{T \in \mathcal{P}(E) \mid T \text{ contains no circuit of } G \text{ as a subset}\}.$$

Then, $(E, \mathcal{I})$ is a matroid; it is called the *graphical matroid* (or the *cycle matroid*) of G. It has the following properties:

- The matroid $(E, \mathcal{I})$ is loopless.
- For each $T \in \mathcal{P}(E)$, we have

$$r_{(E,\mathcal{I})}(T) = |V| - \operatorname{conn}(V, T)$$

(where $\operatorname{conn}(V, T)$ is defined as in Definition 4.3.3).

- The circuits of the matroid $(E, \mathcal{I})$ are precisely the circuits of the graph G.
- The flats of the matroid $(E, \mathcal{I})$ are related to colorings of G. More precisely: For each set X and each X-coloring f of G, the set $E \cap \operatorname{Eqs} f$ is a flat of $(E, \mathcal{I})$. Every flat of $(E, \mathcal{I})$ can be obtained in this way when X is chosen large enough; but often, several distinct X-colorings f lead to one and the same flat $E \cap \operatorname{Eqs} f$.

We recall three basic facts that are used countless times in arguing about matroids:

Lemma 4.5.6. Let $M = (E, \mathcal{I})$ be a matroid. Let $T \in \mathcal{I}$. Then, $r_M(T) = |T|$.

Proof of Lemma 4.5.6. We have $T \in \mathcal{I}$ and $T \subseteq T$. Thus, T is a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$. Therefore, $|T| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$, so that

$$|T| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \tag{4.19}$$

(since any element of a set of integers is smaller or equal to the maximum of this set).

On the other hand, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}.$$

Hence, (4.19) rewrites as follows:

$$|T| \leq r_M(T).$$

Also,

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of any set belongs to this set). Thus, there exists a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Consider this Z. From $Z \subseteq T$, we obtain $|Z| \leq |T|$, so that $r_M(T) = |Z| \leq |T|$. Combining this with $|T| \leq r_M(T)$, we obtain $r_M(T) = |T|$. This proves Lemma 4.5.6. □

Lemma 4.5.7. Let $M = (E, \mathcal{I})$ be a matroid. Let $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Then, there exists a circuit C of M such that $C \subseteq Q$.

Proof of Lemma 4.5.7. We have $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Thus, there exists at least one $C \in \mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$ (namely, $C = Q$). Thus, there also exists a **minimal** such C. Consider this minimal C. We know that C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$. In other words, C is an element of $\mathcal{P}(E) \setminus \mathcal{I}$ satisfying $C \subseteq Q$, and moreover,

$$\text{every } D \in \mathcal{P}(E) \setminus \mathcal{I} \text{ satisfying } D \subseteq Q \text{ and } D \subseteq C \text{ must satisfy } D = C. \tag{4.20}$$

Thus, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ [20]. In other words, C is a circuit of M (by the definition of a "circuit"). This circuit C satisfies $C \subseteq Q$. Thus, we have constructed a circuit C of M satisfying $C \subseteq Q$. Lemma 4.5.7 is thus proven. □

Lemma 4.5.8. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Let $S \in \mathcal{I}$ be such that $S \subseteq T$. Then, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$.

Proof of Lemma 4.5.8. Clearly, there exists at least one $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ (namely, $S' = S$). Hence, there exists a **maximal** such S'. Let Q be such a maximal S'. Thus, Q is an element of $\mathcal{I}$ satisfying $S \subseteq Q \subseteq T$.

[20] *Proof.* We need to show that every $D \in \mathcal{P}(E) \setminus \mathcal{I}$ satisfying $D \subseteq C$ must satisfy $D = C$ (since we already know that $C \in \mathcal{P}(E) \setminus \mathcal{I}$).

So let $D \in \mathcal{P}(E) \setminus \mathcal{I}$ be such that $D \subseteq C$. Then, $D \subseteq C \subseteq Q$. Hence, (4.20) shows that $D = C$. This completes our proof.

Recall that

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} && \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of any set must belong to this set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Denote such a Z by W. Thus, W is an element of $\mathcal{I}$ satisfying $W \subseteq T$ and $r_M(T) = |W|$.

We have $|Q| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$ (since $Q \in \mathcal{I}$ and $Q \subseteq T$). Since any element of a set is smaller or equal to the maximum of this set, this entails that $|Q| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} = r_M(T) = |W|$.

Now, assume (for the sake of contradiction) that $|Q| \neq |W|$. Thus, $|Q| < |W|$ (since $|Q| \leq |W|$). Hence, Matroid axiom 3 (applied to $Y = Q$ and $Z = W$) shows that there exists some $x \in W \setminus Q$ such that $Q \cup \{x\} \in \mathcal{I}$. Consider this x. We have $x \in W \setminus Q \subseteq W \subseteq T$, so that $Q \cup \{x\} \subseteq T$ (since $Q \subseteq T$). Also, $x \notin Q$ (since $x \in W \setminus Q$).

Recall that Q is a **maximal** $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$. Thus, if some $S' \in \mathcal{I}$ satisfies $S \subseteq S' \subseteq T$ and $S' \supseteq Q$, then $S' = Q$. Applying this to $S' = Q \cup \{x\}$, we obtain $Q \cup \{x\} = Q$ (since $S \subseteq Q \subseteq Q \cup \{x\} \subseteq T$ and $Q \cup \{x\} \supseteq Q$). Thus, $x \in Q$. But this contradicts $x \notin Q$. This contradiction shows that our assumption (that $|Q| \neq |W|$) was wrong. Hence, $|Q| = |W| = r_M(T)$. Thus, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$ (namely, $S' = Q$). This proves Lemma 4.5.8. □

4.5.2 The lattice of flats

We shall now show a lemma that can be regarded as an alternative criterion for a subset of E to be a flat:

Lemma 4.5.9. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Then, the following statements are equivalent:

Statement $\mathfrak{F}_1$*:* The set T is a flat of M.

Statement $\mathfrak{F}_2$*:* If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq T$, then $C \subseteq T$.

Proof of Lemma 4.5.9. Proof of the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$*:* Assume that Statement $\mathfrak{F}_1$ holds. We must prove that Statement $\mathfrak{F}_2$ holds.

Let C be a circuit of M. Let $e \in C$ be such that $C \setminus \{e\} \subseteq T$. We must prove that $C \subseteq T$.

Assume the contrary. Thus, $C \not\subseteq T$. Combining this with $C \setminus \{e\} \subseteq T$, we obtain $e \notin T$. Hence, T is a proper subset of $T \cup \{e\}$.

We have assumed that Statement $\mathfrak{F}_1$ holds. In other words, the set T is a flat of M. In other words, there exists some $k \in \mathbb{N}$ such that T is a k-flat of M. Consider this k.

The set T is a k-flat of M, thus a subset of E which has rank k and is maximal among all such subsets. In other words, $r_M(T) = k$, but every subset S of E for which T is a proper subset of S must satisfy

$$r_M(S) \neq k. \tag{4.21}$$

Applying (4.21) to $S = T \cup \{e\}$, we obtain $r_M(T \cup \{e\}) \neq k$. Since $T \cup \{e\} \supseteq T$ (and since the function $r_M : \mathcal{P}(E) \to \mathbb{N}$ is weakly increasing), we have $r_M(T \cup \{e\}) \geq r_M(T) = k$. Combined with $r_M(T \cup \{e\}) \neq k$, this yields $r_M(T \cup \{e\}) > k$. Thus, $r_M(T \cup \{e\}) \geq k + 1$.

Notice that $C \setminus \{e\}$ is a proper subset of C (since $e \in C$). The set C is a circuit of M, thus a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, no proper subset of C belongs to $\mathcal{P}(E) \setminus \mathcal{I}$ (because C is minimal). In other words, every proper subset of C belongs to $\mathcal{I}$. Applying this to the proper subset $C \setminus \{e\}$ of C, we conclude that $C \setminus \{e\}$ belongs to $\mathcal{I}$. Hence, Lemma 4.5.8 (applied to $S = C \setminus \{e\}$) shows that there exists an $S' \in \mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S' \subseteq T$ and $|S'| = r_M(T)$. Denote this S' by S. Thus, S is an element of $\mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S \subseteq T$ and $|S| = r_M(T)$.

Furthermore, $S \subseteq T \subseteq T \cup \{e\}$. Thus, Lemma 4.5.8 (applied to $T \cup \{e\}$ instead of T) shows that there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T \cup \{e\}$ and $|S'| = r_M(T \cup \{e\})$. Consider this S'.

We have $|S'| = r_M(T \cup \{e\}) > r_M(T)$. Hence, $S' \not\subseteq T$ [21]. Combining this with $S' \subseteq T \cup \{e\}$, we obtain $e \in S'$. Combining this with $C \setminus \{e\} \subseteq S'$, we find that $(C \setminus \{e\}) \cup \{e\} \subseteq S'$. Thus, $C = (C \setminus \{e\}) \cup \{e\} \subseteq S'$. Since $S' \in \mathcal{I}$, this entails that $C \in \mathcal{I}$ (by Matroid axiom 2). But $C \in \mathcal{P}(E) \setminus \mathcal{I}$ (since C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$), so that $C \notin \mathcal{I}$. This contradicts $C \in \mathcal{I}$. This contradiction shows that our assumption was wrong. Hence, $C \subseteq T$ is proven. Therefore, Statement $\mathfrak{F}_2$ holds. Thus, the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ is proven.

Proof of the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$: Assume that Statement $\mathfrak{F}_2$ holds. We must prove that Statement $\mathfrak{F}_1$ holds.

Let $k = r_M(T)$. We shall show that T is a k-flat of M.

Let W be a subset of E which has rank k and satisfies $T \subseteq W$. We shall show that $T = W$.

Indeed, assume the contrary. Thus, $T \neq W$. Combined with $T \subseteq W$, this shows that T is a proper subset of W. Thus, there exists an $e \in W \setminus T$. Consider this e. We have $e \notin T$ (since $e \in W \setminus T$).

We have

$$\begin{aligned} k = r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad \text{(by the definition of } r_M) \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of a set must belong to that set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $k = |Z|$. Denote this Z by K. Thus, K is an element of $\mathcal{I}$ and

[21] *Proof.* Assume the contrary. Thus, $S' \subseteq T$. Hence, S' is an element of $\mathcal{I}$ and satisfies $S' \subseteq T$. Thus, $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$.

Now, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \geq |S'|$$

(since $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$). This contradicts $|S'| > r_M(T)$. This contradiction proves that our assumption was wrong, qed.

satisfies $K \subseteq T$ and $k = |K|$. Notice that $e \notin T$, so that $e \notin K$ (since $K \subseteq T$).

We have $r_M(W) = k$ (since W has rank k). Hence, $K \cup \{e\} \notin \mathcal{I}$ [22]. In other words, $K \cup \{e\} \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, Lemma 4.5.7 (applied to $Q = K \cup \{e\}$) shows that there exists a circuit C of M such that $C \subseteq K \cup \{e\}$. Consider this C. From $C \subseteq K \cup \{e\}$, we obtain $C \setminus \{e\} \subseteq K \subseteq T$.

From $C \setminus \{e\} \subseteq K$, we conclude (using Matroid axiom 2) that $C \setminus \{e\} \in \mathcal{I}$ (since $K \in \mathcal{I}$). On the other hand, C is a circuit of M. In other words, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, $C \in \mathcal{P}(E) \setminus \mathcal{I}$, so that $C \notin \mathcal{I}$. Hence, $e \in C$ (since otherwise, we would have $C \setminus \{e\} = C \notin \mathcal{I}$, which would contradict $C \setminus \{e\} \in \mathcal{I}$). Now, Statement $\mathfrak{F}_2$ shows that $C \subseteq T$. Hence, $e \in C \subseteq T$, which contradicts $e \notin T$.

This contradiction shows that our assumption was wrong. Hence, $T = W$ is proven.

Now, forget that we fixed W. Thus, we have shown that if W is a subset of E which has rank k and satisfies $T \subseteq W$, then $T = W$. In other words, T is a subset of E which has rank k and is maximal among all such subsets (because we already know that T has rank $r_M(T) = k$). In other words, T is a k-flat of M (by the definition of a "k-flat"). Thus, T is a flat of M. In other words, Statement $\mathfrak{F}_1$ holds. This proves the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$.

We have now proven the implications $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ and $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$. Together, these implications show that Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent. This proves Lemma 4.5.9. □

Corollary 4.5.10. Let $M = (E, \mathcal{I})$ be a matroid. Let $F_1, F_2, \ldots, F_k$ be flats of M. Then, $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. (Notice that k is allowed to be 0 here; in this case, the empty intersection $F_1 \cap F_2 \cap \cdots \cap F_k$ is to be interpreted as E.)

Proof of Corollary 4.5.10. Lemma 4.5.9 gives a necessary and sufficient criterion for

[22] *Proof.* Assume the contrary. Thus, $K \cup \{e\} \in \mathcal{I}$. Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}|$ (by Lemma 4.5.6). Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}| > |K|$ (since $e \notin K$).

But $K \cup \{e\} \subseteq W$ (since $K \subseteq T \subseteq W$ and $e \in W \setminus T \subseteq W$). Since the function r_M is weakly increasing, this yields $r_M(K \cup \{e\}) \leq r_M(W) = k = |K|$. This contradicts $r_M(K \cup \{e\}) > |K|$. This contradiction proves that our assumption was wrong, qed.

a subset T of E to be a flat of M. It is easy to see that if this criterion is satisfied for $T = F_1$, for $T = F_2$, etc., and for $T = F_k$, then it is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, if $F_1, F_2, \ldots, F_k$ are flats of M, then $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. [23] This proves Corollary 4.5.10. □

Corollary 4.5.10 (a well-known fact, which is left to the reader to prove in [Stanley06, §3.1]) allows us to define the *closure* of a set in a matroid:

Definition 4.5.11. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. The *closure* of T is defined to be the intersection of all flats of M which contain T as a subset. In other words, the closure of T is defined to be $\bigcap_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. The closure of T is denoted by $\overline{T}$.

The following proposition gathers some simple properties of closures in matroids:

Proposition 4.5.12. Let $M = (E, \mathcal{I})$ be a matroid.

(a) If T is a subset of E, then $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$.

(b) If G is a flat of M, then $\overline{G} = G$.

(c) If T is a subset of E and if G is a flat of M satisfying $T \subseteq G$, then $\overline{T} \subseteq G$.

(d) If S and T are two subsets of E satisfying $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

(e) If the matroid M is loopless, then $\overline{\varnothing} = \varnothing$.

---

[23] Here is this argument in slightly more detail:

For every $i \in \{1, 2, \ldots, k\}$, the following statement holds: If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then

$$C \subseteq F_i. \tag{4.22}$$

Proof of (4.22): Let $i \in \{1, 2, \ldots, k\}$. Then, the set F_i is a flat of M. In other words, Statement $\mathfrak{F}_1$ of Lemma 4.5.9 is satisfied for $T = F_i$. Therefore, Statement $\mathfrak{F}_2$ of Lemma 4.5.9 must also be satisfied for $T = F_i$ (since Lemma 4.5.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In other words, if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then $C \subseteq F_i$. This proves (4.22).

Now, let C be a circuit of M, and let $e \in C$ be such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. For every $i \in \{1, 2, \ldots, k\}$, we have $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k \subseteq F_i$, and therefore $C \subseteq F_i$ (by (4.22)). So we have shown the inclusion $C \subseteq F_i$ for each $i \in \{1, 2, \ldots, k\}$. Combining these k inclusions, we obtain $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$.

Now, forget that we fixed C. We thus have shown that if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$, then $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, Statement $\mathfrak{F}_2$ of Lemma 4.5.9 is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. Therefore, Statement $\mathfrak{F}_1$ of Lemma 4.5.9 must also be satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$ (since Lemma 4.5.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In other words, the set $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. Qed.

(f) Every subset T of E satisfies $r_M(T) = r_M(\overline{T})$.

(g) If T is a subset of E and if G is a flat of M, then the conditions $(\overline{T} \subseteq G)$ and $(T \subseteq G)$ are equivalent.

Proof of Proposition 4.5.12. **(a)** The set $\operatorname{Flats} M$ is a subset of the finite set $\mathcal{P}(E)$, and thus itself finite.

Let T be a subset of E. The closure $\overline{T}$ of T is defined as $\bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F$. Now, Corollary 4.5.10 shows that any intersection of finitely many flats of M is a flat of M. Hence, $\bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F$ (being an intersection of finitely many flats of M [24]) is a flat of M. In other words, $\overline{T}$ is a flat of M (since $\overline{T} = \bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F$).

Also, $T \subseteq F$ for every $F \in \operatorname{Flats} M$ satisfying $T \subseteq F$. Hence, $T \subseteq \bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F = \overline{T}$. This completes the proof of Proposition 4.5.12 **(a)**.

(c) Let T be a subset of E, and let G be a flat of M satisfying $T \subseteq G$. Then, G is an element of $\operatorname{Flats} G$ satisfying $T \subseteq G$. Hence, G is one term in the intersection $\bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F$. Thus, $\bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F \subseteq G$. But the definition of $\overline{T}$ yields $\overline{T} = \bigcap_{F \in \operatorname{Flats} M;\ T \subseteq F} F \subseteq G$. This proves Proposition 4.5.12 **(c)**.

(b) Let G be a flat of M. Proposition 4.5.12 **(b)** (applied to $T = G$) yields $\overline{G} \subseteq G$. But Proposition 4.5.12 **(a)** (applied to $T = G$) shows that $\overline{G}$ is a flat of M satisfying $G \subseteq \overline{G}$. Combining $G \subseteq \overline{G}$ with $\overline{G} \subseteq G$, we obtain $\overline{G} = G$. This proves Proposition 4.5.12 **(b)**.

(d) Let S and T be two subsets of E satisfying $S \subseteq T$. Proposition 4.5.12 **(a)** shows that $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$. Now, $S \subseteq T \subseteq \overline{T}$. Hence, Proposition 4.5.12 **(b)** (applied to S and $\overline{T}$ instead of T and G) shows $\overline{S} \subseteq \overline{T}$. This proves Proposition 4.5.12 **(d)**.

(e) Assume that the matroid M is loopless. In other words, no loops (of M) exist.

The definition of r_M quickly yields $r_M(\varnothing) = 0$. In other words, the set $\varnothing$ has rank 0. We shall now show that $\varnothing$ is a 0-flat of M.

[24] "Finitely many" since the set $\operatorname{Flats} M$ is finite.

Indeed, let W be a subset of E which has rank 0 and satisfies $\varnothing \subseteq W$. We shall show that $\varnothing = W$.

Assume the contrary. Thus, $\varnothing \neq W$. Hence, W has an element w. Consider this w. The element w of E is not a loop (since no loops exist). In other words, we cannot have $\{w\} \notin \mathcal{I}$ (since w is a loop if and only if $\{w\} \notin \mathcal{I}$ (by the definition of a loop)). In other words, we must have $\{w\} \in \mathcal{I}$. Clearly, $\{w\} \subseteq W$ (since $w \in W$). Thus, $\{w\}$ is a $Z \in \mathcal{I}$ satisfying $Z \subseteq W$. Thus, $|\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}$.

But W has rank 0. In other words,

$$\begin{aligned} 0 = r_M(W) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\} \qquad \text{(by the definition of } r_M\text{)} \\ &\geq |\{w\}| \qquad \left(\text{since } |\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}\right) \\ &= 1, \end{aligned}$$

which is absurd. This contradiction shows that our assumption was wrong. Hence, $\varnothing = W$ is proven.

Let us now forget that we fixed W. We thus have proven that if W is any subset of E which has rank 0 and satisfies $\varnothing \subseteq W$, then $\varnothing = W$. Thus, $\varnothing$ is a subset of E which has rank 0 and is maximal among all such subsets (because we already know that $\varnothing$ has rank 0). In other words, $\varnothing$ is a 0-flat of M (by the definition of a "0-flat"). Thus, $\varnothing$ is a flat of M. Thus, Proposition 4.5.12 **(b)** (applied to $G = \varnothing$) yields $\overline{\varnothing} = \varnothing$. This proves Proposition 4.5.12 **(e)**.

(f) Let T be a subset of E. We have $T \subseteq \overline{T}$ (by Proposition 4.5.12 **(a)**), and thus $r_M(T) \leq r_M\left(\overline{T}\right)$ (since the function r_M is weakly increasing).

Let $k = r_M(T)$. Thus, there exists a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$ (namely, $Q = T$). Hence, there exists a **maximal** such Q. Denote this Q by R. Thus, R is a maximal $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. In particular, R is an element of $\mathcal{P}(E)$ and satisfies $T \subseteq R$ and $k = r_M(R)$.

Now, R is a subset of E (since $R \in \mathcal{P}(E)$) and has rank $r_M(R) = k$. Thus, R is a subset of E which has rank k. Furthermore, R is maximal among all such subsets[25].

