Essays on Macroeconomics and Risk Premium
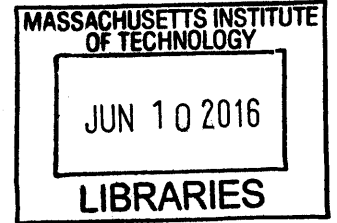
by

Dejanir Henrique Silva

Submitted to the Department of Economics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Economics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2016

© Dejanir Henrique Silva, MMXVI. All rights reserved.

Author . Signature redacted ....
Department of Economics
May 8, 2016

Certified by.. Signature redacted ....
Robert M. Townsend
Elizabeth & James Killian Professor of Economics
Thesis Supervisor

Certified by.. Signature redacted. ....
Iván Werning
Robert M. Solow Professor of Economics
Thesis Supervisor

Accepted by . Signature redacted ....
Ricardo Caballero
Ford International Professor of Economics
Chairman, Department Committee on Graduate Theses

Essays on Macroeconomics and Risk Premium

by

Dejanir Henrique Silva

Submitted to the Department of Economics
on May 8, 2016, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Economics

Abstract

The thesis consists of three essays on how macroeconomic policy can be an important determinant of risk premium and how variations in risk premium may affect macroeconomic policy.

Unconventional monetary policy represents a main example of how the transmission of macroeconomic policy is mediated by movements in risk premium. In the first essay, I examine how unconventional monetary policy affects asset prices by reallocating risk in the economy. I consider an environment with heterogeneity in risk tolerance and limited asset market participation. Risk-tolerant investors take leveraged positions, exposing the economy to balance sheet recessions. Limited asset market participation implies the balance sheet of the central bank is non-neutral. Unconventional monetary policy reduces the risk premium and endogenous volatility. During balance sheet recessions, asset purchases boost investment and growth. In contrast, during normal times, the expectation of future interventions reduces growth. Leveraged institutions respond to the policy by reducing risk-taking relatively more than risk-averse investors. As risk concentration falls, the probability of negative tail-events is reduced, enhancing financial stability.

An important determinant of entrepreneurial activity in developing countries is the amount of risk the entrepreneur must bear. The second essay, joint with Robert M. Townsend, analyzes the risk-taking behavior of entrepreneurs. Using data from a survey conducted in villages in Thailand, we document substantial heterogeneity in entrepreneurial activity. The fraction of net worth invested by entrepreneurs in risky activities decreases over the life cycle. Consumption-to-wealth ratio is U-shaped, being high for young and old entrepreneurs. We propose a model that captures both the life cycle patterns and limited idiosyncratic insurance observed in the Thai data. An expansion in idiosyncratic insurance will reduce the idiosyncratic risk premium, increasing the proportion of wealth invested in risky activities and aggregate output. However, as the return on the project falls, entrepreneurs accumulate less wealth, reducing their welfare in the long-run.

The third essay studies the optimal response of fiscal policy to a risk premium shock when a country is in a currency union. In the context of an open economy New

Keynesian model, I show that the government should *not* deviate from the optimal provision of public goods at an attempt to stabilize the economy. A consumption tax is used to *lean against the wind* and reduce the real interest rate in the presence of a positive risk premium shock. A VAT tax allows the government to independently influence the terms of trade. Optimal fiscal policy has the property of being revenue-generating. Therefore, there is not necessarily a trade off between stabilization policy and fiscal consolidation.

Thesis Supervisor: Robert M. Townsend
Title: Elizabeth & James Killian Professor of Economics

Thesis Supervisor: Iván Werning
Title: Robert M. Solow Professor of Economics

Acknowledgments

First and foremost, I am deeply indebted to my advisor, Robert M. Townsend, for my development throughout my graduate years. Rob has always been an incredible advisor, always open to hear my oftentimes vague ideas and pushing me in the right direction. Having the chance of being his student, research assistant, teaching assistant, and coauthor enriched my whole experience at MIT and it will have a long-lasting impact on my way of viewing this profession.

I would like to also thank my advisors, Iván Werning, Alp Simsek, and Leonid Kogan. Their unique perspective on my work was extremely enriching and each one of them had a great influence on the final outcome of this thesis. Iván has always been an inspiration and his clarity of thinking represents a model to be emulated. Alp always had an insightful comment to offer and his guidance was very important throughout this whole process. Leonid's help was invaluable in helping me to substantially improve both the presentation and substantive aspects of this thesis.

I am also grateful to Marios Angeletos, Ricardo Caballero, Daron Acemoglu, and Adrien Verdelhan for their guidance and support. I would like to thank also my fellow MIT graduate students, in particular, Adrien Auclert, Nico Caramp, Felipe Iachan, Sebastian Fanelli, Ameya Mulley, Juan Passadore, Matt Rognlie, Ludwig Straub, among many others who made an integral part of my experience at MIT. I owe a special thank to Rodrigo Adao for the constant exchange of ideas which certainly made this period much more interesting.

Last, but not least, I would like to thank my wife and son, Luciene and Gael. This thesis is a culmination of almost fifteen years of investment and preparation, from undergraduate to my doctoral studies. Luciene was on my side at every single one of these moments. For all her love and sacrifice during these years, I owe her more than I can possibly repay in a lifetime. Gael is my pride and joy and brought immense happiness to my graduate years. I would like also to thank my parents, Zuleide and Dejanir, the ones who inculcated in me an intellectual curiosity from an early age. I am thankful to my brother and sister, Lucas and Patricia, for their love and support.

Contents

List of Figures

11

List of Tables

Chapter 1

The Risk Channel of Unconventional Monetary Policy

1.1 Introduction

Unconventional monetary policy has been at the center of policy debate since the onset of the Great Recession. The policy consisted of large-scale asset purchases (LSAPs) in an attempt to compress risk premium and ease credit conditions.[1] Given the unprecedented character of such policies, the transmission mechanism of LSAPs as well as the potential side-effects of these policies remains a source of debate. What is the effect of different "exit strategies"? Does unconventional monetary policy induce more risk-taking in the financial sector? Can the expectation of future interventions affect the economy even after the central bank unwind its portfolio? In this essay, I provide a framework for the analysis of the transmission mechanism of LSAPs which allow us to address these questions.

I propose a *risk channel* of unconventional monetary policy. This channel operates through changes in the supply of risk to marginal investors, resulting in changes in risk premium and ultimately affecting risk-taking and economic growth. Key for

---

[1]See Fawley and Neely (2013) for a detailed discussion of large-scale asset purchases by the Federal Reserve, its motivation, and the comparison with the experience of other central banks. LSAPs is typically referred to as *quantitative easing*.

my results are two forms of heterogeneity: differences in market access and, among market participants, differences in risk tolerance. Limited asset market participation is important to guarantee that changes in the central bank balance sheet have real effects. Heterogeneity in risk tolerance will lead to a *countercyclical aggregate risk aversion*, exposing the economy to *balance sheet recessions*, i.e., a drop in asset prices and growth associated with a weak balance sheet of (risk-tolerant) financial intermediaries. In this context, LSAPs can be used to counteract the effects of balance sheet recessions, with consequences to portfolio and savings decisions of investors.

The environment consists of a stochastic growth model in continuous-time with heterogeneous agents. The economy is subject to limited asset market participation and market participants have heterogeneous preferences. Market participants (investors) can trade without any frictions while non-participants simply consume their endowment plus any transfers from the government. A group of relatively risk-tolerant investors (financial intermediaries) obtain short-term funding from a group of more risk-averse investors (savers) to finance risky investments.[2] The central bank invests in risky and riskless assets (reserves) and rebates the proceeds of the investment to non-participants according to given policy rules. In order to isolate the role of the risk channel, I abstract from features present in other theories of LSAPs, like liquidity frictions in the financial sector, limited commitment by the central bank, or a special role for the central bank's liability.[3] Importantly, even in the absence of these features, asset purchases by the central bank can have real effects.

In the first part of the essay, I show the laissez-faire economy is subject to balance sheet recessions. The main feature driving this result is countercyclical aggregate risk aversion. Given the high demand for safe assets from risk-averse savers, financial intermediaries issue riskless assets and invest in risky assets. Financial intermediaries will expose themselves to *risk-mismatch* as their assets are riskier than their liabilities. After a negative shock, their share of wealth will fall, increasing the aver-

---

[2]Intermediaries rely on short-term borrowing to finance risky investment. They should be interpret as leveraged institutions in general, like commercial bank, investment banks, and hedge funds. Savers correspond generically to funding institutions. Gertler et al. (2015) in a related setting focus instead on the distinction between retail and wholesale banks.

[3]I discuss the alternative theories in more detail in the literature review at end of this section.

age (wealth-weighted) risk aversion. A countercyclical aggregate risk aversion implies that risk premium rises and the interest rate falls after a negative shock. The increase in risk premium will depress asset prices and ultimately will reduce investment and growth. Heterogeneity in risk-tolerance is key for this result. Under homogeneous preferences, intermediaries and savers would choose the same exposure to risk and their relative wealth position would not respond to shocks. The economy would jump to a *balance growth path* with no variation in returns or investment. Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012) obtain balance sheet recessions in a setting with homogeneous preferences by limiting trade in aggregate risk. Di Tella (2012) allows for aggregate risk sharing, but a balance sheet channel only arises in the presence of idiosyncratic uncertainty shocks and moral hazard frictions. Preference heterogeneity allows me to keep the tractability of Brownian investment shocks without imposing any restriction on the ability of investors to trade aggregate risk.

The assumption of limited asset market participation is crucial to guarantee LSAPs can affect asset prices. Under full participation, a result akin to Modigliani-Miller/Ricardian Equivalence holds, and investors will exactly offset any changes in asset holdings of the central bank.[4] The assumption of limited participation captures the fact that the central bank intervened in relatively sophisticated markets, like MBSs, which are not readily accessible to everyone. As the central bank rebates the proceeds from its investment to workers, less risk will be held by the marginal investors, affecting asset prices.

I follow the tradition in the analysis of conventional monetary policy, and I specify the policy instruments of the central bank by *policy rules*. In particular, the portfolio of the central bank is a function of the balance sheet position of financial intermediaries. I focus a rule where central bank intervenes only when financial intermediaries' balance sheet is sufficiently weak, i.e., when aggregate risk aversion is high and asset prices are depressed.[5] This corresponds to an *unconventional Greenspan's Put*, where

---

[4]Wallace (1981) was the first to derive such neutrality result. Eggertsson and Woodford (2003) derives a similar result in an economy with sticky prices.

[5]In parallel work, Silva (2015), I discuss the determination of optimal policy in a two-period

instead of easing conventional monetary policy when asset prices are low, the central bank expands its balance sheet in states where asset prices are depressed. Given the specification of the policy rule, I solve the model numerically and calibrate it to US data. To capture the nonlinearities involved in movements in risk premium and in risk-taking of investors, it is important to use global solution methods instead of local approximations around a steady state. One advantage of the continuous-time setting is to allow for an effective solution method. The equilibrium can be obtained as the solution to a system of partial differential equations (PDEs) in the two state variables: the share of wealth of intermediaries and of the central bank.

Asset purchases by the central bank reduce the risk premium. LSAPs reduce the net supply of risk to investors, so in equilibrium a smaller return per unit of risk (Sharpe ratio) is required to clear the market. Endogenous volatility also fall in response to policy, so risk premium falls. Interest rates rise due to a combination of two effects. First, as the purchase of risky assets is financed by an increase in reserves, a higher interest rate is required to induce investors to hold the larger supply of safe assets. Second, as the intervention reduces volatility, investors have less of a precautionary savings motive. Importantly, this effect is present even when the central bank is not currently intervening. The expectation of intervention during crises reduce savings in normal times. In equilibrium, investment will fall to accommodate higher consumption. Hence, expectation of future intervention will reduce economic growth in normal times.

One concern with unconventional monetary policy is that it could create financial stability risks. In contrast, I find no support for such concerns. I identify two effects of LSAPs on the concentration of risk in the hands of financial intermediaries. First, a *hegding effect*. Since returns are countercyclical, intermediaries tilt their portfolio toward states with high returns, i.e., they hedge variations in returns. After the intervention returns are less countercyclical. In response, intermediaries tend to increase exposure to risk, given the weaker incentive to hedge. This argument is in line

setting. I introduce a moral hazard friction in the financial sector, and find the optimal policy would balance the pecuniary externality with the limited participation problem. Optimal policy would limit variations in price, similar to the effects of the policy proposed here.

with the concerns of critics. However, I identify a second effect, a *return sensitivity effect*. Since intermediaries are more risk tolerant, they are also more sensitive to returns. As the central bank compress returns, intermediaries are the ones with the stronger incentive to sell assets to the central bank, reducing their exposure to risk relative to savers. I find the return sensitivity effect dominates, reducing the concentration of risk in equilibrium. Endogenous volatility is related to the concentration of risk. LSAPs will reduce endogenous volatility and, in the stationary distribution, the probability of large drops in asset prices and growth.

Another source of debate was the role of different exit strategies. I capture the effect of (state-contingent) exit strategies by comparing two policy rules that are identical in states where the balance sheet of intermediaries are weak, but the policies differ as the economy recovers. Perhaps surprisingly, I find that the policy rule where the central bank sells more aggressively as the economy recovers, the "early-exit", *amplifies* the effects of LSAPs during crises. The intuition for this result is that returns are less countercyclical under the early-exit policy, inducing intermediaries to take more risk. Given the higher demand for risk, the market price of risk falls by more under early-exit.

I consider two additional issues: the effectiveness of LSAPs and its impact on the term premium. I find the marginal effect of the policy is higher when intermediaries have a weak balance sheet. The reason is that in those states savers have a greater weight in the demand for risk. Since savers are relatively insensitive to returns, the risk premium falls by more to induce them to sell their assets. The marginal effect increases with the size of the intervention. This suggests that measuring the effect of the policy when the intervention is still small will not capture the impact of the policy after it is scaled up. I also consider the effect on long-term bonds. Since bonds increase in value after a negative shock, they act as an insurance. Since LSAPs reduce endogenous volatility, the demand for insurance falls, rising the term premium and reducing the price of bonds. Hence, the policy will have a differential effect on assets depending on the relative importance of term and risk premium to price that asset.

Literature review. This essay is related to several strands of the literature in

macroeconomics and finance. A large literature evaluates empirically LSAPs (Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), D'Amico and King (2013), Greenwood and Vayanos (2014), see Joyce et al. (2012) for a survey). The main conclusion of these studies is that unconventional measures affect asset prices, even though there is a debate about the precise mechanism. Several channels have been proposed in the literature, e.g. a *liquidity channel* (Gertler and Kiyotaki (2010), Del Negro et al. (2011), Curdia and Woodford (2011); Gertler and Karadi (2011, 2013), Williamson (2012), Araújo et al. (2015)), a *signalling channel* (Bhatarai et al. (2014), Berriel and Mendes (2015)), an *asset scarcity/safety premium channel* (Krishnamurthy and Vissing-Jorgensen (2012), Caballero and Farhi (2014)).[6] The liquidity channel emphasizes the role of the central bank in performing intermediation when banks are liquidity constrained. The signaling channel corresponds to the effect of changes in the portfolio of the central bank on the expectation of the path of future interest rates. Asset scarcity theories emphasize the special role of the central bank liability as a safe asset. My research focus on a *risk channel* of unconventional monetary policy and shows how the balance sheet of the central bank have real effects even if banks are unconstrained, the central bank has full commitment, and its liability plays no special role.

This essay is related to the literature on the macroeconomic effects of shocks to balance sheets of firms and banks (e.g. Holmstrom and Tirole (1997), Bernanke et al. (1999), Adrian et al. (2010), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014)). In contrast to this literature, I assume intermediaries can trade aggregate risk without frictions. Di Tella (2012) adopts the same assumption, but balance sheet recessions arise by a combination of idiosyncratic uncertainty shocks and strong income effects on portfolio choice. In his setting, experts tilt their portfolio toward states with *low* returns, as those states require more resources to achieve a given utility level (an income effect). Since idiosyncratic returns decrease after a positive shock, experts have an incentive to take more risk, explaining why risk is

---

[6]A related literature looks at how quantitative easing can potentially affect the solvency of the central bank (see Hall and Reis (2013), Reis (2015)).

concentrated in his model. In my setting, balance sheet recessions arise due to differences in risk aversion, regardless of whether income or substitution effects dominate on portfolio choice.

A recent literature has considered the impact of monetary policy on financial stability and intermediaries' risk-taking. Diamond and Rajan (2012) and Brunnermeier and Sannikov (2015) illustrate how the expectation of central bank intervention can induce banks to take more risk. I show that unconventional monetary actually *reduces* the concentration of risks in intermediaries, despite the "stealth recapitalization" of banks generated by this policy.

An old literature emphasized the role of *portfolio balance effects* (Gurley et al. (1960),Tobin and Brainard (1963), Tobin (1969), Brunner and Meltzer (1973)). The key assumption of this literature was that assets are imperfect substitutes (usually for unmodelled reasons), so the central bank can affect the return of different assets by affecting their relative supply. I explore a similar mechanism, with risk being the source of differentiation, and emphasize that limited participation is needed in addition to imperfect substitutability.[7]

My analysis is also related to the literature on the effects of limited asset market participation. There is a long tradition of models of limited participation to study conventional monetary policy (Grossman and Weiss (1983), Rotemberg (1984), Alvarez et al. (2002)). I follow this tradition and study unconventional monetary policy by considering a tractable form of limited participation. A different literature focused on the effects of limited participation on risk premia and volatility (Mankiw and Zeldes (1991),Allen and Gale (1994), Basak and Cuoco (1998),Brav et al. (2002), Guvenen (2009)). Limited participation plays a different role in my analysis than in this literature, as it allows for real effects of changes in the central bank balance sheet, instead of acting as factor contributing to the concentration of risk on market participants.

A different form of limited participation appears in preferred habitat models.

---

[7]See Andrés et al. (2004) and Chen et al. (2012) for attempts to incorporate portfolio balance effects into modern DSGEs.

Effects of bond purchases by the central bank are usually interpreted using such theories. Vayanos and Vila (2009) provide a formalization in a setting with risk-averse arbitrageurs and investors that invest only in specific maturities of bonds and explore the implications for bond premia. In contrast, I emphasize that bond purchases can affect bond prices even if market participants trade in all maturities. Chien et al. (2012) provide another example of limited access to financial markets, where a set of investors does not rebalance their portfolio frequently.

Methodologically, my research draws on the asset pricing literature on heterogeneous investors (Constantinides (1982), Dumas (1989), Wang (1996), Chan and Kogan (2002), Gomes and Michaelides (2005), Gârleanu and Pedersen (2011), Longstaff and Wang (2012), Gârleanu and Panageas (2015)). My work is close to the papers by Drechsler et al. (2014) and Barro and Mollerus (2014). Both papers adopt Epstein-Zin preferences and follow Longstaff and Wang (2012) in modeling intermediaries as (relatively) risk-tolerant agents. Barro and Mollerus (2014) study the creation of safe assets but abstracts from limited participation, so the government's balance sheet position is neutral. Drechsler et al. (2014) study conventional monetary policy by assuming a special role for the central bank liability. I assume perfect substitutability between intermediaries' and central bank's liability and focus instead on unconventional monetary policy. [8]

Layout. The remainder of the essay is organized as follows. Section 1.2 presents the baseline model, and section 1.3 describes the characterization of the equilibrium. Section 1.4 discuss the effects of LSAPs during a balance sheet recession. Section 1.5 shows the impact of policy intervention to financial stability and section 1.6 discuss exit strategies. Section 1.7 presents the extensions: a discussion of the effectiveness of unconventional monetary policy, and the effects of policy on the term premium. Section 1.8 concludes.

---

[8]Caballero and Farhi (2014) study QE in a setting with risk neutral and infinitely risk aversion investors and where the central bank's liability plays a special role as a safe asset. Gennaioli et al. (2012) adopts a similar setting to study financial innovation.

1.2 The Model

I consider a continuous-time stochastic growth model with two goods, consumption and capital goods. The economy is populated by final good producers (firms), two types of market participants (savers and financial intermediaries), non-participants (households), and a government (central bank). Final good producers use capital to produce output subject to investment adjustment costs and finance their operations by issuing state contingent liabilities. Investment technology is subject to aggregate shocks. Financial intermediaries (or simply "bankers") and savers trade in dynamically complete financial markets.[9] The distinguishing feature of financial intermediaries is that they are relatively more risk tolerant than savers. Non-participants, or simply households, do not have access to financial markets, i.e., they consume their income plus any transfers from the government. The central bank invests in risky assets financed by its own net worth and riskless reserves. The central bank rebates the proceeds from investment to non-participants. Central bank risk exposure and rebates are defined by policy rules.

In the remainder of this section, I discuss the decision problem of each agent in detail and define the competitive equilibrium. Aggregate conditions are summarized by the vector of aggregate state variables X_t to be explicitly defined later on.

1.2.1 Final good producers

Firms use capital to produce final goods according to the linear technology

$$Y_t = AK_t \tag{1.1}$$

Capital evolves according to the law of motion

$$\frac{dK_t}{K_t} = g_t dt + \sigma dZ_t \tag{1.2}$$

---

[9]By an abuse of terminology, I use the expression to mean that dynamic trading spans aggregate risks. I later introduce idiosyncratic mortality shocks that are not spanned by financial markets.

Capital grows at the expected growth rate g_t. In order to achieve a given growth rate g_t the firm must invest $\iota(g_t)K_t$, where $\iota'(\cdot) > 0, \iota''(\cdot) > 0$. This captures the presence of adjustment costs for capital. Importantly, capital accumulation is subject to *investment shocks* with volatility σ. [10]

Firms pay output net of investment as dividends to its shareholders:

$$D_t \equiv AK_t - \iota(g_t)K_t \qquad (1.3)$$

Firms will choose investment to maximize the expected discounted value of dividends. Firms discount future dividends using the *state price density* π_t. The evolution of π_t can be written as

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \eta_t dZ_t \qquad (1.4)$$

where r_t is the instantaneous interest rate and η_t is the market price of risk.

The state price density, interest rate and the market price of risk are all functions of the aggregate state variable X_t: $\pi_t = \pi(X_t)$, $r_t = r(X_t)$, and $\eta_t = \eta(X_t)$. These functions will be determined in equilibrium.

The problem of the firm can then be written as

$$S_t = \max_g \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right] \qquad (1.5)$$

subject to (1.1), (1.2), (1.3), (1.4) and $K_t > 0$.

A version of Hayashi's (1982) hold in this environment, so the value of the firm can be written as $S_t = q_t K_t$, where $q_t = q(X_t)$ evolves according to

$$\frac{dq_t}{q_t} = \mu_{q,t} dt + \sigma_{q,t} dZ_t$$

where $\mu_{q,t}$ and $\sigma_{q,t}$ will be determined in equilibrium.

The return of holding claims in the firm is given by the dividend yield, $\frac{D_t}{S_t}$, plus

---

[10]Investment shocks have been identified by the DSGE literature as a main driver of business cycles fluctuations. See Justiniano et al. (2010).

capital gains, $\frac{dS_t}{S_t}$:[11]

$$dR_t = \underbrace{\left[\frac{A - \iota(g_t)}{q_t} + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \right]}_{\equiv \mu_{R,t}} dt + \underbrace{(\sigma + \sigma_{q,t})}_{\equiv \sigma_{R,t}} dZ_t$$

Notice that the volatility of returns has an exogenous component (σ), due to the volatility of investment shocks, and an endogenous component ($\sigma_{q,t}$), due to endogenous variations in q_t.

1.2.2 Financial intermediaries and savers

Private investors, financial intermediaries and savers, start with net worth $n_{j,0} > 0$, for $j \in \{b, s\}$ ("b" stands for "bankers"). Investors trade risky claims on firms and a riskless asset, choose how much to consume ($c_{j,t}$) and the share invested in the risky asset ($\alpha_{j,t}$). I focus on the case where investors do not receive transfers from the government. In proposition 3, I discuss the consequences of allowing investors to receive public transfers.

The net worth of investors evolves according to

$$\frac{dn_{j,t}}{n_{j,t}} = \left[r_t + \alpha_{j,t}(\mu_{R,t} - r_t) - \hat{c}_{j,t} \right] dt + \alpha_{j,t}\sigma_{R,t} dZ_t$$

where $\hat{c}_{j,t} \equiv \frac{c_{j,t}}{n_{j,t}}$.

I will consider a change of variable that will prove useful when characterizing the equilibrium. First, notice that no-arbitrage imply that the excess return on the risky asset is given by

$$\mathbb{E}_t[dR_t] - r_t dt = -\mathbf{Cov}_t \left(\frac{d\pi_t}{\pi_t}, dR_t \right) \iff \mu_{R,t} - r_t = \eta_t \sigma_{R,t} \qquad (1.6)$$

where η_t is (minus) the diffusion term in $\frac{d\pi_t}{\pi_t}$ (see equation (1.4)).

Define the *risk exposure* of investor j as $\sigma_{j,t} \equiv \alpha_{j,t}\sigma_{R,t}$. The decision problem of

---

[11]In order to compute the expected capital gain $\mathbb{E}_t\left[\frac{d(q_t K_t)}{q_t K_t} \right]$, I applied Ito's product rule: $\frac{d(q_t K_t)}{q_t K_t} = \frac{dq_t}{q_t} + \frac{dK_t}{K_t} + \frac{dq_t}{q_t}\frac{dK_t}{K_t}$.

investor j at time t_0 can be written directly in terms of $\sigma_{j,t}$ instead of $\alpha_{j,t}$:

$$V_{j,t_0} = \max_{(c_j,\sigma_j)} U_{j,t_0}(c_j) \tag{1.7}$$

subject to

$$\frac{dn_{j,t}}{n_{j,t}} = [r_t + \sigma_{j,t}\eta_t - \hat{c}_{j,t}]\,dt + \sigma_{j,t}dZ_t; \qquad n_{j,t} \geq 0 \tag{1.8}$$

given $n_{j,0} > 0$, and equation (1.6) was used to eliminate $\mu_{R,t} - r_t$.

The formulation highlights the investors care about its risk exposure, regardless if it invests a high share on a low volatility asset or a low share in a high volatility asset, and the excess return it gets per unit of risk. This justifies the term *market price of risk* to denote η_t, which is given to the (instantaneous) Sharpe ratio on the risky investment.[12]

In order to guarantee the existence of a non-degenerate stationary distribution of wealth, investors are subject to mortality risk. Each investor faces a constant hazard rate of death $\kappa > 0$. A mass κ of agents is born every period, so total population is kept constant. Newborn investors are of type b or type s with equal probability, and inherit the net worth of their "parents". Mortality risk will imply the (effective) discount rate is given by $\rho \equiv \hat{\rho} + \kappa$, where $\hat{\rho}$ captures impatience and κ the effect of mortality risk.[13]

Investors have the analogous in continuous-time of Epstein-Zin recursive preferences, as defined by Duffie and Epstein (1992):

$$U_{j,t} = \mathbb{E}_t\left[\int_t^\infty f^j(c_{j,s}, U_{j,s})ds\right] \tag{1.9}$$

---

[12]Another advantage of this formulation is that it encompasses different market structures. For instance, instead of assuming investors trade in a riskless asset and firm's equity, firms could be entirely financed by intermediaries and savers would hold riskless "deposits" and a risky asset issued by intermediaries (an asset-backed security). The important aspect is that investors can freely trade aggregate risk.

[13]See Gârleanu and Panageas (2015) for a derivation of the Epstein-Zin preferences under mortality risk.

where

$$f^j(c, U) = \rho \frac{(1 - \gamma_j)U}{1 - \psi^{-1}} \left\{ \left(\frac{c}{((1 - \gamma_j)U)^{\frac{1}{1-\gamma_j}}} \right)^{1-\psi^{-1}} - 1 \right\} \tag{1.10}$$

The coefficient of relative risk aversion is given by γ_j and the elasticity of intertemporal substitution (EIS) is ψ. Intermediaries are assumed to be relatively less risk averse than savers: $\gamma_b \leq \gamma_s$. Notice that while the coefficient of risk aversion depends on the type $j \in \{b, s\}$, the EIS is the same for both groups. Epstein-Zin preferences allow us to focus on heterogeneity of risk aversions while abstracting from differences in the EIS.[14]

1.2.3 Households

Households receive an endowment $(Y_{h,t})$, transfers from the government $(T_{h,t})$ and consume $(c_{h,t})$. Households are *hand-to-mouth*, i.e., they simply consume their total income

$$c_{h,t} = Y_{h,t} + T_{h,t} \tag{1.11}$$

For completeness, I assume that households have the same preferences as savers and the endowment $Y_{h,t}$ is constant, however these assumptions have no bearing in the positive implications of the model. As discussed in section 1.3, the presence of non-participant households will be important to understand the role of the central bank balance sheet on affecting asset prices and the macroeconomy.

1.2.4 Central Bank

Central bank starts with net worth $n_{cb,0} \geq 0$ and it chooses exposure to aggregate risk $\sigma_{cb,t}$. The central bank rebates the proceeds from its investment to households

---

[14]As discussed in section 1.4, Epstein-Zin preferences are also important to obtain the right co-movement between asset prices and risk premium.

$(T_{h,t})$. Central bank's balance sheet evolves according to[15]

$$dn_{cb,t} = n_{cb,t} \left[r_t + \sigma_{cb,t}\eta_t - \hat{T}_t \right] dt + n_{cb,t}\sigma_{cb,t}dZ_t \qquad (1.12)$$

where $n_{cb,t}\hat{T}_t = T_t$.

Central bank is subject to a No-Ponzi condition:

$$\lim_{T \to \infty} \mathbb{E}_t \left[\frac{\pi_T}{\pi_t} n_{cb,T} \right] \geq 0 \qquad (1.13)$$

I will focus on the case where the central bank chooses policy rules for risk exposure and transfers conditional on the aggregate state variable X_t, i.e., $\sigma_{cb,t} = \sigma_{cb}(X_t)$ and $T_t = T(X_t)$.[16]

1.2.5 Equilibrium

Definition 1. *An equilibrium is a set of stochastic processes for the interest rate $r = \{r_t : t \geq 0\}$, market price of risk $\eta = \{\eta_t : t \geq 0\}$, state price density $\pi = \{\pi_t : t \geq 0\}$, and the value of the firm $S_t = \{S_t : t \geq 0\}$; aggregate output $Y = \{Y_t : t \geq 0\}$, capital $K = \{K_t : t \geq 0\}$ and investment rate $g = \{g_t : t \geq 0\}$; consumption and risk exposure of sophisticated investors $c_j = \{c_{j,t} : t \geq 0\}$, $\sigma_j = \{\sigma_{j,t} : t \geq 0\}$, $j \in \{b, s\}$; consumption of workers $c_w = \{c_{w,t} : t \geq 0\}$; transfers and risk exposure for the central bank $T_j = \{T_{j,t} : t \geq 0\}$, $j \in \{b, s, w\}$, $\sigma_{cb} = \{\sigma_{cb,t} : t \geq 0\}$ such that*

i) *g_t solves problem (1.5) and the value of the objective is S_t.*

ii) *(c_j, σ_j) solves (1.7), given (r, η).*

---

[15]To isolate the role of the central bank in reallocating risk in the economy, I assume intermediaries' liability and the central bank's liability (reserves) are perfect substitutes. Hence, the central bank pay interest on reserves r_t in equilibrium. See Drechsler et al. (2014) for a model where reserves play a special role as safe assets.

[16]The central bank effectively acts as an intermediary for households by choosing its risk exposure and rebating the proceeds from the investment. In Silva (2015), I argue that it is not optimal for the central bank to simply replicate the full participation equilibrium in the presence of frictions. When the financial sector is subject to a moral hazard problem, the central bank would deviate from the full participation equilibrium to correct a pecuniary externality. A similar logic applies to environments with aggregate demand externalities.

iii) c_w satisfies (1.11) given (Y_w, T).

iv) (σ_{cb}, T) satisfies (1.12) given (r, η).

v) Markets clear:

$$c_{b,t} + c_{s,t} + c_{h,t} = Y_t - \iota(g_t)K_t + Y_{h,t}$$

$$n_{b,t} + n_{s,t} + n_{cb,t} = S_t$$

$$n_{b,t}\sigma_{b,t} + n_{s,t}\sigma_{s,t} + n_{cb,t}\sigma_{cb,t} = \sigma_{S,t}S_t$$

1.3   Equilibrium characterization

In this section, I characterize the equilibrium and provide the . First, I discuss the role of the two key assumption, heterogeneity in risk-tolerance and in market access.

1.3.1   The two dimensions of heterogeneity

Before describing the equilibrium conditions in detail, let's consider the role of the two dimensions of heterogeneity: risk tolerance and market participation.

Homogeneous risk tolerances. The following proposition shows that in the absence of heterogeneity in risk tolerance there are no fluctuations in returns or macroeconomic variables (up to scale):

Proposition 1. *Consider an economy with no central bank intervention, i.e., $\sigma_{cb,t} = T_t = n_{cb,t} = 0$ for all $t \geq 0$. Suppose $n_{b,0} = n_{s,0}$. If $\gamma_b = \gamma_s \equiv \gamma$, then*

1. Market price of risk and risk exposures are given by:

$$\eta_t = \gamma\sigma; \qquad \sigma_{b,t} = \sigma_{s,t} = \sigma$$

2. Growth rate and the price of capital satisfy the conditions:

$$\frac{A - \iota(g_t)}{p_t} = \rho - (1 - \psi^{-1})\left(g_t - \frac{\gamma\sigma^2}{2}\right); \qquad \iota'(g_t) = p_t$$

3. Interest rate and consumption-wealth ratios are given by:

$$r_t = \rho + \psi^{-1} g_t - (1 + \psi^{-1})\frac{\gamma\sigma^2}{2}; \qquad \hat{c}_{b,t} = \hat{c}_{s,t} = \rho - (1 - \psi^{-1})\left(g_t - \frac{\gamma\sigma^2}{2}\right)$$

The appendix contains closed-form expressions for the price-dividend ratio and the growth rate and provides parameter restrictions for the existence of equilibrium for the case with quadratic adjustment costs.

In the absence of differences in risk tolerance, the economy is essentially deterministic. Of course, aggregate capital is still subject to shocks, so output, consumption, and investment all move in proportion to the capital stock, but scaled variables do not respond to shocks. The assumption that we start at the steady state level of wealth, $n_{b,0} = n_{s,0}$, imply that scaled variables are constant. If we start at a different initial condition, there would be *deterministic* dynamics as the economy converges to the balanced growth path, but still scaled variables would not respond to shocks. The proposition also assumes the central bank does not intervene in the economy. An active central bank is able to affect returns and the macroeconomy provided there is variation in market participation.[17]

The result that risk exposures are the same for both types is more general than stated here. For instance, it does *not* rely on the existence of a representative agent. Scaled variables would still not respond to aggregate shocks if investors had different EIS or if they had access to different idiosyncratic investment opportunities.[18] The key to this result is a combination of the ability of agents to trade aggregate risk with the assumption of equal risk aversion. In this case, both agents will choose the same exposure to aggregate risk. Hence, aggregate shocks will affect the balance sheet of both investors equally, so their relative wealth does not change and the same is true for other scaled variables.

Balance sheet recessions, a fall in the growth rate of the economy due to a weakened

---

[17]However, even in this case there is no balance sheet recession, in the sense the relative net worth of intermediaries and savers will not respond to aggregate shocks.

[18]Gârleanu and Panageas (2015) found a version of this result for the case with different EIS, but no physical investment or differences in investment opportunities. Di Tella (2012) provides a similar result for the case without differences in EIS.

balance sheet of intermediaries, cannot arise under these assumptions. Brunnermeier and Sannikov (2014) and He and Krishnamurthy (2012) are able to generate balance sheet recessions by restricting aggregate risk sharing. Di Tella (2012) allows for trade in aggregate risk, but balance sheet recessions only arise in the presence of uncertainty shocks.

Proposition 2 shows that risk will be concentrated on the hands of intermediaries, $\sigma_{b,t} > \sigma_{s,t}$, when intermediaries are more risk tolerant. The next section will show how the fact that risk is concentrated on intermediaries can generate balance sheet recessions, even maintaining the assumption of Brownian investment shocks and perfect aggregate risk sharing.

Proposition 2 (Risk concentration). *If $\gamma_s = \gamma_b + \epsilon$, for $\epsilon > 0$ small, then*

$$\sigma_{b,t} - \sigma_{s,t} > 0 \tag{1.14}$$

Full participation benchmark. The next proposition describes the effect of the balance sheet of the central bank in an economy populated only by intermediaries and savers, and the central bank rebates the profits from investment to savers (denoted by $T_{s,t}$). I discuss the case with transfers to both investors in the appendix.

Proposition 3 (Neutrality result). *Suppose $T_h = 0$, and fix an initial policy rule (σ_{cb}, T_s) and corresponding equilibrium allocation. Consider an alternative policy rule (σ_{cb}^*, T_s^*) that satisfy the central bank's budget constraint.*

1. *Prices, consumption, and investment do not change with the central bank's portfolio:*

$$(r^*, \eta^*, S^*, c_b^*, c_s^*, g^*) = (r, \eta, S, c_b, c_s, g) \tag{1.15}$$

2. *Savers exactly offset the portfolio position of the central bank:*

$$\sigma_{s,t}^* n_{j,t}^* - \sigma_{s,t} n_{j,t} = - \left(\sigma_{cb,t}^* n_{cb,t}^* - \sigma_{cb,t} n_{cb,t} \right) \tag{1.16}$$

This result is reminiscent of the neutrality result for open market operations de-

rived by Wallace (1981) .[19] To gain intuition for this result, notice the relevant notion of wealth to savers is given by the sum of financial wealth and the present value of transfers, which is given by the net worth of the central bank in this case. Total wealth is given by $\tilde{n}_{s,t} \equiv n_{s,t} + n_{cb,t}$ and it evolves according to

$$\frac{d\tilde{n}_{s,t}}{\tilde{n}_{s,t}} = [r_t + \tilde{\sigma}_{s,t}\eta_t - \tilde{c}_{s,t}] \, dt + \tilde{\sigma}_{s,t} dZ_t \tag{1.17}$$

where

$$\tilde{\sigma}_{s,t} \equiv \frac{n_{s,t}}{n_{s,t} + n_{cb,t}} \sigma_{s,t} + \frac{n_{cb,t}}{n_{s,t} + n_{cb,t}} \sigma_{cb,t}; \qquad \tilde{c}_{j,t} \equiv \frac{c_{j,t}}{\tilde{n}_{j,t}}$$

Hence, the relevant risk exposure to the savers is $\tilde{\sigma}_{s,t}$ which includes both financial risk and the riskiness coming from transfers. If government transfers become more risky, perhaps because the central bank increases its exposure to risky assets, this will imply that investors must hold less *financial* risk in order to achieve a desired total risk exposure. Since total risk exposure does not change, then asset prices and macroeconomic variables will not change as well. Importantly, the result does *not* rely on the assumption of complete markets. Even in the presence of risks unspanned by financial markets the neutrality result would hold, provided transfers belong to the space of tradeable assets.

Limited asset market participation breaks this result. If a fraction of agents in the economy is unable to trade in financial markets, then the risk exposure of their total wealth will respond to changes in the central bank portfolio. Hence, unconventional monetary policy works by *redistributing risks* from (marginal) investors to non-participants.[20] If the central bank were to rebate to a fraction of its profits (or losses) to all agents, only the fraction going to non-participants would be non-neutral. Given the neutrality result, I focus on the case where the central bank rebates the

---

[19]Wallace's result is derived in a monetary overlapping generation economy. Eggertsson and Woodford (2003) shows a similar result in an economy with sticky prices.

[20]The mechanism is similar to the one in Alvarez et al. (2002) with fixed costs to access asset markets. Borrowing constraints, by impeding the investor to adjust his portfolio at the margin, would work in a similar way. I conjecture that several other frictions would break the neutrality result, for instance, intermittent portfolio rebalancing Chien et al. (2012), rational inattention Sims (2003), and bounded rationality Gabaix (2014).

proceeds from investment entirely to households.

1.3.2 Solving the model

Let's go back to the case with differences in risk aversion and in market participation. I will now discuss the solution to the problem of firms, investors, and the determination of prices in equilibrium.

Firms

The HJB equation for the final goods producer gives the pricing condition for capital:

$$r_t + (\sigma + \sigma_{q,t})\eta_t = \max_{g_t} \left\{ \frac{A - \iota(g_t)}{q_t} + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \right\} \tag{1.18}$$

The optimal investment rate satisfies

$$\iota'(g_t) = q_t \tag{1.19}$$

Investment is an increasing function of q_t: $g_t = (\iota')^{-1}(q_t)$. The marginal cost of increasing g_t is given by $\iota'(g_t)$ and the marginal benefit is given by increase in the value of the firm, summarized by q_t. This will represent the key mechanism connecting asset prices to investment and economic growth.

Investors

Given the homotheticity assumption, the value function of market participants $V_{j,t} = V_j(n_{j,t}, X_t)$ have a power form:

$$V_b(n_{b,t}, X_t) = \frac{(\zeta_t n_{b,t})^{1-\gamma_b}}{1 - \gamma_b}; \qquad V_s(n_{s,t}, X_t) = \frac{(\xi_t n_{s,t})^{1-\gamma_s}}{1 - \gamma_s} \tag{1.20}$$

where $\zeta_t = \zeta(X_t)$ and $\xi_t = \xi(X_t)$ follow a diffusion process (to be determined in equilibrium):

$$\frac{d\zeta_t}{\zeta_t} = \mu_{\zeta,t}dt + \sigma_{\zeta,t}dZ_t; \qquad \frac{d\xi_t}{\xi_t} = \mu_{\xi,t}dt + \sigma_{\xi,t}dZ_t;$$

I refer to (ζ_t, ξ_t) as *net worth multipliers*. Net worth multipliers are related to the marginal utility of wealth. They capture the fact that the marginal utility of wealth depends on the level of returns. For instance, if returns are expected to be low for a reasonable amount of time in the future, this will hurt those who rely on financial assets to finance future consumption. Hence, an additional unity of wealth in those states will not increase utility by the same amount compared to states where returns are high.[21] This will be reflected in a lower value for the net worth multiplier.

After some rearrangement, the HJB equation for intermediaries can be written as:

$$0 = \max_{\hat{c}_b, \sigma_b} \frac{\rho}{1 - \psi^{-1}} \left[\left(\frac{\hat{c}_{b,t}}{\zeta_t} \right)^{1 - \psi^{-1}} - 1 \right] + r_t + \sigma_{b,t} \eta_t - \hat{c}_{b,t} + \mu_{\zeta,t} - \frac{\gamma_b}{2} \left(\sigma_{b,t}^2 - 2 \frac{1 - \gamma_b}{\gamma_b} \sigma_{\zeta,t} \sigma_{b,t} + \sigma_{\zeta,t}^2 \right)$$

$$(1.21)$$

and an analogous condition holds for savers.

The optimal risk exposure is given by

$$\sigma_{b,t} = \underbrace{\frac{\eta_t}{\gamma_b}}_{\text{myopic}} + \underbrace{\frac{1 - \gamma_b}{\gamma_b} \sigma_{\zeta,t}}_{\text{hedging}};$$

$$(1.22)$$

The optimal portfolio decision has two components: a *myopic demand* and a *hedging demand*. The myopic demand coincides with the portfolio of a one-period mean-variance investor (hence the name "myopic"). It equals the market price of risk (or Sharpe ratio) times the risk tolerance. The fact that intermediaries are more sensitive to changes in the market price of risk than savers will be important when considering the effects of unconventional monetary policy.

Hedging demand captures deviations from the mean-variance portfolio due to variations in investment opportunities. If returns were constant, then $\sigma_{\zeta,t} = 0$ and intermediaries would act as mean-variance investors. In equilibrium, returns will actually be *countercyclical*, so $\sigma_{\zeta,t} < 0$.[22] This captures the fact that after a negative

---

[21] It is usually argued that zero interest rates combined with quantitative easing hurt savers and those planning future retirement (see Jeff Cox, "FED policies have cost savers \$470 billion: Report", CNBC, March 26, 2015).

[22] See discussion in section 1-2 for a discussion of the countercyclicality of returns.

shock asset prices fall, so the marginal utility of wealth increases for intermediaries. If we assume $\gamma_b < 1$, intermediaries react to countercyclical returns by reducing risk-taking. Since returns are high after a negative shock, intermediaries shift resources to those states, taking less risk ex-ante.

If $\gamma_b > 1$, then intermediaries would take more risk than a mean-variance investor. The reason is that instead of trying to send resources to states of nature where returns are high, intermediaries would do the opposite. Returns are already high after a negative shock, so less resources are needed to achieve the same level of utility. This *income effect* will dominate when $\gamma_b > 1$.[23]

Consider now the optimal consumption-wealth ratio:

$$\hat{c}_{b,t} = \rho^\psi \zeta_t^{1-\psi}; \qquad \hat{c}_{s,t} = \rho^\psi \xi_t^{1-\psi}; \qquad (1.23)$$

If EIS is equal to one, $\psi = 1$, then the consumption-wealth ratio is constant and equal to ρ. When $\psi \neq 1$, consumption-wealth ratio will respond to changes in investment opportunities. Plugging (1.23) into (1.21), we obtain

$$\rho^\psi \zeta_t^{1-\psi} = \psi\rho + (1-\psi)\left[\underbrace{r_t + \sigma_{b,t}\eta_t}_{\text{return on portfolio}} + \underbrace{\mu_{\zeta,t}}_{\text{change in inv. opp.}} - \underbrace{\Phi_{b,t}}_{\text{precautionary sav.}}\right] \qquad (1.24)$$

where

$$\Phi_{b,t} \equiv \frac{\gamma_b}{2}\left[\sigma_{b,t}^2 + 2\frac{\gamma_b - 1}{\gamma_b}\sigma_{\zeta,t}\sigma_{b,t} + \sigma_{\zeta,t}^2\right]$$

As in the traditional Fisherian analysis, the effect of interest rates on consumption generates income and substitution effects and the EIS determines which effect dominates. The expression above shows that a similar logic apply not only to the riskless return, but to the expected return on the portfolio adjusting for risk and expectations of changes in future returns.[24]

---

[23]The available empirical evidence on the risk-management of banks, in particular on the use of interest rate derivatives, seems to indicate the substitution effect dominates (see Begenau et al. (2013)).

[24]For instance, suppose returns are expected to improve and $\mu_{\zeta,t} > 0$. On the one hand, the investor may save more to have resources in the future when returns are high (substitution effect).

Market price of risk and interest rate

Define the share of *private wealth* held by intermediaries, x_t, and the share of *total wealth* held by the central bank, w_t:

$$x_t = \frac{n_{b,t}}{n_{b,t} + n_{s,t}}; \qquad w_t = \frac{n_{cb,t}}{n_{b,t} + n_{s,t} + n_{cb,t}}$$

Using these definitions, we can write the market clearing condition for risk exposures as

$$x_t \sigma_{b,t} + (1 - x_t) \sigma_{s,t} = \omega_t^r (\sigma + \sigma_{q,t}) \tag{1.25}$$

where

$$\omega_t^r \equiv \frac{1 - \frac{n_{cb,t} \sigma_{cb,t}}{\sigma_{S,t} S_t}}{1 - w_t}$$

The term ω_t^r measures the *net asset supply* to market participants. It equals the share of aggregate risk (per unit of wealth) held by private agents. In the absence of a central bank ($n_{cb,t} = 0$) or when the central bank's share of aggregate risk equals its share of wealth ($\sigma_{cb,t} = \sigma_{S,t}$), net asset supply equals one. If the central bank decides to hold proportionally more risk than its wealth share, then market participants will hold relatively less risk ($\omega_t^r < 1$). Hence, an expansion of the balance sheet where the central bank buys risky assets by issuing reserves will reduce the net asset supply to sophisticated investors.

Plugging in risk exposures from (1.22) into (1.25), we obtain an expression for the market price of risk:

$$\eta_t = \underbrace{\gamma_t}_{\text{agg. risk aversion}} \left[\underbrace{\omega_t^r (\sigma + \sigma_{q,t})}_{\text{net supply of risk}} - \underbrace{\left(x_t \frac{1 - \gamma_b}{\gamma_b} \sigma_{\zeta,t} + (1 - x_t) \frac{1 - \gamma_s}{\gamma_s} \sigma_{\xi,t} \right)}_{\text{avg. hedging demand}} \right] \tag{1.26}$$

On the other hand, it may save less since less resources are necessary to achieve the same level of utility in the future (income effect). If $\psi > 1$ the substitution effect dominates.

where γ_t is the aggregate risk aversion

$$\gamma_t = \left(\frac{x_t}{\gamma_b} + \frac{1 - x_t}{\gamma_s}\right)^{-1} \tag{1.27}$$

The market price of risk is the product of aggregate risk aversion with the difference between the net supply of risk and the average hedging demand. Hence, periods where intermediaries are relatively less capitalized, so aggregate risk aversion is high, will tend to have a high market price of risk. Similarly, this suggests that reductions in the supply of risk associated with the expansion of the balance sheet of the central bank will reduce the price of risk. Finally, if average hedging demand is high, then a smaller market price of risk will be required to induce investors to hold the net supply of risk.

Consider now the market clearing condition for goods:

$$x_t \hat{c}_{b,t} + (1 - x_t)\hat{c}_{s,t} = \omega_t^d \frac{A - \iota(g_t)}{q_t} \tag{1.28}$$

where

$$\omega_t^d \equiv \frac{1 - \frac{T_t}{D_t}}{1 - w_t}$$

The term ω_t^d captures the impact of central bank policy on the goods market. In the absence of a central bank or if the central bank rebates to workers all dividends received, then $\omega_t^d = 1$. If the central bank decides to reduce the size of its balance sheet by transferring relatively more resources to workers, this reduce the supply of goods to sophisticated investors $(\omega_t^d < 1)$.

Plugging in the expression for the consumption-wealth ratio into (1.28), we obtain an expression for the interest rate:

$$\psi \rho + (1 - \psi)\left[r_t + \omega_t^r(\sigma + \sigma_{q,t})\eta_t + \mu_t - \Phi_t\right] = \omega_t^d \frac{A - \iota(g_t)}{q_t} \tag{1.29}$$

where

$$\mu_t \equiv x_t \mu_{\zeta,t} + (1 - x_t)\mu_{\xi,t}; \qquad \Phi_t \equiv x_t \Phi_{b,t} + (1 - x_t)\Phi_{s,t};$$

37

The equation for the interest rate can be interpreted as representing the aggregate demand (in the left-hand side) and the aggregate supply for goods (in the right-hand side), both normalized by total wealth. The effect of interest rate on aggregate demand depends crucially on the EIS: if $\psi > 1$, then an increase in the interest rate decreases aggregate demand (the opposite is true when $\psi < 1$).

1.3.3 Markov Equilibrium

I will focus on a Markov equilibrium on the state variable $X = (x, w) \in [0, 1]^2$, so equilibrium prices $(r(X), \eta(X), q(X))$ and net worth multipliers $(\xi(X), \zeta(X))$ are all functions of X_t. Moreover, in a Markov equilibrium the central bank chooses policy rules that depends only on X_t. Instead of specifying the central bank policy in terms of $(\sigma_{cb,t}, T_t)$, I will assume without loss of generality the central chooses directly (ω_t^r, ω_t^d):

$$\omega_t^r = \omega^r(X_t); \qquad \omega_t^d = \omega^d(X_t) \tag{1.30}$$

I impose two constraints on policy rules: i) $\omega^d(x, w) > 0$ for all $(x, w) \in [0, 1]^2$; ii) $\omega^r(x, 0) = 1$ for all $x \in [0, 1]$. The first condition guarantees that consumption of sophisticated investors is always positive. The second constraint says the central bank cannot have infinite leverage, i.e., if the net worth of the central bank goes to zero, then asset holdings must also go to zero.

The next proposition characterizes the law of motion of X:

Proposition 4. *The state variables (x, w) evolve according to*

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t; \qquad dw_t = \mu_{w,t}dt + \sigma_{w,t}dZ_t;$$

1. The drift of x and w is given by

$$\mu_{x,t} = x_t(1 - x_t)\left[(\sigma_{b,t} - \sigma_{s,t})(\eta_t - \omega_t^r(\sigma + \sigma_{q,t})) + \hat{c}_{s,t} - \hat{c}_{b,t}\right] - \kappa(x_t - 0.5)$$

$$\mu_{w,t} = (1 - w_t)\left[(1 - \omega_t^r)(\sigma + \sigma_{q,t})(\eta_t - (\sigma + \sigma_{p,t})) - (1 - \omega_t^d)\frac{A - \iota(g_t)}{q_t}\right]$$

38

2. The diffusion of x and w is given by

$$\sigma_{x,t} = x_t(1 - x_t)(\sigma_{b,t} - \sigma_{s,t}); \qquad \sigma_{w,t} = (1 - w_t)(1 - \omega_t^r)(\sigma + \sigma_{q,t}); \quad (1.31)$$

The appendix provides the expression for (σ_x, σ_w) and (μ_x, μ_w) in terms of (ζ, ξ) and its derivatives.

The proposition describes the law of motion of the state variables. The assumption that investors have finite lives guarantees the extremes $x_t = 0$ or $x_t = 1$ are not absorbing states. In particular, $\mu_{x,t} < 0$ for $x_t = 1$ and $\mu_{x,t} > 0$ for $x_t = 0$. The assumption that future generations do not necessarily inherit the type of their "parents" imply that no type will eventually hold all the wealth in the economy.

The proposition shows that the diffusion of x depends on the relative risk exposure of intermediaries and savers $\sigma_{b,t} - \sigma_{s,t}$. Since the volatility of q_t is qiven by $\sigma_{q,t} = \frac{q_{x,t}}{q_t}\sigma_{x,t}$, the amount of endogenous volatility depends on the degree of risk concentration.

A corollary of the proposition above is that if $\omega_t^r = \omega_t^d = 1$, then w_t is constant, and the equilibrium allocation is identical to an economy without the central bank. This suggests the importance of the central bank operating leveraged in order to affect the economy.

Corollary 1. *Suppose $\omega^r(x, w_0) = \omega^d(x, w_0) = 1$ for all $x \in [0, 1]$, then $w_t = w_0$ for all $t \geq 0$. Moreover, equilibrium conditions coincide with the ones in a unregulated equilibrium without the central bank ($w_0 = 0$).*

1.3.4 Numerical solution and calibration

The functions $\zeta(x, w)$ and $\xi(x, w)$ can be obtained by solving a system of partial differential equations (PDEs). For this we need two conditions that can be expressed only in terms of the net worth multipliers and its derivatives. Hence, we need to compute the remaining equilibrium conditions as functions of (ζ, ξ). In the appendix 1.A.2, I discuss the algorithm used to solve the system of PDEs.

I adopt the following calibration. The level of technology A is set to $1/3$ in order to match a capital-output ratio of 3. Depreciation rate is set to $\delta = 0.05$. Investment adjustment costs are assumed to be quadratic $\iota(g) = \phi_0(g+\delta) + \frac{\phi_1}{2}(g+\delta)^2$, where ϕ_0 and ϕ_1 are chosen to match an average investment rate of 20% and average growth rate of 2%.[25] Volatility of aggregate output is set to match the (time-integrated) one-year volatility of output 2.33% in a post-war sample period. I set the EIS to $\psi = 2$, a common value found in the literature.[26] There is little guide for the choice of the risk aversion of intermediaries. I will focus on the case where the substitution effect dominates in the portfolio choice ($\gamma_b < 1$) and intermediaries have preferences for early resolution of uncertainty ($\gamma_b > \psi^{-1} = 1/2$). I chose $\gamma_b = 0.7$ but other values on this range generate similar results. The risk aversion of savers is set to $\gamma_s = 30$. This will generate a value for the average risk aversion around 1.4 and 3.6 during 90% of the time at the stationary distribution for x_t in the laissez-faire. Mortality rate is given by $\kappa = 0.02$ and $\tilde{\rho} = 0.001$ such that $\rho \approx 0.02$. For the numerical solution, I extend the model to allow for different sizes of intermediaries and savers and I set the share of intermediaries to $\theta_b = 0.1\%$.

1.4 Balance Sheet Recessions

In this section, I consider how unconventional monetary policy can be used to counteract the effects of balance sheet recessions. I first show how balance sheet recessions can emerge in a laissez-faire equilibrium. I discuss then the choice of policy rules for the central bank, the impact of central bank policy on asset prices and growth, and how the effectiveness of the policy vary with the state of the economy and the size of the intervention.

---

[25]The search for the parameters is subject to the conditions (1.A.1) for $\gamma = \gamma_b$ and $\gamma = \gamma_s$. These conditions guarantee the existence of equilibrium in the polar cases where a single type of sophisticated investor holds all private wealth.

[26]It is also a value commonly found in empirical studies that focus on the EIS for market participants. Vissing-Jørgensen and Attanasio (2003) explicitly distinguishes between EIS and risk aversion and find values between 1 and 2. Gruber (2013) estimates an EIS of 2 using tax data. Kapoor and Ravi (2010) estimates an EIS of 2.2 exploring a change in banking regulation in India.

1.4.1 The Laissez-Faire Equilibrium

Let's consider initially the case without the central bank, i.e., $\omega_t^r = \omega_t^d = 1$ and $w_0 = 0$. The evolution of the state variable x_t is given in figure 1-1, where $\mu_{x,t}$ and $\sigma_{x,t}$ are plotted as functions of x_t. The drift of x_t is positive for low levels of x_t and negative for high values of x_t. The point where it crosses zero is the *stochastic steady state*, the point of attraction of the system in the absence of shocks. Importantly, the diffusion term is always positive. Hence, from (1.31) and consistent with proposition 2, intermediaries are always more exposed to risk than savers.

Figure 1-1: Law of Motion of x_t
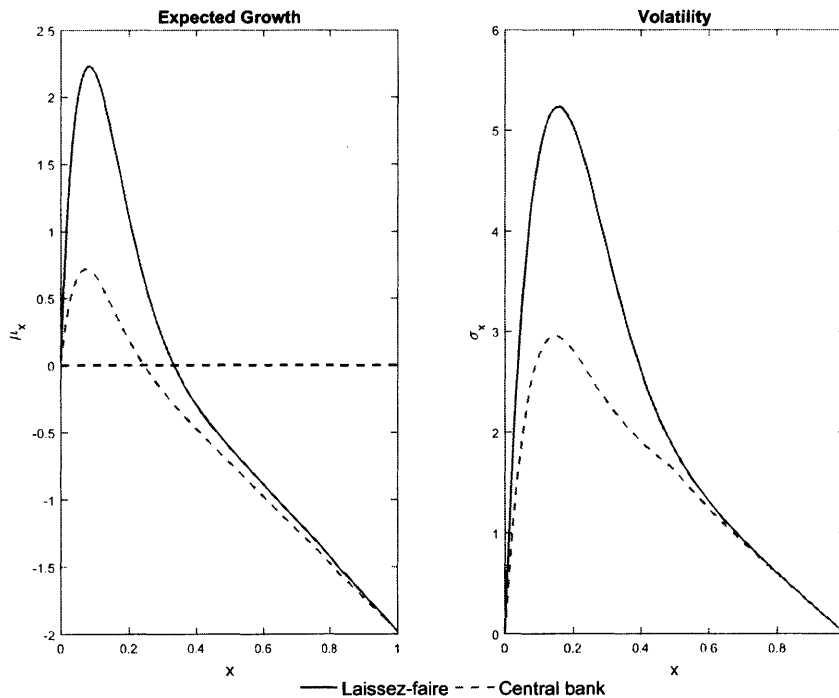
Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Drift and diffusion of x_t in percentage points.

The fact that intermediaries operate leveraged in equilibrium, so x_t is positively exposed to risk, will imply the economy is subject to *balance sheet recessions*. Figure 1-2 shows the price of capital q_t, the market price of risk η_t, interest rate r_t, and volatility of returns $\sigma + \sigma_{q,t}$ as functions of the relative strength of intermediaries' balance sheet (x_t).

41

The key feature driving variation in assets prices is a *countercyclical aggregate risk aversion*. Since intermediaries are more exposed to risk than savers, their share of wealth fall after a negative shock. Average risk aversion in the economy rises (see (1.27)). This will push the market price of risk up and interest rates down.

Here is the intuition behind figure 1-2. After a negative aggregate shock, the share of wealth of intermediaries will fall, given their higher exposure to risk. Savers will have to absorb a higher fraction of the risk and, since they are more risk-averse, the market price of risk will have to increase. The interest rate will fall, as the average precautionary motive becomes stronger as savers become relatively more important. The effect on the price is, in principle, ambiguous. The assumption that $\psi > 1$ plays a role to determine which effect dominates. Given a high elasticity of substitution, a small movement on interest rates is enough to restore equilibrium, so the price of capital will fall after a negative shock. If $\psi < 1$, a stronger response of interest rates would be required and the price of capital would actually fall after a reduction in x_t.

Volatility is a non-monotonic function of x_t. It inherits this pattern from the diffusion of the state variable. The reason is that if the net worth of an agent is sufficiently close to zero, then her wealth will not respond much to shocks (in the limit as the net worth is zero, there is no response to shocks). Even though the response is non-monotonic, for values of x_t around the stochastic steady state, a balance sheet recession (a reduction in x_t) will be associated with an increase in endogenous volatility.

Balance sheet recessions can be understood as the consequence of a *financial fire sale*. As the risk bearing capacity of intermediaries is reduced after a negative shock, high risk aversion agents will hold more of the risk in equilibrium, requiring a higher risk compensation. Savers make the role of a *second-best owner* of the asset, similar to farmers in Kiyotaki and Moore (1997). Importantly, the fire sale affects asset prices through changes in the *discount rate*. In contrast to the fire sales in Kiyotaki and Moore (1997) and Brunnermeier and Sannikov (2014), where the physical asset changes hands and asset prices are affected due to a direct reduction in the dividend.[27]

---

[27]See Cochrane (2011) for a review of the literature documenting the importance of variations in

Figure 1-2: Balance Sheet Recessions
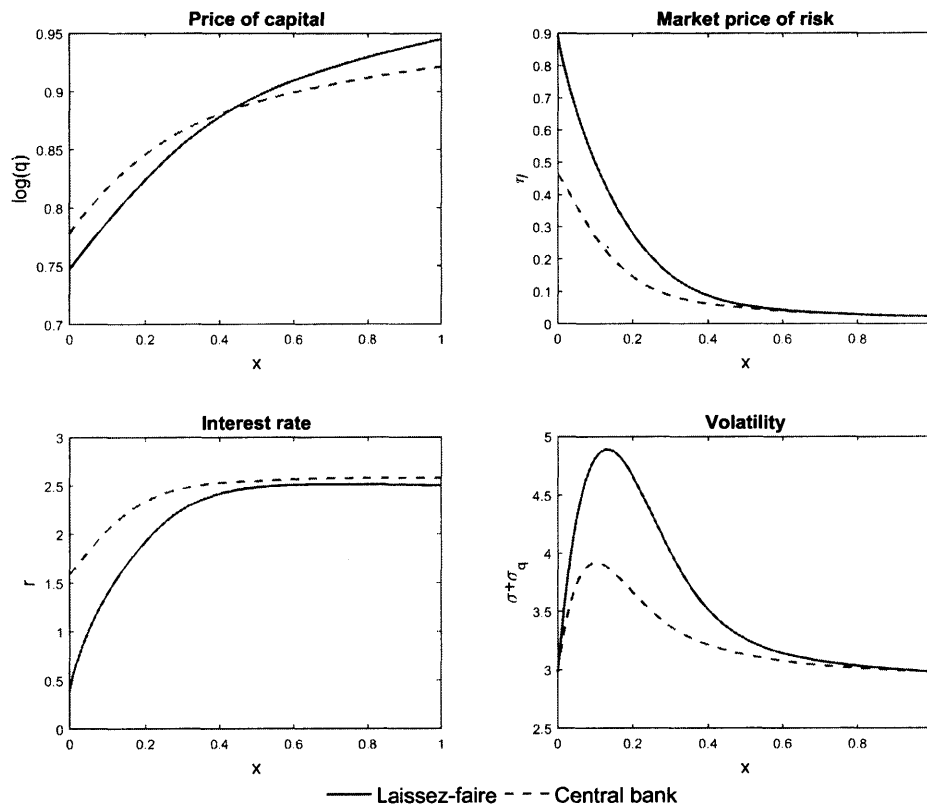
Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Interest rate and volatility in percentage points.

1.4.2 The Effects of Unconventional Monetary Policy

Policy rules

Consider the economy with a central bank. To capture the idea that unconventional monetary policy is a policy instrument typically used during crises, $\omega^r(\cdot, w)$ is assumed to be increasing in x:

$$\omega^r(x, w) = \beta_0^r(w) + \beta_1^r(w) \min\{x, x^*\} \tag{1.32}$$

where $\beta_0^r(w) \leq 1$ and $\beta_0^r(w) + \beta_1^r(w)x^* = 1$.

Remember a low of value of $\omega^r(\cdot)$ means the central bank is holding proportionally

---

discount rates to explain movements in asset prices.

a large fraction of risk.[28] Hence, when intermediaries are relatively less capitalized ($x_t < x^*$) the central bank will intervene and reduce the net asset supply ω^r. When intermediaries are relatively well capitalized, the central bank will keep the net asset supply at the laissez-faire value $\omega^r = 1$.

Since the central bank does not intervene if it has no net worth, the coefficients in the policy rule satisfy the condition $\beta_0^r(0) = 1$ and $\beta_1^r(0) = 0$. In the calibrated example, the coefficients in the policy rule satisfy $\beta_0^r(w) = 0.5$ and $\beta^r(w) = 1$ for $w \geq w^*$, where $w^* = 0.01$. For $w < w^*$, the coefficients are linearly interpolated: $\beta_0^r(w) = 1 - 0.5\frac{w}{w^*}$ and $\beta_1^r(w) = \frac{w}{w^*}$. The precise value of w^* and specification of the coefficients for $w < w^*$ have only a minor impact in the solution provided w^* is sufficiently small.

The proposed policy is meant to illustrate the effects of an aggressive policy, in particular, in very low probability states with low values of x_t. In a stationary distribution of the laissez-faire equilibrium, the first quartile of x_t is 0.3 and the third quartile is 0.42. Hence, the system spends most of the time around moderate values of x_t. Even extreme events do not reach values of x around 0. For instance, the 5th percentile is equal to 0.18. However, promises to intervene in these extreme events will have an impact on prices.

The policy rule $\omega^d(x, w)$ is assumed to be linear in w and independent of x:

$$\omega^d(x, w) = \beta_0^d + \beta_1^d w \tag{1.33}$$

where $\beta_0^d = 1.01$ and $\beta_1^d = -0.02$ in the calibrated example.

The role of the variation in ω^d is to guarantee that wealth of the central bank will return to the interior of the state space if ever reaches the boundaries of the system.

Unconventional Monetary Policy and Balance Sheet Recessions

Given the specified policy rules, we can compute the equilibrium in the presence of the central bank. Figure 1-1 shows the impact of unconventional monetary policy in

---

[28]More precisely, $\omega^r(\cdot) < 1$ if and only if the fraction of risk held by the central bank exceeds w_t.

the law of motion of x_t. Both the drift and the diffusion are uniformly reduced by the policy. The effect on the volatility of x_t and its connection with the concentration of risks in the financial sector will be discussed in detail in the next section.

Figure 1-2 shows how the policy of the central bank affects asset prices. Unconventional monetary policy reduces the market price of risk. As the central bank expands its balance sheet, the net supply of risk to sophisticated investors falls and, from (1.26), contributes to the reduction in η. Volatility of returns is reduced as the volatility of the state variable goes down. This reduction in volatility will contribute to the reduction in the market price of risk even in periods where $\omega^r = 1$.

The effect of asset purchases will be stronger the weaker the balance sheet position of intermediaries. There are two reasons for this. First, given the assumed policy rule, the central bank will intervene more in bad times. Second, demand for assets is more elastic when intermediaries are relatively well capitalized. Intermediaries tend to respond more strongly to changes in returns than savers. When intermediaries are undercapitalized, savers must bear most of the risk, and aggregate demand for risk will be relatively insensitive to returns. Hence, a higher reduction in returns is required to accommodate a reduction in the supply of risk caused by central bank policy.

The reduction in the return to the risky asset, by the intertemporal substitution channel, and the reduction in volatility, by the precautionary savings channel, will both tend to increase aggregate demand (or equivalently, reduce the incentive to save). In order to restore equilibrium, the riskless interest rate goes up. For low levels of x_t, the reduction in the risk premium will dominate the increase in the riskless rate and the price of capital will go up. However, for high values of x_t the interest rate will dominate and the price of the asset will go down.

Hence, unconventional monetary policy ameliorates the effects of balance sheet recessions in crisis, at the cost of reducing the growth rate in normal times. The central bank is able to reduce risk premium and volatility while it boosts investment and growth during a crisis. However, it reduces economic growth in booms compared to the laissez-faire economy.

1.5 Financial Stability

This section shows how unconventional monetary policy affect financial stability. First, I discuss how the concentration of risk in the financial sector responds to changes in central bank policy. I compute the stationary distribution for asset prices and show how the model generates endogenous "disasters", i.e, a relatively high probability of negative tail events. I show that unconventional monetary policy reduces tail risk but it reduces average growth rate in the economy.

1.5.1 Intermediaries' Risk Taking Decision

Unconventional monetary policy will affect the leverage decision of intermediaries, as the incentives to hold risky assets will respond to central bank's asset purchase.

Figure 1-3 shows intermediaries' leverage and risk concentration as a function of x_t for the laissez-faire economy and the economy with the central bank.[29] Notice that, compared to the homogeneous preferences benchmark (where leverage is always equal to one), the heterogeneous agent economy generates significant concentration of risk. The figure also shows the myopic and hedging component, as defined in (1.22), as the central bank policy will have different effects in the different components.

The hedging component is negative for all $x \in [0,1]$. The reason is that intermediaries anticipate that after a negative shock returns will increase. Intermediaries hedge against these changes in returns and reduce risk taking ex-ante. Consider now the effect of asset purchases by the central bank. Central bank policy reduces returns relatively more in bad times. The incentives for intermediaries to hedge will then be reduced, causing them to take more risk. Since risk-taking increases after a reduction in returns, I refer to this effect as the *hedging effect*.[30]

Consider now the myopic component. As the market price of risk is reduced, myopic demand falls with the central bank intervention. Importantly, myopic demand

---

[29]Leverage in this setting is simply risk exposure divided by volatility $\frac{\sigma_{b,t}}{\sigma+\sigma_{q,t}}$. Risk concentration is the difference between the exposure of intermediaries and savers: $\sigma_{b,t} - \sigma_{s,t}$.

[30]The leverage decision of savers have a similar pattern, but it is quantitatively smaller. Hence, the *relative* hedging demand behaves similarly to the hedging demand of intermediaries.