[25] *Proof.* Let W be any subset of E which has rank k and satisfies $W \supseteq R$. We must prove that

Thus, R is a k-flat of M (by the definition of a "k-flat"), and therefore a flat of M. Now, Proposition 4.5.12 **(c)** (applied to $G = R$) shows that $\overline{T} \subseteq R$. Since the function r_M is weakly increasing, this yields $r_M\left(\overline{T}\right) \leq r_M(R) = k$. Combining this with $k = r_M(T) \leq r_M\left(\overline{T}\right)$, we obtain $r_M\left(\overline{T}\right) = k = r_M(T)$. This proves Proposition 4.5.12 **(f)**.

(g) Let T be a subset of E. Let G be a flat of M. Proposition 4.5.12 **(a)** shows that $T \subseteq \overline{T}$. Hence, if $\overline{T} \subseteq G$, then $T \subseteq \overline{T} \subseteq G$. Thus, we have proven the implication $\left(\overline{T} \subseteq G\right) \Longrightarrow (T \subseteq G)$. The reverse implication (i.e., the implication $(T \subseteq G) \Longrightarrow \left(\overline{T} \subseteq G\right)$) follows from Proposition 4.5.12 **(c)**. Combining these two implications, we obtain the equivalence $\left(\overline{T} \subseteq G\right) \Longleftrightarrow (T \subseteq G)$. This proves Proposition 4.5.12 **(g)**. □

We shall now recall a few more classical notions related to posets:

Definition 4.5.13. Let P be a poset.

(a) An element $p \in P$ is said to be a *global minimum* of P if every $q \in P$ satisfies $p \leq q$. Clearly, a global minimum of P is unique if it exists.

(b) An element $p \in P$ is said to be a *global maximum* of P if every $q \in P$ satisfies $p \geq q$. Clearly, a global maximum of P is unique if it exists.

(c) Let x and y be two elements of P. An *upper bound* of x and y (in P) means an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A *join* (or *least upper bound*) of x and y (in P) means an upper bound z of x and y such that every upper bound z' of x and y satisfies $z' \geq z$. In other words, a join of x and y is a global minimum of the subposet $\{w \in P \mid w \geq x \text{ and } w \geq y\}$ of P. Thus, a join of x and y is unique if it exists.

(d) Let x and y be two elements of P. A *lower bound* of x and y (in P) means an element $z \in P$ satisfying $z \leq x$ and $z \leq y$. A *meet* (or *greatest lower bound*) of x and y (in P) means a lower bound z of x and y such that every lower bound z'

$W = R$.

We have $W \in \mathcal{P}(E)$, $T \subseteq R \subseteq W$ and $k = r_M(W)$ (since W has rank k). Thus, W is a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. But recall that R is a **maximal** such Q. Hence, if $W \supseteq R$, then $W = R$. Therefore, $W = R$ (since we know that $W \supseteq R$). Qed.

of x and y satisfies $z' \leq z$. In other words, a meet of x and y is a global maximum of the subposet $\{w \in P \mid w \leq x \text{ and } w \leq y\}$ of P. Thus, a meet of x and y is unique if it exists.

(e) The poset P is said to be a *lattice* if and only if it has a global minimum and a global maximum, and every two elements of P have a meet and a join.

Proposition 4.5.14. Let $M = (E, \mathcal{I})$ be a matroid. The subposet $\operatorname{Flats} M$ of the poset $\mathcal{P}(E)$ is a lattice.

Proof of Proposition 4.5.14. By the definition of a lattice, it suffices to check the following four claims:

Claim 1: The poset $\operatorname{Flats} M$ has a global minimum.

Claim 2: The poset $\operatorname{Flats} M$ has a global maximum.

Claim 3: Every two elements of $\operatorname{Flats} M$ have a meet (in $\operatorname{Flats} M$).

Claim 4: Every two elements of $\operatorname{Flats} M$ have a join (in $\operatorname{Flats} M$).

Proof of Claim 1: Applying Proposition 4.5.12 **(a)** to $T = \varnothing$, we see that $\overline{\varnothing}$ is a flat of M satisfying $\varnothing \subseteq \overline{\varnothing}$. In particular, $\overline{\varnothing}$ is a flat of M, so that $\overline{\varnothing} \in \operatorname{Flats} M$. If G is a flat of M, then $\overline{\varnothing} \subseteq G$ (by Proposition 4.5.12 **(c)**, applied to $T = \varnothing$). Hence, $\overline{\varnothing}$ is a global minimum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global minimum. This proves Claim 1.

Proof of Claim 2: Applying Proposition 4.5.12 **(a)** to $T = E$, we see that $\overline{E}$ is a flat of M satisfying $E \subseteq \overline{E}$. From $E \subseteq \overline{E}$, we conclude that $\overline{E} = E$. Thus, E is a flat of M (since $\overline{E}$ is a flat of M). In other words, $E \in \operatorname{Flats} M$. If G is a flat of M, then $E \supseteq G$ (obviously). Hence, E is a global maximum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global maximum. This proves Claim 2.

Proof of Claim 3: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a meet.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Hence, Corollary 4.5.10 shows that $F \cap G$ is a flat of M. In other words, $F \cap G \in \operatorname{Flats} M$. Clearly, $F \cap G \subseteq F$ and $F \cap G \subseteq G$; thus, $F \cap G$ is a lower bound of F and G in $\operatorname{Flats} M$.

Also, every lower bound H of F and G in $\operatorname{Flats} M$ satisfies $H \subseteq F \cap G$ [26]. Hence, $F \cap G$ is a meet of F and G. Thus, F and G have a meet. This proves Claim 3.

Proof of Claim 4: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a join.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Proposition 4.5.12 **(a)** (applied to $T = F \cup G$) shows that $\overline{F \cup G}$ is a flat of M satisfying $F \cup G \subseteq \overline{F \cup G}$. Now, $\overline{F \cup G} \in \operatorname{Flats} M$ (since $\overline{F \cup G}$ is a flat of M). Clearly, $F \subseteq F \cup G \subseteq \overline{F \cup G}$ and $G \subseteq F \cup G \subseteq \overline{F \cup G}$; thus, $\overline{F \cup G}$ is an upper bound of F and G in $\operatorname{Flats} M$. Also, every upper bound H of F and G in $\operatorname{Flats} M$ satisfies $H \supseteq \overline{F \cup G}$ [27]. Hence, $\overline{F \cup G}$ is a join of F and G. Thus, F and G have a join. This proves Claim 4.

We have now proven all four Claims 1, 2, 3, and 4. Thus, Proposition 4.5.14 is proven. □

Definition 4.5.15. Let $M = (E, \mathcal{I})$ be a matroid. Proposition 4.5.14 shows that the subposet $\operatorname{Flats} M$ of the poset $\mathcal{P}(E)$ is a lattice. This subposet $\operatorname{Flats} M$ is called the *lattice of flats* of M. (Beware: It is a subposet, but not a sublattice of $\mathcal{P}(E)$, since its join is not a restriction of the join of $\mathcal{P}(E)$.)

The lattice of flats $\operatorname{Flats} M$ of a matroid M is denoted by $L(M)$ in [Stanley06, §3.2].

Next, we recall the definition of the Möbius function of a poset (see, e.g., [Stanley06, Definition 1.2] or [Martin15, §5.2]):

Definition 4.5.16. Let P be a poset.

(a) If x and y are two elements of P satisfying $x \leq y$, then the set $\{z \in P \mid x \leq z \leq y\}$ is denoted by $[x, y]$.

(b) A subset of P is called a *closed interval* of P if it has the form $[x, y]$ for two elements x and y of P satisfying $x \leq y$.

[26] *Proof.* Let H be a lower bound of F and G in $\operatorname{Flats} M$. Thus, $H \subseteq F$ and $H \subseteq G$. Combining these two inclusions, we obtain $H \subseteq F \cap G$, qed.

[27] *Proof.* Let H be an upper bound of F and G in $\operatorname{Flats} M$. Thus, $H \supseteq F$ and $H \supseteq G$. Combining these two inclusions, we obtain $H \supseteq F \cup G$. But $H \in \operatorname{Flats} M$; thus, H is a flat of M. Since H satisfies $F \cup G \subseteq H$, we therefore obtain $\overline{F \cup G} \subseteq \overline{H}$ (by Proposition 4.5.12 **(c)**, applied to $F \cup G$ and H instead of T and G). In other words, $H \supseteq \overline{F \cup G}$, qed.

(c) We denote by $\operatorname{Int} P$ the set of all closed intervals of P.

(d) If $f : \operatorname{Int} P \to \mathbb{Z}$ is any map, then the image $f([x,y])$ of a closed interval $[x,y] \in \operatorname{Int} P$ under f will be abbreviated by $f(x,y)$.

(e) Assume that every closed interval of P is finite. The *Möbius function* of the poset P is defined to be the unique function $\mu : \operatorname{Int} P \to \mathbb{Z}$ having the following two properties:

- We have
$$\mu(x,x) = 1 \qquad \text{for every } x \in P. \tag{4.23}$$

- We have
$$\mu(x,y) = -\sum_{\substack{z\in P;\\ x\leq z<y}} \mu(x,z) \qquad \text{for all } x,y \in P \text{ satisfying } x<y. \tag{4.24}$$

(It is easy to see that these two properties indeed determine μ uniquely.) This Möbius function is denoted by μ.

We can now define the characteristic polynomial of a matroid M, following [Stanley06, (22)][28]:

Definition 4.5.17. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. The *characteristic polynomial* χ_M of the matroid M is defined to be the polynomial

$$\sum_{F\in\operatorname{Flats} M} \mu\left(\overline{\varnothing}, F\right) x^{m-r_M(F)} \in \mathbb{Z}[x]$$

(where μ is the Möbius function of the lattice $\operatorname{Flats} M$). We further define a polynomial $\widetilde{\chi}_M \in \mathbb{Z}[x]$ by $\widetilde{\chi}_M = [\overline{\varnothing} = \varnothing]\, \chi_M$. Here, we are using the Iverson

[28] Our notation slightly differs from that in [Stanley06, (22)]. Namely, we use x as the indeterminate, while Stanley instead uses t. Stanley also denotes the global minimum $\overline{\varnothing}$ of $\operatorname{Flats} M$ by $\widehat{0}$.

bracket notation (as in Definition 4.2.6). If the matroid M is loopless, then

$$\widetilde{\chi}_M = \underbrace{[\varnothing = \varnothing]}_{\substack{=1 \\ \text{(by Proposition 4.5.12 } \textbf{(e)}\text{)}}} \chi_M = \chi_M.$$

Example 4.5.18. Let $G = (V, E)$ be a finite graph. Consider the graphical matroid $(E, \mathcal{I})$ defined as in Example 4.5.5. Then, the characteristic polynomial $\chi_{(E,\mathcal{I})}$ of this matroid is connected to the chromatic polynomial χ_G of the graph G as follows:

$$x^{\operatorname{conn} G} \cdot \chi_{(E,\mathcal{I})}(x) = \chi_G(x).$$

4.5.3 Generalized formulas

Let us next define broken circuits of a matroid $M = (E, \mathcal{I})$. Stanley, in [Stanley06, §4.1], defines them in terms of a total ordering $\mathcal{O}$ on the set E, whereas we shall use a "labeling function" $\ell : E \to X$ instead (as in the case of graphs); our setting is slightly more general than Stanley's.

Definition 4.5.19. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every $e \in E$, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of M means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C). Of course, the notion of a broken circuit of M depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

We shall now state analogues (and, in light of Example 4.5.18, generalizations, although we shall not elaborate on the few minor technicalities of seeing them as such) of Theorem 4.3.5, Theorem 4.3.4, Corollary 4.3.6, Corollary 4.3.7 and Corollary 4.3.13:

Theorem 4.5.20. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) x^{m - r_M(F)}.$$

Theorem 4.5.21. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 4.5.22. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 4.5.23. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E;\\ F \text{ contains no broken}\\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 4.5.24. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m-|F|}.$$

We notice that Corollary 4.5.24 is equivalent to [Stanley06, Theorem 4.12] (at least when M is loopless).

Before we prove these results, let us state a lemma which will serve as an analogue of Lemma 4.2.7:

Lemma 4.5.25. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let F be any flat of M. Then,

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [F = \varnothing]. \tag{4.25}$$

(Again, we are using the Iverson bracket notation as in Definition 4.2.6.)

Proof of Lemma 4.5.25. Our proof will imitate the proof of Lemma 4.2.7 much of the time (with $E \cap \operatorname{Eqs} f$ replaced by F); thus, we will allow ourselves some more brevity.

We WLOG assume that $F \neq \varnothing$ (since otherwise, the claim is obvious[29]). Thus, $[F = \varnothing] = 0$.

[29] *Proof.* Assume that $F = \varnothing$. We must show that the claim is obvious.

Let us first show that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. Indeed, assume the contrary. Thus, there exists a $K \in \mathfrak{K}$ satisfying $K = \varnothing$. In other words, $\varnothing \in \mathfrak{K}$. Thus, $\varnothing$ is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M). Therefore, $\varnothing$ is obtained from a circuit of M by removing one element (by the definition of a broken circuit). This latter circuit must therefore be a one-element set, i.e., it has the form $\{e\}$ for some $e \in E$. Consider this e. Thus, $\{e\}$ is a circuit of M.

But F is a flat of M. In other words, Statement $\mathfrak{F}_1$ (of Lemma 4.5.9) holds for $T = F$. Hence, Statement $\mathfrak{F}_2$ (of Lemma 4.5.9) also holds for $T = F$ (since Lemma 4.5.9 shows that these two statements are equivalent). Applying Statement $\mathfrak{F}_2$ to $T = F$ and $C = \{e\}$, we thus obtain $\{e\} \subseteq F$ (because $\{e\} \setminus \{e\} = \varnothing \subseteq F$). Thus, $e \in \{e\} \subseteq F = \varnothing$, which is absurd. This contradiction proves that our assumption was wrong.

Hence, we have shown that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. But from $F = \varnothing$, we see that the sum

Pick any $d \in F$ with maximum $\ell(d)$ (among all $d \in F$). (This is clearly possible, since $F \neq \varnothing$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(F)$ as follows:

$$\mathcal{U} = \{T \in \mathcal{P}(F) \mid d \notin T\};$$
$$\mathcal{V} = \{T \in \mathcal{P}(F) \mid d \in T\}.$$

Thus, we have $\mathcal{P}(F) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

This map Φ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ [30]) and a bijection[31]. Moreover, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|} \tag{4.26}$$

[32].

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{4.27}$$

Proof of (4.27): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We must prove the equivalence (4.27). The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$. Hence, if $K \subseteq B$,

---

$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K$ has only one addend (namely, the addend for $B = \varnothing$), and thus simplifies to

$$\underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \underbrace{\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq \varnothing}}}_{= \prod_{\substack{K \in \mathfrak{K}; \\ K = \varnothing}}} a_K = \prod_{\substack{K \in \mathfrak{K}; \\ K = \varnothing}} a_K = (\text{empty product}) \qquad (\text{since no } K \in \mathfrak{K} \text{ satisfies } K = \varnothing)$$
$$= 1 = [F = \varnothing] \qquad (\text{since } F = \varnothing).$$

Thus, Lemma 4.5.25 is proven.

[30]This follows from the fact that $d \in F$.

[31]Its inverse is the map $\Psi : \mathcal{V} \to \mathcal{U}$ defined by $(\Psi(B) = B \setminus \{d\} \qquad \text{for every } B \in \mathcal{V})$.

[32]*Proof.* This is proven exactly like we proved (4.9).

then $K \subseteq B \subseteq \Phi(B)$. Therefore, the forward implication of the equivalence (4.27) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if $K \subseteq \Phi(B)$, then $K \subseteq B$. So let us assume that $K \subseteq \Phi(B)$.

We want to prove that $K \subseteq B$. Assume the contrary. Thus, $K \not\subseteq B$. We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C) (because this is how a broken circuit is defined). Consider these C and e. Thus, $K = C \setminus \{e\}$.

The element e is the unique element of C having maximum label (among the elements of C). Thus, if e' is any element of C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{4.28}$$

But $\underbrace{K}_{\subseteq \Phi(B) = B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq F$ (since $\Phi(B) \in \mathcal{P}(F)$). On the other hand, Statement $\mathfrak{F}_1$ (of Lemma 4.5.9) holds for $T = F$ (since F is a flat of M). Hence, Statement $\mathfrak{F}_2$ (of Lemma 4.5.9) also holds for $T = F$ (since Lemma 4.5.9 shows that these two statements are equivalent). Thus, from $C \setminus \{e\} \subseteq F$, we obtain $C \subseteq F$. Thus, $e \in C \subseteq F$. Consequently, $\ell(d) \geq \ell(e)$ (since d was defined to be an element of F with maximum $\ell(d)$ among all $d \in F$).

Also, $d \in C$. Since $\ell(d) \geq \ell(e)$, we can therefore apply (4.28) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven. Thus, we have proven the backward implication of the equivalence (4.27); this completes the proof of (4.27).

Now, proceeding as in the proof of (4.12), we can show that

$$\sum_{B\subseteq F} (-1)^{|B|} \prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K = [F = \varnothing] .$$

This proves Lemma 4.5.25. □

We shall furthermore use a classical and fundamental result on the Möbius function of any finite poset:

Proposition 4.5.26. Let P be a finite poset. Let μ denote the Möbius function of P.

(a) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(x, z) = [x = y] . \tag{4.29}$$

(b) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z, y) = [x = y] . \tag{4.30}$$

(c) Let $\mathbf{k}$ be a $\mathbb{Z}$-module. Let $(\beta_x)_{x\in P}$ be a family of elements of $\mathbf{k}$. Then, every $z \in P$ satisfies

$$\beta_z = \sum_{\substack{y\in P;\\ y\leq z}} \mu(y, z) \sum_{\substack{x\in P;\\ x\leq y}} \beta_x .$$

For the sake of completeness, let us give a self-contained proof of this proposition (slicker arguments appear in the literature[33]):

Proof of Proposition 4.5.26. **(a)** Let $x \in P$ and $y \in P$. We must prove the equality (4.29). We are in one of the following three cases:

Case 1: We have $x = y$.

[33] For example, Proposition 4.5.26 **(c)** is equivalent to the $\Longrightarrow$ implication of [Martin15, (5.1a)].

Case 2: We have $x < y$.

Case 3: We have neither $x = y$ nor $x < y$.

Let us first consider Case 1. In this case, we have $x = y$. Hence, the sum $\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x, z)$ contains only one addend – namely, the addend for $z = x$. Thus,

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x, z) = \mu(x, x) = 1 \qquad \text{(by the definition of the Möbius function)}$$
$$= [x = y] \qquad (\text{since } x = y).$$

Thus, (4.29) is proven in Case 1.

Let us now consider Case 2. In this case, we have $x < y$. Hence, $x \neq y$, so that $[x = y] = 0$. Now, y is an element of P satisfying $x \leq y \leq y$. Thus, the sum $\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x, z)$ contains an addend for $z = y$. Splitting off this addend, we obtain

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x, z) = \underbrace{\sum_{\substack{z \in P;\\ x \leq z \leq y;\ z \neq y}}}_{= \sum_{\substack{z \in P;\\ x \leq z < y}}} \mu(x, z) + \underbrace{\mu(x, y)}_{\substack{= - \sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x, z) \\ \text{(by (4.24))}}}$$
$$= \sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x, z) + \left(- \sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x, z) \right) = 0 = [x = y].$$

Hence, (4.29) is proven in Case 2.

Finally, let us consider Case 3. In this case, we have neither $x = y$ nor $x < y$. Thus, we do not have $x \leq y$. Hence, there exists no $z \in P$ satisfying $x \leq z \leq y$. Thus,

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x, z) = (\text{empty sum}) = 0 = [x = y]$$

(since we do not have $x = y$). Thus, (4.29) is proven in Case 3.

Hence, (4.29) is proven in all three cases. This proves Proposition 4.5.26 **(a)**.

(b) For any two elements u and v of P, we define a subset $[u, v]$ of P by

$$[u, v] = \{w \in P \mid u \leq w \leq v\}.$$

Thus subset $[u, v]$ is finite (since P is finite), and thus its size $|[u, v]|$ is a nonnegative integer.

We shall now prove Proposition 4.5.26 **(b)** by strong induction on $|[x, y]|$:

Induction step: Let $N \in \mathbb{N}$. Assume that Proposition 4.5.26 **(b)** holds whenever $|[x, y]| < N$. We must now prove that Proposition 4.5.26 **(b)** holds whenever $|[x, y]| = N$.

We have assumed that Proposition 4.5.26 **(b)** holds whenever $|[x, y]| < N$. In other words, we have assumed the following claim:

Claim 1: For any $x \in P$ and $y \in P$ satisfying $|[x, y]| < N$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z, y) = [x = y].$$

Now, let x and y be two elements of P satisfying $|[x, y]| = N$. We are going to prove that

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z, y) = [x = y]. \tag{4.31}$$

We are in one of the following three cases:

Case 1: We have $x = y$.

Case 2: We have $x < y$.

Case 3: We have neither $x = y$ nor $x < y$.

In Case 1 and in Case 3, we can prove (4.31) in exactly the same way as (in our above proof of Proposition 4.5.26 **(a)**) we have proven (4.29). Thus, it remains only to prove (4.31) in Case 2. In other words, we can WLOG assume that we are in Case 2.

Assume this. Hence, $x < y$, so that $[x = y] = 0$.

For every $t \in P$ satisfying $x \leq t < y$, we have

$$|[x,t]| < N \tag{4.32}$$

[34]. Therefore, for every $t \in P$ satisfying $x \leq t < y$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq t}} \mu(z,t) = [x = t] \tag{4.33}$$

(by Claim 1, applied to t instead of y). Also, for every $u \in P$ and $v \in P$, we have

$$\sum_{\substack{t \in P; \\ u \leq t \leq v}} \mu(u,t) = [u = v] \tag{4.34}$$

[35].

---

[34] *Proof of (4.32):* Let $t \in P$ be such that $x \leq t < y$. We shall proceed in several steps:

- We have

$$\begin{aligned}
[x,t] &= \{w \in P \mid x \leq w \leq t\} \qquad \text{(by the definition of } [x,t]) \\
&\subseteq \{w \in P \mid x \leq w \leq y\} \qquad \left(\begin{array}{c}\text{because every } w \in P \text{ satisfying } w \leq t \\ \text{must also satisfy } w \leq y \text{ (since } t < y)\end{array}\right) \\
&= [x,y] \qquad \text{(by the definition of } [x,y]).
\end{aligned}$$

- We have $t < y$. Thus, we do not have $y \leq t$. Hence, we do not have $x \leq y \leq t$. Hence, $y \notin [x,t]$. But $y \in [x,y]$ (since $x \leq y \leq y$). Hence, the sets $[x,t]$ and $[x,y]$ are distinct (since the latter contains y but the former does not). Combining this with $[x,t] \subseteq [x,y]$, we conclude that $[x,t]$ is a proper subset of $[x,y]$. Hence, $|[x,t]| < |[x,y]| = N$. This proves (4.32).

[35] *Proof of (4.34):* Let $u \in P$ and $v \in P$. Proposition 4.5.26 **(a)** (applied to $x = u$ and $y = v$) shows that $\sum_{\substack{z \in P; \\ u \leq z \leq v}} \mu(u,t) = [u = v]$. Now,

$$\begin{aligned}
\sum_{\substack{t \in P; \\ u \leq t \leq v}} \mu(u,t) &= \sum_{\substack{z \in P; \\ u \leq z \leq v}} \mu(u,t) \qquad \text{(here, we have substituted } z \text{ for } t \text{ in the sum)} \\
&= [u = v].
\end{aligned}$$

This proves (4.34).

Now,

$$\begin{aligned}
&\underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq t\leq y}}}_{=\sum_{\substack{z\in P;\\ x\leq z\leq y}}\sum_{\substack{t\in P;\\ z\leq t\leq y}}} \mu\left(z,t\right)\\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \underbrace{\sum_{\substack{t\in P;\\ z\leq t\leq y}} \mu\left(z,t\right)}_{\substack{=[z=y]\\ \text{(by (4.34)}\\ \text{(applied to } u=z \text{ and } v=y\text{))}}} = \sum_{\substack{z\in P;\\ x\leq z\leq y}} [z=y]\\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}} \underbrace{[z=y]}_{\substack{=1\\ \text{(since } z=y\text{)}}} + \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \underbrace{[z=y]}_{\substack{=0\\ \text{(since } z\neq y\text{)}}}\\
&\qquad \left(\text{since every } z\in P \text{ satisfies either } z=y \text{ or } z\neq y \text{ (but not both)}\right)\\
&= \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}}}_{=\sum_{z\in\{w\in P \ \mid\ x\leq w\leq y \text{ and } w=y\}}} 1 + \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} 0}_{=0} = \sum_{z\in\{w\in P \ \mid\ x\leq w\leq y \text{ and } w=y\}} 1\\
&= \left|\underbrace{\{w\in P \ \mid\ x\leq w\leq y \text{ and } w=y\}}_{=\{y\}}\right| = \left|\{y\}\right| = 1.
\end{aligned}$$

Hence,

$$1 = \underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq t\leq y}}}_{=\sum_{\substack{t\in P;\\ x\leq t\leq y}}\sum_{\substack{z\in P;\\ x\leq z\leq t}}} \mu(z,t) = \sum_{\substack{t\in P;\\ x\leq t\leq y}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t)$$

$$= \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t=y}}}_{\substack{=\sum_{t\in\{w\in P \mid x\leq w\leq y \text{ and } w=y\}} = \sum_{t\in\{y\}} \\ (\text{since } \{w\in P \mid x\leq w\leq y \text{ and } w=y\}=\{y\})}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t) + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t)}_{\substack{=[x=t] \\ (\text{by } (4.33) \\ (\text{since } t<y \text{ (because } t\leq y \\ \text{and } t\neq y)) \text{ and } x\leq t)}}$$

$$(\text{since every } t\in P \text{ satisfies either } t=y \text{ or } t\neq y \text{ (but not both)})$$

$$= \underbrace{\sum_{t\in\{y\}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t)}_{=\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)} + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t]$$

$$= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t].$$

Subtracting $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)$ from both sides of this equality, we obtain

$$\begin{aligned}
&1-\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)\\
&=\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t]\\
&=\underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t=x \text{ and } t\neq y}}}_{\substack{=\sum_{t\in\{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}} = \sum_{t\in\{x\}}\\ (\text{since } \{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}=\{x\})}} \underbrace{[x=t]}_{\substack{=1\\ (\text{since } x=t)}} + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq x \text{ and } t\neq y}} \underbrace{[x=t]}_{\substack{=0\\ (\text{since } x\neq t)}}\\
&\qquad (\text{since every } t\in P \text{ satisfies either } t=x \text{ or } t\neq x \text{ (but not both)})\\
&=\sum_{t\in\{x\}} 1+\underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq x \text{ and } t\neq y}} 0}_{=0}=\sum_{t\in\{x\}} 1=1.
\end{aligned}$$

Solving this equality for $\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)$, we obtain

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)=1-1=0=[x=y]$$

(since $x<y$). Thus, (4.31) is proven.

Let us now forget that we fixed x and y. We thus have proven that for any $x\in P$ and $y\in P$ satisfying $|[x,y]|=N$, we have

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)=[x=y].$$

In other words, Proposition 4.5.26 **(b)** holds whenever $|[x,y]|=N$. This completes the induction step. Thus, Proposition 4.5.26 **(b)** is proven by induction.

(c) For every $v \in P$, we have

$$\sum_{\substack{y \in P; \\ y \leq v}} \mu(y,v) \sum_{\substack{x \in P; \\ x \leq y}} \beta_x$$

$$= \sum_{\substack{z \in P; \\ z \leq v}} \mu(z,v) \sum_{\substack{x \in P; \\ x \leq z}} \beta_x \qquad \left(\begin{array}{c} \text{here, we have renamed the summation} \\ \text{index } y \text{ as } z \text{ in the outer sum} \end{array} \right)$$

$$= \underbrace{\sum_{\substack{z \in P; \\ z \leq v}} \sum_{\substack{x \in P; \\ x \leq z}}}_{= \sum_{x \in P} \sum_{\substack{z \in P; \\ x \leq z \leq v}}} \mu(z,v) \beta_x = \sum_{x \in P} \underbrace{\sum_{\substack{z \in P; \\ x \leq z \leq v}} \mu(z,v)}_{\substack{= [x=v] \\ \text{(by Proposition 4.5.26 (b)} \\ \text{(applied to } y=v\text{))}}} \beta_x$$

$$= \sum_{x \in P} [x=v] \beta_x = \sum_{\substack{x \in P; \\ x = v}} \underbrace{[x=v]}_{\substack{=1 \\ \text{(since } x=v\text{)}}} \beta_x + \sum_{\substack{x \in P; \\ x \neq v}} \underbrace{[x=v]}_{\substack{=0 \\ \text{(since } x\neq v\text{)}}} \beta_x$$

$$\text{(since every } x \in P \text{ satisfies either } x = v \text{ or } x \neq v \text{ (but not both))}$$

$$= \sum_{\substack{x \in P; \\ x = v}} \beta_x + \underbrace{\sum_{\substack{x \in P; \\ x \neq v}} 0 \beta_x}_{=0} = \sum_{\substack{x \in P; \\ x = v}} \beta_x = \beta_v \qquad (\text{since } v \in P).$$

Renaming v as z in this result, we obtain precisely Proposition 4.5.26 **(c)**. □

Proof of Theorem 4.5.20. If T is a subset of E, then $\overline{T}$ is a flat of M (by Proposition 4.5.12 **(a)**). In other words, if T is a subset of E, then $\overline{T} \in \operatorname{Flats} M$. Renaming T as B in this statement, we conclude that if B is a subset of E, then $\overline{B} \in \operatorname{Flats} M$.

For every $F \in \operatorname{Flats} M$, define an element $\beta_F \in \mathbf{k}$ by

$$\beta_F = \sum_{\substack{B \subseteq E; \\ \overline{B} = F}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K \right).$$

Now, using Lemma 4.5.25, we can easily see that

$$\sum_{\substack{G \in \operatorname{Flats} M; \\ G \subseteq F}} \beta_G = [F = \varnothing] \qquad \text{for every } F \in \operatorname{Flats} M \tag{4.35}$$

[36].

Let μ be the Möbius function of the lattice $\operatorname{Flats} M$. The element $\overline{\varnothing}$ is the global minimum of the poset $\operatorname{Flats} M$. [37] In particular, $\overline{\varnothing} \in \operatorname{Flats} M$ and $\overline{\varnothing} \subseteq F$. Hence, $\mu\left(\overline{\varnothing}, F\right)$ is well-defined.

Now, fix $F \in \operatorname{Flats} M$. Proposition 4.5.26 **(c)** (applied to $P = \operatorname{Flats} M$ and $z = F$)

---

[36]*Proof of (4.35):* Let $F \in \operatorname{Flats} M$. Thus, F is a flat of M.

If B is a subset of E, then the statements $\left(\overline{B} \subseteq F\right)$ and $(B \subseteq F)$ are equivalent. (This follows from Proposition 4.5.12 **(g)**, applied to $T = B$ and $G = F$.)

Now,

$$\sum_{\substack{G \in \operatorname{Flats} M;\\ G \subseteq F}} \underbrace{\beta_G}_{\substack{= \sum\limits_{\substack{B \subseteq E;\\ \overline{B} = G}} (-1)^{|B|} \left(\prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) \\ \text{(by the definition of } \beta_G\text{)}}}$$

$$= \underbrace{\sum_{\substack{G \in \operatorname{Flats} M;\\ G \subseteq F}} \sum_{\substack{B \subseteq E;\\ \overline{B} = G}}}_{\substack{= \sum\limits_{\substack{B \subseteq E;\\ \overline{B} \subseteq F}} \\ \text{(because if } B \text{ is a subset of } E\text{,} \\ \text{then } \overline{B} \in \operatorname{Flats} M\text{)}}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right)$$

$$= \underbrace{\sum_{\substack{B \subseteq E;\\ \overline{B} \subseteq F}}}_{\substack{= \sum\limits_{\substack{B \subseteq E;\\ B \subseteq F}} \\ \text{(because if } B \text{ is a subset of } E\text{, then} \\ \text{the statements } \left(\overline{B} \subseteq F\right) \text{ and } (B \subseteq F) \text{ are} \\ \text{equivalent)}}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) = \underbrace{\sum_{\substack{B \subseteq E;\\ B \subseteq F}}}_{= \sum\limits_{B \subseteq F}} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right)$$

$$= \sum_{B \subseteq F} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) = [F = \varnothing] \qquad \text{(by (4.25))}.$$

This proves (4.35).

[37]This was proven during our proof of Proposition 4.5.14.

shows that

$$\begin{aligned}
\beta_F &= \sum_{\substack{y \in \operatorname{Flats} M; \\ y \subseteq F}} \mu(y, F) \sum_{\substack{x \in \operatorname{Flats} M; \\ x \subseteq y}} \beta_x \\
&\qquad \left(\text{since the relation } \leq \text{ of the poset } \operatorname{Flats} M \text{ is } \subseteq\right) \\
&= \sum_{\substack{H \in \operatorname{Flats} M; \\ H \subseteq F}} \mu(H, F) \underbrace{\sum_{\substack{G \in \operatorname{Flats} M; \\ G \subseteq H}} \beta_G}_{\substack{= [H = \varnothing] \\ \text{(by (4.35), applied to} \\ H \text{ instead of } F)}} \\
&\qquad \left(\text{here, we renamed the summation indices } y \text{ and } x \text{ as } H \text{ and } G\right) \\
&= \sum_{\substack{H \in \operatorname{Flats} M; \\ H \subseteq F}} \mu(H, F) [H = \varnothing] \\
&= \sum_{\substack{H \in \operatorname{Flats} M; \\ H \subseteq F; \\ H = \varnothing}} \mu(H, F) \underbrace{[H = \varnothing]}_{\substack{=1 \\ \text{(since } H = \varnothing)}} + \sum_{\substack{H \in \operatorname{Flats} M; \\ H \subseteq F; \\ H \neq \varnothing}} \mu(H, F) \underbrace{[H = \varnothing]}_{\substack{=0 \\ \text{(since } H \neq \varnothing)}} \\
&= \underbrace{\sum_{\substack{H \in \operatorname{Flats} M; \\ H \subseteq F; \\ H = \varnothing}}}_{\substack{= \sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \\ \text{(since the condition } H \subseteq F \\ \text{is automatically implied by} \\ \text{the condition } H = \varnothing)}} \mu(H, F) \\
&= \sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F). \qquad (4.36)
\end{aligned}$$

Now, we shall prove that

$$\beta_F = \left[\overline{\varnothing} = \varnothing\right] \mu\left(\overline{\varnothing}, F\right). \qquad (4.37)$$

Proof of (4.37): We are in one of the following two cases:

Case 1: We have $\overline{\varnothing} = \varnothing$.

Case 2: We have $\overline{\varnothing} \neq \varnothing$.

Let us consider Case 1 first. In this case, we have $\overline{\varnothing} = \varnothing$. Hence, $\varnothing = \overline{\varnothing} \in \operatorname{Flats} M$. Thus, the sum $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F)$ has exactly one addend: namely, the addend for $H = \varnothing$. Thus, $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F) = \mu\left(\underbrace{\varnothing}_{=\overline{\varnothing}}, F\right) = \mu(\overline{\varnothing}, F)$. Thus, (4.36) becomes $\beta_F = \sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F) = \mu(\overline{\varnothing}, F)$. Comparing this with $\underbrace{[\overline{\varnothing} = \varnothing]}_{\substack{=1 \\ (\text{since } \overline{\varnothing} = \varnothing)}} \mu(\overline{\varnothing}, F) = \mu(\overline{\varnothing}, F)$, we obtain $\beta_F = [\overline{\varnothing} = \varnothing] \mu(\overline{\varnothing}, F)$. Thus, (4.37) is proven in Case 1.

Let us now consider Case 2. In this case, we have $\overline{\varnothing} \neq \varnothing$. Thus, there exists no $H \in \operatorname{Flats} M$ such that $H = \varnothing$ [38]. Hence, the sum $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F)$ is empty. Thus, $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F) = (\text{empty sum}) = 0$, so that (4.36) becomes $\beta_F = \sum_{\substack{H \in \operatorname{Flats} M; \\ H = \varnothing}} \mu(H, F) = 0$. Comparing this with $\underbrace{[\overline{\varnothing} = \varnothing]}_{\substack{=0 \\ (\text{since } \overline{\varnothing} \neq \varnothing)}} \mu(\overline{\varnothing}, F) = 0$, we obtain $\beta_F = [\overline{\varnothing} = \varnothing] \mu(\overline{\varnothing}, F)$. Thus, (4.37) is proven in Case 2.

Now, we have proven (4.37) in both possible Cases 1 and 2. Thus, (4.37) always holds.

Now, let us forget that we fixed F. We thus have proven (4.37) for each $F \in \operatorname{Flats} M$.

[38] *Proof.* Assume the contrary. Thus, there exists some $H \in \operatorname{Flats} M$ such that $H = \varnothing$. In other words, $\varnothing \in \operatorname{Flats} M$. Hence, $\varnothing$ is a flat of M. Proposition 4.5.12 **(b)** (applied to $G = \varnothing$) thus shows that $\overline{\varnothing} = \varnothing$. This contradicts $\overline{\varnothing} \neq \varnothing$. This contradiction proves that our assumption was wrong, qed.

Now,

$$\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}$$

$$=\underbrace{\sum_{B\subseteq E}}_{\substack{=\sum\limits_{F\in\operatorname{Flats}M}\ \sum\limits_{\substack{B\subseteq E;\\ \overline{B}=F}}\\ \text{(because if } B \text{ is a subset of } E\text{,}\\ \text{then } \overline{B}\in\operatorname{Flats}M)}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M(B)}}_{\substack{=x^{m-r_M\left(\overline{B}\right)}\\ \text{(since Proposition 4.5.12 \textbf{(f)} (applied to } T=B\text{)}\\ \text{shows that } r_M(B)=r_M\left(\overline{B}\right)\text{)}}}$$

(here, we have renamed the summation index F as B)

$$=\sum_{F\in\operatorname{Flats}M}\sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M\left(\overline{B}\right)}}_{\substack{=x^{m-r_M(F)}\\ \text{(since } \overline{B}=F\text{)}}}$$

$$=\sum_{F\in\operatorname{Flats}M}\underbrace{\sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)}_{\substack{=\beta_F=\left[\overline{\varnothing}=\varnothing\right]\mu\left(\overline{\varnothing},F\right)\\ \text{(by (4.37))}}}x^{m-r_M(F)}$$

$$=\sum_{F\in\operatorname{Flats}M}\left[\overline{\varnothing}=\varnothing\right]\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}$$

$$=\left[\overline{\varnothing}=\varnothing\right]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}. \tag{4.38}$$

But the definition of χ_M yields $\chi_M=\sum\limits_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}$. The definition of $\widetilde{\chi}_M$ yields

$$\widetilde{\chi}_M=\left[\overline{\varnothing}=\varnothing\right]\underbrace{\chi_M}_{=\sum\limits_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}}=\left[\overline{\varnothing}=\varnothing\right]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}$$

$$=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}\qquad\text{(by (4.38))}.$$

This proves Theorem 4.5.20. $\square$

Proof of Corollary 4.5.22. Corollary 4.5.22 can be derived from Theorem 4.5.20 in the same way as Corollary 4.1.13 was derived from Theorem 4.1.11. $\square$

Proof of Theorem 4.5.21. Theorem 4.5.21 can be derived from Theorem 4.5.20 in the same way as Theorem 4.1.8 was derived from Theorem 4.1.11. $\square$

Proof of Corollary 4.5.23. Corollary 4.5.23 follows from Corollary 4.5.22 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of M. $\square$

Proof of Corollary 4.5.24. If F is a subset of E such that F contains no broken circuit of M as a subset, then

$$r_M(F) = |F| \qquad (4.39)$$

[39]. Now, Corollary 4.5.23 yields

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} \underbrace{x^{m - r_M(F)}}_{\substack{= x^{m-|F|} \\ \text{(by (4.39))}}} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m-|F|}.$$

This proves Corollary 4.5.24. $\square$

[39] *Proof of (4.39):* Let F be a subset of E such that F contains no broken circuit of M as a subset. We shall show that $F \in \mathcal{I}$. Indeed, assume the contrary. Thus, $F \notin \mathcal{I}$, so that $F \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, there exists a circuit C of M such that $C \subseteq F$ (according to Lemma 4.5.7, applied to $Q = F$). Consider this C. The set C is a circuit, and thus nonempty (because the empty set is in $\mathcal{I}$). Let e be the unique element of C having maximum label. (This is clearly well-defined, since the labeling function ℓ is injective). Then, $C \setminus \{e\}$ is a broken circuit of M (by the definition of a broken circuit). Thus, F contains a broken circuit of M as a subset (since $C \setminus \{e\} \subseteq C \subseteq F$). This contradicts the fact that F contains no broken circuit of M as a subset. This contradiction shows that our assumption was wrong. Hence, $F \in \mathcal{I}$ is proven.

Thus, Lemma 4.5.6 (applied to $T = F$) shows that $r_M(F) = |F|$, qed.

Chapter 5

Proof of a conjecture of Bergeron, Ceballos and Labbé (joint work with Alexander Postnikov)

Abstract

The reduced expressions for a given element w of a Coxeter group (W, S) can be regarded as the vertices of a directed graph $\mathcal{R}(w)$; its arcs correspond to the braid moves. Specifically, an arc goes from a reduced expression $\overrightarrow{a}$ to a reduced expression $\overrightarrow{b}$ when $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by replacing a contiguous subword of the form $stst\cdots$ (for some distinct $s, t \in S$) by $tsts\cdots$ (where both subwords have length $m_{s,t}$, the order of $st \in W$). We prove a strong bipartiteness-type result for this graph $\mathcal{R}(w)$: Not only does every cycle of $\mathcal{R}(w)$ have even length; actually, the arcs of $\mathcal{R}(w)$ can be colored (with colors corresponding to the type of braid moves used), and to every color c corresponds an "opposite" color c^{op} (corresponding to the reverses of the braid moves with color c), and for any color c, the number of arcs in any given cycle of $\mathcal{R}(w)$ having color in $\{c, c^{\mathrm{op}}\}$ is even. This is a generalization and strengthening of a 2014 result by Bergeron, Ceballos and Labbé.