Figure 1-3: Intermediaries' Leverage and Risk Concentration
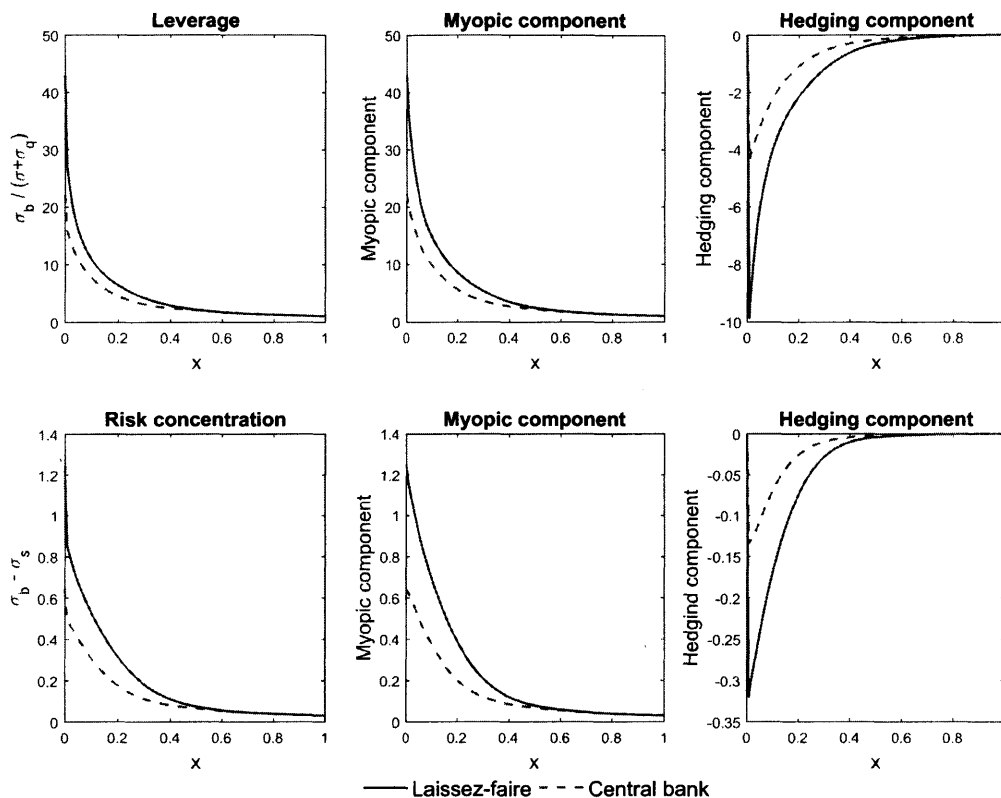
Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t. Leverage (assets over net worth) equals risk exposure over volatility.

falls *more* to intermediaries than savers. The reason is that intermediaries are relatively more risk tolerant, which imply they are also more sensitive to changes in η_t. Therefore, risk concentration will tend to fall with the central bank policy. I refer to this effect as the *return sensitivity effect*.

Another way of looking at the return sensitivity effect is to consider who will the central bank buy assets from. Since savers have high risk aversion, they don't respond very strongly to changes in returns. As the central bank buys risky assets, a given drop in returns would not be enough to induce savers to change its portfolio by much, but it would be enough to induce intermediaries to sell. Hence, most of the assets will flow from intermediaries to the central bank. This will tend to reduce risk exposure of intermediaries relative to savers.

Figure 1-3 shows that the return sensitivity effect dominates, so leverage of intermediaries and risk concentration falls. One intuition for this result is that the hedging effect arises because the volatility falls, so returns will not increase as much after a negative shock. But if the hedging effect were to dominate, risk concentration and volatility would increase, contradicting the fact that volatility must go down to generate the hedging effect.[31] This suggests that the return sensitivity effect should dominate the hedging effect.

1.5.2 Stationary distributions

So far we considered how unconventional monetary policy can affect the economy if the economy is in a crisis. I will now focus on how the central bank balance sheet can affect the *likelihood* of future crisis.

Figure 1-4 shows the stationary distribution of the growth rate of capital for the laissez-faire economy and for the economy with the central bank. The left panel shows the probability density function (PDF) for the two economies and the right panel shows the tail behavior of the stationary distribution measured by the probability of being a given number of standard-deviations below the mean.

An important feature of these distributions is that they present *negative skewness* and *excess kurtosis*, as can be seen in table 1.1. This means the economy is subject to (left) tail risk. [32]

Table 1.1: Summary Statistics of Stationary Distribution for g_t

Economy	Mean	Std. Dev.	Skewness	Kurtosis
Laissez-faire	1.3%	0.2%	-1.45	5.64
Central bank	1.2%	0.1%	-1.09	3,98

The economic mechanism that generates tail risk is related to the return sensitivity effect described in section 1.5.1. Suppose that initially intermediaries are

---

[31]This argument is incomplete since it ignores the effect of the volatility of w_t on returns.

[32]In particular, extreme negative events are much more likely than in a normal distribution. The probability of a negative 2 standard-deviation event is more than twice the one for the normal distribution and the difference is even higher for more extreme events.

Figure 1-4: Stationary Distribution
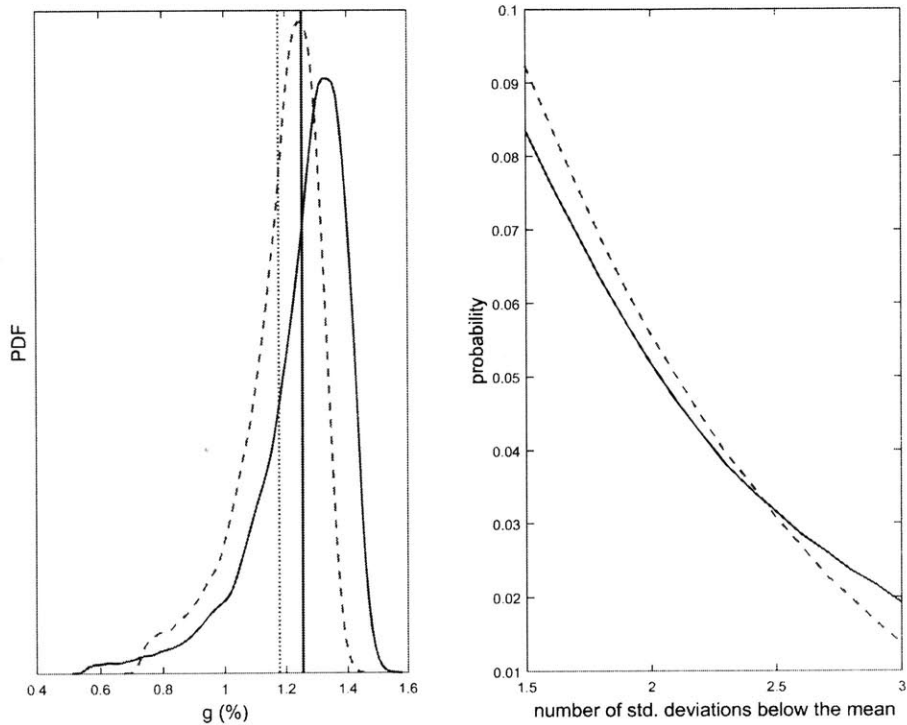
Note: Solid (dashed) line represents laissez-faire (central bank) equilibrium. Solution with the central bank evaluated at the ergodic mean of w_t.

relatively well capitalized and the price of risk is low. After a negative shock, wealth is redistributed towards savers and the aggregate demand for risk falls generating an increase in the price of risk. However, when intermediaries are initially well capitalized the effect on the aggregate demand for risk is small, since the difference between the portfolio of intermediaries and savers is also small.[33] Suppose now that intermediaries have low risk bearing capacity (low x_t). The same redistribution of wealth between intermediaries and savers have now a big effect on the aggregate demand for risk. The reason is that when returns are high, intermediaries have a greater incentive to hold risk so a reduction in their risk bearing capacity has a big impact on the demand for risk. In this situation, the same redistribution between intermediaries and savers will have a big impact on the price of risk and the price of capital. This asymmetric response of the price of capital will translate in an asymmetric response

---

[33]As shown in figure 1-3, risk concentration falls with share of wealth of intermediaries x_t.

of the growth rate of capital through equation (1.19).

Consider now the economy with a central bank. As described above, risk concentration falls with central bank policy, especially for low levels of x_t. Hence, the system will present less of a asymmetric response to shocks and central bank policy reduces skewness and kurtosis. Hence, if we measure financial stability by left-tail risk, we can conclude that asset purchases by the central bank *enhances* financial stability.

In contrast, the average growth rate in the economy falls. As discussed in section 1.4, asset purchases have an ambiguous effect on the price of risky assets. In particular, the price of the risky asset falls if intermediaries are relatively well capitalized. As the incentive to save falls with the policy, interest rates will increase, reducing the incentive to invest. Hence, asset purchases by the central bank reduces average growth rate in the economy.

1.6   Exit Strategies

Consider now the role of different exit strategies. I will focus on state-contingent rules where the central bank unwind its portfolio according to the balance sheet position of intermediaries. In particular, there is *reference strategy*, which correspond to the policy rule we have been analyzing so far, and two alternative rules, an *early exit* and a *late exit* strategy. The first will completely unwind its portfolio when $x = 0.3$ and the second when $x = 0.7$. Figure 1-5 plots the policy rules.

Figure 1-6 shows the market price of risk and the price of capital for early and late strategies *relative* to the reference strategy. Over the region $0 \leq x \leq 0.2$ where all policy rules coincide, the price of risk is smaller for the early strategy and higher for the late strategy. Here is the intuition: by promising to sell faster, conditional on the balance sheet of intermediaries, the central bank makes returns in good times relatively more attractive. Hence, intermediaries will have an incentive to take more risk. Given this higher demand for risk, the price of risk must fall. Therefore, risk premium is smaller under the early strategy, for the region policies coincide.

In the region the central bank is actually selling faster, for $0.2 \leq x \leq 0.3$, the

market price of risk is increasing and it surpasses the risk premium under the reference strategy. As more assets are sold in the reference strategy in the region $0.3 \leq x \leq 0.5$, the difference between the early strategy and the reference strategy falls.

Figure 1-5: Policy Rules - Exit Strategies
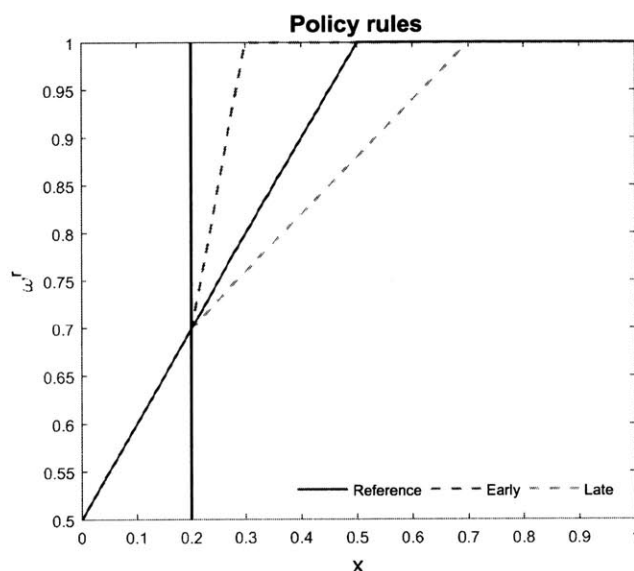
This illustrates the fact that the effects of the central bank intervention on asset prices will depend in a subtle way on the incentives of financial intermediaries to take risk. By inducing the a hedging demand over the region $0 \leq x \leq 0.2$, the central bank is able to increase the price of capital, stimulating investment and growth. The same is true for higher levels of x. The reason is that now incentive to save does not fall as much as in the reference strategy. The only region where the early strategy obtains a smaller value of the price of capital is when the risky asset is effectively being sold.

1.7 Extensions

1.7.1 Effectiveness of Asset Purchases

We have seen in previous sections that asset purchases by the central bank affect the price of risk. However, the effect is not constant, in particular, it is state-dependent

Figure 1-6: Exit Strategies
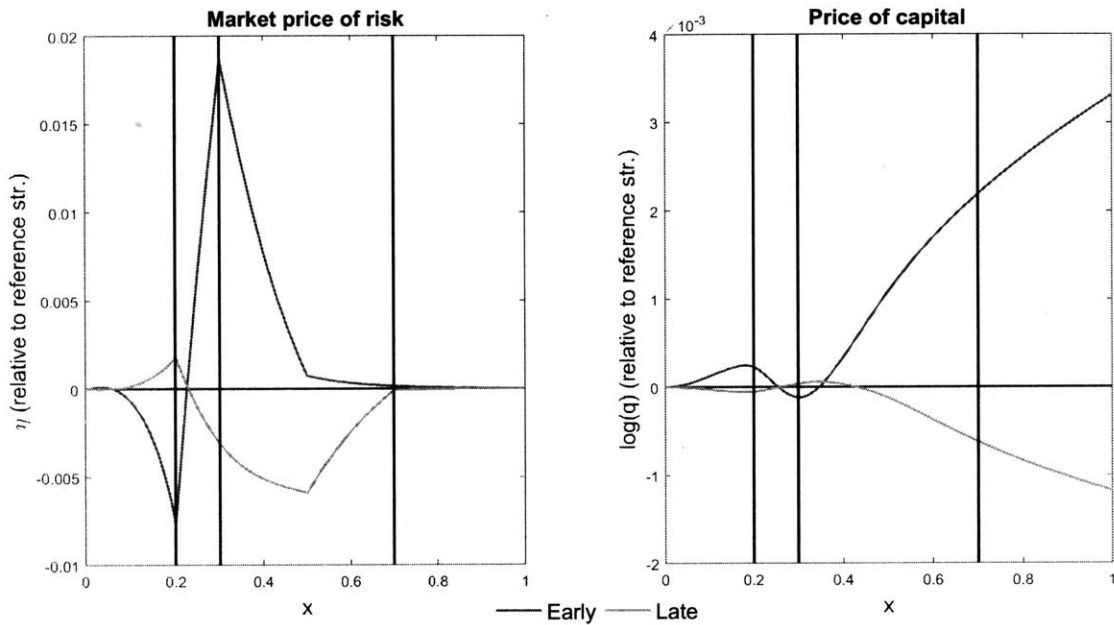
Note: Red (orange) line indicates the difference between variable under early (late) exit and the reference strategy. Vertical lines indicates the point $x = 0.2$, $x = 0.3$, and $x = 0.7$. Solution evaluated at the ergodic mean of w_t.

and non-linear. The effectiveness of unconventional monetary policy depends both on the strength of intermediaries' balance sheet and on the size of the intervention.

In order to isolate these effects, I will consider a simpler policy rule with $\beta_1^\tau(w) = 0$, so that policy does not respond to variations in x_t. Hence, the effect of the policy will vary with x_t only through the internal propagation mechanisms of the model.

The left panel on figure 1-7 shows the effect of assets purchases on the market price of risk for different levels of intervention. First, notice that effect gets smaller as intermediaries get better capitalized. Hence, unconventional monetary policy is more effective in crisis. The reason is that when savers are relatively more important, aggregate demand for risk becomes less elastic, so a larger change in returns is necessary to induce private agents to sell their risk assets to the central bank.

The effect of asset purchases also depend on the size of the intervention. The right panel on figure 1-7 shows the semi-elasticity of the price of risk with respect

Figure 1-7: Effectiveness of Asset Purchases
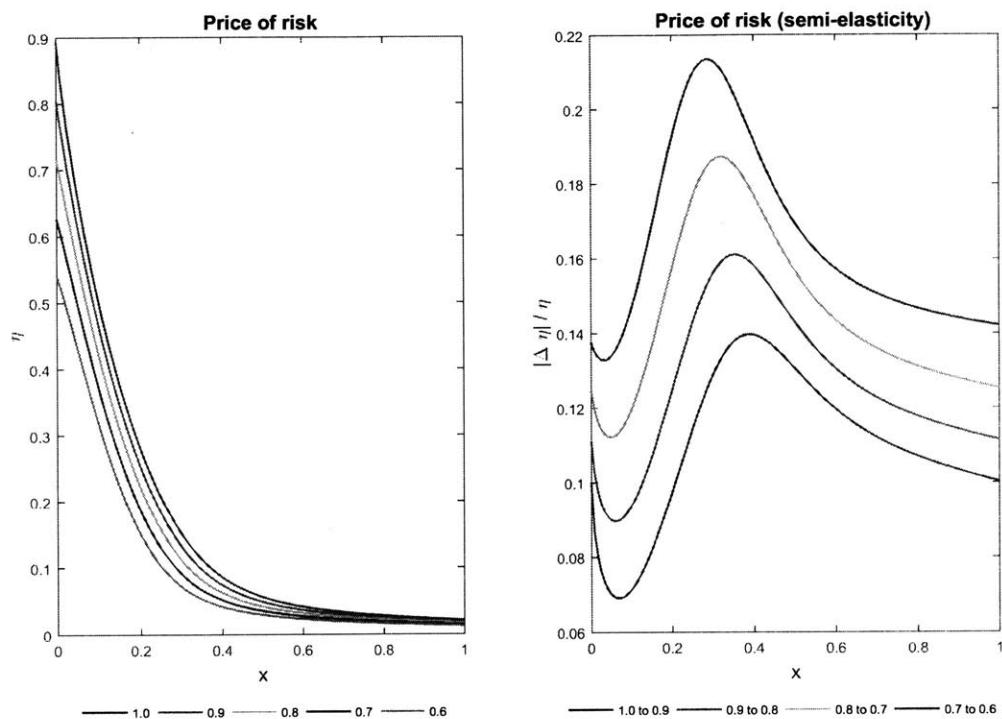
Note: Left panel shows the price of risk for the values of $\omega^r \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$. Right panel shows the proportional change in η for the specified changes in ω^r. Solution with the central bank evaluated at the ergodic mean of w_t.

to the net supply of risk for different levels of intervention.[34] Perhaps surprisingly, asset purchases become more effective as the size of the intervention increases. For small interventions, the level of endogenous volatility is relatively high, so the hedging effect is stronger. This attenuates the impact of the policy. For large interventions, endogenous volatility is low, so the hedging effect is weaker, and the policy becomes more powerful.

These two results have implications for the interpretation of the empirical evidence on the effects of QE. Researchers have found stronger effects for early interventions of the FED, exactly when intermediaries were less capitalized, consistent with the state-dependent effects described above.[35] However, the observation that the effectiveness of the policy increases with the size of the intervention indicates these estimates can

---

[34]The graph can be read as follows: a reduction of 0.1 in the net asset supply when $x = 0.1$ will reduce the price of risk by about 8% (14%) the price of risk if we start at $\omega^r = 1.0$ ($\omega^r = 0.7$).

[35]See (Joyce et al., 2012) for a discussion of the empirical evidence.

be a poor guide of the potential impacts of unconventional monetary policy. The effect of large intervention can be significantly larger than captured by the initial estimates. The calibrated example indicates the effects of large policies can be up to 80% higher then the effects of small interventions.

1.7.2 Long-term bonds and term premium

The focus so far has been on the effects of asset purchases on the risk premium and the price of risky assets. Asset purchases also have implications for the price of long-term bonds and the term premium, even if the central bank does not buy long-term bonds directly.

Instead of considering the whole term structure and how it varies with the state variables, I will focus on the price of a long-term bond with exponentially decaying coupons $e^{-\delta_b t}$. This will provide a parsimonious way of capturing the responses of the term structure to the central bank policy.

The price of the bond, denoted by p_t, can be written as

$$p_t = \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} e^{-\delta_b(s-t)} ds \right] = \int_t^\infty e^{-\delta_b(s-t)} p_{t,s} ds \qquad (1.34)$$

where $p_{t,s} = \mathbb{E}_t \left[\frac{\pi_s}{\pi_t} \right]$ denotes the period t price of a zero-coupon bond maturing at date s.

The *yield* of the long-term bond is defined as the value of y_t satisfying

$$p_t = \int_t^\infty e^{-(\delta_b+y_t)(s-t)} ds \Rightarrow y_t = \frac{1}{p_t} - \delta_b$$

and similarly for zero-coupon bonds: $p_{t,s} = e^{-y_{t,s}(s-t)}$.

It can be shown that yield on the bond with decaying coupons is, up to first-order,

an average of the yield on the zero-coupon bonds: [36]

$$y_t = \int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} y_{t,s} ds$$

We can define the term premium in a zero-coupon bond as:

$$\tau_{t,s} \equiv y_{t,s} - r_{t,s}^e$$

where $r_{t,s}^e = \frac{1}{s-t}\mathbb{E}\left[\int_t^s r_u du\right]$ is average expected interest rate between t and s.

The yield on the long-term bond can then be decomposed as an average of future expected short interest rates and an average term premium:

$$y_t = \underbrace{\int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} r_{t,s}^e ds}_{\equiv r_t^e} + \underbrace{\int_t^\infty (s-t)\delta_b^2 e^{-\delta_b(s-t)} \tau_{t,s} ds}_{\equiv \tau_t} \qquad (1.35)$$

The next proposition shows how to obtain the price of the bond and the average expected interest rate by solving a partial differential equation:

Proposition 5. *The price of the bond* $p_t = p(x_t, w_t)$ *satisfy the condition:*

$$0 = \frac{\sigma_{x,t}^2}{2} p_{xx,t} + \sigma_{x,t}\sigma_{w,t} p_{xw,t} + \frac{\sigma_{w,t}^2}{2} p_{ww,t} + (\mu_{x,t} - \sigma_{x,t}\eta, t)p_{x,t} + (\mu_{w,t} - \sigma_{w,t}\eta, t)p_{w,t} + 1 - (r_t + \delta_b)p_t$$

The average expected interest rate $r_t^e = r^e(x_t, w_t)$ *satisfy the condition:*

$$0 = \frac{\sigma_{x,t}^2}{2} r_{xx,t}^e + \sigma_{x,t}\sigma_{w,t} r_{xw,t}^e + \frac{\sigma_{w,t}^2}{2} r_{ww,t}^e + \mu_{x,t} r_{x,t}^e + \mu_{w,t} r_{w,t}^e + \delta r_t - \delta r_t^e$$

Figure 1-8 shows the yield, the expected future interest rate, and the term premium as a function of the state variable x_t. The yield of the bond is increasing in x_t. Hence, the bond increases in value in bad times, providing an insurance against aggregate shocks. Given the insurance properties of bonds, the term premium is

---

[36] To obtain the result, combine the expressions $p_t = \int_t^\infty e^{-\delta(s-t)} e^{-y_{t,s}(s-t)} ds \approx \frac{1}{\delta} - \int_t^\infty (s-t)e^{-\delta(s-t)} y_{t,s} ds$ and $p_t \approx \frac{1}{\delta_b} - \frac{1}{\delta_b^2} y_t$. Notice that $\delta^2 \int_t^\infty (s-t)e^{-\delta(s-t)} ds = 1$, so the weights integrate to one.

negative.[37]

Figure 1-8: Long-Term Bonds
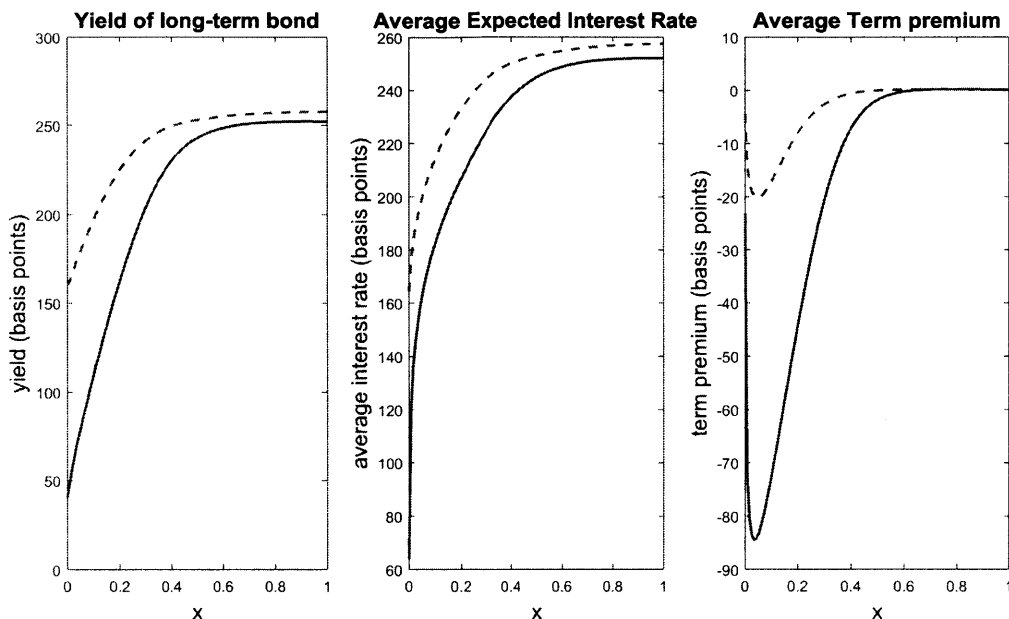
Purchases of risky assets *increases* the term premium. The intuition is simply that by reducing the amount of endogenous volatility in the economy, purchases of risky assets reduces the demand for insurance leading to an increase in the term premium. Given that the purchase of risky assets reduces the risk premium, this result indicates there is a trade-off between risk and term premium: as the central bank reduces the risk premium, it increases the term premium.

1.8 Conclusion

In this essay, I studied the macroeconomic effects of large-scale asset purchases by central banks. I consider not only the immediate impact of the intervention, but also how expectation of future interventions affect asset prices and financial risk-

---

[37]Relatedly, the yield curve is downward-sloping, a common feature in models lacking inflation risk or (conventional) monetary shocks. One form of obtaining an upward-sloping yield curve would be to introduce preferences shocks to the model.

taking. In line with the empirical evidence, I find that asset purchases reduces the risk premium and increase asset prices during crisis. In contrast, the expectation of future intervention have a negative impact on growth in normal times. As crisis gets less severe, investors have a weaker incentive to save, reducing growth. In contrast to what is typically argued in the popular press, I find that asset purchases reduce the concentration of risk on the hands of financial intermediaries. The reason is that intermediaries are more sensitive to variations in returns, so as returns fall with the intervention, they have a stronger incentive to sell than savers. I also consider the role of exit strategies. A commitment by the central bank to sell more of its assets in the future, conditional on the recovery of intermediaries' balance sheet, amplify the effects of asset purchases during crisis. By making the exit strategy conditional on the recovery, the central bank induces intermediaries to take more risk, reducing the risk premium.

This analysis suggests a few avenues for future research. First, a more detailed analysis of long-term bonds. This would require an extension with multiple shocks, as purchases of long-term bonds are redundant in the current setting. Multiple shocks would also allow for an analysis of transmission of asset purchases across different asset classes, as the increase in the exposure of the central bank to one risk factor may affect the premium for holding different risks. Second, the interaction between conventional and unconventional monetary policy. The results in this essay can be understood as the characterization of the natural (flexible price) allocation. In particular, my results show that the natural interest rate respond to the asset purchases of the central bank. This becomes particularly important under a binding zero lower bound, as an increase in the natural interest rates would stimulate the economy, as the interest gap would fall. Another important issue is the analysis of optimal policy. The presence of sticky prices would imply the central bank faces a trade-off: asset purchases stimulate the economy during crisis, but it distorts the allocation of risk to workers.

1.A Appendices

1.A.1 Proofs

Proof of proposition 1

Proof. The assumptions on the central bank imply $w_t = 0$ for all $t \geq 0$. Since $\gamma_b = \gamma_s$, then $\sigma_{x,t} = 0$, what imply $\sigma_{\zeta,t} = \sigma_{\xi,t} = \sigma_{q,t} = 0$. Hence, up to scale, the economy is non-stochastic. The assumption on the initial net worth imply that $\mu_{x,t} = 0$.

The market price of risk is given by $\eta_t = \gamma\sigma$. Risk exposures are given by $\sigma_{b,t} = \sigma_{s,t} = \sigma$. Combining the market clearing condition for consumption and the pricing condition for capital, we obtain:

$$r_t = \rho + \psi^{-1}g_t - \left(1 + \psi^{-1}\right)\frac{\gamma\sigma^2}{2}$$

Plugging the expression above into the pricing condition for capital:

$$\frac{A - \iota(g_t)}{q_t} = \rho - \left(1 - \psi^{-1}\right)\left(g_t - \frac{\gamma\sigma^2}{2}\right)$$

The condition above combined with $\iota'(g_t) = q_t$ determine g_t and q_t. Consider the quadratic adjustment costs case:

$$\iota(g) = \phi_0(g + \delta) + \frac{\phi_1}{2}(g + \delta)^2$$

Growth rate is given by $g_t = \frac{q - \phi_0}{\phi_1} - \delta$. Plugging into the expression above, we obtain $\iota(g(q)) = \frac{q^2 - \phi_0^2}{2\phi_1}$. The pricing condition for capital can then be written as

$$(1 - 2\psi^{-1})q_t^2 - 2\phi_1\left[\rho + \left(1 - \psi^{-1}\right)\left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma\sigma^2}{2}\right)\right]q_t + 2\phi_1 A + \phi_0^2 = 0$$

It is instructive to consider first the limit $\phi_1 \to \infty$. In this case $g_t = -\delta$. Price is given by

$$q_t = \frac{A}{\rho - \left(1 - \psi^{-1}\right)\left(g_t - \frac{\gamma\sigma^2}{2}\right)}$$

and the existence of a positive price requires

$$\rho - (1 - \psi^{-1}) \left(g_t - \frac{\gamma \sigma^2}{2} \right) > 0$$

This is the usual condition for existence of equilibrium in a Lucas tree model with Epstein-Zin preferences. For the case with finite ϕ_1 the condition is given by

$$\rho - (1 - \psi^{-1}) \left(\bar{g} - \frac{\gamma \sigma^2}{2} \right) > 0 \qquad (1.A.1)$$

where

$$\bar{g} \equiv \frac{\sqrt{2\phi_1 A + \phi_0^2} - \phi_0}{\phi_1} - \delta$$

Let's now check this is indeed the case. The consumption-wealth ratio will be positive if

$$\phi_1 \left[\rho + (1 - \psi^{-1}) \left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma \sigma^2}{2} \right) \right] - (1 - \psi^{-1}) q_t > 0 \qquad (1.A.2)$$

If $\psi = 2$, then the price is given by

$$q_t = \frac{A + \frac{\phi_0^2}{2\phi_1}}{\rho + \frac{1}{2} \left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma \sigma^2}{2} \right)}$$

Plugging the expression above into (1.A.2), we obtain (1.A.1) for $\psi = 2$. Define the following coefficients:

$$B = \phi_1 \left[\rho + (1 - \psi^{-1}) \left(\frac{\phi_0}{\phi_1} + \delta + \frac{\gamma \sigma^2}{2} \right) \right]; \qquad C = 2\phi_1 A + \phi_0^2;$$

If $\psi < 2$, then the quadratic equation for p_t has a unique positive root:

$$q_t = \frac{\sqrt{B^2 + (2\psi^{-1} - 1)C} - B}{2\psi^{-1} - 1}$$

Condition (1.A.2) can be written as

$$B + (1 - \psi)\sqrt{B^2 + (2\psi^{-1} - 1)C} > 0 \iff B - (1 - \psi^{-1})\sqrt{C} > 0$$

59

and the second inequality is equivalent to (1.A.1).

In order to see the equivalence between the two inequalities, notice that for a given C, the two functions of B at the left-hand side are strictly increasing and have a zero at the same point, so the two functions are positive for the same set of values of B (given C).

If $\psi > 2$ and the quadratic equation has real roots, then there is a single root consistent with positive consumption-wealth ratio:

$$q_t = \frac{B - \sqrt{B^2 - (1 - 2\psi^{-1})C}}{1 - 2\psi^{-1}}$$

The consumption-wealth ratio will be positive if

$$B - (\psi - 1)\sqrt{B^2 - (1 - 2\psi^{-1})C} < 0 \text{ and } B^2 - (1 - 2\psi^{-1})C \geq 0 \iff B - (1 - \psi^{-1})\sqrt{C} > 0$$

and the second inequality is equivalent to (1.A.1).

A similar argument applies: for the region where the term inside the square root is non-negative, the function on the left of the first inequality is decreasing in B and it is equal to zero and the function on the left of the last inequality is zero. Hence, the two sets of inequalities are equivalent. $\square$

Proof of proposition 2

Proof. First, notice the market price of risk can be written as

$$\eta_t = \gamma_b \sigma_{b,t} - (1 - \gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\sigma_{x,t}$$

Using the market clearing condition for risk, we obtain

$$x_t \sigma_{b,t} + (1 - x_t)\frac{1}{\gamma_s}\left[\gamma_b \sigma_{b,t} + \left((1 - \gamma_s)\frac{\xi_{x,t}}{\xi_t} - (1 - \gamma_b)\frac{\zeta_{x,t}}{\zeta_t}\right)\sigma_{x,t}\right] = \omega_t^r(\sigma + \sigma_{q,t})$$

Using the market clearing condition for risk and (1.31), we can write

$$\sigma_{x,t} = x_t \left[\sigma_{b,t} - \omega_t^r (\sigma + \sigma_{q,t}) \right]$$

Combining the previous two expressions, we get

$$x_t \tilde{\sigma}_{b,t} + (1 - x_t) \frac{1}{\gamma_s} \left[\gamma_b \tilde{\sigma}_{b,t} + \left((1 - \gamma_s) \frac{\xi_{x,t}}{\xi_t} - (1 - \gamma_b) \frac{\zeta_{x,t}}{\zeta_t} \right) x_t \left[\tilde{\sigma}_{b,t} - 1 \right] \right] = 1$$

where $\tilde{\sigma}_{b,t} \equiv \frac{\sigma_{b,t}}{\omega_t^r (\sigma + \sigma_{q,t})}$.

Rearranging the expression the above,

$$\tilde{\sigma}_{b,t} = \frac{1 + \frac{x_t(1-x_t)}{\gamma_s} \left((1 - \gamma_s) \frac{\xi_{x,t}}{\xi_t} - (1 - \gamma_b) \frac{\zeta_{x,t}}{\zeta_t} \right)}{1 + \frac{x_t(1-x_t)}{\gamma_s} \left((1 - \gamma_s) \frac{\xi_{x,t}}{\xi_t} - (1 - \gamma_b) \frac{\zeta_{x,t}}{\zeta_t} \right) - \frac{\gamma_s - \gamma_b}{\gamma_s} (1 - x_t)} \tag{1.A.3}$$

Notice that $\tilde{\sigma}_{b,t}$ is increasing in $\frac{\gamma_s - \gamma_b}{\gamma_s} \frac{1-x_t}{1+\mathcal{H}_t}$, where $\mathcal{H}_t \equiv \frac{x_t(1-x_t)}{\gamma_s} \left((1 - \gamma_s) \frac{\xi_{x,t}}{\xi_t} - (1 - \gamma_b) \frac{\zeta_{x,t}}{\zeta_t} \right)$. Let's compute how this term responds to γ_s:

$$\left. \frac{\partial \left[\frac{\gamma_s - \gamma_b}{\gamma_s} \frac{1-x_t}{1+\mathcal{H}_t} \right]}{\partial \gamma_s} \right|_{\gamma_s = \gamma_b} = \frac{1 - x_t}{\gamma_b} \tag{1.A.4}$$

where I used the fact that $\mathcal{H}_t = 0$ for $\gamma_s = \gamma_b$.

Since $\tilde{\sigma}_{b,t} = 1$ for $\gamma_s = \gamma_b$ and it is (locally) increasing in γ_s, then $\tilde{\sigma}_{b,t} > 1$ for $\gamma_s = \gamma_b + \epsilon$, $\epsilon > 0$ sufficiently small. Market clearing imply $\tilde{\sigma}_{s,t} < 1$, so $\sigma_{b,t} > \sigma_{s,t}$.[38] $\quad \square$

Proof of proposition 3

Define $\mathcal{T}_{j,t}$ as the expected discounted value of transfers:

$$\mathcal{T}_{j,t} \equiv \mathbb{E} \left[\int_t^\infty \frac{\pi_s}{\pi_t} T_{j,s} ds \right]$$

---

[38] $\tilde{\sigma}_{b,t}$ is locally increasing for $x_t < 1$, but we can check directly that $\sigma_{b,t} > \sigma_{s,y}$ as x_t approaches.

Define the martingale process $G_{j,t}$:

$$G_{j,t} = \mathbb{E}_t\left[\int_0^\infty \frac{\pi_s}{\pi_0}T_{j,s}ds\right] = \int_0^t \frac{\pi_s}{\pi_0}T_{j,s}ds + \frac{\pi_t}{\pi_0}\mathcal{T}_{j,t}$$

The martingale representation theorem implies that there exists a process $\sigma_{G_j,t}$ such that

$$dG_{j,t} = \frac{\pi_t}{\pi_0}\sigma_{G_j,t}dZ_t$$

Combining the previous two expressions, we get

$$T_{j,t}dt + \frac{d\left(\pi_t\mathcal{T}_{j,t}\right)}{\pi_t} = \sigma_{G_j,t}dZ_t$$

Applying Ito's lemma, we obtain

$$\mu_{\mathcal{T}_j,t} = r_t\mathcal{T}_{j,t} + \eta_t\sigma_{\mathcal{T}_j,t} - T_{j,t}$$

where $\sigma_{H_j,t} = \sigma_{G_j,t} + \mathcal{T}_{j,t}\eta_t$.

Define total wealth as the sum of financial wealth and the value of transfers:

$$\tilde{n}_{j,t} = n_{j,t} + \mathcal{T}_{j,t}$$

which evolves according to

$$\frac{d\tilde{n}_{j,t}}{\tilde{n}_{j,t}} = \left[r_t + \tilde{\sigma}_{j,t}\eta_t - \tilde{c}_{j,t}\right]dt + \tilde{\sigma}_{j,t}dZ_t \qquad (1.A.5)$$

where

$$\tilde{\sigma}_{j,t} = \frac{n_{j,t}}{\tilde{n}_{j,t}}\sigma_{j,t} + \frac{\sigma_{\mathcal{T}_j,t}}{\tilde{n}_j,t}; \qquad \tilde{c}_{j,t} = \frac{c_{j,t}}{\tilde{n}_{j,t}}$$

The problem of investor j can alternatively be written as choosing $(c_j, \tilde{\sigma}_j)$ subject to (1.A.5), given $\tilde{n}_{j,0} > 0$.

Fix a set of policy rules $\sigma_{cb,t}$ and $T_{j,t}$ and the corresponding equilibrium prices (r_t, η_t, S_t). Consider an alternative set of policy rules $\sigma^*_{cb,t}$ and $T^*_{j,t}$ such that total wealth at period 0 is unchanged for both types when computed using the state

62

price density from the *initial* equilibrium. I will conjecture that prices, investment, and consumption are unchanged $(r_t^*, \eta_t^*, \pi_t^*, S_t^*, g_t^*, c_t^*) = (r_t, \eta_t, \pi_t, S_t, g_t, c_t)$ and risk exposures are given by

$$n_{j,t}^* \sigma_{j,t}^* = n_{j,t} \sigma_{j,t} - \left(\sigma_{\mathcal{T}_j,t}^* - \sigma_{\mathcal{T}_j,t} \right) \tag{1.A.6}$$

Let's guess and verify this is an equilibrium. By assumption $\tilde{n}_{j,0}^* = \tilde{n}_{j,t}$. The budget set for investor j is unchanged, then $(\tilde{\sigma}_j^*, c_j^*) = (\tilde{\sigma}_j, c_j)$ and equation (1.A.6) hold. Since the state price density is the same, $g^* = g$. Consumption did not change, so the market clearing condition for consumption hold. It remains to determine the remainder market clearing conditions hold.

The budget of the central bank can be written as

$$n_{cb,t} = \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} T_s ds \right] = \mathcal{T}_{b,t} + \mathcal{T}_{s,t}$$

This imply the following chain of equalities

$$n_{b,t}^* + n_{s,t}^* + n_{cb,t}^* = \tilde{n}_{b,t}^* + \tilde{n}_{s,t}^* = \tilde{n}_{b,t} + \tilde{n}_{s,t} = n_{b,t} + n_{s,t} + n_{cb,t} = S_t$$

where I used the fact that total wealth did not change with the new policy.

Since $n_{cb,t} = \mathcal{T}_{b,t} + \mathcal{T}_{s,t}$, we have that

$$\sigma_{cb,t} n_{cb,t} = \sigma_{\mathcal{T}_b,t} + \sigma_{\mathcal{T}s,t}$$

We can use this fact to show the remaining market clearing condition hold:

$$\sigma_{b,t}^* n_{b,t}^* + \sigma_{s,t}^* n_{s,t}^* + \sigma_{cb,t}^* n_{cb,t}^* = \tilde{\sigma}_{b,t}^* \tilde{n}_{b,t}^* + \tilde{\sigma}_{s,t}^* \tilde{n}_{s,t}^* = \tilde{\sigma}_{b,t} \tilde{n}_{b,t} + \tilde{\sigma}_{s,t} \tilde{n}_{s,t} = \sigma_{b,t} n_{b,t} + \sigma_{s,t} n_{s,t} + \sigma_{cb,t} n_{cb,t} = \sigma_{S,t} S_t$$

This concludes the proof that the conjectured allocation is an equilibrium. $\square$

63

Proof of proposition 4

Proof. Law of motion of X_t: Applying Ito's lemma to the definition of x_t and w_t we obtain:

$$
dx_t = x_t \left[\frac{dn_{b,t}}{n_{b,t}} - \frac{d(n_{b,t} + n_{s,t})}{n_{b,t} + n_{s,t}} + \left(\frac{d(n_{b,t} + n_{s,t})}{n_{b,t} + n_{s,t}} \right)^2 - \frac{d(n_{b,t} + n_{s,t})}{n_{b,t} + n_{s,t}} \frac{dn_{b,t}}{n_{b,t}} \right]
$$

$$
= \underbrace{x_t(1 - x_t) \left[(\sigma_{b,t} - \sigma_{s,t})(\eta_t - \omega_t^r(\sigma + \sigma_{q,t})) + \hat{c}_{s,t} - \hat{c}_{b,t} \right] - \kappa (x_t - \theta_b)}_{\mu_{x,t}} dt +
$$

$$
+ \underbrace{x_t(1 - x_t)(\sigma_{b,t} - \sigma_{s,t})}_{\sigma_{x,t}} dZ_t
$$

$$
dw_t = w_t \left[\frac{dn_{cb,t}}{n_{cb,t}} - \frac{d(q_t Y_t)}{q_t Y_t} + \frac{d(q_t K_t)^2}{(q_t K_t)^2} - \frac{dn_{cb,t}}{n_{cb,t}} \frac{d(q_t K_t)}{q_t K_t} \right]
$$

$$
= \underbrace{w_t \left[r_t + \sigma_{cb,t} \eta_t - \hat{T}_t - (\mu_{q,t} + g_t + \sigma \sigma_{p,t}) \right] - (1 - w_t)(1 - \omega_t^r)(\sigma + \sigma_{p,t})^2}_{\mu_{w,t}} dt +
$$

$$
+ \underbrace{(1 - w_t)(1 - \omega_t^r)(\sigma + \sigma_{q,t})}_{\sigma_{w,t}} dZ_t
$$

using the fact $\sigma_{cb,t} = \frac{1 - \omega_t^r(1 - w_t)}{w_t}(\sigma + \sigma_{q,t})$.

Let's solve for the drift and diffusion terms as a function of the net worth multipliers and their derivatives.

Diffusion: The diffusion of X_t is given by

$$
\sigma_{x,t} = x_t(1 - x_t)(\sigma_{b,t} - \sigma_{s,t}); \qquad \sigma_{w,t} = (1 - w_t)(1 - \omega_t^r)(\sigma + \sigma_{q,t});
$$

The risk exposure of intermediaries and savers can be written as a function of $\sigma_{x,t}$:

$$
\sigma_{s,t} = \frac{\eta_t}{\gamma_s} - \frac{1}{\gamma_s} \left(\hat{\xi}_x \sigma_{x,t} + \hat{\xi}_w \sigma_{w,t} \right); \qquad \sigma_{b,t} = \frac{\eta_t}{\gamma_b} - \frac{1}{\gamma_b} \left(\hat{\zeta}_x \sigma_{x,t} + \hat{\zeta}_w \sigma_{w,t} \right); \qquad (1.A.7)
$$

where $\hat{\xi}_t = (\gamma_s - 1) \log \xi_t$ and $\hat{\zeta}_t = (\gamma_b - 1) \log \zeta_t$ (for instance, $\hat{\xi}_x = (\gamma_s - 1)\frac{\xi_x}{\xi}$).

Plugging in the expressions for $(\sigma_{\xi,t}, \sigma_{\zeta,t})$ into the market clearing condition for

capital, we obtain:

$$\eta_t = \gamma_t \left[\omega_t^r (\sigma + \sigma_{q,t}) + \frac{x_t}{\gamma_b} \left(\hat{\zeta}_x \sigma_{x,t} + \hat{\zeta}_w \sigma_{w,t} \right) + \frac{1 - x_t}{\gamma_s} \left(\hat{\xi}_x \sigma_{x,t} + \hat{\xi}_w \sigma_{w,t} \right) \right] \quad (1.A.8)$$

The risk exposures can then be written as

$$\sigma_{b,t} = \frac{\gamma_t}{\gamma_b} \left[\omega_t (\sigma + \sigma_{q,t}) + \frac{1 - x_t}{\gamma_s} \left[\left(\hat{\xi}_x - \hat{\zeta}_x \right) \sigma_{x,t} + \left(\hat{\xi}_w - \hat{\zeta}_w \right) \sigma_{w,t} \right] \right]$$

$$\sigma_{s,t} = \frac{\gamma_t}{\gamma_s} \left[\omega_t (\sigma + \sigma_{q,t}) - \frac{x_t}{\gamma_b} \left[\left(\hat{\xi}_x - \hat{\zeta}_x \right) \sigma_{x,t} + \left(\hat{\xi}_w - \hat{\zeta}_w \right) \sigma_{w,t} \right] \right]$$

Plugging in the expression above into the equation for $\sigma_{x,t}$, we get

$$\sigma_{x,t} = x_t (1 - x_t) \frac{\gamma_t}{\gamma_b \gamma_s} \left[(\gamma_s - \gamma_b) \omega_t^r (\sigma + \sigma_{q,t}) + \left(\hat{\xi}_x - \hat{\zeta}_x \right) \sigma_{x,t} + \left(\hat{\xi}_w - \hat{\zeta}_w \right) \sigma_{w,t} \right]$$

$$\sigma_{w,t} = w_t (l_{cb,t} - 1) (\sigma + \sigma_{q,t})$$

Consider first the special case where there is no intervention, i.e., $w_t = 0$. In this case, we can easily solve for $\sigma_{x,t}$:

$$\sigma_{x,t} = \frac{\gamma_t \frac{x_t}{\gamma_b} \frac{1 - x_t}{\gamma_s} (\gamma_s - \gamma_b)}{1 - \gamma_t \frac{x_t}{\gamma_b} \frac{1 - x_t}{\gamma_s} \left[(\gamma_s - \gamma_b) \frac{q_x}{q} + \left(\hat{\xi}_x - \hat{\zeta}_x \right) \right]} \sigma \quad (1.A.9)$$

Let's go back to the general case. The diffusion of $w_{cb,t}$ can be written as

$$\sigma_{w,t} = \frac{w_{cb,t} (l_{cb,t} - 1) \left(\sigma + \frac{q_x}{q} \sigma_{x,t} \right)}{1 - w_{cb,t} (l_{cb,t} - 1) \frac{q_w}{q}} \quad (1.A.10)$$

and $\sigma + \sigma_{p,t}$ is given by

$$\sigma + \sigma_{q,t} = \frac{\sigma + \frac{q_x}{q} \sigma_{x,t}}{1 - w_t (l_{cb,t} - 1) \frac{p_w}{p}} \quad (1.A.11)$$

65

The diffusion of x_b can be written as

$$\sigma_{x,t} = \frac{\gamma_t \frac{x_t}{\gamma_b} \frac{1-x_t}{\gamma_s} \left[\frac{(\gamma_s - \gamma_b)\omega_t + w_t(l_{cb,t}-1)(\hat\xi_w - \hat\zeta_w)}{1 - w_t(l_{cb,t}-1)\frac{q_w}{q}} \right]}{1 - \gamma_t \frac{x_t}{\gamma_b} \frac{1-x_t}{\gamma_s} \left[\frac{(\gamma_s - \gamma_b)\omega_t \frac{q_x}{q} + w_t(l_{cb,t}-1)\frac{q_x}{q}(\hat\xi_w - \hat\zeta_w)}{1 - w_t(l_{cb,t}-1)\frac{p_w}{p}} + \left(\hat\xi_x - \hat\zeta_x \right) \right]} \sigma \qquad (1.A.12)$$

The expression above depends on the derivatives of the net worth multipliers, but it depends on the derivatives of the price-output ratio as well. However, we can use the market clearing condition for goods to eliminate the derivatives involving p. The following expressions show how to obtain p and its derivatives from the net worth multipliers (and their derivatives):

$$\frac{\alpha(x,w)}{q(x,w)} = x\rho^\psi \zeta(x,w)^{1-\psi} + (1-x)\rho^\psi \xi(x,w)^{1-\psi}$$

$$\frac{p_w}{p} = \frac{\alpha_w}{\alpha} - (1-\psi)\frac{p}{\alpha}\left[x\rho^\psi \zeta^{1-\psi}\frac{\zeta_w}{\zeta} + (1-x)\rho^\psi \xi^{1-\psi}\frac{\xi_w}{\xi} \right]$$

$$\frac{q_x}{q} = \frac{\alpha_x}{\alpha} - (1-\psi)\frac{p}{\alpha}\left[x\rho^\psi \zeta^{1-\psi}\frac{\zeta_x}{\zeta} + (1-x)\rho^\psi \xi^{1-\psi}\frac{\xi_x}{\xi} + \frac{\rho^\psi \zeta^{1-\psi} - \rho^\psi \xi^{1-\psi}}{1-\psi} \right]$$

$$\frac{q_{xx}}{q} = \frac{\alpha_{xx}}{\alpha} - 2\frac{\alpha_x}{\alpha}\frac{q_x}{q} + 2\left(\frac{q_x}{q}\right)^2 - (1-\psi)\frac{q}{\alpha}\left[x\rho^\psi \zeta^{1-\psi}\left(\frac{\zeta_{xx}}{\zeta} + \frac{2\zeta_x}{x\zeta} - \psi\left(\frac{\zeta_x}{\zeta}\right)^2 \right) + (1-x)\rho^\psi \xi^{1-\psi}\left(\frac{\xi_{xx}}{\xi} - \frac{2\xi_x}{(1-x)\xi} - \psi\left(\frac{\xi_x}{\xi}\right)^2 \right) \right]$$

$$\frac{q_{ww}}{q} = \frac{\alpha_{ww}}{\alpha} - 2\frac{\alpha_w}{\alpha}\frac{q_w}{q} + 2\left(\frac{q_w}{q}\right)^2 - (1-\psi)\frac{q}{\alpha}\left[x\rho^\psi \zeta^{1-\psi}\left(\frac{\zeta_{ww}}{\zeta} - \psi\left(\frac{\zeta_w}{\zeta}\right)^2 \right) + (1-x)\rho^\psi \xi^{1-\psi}\left(\frac{\xi_{ww}}{\xi} - \psi\left(\frac{\xi_w}{\xi}\right)^2 \right) \right]$$

$$\frac{q_{wx}}{q} = \frac{\alpha_{wx}}{\alpha} - \frac{\alpha_w}{\alpha}\frac{q_x}{q} - \frac{\alpha_x}{\alpha}\frac{q_w}{q} + 2\frac{q_x}{q}\frac{q_w}{q} - (1-\psi)\frac{q}{\alpha}\left[x\rho^\psi \zeta^{1-\psi}\left(\frac{\zeta_{wx}}{\zeta} + \frac{\zeta_w}{x\zeta} - \psi\frac{\zeta_x}{\zeta}\frac{\zeta_w}{\zeta} \right) + (1-x)\rho^\psi \xi^{1-\psi}\left(\frac{\xi_{wx}}{\xi} - \frac{\xi_w}{(1-x)\xi} - \psi\frac{\xi_x}{\xi}\frac{\xi_w}{\xi} \right) \right]$$

Drift: The drift of X_t is given by

$$\mu_{x,t} = x_t(1-x_t)\left[(\sigma_{b,t} - \sigma_{s,t})(\eta_t - \omega_t(\sigma + \sigma_{q,t})) + \rho^{\psi_s}\xi_t^{1-\psi_s} - \rho^{\psi_b}\zeta_t^{1-\psi_b} \right] - \kappa(x_t - \theta_b)$$

$$(1.A.13)$$

$$\mu_{w,t} = w_t\left[r_t + \sigma_{cb,t}\eta_t - \hat{T}_t - (\mu_{q,t} + g_t + \sigma\sigma_{p,t}) \right] - (1-w_t)(1-\omega_t^r)(\sigma + \sigma_{p,t})^2$$

Notice that $\mu_{x,t}$ can be computed given the diffusion terms derived above. Hence, we only need to solve for $\mu_{w,t}$. First, from (1.18) we obtain an expression for r_t:

$$r_t = \frac{A - \iota(g(q_t))}{q_t} - (\sigma + \sigma_{q,t})\eta_t + \mu_{q,t} + g_t + \sigma\sigma_{q,t} \qquad (1.A.14)$$

Combining the previous two expressions, we get

$$\mu_{w,t} = (1-w_t)(1-\omega_t^r)(\sigma + \sigma_{q,t})(\eta_t - (\sigma + \sigma_{q,t})) - (1-w_t)(1-\omega_t^d)\frac{A - \iota(g(q_t))}{q_t} \qquad (1.A.15)$$

using the fact $\hat{T}_t = \frac{1-\omega_t^d(1-w_t)}{w_t} \frac{A-\iota(g_t)}{q_t}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of proposition 5

Proof. Define the martingale G_t:

$$G_t = \int_0^t \frac{\pi_s}{\pi_0} e^{-\delta_b s} ds + \mathbb{E}_t \left[\int_t^\infty \frac{\pi_s}{\pi_0} e^{-\delta_b s} ds \right] = \int_0^t \frac{\pi_s}{\pi_0} e^{-\delta_b s} ds + e^{-\delta_b t} \frac{\pi_t}{\pi_0} p_t \qquad (1.A.16)$$

where the second equality uses (1.34).

Computing the drift of the expression above and setting it to zero, we obtain the no-arbitrage condition:

$$\underbrace{\frac{1}{p_t} + \mu_{p,t} - \delta_b - r_t}_{\text{expected excess return}} = \sigma_{p,t} \eta_t \qquad (1.A.17)$$

Applying Ito's lemma to $p_t = p(x_t, w_t)$, we obtain

$$\mu_{p,t} = \frac{p_{x,t}}{p_t} \mu_{x,t} + \frac{p_{w,t}}{p_t} \mu_{w,t} + \frac{\sigma_{x,t}^2}{2} \frac{p_{xx,t}}{p_t} + \sigma_{x,t}\sigma_{w,t} \frac{p_{xw,t}}{p_t} + \frac{\sigma_{w,t}^2}{2} \frac{p_{ww,t}}{p_t}$$

$$\sigma_{p,t} = \frac{p_{x,t}}{p_t} \sigma_{x,t} + \frac{p_{w,t}}{p_t} \sigma_{w,t}$$

Using the no-arbitrage condition, we obtain the PDE for the price of the bond:

$$0 = \frac{\sigma_{x,t}^2}{2} p_{xx,t} + \sigma_{x,t}\sigma_{w,t} p_{xw,t} + \frac{\sigma_{w,t}^2}{2} p_{ww,t} + (\mu_{x,t} - \sigma_{x,t}\eta, t) p_{x,t} + (\mu_{w,t} - \sigma_{w,t}\eta, t) p_{w,t} + 1 - (r_t + \delta_b) p_t$$

$$(1.A.18)$$

The average expected interest rate can be written as

$$r_t^e \equiv \delta_b^2 \mathbb{E}_t \left[\int_t^\infty \int_t^s e^{-\delta_b(s-t)} r_u \, du \, ds \right] = \delta_b \mathbb{E}_t \left[\int_t^\infty e^{-\delta_b(s-t)} r_s \, ds \right] \qquad (1.A.19)$$

The expected path of interest rate can be compute by solving the following PDE:

$$0 = \frac{\sigma_{x,t}^2}{2} r_{xx,t}^e + \sigma_{x,t}\sigma_{w,t} r_{xw,t}^e + \frac{\sigma_{w,t}^2}{2} r_{ww,t}^e + \mu_{x,t} r_{x,t}^e + \mu_{w,t} r_{w,t}^e + \delta r_t - \delta r_t^e \qquad (1.A.20)$$

Consider also the expected average interest rate up to period T:

$$r_t^e(x, w; T) \equiv \frac{\delta}{1 - e^{-\delta(T-t)}} \mathbb{E}\left[\int_t^T e^{-\delta(s-t)} r_s ds\right] \qquad (1.A.21)$$

The expression above can be written as:

$$\begin{aligned}
r_t^e(x, w; T) &= \mathbb{E}_t\left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds + e^{-\delta \Delta t} \frac{1 - e^{-\delta(T-t-\Delta t)}}{1 - e^{-\delta(T-t)}} \mathbb{E}_{t+\Delta t}\left[\int_{t+\Delta t}^T \frac{\delta e^{-\delta(s-t-\Delta t)}}{1 - e^{-\delta(T-t-\Delta t)}} r_s ds\right]\right] \\
&= \mathbb{E}_t\left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds\right] + \frac{e^{-\delta \Delta t} - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \mathbb{E}_t\left[r_{t+\Delta}^e(x_{t+\Delta t}, w_{t+\Delta t}; T)\right] \\
&= \mathbb{E}_t\left[\int_t^{\Delta t} \frac{\delta e^{-\delta(s-t)}}{1 - e^{-\delta(T-t)}} r_s ds\right] + \frac{e^{-\delta \Delta t} - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \left[r_t^e(x_t, w_t; T) + \int_t^{t+\Delta t} (\dot{r}_s^e + \mathcal{D}r_s^e) ds\right] \\
&\approx \frac{\delta r_t \Delta t}{1 - e^{-\delta(T-t)}} + \frac{1 - \delta \Delta t - e^{-\delta(T-t)}}{1 - e^{-\delta(T-t)}} \left[r_t^e(x_t, w_t; T) + (\dot{r}_s^e + \mathcal{D}r_s^e) \Delta t\right]
\end{aligned}$$

where we denote a time derivative with a dot and $\mathcal{D}r_s^e$ denotes

$$\mathcal{D}r_s^e \equiv \mu_x r_x^e + \mu_w r_w^e + \frac{\sigma_x^2}{2} r_{xx}^e + \sigma_x \sigma_w r_{xw}^e + \frac{\sigma_w^2}{2} r_{ww}^e \qquad (1.A.22)$$

Rearranging the expression above, we obtain

$$-\dot{r}_t^e = \frac{\delta(r_t - r_t^e)}{1 - e^{-\delta(T-t)}} + \mathcal{D}r_t^e \qquad (1.A.23)$$

It is convenient to work $\tau = T - t$ instead of t directly. The expression above can be rewritten as

$$\frac{\partial r^e(x, w, \tau)}{\partial \tau} = \frac{\delta(r(x, w) - r^e(x, w, \tau))}{1 - e^{-\delta \tau}} + \mathcal{D}r^e(x, w, \tau) \qquad (1.A.24)$$

subject to the boundary condition:

$$r^e(x, w, 0) = r(x, w) \qquad (1.A.25)$$

68

Taking the limit as τ goes to zero, we have

$$\frac{\partial r^e(x, w, 0)}{\partial \tau} = \frac{1}{2} \mathcal{D} r^c(x, w, 0) \tag{1.A.26}$$

where I used the fact

$$\lim_{\tau \to 0} \frac{\delta(r(x, w) - r^e(x, w, \tau))}{1 - e^{-\delta \tau}} = -\frac{\partial r^e(x, w, 0)}{\partial \tau} \tag{1.A.27}$$

$\square$

1.A.2 Numerical Solution

The computation of equilibrium is reduced to the solution of a system of partial differential equations (PDEs) involving $(\zeta(x, w), \xi(x, w))$. The following procedure shows how to obtain a pair of conditions involving (ζ, ξ) and its derivatives:

1. Compute $q(x, w)$ using the condition:

$$x \rho^{\psi} \zeta(x, w)^{1-\psi} + (1 - x) \rho^{\psi} \xi(x, w)^{1-\psi} = \omega^d(x, w) \frac{A - \iota(g(q(x, w)))}{q(x, w)}$$

and differentiate the condition above to obtain the derivatives of $q(x, w)$.

2. Compute (σ_x, σ_w) using (1.A.12) and (1.A.10) in the appendix.

3. Applying Ito's lemma, compute $(\sigma_{q,t}, \sigma_{\zeta,t}, \sigma_{\xi,t})$

$$\sigma_{q,t} = \frac{q_{x,t}}{q_t} \sigma_{x,t} + \frac{q_{w,t}}{q_t} \sigma_{w,t}; \qquad \sigma_{\zeta,t} = \frac{\zeta_{x,t}}{\zeta_t} \sigma_{x,t} + \frac{\zeta_{w,t}}{\zeta_t} \sigma_{w,t}; \qquad \sigma_{\xi,t} = \frac{\xi_{x,t}}{\xi_t} \sigma_{x,t} + \frac{\xi_{w,t}}{\xi_t} \sigma_{w,t}$$

4. Compute η_t using (1.26) and $(\sigma_{b,t}, \sigma_{s,t})$ using (1.22) and the analogous condition for $\sigma_{s,t}$.

5. Compute $(\mu_{x,t}, \mu_{w,t})$ using conditions (1.A.13) and (1.A.15) in the appendix.

69

6. Applying Ito's lemma, compute $(\mu_{q,t}, \mu_{\zeta,t}, \mu_{\xi,t})$

$$\mu_{q,t} = \frac{q_{x,t}}{q_t}\mu_{x,t} + \frac{q_{w,t}}{q_t}\mu_{w,t} + \frac{1}{2}\left[\frac{q_{xx,t}}{q_t}\sigma_{x,t}^2 + 2\frac{q_{xw,t}}{q_t}\sigma_{x,t}\sigma_{w,t} + \frac{q_{ww,t}}{q_t}\sigma_{w,t}^2\right];$$

$$\mu_{\zeta,t} = \frac{\zeta_{x,t}}{\zeta_t}\mu_{x,t} + \frac{\zeta_{w,t}}{\zeta_t}\mu_{w,t} + \frac{1}{2}\left[\frac{\zeta_{xx,t}}{\zeta_t}\sigma_{x,t}^2 + 2\frac{\zeta_{xw,t}}{\zeta_t}\sigma_{x,t}\sigma_{w,t} + \frac{\zeta_{ww,t}}{\zeta_t}\sigma_{w,t}^2\right];$$

$$\mu_{\xi,t} = \frac{\xi_{x,t}}{\xi_t}\mu_{x,t} + \frac{\xi_{w,t}}{\xi_t}\mu_{w,t} + \frac{1}{2}\left[\frac{\xi_{xx,t}}{\xi_t}\sigma_{x,t}^2 + 2\frac{\xi_{xw,t}}{\xi_t}\sigma_{x,t}\sigma_{w,t} + \frac{\xi_{ww,t}}{\xi_t}\sigma_{w,t}^2\right];$$

7. Compute r_t using (1.18).

8. Plug $(r_t, \eta_t, \sigma_{b,t}, \sigma_{\zeta,t})$ into (1.21), analogously for savers, to obtain the system of PDEs.

The boundary conditions for the PDEs can be obtained by the behavior of the diffusion for (x_t, w_t) at the boundaries:[39]

$$\lim_{x\to 0}\sigma_{x,t} = 0, \ \forall w \in [0,1]; \qquad \lim_{x\to 1}\sigma_{x,t} = 0, \ \forall w \in [0,1];$$

$$\lim_{w\to 0}\sigma_{w,t} = 0, \ \forall x \in [0,1]; \qquad \lim_{w\to 1}\sigma_{w,t} = 0, \ \forall x \in [0,1];$$

The numerical solution is a finite-difference implementation of a *method of lines with false transient*. The method consists of introducing a "false" time dimension (or considering the finite horizon version of the problem) and discretizing the derivatives involving (x, w) using finite differences. The time dimension is kept continuous, so we convert the problem from a two-dimensional system of PDEs to a 2N-dimensional system of ODE (where N is the number of points in the grid for (x, w)). The system of ODEs is solved using MATLAB's ODE suite.

---

[39]See Schiesser (1996) for a discussion of boundary conditions involving order reduction of PDEs.

Chapter 2

Risk-Taking over the Life Cycle in Village Economies

* Joint with Robert M. Townsend (MIT).

2.1 Introduction

We study the risk-taking and savings behavior of entrepreneurs over the life cycle. Using data from a household survey of rural and semi-urban households in four provinces in Thailand over a period of 156 months, we document substantial amount of heterogeneity across and withing demographic groups. Across demographic groups, we find that young entrepreneurs invest proportionally more in risky assets than older entrepreneurs. Consumption, as a function of financial wealth, is a U-shaped function of age, with households consuming proportionally more during young and old age. Differences in behavior of different demographic groups is economically significant, with the portfolio share of young entrepreneurs being up to 60% higher than the portfolio share of old entrepreneurs.

There is also a lot of heterogeneity within demographic groups. Even controlling for age and type of occupation, we observe large differences in returns. (Samphantharak and Townsend, 2013) documents an important idiosyncratic component on the volatility of returns. They also document that idiosyncratic risk explains a relatively

small fraction of the risk premium. The fact that these entrepreneurs have access to informal risk sharing networks partially mitigate the impact of idiosyncratic shocks. This suggests that entrepreneurs have access to some form of (partial) idiosyncratic insurance.

We propose a theoretical model that accounts for the heterogeneity in behavior over the life cycle as well as to the fact that entrepreneurs are (partiallly) exposed to idiosyncratic risk. The model has two main ingredients. First, a rich demographic structure, with entrepreneurs of different generations overlapping at each point in time. Second, imperfect idiosyncratic insurance, which will be generated by a moral hazard problem.

Formally, we propose an overlapping generations model in continuous time with imperfect altruism. There are two types of agents, entrepreneurs and financiers. Entrepreneurs live for T periods and, after they die, they are replaced by their heir. Entrepreneurs derive utility from bequests, but to a lesser extent than a perfect altruism benchmark (which effectively coincides with a infinite horizon economy). This will create intergenerational links, but also life-cycle patterns. At each point in each time, there are entrepreneurs with age ranging from 0 to T, in contrast to typical overlapping generations model where there is only two types of agents, the young and old (see Samuelson (1958) and Diamond (1965)).

Entrepreneurs' income come from wage income and the proceeds from a risky project. The project is subject to both aggregate and idiosyncratic shocks. Entrepreneurs can divert some of the capital and sell it privately in the market. The optimal contract between the entrepreneur and the financier can be implemented using simple financial instruments: a riskless asset, aggregate insurance, and idiosyncratic insurance. However, the entrepreneur will be limited in how much he can buy of idiosyncratic insurance.

The imperfect risk sharing will allow us to capture the idiosyncratic risk premium observed in the data. Idiosyncratic volatility can be high, but if part of it is insured by financiers, then the idiosyncratic risk premium. Using the model, we can infer the amount of idiosyncratic insurance in the data by comparing idiosyncratic returns and

idiosyncratic volatility. This requires using the model to factor out the component of the "price" of risk which is not related to the amount of idiosyncratic insurance.

Entrepreneurs also receive labor income, which can vary over the life cycle. Given the finite horizon, the value of human wealth, the present discount value of labor income, will vary over the life cycle. This will have implications for the consumption-wealth ratio and the portfolio share. An entrepreneur with a large stock of human wealth is effectively richer than what indicates its financial position. Hence, the entrepreneur will be willing to be more exposed to both aggregate and idiosyncratic risk. Human wealth is high relatively early in life and tend to decrease over time. Hence, both the consumption-wealth ratio and the portfolio share will be high early in life.

The connection between human wealth and portfolio choice has long been recognized in the finance literature (see Bodie et al. (1992), Heaton and Lucas (1997), Viceira (2001)). The focus of these studies is typically in a partial equilibrium analysis of a portfolio problem. Here we consider the challenging problem of having human wealth as a general equilibrium object in a setting with imperfect risk sharing.

Given the discipline of having the model to match the cross-sectional implications as well as the aggregate predictions about (aggregate and idiosyncratic) returns, we use the model to study the effect of innovations that expand insurance to entrepreneurs, or equivalently an increase in the level of financial development. An increase in idiosyncratic insurance will reduce the idiosyncratic risk premium. This reduction in the premium will lead to an expansion of the scale of the project, which reduces the marginal product of capital, in order to be consistent with the new level of the risk premium.

Another consequence of the reduction of the risk premium is that entrepreneurs will tend to accumulate less wealth. In the long-run, the share of wealth of entrepreneurs falls, with adverse effects on entrepreneurs' welfare. Finally, we consider the transitional dynamics in a tractable special case of the model with endogenous growth. We find that the risk premium overshoots its long-run level. The initial drop in risk premium is higher on impact than it is in a stationary equilibrium. The

reason is that in the short-run the effect of an increase in idiosyncratic insurance is amplified by an endogenous response of the price of risk. As the share of wealth of entrepreneurs start to fall, the risk premium converges to its long-run level.

The essay proceeds as follows. Section 2.3 describes the environment. Section 2.4 discuss the equilibrium characterization, including the connection between human wealth and risk-taking behavior. Section 2.5 discuss a stationary equilibrium and the effects of improvements in idiosyncratic risk premium. Section 2.6 discuss the transitional dynamics of the model in a tractable special case.

2.2 Life cycle patterns in village economies

We use data from the Townsend Thai Monthly Survey, an ongoing monthly survey initiated in 1998 in four provinces of Thailand. Two provinces, Chachoengsao and Lopburi, are semi-urban and located in a more developed central region near the capital city, Bangkok. Two provinces, Buriram and Srisaket, are rural and located in the less developed northeastern region. In each province, the survey is conducted in four villages, chosen at random with a given subdistrict. The survey covers both consumption and production decisions of households, including the value of fixed assets and inventories. There are five main occupations in the survey: cultivation, livestock raisin, fish and shrimp farming, non-farm business, and wage earning. Rates of return are computed by the household's accrued net income divided by household's average total assets (net of total liabilities). Samphantharak and Townsend (2013) discuss in more detail the survey.

We start documenting the life cycle pattern on consumption and risk-taking. The first panel of figure 2-1 shows the profile of consumption to net worth ratio over the life cycle. It is convenient to look to the consumption to net worth ratio, as the theory will have clear predictions about this ratio. Consumption is decreasing over most of the cycle, and it is increasing at old age. The share of wealth invest in risky (physical) assets is decreasing over most of life, being stable at the end of the cycle. In both cases, the variations over the cycle are economically significant. Young households

portfolio share of the risky asset it is up to 60% higher than the same share for older households. A -similar number holds for the consumption rate.