Introduction

Let (W, S) be a Coxeter group[1] with Coxeter matrix $(m_{s,s'})_{(s,s') \in S \times S}$, and let $w \in W$. Consider a directed graph $\mathcal{R}(w)$ whose vertices are the reduced expressions for w, and whose arcs are defined as follows: The graph $\mathcal{R}(w)$ has an arc from a reduced expression $\overrightarrow{a}$ to a reduced expression $\overrightarrow{b}$ whenever $\overrightarrow{b}$ can be obtained from $\overrightarrow{a}$ by replacing some contiguous subword of the form $\underbrace{(s, t, s, t, \ldots)}_{m_{s,t} \text{ factors}}$ by $\underbrace{(t, s, t, s, \ldots)}_{m_{s,t} \text{ factors}}$, where s and t are two distinct elements of S. (This replacement is called an (s, t)-*braid move.*)

The directed graph $\mathcal{R}(w)$ (or, rather, its undirected version) has been studied many times; see, for example, [ReiRoi11] and the references therein. In this note, we shall prove a bipartiteness-type result for $\mathcal{R}(w)$. Its simplest aspect (actually, a corollary) is the fact that $\mathcal{R}(w)$ is bipartite (i.e., every cycle of $\mathcal{R}(w)$ has even length); but we shall concern ourselves with stronger statements. We can regard $\mathcal{R}(w)$ as an edge-colored directed graph: Namely, whenever a reduced expression $\overrightarrow{b}$ is obtained from a reduced expression $\overrightarrow{a}$ by an (s, t)-braid move, we color the arc from $\overrightarrow{a}$ to $\overrightarrow{b}$ with the conjugacy class[2] $[(s, t)]$ of the pair $(s, t) \in S \times S$. Our result (Theorem 5.2.3) then states that, for every such color $[(s, t)]$, every cycle of $\mathcal{R}(w)$ has as many arcs colored $[(s, t)]$ as it has arcs colored $[(t, s)]$, and that the total number of arcs colored $[(s, t)]$ and $[(t, s)]$ in any given cycle is even. This generalizes and strengthens a result of Bergeron, Ceballos and Labbé [BeCeLa14, Theorem 3.1].

5.1 A motivating example

Before we introduce the general setting, let us demonstrate it on a simple example. This example is not necessary for the rest of this note (and can be skipped by the

[1] All terminology and notation that appears in this introduction will later be defined in more detail.

[2] A *conjugacy class* here means an equivalence class under the relation $\sim$ on the set $S \times S$, which is given by

$$\left((s, t) \sim (s', t') \iff \text{there exists a } q \in W \text{ such that } qsq^{-1} = s' \text{ and } qtq^{-1} = t'\right).$$

The conjugacy class of an $(s, t) \in S \times S$ is denoted by $[(s, t)]$.

reader[3]); it merely provides some intuition and motivation for the definitions to come.

For this example, we fix an integer $n \geq 1$, and we let W be the symmetric group S_n of the set $\{1, 2, \ldots, n\}$. For each $i \in \{1, 2, \ldots, n-1\}$, let $s_i \in W$ be the transposition which switches i with $i+1$ (while leaving the remaining elements of $\{1, 2, \ldots, n\}$ unchanged). Let $S = \{s_1, s_2, \ldots, s_{n-1}\} \subseteq W$. The pair (W, S) is an example of what is called a *Coxeter group* (see, e.g., [Bourba81, Chapter 4] and [Lusztig14, §1]); more precisely, it is known as the Coxeter group A_{n-1}. In particular, S is a generating set for W, and the group W can be described by the generators $s_1, s_2, \ldots, s_{n-1}$ and the relations

$$s_i^2 = \text{id} \qquad \text{for every } i \in \{1, 2, \ldots, n-1\}; \tag{5.1}$$

$$s_i s_j = s_j s_i \qquad \text{for every } i, j \in \{1, 2, \ldots, n-1\} \text{ such that } |i-j| > 1; \tag{5.2}$$

$$s_i s_j s_i = s_j s_i s_j \qquad \text{for every } i, j \in \{1, 2, \ldots, n-1\} \text{ such that } |i-j| = 1. \tag{5.3}$$

This is known as the *Coxeter presentation* of S_n, and is due to Moore (see, e.g., [Willia03, Theorem 1.2.4]).

Given any $w \in W$, there exists a tuple $(a_1, a_2, \ldots, a_k)$ of elements of S such that $w = a_1 a_2 \cdots a_k$ (since S generates W). Such a tuple is called a *reduced expression* for w if its length k is minimal among all such tuples (for the given w). For instance, when $n = 4$, the permutation $\pi \in S_4 = W$ that is written as $(3, 1, 4, 2)$ in one-line notation has reduced expressions (s_2, s_1, s_3) and (s_2, s_3, s_1); in fact, $\pi = s_2 s_1 s_3 = s_2 s_3 s_1$. (We are following the convention by which the product $u \circ v = uv$ of two permutations $u, v \in S_n$ is defined to be the permutation sending each i to $u(v(i))$.)

Given a $w \in W$, the set of reduced expressions for w has an additional structure of a directed graph. Namely, the equalities (5.2) and (5.3) show that, given a reduced expression $\overrightarrow{a} = (a_1, a_2, \ldots, a_k)$ for $w \in W$, we can obtain another reduced expression in any of the following two ways:

- Pick some $i, j \in \{1, 2, \ldots, n-1\}$ such that $|i-j| > 1$, and pick any factor of

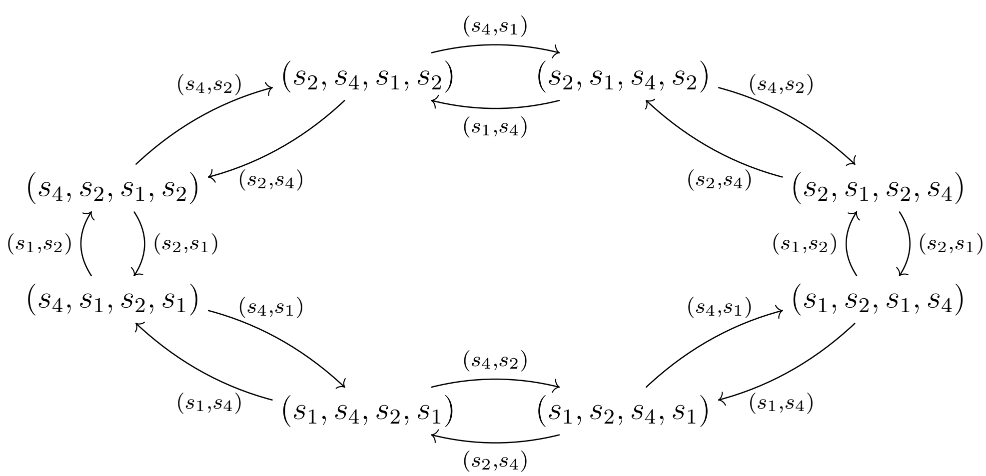
[3] All notations introduced in Section 5.1 should be understood as local to this section; they will not be used beyond it (and often will be replaced by eponymic notations for more general objects).

the form (s_i, s_j) in $\overrightarrow{a}$ (that is, a pair of adjacent entries of $\overrightarrow{a}$, the first of which is s_i and the second of which is s_j), provided that such a factor exists, and replace this factor by (s_j, s_i).

- Alternatively, pick some $i, j \in \{1, 2, \ldots, n-1\}$ such that $|i - j| = 1$, and pick any factor of the form (s_i, s_j, s_i) in $\overrightarrow{a}$, provided that such a factor exists, and replace this factor by (s_j, s_i, s_j).

In both cases, we obtain a new reduced expression for w (provided that the respective factors exist). We say that this new expression is obtained from $\overrightarrow{a}$ by an (s_i, s_j)-*braid move*, or (when we do not want to mention s_i and s_j) by a *braid move*. For instance, the reduced expression (s_2, s_1, s_3) for $\pi = (3, 1, 4, 2) \in S_4$ is obtained from the reduced expression (s_2, s_3, s_1) by an (s_3, s_1)-braid move, and conversely (s_2, s_3, s_1) is obtained from (s_2, s_1, s_3) by an (s_1, s_3)-braid move.

Now, we can define a directed graph $\mathcal{R}_0(w)$ whose vertices are the reduced expressions for w, and which has an edge from $\overrightarrow{a}$ to $\overrightarrow{b}$ whenever $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by a braid move (of either sort). For instance, let $n = 5$, and let w be the permutation written in one-line notation as $(3, 2, 1, 5, 4)$. Then, $\mathcal{R}_0(w)$ looks as follows:



Here, we have "colored" (i.e., labelled) every arc $\left(\overrightarrow{a}, \overrightarrow{b}\right)$ with the pair (s_i, s_j) such that $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s_i, s_j)-braid move.

In our particular case, the graph $\mathcal{R}_0(w)$ consists of a single bidirected cycle. This is not true in general, but certain things hold in general. First, it is clear that whenever an arc from some vertex $\overrightarrow{a}$ to some vertex $\overrightarrow{b}$ has color (s_i, s_j), then there is an arc with color (s_j, s_i) from $\overrightarrow{b}$ to $\overrightarrow{a}$. Thus, $\mathcal{R}_0(w)$ can be regarded as an undirected graph (at the expense of murkying up the colors of the arcs). Furthermore, every reduced expression for w can be obtained from any other by a sequence of braid moves (this is the Matsumoto-Tits theorem; it appears, e.g., in [Lusztig14, Theorem 1.9]). Thus, the graph $\mathcal{R}_0(w)$ is strongly connected.

What do the cycles of $\mathcal{R}_0(w)$ have in common? Walking down the long cycle in the graph $\mathcal{R}_0(w)$ for $w = (3, 2, 1, 5, 4) \in S_5$ counterclockwise, we observe that the (s_1, s_2)-braid move is used once (i.e., we traverse precisely one arc with color (s_1, s_2)), the (s_2, s_1)-braid move once, the (s_1, s_4)-braid move twice, the (s_4, s_1)-braid move once, the (s_2, s_4)-braid move once, and the (s_4, s_2)-braid move twice. In particular:

- The total number of (s_i, s_j)-braid moves with $|i - j| = 1$ used is even (namely, 2).
- The total number of (s_i, s_j)-braid moves with $|i - j| > 1$ used is even (namely, 6).

This example alone is scant evidence of any general result, but both evenness patterns persist for general n, for any $w \in S_n$ and any directed cycle in $\mathcal{R}_0(w)$. We can simplify the statement if we change our coloring to a coarser one. Namely, let $\mathfrak{M}$ denote the subset $\{(s, t) \in S \times S \mid s \neq t\} = \{(s_i, s_j) \mid i \neq j\}$ of $S \times S$. We define a binary relation $\sim$ on $\mathfrak{M}$ by

$$\left((s, t) \sim (s', t') \iff \text{there exists a } q \in W \text{ such that } qsq^{-1} = s' \text{ and } qtq^{-1} = t'\right).$$

This relation $\sim$ is an equivalence relation; it thus gives rise to a quotient set $\mathfrak{M}/\sim$. It is easy to see that the quotient set $\mathfrak{M}/\sim$ has exactly two elements (for $n \geq 4$): the equivalence class of all (s_i, s_j) with $|i - j| = 1$, and the equivalence class of all (s_i, s_j) with $|i - j| > 1$. Let us now define an edge-colored directed graph $\mathcal{R}(w)$ by starting with $\mathcal{R}_0(w)$, and replacing each color (s_i, s_j) by its equivalence class $[(s_i, s_j)]$. Thus, in $\mathcal{R}(w)$, the arcs are colored with the (at most two) elements of $\mathfrak{M}/\sim$. Now, our evenness patterns can be restated as follows: For any $n \in \mathbb{N}$, any $w \in S_n$ and any color $c \in \mathfrak{M}/\sim$, any directed cycle of $\mathcal{R}(w)$ has an even number of arcs with color c.

This can be generalized further to every Coxeter group, with a minor caveat. Namely, let (W, S) be a Coxeter group with Coxeter matrix $(m_{s,s'})_{(s,s') \in S \times S}$. Notions such as reduced expressions and braid moves still make sense (see below for references and definitions). We redefine $\mathfrak{M}$ as $\{(s,t) \in S \times S \mid s \neq t \text{ and } m_{s,t} < \infty\}$ (since pairs (s,t) with $m_{s,t} = \infty$ do not give rise to braid moves). Unlike in the case of $W = S_n$, it is not necessarily true that $(s,t) \sim (t,s)$ for every $(s,t) \in \mathfrak{M}$. We define $[(s,t)]^{\text{op}} = [(t,s)]$. The evenness pattern now has to be weakened as follows: For every $w \in W$ and any color $c \in \mathfrak{M}/\sim$, any directed cycle of $\mathcal{R}(w)$ has an even number of arcs whose color belongs to $\{c, c^{\text{op}}\}$. (For $W = S_n$, we have $c = c^{\text{op}}$, and thus this recovers our old evenness patterns.) This is part of the main theorem we will prove in this note – namely, Theorem 5.2.3 **(b)**; it extends a result [BeCeLa14, Theorem 3.1] obtained by Bergeron, Ceballos and Labbé by geometric means. The other part of the main theorem (Theorem 5.2.3 **(a)**) states that any directed cycle of $\mathcal{R}(w)$ has as many arcs with color c as it has arcs with color c^{op}.

5.2 The theorem

In the following, we shall use the notations of [Lusztig14, §1] concerning Coxeter groups. (These notations are compatible with those of [Bourba81, Chapter 4], except that Bourbaki writes $m(s,s')$ instead of $m_{s,s'}$, and speaks of "Coxeter systems" instead of "Coxeter groups".)

We fix a Coxeter group[4] (W, S) with Coxeter matrix $(m_{s,s'})_{(s,s') \in S \times S}$. Thus, W is

[4] Let us give a brief definition of Coxeter groups and Coxeter matrices:
A *Coxeter group* is a pair (W, S), where W is a group, and where S is a finite subset of W having the following property: There exists a matrix $(m_{s,s'})_{(s,s') \in S \times S} \in \{1, 2, 3, \ldots, \infty\}^{S \times S}$ such that

- every $s \in S$ satisfies $m_{s,s} = 1$;
- every two distinct elements s and t of S satisfy $m_{s,t} = m_{t,s} \geq 2$;
- the group W can be presented by the generators S and the relations

$$(st)^{m_{s,t}} = 1 \qquad \text{for all } (s,t) \in S \times S \text{ satisfying } m_{s,t} \neq \infty.$$

a group, and S is a set of elements of order 2 in W such that for every $(s, s') \in S \times S$, the element $ss' \in W$ has order $m_{s,s'}$. (See, e.g., [Lusztig14, Proposition 1.3(b)] for this well-known fact.)

We let $\mathfrak{M}$ denote the subset

$$\{(s,t) \in S \times S \mid s \neq t \text{ and } m_{s,t} < \infty\}$$

of $S \times S$. (This is denoted by I in [Bourba81, Chapter 4, n° 1.3].) We define a binary relation $\sim$ on $\mathfrak{M}$ by

$$\left((s,t) \sim (s',t') \iff \text{there exists a } q \in W \text{ such that } qsq^{-1} = s' \text{ and } qtq^{-1} = t'\right).$$

It is clear that this relation $\sim$ is an equivalence relation; it thus gives rise to a quotient set $\mathfrak{M}/\sim$. For every pair $P \in \mathfrak{M}$, we denote by $[P]$ the equivalence class of P with respect to this relation $\sim$.

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$.

A *word* will mean a k-tuple for some $k \in \mathbb{N}$. A *subword* of a word $(s_1, s_2, \ldots, s_k)$ will mean a word of the form $(s_{i_1}, s_{i_2}, \ldots, s_{i_p})$, where $i_1, i_2, \ldots, i_p$ are elements of $\{1, 2, \ldots, k\}$ satisfying $i_1 < i_2 < \cdots < i_p$. For instance, (1), $(3,5)$, $(1,3,5)$, $()$ and $(1,5)$ are subwords of the word $(1,3,5)$. A *factor* of a word $(s_1, s_2, \ldots, s_k)$ will mean a word of the form $(s_{i+1}, s_{i+2}, \ldots, s_{i+m})$ for some $i \in \{0, 1, \ldots, k\}$ and some $m \in \{0, 1, \ldots, k-i\}$. For instance, (1), $(3,5)$, $(1,3,5)$ and $()$ are factors of the word $(1,3,5)$, but $(1,5)$ is not.

We recall that a *reduced expression* for an element $w \in W$ is a k-tuple $(s_1, s_2, \ldots, s_k)$ of elements of S such that $w = s_1 s_2 \cdots s_k$, and such that k is minimum (among all such tuples). The length of a reduced expression for w is called the *length* of w, and is denoted by $l(w)$. Thus, a reduced expression for an element $w \in W$ is a k-tuple

In this case, the matrix $(m_{s,s'})_{(s,s') \in S \times S}$ is called the *Coxeter matrix* of (W, S). It is well-known (see, e.g., [Lusztig14, §1]) that any Coxeter group has a unique Coxeter matrix, and conversely, for every finite set S and any matrix $(m_{s,s'})_{(s,s') \in S \times S} \in \{1, 2, 3, \ldots, \infty\}^{S \times S}$ satisfying the first two of the three requirements above, there exists a unique (up to isomorphism preserving S) Coxeter group (W, S).

$(s_1, s_2, \ldots, s_k)$ of elements of S such that $w = s_1 s_2 \cdots s_k$ and $k = l(w)$.

Definition 5.2.1. Let $w \in W$. Let $\overrightarrow{a} = (a_1, a_2, \ldots, a_k)$ and $\overrightarrow{b} = (b_1, b_2, \ldots, b_k)$ be two reduced expressions for w.

Let $(s,t) \in \mathfrak{M}$. We say that $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s,t)-*braid move* if $\overrightarrow{b}$ can be obtained from $\overrightarrow{a}$ by finding a factor of $\overrightarrow{a}$ of the form $\underbrace{(s,t,s,t,s,\ldots)}_{m_{s,t} \text{ elements}}$ and replacing it by $\underbrace{(t,s,t,s,t,\ldots)}_{m_{s,t} \text{ elements}}$.

We notice that if $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s,t)-braid move, then $\overrightarrow{a}$ is obtained from $\overrightarrow{b}$ by an (t,s)-braid move.

Definition 5.2.2. Let $w \in W$. We define an edge-colored directed graph $\mathcal{R}(w)$, whose arcs are colored with elements of $\mathfrak{M}/\sim$, as follows:

- The vertex set of $\mathcal{R}(w)$ shall be the set of all reduced expressions for w.
- The arcs of $\mathcal{R}(w)$ are defined as follows: Whenever $(s,t) \in \mathfrak{M}$, and whenever $\overrightarrow{a}$ and $\overrightarrow{b}$ are two reduced expressions for w such that $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s,t)-braid move, we draw an arc from s to t with color $[(s,t)]$.

Theorem 5.2.3. Let $w \in W$. Let C be a (directed) cycle in the graph $\mathcal{R}(w)$. Let $c = [(s,t)] \in \mathfrak{M}/\sim$ be an equivalence class with respect to $\sim$. Let c^{op} be the equivalence class $[(t,s)] \in \mathfrak{M}/\sim$. Then:

(a) The number of arcs colored c appearing in the cycle C equals the number of arcs colored c^{op} appearing in the cycle C.

(b) The number of arcs whose color belongs to $\{c, c^{\text{op}}\}$ appearing in the cycle C is even.

None of the parts **(a)** and **(b)** of Theorem 5.2.3 is a trivial consequence of the other: When $c = c^{\text{op}}$, the statement of Theorem 5.2.3 **(a)** is obvious and does not imply part **(b)**.

Theorem 5.2.3 **(b)** generalizes [BeCeLa14, Theorem 3.1] in two directions: First, Theorem 5.2.3 is stated for arbitrary Coxeter groups, rather than only for finite

Coxeter groups as in [BeCeLa14]. Second, in the terms of [BeCeLa14, Remark 3.3], we are working with sets Z that are "stabled by conjugation instead of automorphism".

5.3 The strategy

We shall now introduce some notations and state some auxiliary results that will be used to prove Theorem 5.2.3. Our strategy of proof is inspired by that used in [BeCeLa14, §3.4] and thus (indirectly) also by that in [ReiRoi11, §3, and proof of Corollary 5.2]; however, we shall avoid any use of geometry (such as roots and hyperplane arrangements), and work entirely with the Coxeter group itself.

We denote the subset $\bigcup_{x \in W} xSx^{-1}$ of W by T. The elements of T are called the *reflections* (of W). They all have order 2. (The notation T is used here in the same meaning as in [Lusztig14, §1] and in [Bourba81, Chapter 4, n° 1.4].)

Definition 5.3.1. For every $k \in \mathbb{N}$, we consider the set W^k as a left W-set by the rule

$$w\left(w_1, w_2, \ldots, w_k\right) = \left(ww_1, ww_2, \ldots, ww_k\right),$$

and as a right W-set by the rule

$$\left(w_1, w_2, \ldots, w_k\right) w = \left(w_1 w, w_2 w, \ldots, w_k w\right).$$

Definition 5.3.2. Let s and t be two distinct elements of T. Let $m_{s,t}$ denote the order of the element $st \in W$. (This extends the definition of $m_{s,t}$ for $s, t \in S$.) Assume that $m_{s,t} < \infty$. We let $D_{s,t}$ denote the subgroup of W generated by s and t. Then, $D_{s,t}$ is a dihedral group (since s and t are two distinct nontrivial involutions, and since any group generated by two distinct nontrivial involutions is dihedral). We denote by $\rho_{s,t}$ the word

$$\left((st)^0 s, (st)^1 s, \ldots, (st)^{m_{s,t}-1} s\right) = \left(s, sts, ststs, \ldots, \underbrace{ststs \cdots s}_{2m_{s,t}-1 \text{ factors}}\right) \in \left(D_{s,t}\right)^{m_{s,t}}.$$

The *reversal* of a word $(a_1, a_2, \ldots, a_k)$ is defined to be the word $(a_k, a_{k-1}, \ldots, a_1)$.

The following proposition collects some simple properties of the words $\rho_{s,t}$. We delay its proof until the next section, to avoid cluttering up this part of the note.

Proposition 5.3.3. Let s and t be two distinct elements of T such that $m_{s,t} < \infty$. Then:

(a) The word $\rho_{s,t}$ consists of reflections in $D_{s,t}$, and contains every reflection in $D_{s,t}$ exactly once.

(b) The word $\rho_{t,s}$ is the reversal of the word $\rho_{s,t}$.

(c) Let $q \in W$. Then, the word $q\rho_{t,s}q^{-1}$ is the reversal of the word $q\rho_{s,t}q^{-1}$.

Definition 5.3.4. Let $\overrightarrow{a} = (a_1, a_2, \ldots, a_k) \in S^k$. Then, $\operatorname{Invs} \overrightarrow{a}$ is defined to be the k-tuple $(t_1, t_2, \ldots, t_k) \in T^k$, where we set

$$t_i = (a_1 a_2 \cdots a_{i-1})\, a_i\, (a_1 a_2 \cdots a_{i-1})^{-1} \qquad \text{for every } i \in \{1, 2, \ldots, k\}.$$

Remark 5.3.5. Let $w \in W$. Let $\overrightarrow{a} = (a_1, a_2, \ldots, a_k)$ be a reduced expression for w. The k-tuple $\operatorname{Invs} \overrightarrow{a}$ is denoted by $\Phi(\overrightarrow{a})$ in [Bourba81, Chapter 4, n° 1.4], and is closely connected to various standard constructions in Coxeter group theory. A well-known fact states that the set of all entries of $\operatorname{Invs} \overrightarrow{a}$ depends only on w (but not on $\overrightarrow{a}$); this set is called the *(left) inversion set* of w. The k-tuple $\operatorname{Invs} \overrightarrow{a}$ contains each element of this set exactly once (see Proposition 5.3.6 below); it thus induces a total order on this set.

Proposition 5.3.6. Let $w \in W$.

(a) If $\overrightarrow{a}$ is a reduced expression for w, then all entries of the tuple $\operatorname{Invs} \overrightarrow{a}$ are distinct.

(b) Let $(s, t) \in \mathfrak{M}$. Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two reduced expressions for w such that $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s, t)-braid move. Then, there exists a $q \in W$ such that $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal[5].

Again, we refer to the next section for the proof of Proposition 5.3.6.

The following fact is rather easy (but will be proven in detail in the next section):

Proposition 5.3.7. Let $w \in W$. Let s and t be two distinct elements of T such that $m_{s,t} < \infty$. Let $\overrightarrow{a}$ be a reduced expression for w.

(a) The word $\rho_{s,t}$ appears as a subword of $\operatorname{Invs} \overrightarrow{a}$ at most one time.

(b) The words $\rho_{s,t}$ and $\rho_{t,s}$ cannot both appear as subwords of $\operatorname{Invs} \overrightarrow{a}$.

We now let $\mathfrak{N}$ denote the subset $\bigcup\limits_{x \in W} x\mathfrak{M}x^{-1}$ of $T \times T$. Clearly, $\mathfrak{M} \subseteq \mathfrak{N}$. Moreover, for every $(s,t) \in \mathfrak{N}$, we have $s \neq t$ and $m_{s,t} < \infty$ (because $(s,t) \in \mathfrak{N} = \bigcup\limits_{x \in W} x\mathfrak{M}x^{-1}$, and because these properties are preserved by conjugation). Thus, for every $(s,t) \in \mathfrak{N}$, the word $\rho_{s,t}$ is well-defined and has length $m_{s,t}$.

We define a binary relation $\approx$ on $\mathfrak{N}$ by

$$\left((s,t) \approx (s',t') \iff \text{there exists a } q \in W \text{ such that } qsq^{-1} = s' \text{ and } qtq^{-1} = t'\right).$$

It is clear that this relation $\approx$ is an equivalence relation; it thus gives rise to a quotient set $\mathfrak{N}/\approx$. For every pair $P \in \mathfrak{N}$, we denote by $[[P]]$ the equivalence class of P with respect to this relation $\approx$.

The relation $\sim$ on $\mathfrak{M}$ is the restriction of the relation $\approx$ to $\mathfrak{M}$. Hence, every equivalence class c with respect to $\sim$ is a subset of an equivalence class with respect to $\approx$. We denote the latter equivalence class by $c_{\mathfrak{N}}$. Thus, $[P]_{\mathfrak{N}} = [[P]]$ for every $P \in \mathfrak{M}$.

We notice that the set $\mathfrak{N}$ is invariant under switching the two elements of a pair (i.e., for every $(u,v) \in \mathfrak{N}$, we have $(v,u) \in \mathfrak{N}$). Moreover, the relation $\approx$ is preserved under switching the two elements of a pair (i.e., if $(s,t) \approx (s',t')$, then $(t,s) \approx (t',s')$). This shall be tacitly used in the following proofs.

[5] See Definition 5.3.1 for the meaning of $q\rho_{s,t}q^{-1}$.

Definition 5.3.8. Let $w \in W$. Let $\overrightarrow{a}$ be a reduced expression for w.

(a) For any $(s,t) \in \mathfrak{N}$, we define an element $\operatorname{has}_{s,t} \overrightarrow{a} \in \{0,1\}$ by

$$\operatorname{has}_{s,t} \overrightarrow{a} = \begin{cases} 1, & \text{if } \rho_{s,t} \text{ appears as a subword of } \operatorname{Invs} \overrightarrow{a}; \\ 0, & \text{otherwise} \end{cases}.$$

(Keep in mind that we are speaking of subwords, not just factors, here.)

(b) Consider the free $\mathbb{Z}$-module $\mathbb{Z}[\mathfrak{N}]$ with basis $\mathfrak{N}$. We define an element $\operatorname{Has} \overrightarrow{a} \in \mathbb{Z}[\mathfrak{N}]$ by

$$\operatorname{Has} \overrightarrow{a} = \sum_{(s,t) \in \mathfrak{N}} \operatorname{has}_{s,t} \overrightarrow{a} \cdot (s,t)$$

(where the (s,t) stands for the basis element $(s,t) \in \mathfrak{N}$ of $\mathbb{Z}[\mathfrak{N}]$).

We can now state the main result that we will use to prove Theorem 5.2.3:

Theorem 5.3.9. Let $w \in W$. Let $(s,t) \in \mathfrak{M}$. Let $\overrightarrow{a}$ and $\overrightarrow{b}$ be two reduced expressions for w such that $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s,t)-braid move.

Proposition 5.3.6 **(b)** shows that there exists a $q \in W$ such that $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal. Consider this q. Set $s' = qsq^{-1}$ and $t' = qtq^{-1}$; thus, s' and t' are reflections and satisfy $m_{s',t'} = m_{s,t} < \infty$. Also, the definitions of s' and t' yield $(s',t') = q \underbrace{(s,t)}_{\in \mathfrak{M}} q^{-1} \in q\mathfrak{M}q^{-1} \subseteq \mathfrak{N}$. Similarly, $(t',s') \in \mathfrak{N}$ (since $(t,s) \in \mathfrak{M}$).

Now, we have

$$\operatorname{Has} \overrightarrow{b} = \operatorname{Has} \overrightarrow{a} - (s',t') + (t',s') . \qquad (5.4)$$

5.4 The proof

By now, we owe the reader several proofs. Let us first see how Theorem 5.2.3 follows from Theorem 5.3.9:

Proof of Theorem 5.2.3. We shall use the *Iverson bracket notation*: i.e., if $\mathcal{A}$ is any logical statement, then we shall write $[\mathcal{A}]$ for the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$.

For every $z \in \mathbb{Z}[\mathfrak{N}]$ and $n \in \mathfrak{N}$, we let $\operatorname{coord}_n z \in \mathbb{Z}$ be the n-coordinate of z (with respect to the basis $\mathfrak{N}$ of $\mathbb{Z}[\mathfrak{N}]$).

For every $z \in \mathbb{Z}[\mathfrak{N}]$ and $N \subseteq \mathfrak{N}$, we set $\operatorname{coord}_N z = \sum_{n \in N} \operatorname{coord}_n z$.

We have $c = [(s,t)]$, thus $c_{\mathfrak{N}} = [[(s,t)]]$ and $c^{\mathrm{op}} = [(t,s)]$. From the latter equality, we obtain $(c^{\mathrm{op}})_{\mathfrak{N}} = [[(t,s)]]$.

Let $\overrightarrow{c_1}, \overrightarrow{c_2}, \ldots, \overrightarrow{c_k}, \overrightarrow{c_{k+1}}$ be the vertices on the cycle C (listed in the order they are encountered when we traverse the cycle, starting at some arbitrarily chosen vertex on the cycle and going until we return to the starting point). Thus:

- We have $\overrightarrow{c_{k+1}} = \overrightarrow{c_1}$.
- There is an arc from $\overrightarrow{c_i}$ to $\overrightarrow{c_{i+1}}$ for every $i \in \{1, 2, \ldots, k\}$.

Fix $i \in \{1, 2, \ldots, k\}$. Then, there is an arc from $\overrightarrow{c_i}$ to $\overrightarrow{c_{i+1}}$. In other words, there exists some $(s_i, t_i) \in \mathfrak{M}$ such that $\overrightarrow{c_{i+1}}$ is obtained from $\overrightarrow{c_i}$ by an (s_i, t_i)-braid move. Consider this (s_i, t_i). Thus,

$$\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}} \text{ is } [(s_i, t_i)]. \tag{5.5}$$

Proposition 5.3.6 **(b)** (applied to $\overrightarrow{c_i}$, $\overrightarrow{c_{i+1}}$, s_i and t_i instead of $\overrightarrow{a}$, $\overrightarrow{b}$, s and t) shows that there exists a $q \in W$ such that $\operatorname{Invs} \overrightarrow{c_{i+1}}$ is obtained from $\operatorname{Invs} \overrightarrow{c_i}$ by replacing a particular factor of the form $q \rho_{s_i, t_i} q^{-1}$ by its reversal. Let us denote this q by q_i. Set $s_i' = q_i s_i q_i^{-1}$ and $t_i' = q_i t_i q_i^{-1}$. Thus, $s_i' \neq t_i'$ (since $s_i \neq t_i$) and $m_{s_i', t_i'} = m_{s_i, t_i} < \infty$ (since $(s_i, t_i) \in \mathfrak{M}$). Also, the definitions of s_i' and t_i' yield $(s_i', t_i') = \left(q_i s_i q_i^{-1}, q_i t_i q_i^{-1}\right) = q_i \underbrace{(s_i, t_i)}_{\in \mathfrak{M}} q_i^{-1} \in q_i \mathfrak{M} q_i^{-1} \subseteq \mathfrak{N}$. From $s_i' = q_i s_i q_i^{-1}$ and $t_i' = q_i t_i q_i^{-1}$, we obtain $(s_i', t_i') \approx (s_i, t_i)$.

We shall now show that

$$\operatorname{coord}_{c_{\mathfrak{N}}} \left(\operatorname{Has} \overrightarrow{c_{i+1}} - \operatorname{Has} \overrightarrow{c_i}\right) = [[(s_i, t_i)] = c^{\mathrm{op}}] - [[(s_i, t_i)] = c]. \tag{5.6}$$

Proof of (5.6): We have the following chain of logical equivalences:

$$\begin{aligned}
&\left((t_i', s_i') \in \underbrace{c_{\mathfrak{N}}}_{=[[(s,t)]]}\right) \\
\Longleftrightarrow\ & ((t_i', s_i') \in [[(s,t)]]) \Longleftrightarrow ((t_i', s_i') \approx (s,t)) \Longleftrightarrow ((s_i', t_i') \approx (t,s)) \\
\Longleftrightarrow\ & ((s_i, t_i) \approx (t,s)) \qquad \text{(since } (s_i', t_i') \approx (s_i, t_i)\text{)} \\
\Longleftrightarrow\ & ((s_i, t_i) \sim (t,s)) \qquad \text{(since the restriction of the relation } \approx \text{ to } \mathfrak{M} \text{ is } \sim\text{)} \\
\Longleftrightarrow\ & \left((s_i, t_i) \in \underbrace{[(t,s)]}_{=c^{\mathrm{op}}}\right) \Longleftrightarrow ((s_i, t_i) \in c^{\mathrm{op}}) \Longleftrightarrow ([(s_i, t_i)] = c^{\mathrm{op}}) .
\end{aligned}$$

Hence,

$$[(t_i', s_i') \in c_{\mathfrak{N}}] = [[(s_i, t_i)] = c^{\mathrm{op}}] . \tag{5.7}$$

Also, we have the following chain of logical equivalences:

$$\begin{aligned}
&\left((s_i', t_i') \in \underbrace{c_{\mathfrak{N}}}_{=[[(s,t)]]}\right) \\
\Longleftrightarrow\ & ((s_i', t_i') \in [[(s,t)]]) \Longleftrightarrow ((s_i', t_i') \approx (s,t)) \\
\Longleftrightarrow\ & ((s_i, t_i) \approx (s,t)) \qquad \text{(since } (s_i', t_i') \approx (s_i, t_i)\text{)} \\
\Longleftrightarrow\ & ((s_i, t_i) \sim (s,t)) \qquad \text{(since the restriction of the relation } \approx \text{ to } \mathfrak{M} \text{ is } \sim\text{)} \\
\Longleftrightarrow\ & \left((s_i, t_i) \in \underbrace{[(s,t)]}_{=c}\right) \Longleftrightarrow ((s_i, t_i) \in c) \Longleftrightarrow ([(s_i, t_i)] = c) .
\end{aligned}$$

Hence,

$$[(s_i', t_i') \in c_{\mathfrak{N}}] = [[(s_i, t_i)] = c] . \tag{5.8}$$

Applying (5.4) to $\overrightarrow{c_i}$, $\overrightarrow{c_{i+1}}$, s_i, t_i, q_i, s_i' and t_i' instead of $\overrightarrow{a}$, $\overrightarrow{b}$, s, t, q, s' and t', we obtain $\operatorname{Has}\overrightarrow{c_{i+1}} = \operatorname{Has}\overrightarrow{c_i} - (s_i', t_i') + (t_i', s_i')$. In other words, $\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i} =$

$(t_i', s_i') - (s_i', t_i')$. Thus,

$$\begin{aligned}
&\operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i}\right) \\
&= \operatorname{coord}_{c_{\mathfrak{N}}}\left((t_i', s_i') - (s_i', t_i')\right) = \underbrace{\operatorname{coord}_{c_{\mathfrak{N}}}(t_i', s_i')}_{\substack{=\left[\left(t_i', s_i'\right) \in c_{\mathfrak{N}}\right] \\ =[[(s_i,t_i)]=c^{\mathrm{op}}] \\ \text{(by (5.7))}}} - \underbrace{\operatorname{coord}_{c_{\mathfrak{N}}}(s_i', t_i')}_{\substack{=\left[\left(s_i', t_i'\right) \in c_{\mathfrak{N}}\right] \\ =[[(s_i,t_i)]=c] \\ \text{(by (5.8))}}} \\
&= [[(s_i, t_i)] = c^{\mathrm{op}}] - [[(s_i, t_i)] = c]\,.
\end{aligned}$$

This proves (5.6).

Now, let us forget that we fixed i. Thus, for every $i \in \{1, 2, \ldots, k\}$, we have defined $(s_i, t_i) \in \mathfrak{M}$ satisfying (5.5) and (5.6).

Now,

$$\begin{aligned}
&\sum_{i=1}^{k} \underbrace{\operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i}\right)}_{=\operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_{i+1}}\right) - \operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_i}\right)} \\
&= \sum_{i=1}^{k} \left(\operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_{i+1}}\right) - \operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_i}\right)\right) = 0
\end{aligned}$$

(by the telescope principle). Hence,

$$\begin{aligned}
0 &= \sum_{i=1}^{k} \underbrace{\operatorname{coord}_{c_{\mathfrak{N}}}\left(\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i}\right)}_{\substack{=[[(s_i,t_i)]=c^{\mathrm{op}}] - [[(s_i,t_i)]=c] \\ \text{(by (5.6))}}} \\
&= \sum_{i=1}^{k} \left([[(s_i, t_i)] = c^{\mathrm{op}}] - [[(s_i, t_i)] = c]\right)
\end{aligned}$$

$$= \sum_{i=1}^{k} \left[\underbrace{[(s_i, t_i)]}_{\substack{=\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) \\ \text{(by (5.5))}}} = c^{\text{op}} \right]$$

$$- \sum_{i=1}^{k} \left[\underbrace{[(s_i, t_i)]}_{\substack{=\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) \\ \text{(by (5.5))}}} = c \right]$$

$$= \underbrace{\sum_{i=1}^{k} \left[\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) = c^{\text{op}} \right]}_{=(\text{the number of arcs colored } c^{\text{op}} \text{ appearing in } C)}$$

$$- \underbrace{\sum_{i=1}^{k} \left[\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) = c \right]}_{=(\text{the number of arcs colored } c \text{ appearing in } C)}$$

$$= (\text{the number of arcs colored } c^{\text{op}} \text{ appearing in } C)$$

$$- (\text{the number of arcs colored } c \text{ appearing in } C)\,.$$

In other words, the number of arcs colored c appearing in C equals the number of arcs colored c^{op} appearing in C. This proves Theorem 5.2.3 **(a)**.

(b) If $c \neq c^{\text{op}}$, then Theorem 5.2.3 **(b)** follows immediately from Theorem 5.2.3 **(a)**. Thus, for the rest of this proof, assume that $c = c^{\text{op}}$ (without loss of generality).

We have $[(s,t)] = c = c^{\text{op}} = [(t,s)]$, so that $(t,s) \sim (s,t)$. Hence, $(t,s) \approx (s,t)$ (since $\sim$ is the restriction of the relation $\approx$ to $\mathfrak{M}$).

Fix some total order on the set S. Let d be the subset $\{(u,v) \in c_{\mathfrak{N}} \mid u < v\}$ of $c_{\mathfrak{N}}$.

Fix $i \in \{1, 2, \ldots, k\}$. We shall now show that

$$\operatorname{coord}_d \left(\operatorname{Has} \overrightarrow{c_{i+1}} - \operatorname{Has} \overrightarrow{c_i}\right) \equiv [[(s_i, t_i)] = c] \bmod 2. \tag{5.9}$$

Proof of (5.9): Define q_i, s'_i and t'_i as before. We have $s'_i \neq t'_i$. Hence, either $s'_i < t'_i$ or $t'_i < s'_i$.