Figure 2-1: Savings and risk taking behavior over life cycle
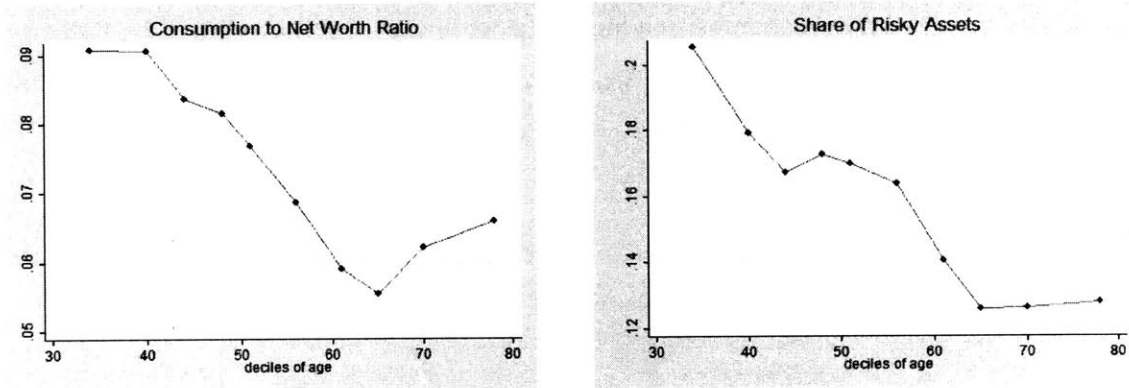
The fact that young entrepreneurs have a relatively high consumption rate makes harder for them to accumulate wealth. In contrast, the higher portfolio share of the risky asset implies returns on the portfolio are higher for young households, stimulating wealth accumulation. Figure 2-2 shows that total assets increase over most of the cycle, indicating the second effect, together with the fact income is relatively high at beginning of life, tends to be more preponderant.

Figure 2-2: Income and assets over life cycle
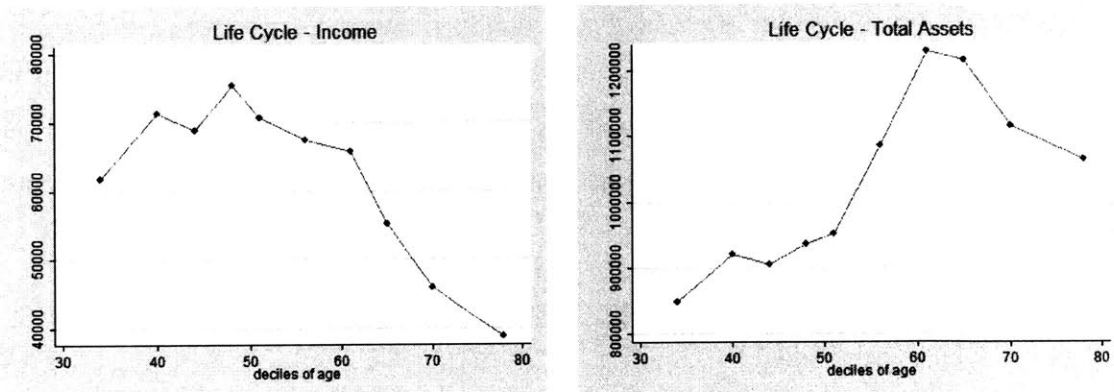
We can capitalize the flow of labor income income to obtain a measure of human

wealth. The value of human wealth is decreasing during most of the life cycle. In section 2.3, we propose a model that ties together these different facts. The value of human wealth will play an important role. In the model, a decreasing portfolio share comes a decreasing human capital to net worth ratio over the life cycle. The intuition is that entrepreneurs with a high value of human capital (relative to financial wealth) is proportionally richer than what it is indicated exclusively by their financial position. This will induce entrepreneurs to take a riskier portfolio position.

Figure 2-3: Human capital-net worth ratio
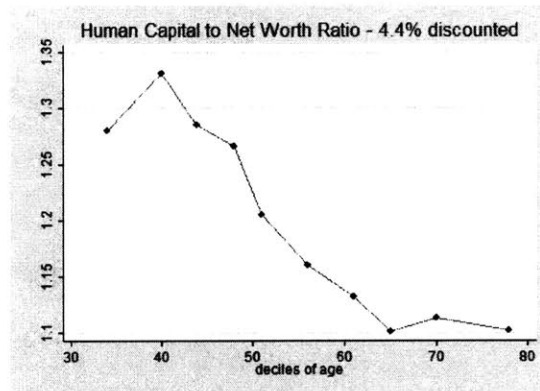
In the next section, we present a model that will try to capture the life cycle patterns described above as well as the behavior of risk and return described by (Samphantharak and Townsend, 2013). The authors show that the risk on entrepreneurial activity have an important idiosyncratic component. However, transfers among village members effectively mitigate the impact of idiosyncratic shocks. Despite of the large idiosyncratic risk, the idiosyncratic *risk premium* accounts for a smaller fraction of returns. This motivate us to propose a model with imperfect risk sharing, where entrepreneurs can insure idiosyncratic risk, but imperfectly.

2.3 The model

Time is continuous and there are three goods in this economy, consumption goods, capital, and labor. The economy is populated by two types of agents, entrepreneurs

and financiers. Entrepreneurs operate the production technology, but they face a moral hazard problem. Financiers can trade among themselves and write contracts with entrepreneurs.

2.3.1 Entrepreneurs

There is a double continuum of *dynasties* of entrepreneurs in the economy. Entrepreneurs live for (a measure) T periods and when an entrepreneur dies it is replaced by a new entrepreneur, keeping total population is constant. At each point in time, there is a continuum with mass one of entrepreneurs with age a, for all $a \in [0, T]$. Hence, all generations of entrepreneurs overlap at each instant. This rich demographics will be important to capture the empirical patterns described in the previous section.

Production Technology

Entrepreneurs can invest in a project that produces final goods using a CES constant returns to scale technology combining capital and labor. Let $k_{s,t}(i)$ denote capital used by entrepreneur i born at period s at period t, and similarly for the labor input $l_{s,t}(i)$. Output generate by entrepreneur (s, i) is given by

$$y_{s,t}(i) = A_t \left[\alpha k_{s,t}(i)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha) l_{s,t}(i)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} \tag{2.3.1}$$

where ϵ is the elasticity of substitution between capital and labor.

Productivity follows a geometric Brownian motion:

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_t \tag{2.3.2}$$

Aggregate productivity A_t can be written as $A_t = A_0 e^{\left(\mu_A - \frac{\sigma_A^2}{2}\right)t + \sigma_A Z_t}$, where $Z_t \sim \mathcal{N}(0, t)$. Hence, μ_A is the expected growth rate of productivity and σ_A is the volatility of the growth rate of productivity.

The stochastic process Z_t is the only source of *aggregate* uncertainty. Capital

accumulation is also subject to *idiosyncratic* shocks:

$$\frac{dk_{s,t}(i)}{k_{s,t}(i)} = g_{s,t}(i)dt + \sigma_k dW_{s,t}(i) \qquad (2.3.3)$$

where $g_{s,t}(i)$ represents the net investment rate.

The stochastic process $W_{s,t}(i)$ represents an *idiosyncratic* shock, i.e., $\int_0^1 dW_{s,t}(i) = 0$. Importantly, Z_t is perfectly observable, but $W_{s,t}(i)$ is private information to the entrepreneur. As described below, this will give rise to a moral hazard problem and a consequent inability of entrepreneurs to share idiosyncratic risk.

The aggregate capital stock is given by

$$k_t = \frac{1}{T} \int_0^T \int_0^1 k_{t-a,t}(i)dida \qquad (2.3.4)$$

Entrepreneurs can hire labor at the wage rate w_t and buy capital at the price q_t. The relative price of capital evolves according to $dq_t = \mu_{q,t}q_t dt + \sigma_{q,t}q_t dZ_t$, where $(\mu_{q,t}, \sigma_{q,t})$ are determined in equilibrium. Investment is subject to adjustment costs $\iota(g_t)A_t k_{s,t}(i)$, where $\iota'(\cdot) > 0$ and $\iota''(\cdot) > 0$. Adjustment costs depend on the amount of capital in *efficiency units*. This will be important to guarantee the economy has a balanced growth path.

The return of investing in the project can be written as

$$dR_{s,t}(i) = \underbrace{\frac{y_{s,t}(i) - w_t l_{s,t}(i) - \iota(g_{s,t}(i))A_t k_{s,t}(i)}{q_t k_{s,t}(i)}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t k_{s,t}(i))}{q_t k_{s,t}(i)}}_{\text{capital gain}}$$

The capital gain can be expressed as

$$\frac{d(q_t k_{s,t}(i))}{q_t k_{s,t}(i)} = \frac{dq_t}{q_t} + \frac{dk_{s,t}(i)}{k_{s,t}(i)} = (\mu_{q,t} + g_{s,t}(i))dt + \sigma_{q,t}dZ_t + \sigma_k dW_{s,t}(i)$$

More concisely, we can write $dR_{s,t}(i) = \mu_{s,t}(i)dt + \sigma_{q,t}dZ_t + \sigma_k dW_{s,t}(i)$, where

$$\mu_{s,t}(i) \equiv \frac{y_{s,t}(i) - w_t l_{s,t}(i) - \iota(g_{s,t}(i))A_t k_{s,t}(i)}{q_t k_{s,t}(i)} + \mu_{q,t} + g_{s,t}(i) \qquad (2.3.5)$$

78

To ease notation, we will drop the explicit dependence to index i when there is no risk of confusion, i.e., $k_{s,t}(i)$ will be denoted by $k_{s,t}$, similarly for all other variables.

Preferences and Bequest Motive

Entrepreneurs live for T periods and derives utility of a bequest to his heir. Preferences are time-separable with logarithmic instantaneous utility and utility of bequest $B_t(n)$

$$U_s(c_s, n_s) = \mathbb{E}_t \left[\int_s^{s+T} e^{-\rho(t-s)} \log c_{s,t} dt + e^{-\rho T} B_{s+T}(n_{s,s+T}) \right] \qquad (2.3.6)$$

for an entrepreneur born at period s.

Let $V_s(t, n)$ denote the value function of an entrepreneur born at instant s with net worth n. The bequest function is given by

$$B_t(n) = e^{-\theta T} V_t(t, n) \qquad (2.3.7)$$

where $\theta \geq 0$.

The coefficient θ measures the inverse of the bequest motive. If $\theta = 0$, then entrepreneurs give full weight to the next generation and they behave effectively as infinite-horizon agents. If $\theta > 0$, there is imperfect altruism and the behavior of entrepreneurs will deviate from the infinite-horizon benchmark.

Labor is supplied inelastically and can vary (deterministically) over the life cycle, $\bar{l}_{s,t}$, in order to capture the effects of experience and on-the-job learning.[1] The profile of labor supply over the life cycle will follow a flexible functional form in order to capture the pattern observed in figure 2-2:

$$\bar{l}_{s,t} = \beta_1 e^{\gamma_1(t-s)} + \beta_2 e^{\gamma_2(t-s)} \qquad (2.3.8)$$

---

[1]Notice that $\bar{l}_{s,t}$ denotes the amount of labor supplied by an entrepreneur born at s, while $l_{s,t}$ is the amount of labor demanded by an entrepreneur born at s to run his project.

Aggregate labor supply is fixed at one:

$$\frac{1}{T} \int_0^T \bar{l}_{t-z,t} dz = 1 \tag{2.3.9}$$

Entrepreneur's problem

Following recent work in dynamic moral hazard problems, we will assume that entrepreneurs can divert capital and sell it in the market. However, this process is inefficient, as entrepreneurs can sell only a fraction $\phi \in (0,1)$ of capital diverted. Entrepreneurs can meet with financiers and make a take-it-or-leave-it for a risk-sharing contract. Contracts are short-term, as after period $t + dt$ the entrepreneur will meet with a another financier. The derivation of the optimal contract is similar to the one in previous moral hazard problems in continuous-time (see DeMarzo and Sannikov (2006) and Sannikov (2008)). The detailed calculations are provided in the appendix and follow closely the work of Di Tella (2012).

In appendix 2.A.1, we show that the optimal contract can be implemented by the entrepreneur having access to three financial instruments: a riskless asset, aggregate insurance, and idiosyncratic insurance. Notice this is one possible *representation* of the optimal contract. We could instead only specify a process for transfers between entrepreneurs and financiers, depending on aggregate and idiosyncratic shocks, without a direct reference to these markets. However, it is instructive to think of the contract as consisting of these three simple financial instruments. Effectively, the moral hazard friction is determining a market structure, including the access of entrepreneurs to each asset.

Net worth of an entrepreneur born at instant s evolves according to

$$dn_{s,t} = \left[r_t n_{s,t} + q_t k_{s,t} (\mu_{s,t} - r_t) - \kappa_{s,t} \eta_t + w_t \bar{l}_{s,t} - c_{s,t} \right] dt$$
$$+ \left[q_t k_{s,t} \sigma_{q,t} - \kappa_{s,t} \right] dZ_t + \left(\nu q_t k_{s,t} - \kappa_{s,t}^{id} \right) dW_t \tag{2.3.10}$$

where $\kappa_{s,t}$ $\left(\kappa_{s,t}^{id} \right)$ denotes the amount of aggregate (idiosyncratic) insurance.

Notice the entrepreneur has access to four different assets. First, a riskless asset

80

that pays interest rate r_t. Second, the physical investment in the project, which generates expected return $\mu_{s,t}$. Third, aggregate insurance, where η_t denotes the premium the entrepreneur pays for reducing his exposure to aggregate risk. The term η_t is also referred to as the *market price of aggregate risk* and it is determined in equilibrium, as the other prices. Fourth, idiosyncratic insurance. Since idiosyncratic risk can be perfectly diversified, we are already imposing that the premium for idiosyncratic will be zero in equilibrium.

In order to provide appropriate incentives to entrepreneurs, there is limited idiosyncratic risk sharing. Entrepreneurs must hold a fraction ϕ of idiosyncratic risk, i.e., they must have "*skin in the game*":

$$\kappa_{s,t}^{id} \leq (1 - \phi)\nu q_t k_{s,t} \tag{2.3.11}$$

It is easy to see that in the absence of the skin in the game constraint (perfect risk sharing), the entrepreneur would simply choose $\kappa_{s,t}^{id} = \nu q_t k_{s,t}$, since the equilibrium premium for idiosyncratic risk is equal to zero. Hence, we should expect this constraint to be always binding in equilibrium.

The entrepreneur is also subject to a natural borrowing limit, as the entrepreneur cannot borrow more than the present discounted value of its labor income, i.e., the value of his *human wealth*. Let π_t denote the *stochastic discount factor* for this economy (state prices normalized by probabilities), where $d\pi_t = -r_t\pi_t dt - \eta_t\pi_t dZ_t$.[2] The value of human wealth is given by

$$h_{s,t} \equiv \mathbb{E}_t \left[\int_t^{s+T} \frac{\pi_z}{\pi_t} w_z \bar{l}_{s,z} dz \right] \tag{2.3.12}$$

for $s \leq t \leq s + T$ and $h_{s,t} = 0$ otherwise.

The natural borrowing limit is then given by

$$n_{s,t} \geq -h_{s,t} \tag{2.3.13}$$

---

[2]Intuitively, a higher interest rate r_t implies future payoffs are more heavily discounted, and a higher market price of risk implies risky payoffs are also more heavily discounted. Formally, we show in the appendix how to obtain the law of motion of π_t given (r, η).

The entrepreneur's problem is to choose a vector of process $(c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id})$, taking the process for prices (r, η, q, w) as given, to solve the following program:

$$V_s(t_0, n_{s,t}) = \max_{c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id}} \mathbb{E}_t \left[\int_{t_0}^{s+T} e^{-\rho(t-t_0)} \log c_{s,t} dt + e^{-(\rho+\theta)T} V_{s+T}(s+T, n_{s,s+T}) \right]$$
(P1)

subject to

$$dn_{s,t} = \left[r_t n_{s,t} + q_t k_{s,t}(\mu_{s,t} - r_t) - \kappa_{s,t}\eta_t + w_t \bar{l}_{s,t} - c_{s,t} \right] dt + \left(q_t k_{s,t}(\sigma + \sigma_{q,t}) - \kappa_{s,t} \right) dZ_t$$
$$+ \left(\nu q_t k_{s,t} - \kappa_{s,t}^{id} \right) dW_t$$

$$\mu_{s,t} \equiv \frac{y_{s,t} - w_t l_{s,t} - \iota(g_{s,t}) A_t k_{s,t}}{q_t k_{s,t}} + \mu_{q,t} + g_{s,t}$$

$$\kappa_{s,t}^{id} \leq (1-\phi)\nu q_t k_{s,t}; \qquad n_{s,t} \geq -h_{s,t}; \qquad c_{s,t}, k_{s,t} \geq 0$$

for $t \in [t_0, s+T]$, given $n_{s,t_0} > -h_{s,t_0}$.

2.3.2 Financiers

Financiers have logarithmic time-separable utility and choose a path of consumption, c_f, and risk exposure, σ_f, given initial net worth n_f. A financier can be exposed to aggregate risk by trading with another financier or by providing aggregate insurance to entrepreneurs.

Financiers solve the following portfolio problem

$$U(n_{f,0}; X_0) = \max_{c_f, \sigma_f} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_{f,t} dt \right]$$
(2.3.14)

subject to

$$\frac{dn_{f,t}}{n_{f,t}} = \left[r_t + \sigma_{f,t}\eta_t - \frac{c_{f,t}}{n_{f,t}} \right] dt + \sigma_{f,t} dZ_t$$
(2.3.15)

a natural borrowing constraint $n_{f,t} \geq 0$ and given initial net worth $n_{f,0} > 0$.

Notice the return the financier obtains by increasing her risk exposure by one unit is exactly η_t, the premium for aggregate insurance payed by the entrepreneur. Since idiosyncratic risk can be diversified, it commands no premium in equilibrium, so the

net exposure of a financier to idiosyncratic risk is equal to zero.

2.3.3 Competitive Equilibrium

A *competitive equilibrium* is a sequence of allocations $(c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id}, c_f, \sigma_f)$ and prices (r, η, q, w) such that

(a) $(c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id})$ solves the entrepreneur's problem (P1), given (r, η, q, w), for every $s \geq -T$

(b) (c_f, σ_f) solves the financier's problem (2.3.14), given (r, η)

(c) Markets clear:

$$\frac{1}{T}\int_0^T \int_0^1 [c_{t-z,t}(i) + \iota(g_{t-z,t}(i))A_t k_{t-z,t}(i)]\,didz + c_{f,t} = \frac{1}{T}\int_0^T \int_0^1 Y_{t-z,t}(i)didz$$
$$(2.3.16)$$

$$\frac{1}{T}\int_0^T \int_0^1 n_{t-z,t}(i)didz + n_{f,t} = \frac{1}{T}\int_0^T \int_0^1 q_t k_{t-z,t}(i)didz$$
$$(2.3.17)$$

$$\frac{1}{T}\int_0^T \int_0^1 \kappa_{t-z,t}(i)didz = \sigma_{f,t} n_{f,t} \qquad (2.3.18)$$

$$\frac{1}{T}\int_0^T \int_0^1 k_{t-z,t}(i)didz = k_t \qquad (2.3.19)$$

$$\frac{1}{T}\int_0^T \int_0^1 l_{t-z,t}(i)didz = 1 \qquad (2.3.20)$$

The first market clearing condition corresponds to the goods market, implying that the value of consumption plus investment made by entrepreneurs and the consumption of financiers must equal total output produced in the economy. The second condition is the market clearing for the riskless asset. Since bonds are in zero net supply, the net worth of entrepreneurs and financiers must add to the asset in positive net supply, the capital stock. The next condition is the market clearing condition for aggregate insurance, as aggregate insurance must match the supply of insurance by financiers. Finally, we have the market clearing condition for capital and labor.

2.4 Equilibrium characterization

Let's now present the equilibrium characterization. First, we will present the solution to the financier's problem, then the a reformulation and solution to the entrepreneur's problem. Finally, we discuss aggregation and equilibrium prices. Proofs and detailed calculations are provided in the appendix.

2.4.1 Financier's problem

The financier's problem is a standard portfolio problem with log-utility. Log-utility corresponds to the case of unity elasticity of intertemporal substitution (EIS) and unity risk aversion. Unity EIS implies that income and substitution effects cancel out in the savings decision, so the consumption-wealth ratio is independent of returns:

$$\frac{c_{f,t}}{n_{f,t}} = \rho \tag{2.4.1}$$

Similarly, unity risk aversion imply that income and substitution effects cancel out in the portfolio decision, so the risk exposure of financier's depends only on the current market price of aggregate risk:[3]

$$\sigma_{f,t} = \eta_t \tag{2.4.2}$$

It can be shown that the flow budget constraint for the financier can be integrated to obtain an intertemporal budget constraint:

$$\mathbb{E}_t \left[\int_t^\infty \frac{\pi_z}{\pi_t} c_{f,z} dz \right] = n_{f,t} \tag{2.4.3}$$

The state-price density π_t in equilibrium must be proportional to the marginal utility of the financier:

$$\pi_t = e^{-\rho t} \frac{1}{c_{f,t}} \tag{2.4.4}$$

---

[3]In case of general risk aversion γ, the demand for risk is given by $\frac{\sigma_{f,t}}{n_{f,t}} = \frac{\eta_t}{\gamma} + \frac{1-\gamma}{\gamma}\sigma_{\xi,t}$, where the term $\sigma_{\xi,t}$ captures how the marginal utility of wealth respond to shocks. Hence, portfolio decisions respond to how returns vary in response to aggregate shocks.

2.4.2 Expected return on the project

Consider now the entrepreneur's problem. Variables $(l_{s,t}, g_{s,t})$ only enter into the problem through $\mu_{s,t}$, so they will be chosen to maximize expected returns. Labor demand assumes the usual form:

$$w_t = (1-\alpha)A_t \left(\frac{y_{s,t}}{A_t l_{s,t}} \right)^{\frac{1}{\epsilon}} \tag{2.4.5}$$

The net investment rate $g_{s,t}$ is given by

$$\iota'(g_{s,t})A_t = q_t \Rightarrow g_t = \frac{q_t/A_t - \bar{\iota}_0}{\bar{\iota}_1} - \delta \equiv g\left(\frac{q_t}{A_t} \right) \tag{2.4.6}$$

using the quadratic cost specification $\iota(g) = \bar{\iota}_0(g+\delta) + \frac{\bar{\iota}_1}{2}(g+\delta)^2$.

Plugging the previous two equations into (2.3.5), we obtain

$$\mu_{s,t} = \frac{\alpha(w_t/A_t) - \iota(g(q_t/A_t))}{q_t/A_t} + \mu_{q,t} + g(q_t/A_t) \tag{2.4.7}$$

where $\alpha(w) \equiv \alpha^{\frac{\epsilon}{\epsilon-1}} [1 - (1-\alpha)^{\epsilon} w^{1-\epsilon}]^{\frac{1}{1-\epsilon}}$, $\alpha'(w) < 0$ as shown in appendix (2.A.3).

The expression above is independent of the level of capital $k_{s,t}$. Moreover, it is the same for all s, allowing us to write $\mu_{s,t} = \mu_t$. The term $\alpha(w_t/A_t)$ equals the marginal product of capital and it is decreasing in the wage per efficiency unit. The net investment rate $g_{s,t}$ is a function of the relative price of capital (per efficiency units) q_t/A_t, reflecting a standard q−theory logic where q_t captures the marginal benefit of investing.

2.4.3 Aggregate and idiosyncratic risk exposure

Before solving the entrepreneur's problem, it will be convenient to reformulate the problem. It will be shown that the problem can be written in terms of entrepreneur's total aggregate and idiosyncratic risk exposure.

First, notice the skin-in-the-game constraint is always binding, as an increase in $\kappa_{s,t}^{id}$ reduces the amount of idiosyncratic risk without affecting returns. Hence,

$\kappa_{s,t}^{id} = (1 - \phi)\nu q_t k_{s,t}$. Consider now the evolution of human wealth $h_{s,t}$ and total wealth, $\bar{n}_{s,t} \equiv n_{s,t} + h_{s,t}$:

Lemma 1. *Consider human wealth $h_{s,t}$ and total wealth $\bar{n}_{s,t}$ for entrepreneur born at s.*

(a) *Human wealth evolves according to*

$$dh_{s,t} = \left[r_t h_{s,t} + \sigma_{s,t}^h h_{s,t}\eta_t - w_t \bar{l}_{s,t}\right] dt + \sigma_{s,t}^h h_{s,t} dZ_t \qquad (2.4.8)$$

for some process $\sigma_{s,t}^h$.

(b) *Total wealth evolves according to*

$$d\bar{n}_{s,t} = \left[r_t \bar{n}_{s,t} + q_t k_{s,t}(\mu_{s,t} - r_t) + (\sigma_{s,t}^h h_{s,t} - \kappa_{s,t})\eta_t - c_{s,t}\right] dt + \left[q_t k_{s,t}\sigma_{q,t} + \sigma_{s,t}^h h_{s,t} - \kappa_{s,t}\right] dZ_t + \phi\nu q_t \prime$$

imposing the skin-in-the-game constraint (2.3.11) binds.

The first part of lemma 1 gives the law of motion of human capital. For a given process for the exposure of human capital to aggregate risk, $\sigma_{s,t}^h$, we obtain the expected growth rate of human capital. The term $\sigma_{s,t}^h$ depends on the process for wages and need to be solved in equilibrium. Human capital grows according to the riskless interest rate r_t plus a remuneration for the riskness of human capital $\sigma_{s,t}^h \eta_t$. Notice the return per unit of aggregate risk η_t is exactly the premium for aggregate insurance payed by the entrepreneur. Finally, since human capital is the discounted value of wage income going forward, the value of human capital in the next instant $h_{s,t} + dh_{s,t}$ is *decreased* by the current value of wages $w_t \bar{l}_{s,t}$.

The second part of lemma 1 gives the law of motion of total wealth, $d\bar{n}_{s,t} = dn_{s,t} + dh_{s,t}$. Importantly, wage income does not enter directly in the budget constraint anymore. Intuitively, it is as if the entrepreneurs sells the rights to all future wage income to financiers and receive in return an asset with value $h_{s,t}$ and risk exposure $\sigma_{s,t}^h$. Exposure to aggregate risk now comes from the project $q_t k_{s,t}\sigma_{q,t}$, from human capital $\sigma_{s,t}^h h_{s,t}$ and from aggregate insurance κ_t. Exposure to idiosyncratic risk is given

by the residual risk of the project after taking the maximum possible idiosyncratic insurance from financiers.

By choosing the scale of the project, the entrepreneur is effectively choosing his exposure to idiosyncratic risk, while aggregate insurance can be used to obtain any desired exposure to aggregate risk. We can use this observation to rewrite the entrepreneur's problem in more convenient form. First, define the *market price of idiosyncratic risk* as the excess return on the project not accounted for aggregate risk per unit of idiosyncratic risk:

$$\eta_t^{id} \equiv \frac{\mu_t - r_t - (\sigma + \sigma_{q,t})\eta_t}{\phi\nu} \tag{2.4.9}$$

The next proposition shows how to write the entrepreneur's problem in terms of choosing total exposure to aggregate and idiosyncratic risk.

Proposition 6. *Suppose* $(c_s, \overline{\sigma}_s, \overline{\sigma}_s^{id})$ *solves the following problem:*

$$\overline{V}_s(t_0, \overline{n}_{s,t}) = \max_{c_s, \sigma_s, \sigma_s^{id}} \mathbb{E}_t \left[\int_{t_0}^{s+T} e^{-\rho(t-t_0)} \log c_{s,t} dt + e^{-(\rho+\theta)T} \overline{V}_{s+T}(s+T, \overline{n}_{s,s+T}) \right] \tag{P2}$$

subject to $\overline{n}_{s,t} \geq 0; c_t \geq 0,$ *and*

$$\frac{d\overline{n}_{s,t}}{\overline{n}_{s,t}} = \left[r_t + \overline{\sigma}_{s,t}\eta_t + \overline{\sigma}_{s,t}^{id}\eta_t^{id} - \frac{c_{s,t}}{\overline{n}_{s,t}} \right] dt + \overline{\sigma}_{s,t} dZ_t + \overline{\sigma}_{s,t}^{id} dW_t \tag{2.4.10}$$

for $t \in [t_0, s+T]$, *given* $\overline{n}_{s,t_0} > 0$.

Then, $(c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id})$ *solves* (P1), *where* $l_{s,t}$ *satisfies* (2.4.5), $g_{s,t}$ *satisfies* (2.4.6), $\kappa_{s,t}^{id}$ *satisfies* (2.3.11) *as equality,* $(k_{s,t}, \kappa_{s,t})$ *solve the equations*

$$\overline{\sigma}_{s,t} = \left(\frac{q_t k_{s,t}}{n_{s,t}} \sigma_{q,t} - \frac{\kappa_{s,t}}{n_{s,t}} \right) \frac{n_{s,t}}{n_{s,t} + h_{s,t}} + \sigma_{s,t}^h \frac{h_{s,t}}{n_{s,t} + h_{s,t}} \qquad \overline{\sigma}_{s,t}^{id} = \phi\nu \frac{q_t k_{s,t}}{n_{s,t}} \frac{n_{s,t}}{n_{s,t} + h_{s,t}} \tag{2.4.11}$$

where $n_{s,t}$ *satisfy* (2.3.10).

The proposition above highlights the fact that the risk-taking decision of entrepreneurs can be taken in two steps. First, entrepreneurs choose how much of

total wealth will be exposed to aggregate and idiosyncratic risk. Second, given the desired exposure of total wealth to risk, entrepreneurs choose the necessary exposure of *financial wealth* to risk.

2.4.4 Savings behavior and risk-taking over the life cycle

The next proposition characterizes the solution to the portfolio problem (P2)

Proposition 7. *Consider problem* (P2):

(a) *The value function $V_s(t, n)$ can be expressed as*

$$\overline{V}_s(t, n) = \frac{1 - \Theta e^{-\rho(T-a)}}{\rho} \log n + B_{s,t} \tag{2.4.12}$$

for some process $B_{s,t}$, where $a = t - s$.

(b) *Consumption to (financial) net worth ratio is given by*

$$\frac{c_{s,t}}{n_{s,t}} = \frac{\rho}{1 - \Theta e^{-\rho(T-a)}} \left(1 + \frac{h_{s,t}}{n_{s,t}}\right) \tag{2.4.13}$$

where $\Theta \equiv \frac{1-e^{-\theta T}}{1-e^{-(\rho+\theta)T}}$.

(c) *Exposure to aggregate risk is given by*

$$\sigma_{s,t} = \eta_t \left(1 + \frac{h_{s,t}}{n_{s,t}}\right) - \sigma_{h,s,t}\frac{h_{s,t}}{n_{s,t}} \tag{2.4.14}$$

where $\sigma_{s,t} \equiv \frac{q_t k_{s,t}}{n_{s,t}}(\sigma + \sigma_{q,t}) - \frac{\kappa_{s,t}}{n_{s,t}}$.

(d) *Demand for capital is given by*

$$\frac{q_t k_{s,t}}{n_{s,t}} = \frac{\eta_t^{id}}{\phi \nu} \left(1 + \frac{h_{s,t}}{n_{s,t}}\right) \tag{2.4.15}$$

Proposition 7 establishes the link between the ratio of human wealth to (financial) net worth and the savings and risk-taking behavior. The higher the human wealth to

net worth ratio, the more the entrepreneur will consume out of his financial wealth. As seen in figure 2-3, human wealth is decreasing over most of the life cycle and stabilize at a low level later in life. The term $\frac{\rho}{1-\Theta e^{-\rho(T-a)}}$ is increasing, provided $\theta > 0$ and captures the fact that entrepreneurs have an incentive to run down his assets as the end of life approaches, reflecting imperfect altruism. This second effect may help explain the increasing consumption to net worth ratio late in the cycle, as shown in figure 2-1.

Similarly, a decreasing human wealth to net worth helps explain the decreasing share of wealth invested in the risky asset. The higher the value of human wealth, the more the entrepreneur is willing to hold of the project and the associated idiosyncratic risk.

2.4.5 Stationary Distribution of Relative Net Worth

Define the average net worth for entrepreneurs born at instant s and the aggregate net worth of entrepreneurs:

$$\vartheta_{s,t} \equiv \frac{\int_0^1 \overline{n}_{s,t}(i)di}{\frac{1}{T}\int_0^T \int_0^1 \overline{n}_{t-z,t}(i)didz} \qquad \overline{n}_{e,t} \equiv \frac{1}{T}\int_0^T \int_0^1 \overline{n}_{t-z,t}(i)didz \qquad (2.4.16)$$

Since all entrepreneurs choose the same exposure of their total wealth to risk, the share of wealth held by each demographic group does not respond to shocks:

$$\frac{d\vartheta_{s,t}}{\vartheta_{s,t}} = \left[\frac{1}{T}\int_0^T \frac{\rho \vartheta_{t-z,t}}{1-\Theta e^{-\rho(T-z)}}dz - \frac{\rho}{1-\Theta e^{-\rho(T-(t-s))}} \right] dt \qquad (2.4.17)$$

with the two boundary conditions $\vartheta_{t-T,t} = \vartheta_{t,t}$ and $\frac{1}{T}\int_0^T \vartheta_{t-a,t}da = 1$.

In a stationary equilibrium, we are able to solve the system of differential equations and obtain the distribution of net worth across different ages in closed form. The assumption of stationarity implies the wealth shares depend on (s,t) only through the age of the group $a = t - s$, allowing us to write $\vartheta_{s,t} = \vartheta_a$, abusing notation.

Proposition 8. *Let ϑ_a denote the stationary distribution of wealth for entrepreneurs.*

(a) *The share (density) of total wealth held by entrepreneurs of $a \in [0, T]$ is given by*

$$\vartheta_a = \frac{\theta(\rho + \theta)T}{\rho} \left[\frac{e^{\theta a}}{e^{\theta T} - 1} - \frac{e^{-(\rho+\theta)(T-a)}}{1 - e^{-(\rho+\theta)T}} \right] \qquad (2.4.18)$$

and the average consumption-wealth ratio is given by

$$\frac{1}{T} \int_0^T \frac{\rho \vartheta_z}{1 - \Theta e^{-\rho(T-z)}} dz = \rho + \theta \qquad (2.4.19)$$

(b) ϑ_a *is increasing around $a = 0$, decreasing around $a = T$, and achieves a maximum at an intermediate age.*

The proposition gives the distribution of net worth across different ages in a stationary equilibrium. It starts and ends at the same point since we assumed the entrepreneur at age T leaves his net worth to the newborn agent and this is his only source of wealth. At the beginning of life savings rate is high, since the net worth must finance consumption throughout his life, what leads the entrepreneur to accumulate wealth at the beginning of life. As the end of life approaches, the savings rate falls (given the imperfect altruism), depleting most of the net worth.

The stationary distribution depends on two variables: the degree of altruism θ and the discount rate ρ. Hence, the stationary distribution will be unaffected by changes in endogenous variables. Hence, if the economy starts with a stationary distribution of wealth among entrepreneurs, it will remain to do so even in the returns move over time. While variations in returns can affect the distribution of wealth between financiers and entrepreneurs, it will not affect how wealth is distributed among entrepreneurs.

We will assume the economy starts at the stationary distribution of relative wealth.

Assumption 1. *The initial distribution of wealth $\overline{n}_{s,0}(i)$ satisfies the condition*

$$\frac{\int_0^1 \overline{n}_{-a,0}(i)di}{\frac{1}{T} \int_0^T \int_0^1 \overline{n}_{-z,0}(i)didz} = \vartheta_a \qquad (2.4.20)$$

for all $a \in [0, T]$.

90

Figure 2-4: Distribution of net worth across age
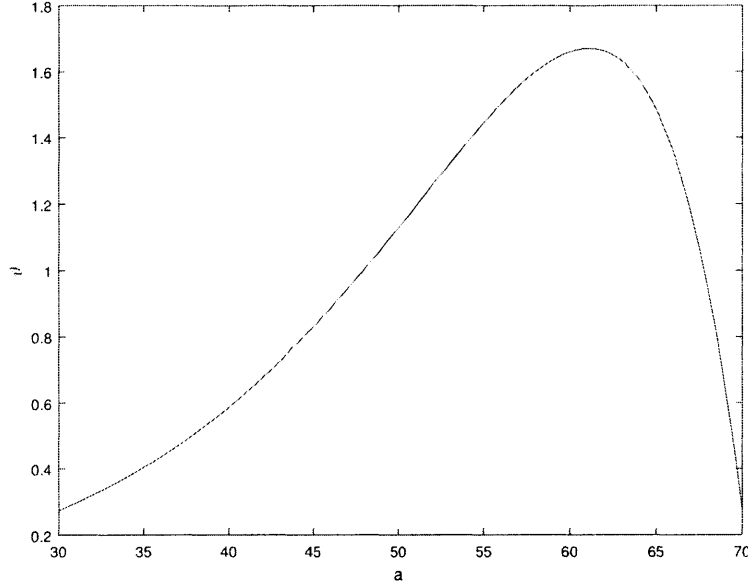
Even under assumption (1), we can analyze the transitional dynamics when the wealth distribution between entrepreneurs and financiers is not at its stationary value or the capital stock is not at your long-run level.

2.4.6 Pricing of capital and idiosyncratic risk

In equilibrium, the wage rate will be a function of the aggregate capital stock k_t. To derive the expression for the wage, notice that all entrepreneurs choose the same capital-labor ratio equal to the aggregate ratio k_t. Hence, from (2.4.5), we obtain

$$\frac{w_t}{A_t} = (1-\alpha)\left[\alpha k_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)\right]^{\frac{1}{\epsilon-1}} \equiv w(k_t) \tag{2.4.21}$$

From the demand for capital (2.4.15), we conclude that every entrepreneur will choose the same scale of the project as a fraction of their total wealth:

$$\frac{q_t k_{s,t}(i)}{\bar{n}_{s,t}(i)} = \frac{\eta_t^{id}}{\phi\nu} \Rightarrow \frac{\bar{n}_{s,t}(i)}{q_t k_{s,t}(i)} = \frac{\bar{n}_{e,t}}{q_t k_t} = x_t\left(1 + \frac{h_t}{q_t k_t}\right) \tag{2.4.22}$$

91

where $\overline{n}_{e,t} \equiv \frac{1}{T} \int_0^T \int_0^1 \overline{n}_{t-z,t}(i) di dz$.

Hence, the market price of idiosyncratic risk can be expressed as

$$\eta_t^{id} = \frac{\phi\nu}{x_t} \frac{q_t k_t}{q_t k_t + h_t} \tag{2.4.23}$$

From the definition of the market price for idiosyncratic risk, we obtain an expression for the expected return on the project:

$$\frac{\alpha(w(k_t)) - \iota(g(q_t/A_t))}{q_t/A_t} + g(q_t/A_t) + \mu_{q,t} = r_t + \sigma_{q,t}\eta_t + \frac{(\phi\nu)^2}{x_t} \frac{q_t k_t}{q_t k_t + h_t} \tag{2.4.24}$$

The expression above says that the entrepreneur is remunerated for investing in the project by the riskless interest rate plus a premium for holding aggregate risk, $\sigma_{q,t}\eta_t$, and a premium for holding idiosyncratic risk $\frac{(\phi\nu)^2}{x_t}$.

2.4.7 Equilibrium Prices

Let's now solve for (q_t, η_t). The following lemma show how to rewrite the market clearing conditions for consumption and aggregate risk taking.

Lemma 2. *Suppose assumption 1 holds. The market clearing condition for the riskless asset, consumption and aggregate risk can be written as*

(a) *Market clearing for riskless asset:*

$$\overline{n}_{e,t} + n_{f,t} = q_t k_t + h_t \tag{2.4.25}$$

where $h_t \equiv \frac{1}{T} \int_0^T h_{t-z,t} dz$.

(b) *Market clearing for consumption:*

$$(\rho + \theta)x_t + \rho(1 - x_t) = \frac{y_t - \iota(g(q_t/A_t))A_t k_t}{q_t k_t + h_t} \tag{2.4.26}$$

where

$$x_t \equiv \frac{\overline{n}_{e,t}}{q_t k_t + h_t} \tag{2.4.27}$$

92

(c) *Market clearing for aggregate risk*

$$x_t \overline{\sigma}_{e,t} + (1 - x_t)\sigma_{f,t} = \frac{q_t k_t}{q_t k_t + h_t}\sigma_{q,t} + \frac{h_t}{q_t k_t + h_t}\sigma_{h,t} \qquad (2.4.28)$$

where $\overline{\sigma}_{e,t} = \frac{1}{T}\int_0^T \int_0^1 \overline{\sigma}_{t-z,t}(i)didz$ *and* $\sigma_{h,t} \equiv \frac{1}{T}\int_0^T \sigma_{t-z,t}^h \frac{h_{t-z,t}}{h_t}dz$.

Equation (2.4.25) results from bonds being in zero net supply in this economy, so total wealth must equal the value of physical and human wealth. The left-hand side of equation (2.4.26) equals consumption by entrepreneurs and financiers normalized by physical and human wealth. Assumption 1 is used to obtain the average consumption-wealth ratio for entrepreneurs as simply $\rho + \theta$. Finally, equation (2.4.28) gives demand for aggregate risk on the left-hand side and supply of aggregate risk, coming from physical and human wealth.

The next proposition gives (r, q, η) as a function of (x, k, h).

Proposition 9. *Suppose assumption 1 holds. Consider the prices* (q_t, η_t, r_t):

(a) *Relative price of capital:* $q_t = A_t q(x_t, k_t, \frac{h_t}{A_t})$, *where*

$$q(x, k, h) \equiv \sqrt{\bar{\iota}_1^2 \left[\theta x + \rho\right]^2 + \bar{\iota}_1 \left(\left[\alpha + (1-\alpha)k^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} + \frac{\bar{\iota}_0^2}{2\bar{\iota}_1} - [\theta x + \rho]\frac{h}{k}\right)} - \bar{\iota}_1 \left[\theta x_t + \rho\right]$$

$$(2.4.29)$$

(b) *Market price of aggregate risk:*

$$\eta_t = \frac{q_t k_t}{q_t k_t + h_t}\sigma_{q,t} + \frac{h_t}{q_t k_t + h_t}\sigma_{h,t} \qquad (2.4.30)$$

(c) *Interest rate:*

$$r_t = \frac{\alpha \left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{1}{1-\epsilon}} - \frac{1}{2\bar{\iota}_1}\left[q(x_t, k_t, \frac{h_t}{A_t})^2 - \bar{\iota}_0^2\right]}{q(x_t, k_t, \frac{h_t}{A_t})} + \frac{q(x_t, k_t, \frac{h_t}{A_t}) - \bar{\iota}_0}{\bar{\iota}_1} - \delta$$

$$+ \mu_{q,t} - \sigma_{q,t}\eta_t - \frac{1}{x_t}\frac{k_t(\phi\nu)^2}{k_t + \frac{h_t/A_t}{q(x_t, k_t, \frac{h_t}{A_t})}} \qquad (2.4.31)$$

The proposition above shows how to obtain prices given (x_t, k_t, h_t). We will focus in a Markov equilibrium where human wealth is a function of the aggregate state variables (x_t, k_t), i.e., $h_t = A_t h(x_t, k_t)$ for some function $h(x, k)$. Plugging the value of $h(x_t, k_t)$, and abusing notation, we can write $q(x_t, k_t, h(x_t, k_t)) = q(x_t, k_t)$. The following proposition shows how to compute human wealth and derives the law of motion of the state variables.

Proposition 10. *Suppose assumption 1 holds and equilibrium is Markov in the state variable (x_t, k_t).*

(a) *Human wealth satisfies $h_t = A_t h(x_t, k_t)$:*

$$h(x_t, k_t) = \mathbb{E}_t \left[\int_0^T \frac{\pi_{t+z}}{A_t \pi_t} w_{t+z} \ell_{t,z} dz \right] \tag{2.4.32}$$

where $\ell_{t,z} \equiv \frac{1}{T} \int_0^{T-z} \bar{l}_{t-a,t+z} da$ and

$$\frac{\pi_{t+z}}{\pi_t} = \frac{e^{-\rho z}(1 - x_t)(q(x_t, k_t)k_t + h(x_t, k_t))}{(1 - x_{t+z})(q(x_{t+z}, k_{t+z})k_{t+z} + h(x_{t+z}, k_{t+z}))} \frac{A_t}{A_{t+z}} \tag{2.4.33}$$

(b) *The law of motion of x_t is given by*

$$dx_t = x_t(1 - x_t) \left[\left(\eta_t^{id} \right)^2 - \theta \right] dt \tag{2.4.34}$$

where $\eta_t^{id} \equiv \frac{\phi \nu}{x_t} \frac{q(x_t, k_t)k_t}{q(x_t, k_t)k_t + h(x_t, k_t)}$.

(c) *The law of motion of k_t is given by*

$$dk_t = \left[\frac{q(x_t, k_t) - \bar{\iota}_0}{\bar{\iota}_1} - \delta \right] k_t dt \tag{2.4.35}$$

Proposition 10 implies that, despite the presence of aggregate shocks, the economy presents *deterministic dynamics* for the state variables (x_t, k_t). Aggregate shocks affect the scale of the economy by changing the efficiency units of both capital and labor, but it does not affect properly normalized variables. Hence, if the economy starts at a point where $dx_t = dk_t = 0$ it will remain there, even after aggregate shocks

hit the economy. In the next session we study in detail the stationary equilibrium of this economy.

2.4.8 Calibration

We adopt the following calibration. The elasticity of substitution is set to $\epsilon = 0.7$, following the evidence in Oberfield and Raval (2014). The capital share is set to $\alpha = 0.5$ and the average growth rate of productivity is set to $\mu = 0.023$, following the evidence provided by Jeong and Townsend (2007) for Thailand. Aggregate and idiosyncratic volatility are set to $\sigma = 0.04$ and $\phi = 0.13$, using the estimates in Samphantharak and Townsend (2013). Adjustment costs are assumed to be quadratic $\iota(g) = \bar{\iota}_0(g+\delta) + \frac{\bar{\iota}_1}{2}(g+\delta)^2$. The depreciation rate is set to $\delta = 0.05$ and the coefficients of the adjustment cost function are set to match an average investment rate of 20% and capital-output ratio of 3. The discount rate is set to $\rho = 0.01$ to match a near zero riskless interest rate and the bequest motive coefficient $\theta = 0.10$ to match the low savings rate of entrepreneurs. The life span is set to $T = 40$, from 30 to 70. The moral hazard parameter ϕ can be identified from information on the idiosyncratic risk premium. We set it to $\phi = 0.4$ to capture the level of idiosyncratic risk premium. For simplicity, we will assume a flat labor supply profile $\bar{l}_{s,t} = 1$. In future versions, the coefficients $(\beta_1, \gamma_1, \beta_2, \gamma_2)$ will be estimated from the empirical labor income profile.

2.5 Stationary Equilibrium

Consider a stationary equilibrium where $x_0 = x^*$ and $k_t = k^*$ such that $dx_t = dk_t = 0$ for all $t \geq 0$. Notice the economy is still subject to aggregate and idiosyncratic shocks. However, *normalized* variables are non-stochastic and, in the stationary equilibrium, constant.

Applying the results of proposition 10 for a stationary equilibrium, we can solve for (x^*, k^*). In a stationary equilibrium, the relative price of capital is determined by

the depreciation rate and the parameters of the adjustment cost function:

$$q^* = \bar{\iota}_0 + \bar{\iota}_1 \delta \tag{2.5.1}$$

Using the expression for $q(x^*, k^*)$, we obtain

$$q^* = \sqrt{\bar{\iota}_1^2 \left[\theta x^* + \rho\right]^2 + \bar{\iota}_1 \left(\left[\alpha + (1-\alpha)(k^*)^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} + \frac{\bar{\iota}_0^2}{2\bar{\iota}_1} - \left[\theta x^* + \rho\right]\frac{h^*}{k^*} \right)} - \bar{\iota}_1 \left[\theta x^* + \rho\right] \tag{2.5.2}$$

Human wealth satisfy the condition:

$$\frac{h^*}{k^*} = (1-\alpha)(k^*)^{\frac{1-\epsilon}{\epsilon}} \left[\alpha + (1-\alpha)(k^*)^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{1}{1-\epsilon}} \ell^* \tag{2.5.3}$$

where $\ell^* = \int_0^T e^{-\rho z} \ell_{t,z} dz$.

The share of wealth of entrepreneurs satisfy the condition

$$x^* = \frac{\phi \nu}{\sqrt{\theta}} \frac{q^* k^*}{q^* k^* + h^*} \tag{2.5.4}$$

The riskless interest rate is given by

$$r^* = \frac{\alpha \left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{1}{1-\epsilon}} - \left[\bar{\iota}_0 \delta + \frac{\bar{\iota}_1}{2}\delta^2\right]}{\bar{\iota}_0 + \bar{\iota}_1 \delta} + \mu - \sigma^2 - \sqrt{\theta}\phi\nu$$

Figure 2-5 shows the stationary equilibrium for different values of ϕ. We consider three different values of the elasticity of substitution between capital and labor, ϵ. A low level, $\epsilon = 0.7$, consistent with the evidence for the US presented by Oberfield and Raval (2014). An intermediary value $\epsilon = 1$, corresponding to the Cobb-Douglass production function, and a value of elasticity higher than Cobb-Douglas, which we choose to be $\epsilon = 1.3$. For each value of ϵ, we adopt a calibration of the adjustment cost parameters to have a capital-output ratio equal to 3 and investment rate of 20% when $\phi = 0.2$. The value of ϵ will be important to determine the impact of financial development and the consequent reduction in financial frictions.

Consider the long-run impact of reducing the moral hazard parameter from $\phi =$

Figure 2-5: Stationary Equilibrium

Idiosyncratic risk premium · Interest rate · Cost of funds · Capital-output ratio · Output per efficiency unit · Share of wealth of entrepreneurs

$\epsilon = 0.7$ —— $\epsilon = 1.0$ ····· $\epsilon = 1.3$
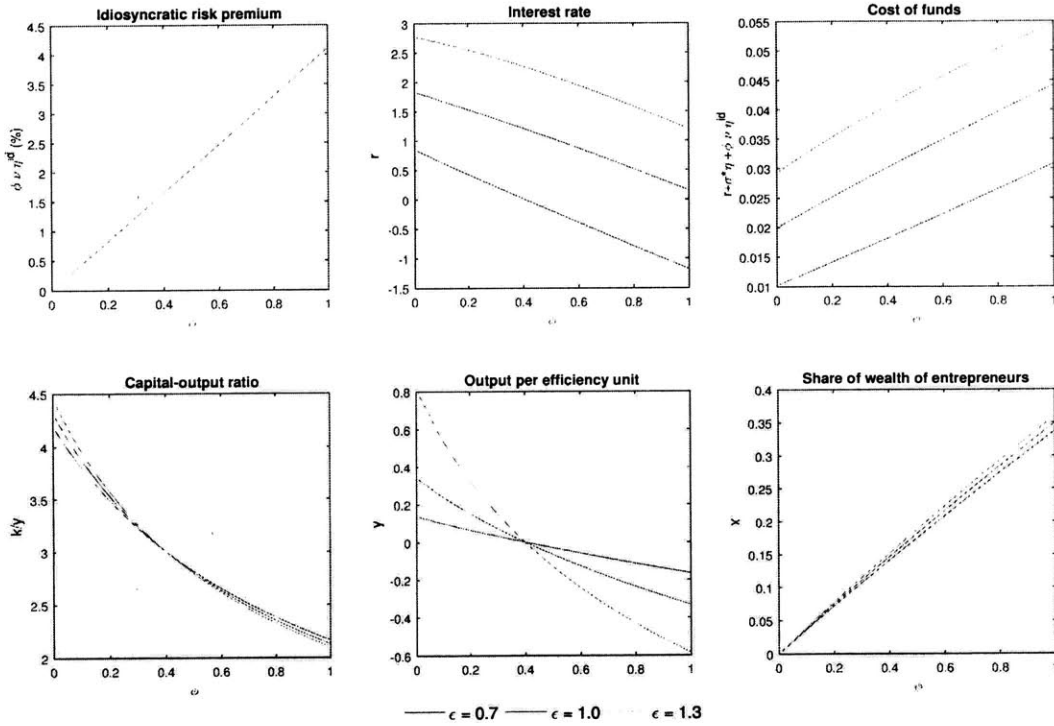
0.4, the value used to match the observed idiosyncratic risk premium, to $\phi = 0.2$, implying a reduction in the idiosyncratic risk premium in 100 basis point. In response, the capital-output ratio would increase to about 3.5. To see why, consider the specialization of equation (2.4.24) to a stationary equilibrium:

$$\underbrace{\frac{MPK(k^*) - \iota(0)}{q^*} + \mu}_{\text{expected return}} = \underbrace{r^* + \sigma\eta^* + \phi\nu(\eta^{id})^*}_{\text{cost of funds}} \qquad (2.5.5)$$

where $MPK(k) = \alpha(w(k))$ is the marginal product of capital.

In a stationary equilibrium, q^* is determined by technological parameters. Hence, the capital stock will be inversely related to the cost of funds, i.e., the riskless interest rate plus the aggregate and idiosyncratic risk premium. The market price of both aggregate and idiosyncratic risk are independent of ϕ in a stationary equilibrium: $\eta^* = \sigma$ and $(\eta^{id})^* = \sqrt{\theta}$. Hence, a reduction in ϕ will reduce the idiosyncratic

97

risk premium. The interest rate will increase, as entrepreneurs reduce precautionary savings as they hold less idiosyncratic risk, dampening (but not overturning) the effect of the reduction in the risk premium. As the cost of funds fall, the marginal product of capital goes down, increasing the capital stock.

The effect of the reduction in the cost of funds in output depends crucially on the elasticity of substitution between capital and labor. If it is relatively easy to substitute capital and labor, then the reduction in the cost of funds will have a large impact in output. If it is relatively difficult to substitute capital and labor, then the effect will be much less pronounced.

A reduction in the moral hazard parameter also has implications for the distribution of wealth between entrepreneurs and financiers. Extending insurance to entrepreneurs has the effect of *reducing* their share of wealth. The reason is that a reduction in ϕ reduces the return on the portfolio of entrepreneurs, making harder for them to accumulate in the long-run. The reduction in share of wealth of entrepreneurs helps explain why the price of idiosyncratic risk is independent of ϕ in the long run. In the short run, the market price of idiosyncratic risk is given by $\eta_t^{id} = \frac{\phi\nu}{x_t} \frac{q_t k_t}{q_t k_t + h_t}$. As x_t is fixed in the short-run, a reduction in ϕ would tend to reduce η_t^{id}. In the long-run, as x_t converges to a lower level, the price of idiosyncratic would go back to its long-level $\sqrt{\theta}$.

The fact that the share of wealth of entrepreneurs is reduced as they more access to idiosyncratic insurance it is indicative of the welfare impact of changes in ϕ. Formally, we can measure the welfare associated with a given stationary equilibrium, denoted by $*$, as the proportional increase in consumption in all dates and state, $1 + \omega_e$, in a reference stationary equilibrium, denoted by o. In the appendix, we show the correct welfare measure is given by

$$1 + \omega_e = \frac{x^*(q^* k^* + h^*)}{x^o(q^o k^o + h^o)} \qquad (2.5.6)$$

and for financiers $1 + \omega_f = \frac{(1-x^*)(q^* k^* + h^*)}{(1-x^o)(q^o k^o + h^o)}$.

In figure 2-6 we plot welfare for entrepreneurs and financiers, where the reference

Figure 2-6: Welfare of entrepreneurs and financiers
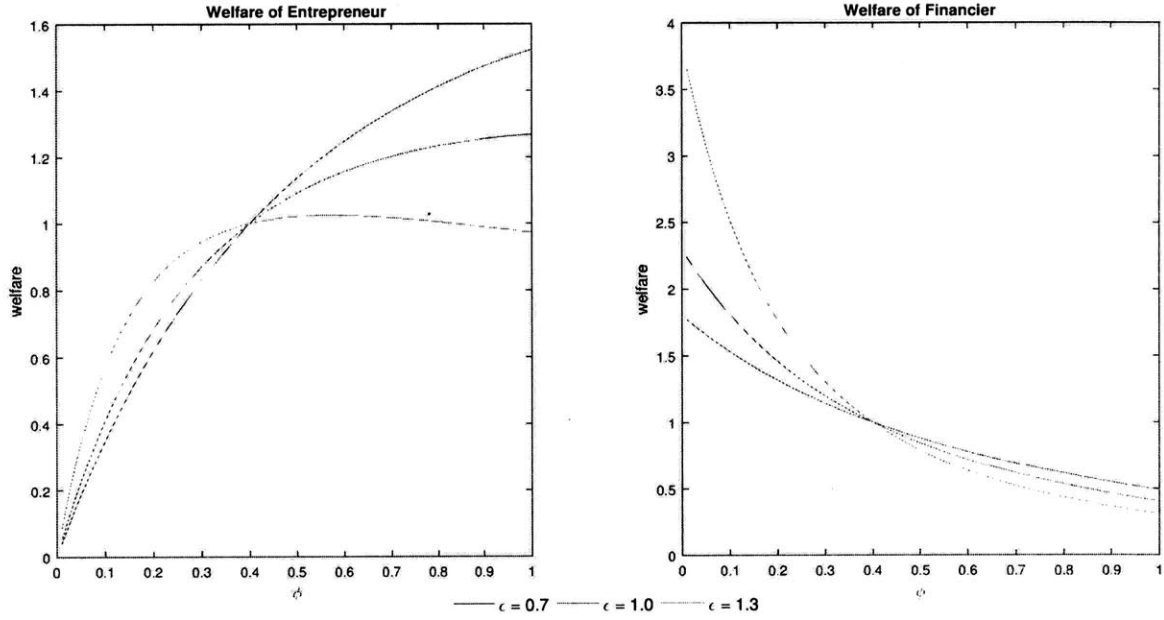
equilibrium correspond to the stationary equilibrium with $\phi = 0.4$. The welfare measure ω_e is reduced as we reduce ϕ. The reason is that the share of wealth of entrepreneurs x^* falls by more than total wealth increases $q^*k^* + h^*$. However, this does not mean all entrepreneurs will be worse off after a reduction in ϕ, since we are not taking the transitional dynamics into account. The distinction is analogous to the comparison between the golden rule and modified golden rule in growth theory.

2.6 Transitional Dynamics and Financial Development

2.6.1 An Economy with Endogenous Growth

Consider the special case of no human wealth, $\alpha = 1$. This corresponds to the case of an AK economy and the absence of decreasing returns will give rise to endogenous growth. In contrast to the case of $0 < \alpha < 1$, changes in the moral hazard parameter

will now have on the growth of the economy. The next proposition shows how to derive the stationary equilibrium and the transitional dynamics for this special case.

Proposition 11. *Suppose assumption 1 holds and $\alpha = 1$.*

(a) *The share of wealth of entrepreneurs evolves according to*

$$\dot{x}_t = \theta x_t (1 - x_t) \left[\left(\frac{x^*}{x_t} \right)^2 - 1 \right] \tag{2.6.1}$$

where

$$x^* = \frac{\phi \sigma_k}{\sqrt{\theta}} \tag{2.6.2}$$

(b) *The market price of aggregate and idiosyncratic risk are given by*

$$\eta = \sigma \qquad \eta_{id}(x) = \frac{\phi \sigma_k}{x} \tag{2.6.3}$$

(c) *The growth rate of output is given by*

$$g(x_t) = \sqrt{(\theta x + \rho)^2 + \frac{1}{\bar{\iota}_1} \left(1 + \frac{\bar{\iota}_0^2}{2\bar{\iota}_1} \right)} - [\theta x_t + \rho] - \frac{\bar{\iota}_0 + \delta \bar{\iota}_1}{\bar{\iota}_1} \tag{2.6.4}$$

(d) *The interest rate is given by*

$$r(x) = \rho + g(x) - \sigma^2 + \theta x \frac{g(x) + \theta + \rho}{g(x) + \theta x + \rho} \left[1 - \left(\frac{x^*}{x} \right)^2 \right] \tag{2.6.5}$$

An important implication of the assumption $\alpha = 1$ is not a state variable anymore. The stock of capital will depend on the initial condition and the path for $g_t = g(x_t)$. The stationary value of x_t is given by $x^* = \phi \sigma_k / \sqrt{\theta}$. Hence, an increase in idiosyncratic insurance will reduce the long-run level of the share of wealth of entrepreneurs. The intuition is again that a reduction in ϕ will lead to a drop in the idiosyncratic premium, reducing the pace entrepreneurs accumulate wealth.

Consider a reduction in ϕ from 0.4 to 0.2. Initially, this will cause a large reduction in the idiosyncratic premium, as a drop in the market price of idiosyncratic, $\eta_t^{id} =$

$\phi\nu/x_t$ risk amplifies the initial reduction in ϕ. However, as time goes by, the share of wealth of entrepreneurs will tend to go down, due to the reduction in returns, pushing the idiosyncratic back up. Hence, the reduction in the idiosyncratic premium initially overshoots its long-run level. Interest rate increases, in response to a reduction in desired precautionary savings by entrepreneurs. As with the idiosyncratic premium, the short-run increase in interest rate exceed its long-run.

Figure 2-7: Transitional Dynamics
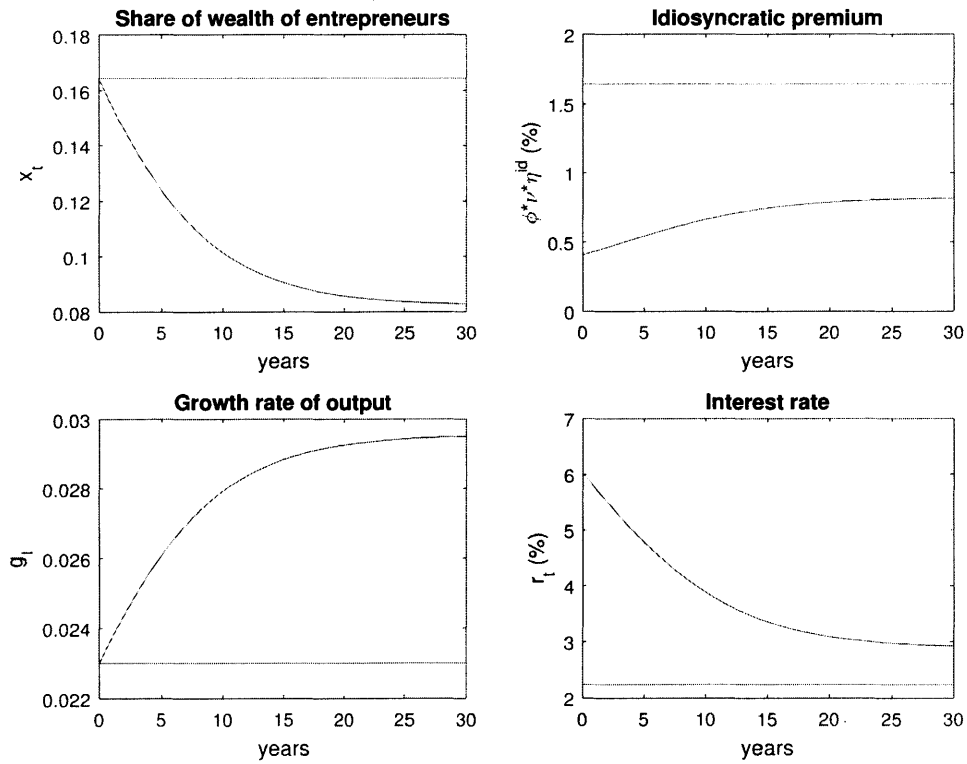
The growth rate of the economy increases during this process. The higher growth rate is related to the lower share of wealth of entrepreneurs. Since entrepreneurs are the agents in the economy with low savings rate, due to imperfect altruism, a shift of wealth towards financiers will increase desired aggregate savings in the economy. This increase in savings will be met a higher investment rate and consequent higher growth rate of the economy.

2.A Appendices

2.A.1 Derivation of Optimal Contract

From discrete to continuous time

Prices: The relative price of capital q_t and the stochastic discount factor can be written as:

$$\frac{q_{t+\Delta}}{q_t} = 1 + \mu_{q,t}\Delta + \sigma_{q,t}z_{t+\Delta}\sqrt{\Delta}; \qquad \frac{\pi_{t+\Delta}}{\pi_t} = 1 - r_t\Delta - \eta_t z_{t+\Delta}\sqrt{\Delta};$$

or in levels

$$q_{t+\Delta} = q_0 + \sum_{j=0}^{t}\mu_{q,j}q_j\Delta + \sum_{j=0}^{t}\sigma_{q,j}q_j z_{t+j}\sqrt{\Delta}; \qquad \pi_{t+\Delta} = \pi_0 - \sum_{j=0}^{t}r_{,j}\pi_j\Delta - \sum_{j=0}^{t}\eta_j q_j z_{t+j}\sqrt{\Delta};$$

Taking the limit as Δ goes to zero:

$$q_t = q_0 + \int_0^t \mu_{q,u}q_u du + \int_0^t \sigma_{q,u}q_u dZ_u; \qquad \pi_t = \pi_0 - \int_0^t r_u\pi_u du - \int_0^t \eta_u\pi_u dZ_u;$$

Using the usual more compact notation, we obtain

$$dq_t = \mu_{q,t}dt + \sigma_{q,t}dZ_t; \qquad d\pi_t = -r_t dt - \eta_t dZ_t \qquad (2.A.1)$$

where $Z = \{Z_t : 0 \leq t \leq T\}$ is a Brownian motion.

In order to see the intuition for the approximation above, notice we can write

$$\lim_{\Delta\to 0}\sum_{j=0}^{t}z_j\sqrt{\Delta} = \sqrt{t}\lim_{N\to\infty}\frac{1}{\sqrt{N}}\sum_{j=0}^{t}z_j \equiv Z_t \sim \mathcal{N}(0,t) \qquad (2.A.2)$$

where $\Delta = t/N$.

Returns: Define the cumulative log return:

$$\tilde{R}_{a,t+\Delta} \equiv \log\prod_{j=0}^{t}R_{i,j+\Delta} \qquad (2.A.3)$$

102

Changes in cumulative return are given by[4]

$$\tilde{R}_{a,t+\Delta} - \tilde{R}_{a,t} = \left[\frac{ak_{a,t}^{\alpha}l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_{a,t}k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_{a,t}\right]\Delta + (\sigma+\sigma_{q,t})z_{t+\Delta}\sqrt{\Delta} + \nu w_{a,t+\Delta}\sqrt{\Delta} + o(\Delta)$$

where $o(\Delta)$ collects terms such that satisfy $\lim_{\Delta\to 0}\frac{o(\Delta)}{\Delta} = 0$.

Taking the limit as $\Delta \to 0$:

$$d\tilde{R}_{a,t} = \left[\underbrace{\frac{ak_{a,t}^{\alpha}l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_t k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_{a,t}}_{\mu_{a,t}}\right]dt + (\sigma + \sigma_{q,t})dZ_t + \nu dW_{a,t}$$

$$(2.A.4)$$

I will abuse notation and write $dR_{a,t}$ instead of $d\tilde{R}_{a,t}$ and similarly for the market return.

State contingent liability: The participation constraint can be written as

$$0 = \mathbb{E}\left[\frac{\pi_{t+\Delta}}{\pi_t}\left(\overline{m}_t\Delta + \beta_{t,z}z_{t+\Delta}\sqrt{\Delta} + (1-\phi)q_{t+\Delta}k_t w_{t+\Delta}\sqrt{\Delta}\right)\right]$$

Using the expression for the stochastic discount factor and ignoring higher order terms in Δ:

$$\overline{m} = \beta_{t,z}\eta_t$$

For the continuous time analysis it is more convenient to define the cumulative payment to the principal $M_t = \sum_{j=0}^{t} m_j$ such that:

$$dM_t = \overline{m}dt + \beta_{t,z}dZ_t + (1-\phi)q_t k_t dW_t$$

Net worth evolution: From the flow budget constraint and plugging in the expression for $m_{t+\Delta}$, we obtain

$$\frac{n_{t+\Delta} - n_t}{n_t} = \left[r_t - \frac{c_t}{n_t} - \frac{\beta_{t,z}}{n_t}\eta_t + \frac{q_t k_t}{n_t}(\mu_t - r_t)\right]\Delta - \frac{\beta_{t,z}}{n_t}z_{t+\Delta}\sqrt{\Delta} + \frac{q_{t+\Delta}k_t}{n_t}\frac{q_{t+\Delta}}{q_t}\left[1 + \sigma z_{t+\Delta}\sqrt{\Delta} + \phi\nu w_{t+}\right.$$

---

[4]The expression for the return already imposes $s_{a,t+\Delta} = 0$.

Taking the limit as Δ goes to zero:

$$\frac{dn_t}{n_t} = \left[r_t - \frac{c_t}{n_t} + \sigma_{agg,t}\eta_t + \phi\nu\frac{q_t k_t}{n_t}\eta_{id,t} \right] dt + \sigma_{agg,t}dZ_t + \phi\nu\frac{q_t k_t}{n_t}dW_t$$

where

$$\sigma_{agg,t} \equiv \frac{q_t k_t}{n_t}(\sigma + \sigma_{q,t}) - \frac{\beta_{t,z}}{n_t}; \qquad \eta_{id,t} \equiv \frac{\mu_t - r_t - (\sigma + \sigma_{q,t})\eta_t}{\phi\nu}$$

Optimal contract in continuous time:

$$\rho V_t(n)dt = \max_{c,l,g,k,\sigma_{agg}} u(c,l)dt + \mathbb{E}\left[dV_t(n)\right]$$

subject to

$$\frac{dn_t}{n_t} = \left[r_t + \sigma_{agg,t}\eta_t + \phi\nu\frac{q_t k_t}{n_t}\eta_{id,t} - \frac{c_t}{n_t} \right] dt + \sigma_{agg,t}dZ_t + \phi\nu\frac{q_t k_t}{n_t}dW_t$$

$$\eta_{id,t} = \frac{\mu_t - r_t - (\sigma + \sigma_{q,t})\eta_t}{\phi\nu}; \qquad \mu_t = \frac{ak_{a,t}^{\alpha}l_{a,t}^{1-\alpha} - \iota(g_{a,t})k_{a,t}}{q_t k_{a,t}} + \mu_{q,t} + \sigma\sigma_{q,t} + g_t; \qquad n_t \geq 0$$

and the law of motion of (r_t, η_t, q_t).

2.A.2 Proofs

Proof of lemma 1

Proof. **(a)** Define the discounted value of labor income for an entrepreneur since he was born:

$$G_{s,t} = \int_s^t \pi_z w_z \bar{l}_{s,z}dz + \underbrace{\mathbb{E}_t\left[\int_t^{s+T} \pi_z w_z \bar{l}_{s,z}dz \right]}_{=\pi_t h_{s,t}} \qquad (2.A.5)$$

for $t \leq s + T$, and $G_{s,t} = \int_s^{s+T} \pi_z w_z \bar{l}_{s,z}dz$ for $t > s + T$.