Applying (5.4) to $\overrightarrow{c_i}$, $\overrightarrow{c_{i+1}}$, s_i, t_i, q_i, s'_i and t'_i instead of $\overrightarrow{a}$, $\overrightarrow{b}$, s, t, q, s' and t', we obtain $\operatorname{Has}\overrightarrow{c_{i+1}} = \operatorname{Has}\overrightarrow{c_i} - (s'_i, t'_i) + (t'_i, s'_i)$. In other words, $\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i} = (t'_i, s'_i) - (s'_i, t'_i)$. Thus,

$$
\begin{aligned}
&\operatorname{coord}_d\left(\operatorname{Has}\overrightarrow{c_{i+1}} - \operatorname{Has}\overrightarrow{c_i}\right) \\
&= \operatorname{coord}_d\left((t'_i, s'_i) - (s'_i, t'_i)\right) = \underbrace{\operatorname{coord}_d(t'_i, s'_i)}_{=\left[\left(t'_i, s'_i\right)\in d\right]} - \underbrace{\operatorname{coord}_d(s'_i, t'_i)}_{=\left[\left(s'_i, t'_i\right)\in d\right]} \\
&= \left[(t'_i, s'_i) \in d\right] - \left[(s'_i, t'_i) \in d\right] \\
&\equiv \left[\underbrace{(t'_i, s'_i) \in d}_{\substack{\iff \left(\left(t'_i, s'_i\right)\in c_{\mathfrak{N}} \text{ and } t'_i < s'_i\right) \\ \text{(by the definition of } d\text{)}}}\right] + \left[\underbrace{(s'_i, t'_i) \in d}_{\substack{\iff \left(\left(s'_i, t'_i\right)\in c_{\mathfrak{N}} \text{ and } s'_i < t'_i\right) \\ \text{(by the definition of } d\text{)}}}\right] \\
&= \left[(t'_i, s'_i) \in \underbrace{c_{\mathfrak{N}}}_{=[[(s,t)]]} \text{ and } t'_i < s'_i\right] + \left[(s'_i, t'_i) \in \underbrace{c_{\mathfrak{N}}}_{=[[(s,t)]]} \text{ and } s'_i < t'_i\right] \\
&= \left[\underbrace{(t'_i, s'_i) \in [[(s,t)]]}_{\substack{\iff \left(\left(t'_i, s'_i\right)\approx(s,t)\right) \\ \iff \left(\left(s'_i, t'_i\right)\approx(t,s)\right) \\ \iff ((s_i,t_i)\approx(s,t)) \\ \text{(since } \left(s'_i, t'_i\right)\approx(s_i,t_i) \\ \text{and } (t,s)\approx(s,t))}} \text{ and } t'_i < s'_i\right] - \left[\underbrace{(s'_i, t'_i) \in [[(s,t)]]}_{\substack{\iff \left(\left(s'_i, t'_i\right)\approx(s,t)\right) \\ \iff ((s_i,t_i)\approx(s,t)) \\ \text{(since } \left(s'_i, t'_i\right)\approx(s_i,t_i))}} \text{ and } s'_i < t'_i\right] \\
&= \left[(s_i, t_i) \approx (s,t) \text{ and } t'_i < s'_i\right] + \left[(s_i, t_i) \approx (s,t) \text{ and } s'_i < t'_i\right] \\
&= \left[\underbrace{(s_i, t_i) \approx (s,t)}_{\substack{\iff ((s_i,t_i)\sim(s,t)) \\ \text{(since the restriction of the} \\ \text{relation } \approx \text{ to } \mathfrak{M} \text{ is } \sim)}}\right] \qquad \text{(because either } s'_i < t'_i \text{ or } t'_i < s'_i\text{)} \\
&= \left[\underbrace{(s_i, t_i) \sim (s,t)}_{\iff ((s_i,t_i)\in[(s,t)])}\right] = \left[(s_i, t_i) \in \underbrace{[(s,t)]}_{=c}\right] = \left[\underbrace{(s_i, t_i) \in c}_{\iff ([(s_i,t_i)]=c)}\right] \\
&= \left[[(s_i, t_i)] = c\right] \bmod 2.
\end{aligned}
$$

This proves (5.9).

Now,

$$\sum_{i=1}^{k} \underbrace{\operatorname{coord}_d \left(\operatorname{Has} \overrightarrow{c_{i+1}} - \operatorname{Has} \overrightarrow{c_i}\right)}_{=\operatorname{coord}_d\left(\operatorname{Has} \overrightarrow{c_{i+1}}\right) - \operatorname{coord}_d\left(\operatorname{Has} \overrightarrow{c_i}\right)} = \sum_{i=1}^{k} \left(\operatorname{coord}_d \left(\operatorname{Has} \overrightarrow{c_{i+1}}\right) - \operatorname{coord}_d \left(\operatorname{Has} \overrightarrow{c_i}\right)\right) = 0$$

(by the telescope principle). Hence,

$$\begin{aligned} 0 &= \sum_{i=1}^{k} \underbrace{\operatorname{coord}_d \left(\operatorname{Has} \overrightarrow{c_{i+1}} - \operatorname{Has} \overrightarrow{c_i}\right)}_{\substack{\equiv [[(s_i,t_i)]=c] \bmod 2 \\ \text{(by (5.9))}}} \\ &\equiv \sum_{i=1}^{k} \left[\underbrace{[(s_i, t_i)]}_{\substack{=\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) \\ \text{(by (5.5))}}} = c \right] \\ &= \sum_{i=1}^{k} \left[\left(\text{the color of the arc from } \overrightarrow{c_i} \text{ to } \overrightarrow{c_{i+1}}\right) = c\right] \\ &= \left(\text{the number of arcs colored } c \text{ appearing in } C\right) \bmod 2. \end{aligned}$$

Thus, the number of arcs colored c appearing in C is even. In other words, the number of arcs whose color belongs to $\{c\}$ appearing in C is even. In other words, the number of arcs whose color belongs to $\{c, c^{\mathrm{op}}\}$ appearing in C is even (since $\left\{c, \underbrace{c^{\mathrm{op}}}_{=c}\right\} = \{c, c\} = \{c\}$). This proves Theorem 5.2.3 **(b)**. □

Now we shall prove the auxiliary results we have stated without proof in the previous section.

Proof of Proposition 5.3.3. **(a)** We need to prove three claims:

Claim 1: Every entry of the word $\rho_{s,t}$ is a reflection in $D_{s,t}$.

Claim 2: The entries of the word $\rho_{s,t}$ are distinct.

Claim 3: Every reflection in $D_{s,t}$ is an entry of the word $\rho_{s,t}$.

Proof of Claim 1: We must show that $(st)^k s$ is a reflection in $D_{s,t}$ for every

$k \in \{0, 1, \ldots, m_{s,t} - 1\}$. Thus, fix $k \in \{0, 1, \ldots, m_{s,t} - 1\}$. Then,

$$(st)^k s = \underbrace{stst \cdots s}_{2k+1 \text{ factors}} = \begin{cases} \underbrace{stst \cdots t}_{k \text{ factors}} s \underbrace{tsts \cdots s}_{k \text{ factors}}, & \text{if } k \text{ is even;} \\ \underbrace{stst \cdots s}_{k \text{ factors}} t \underbrace{stst \cdots s}_{k \text{ factors}}, & \text{if } k \text{ is odd} \end{cases}$$

$$= \begin{cases} \underbrace{stst \cdots t}_{k \text{ factors}} s \left(\underbrace{stst \cdots t}_{k \text{ factors}} \right)^{-1}, & \text{if } k \text{ is even;} \\ \underbrace{stst \cdots s}_{k \text{ factors}} t \left(\underbrace{stst \cdots s}_{k \text{ factors}} \right)^{-1}, & \text{if } k \text{ is odd} \end{cases}$$

$$\left(\begin{array}{c} \text{since } \underbrace{tsts \cdots s}_{k \text{ factors}} = \left(\underbrace{stst \cdots t}_{k \text{ factors}} \right)^{-1} \text{ if } k \text{ is even,} \\ \text{and } \underbrace{stst \cdots s}_{k \text{ factors}} = \left(\underbrace{stst \cdots s}_{k \text{ factors}} \right)^{-1} \text{ if } k \text{ is odd} \end{array} \right).$$

Hence, $(st)^k s$ is conjugate to either s or t (depending on whether k is even or odd). Thus, $(st)^k s$ is a reflection. Also, it clearly lies in $D_{s,t}$. This proves Claim 1.

Proof of Claim 2: The element st of W has order $m_{s,t}$. Thus, the elements $(st)^0, (st)^1, \ldots, (st)^{m_{s,t}-1}$ are all distinct. Hence, the elements $(st)^0 s, (st)^1 s, \ldots, (st)^{m_{s,t}-1} s$ are all distinct. In other words, the entries of the word $\rho_{s,t}$ are all distinct. Claim 2 is proven.

Proof of Claim 3: The dihedral group $D_{s,t}$ has $2m_{s,t}$ elements[6], of which at most $m_{s,t}$ are reflections[7]. But the word $\rho_{s,t}$ has length $m_{s,t}$, and all its entries are reflections in $D_{s,t}$ (by Claim 1); hence, it contains $m_{s,t}$ reflections in $D_{s,t}$ (by Claim 2). Since $D_{s,t}$ has only at most $m_{s,t}$ reflections, this shows that every reflection in $D_{s,t}$ is an entry of the word $\rho_{s,t}$. Claim 3 is proven.

This finishes the proof of Proposition 5.3.3 **(a)**.

(b) We have $\rho_{s,t} = \left((st)^0 s, (st)^1 s, \ldots, (st)^{m_{s,t}-1} s \right)$ and

---

[6]since it is generated by two distinct involutions $s \neq 1$ and $t \neq 1$ whose product st has order $m_{s,t}$

[7]*Proof.* Consider the group homomorphism $\operatorname{sgn} : W \to \{1, -1\}$ defined in [Lusztig14, §1.1]. The group homomorphism $\operatorname{sgn}|_{D_{s,t}} : D_{s,t} \to \{1, -1\}$ sends either none or $m_{s,t}$ elements of $D_{s,t}$ to -1. Thus, this homomorphism $\operatorname{sgn}|_{D_{s,t}}$ sends at most $m_{s,t}$ elements of $D_{s,t}$ to -1. Since it must send every reflection to -1, this shows that at most $m_{s,t}$ elements of $D_{s,t}$ are reflections.

(Actually, we can replace "at most" by "exactly" here, but we won't need this.)

$\rho_{t,s} = \left((ts)^0 t, (ts)^1 t, \ldots, (ts)^{m_{s,t}-1} t\right)$ (since $m_{t,s} = m_{s,t}$). Thus, in order to prove Proposition 5.3.3 **(b)**, we must merely show that $(st)^k s = (ts)^{m_{s,t}-1-k} t$ for every $k \in \{0, 1, \ldots, m_{s,t} - 1\}$.

So fix $k \in \{0, 1, \ldots, m_{s,t} - 1\}$. Then,

$$(st)^k s \cdot \left((ts)^{m_{s,t}-1-k} t\right)^{-1} = (st)^k s \underbrace{t^{-1}}_{=t} \underbrace{\left((ts)^{m_{s,t}-1-k}\right)^{-1}}_{=(s^{-1}t^{-1})^{m_{s,t}-1-k}} = \underbrace{(st)^k st}_{=(st)^{k+1}} \left(\underbrace{s^{-1}}_{=s} \underbrace{t^{-1}}_{=t}\right)^{m_{s,t}-1-k}$$
$$= (st)^{k+1} (st)^{m_{s,t}-1-k} = (st)^{m_{s,t}} = 1,$$

so that $(st)^k s = (ts)^{m_{s,t}-1-k} t$. This proves Proposition 5.3.3 **(b)**.

(c) Let $q \in W$. Proposition 5.3.3 **(b)** shows that the word $\rho_{t,s}$ is the reversal of the word $\rho_{s,t}$. Hence, the word $q\rho_{t,s}q^{-1}$ is the reversal of the word $q\rho_{s,t}q^{-1}$ (since the word $q\rho_{t,s}q^{-1}$ is obtained from $\rho_{t,s}$ by conjugating each letter by q, and the word $q\rho_{s,t}q^{-1}$ is obtained from $\rho_{s,t}$ in the same way). This proves Proposition 5.3.3 **(c)**. $\square$

Proof of Proposition 5.3.6. Let $\overrightarrow{a}$ be a reduced expression for w. Write $\overrightarrow{a}$ as $(a_1, a_2, \ldots, a_k)$. Then, the definition of $\operatorname{Invs} \overrightarrow{a}$ shows that $\operatorname{Invs} \overrightarrow{a} = (t_1, t_2, \ldots, t_k)$, where the t_i are defined by

$$t_i = (a_1 a_2 \cdots a_{i-1}) a_i (a_1 a_2 \cdots a_{i-1})^{-1} \qquad \text{for every } i \in \{1, 2, \ldots, k\}.$$

Now, every $i \in \{1, 2, \ldots, k\}$ satisfies

$$t_i = (a_1 a_2 \cdots a_{i-1}) a_i \underbrace{(a_1 a_2 \cdots a_{i-1})^{-1}}_{\substack{=a_{i-1}^{-1} a_{i-2}^{-1} \cdots a_1^{-1} = a_{i-1} a_{i-2} \cdots a_1 \\ \text{(since each } a_j \text{ belongs to } S)}} = (a_1 a_2 \cdots a_{i-1}) a_i (a_{i-1} a_{i-2} \cdots a_1)$$
$$= a_1 a_2 \cdots a_{i-1} a_i a_{i-1} \cdots a_2 a_1.$$

But [Lusztig14, Proposition 1.6 (a)] (applied to $q = k$ and $s_i = a_i$) shows that the elements $a_1, a_1 a_2 a_1, a_1 a_2 a_3 a_2 a_1, \ldots, a_1 a_2 \cdots a_{k-1} a_k a_{k-1} \cdots a_2 a_1$ are distinct[8]. In other words, the elements $t_1, t_2, \ldots, t_k$ are distinct (since

[8]This also follows from [Bourba81, Chapter 4, n° 1.4, Lemme 2].

$t_i = a_1 a_2 \cdots a_{i-1} a_i a_{i-1} \cdots a_2 a_1$ for every $i \in \{1, 2, \ldots, k\}$). In other words, all entries of the tuple $\operatorname{Invs} \overrightarrow{a}$ are distinct. Proposition 5.3.6 **(a)** is proven.

(b) We need to prove that there exists a $q \in W$ such that $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q \rho_{s,t} q^{-1}$ by its reversal.

We set $m = m_{s,t}$ (for the sake of brevity).

Write $\overrightarrow{a}$ as $(a_1, a_2, \ldots, a_k)$.

The word $\overrightarrow{b}$ can be obtained from $\overrightarrow{a}$ by finding a factor of $\overrightarrow{a}$ of the form $\underbrace{(s, t, s, t, s, \ldots)}_{m_{s,t} \text{ elements}}$ and replacing it by $\underbrace{(t, s, t, s, t, \ldots)}_{m_{s,t} \text{ elements}}$ (by the definition of an "(s,t)-braid move"). In other words, the word $\overrightarrow{b}$ can be obtained from $\overrightarrow{a}$ by finding a factor of $\overrightarrow{a}$ of the form $\underbrace{(s, t, s, t, s, \ldots)}_{m \text{ elements}}$ and replacing it by $\underbrace{(t, s, t, s, t, \ldots)}_{m \text{ elements}}$ (since $m_{s,t} = m$). In other words, there exists an $p \in \{0, 1, \ldots, k - m\}$ such that $(a_{p+1}, a_{p+2}, \ldots, a_{p+m}) = \underbrace{(s, t, s, t, s, \ldots)}_{m \text{ elements}}$, and the word $\overrightarrow{b}$ can be obtained by replacing the $(p+1)$-st through $(p+m)$-th entries of $\overrightarrow{a}$ by $\underbrace{(t, s, t, s, t, \ldots)}_{m \text{ elements}}$. Consider this p. Write $\overrightarrow{b}$ as $(b_1, b_2, \ldots, b_k)$ (this is possible since $\overrightarrow{b}$ has the same length as $\overrightarrow{a}$). Thus,

$$(a_1, a_2, \ldots, a_p) = (b_1, b_2, \ldots, b_p), \tag{5.10}$$

$$(a_{p+1}, a_{p+2}, \ldots, a_{p+m}) = \underbrace{(s, t, s, t, s, \ldots)}_{m \text{ elements}}, \tag{5.11}$$

$$(b_{p+1}, b_{p+2}, \ldots, b_{p+m}) = \underbrace{(t, s, t, s, t, \ldots)}_{m \text{ elements}}, \tag{5.12}$$

$$(a_{p+m+1}, a_{p+m+2}, \ldots, a_k) = (b_{p+m+1}, b_{p+m+2}, \ldots, b_k). \tag{5.13}$$

Write the k-tuples $\operatorname{Invs} \overrightarrow{a}$ and $\operatorname{Invs} \overrightarrow{b}$ as $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\beta_1, \beta_2, \ldots, \beta_k)$, respectively. Their definitions show that

$$\alpha_i = (a_1 a_2 \cdots a_{i-1}) \, a_i \, (a_1 a_2 \cdots a_{i-1})^{-1} \tag{5.14}$$

and

$$\beta_i = (b_1 b_2 \cdots b_{i-1}) \, b_i \, (b_1 b_2 \cdots b_{i-1})^{-1} \tag{5.15}$$

for every $i \in \{1, 2, \ldots, k\}$.

Now, set $q = a_1 a_2 \cdots a_p$. From (5.10), we see that $q = b_1 b_2 \cdots b_p$ as well. In order to prove Proposition 5.3.6 **(b)**, it clearly suffices to show that $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ – namely, the factor $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$ – by its reversal.

So let us show this. In view of $\operatorname{Invs} \overrightarrow{a} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $\operatorname{Invs} \overrightarrow{b} = (\beta_1, \beta_2, \ldots, \beta_k)$, it clearly suffices to prove the following claims:

Claim 1: We have $\beta_i = \alpha_i$ for every $i \in \{1, 2, \ldots, p\}$.

Claim 2: We have $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q\rho_{s,t}q^{-1}$.

Claim 3: The m-tuple $(\beta_{p+1}, \beta_{p+2}, \ldots, \beta_{p+m})$ is the reversal of $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$.

Claim 4: We have $\beta_i = \alpha_i$ for every $i \in \{p+m+1, p+m+2, \ldots, k\}$.

Proof of Claim 1: Let $i \in \{1, 2, \ldots, p\}$. Then, (5.10) shows that $a_g = b_g$ for every $g \in \{1, 2, \ldots, i\}$. Now, (5.14) becomes

$$\begin{aligned}
\alpha_i &= (a_1 a_2 \cdots a_{i-1})\, a_i\, (a_1 a_2 \cdots a_{i-1})^{-1} = (b_1 b_2 \cdots b_{i-1})\, b_i\, (b_1 b_2 \cdots b_{i-1})^{-1} \\
&\qquad\qquad (\text{since } a_g = b_g \text{ for every } g \in \{1, 2, \ldots, i\}) \\
&= \beta_i \qquad\qquad (\text{by (5.15)}).
\end{aligned}$$

This proves Claim 1.

Proof of Claim 2: We have

$$\rho_{s,t} = \left((st)^0 s, (st)^1 s, \ldots, (st)^{m_{s,t}-1} s\right) = \left((st)^0 s, (st)^1 s, \ldots, (st)^{m-1} s\right)$$

(since $m_{s,t} = m$). Hence,

$$\begin{aligned}
q\rho_{s,t}q^{-1} &= q\left((st)^0 s, (st)^1 s, \ldots, (st)^{m-1} s\right) q^{-1} \\
&= \left(q\,(st)^0 sq^{-1}, q\,(st)^1 sq^{-1}, \ldots, q\,(st)^{m-1} sq^{-1}\right).
\end{aligned}$$

Thus, in order to prove $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q\rho_{s,t}q^{-1}$, it suffices to show that $\alpha_{p+i} = q\,(st)^{i-1} sq^{-1}$ for every $i \in \{1, 2, \ldots, m\}$. So let us fix $i \in \{1, 2, \ldots, m\}$.

We have

$$a_1 a_2 \cdots a_{p+i-1} = \underbrace{(a_1 a_2 \cdots a_p)}_{=q} \underbrace{(a_{p+1} a_{p+2} \cdots a_{p+i-1})}_{\substack{=\underbrace{stst\cdots}_{i-1 \text{ factors}} \\ (\text{by } (5.11))}} = q \underbrace{stst\cdots}_{i-1 \text{ factors}} .$$

Hence,

$$\begin{aligned}(a_1 a_2 \cdots a_{p+i-1})^{-1} &= \left(q \underbrace{stst\cdots}_{i-1 \text{ factors}}\right)^{-1} = \underbrace{\cdots t^{-1}s^{-1}t^{-1}s^{-1}}_{i-1 \text{ factors}} q^{-1} \\ &= \underbrace{\cdots tsts}_{i-1 \text{ factors}} q^{-1} \qquad \left(\text{since } s^{-1} = s \text{ and } t^{-1} = t\right).\end{aligned}$$

Also,

$$(a_1 a_2 \cdots a_{p+i-1})\, a_{p+i} = a_1 a_2 \cdots a_{p+i} = \underbrace{(a_1 a_2 \cdots a_p)}_{=q} \underbrace{(a_{p+1} a_{p+2} \cdots a_{p+i})}_{\substack{=\underbrace{stst\cdots}_{i \text{ factors}} \\ (\text{by } (5.11))}} = q \underbrace{stst\cdots}_{i \text{ factors}} .$$

Now, (5.14) (applied to $p+i$ instead of i) yields

$$\begin{aligned}\alpha_{p+i} &= \underbrace{(a_1 a_2 \cdots a_{p+i-1})\, a_{p+i}}_{=q \underbrace{stst\cdots}_{i \text{ factors}}} \underbrace{(a_1 a_2 \cdots a_{p+i-1})^{-1}}_{=\underbrace{\cdots tsts}_{i-1 \text{ factors}} q^{-1}} = q \underbrace{\underbrace{stst\cdots}_{i \text{ factors}} \underbrace{\cdots tsts}_{i-1 \text{ factors}}}_{=\underbrace{stst\cdots s}_{2i-1 \text{ factors}} = (st)^{i-1} s} q^{-1} \\ &= q\,(st)^{i-1}\, s q^{-1}.\end{aligned}$$

This completes the proof of $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q \rho_{s,t} q^{-1}$. Hence, Claim 2 is proven.

Proof of Claim 3: In our proof of Claim 2, we have shown that $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q \rho_{s,t} q^{-1}$. The same argument (applied to $\overrightarrow{b}$, $(b_1, b_2, \ldots, b_k)$, $(\beta_1, \beta_2, \ldots, \beta_k)$, t and s instead of $\overrightarrow{a}$, $(a_1, a_2, \ldots, a_k)$, $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, s and t) shows that $(\beta_{p+1}, \beta_{p+2}, \ldots, \beta_{p+m}) = q \rho_{t,s} q^{-1}$ (where we now use (5.12) instead of (5.11), and use $q = b_1 b_2 \cdots b_p$ instead of $q = a_1 a_2 \cdots a_p$).

Now, recall that the word $q\rho_{t,s}q^{-1}$ is the reversal of the word $q\rho_{s,t}q^{-1}$. Since $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q\rho_{s,t}q^{-1}$ and $(\beta_{p+1}, \beta_{p+2}, \ldots, \beta_{p+m}) = q\rho_{t,s}q^{-1}$, this means that the word $(\beta_{p+1}, \beta_{p+2}, \ldots, \beta_{p+m})$ is the reversal of $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$. This proves Claim 3.

Proof of Claim 4: Since $m = m_{s,t}$, we have $\underbrace{stst\cdots}_{m \text{ factors}} = \underbrace{tsts\cdots}_{m \text{ factors}}$ (this is one of the braid relations of our Coxeter group). Let us set $x = \underbrace{stst\cdots}_{m \text{ factors}} = \underbrace{tsts\cdots}_{m \text{ factors}}$. Now, (5.11) yields $a_{p+1}a_{p+2}\cdots a_{p+m} = \underbrace{stst\cdots}_{m \text{ factors}} = x$. Similarly, from (5.12), we obtain $b_{p+1}b_{p+2}\cdots b_{p+m} = x$.

Let $i \in \{p+m+1, p+m+2, \ldots, k\}$. Thus,

$$\begin{aligned} a_1a_2\cdots a_{i-1} &= \underbrace{(a_1a_2\cdots a_p)}_{=q}\underbrace{(a_{p+1}a_{p+2}\cdots a_{p+m})}_{=x}\underbrace{(a_{p+m+1}a_{p+m+2}\cdots a_{i-1})}_{\substack{=b_{p+m+1}b_{p+m+2}\cdots b_{i-1} \\ \text{(by (5.13))}}} \\ &= qx\left(b_{p+m+1}b_{p+m+2}\cdots b_{i-1}\right). \end{aligned}$$

Comparing this with

$$\begin{aligned} b_1b_2\cdots b_{i-1} &= \underbrace{(b_1b_2\cdots b_p)}_{=q}\underbrace{(b_{p+1}b_{p+2}\cdots b_{p+m})}_{=x}(b_{p+m+1}b_{p+m+2}\cdots b_{i-1}) \\ &= qx\left(b_{p+m+1}b_{p+m+2}\cdots b_{i-1}\right), \end{aligned}$$

we obtain $a_1a_2\cdots a_{i-1} = b_1b_2\cdots b_{i-1}$. Also, $a_i = b_i$ (by (5.13)). Now, (5.14) becomes

$$\begin{aligned} \alpha_i &= \left(\underbrace{a_1a_2\cdots a_{i-1}}_{=b_1b_2\cdots b_{i-1}}\right)\underbrace{a_i}_{=b_i}\left(\underbrace{a_1a_2\cdots a_{i-1}}_{=b_1b_2\cdots b_{i-1}}\right)^{-1} = (b_1b_2\cdots b_{i-1})\,b_i\,(b_1b_2\cdots b_{i-1})^{-1} \\ &= \beta_i \qquad \text{(by (5.15))}. \end{aligned}$$

This proves Claim 4.

Hence, all four claims are proven, and the proof of Proposition 5.3.6 **(b)** is complete. ◻

Proof of Proposition 5.3.7. **(a)** This follows from the fact that the word $\rho_{s,t}$ has length

$m_{s,t} \geq 2 > 0$, and from Proposition 5.3.6 **(a)**.

(b) Assume the contrary. Then, both words $\rho_{s,t}$ and $\rho_{t,s}$ appear as a subword of Invs $\overrightarrow{a}$. By Proposition 5.3.3 **(b)**, this means that both the word $\rho_{s,t}$ and its reversal appear as a subword of Invs $\overrightarrow{a}$. Since the word $\rho_{s,t}$ has length $m_{s,t} \geq 2$, this means that at least one letter of $\rho_{s,t}$ appears twice in Invs $\overrightarrow{a}$. This contradicts Proposition 5.3.6 **(a)**. This contradiction concludes our proof. $\square$

Before we finally prove Theorem 5.3.9, we show a lemma:

Lemma 5.4.1. Let $(s,t) \in \mathfrak{M}$ and $(u,v) \in \mathfrak{N}$. Let $q \in W$. Assume that $u \in qD_{s,t}q^{-1}$ and $v \in qD_{s,t}q^{-1}$. Then, $m_{s,t} = m_{u,v}$.

Proof of Lemma 5.4.1. Claim 1: Lemma 5.4.1 holds in the case when $(u,v) \in \mathfrak{M}$.

Proof. Assume that $(u,v) \in \mathfrak{M}$. Thus, $u, v \in S$. Let I be the subset $\{s,t\}$ of S. We shall use the notations of [Lusztig14, §9]. In particular, $l(r)$ denotes the length of any element $r \in W$.

We have $W_I = D_{s,t}$. Consider the coset $W_I q^{-1}$ of W_I. From [Lusztig14, Lemma 9.7 (a)] (applied to $a = q^{-1}$), we know that this coset $W_I q^{-1}$ has a unique element of minimal length. Let w be this element. Thus, $w \in W_I q^{-1}$, so that $W_I w = W_I q^{-1}$. Now,

$$\underbrace{q}_{=(q^{-1})^{-1}} \underbrace{W_I}_{=(W_I)^{-1}} = \left(q^{-1}\right)^{-1} (W_I)^{-1} = \left(\underbrace{W_I q^{-1}}_{=W_I w}\right)^{-1} = (W_I w)^{-1} = w^{-1} W_I.$$

Let $u' = wuw^{-1}$ and $v' = wvw^{-1}$.

We have $u \in q \underbrace{D_{s,t}}_{=W_I} q^{-1} = q \underbrace{W_I q^{-1}}_{=W_I w} = \underbrace{q W_I}_{=w^{-1} W_I} w = w^{-1} W_I w$. In other words, $wuw^{-1} \in W_I$. In other words, $u' \in W_I$ (since $u' = wuw^{-1}$). Similarly, $v' \in W_I$.

We have $u' = wuw^{-1}$, hence $u'w = wu$. But [Lusztig14, Lemma 9.7 (b)] (applied to $a = q^{-1}$ and $y = u'$) shows that $l(u'w) = l(u') + l(w)$. Hence,

$$l(u') + l(w) = l\left(\underbrace{u'w}_{=wu}\right) = l(wu) = l(w) \pm 1 \qquad \text{(since } u \in S\text{)}.$$

Subtracting $l(w)$ from this equality, we obtain $l(u') = \pm 1$, and thus $l(u') = 1$, so that $u' \in S$. Combined with $u' \in W_I$, this shows that $u' \in S \cap W_I = I$. Similarly, $v' \in I$.

We have $u \neq v$ (since $(u,v) \in \mathfrak{N}$), thus $wuw^{-1} \neq wvw^{-1}$, thus $u' = wuw^{-1} \neq wvw^{-1} = v'$. Thus, u' and v' are two distinct elements of the two-element set $I = \{s,t\}$. Hence, either $(u',v') = (s,t)$ or $(u',v') = (t,s)$. In either of these two cases, we have $m_{u',v'} = m_{s,t}$. But since $u' = wuw^{-1}$ and $v' = wvw^{-1}$, we have $m_{u',v'} = m_{u,v}$. Hence, $m_{s,t} = m_{u',v'} = m_{u,v}$. This proves Claim 1.

Claim 2: Lemma 5.4.1 holds in the general case.

Proof. Consider the general case. We have $(u,v) \in \mathfrak{N} = \bigcup_{x \in W} x\mathfrak{M}x^{-1}$. Thus, there exists some $x \in W$ such that $(u,v) \in x\mathfrak{M}x^{-1}$. Consider this x. From $(u,v) \in x\mathfrak{M}x^{-1}$, we obtain $x^{-1}(u,v)\,x \in \mathfrak{M}$. In other words, $(x^{-1}ux, x^{-1}vx) \in \mathfrak{M}$. Moreover,

$$x^{-1} \underbrace{u}_{\in qD_{s,t}q^{-1}} x \in x^{-1}qD_{s,t} \underbrace{q^{-1}x}_{=(x^{-1}q)^{-1}} = x^{-1}qD_{s,t}\left(x^{-1}q\right)^{-1},$$

and similarly $x^{-1}vx \in x^{-1}qD_{s,t}\left(x^{-1}q\right)^{-1}$. Hence, Claim 1 (applied to $(x^{-1}ux, x^{-1}vx)$ and $x^{-1}q$ instead of (u,v) and q) shows that $m_{s,t} = m_{x^{-1}ux, x^{-1}vx} = m_{u,v}$. This proves Claim 2, and thus proves Lemma 5.4.1. □

Proof of Theorem 5.3.9. Conjugation by q (that is, the map $W \to W,\ x \mapsto qxq^{-1}$) is a group endomorphism of W. Hence, for every $i \in \mathbb{N}$, we have

$$q(st)^i sq^{-1} = \left(\underbrace{\left(qsq^{-1}\right)}_{=s'}\underbrace{\left(qtq^{-1}\right)}_{=t'}\right)^i \underbrace{\left(qsq^{-1}\right)}_{=s'} = (s't')^i s'. \qquad (5.16)$$

Let $m = m_{s,t}$. We have

$$\rho_{s,t} = \left((st)^0 s, (st)^1 s, \ldots, (st)^{m_{s,t}-1} s\right) = \left((st)^0 s, (st)^1 s, \ldots, (st)^{m-1} s\right)$$

(since $m_{s,t} = m$) and thus

$$\begin{aligned}
q\rho_{s,t}q^{-1} &= q\left((st)^0 s, (st)^1 s, \ldots, (st)^{m-1} s\right) q^{-1} \\
&= \left(q(st)^0 sq^{-1}, q(st)^1 sq^{-1}, \ldots, q(st)^{m-1} sq^{-1}\right) \\
&= \left((s't')^0 s', (s't')^1 s', \ldots, (s't')^{m-1} s'\right) \\
&\qquad \left(\begin{array}{c}\text{since every } i \in \{0, 1, \ldots, m-1\} \text{ satisfies} \\ q(st)^i sq^{-1} = (s't')^i s' \text{ (by (5.16))}\end{array}\right) \\
&= \left((s't')^0 s', (s't')^1 s', \ldots, (s't')^{m_{s',t'}-1} s'\right) \qquad (\text{since } m = m_{s,t} = m_{s',t'}) \\
&= \rho_{s',t'} \qquad (\text{by the definition of } \rho_{s',t'}).
\end{aligned}$$

The word $\overrightarrow{b}$ is obtained from $\overrightarrow{a}$ by an (s,t)-braid move. Hence, the word $\overrightarrow{a}$ can be obtained from $\overrightarrow{b}$ by a (t,s)-braid move.

From $(s', t') \in \mathfrak{N}$, we obtain $s' \neq t'$. Hence, $(s', t') \neq (t', s')$.

From $s' = qsq^{-1}$ and $t' = qtq^{-1}$, we obtain $D_{s',t'} = qD_{s,t}q^{-1}$ (since conjugation by q is a group endomorphism of W).

Proposition 5.3.3 **(c)** shows that the word $q\rho_{t,s}q^{-1}$ is the reversal of the word $q\rho_{s,t}q^{-1}$. Hence, the word $q\rho_{s,t}q^{-1}$ is the reversal of the word $q\rho_{t,s}q^{-1}$.

Recall that $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal. Since this latter reversal is $q\rho_{t,s}q^{-1}$ (as we have previously seen), this shows that $\operatorname{Invs} \overrightarrow{b}$ has a factor of $q\rho_{t,s}q^{-1}$ in the place where the word $\operatorname{Invs} \overrightarrow{a}$ had the factor $q\rho_{s,t}q^{-1}$. Hence, $\operatorname{Invs} \overrightarrow{a}$ can, in turn, be obtained from $\operatorname{Invs} \overrightarrow{b}$ by replacing a particular factor of the form $q\rho_{t,s}q^{-1}$ by its reversal (since the reversal of $q\rho_{t,s}q^{-1}$ is $q\rho_{s,t}q^{-1}$). Thus, our situation is symmetric with respect to s and t; more precisely, we wind up in an analogous situation if we replace s, t, $\overrightarrow{a}$, $\overrightarrow{b}$, s' and t' by t, s, $\overrightarrow{b}$, $\overrightarrow{a}$, t' and s', respectively.

We shall prove the following claims:

Claim 1: Let $(u, v) \in \mathfrak{N}$ be such that $(u, v) \neq (s', t')$ and $(u, v) \neq (t', s')$. Then, $\operatorname{has}_{u,v} \overrightarrow{b} = \operatorname{has}_{u,v} \overrightarrow{a}$.

Claim 2: We have $\operatorname{has}_{s',t'} \overrightarrow{b} = \operatorname{has}_{s',t'} \overrightarrow{a} - 1$.

Claim 3: We have $\operatorname{has}_{t',s'} \overrightarrow{b} = \operatorname{has}_{t',s'} \overrightarrow{a} + 1$.

Proof of Claim 1: Assume the contrary. Thus, $\operatorname{has}_{u,v} \overrightarrow{b} \neq \operatorname{has}_{u,v} \overrightarrow{a}$. Hence, one of the numbers $\operatorname{has}_{u,v} \overrightarrow{b}$ and $\operatorname{has}_{u,v} \overrightarrow{a}$ equals 1 and the other equals 0 (since both $\operatorname{has}_{u,v} \overrightarrow{b}$ and $\operatorname{has}_{u,v} \overrightarrow{a}$ belong to $\{0,1\}$). Without loss of generality, we assume that $\operatorname{has}_{u,v} \overrightarrow{a} = 1$ and $\operatorname{has}_{u,v} \overrightarrow{b} = 0$ (because in the other case, we can replace s, t, $\overrightarrow{a}$, $\overrightarrow{b}$, s' and t' by t, s, $\overrightarrow{b}$, $\overrightarrow{a}$, t' and s', respectively).

The elements u and v are two distinct reflections (since $(u,v) \in \mathfrak{N}$).

Write the tuple $\operatorname{Invs} \overrightarrow{a}$ as $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. The tuple $\operatorname{Invs} \overrightarrow{b}$ has the same length as $\operatorname{Invs} \overrightarrow{a}$, since $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal. Hence, write the tuple $\operatorname{Invs} \overrightarrow{b}$ as $(\beta_1, \beta_2, \ldots, \beta_k)$.

From $\operatorname{has}_{u,v} \overrightarrow{a} = 1$, we obtain that $\rho_{u,v}$ appears as a subword of $\operatorname{Invs} \overrightarrow{a}$. In other words, $\rho_{u,v} = \left(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}\right)$ for some integers $i_1, i_2, \ldots, i_f$ satisfying $1 \leq i_1 < i_2 < \cdots < i_f \leq k$. Consider these $i_1, i_2, \ldots, i_f$. From $\operatorname{has}_{u,v} \overrightarrow{b} = 0$, we conclude that $\rho_{u,v}$ does not appear as a subword of $\operatorname{Invs} \overrightarrow{b}$.

On the other hand, $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal. This factor has length $m_{s,t} = m$; thus, it has the form $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$ for some $p \in \{0, 1, \ldots, k-m\}$. Consider this p. Thus,

$$(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m}) = q\rho_{s,t}q^{-1} = \left((s't')^0 s', (s't')^1 s', \ldots, (s't')^{m-1} s'\right).$$

In other words,

$$\alpha_{p+i} = (s't')^{i-1} s' \qquad \text{for every } i \in \{1, 2, \ldots, m\}. \tag{5.17}$$

We now summarize:

- The word $\rho_{u,v}$ appears as the subword $\left(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}\right)$ of $\operatorname{Invs} \overrightarrow{a}$, but does not appear as a subword of $\operatorname{Invs} \overrightarrow{b}$.

- The word $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing the factor $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$ by its reversal.

Thus, replacing the factor $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$ in $\operatorname{Invs} \overrightarrow{a}$ by its reversal must mess up the subword $\left(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}\right)$ of $\operatorname{Invs} \overrightarrow{a}$ badly enough that it no longer appears as a subword (not even in different positions). This can only happen if at least two of the integers $i_1, i_2, \ldots, i_f$ lie in the interval $\{p+1, p+2, \ldots, p+m\}$.

Hence, at least two of the integers $i_1, i_2, \ldots, i_f$ lie in the interval $\{p+1, p+2, \ldots, p+m\}$. In particular, there must be a $g \in \{1, 2, \ldots, f-1\}$ such that the integers i_g and i_{g+1} lie in the interval $\{p+1, p+2, \ldots, p+m\}$ (since $i_1 < i_2 < \cdots < i_f$). Consider this g. From $i_1 < i_2 < \cdots < i_f$, we obtain $i_g < i_{g+1}$.

We have $i_g \in \{p+1, p+2, \ldots, p+m\}$. In other words, $i_g = p + r_g$ for some $r_g \in \{1, 2, \ldots, m\}$. Consider this r_g.

We have $i_{g+1} \in \{p+1, p+2, \ldots, p+m\}$. In other words, $i_{g+1} = p + r_{g+1}$ for some $r_{g+1} \in \{1, 2, \ldots, m\}$. Consider this r_{g+1}.

We have $p + r_g = i_g < i_{g+1} = p + r_{g+1}$, thus $r_g < r_{g+1}$. Since both r_g and r_{g+1} are elements of $\{1, 2, \ldots, m\}$, this shows that $r_{g+1} - r_g \in \{1, 2, \ldots, m-1\}$. Set $N = r_{g+1} - r_g$; thus, $N = r_{g+1} - r_g \in \{1, 2, \ldots, m-1\}$.

We have $\left(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}\right) = \rho_{u,v} = \left((uv)^0 u, (uv)^1 u, \ldots, (uv)^{m_{u,v}-1} u\right)$ (by the definition of $\rho_{u,v}$). Hence, $\alpha_{i_g} = (uv)^{g-1} u$ and $\alpha_{i_{g+1}} = (uv)^g u$. Now,

$$\begin{aligned} (uv)^{g-1} u = \alpha_{i_g} &= \alpha_{p+r_g} && \text{(since } i_g = p + r_g\text{)} \\ &= (s't')^{r_g - 1} s' && \text{(by (5.17), applied to } i = r_g\text{)} \end{aligned}$$

and

$$\begin{aligned} (uv)^g u = \alpha_{i_{g+1}} &= \alpha_{p+r_{g+1}} && \text{(since } i_{g+1} = p + r_{g+1}\text{)} \\ &= (s't')^{r_{g+1} - 1} s' && \text{(by (5.17), applied to } i = r_{g+1}\text{)}. \end{aligned}$$

Hence,

$$\underbrace{\left((uv)^g u\right)}_{=(s't')^{r_{g+1}-1}s'} \left(\underbrace{(uv)^{g-1} u}_{=(s't')^{r_g-1}s'}\right)^{-1} = \left((s't')^{r_{g+1}-1} s'\right) \left((s't')^{r_g-1} s'\right)^{-1}$$
$$= (s't')^{r_{g+1}-r_g} = (s't')^N$$

(since $r_{g+1} - r_g = N$). Compared with $((uv)^g u) \left((uv)^{g-1} u\right)^{-1} = uv$, this yields

$$uv = (s't')^N \in D_{s',t'}.$$

Now, $(uv)^{-g} (uv)^g u = u$, so that

$$u = \left(\underbrace{uv}_{\in D_{s',t'}}\right)^{-g} \underbrace{(uv)^g u}_{=(s't')^{r_{g+1}-1}s' \in D_{s',t'}} \in (D_{s',t'})^{-g} D_{s',t'} \subseteq D_{s',t'}.$$

Now, both u and uv belong to the subgroup $D_{s',t'}$ of W. Thus, so does $u^{-1}(uv)$. In other words, $u^{-1}(uv) \in D_{s',t'}$, so that $v = u^{-1}(uv) \in D_{s',t'}$.

Furthermore, we have

$$\alpha_{i_1} = u \qquad \text{and} \qquad \alpha_{i_f} = v$$

[9].

Now, we have $i_1 \in \{p+1, p+2, \ldots, p+m\}$ [10] and $i_f \in \{p+1, p+2, \ldots, p+m\}$

[9] *Proof.* From $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}) = \left((uv)^0 u, (uv)^1 u, \ldots, (uv)^{m_{u,v}-1} u\right)$, we obtain $\alpha_{i_1} = \underbrace{(uv)^0}_{=1} u = u$.

We have $(uv)^{m_{u,v}} = 1$, and thus $(uv)^{m_{u,v}-1} = (uv)^{-1} = v^{-1}u^{-1}$.

From $(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}) = \left((uv)^0 u, (uv)^1 u, \ldots, (uv)^{m_{u,v}-1} u\right)$, we obtain $\alpha_{i_f} = \underbrace{(uv)^{m_{u,v}-1}}_{=v^{-1}u^{-1}} u = v^{-1}u^{-1}u = v^{-1} = v$ (since v is a reflection), qed.