Notice the term $G_{s,t}$ is a martingale, since for $t' \leq s + T$

$$\mathbb{E}_t[G_{s,t'}] = \int_s^t \pi_z w_z \bar{l}_{s,z}dz + \mathbb{E}_t\left[\int_t^{t'} \pi_z w_z \bar{l}_{s,z}dz \right] + \mathbb{E}_t\left[\mathbb{E}_{t'}\left[\int_{t'}^{s+T} \pi_z w_z \bar{l}_{s,z}dz \right] \right] = G_{s,t}$$

and for $t' > s + T$

$$\mathbb{E}_t[G_{s,t'}] = \int_s^t \pi_z w_z \bar{l}_{s,z} dz + \mathbb{E}_t\left[\int_t^{s+T} \pi_z w_z \bar{l}_{s,z} dz\right] = G_{s,t}$$

From the martingale representation theorem, there exists a process $\sigma_{s,t}^h$ such that

$$\pi_t w_t \bar{l}_{s,t} dt + d(\pi_t h_{s,t}) = \pi_t(\sigma_{s,t}^h h_{s,t} - \eta_t) dZ_t \qquad (2.A.6)$$

Applying Ito's lemma, we obtain

$$dh_{s,t} = \left[r_t h_{s,t} + \sigma_{s,t}^h h_{s,t} \eta_t - w_t \bar{l}_{s,t}\right] dt + \sigma_{s,t}^h h_{s,t} dZ_t \qquad (2.A.7)$$

(b) The law of motion of total wealth is given by $d\bar{n}_{s,t} = dn_{s,t} + dh_{s,t}$. Adding up the equations in (2.3.10) and (2.4.8), imposing the skin-in-the-game constraint binds, gives the desired expression.

$\square$

Proof of proposition 6

Proof. Let's first show the equivalence of (P1) and the following problem where we impose the optimality condition for $(l_s, g_s, \kappa_s^{id})$:

$$V_s(t_0, n_{s,t}) = \max_{c_s, k_s, \kappa_s} \mathbb{E}_t\left[\int_{t_0}^{s+T} e^{-\rho(t-t_0)} \log c_{s,t} dt + e^{-(\rho+\theta)T} V_{s+T}(s+T, n_{s,s+T})\right] \quad (P1')$$

subject to

$$dn_{s,t} = \left[r_t n_{s,t} + q_t k_{s,t}(\mu_{s,t} - r_t) - \kappa_{s,t}\eta_t + w_t A_t \bar{l}_{s,t} - c_{s,t}\right] dt + (q_t k_{s,t}(\sigma + \sigma_{q,t}) - \kappa_{s,t}) dZ_t + \phi\nu q_t k_{s,t} dW_t$$

$$\mu_{s,t} \equiv \frac{\alpha(w_t/A_t) - \iota(g(q_t/A_t))}{q_t/A_t} + \mu_{q,t} + g(q_t/A_t)$$

$$n_{s,t} \geq -h_{s,t}; \qquad c_{s,t}, k_{s,t} \geq 0$$

for $t \in [t_0, s + T]$, given $n_{s,t_0} > -h_{s,t_0}$.

105

Consider now problem (P1). The HJB equation can be written for $t \in [s, s+T]$ as

$$\rho V_s(t, n_{s,t}) = \max_{c_s, k_s, l_s, g_s, \kappa_s, \kappa_s^{id}} \left\{ \log c_{s,t} + \frac{\partial V_s}{\partial t} + \frac{\partial V_s}{\partial n_s} \left[r_t n_{s,t} + q_t k_{s,t}(\mu_{s,t} - r_t) - \kappa_{s,t} \eta_t + w_t \bar{l}_{s,t} - c_{s,t} \right] + \right.$$
$$\left. + \frac{1}{2} \frac{\partial^2 V_s}{\partial n_s^2} \left[\left(q_t k_{s,t}(\sigma + \sigma_{q,t}) - \kappa_{s,t} \right)^2 + \left(\nu q_t k_{s,t} - \kappa_{s,t}^{id} \right)^2 \right] \right\}$$

subject to

$$\mu_{s,t} \equiv \frac{y_{s,t} - w_t l_{s,t} - \iota(g_{s,t}) A_t k_{s,t}}{q_t k_{s,t}} + \mu_{q,t} + g_{s,t} \qquad \kappa_{s,t}^{id} \le (1-\phi)\nu q_t k_{s,t}; \qquad c_{s,t}, k_{s,t} \ge 0$$

Usual arguments imply that $\frac{\partial V_s}{\partial n_s} > 0$ and $\frac{\partial^2 V_s}{\partial n_s^2} < 0$. Hence, (l_s, g_s) and $\kappa_{s,t}^{id}$ can be chosen to solve the problems:

$$(l_{s,t}, g_{s,t}) \in \arg\max_{l_{s,t}, g_{s,t}} \mu_{s,t} \qquad \kappa_{s,t}^{id} \in \arg\min_{\kappa_{s,t}^{id}} \left(\nu q_t k_{s,t} - \kappa_{s,t}^{id} \right)^2 \; s.t. \; \kappa_{s,t}^{id} \le (1-\phi)\nu q_t k_{s,t}$$

The solution to the second problem is clearly $\kappa_{s,t}^{id} = (1-\phi)q_t k_{s,t}$. The first problem is concave and the first-order conditions are given by (2.4.5) and (2.4.6). From the labor demand, we have

$$\frac{l_{s,t}}{k_{s,t}} = \alpha^{\frac{\epsilon}{\epsilon-1}} \left(\frac{w_t}{A_t(1-\alpha)} \right)^{-\epsilon} \left[1 - (1-\alpha) \left(\frac{w_t}{A_t(1-\alpha)} \right)^{1-\epsilon} \right]^{\frac{\epsilon}{1-\epsilon}} \tag{2.A.8}$$

From the expression above, we can obtain the output-capital ratio:

$$\frac{y_{s,t}}{A_t k_{s,t}} = \left[\alpha + (1-\alpha) \left(\frac{l_{s,t}}{k_{s,t}} \right)^{\frac{\epsilon-1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}} = \alpha^{\frac{\epsilon}{\epsilon-1}} \left[1 - (1-\alpha) \left(\frac{w_t}{A_t(1-\alpha)} \right)^{1-\epsilon} \right]^{\frac{\epsilon}{1-\epsilon}} \tag{2.A.9}$$

The value of output minus the wage bill is given by

$$\frac{y_{s,t} - w_t l_{s,t}}{A_t k_{s,t}} = \alpha^{\frac{\epsilon}{\epsilon-1}} \left[1 - (1-\alpha) \left(\frac{w_t}{A_t(1-\alpha)} \right)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \equiv \alpha \left(\frac{w_t}{A_t} \right) \tag{2.A.10}$$

Plugging the expression above into (2.3.5) and using $g_{s,t} = g(q_t/A_t)$ gives (2.4.7).

106

Hence, problem (P1') corresponds to problem (P1), after imposing the optimal choice of $(l_s, q_s, \kappa_s^{id})$. Let's now show the equivalence between (P1') and (P2).

Suppose (c_s, k_s, κ_s) satisfies the constraints of (P1'). From lemma (1), we have

$$\frac{d\overline{n}_{s,t}}{\overline{n}_{s,t}} = \left[r_t + \frac{q_t k_{s,t}}{n_{s,t}} \frac{n_{s,t}}{n_{s,t} + h_{s,t}} (\mu_{s,t} - r_t) + \left(\sigma_{s,t}^h \frac{h_{s,t}}{n_{s,t} + h_{s,t}} - \frac{\kappa_{s,t}}{n_{s,t}} \frac{n_{s,t}}{n_{s,t} + h_{s,t}} \right) \eta_t - \frac{c_{s,t}}{\overline{n}_{s,t}} \right] dt$$

$$+ \left[q_t k_{s,t}(\sigma + \sigma_{q,t}) + \sigma_{s,t}^h h_{s,t} - \kappa_{s,t} \right] dZ_t + \phi\nu \frac{q_t k_{s,t}}{n_{s,t}} \frac{n_{s,t}}{n_{s,t} + h_{s,t}} dW_t$$

Defining $(\overline{\sigma}_{s,t}, \overline{\sigma}_{s,t}^{id})$ as in (2.4.11), using the fact that $\mu_t - r_t = \eta_t^{id} \phi\nu + (\sigma + \sigma_{q,t})\eta_t$, gives (2.4.10). Conversely, suppose $(c_s, \overline{\sigma}_{s,t}, \overline{\sigma}_{s,t}^{id})$ satisfies the constraints of (P2). Defining $(k_{s,t}, \kappa_{s,t})$ to satisfy equations (2.4.11), then the equation above is satisfied. Using the law of motion of $h_{s,t}$ obtained in lemma (1), we obtain that the constraints of (P1') are satisfied. Hence, if $(c_s, \overline{\sigma}_{s,t}, \overline{\sigma}_{s,t}^{id})$ solves (P2), then $(c_s, k_{s,t}, \kappa_{s,t})$ solves (P1'), where $(k_{s,t}, \kappa_{s,t})$ to satisfy equations (2.4.11). $\qquad\square$

Proof of proposition 7

Proof. Assume that (r, η, η^{id}) are functions of an aggregate state variable X_t. The value function can be written more explicitly as $\overline{V}_s(t, n; X)$. We will guess and verify the value function assumes the form.

$$\overline{V}_s(t, n; X) = A(t - s) \log n + B_s(t, X) \qquad (2.A.11)$$

Applying Ito's lemma to the value function, we get

$$d\overline{V}_{s,t} = A'(t - s) \log n_t dt + A(t - s) \left[\frac{dn_t}{n_t} - \frac{1}{2} \left(\frac{dn_t}{n_t} \right)^2 \right] + dB_{s,t}$$

Taking expectations of the expression above

$$\mathbb{E}\left[d\overline{V}_{s,t} \right] = A(a) \left[\frac{A'(a)}{A(a)} \log n_t + r_t + \overline{\sigma}_{s,t}\eta_t + \overline{\sigma}_{s,t}^{id}\eta_{id,t} - \frac{c_t}{n_t} - \frac{1}{2} \left(\overline{\sigma}_{s,t}^2 + (\overline{\sigma}_{s,t}^{id})^2 \right) \right] dt + \mu_{s,t}^B dt$$

107

The HJB equation can be written as

$$\rho\left[A(a)\log\overline{n}_{s,t} + B_{s,t}\right] = \max_{c_{s,t},\overline{\sigma}_{s,t},\overline{\sigma}_{s,t}^{id}} \log c_{s,t} + A(a)\left[r_t + \overline{\sigma}_{s,t}\eta_t + \overline{\sigma}_{s,t}^{id}\eta_{id,t} - \frac{c_{s,t}}{\overline{n}_{s,t}} - \frac{1}{2}\left(\overline{\sigma}_{s,t}^2 + (\overline{\sigma}_{s,t}^{id})^2\right)\right]$$
$$+ A'(a)\log\overline{n}_t + \mu_{s,t}^B$$

The problem above is concave, so the first order conditions are necessary and sufficient to characterize the solution. From the first-order condition for consumption, we obtain

$$\frac{c_{s,t}}{\overline{n}_{s,t}} = \frac{1}{A(a)} \tag{2.A.12}$$

The coefficient $A(a)$ must satisfy:

$$A'(a) = \rho A(a) - 1 \Rightarrow A(a) = \frac{1 - e^{-\rho(T-a)}}{\rho} + e^{-\rho(T-a)}A(T) \tag{2.A.13}$$

using the boundary condition $A(T) = e^{-\theta T}A(0)$, we obtain

$$A(0) = \frac{1 - e^{-\rho T}}{1 - e^{-(\rho+\theta)T}}\frac{1}{\rho} \Rightarrow A(a) = \frac{1 - \Theta e^{-\rho(T-a)}}{\rho} \tag{2.A.14}$$

where $\Theta \equiv \frac{1 - e^{-\theta T}}{1 - e^{-(\rho+\theta)T}}$.

The consumption to financial wealth ratio can then be written as

$$\frac{c_{s,t}}{n_{s,t}} = \frac{\rho}{1 - \Theta e^{-\rho(T-a)}}\left(\frac{n_{s,t} + h_{s,t}}{n_{s,t}}\right) \tag{2.A.15}$$

corresponding to (2.4.13).

First order condition with respect to aggregate risk gives $\overline{\sigma}_{s,t} = \eta_t$, which can be written as

$$\underbrace{\frac{q_t k_{s,t}}{n_{s,t}}(\sigma + \sigma_{q,t}) - \frac{\kappa_{s,t}}{n_{s,t}}}_{\equiv \sigma_{s,t}} = \eta_t\left(1 + \frac{h_{s,t}}{n_{s,t}}\right) - \sigma_{s,t}^h\frac{h_{s,t}}{n_{s,t}} \tag{2.A.16}$$

corresponding to (2.4.14).

Similarly, demand for idiosyncratic risk is given by $\overline{\sigma}_{s,t}^{id} = \eta^{id}$, which can be written

108

as

$$\frac{q_t k_{s,t}}{n_{s,t}} = \frac{\eta_t^{id}}{\phi \nu}\left(1 + \frac{h_{s,t}}{n_{s,t}}\right) \tag{2.A.17}$$

corresponding to (2.4.15).

The function $B_s(t, X)$ must satisfy the partial differential equation:

$$\rho\left[A(a)\log \overline{n}_{s,t} + B_{s,t}\right] = \max_{c_{s,t}, \overline{\sigma}_{s,t}, \overline{\sigma}_{s,t}^{id}} \log c_{s,t} + A(a)\left[r_t + \overline{\sigma}_{s,t}\eta_t + \overline{\sigma}_{s,t}^{id}\eta_{id,t} - \frac{c_{s,t}}{\overline{n}_{s,t}} - \frac{1}{2}\left(\overline{\sigma}_{s,t}^2 + (\overline{\sigma}_{s,t}^{id})^2\right)\right]$$

$$+ A'(a)\log \overline{n}_t + \mu_{s,t}^B$$

$$\rho B_s(t, X) = -\log A(t-s) - 1 + A(t-s)\overline{r}(X) + \frac{\partial B_s(t,X)}{\partial t} + \frac{\partial B_s(t,X)}{\partial X}\mu_X(X) + \frac{\partial^2 B_s(t,X)}{\partial X^2}\sigma_X(X)^2 \tag{2.A.18}$$

subject to the boundary condition $B_s(s + T, X) = \Theta B_{s+T}(s + T, X)$ and

$$\overline{r}(X) \equiv r(X) + \frac{\eta(X)^2}{2} + \frac{\eta_{id}(X)^2}{2} \tag{2.A.19}$$

$\square$

Proof of proposition 8

Proof. In a stationary equilibrium, $\vartheta_{s,t}$ is a function only of $t - s$. Abusing notation, let's write ϑ_a. The law of motion of ϑ_a can be written as

$$\frac{d\log \vartheta_a}{da} = \left[\overline{c}_e - \frac{\rho}{1 - \Theta e^{-\rho(T-a)}}\right] \tag{2.A.20}$$

where $\overline{c}_e \equiv \frac{1}{T}\int_0^T \frac{\rho\vartheta_{z,t}}{1-\Theta e^{-\rho(T-z)}}dz$.

Solving for the ODE, we obtain

$$\vartheta_a = \left[\frac{1 - \Theta e^{-\rho(T-a)}}{1 - \Theta e^{-\rho T}}\right]e^{[\overline{c}_e - \rho]a}\vartheta_0 \tag{2.A.21}$$

Using the condition $\vartheta_0 = \vartheta_T$, we obtain

$$\frac{1 - \Theta e^{-\rho T}}{1 - \Theta} = e^{(\bar{c}_e - \rho)T} \Rightarrow e^{\theta T} = e^{(\bar{c}_e - \rho)T} \qquad (2.A.22)$$

solving for $\bar{c}_e$:

$$\bar{c}_e = \rho + \theta > \rho \qquad (2.A.23)$$

The initial condition is obtained by integrating the expression above:

$$1 = \frac{1}{T} \int_0^T \left[e^{\theta a} - \Theta e^{-\rho T} e^{(\rho + \theta)a} \right] da \frac{\vartheta_0}{1 - \Theta e^{-\rho T}} \qquad (2.A.24)$$

solving for ϑ_0

$$\frac{\vartheta_0}{1 - \Theta e^{-\rho T}} = \frac{T(\rho + \theta)\theta}{(\rho + \theta)\left(e^{\theta T} - 1\right) - \theta\Theta e^{-\rho T}(e^{(\rho + \theta)T} - 1)} = \frac{T(\rho + \theta)\theta}{\rho(e^{\theta T} - 1)} \qquad (2.A.25)$$

Share of wealth at age a is then given by

$$\vartheta_a = \frac{T(\rho + \theta)\theta}{\rho} \left[\frac{e^{\theta a}}{e^{\theta T} - 1} - \frac{e^{-(\rho + \theta)(T - a)}}{1 - e^{-(\rho + \theta)T}} \right] \qquad (2.A.26)$$

Consider now how the net worth varies with age:

$$\frac{d\vartheta_a}{da} = \frac{T(\rho + \theta)\theta e^{\theta a}}{\rho} \left[\frac{\theta}{e^{\theta T} - 1} - \frac{(\rho + \theta)e^{-(\rho + \theta)T} e^{\rho a}}{1 - e^{-(\rho + \theta)T}} \right] \qquad (2.A.27)$$

The derivative will be non-negative at age a if

$$\frac{(e^{(\rho + \theta)T} - 1)/(\rho + \theta)}{(e^{\theta T} - 1)/\theta} \geq e^{\rho a} \qquad (2.A.28)$$

For $a = 0$, we have

$$\frac{1}{T} \int_0^T e^{(\rho + \theta)a} da \geq \frac{1}{T} \int_0^T e^{\theta a} da \iff \frac{(e^{(\rho + \theta)T} - 1)/(\rho + \theta)}{(e^{\theta T} - 1)/\theta} \geq 1 \qquad (2.A.29)$$

Hence, ϑ_a is increasing at $a = 0$. For $a = T$, we have

$$\frac{(e^{(\rho+\theta)T} - 1)/(\rho + \theta)}{(e^{\theta T} - 1)/\theta} \le e^{\rho T} \iff \frac{1 - e^{-(\rho+\theta)T}}{\rho + \theta} \le (1 - e^{-\theta T})/\theta \iff \frac{1}{T}\int_0^T e^{-(\rho+\theta)a}da \le \frac{1}{T}\int_0^T e^{-\theta a}da \tag{2.A.30}$$

Hence, ϑ is decreasing at $a = T$. Since the derivative of ϑ_a is strictly decreasing, there is a single point $0 < a^* < T$ such that ϑ_a is maximized. $\qquad\square$

Proof of lemma 2

Proof. **(a)** Rewrite the market clearing condition for the riskless asset as follows

$$\frac{1}{T}\int_0^T \int_0^1 [n_{t-z,t}(i) + h_{t-z,t}(i)]\, didz + n_{f,t} = \frac{1}{T}\int_0^T \int_0^1 [q_t k_{t-z,t}(i) + h_{t-z,t}(i)]\, didz \tag{2.A.31}$$

which can be written as

$$\bar{n}_{e,t} + n_{f,t} = q_t k_t + h_t \tag{2.A.32}$$

where

$$h_t \equiv \frac{1}{T}\int_0^T \int_0^1 h_{t-z,t}(i)\, didz \tag{2.A.33}$$

(b) Define the share of physical and human wealth held by entrepreneurs:

$$x_t \equiv \frac{\bar{n}_{e,t}}{q_t k_t + h_t} \tag{2.A.34}$$

The market clearing condition for the riskless bond imply that

$$\frac{n_{f,t}}{q_t k_t + h_t} = 1 - x_t \tag{2.A.35}$$

The market clearing condition for consumption goods can be written as

$$\frac{1}{T}\int_0^T \int_0^1 \left[\frac{\rho \bar{n}_{t-z,t}(i)}{1 - \Theta e^{-\rho(T-z)}} + \iota(g(q_t/A_t))A_t k_{t-z,t}(i) \right] didz + \rho n_{f,t} = \frac{1}{T}\int_0^T \int_0^1 \frac{y_t}{k_t} k_{s,t}(i)\, didz$$

where $y_t = \frac{1}{T}\int_0^T \int_0^1 y_{t-z,t}(i)\, didz$, using the fact that capital-output ratio is equalized across entrepreneurs.

111

From assumption 1 and proposition 8, we can write $\int_0^1 \overline{n}_{t-z,t}(i)di = \vartheta_z \overline{n}_{e,t}$ for all $t \geq 0$ and $z \in [0,T]$. Using again proposition 8, we have that $\frac{1}{T} \int_0^T \frac{\rho\vartheta_z}{1-\Theta e^{-\rho(T-z)}} dz = \rho + \theta$. The expression above can then be written as

$$(\rho + \theta)\overline{n}_{e,t} + \rho n_{f,t} = y_t - \iota(g(q_t/A_t))A_t k_t \tag{2.A.36}$$

Normalizing by $q_t k_t + h_t$:

$$(\rho + \theta)x_t + \rho(1 - x_t) = \frac{y_t - \iota(g(q_t/A_t))A_t k_t}{q_t k_t + h_t} \tag{2.A.37}$$

(c) Consider now the market clearing condition for aggregate risk taking:

$$\frac{1}{T} \int_0^T \int_0^1 \kappa_{t-z,t}(i)didz = \sigma_{f,t} n_{f,t}$$

Demand for aggregate insurance can be written as

$$\kappa_{s,t}(i) = q_t k_{s,t}(i)\sigma_{q,t} + \sigma_{s,t}^h h_{s,t} - \overline{\sigma}_{s,t}(i)\overline{n}_{s,t}(i) \tag{2.A.38}$$

Combining the previous two conditions, we obtain

$$\overline{\sigma}_{e,t}\overline{n}_{e,t} + \sigma_{f,t} n_{f,t} = q_t k_t \sigma_{q,t} + \sigma_{h,t} h_t \tag{2.A.39}$$

where
$$\sigma_{h,t} \equiv \frac{1}{T} \int_0^T \sigma_{t-z,t}^h \frac{h_{t-z,t}}{h_t} dz \tag{2.A.40}$$

Normalizing by physical and human wealth and using the market clearing condition for the riskless asset gives (2.4.28).

$\square$

112

Proof of proposition 9

(a) From (2.4.26), we have

$$\theta x_t + \rho = \frac{\left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} - \frac{\bar{\iota}_0}{\bar{\iota}_1}\left(\frac{q_t}{A_t} - \bar{\iota}_0\right) - \frac{1}{2\bar{\iota}_1}\left(\frac{q_t}{A_t} - \bar{\iota}_0\right)^2}{q_t/A_t + h_t/(A_t k_t)} \qquad (2.A.41)$$

using the fact that output-capital ratio is given by

$$\frac{y_t}{A_t k_t} = \left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} \qquad (2.A.42)$$

$$\left(\frac{q_t}{A_t}\right)^2 + 2\bar{\iota}_1\left[\theta x_t + \rho\right]\frac{q_t}{A_t} - 2\bar{\iota}_1\left(\left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} + \frac{\bar{\iota}_0^2}{2\bar{\iota}_1} - \left[\theta x_t + \rho\right]\frac{h_t}{A_t k_t}\right) = 0$$

The relative price of capital is given by $q_t = A_t q(x_t, k_t, \frac{h_t}{A_t})$, where

$$q(x, k, h) \equiv \sqrt{\bar{\iota}_1^2\left[\theta x + \rho\right]^2 + \bar{\iota}_1\left(\left[\alpha + (1-\alpha)k^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{\epsilon}{1-\epsilon}} + \frac{\bar{\iota}_0^2}{2\bar{\iota}_1} - \left[\theta x + \rho\right]\frac{h}{k}\right)} - \bar{\iota}_1\left[\theta x_t + \rho\right]$$

$$(2.A.43)$$

(b) Immediate from $\overline{\sigma}_{e,t} = \sigma_{f,t} = \eta_t$ and condition (2.4.28).

(c) From expression (2.4.24), we obtain

$$r_t = \frac{\alpha\left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{1}{1-\epsilon}} - \frac{1}{2\bar{\iota}_1}\left[q(x_t, k_t, \frac{h_t}{A_t})^2 - \bar{\iota}_0^2\right]}{q(x_t, k_t, \frac{h_t}{A_t})} + \frac{q(x_t, k_t, \frac{h_t}{A_t}) - \bar{\iota}_0}{\bar{\iota}_1} - \delta + \mu_{q,t} - \sigma_{q,t}\eta_t - \frac{\gamma}{x_t}\frac{k_t}{k_t +}$$

using the conditions

$$\alpha(w(k)) = \alpha\left[\alpha + (1-\alpha)k^{\frac{1-\epsilon}{\epsilon}}\right]^{-\frac{1}{1-\epsilon}} \qquad \iota(g(q)) = \frac{1}{2\bar{\iota}_1}\left[q^2 - \bar{\iota}_0^2\right] \qquad q(q) = \frac{q - \bar{\iota}_0}{\bar{\iota}_1} - \delta$$

$$(2.A.44)$$

Proof of proposition 10

Proof. **(a)** Integrating the law of motion of $h_{s,t}$ (2.4.8), we obtain

$$dh_t = [r_t h_t + \sigma_{h,t} h_t \eta_t - w_t]\,dt + \sigma_{h,t} h_t dZ_t \qquad (2.A.45)$$

The stock of human wealth is given by

$$h_t = \mathbb{E}_t \left[\frac{1}{T} \int_0^T \int_0^{T-a} \frac{\pi_{t+z}}{\pi_t} w_{t+z} \bar{l}_{-a,t+z}\,dz\,da \right] = \mathbb{E}_t \left[\int_0^T \frac{\pi_{t+z}}{\pi_t} w_{t+z} \ell_z dz \right]$$
$$(2.A.46)$$

where $\ell_z \equiv \frac{1}{T} \int_0^{T-z} \bar{l}(z+a)da$.

Let's show the stochastic discount factor assumes this form:

$$\pi_t = e^{-\rho t} \frac{1}{c_{f,t}} = \frac{e^{-\rho t}}{\rho(1-x_t)(q(x_t,k_t)k_t + h(x_t,k_t))} \frac{1}{A_t} \qquad (2.A.47)$$

From Ito's lemma, we have

$$d(\pi_t n_{f,t}) = -\pi_t c_{f,t} dt + (\eta_t \sigma_{f,t})dZ_t \qquad (2.A.48)$$

Integrating the expression above, we obtain the intertemporal budget constraint

$$\mathbb{E}_t \left[\int_t^\infty \frac{\pi_z}{\pi_t} c_{f,z} dz \right] \le n_{f,t} \qquad (2.A.49)$$

Equivalently, we can find (n_f, σ_f) subject to the flow budget constraint and the natural borrowing limit are both satisfied (by an application of the martingale representation theorem to the expected discounted value of consumption). Hence, we can write the financier's problem as maximizing utility subject to the intertemporal budget constraint above. The first-order condition for this new problem is

$$e^{-\rho t} \frac{1}{c_{f,t}} = \lambda \pi_t \qquad (2.A.50)$$

Since the scale of π_t is irrelevant to the problem, we can take $\pi_t = e^{-\rho t} \frac{1}{c_{f,t}}$.

114

Hence, $h(x, k)$ satisfies the functional equation:

$$h(x_t, k_t) = \int_0^T \frac{e^{-\rho z}(1 - x_t)(q(x_t, k_t)k_t + h(x_t, k_t))w_{t+z}\ell_{t,z}}{(1 - x_{t+z})(q(x_{t+z}, k_{t+z})k_{t+z} + h(x_{t+z}, k_{t+z}))}dz \qquad (2.A.51)$$

It remains to compute ℓ_z. Suppose $\bar{l}(a) = \beta e^{\gamma a}$. If $\gamma \neq 0$, then

$$\ell_z = \frac{1}{T}\int_0^{T-z} \beta e^{\gamma(z+a)}da = \beta\frac{e^{\gamma T} - e^{\gamma z}}{\gamma T} \qquad (2.A.52)$$

and $\ell_z = \beta\frac{T-z}{T}$ for $\gamma = 0$.

In a stationary equilibrium, we have

$$h^* = w^* \int_0^T e^{-\rho z}\ell_z dz \qquad (2.A.53)$$

The average value of ℓ_z is given by

$$\ell^* = \int_0^T e^{-\rho z}\ell_z dz = \begin{cases} \frac{\beta}{\gamma T}\left[\frac{e^{\gamma T} - e^{-(\rho-\gamma)T}}{\rho} - \frac{1-e^{-(\rho-\gamma)T}}{(\rho-\gamma)}\right], & \text{if } \gamma \neq 0 \text{ and } \gamma \neq \rho \\ \frac{\beta}{\rho T}\left[T(1 - e^{-\rho T}) - \frac{1-(1+\rho T)e^{-\rho T}}{\rho}\right], & \text{if } \gamma = 0 \\ \frac{\beta}{\rho T}\left[\frac{e^{\rho T}-1}{\rho} - T\right], & \text{if } \gamma = \rho \end{cases}$$
$$(2.A.54)$$

Since $\frac{1}{T}\int_0^T \bar{l}(a)da = 1$, we have

$$\frac{1}{T}\int_0^T \bar{l}(a)da = 1 \Rightarrow \beta = \frac{\gamma}{e^{\gamma T} - 1} \qquad (2.A.55)$$

$$h_t = \beta e^{\gamma T}\mathbb{E}_t\left[\int_t^{t+T} \frac{\pi_z}{\pi_t}w_z\frac{e^{-\gamma(z-t)} - e^{-\gamma T}}{\gamma T}dz\right] \qquad (2.A.56)$$

If $\eta_t = \sigma$, then

$$h_t = \beta w^* e^{\gamma T}\int_t^{t+T} e^{-(r^*-\mu)(z-t)}\frac{e^{-\gamma(z-t)} - e^{-\gamma T}}{\gamma T}dz \qquad (2.A.57)$$

$$(1-\alpha)\left[\alpha k_t^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)\right]^{\frac{1}{\epsilon-1}} \equiv w(k_t) \qquad (2.A.58)$$

(b) The law of motion of x_t is given by

$$\frac{\mu_{x,t}}{x_t} = r_t + (\eta_t)^2 + (\eta_t^{id})^2 - (\rho + \theta) - \frac{q_t k_t}{q_t k_t + h_t}(g_t + \mu_{q,t}) - \frac{h_t}{q_t k_t + h_t}\left[r_t + \sigma_{h,t}\eta_t - \frac{w_t}{h_t}\right] +$$
$$+ \left(\frac{q_t k_t}{q_t k_t + h_t}\sigma_{q,t} + \frac{h_t}{q_t k_t + h_t}\sigma_{h,t}\right)^2 - \eta_t\left(\frac{q_t k_t}{q_t k_t + h_t}\sigma_{q,t} - \frac{h_t}{q_t k_t + h_t}\sigma_{h,t}\right)$$
$$\frac{\sigma_{x,t}}{x_t} = \eta_t - \frac{q_t k_t}{q_t k_t + h_t}\sigma_{q,t} - \frac{h_t}{q_t k_t + h_t}\sigma_{h,t}$$

$$\square$$

Proof of proposition 11

Proof. **(a)** From the market clearing for goods we obtain

$$\frac{\alpha a}{q} - \frac{\omega}{2}q = \theta x + \rho \qquad (2.A.59)$$

Using the fact that $g = \omega q$, we can solve for g

$$g(x) = \sqrt{(\theta x + \rho)^2 + 2\alpha a\omega} - (\theta x + \rho) \qquad (2.A.60)$$

where $g(x) > 0$ and $g'(x) < 0$ for all $x \in [0,1]$.

The elasticity of g with respect to x is given by

$$\frac{g'(x)x}{g(x)} = -\frac{\theta x}{\theta x + g(x) + \rho} \qquad (2.A.61)$$

Using demand for aggregate risk and market clearing, we get $\sigma_{agg,t} = \sigma_{f,t} = \eta_t = (\sigma + \sigma_{q,t})$. From the law of motion of x, we obtain $\sigma_{x,t} = 0$. Hence, the price of capital will not react to aggregate shocks, $\sigma_{q,t} = 0$. We then obtain the market price of risk:

$$\eta_t = \sigma \qquad (2.A.62)$$

116

The market price of idiosyncratic risk is given by

$$\eta_{id}(x) = \frac{\phi\nu}{x} \tag{2.A.63}$$

From the pricing equation for capital, we have

$$r_t + \sigma^2 - g(x_t) - \frac{q_x}{q}\mu_x = \theta x_t + \rho - \frac{(\phi\nu)^2}{x_t} \tag{2.A.64}$$

The state variable x_t evolves according to

$$\dot{x}_t = \theta x_t (1 - x_t)\left[\left(\frac{x^*}{x_t}\right)^2 - 1\right] \tag{2.A.65}$$

where

$$x^* = \frac{\phi\nu}{\sqrt{\theta}} \tag{2.A.66}$$

The interest rate can be written as

$$r(x) = \rho + g(x) - \sigma^2 + \theta x \frac{g(x) + \theta + \rho}{g(x) + \theta x + \rho}\left[1 - \left(\frac{x^*}{x}\right)^2\right] \tag{2.A.67}$$

$\square$

2.A.3   Derivations

Deriving the Law of Motion of Human Wealth

Suppose $\sigma = 0$ and consider a stationary equilibrium. Human capital can be written as

$$H_{s,t} = \int_t^{s+T} e^{-r(z-t)}w\bar{l}dz = \frac{1 - e^{-r(T-a)}}{r}w\bar{l} \tag{2.A.68}$$

Hence, human wealth is decreasing over the life-cycle in a stationary equilibrium.

Entrepreneurial Returns with CES Production

Suppose the production function assumes the CES form:

$$Y = \left[\alpha(A_K K)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)(A_L L)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \qquad (2.A.69)$$

The marginal product of labor can be written as

$$MPL = (1-\alpha)A_L^{\frac{\epsilon-1}{\epsilon}} Y^{\frac{1}{\epsilon}} L^{-\frac{1}{\epsilon}} \qquad (2.A.70)$$

Labor demand can be written as

$$A_L L = (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^\epsilon Y \qquad (2.A.71)$$

We can use the expression above to solve for the capital-labor ratio:

$$\frac{A_L L}{A_K K} = (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^\epsilon \left[\alpha + (1-\alpha)\left(\frac{A_L L}{A_K K}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{\epsilon}{\epsilon-1}} \qquad (2.A.72)$$

solving the equation above

$$\frac{A_L L}{A_K K} = \frac{\alpha^{\frac{\epsilon}{\epsilon-1}}(1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^\epsilon}{\left[1 - (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^{\epsilon-1}\right]^{\frac{\epsilon}{\epsilon-1}}} \qquad (2.A.73)$$

In terms of the wage bill, we have

$$\frac{wL}{A_K K} = \frac{\alpha^{\frac{\epsilon}{\epsilon-1}}(1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^{\epsilon-1}}{\left[1 - (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^{\epsilon-1}\right]^{\frac{\epsilon}{\epsilon-1}}} \qquad (2.A.74)$$

Output can be written as

$$\frac{Y}{A_K K} = \frac{\alpha^{\frac{\epsilon}{\epsilon-1}}}{\left[1 - (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^{\epsilon-1}\right]^{\frac{\epsilon}{\epsilon-1}}} \qquad (2.A.75)$$

Output minus wage bill is then given by

$$Y - wL = \frac{\alpha^{\frac{\epsilon}{\epsilon-1}} A_K}{\left[1 - (1-\alpha)^\epsilon \left(\frac{A_L}{w}\right)^{\epsilon-1}\right]^{\frac{1}{\epsilon-1}}} K \equiv \alpha(w) A_K K \tag{2.A.76}$$

In equilibrium, $L = 1$, so the wage must satisfy

$$w_t = (1-\alpha)A_L \left[\alpha \left(\frac{A_K K}{A_L}\right)^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)\right]^{\frac{1}{\epsilon-1}} \tag{2.A.77}$$

We can compute the following term:

$$\alpha(w(k)) = \alpha \left[2 - \alpha - (1-\alpha)k^{\frac{\epsilon-1}{\epsilon}}\right] \tag{2.A.78}$$

Calibration

Adjustment cost parameters. Capital-output ratio (in efficiency units) is given by

$$\frac{A_t k_t}{y_t} = \left[\alpha + (1-\alpha)k_t^{\frac{1-\epsilon}{\epsilon}}\right]^{\frac{\epsilon}{1-\epsilon}} \tag{2.A.79}$$

and for $\epsilon = 1$ $\frac{A_t k_t}{y_t} = k_t^{1-\alpha}$.