[10] *Proof.* The element u is a reflection and lies in $D_{s',t'}$. Hence, Proposition 5.3.3 **(a)** (applied to s' and t' instead of s and t) shows that the word $\rho_{s',t'}$ contains u. Since $\rho_{s',t'} = q\rho_{s,t}q^{-1} = (\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$, this shows that the word $(\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{p+m})$ contains u. In other words, $u = \alpha_M$ for some $M \in \{p+1, p+2, \ldots, p+m\}$. Consider this M.

But Proposition 5.3.6 **(a)** shows that all entries of the tuple $\operatorname{Invs} \overrightarrow{a}$ are distinct. In other words,

[11]. Thus, all of the integers $i_1, i_2, \ldots, i_f$ belong to $\{p+1, p+2, \ldots, p+m\}$ (since $i_1 < i_2 < \cdots < i_f$).

Now, recall that f is the length of the word $\rho_{u,v}$ (since $\rho_{u,v} = \left(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_f}\right)$), and thus equals $m_{u,v}$. Thus, $f = m_{u,v}$.

But $u \in D_{s',t'} = qD_{s,t}q^{-1}$ and $v \in D_{s',t'} = qD_{s,t}q^{-1}$. Hence, Lemma 5.4.1 yields $m_{s,t} = m_{u,v}$. Since $m = m_{s,t}$ and $f = m_{u,v}$, this rewrites as $m = f$.

Recall that all of the integers $i_1, i_2, \ldots, i_f$ belong to $\{p+1, p+2, \ldots, p+m\}$. Since $i_1 < i_2 < \cdots < i_f$ and $f = m$, these integers $i_1, i_2, \ldots, i_f$ form a strictly increasing sequence of length m. Thus, $(i_1, i_2, \ldots, i_f)$ is a strictly increasing sequence of length m whose entries belong to $\{p+1, p+2, \ldots, p+m\}$. But the only such sequence is $(p+1, p+2, \ldots, p+m)$ (because the set $\{p+1, p+2, \ldots, p+m\}$ has only m elements). Thus, $(i_1, i_2, \ldots, i_f) = (p+1, p+2, \ldots, p+m)$. In particular, $i_1 = p+1$ and $i_f = p+m$.

Now, $\alpha_{i_1} = u$, so that

$$\begin{aligned} u = \alpha_{i_1} &= \alpha_{p+1} && (\text{since } i_1 = p+1) \\ &= \underbrace{(s't')^{1-1}}_{=1} s' && (\text{by (5.17), applied to } i = 1) \\ &= s'. \end{aligned}$$

Also, $\alpha_{i_f} = v$, so that

$$\begin{aligned} v = \alpha_{i_f} &= \alpha_{p+m} && (\text{since } i_f = p+m) \\ &= \underbrace{(s't')^{m-1}}_{\substack{=(s't')^{-1} \\ (\text{since } (s't')^m = 1 \\ (\text{since } m = m_{s,t} = m_{s',t'}))}} s' && (\text{by (5.17), applied to } i = m) \\ &= (s't')^{-1} s' = t'. \end{aligned}$$

the elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ are pairwise distinct (since those are the entries of $\operatorname{Invs} \overrightarrow{a}$). Hence, from $\alpha_{i_1} = u = \alpha_M$, we obtain $i_1 = M \in \{p+1, p+2, \ldots, p+m\}$. Qed.

[11] *Proof.* The proof of this is essentially analogous to the proof of $i_1 \in \{p+1, p+2, \ldots, p+m\}$, with v occasionally replacing u.

Combined with $u = s'$, this yields $(u, v) = (s', t')$, which contradicts $(u, v) \neq (s', t')$. This contradiction proves that our assumption was wrong. Claim 1 is proven.

Proof of Claim 2: The word $\operatorname{Invs} \overrightarrow{b}$ is obtained from $\operatorname{Invs} \overrightarrow{a}$ by replacing a particular factor of the form $q\rho_{s,t}q^{-1}$ by its reversal. Thus, the word $\operatorname{Invs} \overrightarrow{a}$ has a factor of the form $q\rho_{s,t}q^{-1}$. Since $q\rho_{s,t}q^{-1} = \rho_{s',t'}$, this means that the word $\operatorname{Invs} \overrightarrow{a}$ has a factor of the form $\rho_{s',t'}$. Consequently, the word $\operatorname{Invs} \overrightarrow{a}$ has a subword of the form $\rho_{s',t'}$. In other words, $\operatorname{has}_{s',t'} \overrightarrow{a} = 1$.

The same argument (applied to t, s, $\overrightarrow{b}$, $\overrightarrow{a}$, t' and s' instead of s, t, $\overrightarrow{a}$, $\overrightarrow{b}$, s' and t') shows that $\operatorname{has}_{t',s'} \overrightarrow{b} = 1$. In other words, the word $\operatorname{Invs} \overrightarrow{b}$ has a subword of the form $\rho_{t',s'}$. Hence, the word $\operatorname{Invs} \overrightarrow{b}$ has no subword of the form $\rho_{s',t'}$ (because Proposition 5.3.7 **(b)** (applied to $\overrightarrow{b}$, s' and t' instead of $\overrightarrow{a}$, s and t) shows that the words $\rho_{s',t'}$ and $\rho_{t',s'}$ cannot both appear as subwords of $\operatorname{Invs} \overrightarrow{b}$). In other words, $\operatorname{has}_{s',t'} \overrightarrow{b} = 0$.

Combining this with $\operatorname{has}_{s',t'} \overrightarrow{a} = 1$, we immediately obtain $\operatorname{has}_{s',t'} \overrightarrow{b} = \operatorname{has}_{s',t'} \overrightarrow{a} - 1$. Thus, Claim 2 is proven.

Proof of Claim 3: Applying Claim 2 to t, s, $\overrightarrow{b}$, $\overrightarrow{a}$, t' and s' instead of s, t, $\overrightarrow{a}$, $\overrightarrow{b}$, s' and t', we obtain $\operatorname{has}_{t',s'} \overrightarrow{a} = \operatorname{has}_{t',s'} \overrightarrow{b} - 1$. In other words, $\operatorname{has}_{t',s'} \overrightarrow{b} = \operatorname{has}_{t',s'} \overrightarrow{a} + 1$. This proves Claim 3.

Now, our goal is to prove that $\operatorname{Has} \overrightarrow{b} = \operatorname{Has} \overrightarrow{a} - (s', t') + (t', s')$. But the definition of Has $\overrightarrow{b}$ yields

$$\begin{aligned}
&\operatorname{Has}\overrightarrow{b} \\
&= \sum_{(u,v)\in\mathfrak{N}} \operatorname{has}_{u,v}\overrightarrow{b}\cdot(u,v) \\
&= \sum_{\substack{(u,v)\in\mathfrak{N};\\ (u,v)\neq(s',t');\\ (u,v)\neq(t',s')}} \underbrace{\operatorname{has}_{u,v}\overrightarrow{b}}_{\substack{=\operatorname{has}_{u,v}\overrightarrow{a}\\ \text{(by Claim 1)}}}\cdot(u,v) + \underbrace{\operatorname{has}_{s',t'}\overrightarrow{b}}_{\substack{=\operatorname{has}_{s',t'}\overrightarrow{a}-1\\ \text{(by Claim 2)}}}\cdot(s',t') + \underbrace{\operatorname{has}_{t',s'}\overrightarrow{b}}_{\substack{=\operatorname{has}_{t',s'}\overrightarrow{a}+1\\ \text{(by Claim 3)}}}\cdot(t',s') \\
&\qquad\qquad \left(\text{since } (s',t')\neq(t',s')\right) \\
&= \sum_{\substack{(u,v)\in\mathfrak{N};\\ (u,v)\neq(s',t');\\ (u,v)\neq(t',s')}} \operatorname{has}_{u,v}\overrightarrow{a}\cdot(u,v) + \left(\operatorname{has}_{s',t'}\overrightarrow{a}-1\right)\cdot(s',t') + \left(\operatorname{has}_{t',s'}\overrightarrow{a}+1\right)\cdot(t',s') \\
&= \sum_{\substack{(u,v)\in\mathfrak{N};\\ (u,v)\neq(s',t');\\ (u,v)\neq(t',s')}} \operatorname{has}_{u,v}\overrightarrow{a}\cdot(u,v) + \operatorname{has}_{s',t'}\overrightarrow{a}\cdot(s',t') - (s',t') + \operatorname{has}_{t',s'}\overrightarrow{a}\cdot(t',s') + (t',s') \\
&= \underbrace{\sum_{\substack{(u,v)\in\mathfrak{N};\\ (u,v)\neq(s',t');\\ (u,v)\neq(t',s')}} \operatorname{has}_{u,v}\overrightarrow{a}\cdot(u,v) + \operatorname{has}_{s',t'}\overrightarrow{a}\cdot(s',t') + \operatorname{has}_{t',s'}\overrightarrow{a}\cdot(t',s')}_{\substack{=\sum_{(u,v)\in\mathfrak{N}}\operatorname{has}_{u,v}\overrightarrow{a}\cdot(u,v)\\ \text{(since } (s',t')\neq(t',s'))}} - (s',t') + (t',s') \\
&= \underbrace{\sum_{(u,v)\in\mathfrak{N}} \operatorname{has}_{u,v}\overrightarrow{a}\cdot(u,v)}_{=\operatorname{Has}\overrightarrow{a}} - (s',t') + (t',s') = \operatorname{Has}\overrightarrow{a} - (s',t') + (t',s').
\end{aligned}$$

This proves Theorem 5.3.9. □

Bibliography

[Abe77] Eiichi Abe, *Hopf Algebras*, CUP 1977.

[Aguiar13] Marcelo Aguiar, Swapneel Mahajan, *Hopf monoids in the category of species*, Contemporary Mathematics 585 (2013), pp. 17–124.
`http://www.math.cornell.edu/~maguiar/hmonoid.pdf`

[ABS03] Marcelo Aguiar, Nantel Bergeron, Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*, Compositio Mathematica 142 (2006) 1-30.
Updated version 2011 on Aguiar's website: `http://www.math.cornell.edu/~maguiar/pubs.html`

[AFNT13] Jean-Christophe Aval, Valentin Féray, Jean-Christophe Novelli, Jean-Yves Thibon, *Quasi-symmetric functions as polynomial functions on Young diagrams*, Journal of Algebraic Combinatorics, May 2015, Volume 41, Issue 3, pp. 669–706.
A preprint is avaliable as arXiv preprint `arXiv:1312.2727v2`.

[BenSag14] Carolina Benedetti, Bruce Sagan, *Antipodes and involutions*, arXiv:1410.5023v3.
`http://arxiv.org/abs/1410.5023v3`

[BBSSZ13a] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions*, Canadian Journal of Mathematics 66 (2014), pp. 525–565.
Also available as arXiv preprint `arXiv:1208.5191v3`.

[BBSSZ13b] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *Multiplicative structures of the immaculate basis of non-commutative symmetric functions*, arXiv preprint `arXiv:1305.4700v3`.

[BBSSZ13c] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *Indecomposable modules for the dual immaculate basis of quasi-symmetric functions*, Proc. Amer. Math. Soc. 143 (2015), pp. 991–1000.
A preprint is avaliable as arXiv preprint `arXiv:1304.1224v1`.

[BeCeLa14] Nantel Bergeron, Cesar Ceballos, Jean-Philippe Labbé, *Fan realizations of subword complexes and multi-associahedra via Gale duality*, arXiv:1404.7380v2.

[BSOZ13] Nantel Bergeron, Juana Sánchez-Ortega, Mike Zabrocki, *The Pieri rule for dual immaculate quasi-symmetric functions*, arXiv preprint `arXiv:1307.4273v3`.

[BerZab05] Nantel Bergeron, Mike Zabrocki, *The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree*, J. of Algebra and its Applications (8)(2009), No 4, pp. 581–600.
A preprint is also available as arXiv preprint `arXiv:math/0509265v3`.

[BlaSag86] Andreas Blass, Bruce Eli Sagan, *Bijective Proofs of Two Broken Circuit Theorems*, Journal of Graph Theory, Vol. 10 (1986), pp. 15–21. (See also another scan.)

[Bollob79] Béla Bollobás, *Graph Theory: An Introductory Course*, Springer, Softcover reprint of the hardcover 1st edition 1979.

[Bona11] Miklós Bóna, *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory*, 3rd edition, World Scientific 2011.

[Bourba81] N. Bourbaki, *Éléments de Mathématique: Groupes at algèbres de Lie, Chapitres 4, 5 et 6*, Masson 1981.

[DNR01] Sorin Dăscălescu, Constantin Năstăsescu, Şerban Raianu, *Hopf Algebras*, Marcel Dekker 2001.

[DHNT08] Gerard H. E. Duchamp, Florent Hivert, Jean-Christophe Novelli, Jean-Yves Thibon, *Noncommutative Symmetric Functions VII: Free Quasi-Symmetric Functions Revisited*, Annals of Combinatorics 15 (2011), pp. 655–673, arXiv:0809.4479v2.
`http://arxiv.org/abs/0809.4479v2`

[EbrFar08] Kurusch Ebrahimi-Fard, Dominique Manchon, *Dendriform Equations*, Journal of Algebra 322 (2009), pp. 4053–4079.
An older version is also available as arXiv preprint `arXiv:0805.0762v2`.

[Foissy07] Loïc Foissy, *Bidendriform bialgebras, trees, and free quasi-symmetric functions*, Journal of Pure and Applied Algebra, Volume 209, Issue 2, May 2007, pp. 439–459.
A preprint is also available as arXiv preprint `arXiv:math/0505207v1`.

[FoiMal14] Loïc Foissy and Claudia Malvenuto, *The Hopf algebra of finite topologies and T-partitions*, Journal of Algebra, Volume 438, 15 September 2015, pp. 130–169.
A preprint is also available as arXiv preprint `arXiv:1407.0476v2`.

[Fresse14] Benoit Fresse, *Homotopy of operads & Grothendieck-Teichmüller groups, First Volume*, preprint, 30 January 2016.
http://math.univ-lille1.fr/~fresse/OperadHomotopyBook/OperadHomotopy-FirstVolume.pdf

[GaGrLi16] Pavel Galashin, Darij Grinberg, Gaku Liu, *Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions*, arXiv:1509.03803v2.

[GKLLRT95] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, Jean-Yves Thibon, *Noncommutative symmetric functions*, Advances in Mathematics 112 (1995), no 2., pp. 218–348.
A preprint is also available as arXiv preprint arXiv:hep-th/9407124v1.

[Gessel84] Ira M. Gessel, *Multipartite P-partitions and Inner Products of Skew Schur Functions*, Contemporary Mathematics, vol. 34, 1984, pp. 289–301.
http://people.brandeis.edu/~gessel/homepage/papers/multipartite.pdf

[Gessel15] Ira M. Gessel, *A Historical Survey of P-Partitions*, to be published in Richard Stanley's 70th Birthday Festschrift, arXiv:1506.03508v1.
http://arxiv.org/abs/1506.03508v1

[GriRei15] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, August 25, 2015. arXiv preprint arXiv:1409.8356v3.
A version which is more often updated can be found at http://www.math.umn.edu/~reiner/Classes/HopfComb.pdf .

[GriRob15] Darij Grinberg, Tom Roby, *Iterative Properties of Birational Rowmotion, I and II.*
Part 1: Electronic Journal of Combinatorics, Volume 23, Issue 1 (2016), Paper #P1.33.
Part 2: Electronic Journal of Combinatorics, Volume 22, Issue 3 (2015), Paper #P3.40.

[Haz08] Michiel Hazewinkel, *Witt vectors. Part 1*, arXiv:0804.3888v1.
http://arxiv.org/abs/0804.3888v1

[HaGuKi10] Michiel Hazewinkel, Nadiya Gubareni, V. V. Kirichenko, *Algebras, Rings and Modules: Lie Algebras and Hopf Algebras*, AMS 2010.

[Joch13] Katharina Jochemko, *Order polynomials and Pólya's enumeration theorem*, The Electronic Journal of Combinatorics 21(2) (2014), P2.52. See also arXiv:1310.0838v2 for a preprint.

[Lusztig14] George Lusztig, *Hecke algebras with unequal parameters*, arXiv:math/0208154v2.

[Malve93] Claudia Malvenuto, *Produits et coproduits des fonctions quasi-symétriques et de l'algèbre des descentes*, thesis, defended November 1993. `http://www1.mat.uniroma1.it/people/malvenuto/Thesis.pdf`

[MalReu95] Claudia Malvenuto, Christophe Reutenauer, *Duality between Quasi-Symmetric Functions and the Solomon Descent Algebra*, Journal of Algebra **177** (1995), pp. 967–982.

[MalReu98] Claudia Malvenuto, Christophe Reutenauer, *Plethysm and conjugation of quasi-symmetric functions*, Discrete Mathematics, Volume 193, Issues 1–3, 28 November 1998, Pages 225–233. `http://www.sciencedirect.com/science/article/pii/S0012365X98001423`

[MalReu09] Claudia Malvenuto, Christophe Reutenauer, *A self paired Hopf algebra on double posets and a Littlewood-Richardson rule*, Journal of Combinatorial Theory, Series A 118 (2011) 1322-1333. A preprint version appeared as arXiv:0905.3508v1.

[Manchon04] Dominique Manchon, *Hopf algebras, from basics to applications to renormalization*, Comptes Rendus des Rencontres Mathematiques de Glanon 2001 (published in 2003). `http://arxiv.org/abs/math/0408405v2`

[Martin15] Jeremy L. Martin, *Lecture Notes on Algebraic Combinatorics*, February 18, 2016.

[MeNoTh11] Frédéric Menous, Jean-Christophe Novelli, Jean-Yves Thibon, *Mould calculus, polyhedral cones, and characters of combinatorial Hopf algebras*, Advances in Applied Mathematics, Volume 51, Issue 2, July 2013, pp. 177–227.
A preprint is also available as arXiv preprint `arXiv:1109.1634v2`.

[Montg93] Susan Montgomery, *Hopf Algebras and their Actions on Rings*, Regional Conference Series in Mathematics Nr. 82, AMS 1993.

[NovThi06] Jean-Christophe Novelli, Jean-Yves Thibon, *Polynomial realizations of some trialgebras*, extended abstract presented at FPSAC 2006; arXiv preprint `arXiv:math/0605061v1`.

[NovThi05a] Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, C. R. Acad. Sci. Paris Volume 342, 6, 365-446, 2006.
arXiv preprint `arXiv:math/0510218v1`.

[NovThi05] Jean-Christophe Novelli, Jean-Yves Thibon, *Hopf algebras and dendriform structures arising from parking functions*, Fundamenta Mathematicae 193 (2007), 189–241. A preprint also appears on arXiv as arXiv:math/0511200v1.

[Petra02] Emanuela Petracci, *Functional equations and Lie algebras*, thesis at Rome. http://www.iecn.u-nancy.fr/~petracci/tesi.pdf

[ReiRoi11] Victor Reiner, Yuval Roichman, *Diameter of graphs of reduced words and galleries*, Trans. Amer. Math. Soc. 365 (2013), pp. 2779–2802. A preprint version is available at arXiv:0906.4768v3.

[Schock04] Manfred Schocker, *The Descent Algebra of the Symmetric Group*, published in: Vlastimil Dlab, Claus Michael Ringel (eds.): Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 145–161. http://www.mathematik.uni-bielefeld.de/~ringel/schocker-neu.ps

[Schrij13] Alexander Schrijver, *A Course in Combinatorial Optimization*, February 3, 2013.

[Stanle95] Richard P. Stanley, *A Symmetric Function Generalization of the Chromatic Polynomial of a Graph*, Advances in Mathematics, Volume 111, Issue 1, March 1995, pp. 166–194.

[Stan11] Richard P. Stanley, *Enumerative Combinatorics, volume 1*, Cambridge University Press, 2011. http://math.mit.edu/~rstan/ec/ec1/

[Stan99] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press, 1999.

[Stan71] Richard P. Stanley, *Ordered Structures and Partitions*, Memoirs of the American Mathematical Society, No. 119, American Mathematical Society, Providence, R.I., 1972. http://www-math.mit.edu/~rstan/pubs/pubfiles/9.pdf

[Stan84] Richard P. Stanley, *Quotients of Peck posets*, Order, 1 (1984), pp. 29–34. http://dedekind.mit.edu/~rstan/pubs/pubfiles/60.pdf

[Stanley06] Richard Stanley, *An Introduction to Hyperplane Arrangements*, lecture series at the Park City Mathematics Institute, July 12-19, 2004. See also the errata, last updated 2016.

[Sweed69] Moss E. Sweedler, *Hopf Algebras*, W. A. Benjamin 1969.

[Willia03] Geordie Williamson, *Mind your P and Q-symbols: Why the Kazhdan-Lusztig basis of the Hecke algebra of type A is cellular*, B.A. thesis, University of Sydney, 2003.

[Zelevi81] Andrey V. Zelevinsky, *Representations of Finite Classical Groups: A Hopf Algebra Approach*, Lecture Notes in Mathematics #869, Springer 1981.

[Zinbie10] Guillaume W. Zinbiel, *Encyclopedia of types of algebras 2010*, `arXiv:1101.0267v1`.