Notice that the capital-output ratio (not in efficiency units) is not stationary in this model. The reason is that implicitly we are imposing capital-augmenting technology grows at the same rate as the labor-augmenting technology. By assuming that only labor-augmenting technology grows over time, then we would have k_t/y_t being stationary.

Investment per unit of efficiency units of capital is given by

$$\iota(g(q^*)) = \frac{1}{2\bar{\iota}_1} \left[(q^*)^2 - \bar{\iota}_0^2\right] = \bar{\iota}_0 \delta + \frac{\bar{\iota}_1}{2} \delta^2 \tag{2.A.80}$$

From the calibration targets, we have $\frac{\iota(g(q^*))A_t k_t}{y_t} = invrate$ and $\frac{A_t k_t}{y_t} = cor$. This allow us to set $\bar{\iota}_1$:

$$\bar{\iota}_1 = \frac{2}{\delta} \left[\frac{invrate}{\delta cor} - \bar{\iota}_0\right] \tag{2.A.81}$$

119

We must then search for $\bar{\iota}_0 \leq \frac{invrate}{\delta cor}$ that matches the desired capital-output ratio.

Moral hazard parameter. In a stationary equilibrium, idiosyncratic risk premium is given by

$$\text{idriskpremium} = \phi \nu \sqrt{\theta} \Rightarrow \phi = \frac{1}{\sqrt{\theta}} \frac{idriskpremium}{\nu} \tag{2.A.82}$$

Hence, ϕ is proportional to the ratio between idiosyncratic risk premium and idiosyncratic volatility.

Chapter 3

Optimal Fiscal Policy in a Currency Union

3.1 Introduction

Fiscal issues have been at the core of recent developments in Europe. After the increase in sovereign credit spreads and the simultaneous drop in economic activity, a heated debate on fiscal policy started. On the one side, proponents of fiscal austerity defended a fiscal consolidation and greater emphasis on issues of debt sustainability. On the other side, the ones in favor of stimulus policy questioned the timing of these reforms and raised concerns regarding consolidation in a depressed economy.

Despite the obvious importance of those matters for the current public debate and policymaking, we still lack a better understanding of the appropriate policy response or a formal evaluation of the different arguments. The goal of this essay is try to fill this gap by providing an analysis of the optimal fiscal policy in circumstances resembling those in the peripheral of Europe: an increase in sovereign spreads for a member of a currency union. Since these countries lack an independent monetary policy, fiscal policy becomes the first line of defense.

The environment is based on the continuous time version of the open economy New Keynesian model of Gali and Monacelli (2005) proposed by Farhi and Werning (2012). The model is able to generate most of the salient features of a sudden stop,

as a decrease in domestic output and consumption, and an increase in net exports in response to a risk premium shock.

I first analyze spending and tax policy in the context of a flexible price economy. I start by characterizing the optimal labor and intertemporal wedge in this economy. The labor wedge is non-zero and, in general, time varying. The intertemporal wedge will depend on the evolution over time of the labor wedge. The behavior of wedges will be intimately linked to the optimal tax policy.

The reason for the non-zero labor wedge in the flexible price economy is the ability of the planner to exert internationally monopoly power, keeping the price of exports high and the terms of trade appreciated. In a related environment, Costinot et al. (2011) discuss the implication of the terms of trade manipulation motive to the design of capital controls.

I consider then a first-order approximation of the equilibrium conditions and a second-order approximation of the objective. This allows me to obtain a sharp characterization of the dynamics of the equilibrium objects, as allocations, tax policy, and debt dynamics.

Using the linear-quadratic program, I study tax policy under both sticky prices and downward nominal wage rigidity. First, I consider the simpler case of extreme price rigidity, where firms are not allowed to move prices at all. I then consider the more complicate case where prices are sticky, but not fully rigid. Finally, I discuss the case of downward rigid wages.

I first show that government spending should not deviate from the condition for optimal provision of public goods, even in a sticky price economy or under downward wage rigidities. Hence, stabilization should be a focus of the tax policy. Nevertheless, government spending should be countercyclical, since the cost of providing public goods decrease in a recession.[1]

The key for this result is the presence of a tax instrument that can be used in addition to government spending, as a VAT tax, for instance. This suggests that the

---

[1] This result is in line with the findings of Werning (2011) in the context of a closed-economy liquidity trap model.

use of government spending as a stabilization tool is only a result of limited policy instruments.

This result contrasts with the one found by Gali and Monacelli (2008), who emphasize the stabilization role of government spending. The reason for the different results is that Gali and Monacelli (2008) assumes the tax policy cannot adjust, leading the government spending to compensate for the missing instrument.

The tax policy is used mainly to achieve two goals: to influence the real interest and manipulate the terms of trade. Changes in the consumption tax are used to affect the real interest the consumer faces, while the VAT tax is used to affect the terms of trade.

The consumption tax is used to *lean against the wind*, i.e., to reduce the real interest when there is a positive risk premium shock. The behavior of the consumption tax is in line with the results found in the capital control literature (see Farhi and Werning (2013)).

The evolution of the VAT tax depends on the degree of price flexibility. Under flexible prices, the VAT is used to exercise internationally market power and to keep the terms of trade appreciate. Under sticky prices, there is two forces at play. First, the pricing friction generates a preference for a stable terms of trade. This goes in the direction of dampening the process of internal devaluation. Second, a depreciation of the terms of trade would shift demand towards domestic goods, stimulating the economy. If the output response to real exchange rate is strong enough, the second effect dominates and tax policy will be used to depreciate the real exchange rate and further stimulate the economy.

I consider also the effects of downward nominal wage rigidity. In the absence of policy response, the economy will experience a recession just as under rigid prices, but now the fact the wage cannot adjust will prevent the labor market from clearing, creating unemployment.

It turns out fiscal policy is particularly effective to deal with this situation. The optimal allocation under downward wage rigidity coincide with the optimal allocation under flexible prices. However, the implementation is different from the case of flexible

prices (or rigid prices) and require also the use of a labor income tax. In this case, the government is able of completely avoiding the recession and the rise in unemployment.

Finally, I consider the budgetary effects of the optimal policy. First, I show that the optimal policy is revenue generating, i.e., the amount of lump-sum taxation necessary to satisfy the government budget is smaller under the optimal policy than under a passive policy (where spending and taxes are kept at their steady state level).

Under some circumstances, in particular under low initial government debt, this imply that the government should decrease its debt in response to the sovereign spread shock. Therefore, a fiscal consolidation is optimal even though the economy is initially in a recession.

In contrast to the focus in the dichotomy between fiscal consolidation and fiscal stimulus present in most of the public debate, this analysis suggests there is not necessarily a trade off between the two and that properly design stimulus policy could indeed avoid the worsen of public finances, or even to generate a reduction in the level of government debt.

The essay is organized as following. The next section describes the environment and the equilibrium conditions. Section 3 presents the (non-linear) Ramsey problem under flexible prices. Section 4 considers the linear-quadratic version of the problem and section 5 discuss the case of stick prices. Section 6 discuss the case of downward nominal wage rigidities and the last section presents the conclusion.

3.2   Environment

The economy consists of a continuum of countries, indexed by $i \in [0, 1]$, sharing a common currency. Preferences and technology are symmetric across countries. I will focus attention in an specific country which will be called "'Home", with index $i = H$.

I will consider a perfect foresight equilibrium. Since most of the analysis will focus on a first order approximation of the equilibrium condition, certainty equivalence holds and there is no further loss on abstracting from ongoing uncertainty.

3.2.1 Households

Preferences are given by

$$\int_0^\infty e^{-\rho t} \left[\frac{C_t^{1-\sigma}}{1-\sigma} + \chi \frac{G_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1+\phi} \right] dt \qquad (3.2.1)$$

Consumer derives utility from an aggregate of consumption goods C_t and an aggregate of government purchases G_t and there is a disutility from supplying labor N_t.

Aggregate consumption C_t is a composite of domestic and foreign goods:

$$C_t = \left[(1-\alpha)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \alpha^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \qquad (3.2.2)$$

where $C_{H,t}$ is the index of domestic goods

$$C_{H,t} = \left(\int_0^1 C_{H,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon-1}{\epsilon}} \qquad (3.2.3)$$

and $C_{F,t}$ is a composite of foreign goods:

$$C_{F,t} = \left(\int_0^1 \Lambda_{i,t}^{\frac{1}{\gamma}} C_{i,t}^{\frac{\gamma-1}{\gamma}} di \right)^{\frac{\gamma}{\gamma-1}} \qquad (3.2.4)$$

where $C_{i,t}$ is an index of goods produced in country i:

$$C_{i,t} = \left(\int_0^1 C_{i,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \qquad (3.2.5)$$

The parameter α controls the degree of home-bias. As α goes to zero, there is extreme home-bias and the economy barely trades with the rest of the world. As α goes to one, the economy becomes fully open economy and there is no home-bias.

The parameter ϵ represents the price elasticity of goods within a given country, γ represent the elasticity of substitution between goods from different countries, and η represent the elasticity of substitution between a bundle of domestic and foreign

125

goods. The term $\Lambda_{i,t}$ represents a export demand shock for country i.

The per-period budget constraint is given by

$$\dot{B}_t = i_t B_t + (1 - \tau_t^l)W_t N_t + \Pi_t + \tilde{T}_t - (1 + \tau_t^c)\left[\int_0^1 P_{H,t}(j)C_{H,t}(j)dj + \int_0^1 \int_0^1 P_{i,t}(j)C_{i,t}(j)dj\,di\right]$$

(3.2.6)

where i_t represents the nominal interest rate, B_t nominal assets, and Π_t aggregate nominal profits.

Note that the consumer faces two different types of taxes: a labor income tax τ_t^l, and a tax on consumption τ_t^c. The consumer also possibly receives some lump-sum transfers from the government $\tilde{T}_t$.

Households are subject to the usual No-Ponzi condition:

$$\lim_{t \to \infty} e^{-\int_0^t i_s ds} B_t \geq 0$$

(3.2.7)

3.2.2 Terms of Trade and Real Exchange Rate

Let's define now the price indexes associated with the bundles the consumer demands. I will define the *before-tax* price levels.

The domestic producer price index (PPI) is given by

$$P_{H,t} = \left[\int_0^1 P_{H,t}(j)^{1-\epsilon}dj\right]^{\frac{1}{1-\epsilon}}$$

(3.2.8)

Similarly, the producer price index for country i is defined as

$$P_{i,t} = \left[\int_0^1 P_{i,t}(j)^{1-\epsilon}dj\right]^{\frac{1}{1-\epsilon}}$$

(3.2.9)

The price index for the aggregate of foreign goods is

$$P_{F,t} = \left(\int_0^1 \Lambda_{i,t}P_{i,t}(j)^{1-\gamma}dj\right)^{\frac{1}{1-\gamma}}$$

(3.2.10)

126

The consumer price index (CPI) is given by

$$P_t = \left[(1-\alpha)P_{H,t}^{1-\eta} + \alpha(P_{F,t})^{1-\eta} \right]^{\frac{1}{1-\eta}} \tag{3.2.11}$$

I will focus mostly on the case of a symmetric rest of the world, where $P_{i,t} = P_{j,t}$ for $i, j \neq H$. Hence, the price level for any foreign country and the price index of imported goods are equal to each other, denoted by P_t^*.

The terms of trade are given by

$$S_t = \frac{P_{F,t}}{P_{H,t}} = \frac{P_t^*}{P_{H,t}} \tag{3.2.12}$$

The real exchange rate is given by

$$\mathcal{Q}_t = \frac{P_t^*}{P_t} \tag{3.2.13}$$

From equation (3.2.11), we can obtain a relationship between the real exchange rate and the terms of trade:

$$\mathcal{Q}_t^{\eta-1} = (1-\alpha)S_t^{\eta-1} + \alpha \tag{3.2.14}$$

If we assume $\eta = 1$, expression above simplifies to

$$\mathcal{Q}_t = S_t^{1-\alpha} \tag{3.2.15}$$

3.2.3   Firms

Each differentiated good is produced using labor as the only input:

$$Y_t(j) = A_{H,t}N_t(j)^{\frac{1}{\varphi}} \tag{3.2.16}$$

where $\varphi \geq 1$.

The problem of the firm will depend on the degree of price flexibility. For the

case of sticky prices, I assume Calvo pricing, i.e., the periods that firms are allowed to reset their prices are determined by a Poison arrival with intensity ρ_δ.

The problem of the firm is given by:

$$\max_{\overline{P}_t(j)} \int_0^\infty e^{-\int_0^s (i_{t+z}+\rho_\delta)dz} \left[(1 - \tau_{t+s}^v)\overline{P}_t(j)Y_{t+s|t} - W_{t+s} \left(\frac{Y_{t+s|t}}{A_{t+s}} \right)^\varphi \right] ds \qquad (3.2.17)$$

where the firm faces a VAT tax τ_t^v, $Y_{t+s|t}$ represents the demand function the producer faces at period $t + s$:

$$Y_{t+s|t} = \left(\frac{\overline{P}_t(j)}{P_{H,t+s}} \right)^{-\epsilon} Y_{t+s} \qquad (3.2.18)$$

and Y_t denotes aggregate demand for domestic goods.

If prices are completely flexible, the problem of the firm collapses to

$$\max_{\overline{P}_t(j)} \left\{ (1 - \tau_t^v)\overline{P}_t(j)Y_{t|t} - W_t \left(\frac{Y_{t|t}}{A_{t+s}} \right)^\varphi \right\} \qquad (3.2.19)$$

3.2.4   Government

Government consumption is an aggregate of domestically produced goods:

$$G_t = \left(\int_0^1 G_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \qquad (3.2.20)$$

Government purchase of each individual good is made in order to minimize costs, given the aggregate amount G_t.

Government flow budget constraint is given by

$$\dot{D}_t^g = i_t D_t^g + P_{H,t}G_t + \tilde{T}_t - \tau_t^v P_{H,t}Y_t - \tau_t^c P_t C_t - \tau_t^l W_t N_t \qquad (3.2.21)$$

where D_t^g denotes government debt.

3.2.5   Equilibrium Conditions

We can divide the equilibrium conditions into two blocks: a demand and a supply block.

The Demand Block

The solution of the consumer problem involves an intratemporal condition:

$$C_t^\sigma N_t^\phi = \frac{(1 - \tau_t^l)W_t}{(1 + \tau_t^c)P_t} \tag{3.2.22}$$

and an intertemporal condition[2]

$$\frac{\dot{C}_t}{C_t} = \sigma^{-1}\left(i_t - \pi_t - \dot{\hat{\tau}}_t^c - \rho\right) \tag{3.2.23}$$

and the transversality condition

$$\lim_{t \to \infty} e^{-\int_0^t i_s ds} B_t = 0$$

Notice the level of the labor income tax and the consumption tax affects the labor supply condition, while only changes in the consumption tax affect the Euler equation.

An analogous Euler equation holds at the foreign countries:

$$\frac{\dot{C}_t^*}{C_t^*} = \sigma^{-1}\left(i_t^* - \pi_t^* - \rho\right)$$

where an asterisk indicates foreign variables.

For simplicity, I am assuming taxes are constant in the rest of the world. In order to connect the Euler equation in the home country to the Euler equation in the rest of the world, we need to specify how domestic and foreign interest rates are related:

$$i_t = i_t^* + \psi_t$$

I allow for possible deviations from the uncovered interest parity (UIP). The term ψ_t captures in a reduced form way changes in "risk premium". For instance, an increase in ψ_t may indicate a reduction of the ability of foreign investor to hold domestic asset due to a "loss in confidence" or balance sheet problems.

---

[2] The term $\hat{\tau}_t^c$ is defined by $\hat{\tau}_t^c \equiv \log(1 + \tau_t^c)$.

Taking the difference between the two Euler equations, we obtain

$$\frac{\dot{C}_t}{C_t} - \frac{\dot{C}_t^*}{C_t^*} = \sigma^{-1}\left(\psi_t + \frac{\dot{P}_t^*}{P_t^*} - \frac{\dot{P}_t}{P_t} - \dot{\tau}_t^c\right)$$

Integrating the expression above, we get the so-called Backus-Smith condition:

$$C_t = \Theta_t C_t^* \mathcal{Q}_t^{\frac{1}{\sigma}} \tag{3.2.24}$$

where

$$\Theta_t = \Theta_0 \Psi_t \mathcal{T}_t; \qquad \Psi_t = e^{\frac{1}{\sigma}\int_0^t \psi_s ds}; \qquad \mathcal{T}_t = \left(\frac{1 + \tau_0^c}{1 + \tau_t^c}\right)^{\frac{1}{\sigma}}$$

for some constant Θ_0.

Under constant consumption taxes and no risk premium shock, expression (3.2.24) shows that relative consumption respond to changes in relative inflation. In levels, this correspond to a positive relation between consumption and the real exchange rate. The term Θ_t is a relative Pareto weight and it is constant in the absence of consumption taxes and risk premium shocks.

In the appendix, I show the demand for domestic output is given by the expression

$$Y_t = (1 - \alpha)\left(\frac{\mathcal{Q}_t}{S_t}\right)^{-\eta} C_t + G_t + \alpha\Lambda_{H,t}S_t^\gamma C_t^* \tag{3.2.25}$$

The first term represents the demand of domestic agents for domestic output, the second term represents government demand and the third term corresponds to exports.

The demand block is completed by the external solvency condition

$$\frac{C_0^{-\sigma}E_0}{P_0(1 + \tau_0^c)} = \int_0^\infty e^{-\rho t}\frac{C_t^{-\sigma}}{1 + \tau_t^c}\left[\frac{\mathcal{Q}_t}{S_t}(Y_t - G_t) - C_t\right] dt \tag{3.2.26}$$

130

and the government solvency constraint

$$\frac{C_0^{-\sigma} D_0^g}{P_0(1+\tau_0^c)} = \int_0^\infty e^{-\rho t} \frac{C_t^{-\sigma}}{1+\tau_t^c} \left[\frac{Q_t}{S_t} (\tau_t^v Y_t - G_t - T_t) + \tau_t^c C_t + \tau_t^l \frac{W_t N_t}{P_t} \right] dt \quad (3.2.27)$$

where I used the Euler equation to eliminate the nominal interest from both conditions.

The Supply Block

Aggregate demand for labor is given by

$$N_t = \Delta_t \left(\frac{Y_t}{A_{H,t}} \right)^\varphi \quad (3.2.28)$$

where

$$\Delta_t \equiv \int_0^1 \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon\varphi} dj \quad (3.2.29)$$

The term Δ_t captures the effect of price dispersion on aggregate labor demand (and production).

Under sticky prices, the optimal price setting condition is given by

$$\int_0^\infty e^{-(\rho+\rho_\delta)s} \frac{C_{t+s}^{-\sigma}}{P_{t+s}(1+\tau_{t+s}^c)} \left[(1-\tau_{t+s}^v) \frac{\overline{P}_t(j)^{-\epsilon}}{P_{H,t+s}^{-\epsilon}} Y_{t+s} - \frac{\varphi\epsilon}{\epsilon-1} W_{t+s} \frac{\overline{P}_t(j)^{-\varphi\epsilon-1}}{P_{H,t+s}^{-\varphi\epsilon}} \left(\frac{Y_{t+s}}{A_{t+s}} \right)^\varphi \right] ds = 0$$
$$(3.2.30)$$

Equation above indicates the firm will choose prices as an weighted average of all future marginal costs. In the appendix, I derive the aggregate supply condition that shows how current inflation is determine by current and future costs.

By considering the limit $\rho_\delta \to \infty$, the condition above collapses to the flexible price supply condition:

$$(1-\tau_t^v) \frac{P_{H,t}}{P_t} = \frac{\varphi\epsilon}{\epsilon-1} \frac{W_t}{P_t} \left(\frac{Y_t}{A_t} \right)^\varphi \frac{1}{Y_t} \quad (3.2.31)$$

where I used the fact that $P_{H,t}(j) = P_{H,t}$ under flexible prices.

This completes the description of the equilibrium. An equilibrium is a sequence of allocations (C_t, C_t^*, N_t, Y_t), prices (W_t, S_t, Q_t), and government policy $(G_t, \tau_t^c, \tau_t^l, \tau_t^v, T_t)$

such that conditions (3.2.22) to (3.2.27) are satisfied (demand block) and conditions (3.2.28)-(3.2.29) and (3.2.30) under sticky prices or (3.2.31) under flexible price are satisfied.

3.3 Optimal Policy under Flexible Prices

In this section, I will consider the optimal fiscal policy, both taxes and spending, under flexible prices. The next section consider the case of sticky prices.

From now on, I will assume the elasticity between domestic and foreign goods is given by $\eta = 1$. This implies a simple relationship between terms of trade and the real exchange rate:

$$\mathcal{Q}_t = S_t^{1-\alpha} \tag{3.3.1}$$

However, in contrast to most of the previous literature, I do not impose the further restriction $\gamma = \sigma = 1$, the so-called Cole-Obstfeld case. Therefore, the terms of trade elasticities of imports and exports remains unrestricted, what will have implications for optimal policy below.

The goods market clearing condition simplifies to

$$Y_t = \left[(1 - \alpha)\Theta_t S_t^{\xi} + \alpha \Lambda_{H,t} S_t^{\gamma} \right] C_t^* + G_t \tag{3.3.2}$$

where $\xi \equiv \frac{1-\alpha}{\sigma} + \alpha$ is the (general equilibrium) elasticity of imports.

The external solvency constraint is given by

$$\frac{E_0}{P_0^* C_0^*} = \alpha \int_0^{\infty} e^{-\rho t} \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} \left[\Lambda_{H,t} S_t^{\gamma-1} - \Theta_t S_t^{\xi-1} \right] ds \tag{3.3.3}$$

Conditions (3.3.2) and (3.3.3) are sufficient conditions for a competitive equilibrium, in the following sense. If $(Y_t, G_t, \Theta_t, S_t)$ satisfy the conditions above, we can always find taxes and prices that support this allocation as part of an equilibrium. Therefore, the Ramsey problem boils down to maximize utility subject to conditions (3.3.2) and (3.3.3).

3.3.1 The Ramsey Problem

The Ramsey problem can be written as

$$\max_{\{\Theta_t, S_t, Y_t, G_t\}} \left\{ \int_0^\infty e^{-\rho t} \left[\frac{1}{1-\sigma} \left(\Theta_t C_t^* S_t^{\xi-\alpha} \right)^{1-\sigma} + \chi \log G_t - \frac{1}{1+\phi} \left(\frac{Y_t}{A_{H,t}} \right)^{\varphi(1+\phi)} \right] dt \right\} \tag{3.3.4}$$

subject to

$$Y_t = \left[(1-\alpha)\Theta_t S_t^\xi + \alpha \Lambda_{H,t} S_t^\gamma \right] C_t^* + G_t \tag{3.3.5}$$

$$\frac{E_0}{P_0^* C_0^*} = \alpha \int_0^\infty e^{-\rho t} \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} \left[\Lambda_{H,t} S_t^{\gamma-1} - \Theta_t S_t^{\xi-1} \right] ds \tag{3.3.6}$$

Notice the problem takes the perspective of the domestic country, so the optimal policy can be implemented without a need of international cooperation. This contrasts with previous work on fiscal policy in currency unions (see Ferrero (2009) and Gali and Monacelli (2008)) where it was taken the perspective of the whole union, implicitly assuming some form coordination both at monetary and fiscal policy.

The first order condition with respect to output is

$$\lambda_t = \varphi \left(\frac{Y_t}{A_{H,t}} \right)^{\varphi(1+\phi)} \frac{1}{Y_t} = \frac{N^\phi}{\frac{1}{\varphi} A_{H,t} N_t^{\frac{1}{\varphi}-1}} \tag{3.3.7}$$

where λ_t indicates the Lagrange multiplier on the goods market clearing condition.

The Lagrange multiplier λ_t captures the cost (in utility terms) of producing an additional unity of output. The optimality condition for government spending equates the marginal benefit of government consumption to the marginal cost of increasing output:

$$\chi G_t^{-1} = \lambda_t \Rightarrow \frac{G_t}{Y_t} = \frac{\chi}{\varphi} \left(\frac{Y_t}{A_{H,t}} \right)^{-\varphi(1+\phi)} \tag{3.3.8}$$

Notice the condition above implies that the optimal provision of public goods involves a countercyclical government spending. The reason is the cost of providing government goods is smaller when output is relatively low. This condition will also be important when we study optimal policy under price and wage rigidities.

133

In order to characterize the solution, it is useful to define the labor wedge in this environment:

$$1 + \omega_t^l \equiv \frac{C_t^{-\sigma} S_t^{-\alpha}}{\lambda_t} = \frac{P_{H,t}}{P_t} \frac{\frac{1}{\varphi} A_{H,t} N_t^{\frac{1}{\varphi}-1}}{C_t^\sigma N_t^\phi} \tag{3.3.9}$$

Since the real consumption wage and the product wage are different, we need to adjust the ratio of the marginal productivity of labor to the marginal disutility of labor by the relative price of domestic goods.

We can also define the intertemporal wedge, as the wedge between the marginal utility of consumption between two periods, adjusted by the interest rate:

$$e^{\int_t^{t+s} \omega_z^I dz} = \frac{e^{-\rho t} \frac{C_t^{-\sigma}}{P_t}}{e^{-\rho(t+s)} \frac{C_{t+s}^{-\sigma}}{P_{t+s}} e^{\rho s + \int_t^{t+s} \psi_z dz}} \Rightarrow \omega_t^I = \sigma \frac{\dot{\Theta}_t}{\Theta_t} - \psi_t \tag{3.3.10}$$

where, for simplicity, I am assuming C_t^* is constant.

Under no intervention and perfect competition, both the labor wedge and the intertemporal wedge would be equal to zero. However, in general, the optimal policy imply non-zero wedges:

Proposition 12. *(Wedges) Consider the optimal allocation under flexible prices.*

- *Labor wedge:*

$$\omega_t^l = \frac{\alpha \Lambda_{H,t} S_t^\gamma}{(\gamma - 1)\Lambda_{H,t} S_t^\gamma + (1 - \alpha) S_t^\xi \Theta_t} \tag{3.3.11}$$

- *Intertemporal wedge:*

$$\omega_t^I = \frac{1 - \alpha}{\omega_t^l + \alpha} \frac{\dot{\omega}_t^l}{1 + \omega_t^l} \tag{3.3.12}$$

Notice the labor wedge can also be interpreted as the desired mark-up level:

$$\frac{P_{H,t}}{P_t} = (1 + \omega_t^l) \frac{C_t^\sigma N_t^\phi}{\frac{1}{\varphi} A_{H,t} N_t^{\frac{1}{\varphi}-1}} = (1 + \omega_t^l) C_t^\sigma \lambda_t \tag{3.3.13}$$

where $C_t^\sigma \lambda_t$ is the firm (real) marginal cost.

Expression (3.3.11) shows that the optimal mark-up will in general vary over time, depending on the level of the terms of trade and of the Pareto weight Θ_t. This

contrasts with the private calculation, where the desired mark-up is constant and given by $c/(c-1)$.

In order to see where this result comes from, let's consider the simpler case of $\gamma = \xi = \sigma = 1$. The first order condition for S_t in this case is given by

$$(1-\alpha)\frac{1}{S_t} = \lambda_t \left[(1-\alpha)\Theta_t + \alpha\Lambda_{H,t}\right] C_t^* \Rightarrow C_t^{-1}S_t^{-\alpha} = \lambda_t \left[1 + \frac{\alpha\Lambda_{H,t}}{(1-\alpha)\Theta_t}\right] \quad (3.3.14)$$

The expression above compares the cost and benefit of depreciating the terms of trade. The benefit of an increase in the terms of trade is the increase in consumption (through the usual intertemporal substitution channel). The cost comes, in part, from the necessary resources to increase domestic consumption. However, there is also a cost coming from the fact the country is selling goods internationally more cheaply, requiring a greater amount of production for the same revenue. Notice this cost is more important, the bigger exports are relative to domestic consumption.

When $\gamma \neq 1$ there is an additional effect, since changes in the terms of trade can affect export revenue, impacting the payment of foreign debt. If $\gamma > 1$, then an appreciation of the terms of trade will reduce revenue exports. This additional cost will reduce the size of the labor wedge, i.e., the incentive to keep the real exchange rate appreciated is diminished.

In any case, the planner will try to maintain the terms of trade more appreciated compared to an equilibrium without intervention. Moreover, the more important exports are relative to domestic consumption, the higher will be the mark-up chosen by the planner.

This explains, for instance, why the labor wedge is equal to zero when $\Lambda_{H,t} = 0$. In that case, the country will not export anything, regardless of the price. Hence, the effect on exports is missing and the labor wedge will be zero. Similarly, if $\alpha = 0$, exports are again unimportant and the planner has no reason to distort the pricing decision.

Another interesting special case is $\alpha \to 1$, where the markup is constant and equal to $\frac{\gamma}{\gamma-1}$. In this case, domestic consumption plays a negligible role on domestic

production. The planner then acts as a typical monopolist with isoelastic demand and the usual markup rule applies.

The connection between the intertemporal wedge and the labor wedge can be understood by considering again the case $\gamma = \xi = \sigma = 1$. The first order condition for Θ_t is given by

$$\frac{1}{\Theta_t} = \lambda_t(1-\alpha)S_t C_t^* + \alpha\Gamma\Psi_t^{-1} \Rightarrow 1 = \frac{1-\alpha}{1+\omega_t^l} + \alpha\Gamma\Theta_0 e^{\int_0^t \omega_s^l ds} \qquad (3.3.15)$$

where Γ is the Lagrange multiplier on the external solvency constraint.

The expressions relates the marginal benefit and cost of increasing consumption through changes in Θ_t.[3] Notice the cost of increasing cost has a domestic component, determined by the labor wedge, and a foreign component, influenced by the risk premium shock.

Differentiating the expression above we recover (3.3.12). Hence, the intertemporal wedge is positive at periods where the labor wedge increases. The intuition is the following: if the labor wedge is increasing, it means the cost of providing domestic consumption is decreasing, generating incentives to shift consumption into the future (a positive intertemporal wedge).

This discussion also illustrates the role of the term $1-\alpha$ in the expression for the intertemporal wedge. The labor-wedge is related to the cost of producing domestic goods (a fraction $1-\alpha$ of total spending). If only a negligible fraction of consumption consists of domestic goods, then this time varying cost of consumption becomes irrelevant and the intertemporal wedge converges to zero.

In order to provide a sharper characterization of the optimal policy, I will consider a first order approximation around a stationary solution to the Ramsey problem. The next section describes the stationary solution and the following one discuss the first order approximation.

---

[3]Remember the only instrument the government has to influence changes in Θ_t is the consumption tax. This can seen essentially as a condition determining $\hat{\tau}_t^c$.

3.3.2 Stationary Solution

Suppose $C_t^* = \overline{C}^*$, $\Lambda_{H,t} = \overline{\Lambda}_H$, $A_{H,t} = \overline{A}_H$ and $\Psi_t = 1$ for every $t \geq 0$. I will look for a stationary solution of the Ramsey problem.[4]

For simplicity, assume $\overline{E} = 0$, what imply $\overline{\Theta} = \Lambda_H \overline{S}^{\gamma - \xi}$. Plugging this into the labor wedge, we obtain

$$1 + \overline{\omega}^l = \frac{\gamma}{\gamma - \alpha} \tag{3.3.16}$$

From the flexible price condition (3.2.31) and the labor supply condition (3.2.22), we get

$$\frac{(1 - \overline{\tau}^v)(1 - \overline{\tau}^l)}{1 + \overline{\tau}^c} = \frac{\epsilon}{\epsilon - 1} \frac{\overline{\lambda}}{\overline{S}^{-\alpha} \overline{C}^{-\sigma}} = \frac{\epsilon}{\epsilon - 1} \frac{\gamma - \alpha}{\gamma} \tag{3.3.17}$$

The condition above shows that taxation should correct the monopoly distortion, but there should still be a markup over marginal cost that depends on the degree of openness and the elasticity of exports:

$$\frac{\overline{P}_H}{\overline{P}} = \frac{\gamma}{\gamma - \alpha} \overline{C}^\sigma \overline{\lambda} \tag{3.3.18}$$

where $\overline{C}^\sigma \overline{\lambda}$ represents the real marginal cost of the firm.

Consistent with our previous discussion, as α converges to one, the formula above converges to the usual monopoly rule, where γ makes the role of the elasticity of demand.

Consider now the government finances. There are many ways of closing the government budget and still satisfy condition (3.3.17). One specific example would be to set $\overline{\tau}^c = 0$, choose $\overline{\tau}^l$ to satisfy (3.3.17) given $\overline{\tau}^v$, and the level of the VAT tax is chosen to satisfy the government budget:

$$\frac{\overline{D}}{\overline{P}_H \overline{Y}} = \left(\frac{\overline{\tau}^v - \varsigma_g}{\rho} \right) + \frac{1}{\rho} \overline{\tau}^l \frac{\overline{WN}}{\overline{P}_H \overline{Y}} \tag{3.3.19}$$

This will be formulation I will adopt in the numerical simulations below. The appendix provides the complete derivation of the stationary solution.

---

[4] I will denote by an overbar the stationary version of each variable.

3.3.3 A First Order Approximation

Let's consider a first order approximation of the equilibrium conditions around the stationary solution described above. Suppose also that the only non-zero shock is the risk premium shock, i.e., productivity, export demand and foreign consumption are at their steady state levels. I will denote by a lowercase the log deviation of an uppercase variable from the stationary solution, for instance, $c_t \equiv \log C_t - \log \overline{C}$.

The Backus-Smith condition (3.2.24) holds exactly in log-linear form:

$$c_t = \theta_t + (\xi - \alpha)s_t \tag{3.3.20}$$

where

$$\dot{\theta}_t = \frac{1}{\sigma}\left[\psi_t - \hat{\tau}_t^c\right] \tag{3.3.21}$$

A first order approximation of the aggregate demand condition gives

$$y_t = \varsigma_c(c_t + \alpha s_t) + \varsigma_g g_t + \varsigma_x(\lambda_{H,t} + \gamma s_t) \tag{3.3.22}$$

where $\varsigma_c \equiv (1 - \alpha)\overline{CS}^{\alpha}/\overline{Y}$, $\varsigma_g \equiv \overline{G}/\overline{Y}$, $\varsigma_x \equiv \alpha\overline{\Lambda}_H\overline{C}^*\overline{S}^{\gamma}/\overline{Y}$, and $\varsigma_c + \varsigma_g + \varsigma_x = 1$.

The coefficients ς_k, $k \in \{c, g, x\}$, represent the steady state share of *domestic* consumption, government spending and exports on output, respectively.

The supply condition (3.2.31) in log-linear form is given by[5]

$$-s_t = \sigma\theta_t + (\varphi(1 + \phi) - 1)y_t + \left(\hat{\tau}_t^v + \hat{\tau}_t^l + \hat{\tau}_t^c\right) \tag{3.3.23}$$

The condition above captures the fact that supply is upward sloping, i.e., all else given, an increase in output will be associated with an appreciation of the terms of trade.

In contrast to the optimal fiscal policy discussed below, let's define a *passive fiscal policy*, i.e., assume $g_t = \hat{\tau}_t^v + \hat{\tau}_t^l + \hat{\tau}_t^c = \dot{\hat{\tau}}_t^c = 0$. Under the passive policy, government spending is kept at the steady state level and the tax policy does not distort neither

---

[5]I defined $\hat{\tau}_t^v \equiv -\log \frac{(1-\tau_t^v)}{(1-\overline{\tau}^v)}$, $\hat{\tau}_t^l \equiv -\log \frac{(1-\tau_t^l)}{(1-\overline{\tau}^l)}$ and $\hat{\tau}_t^c \equiv \log \frac{(1+\tau_t^c)}{(1+\overline{\tau}^c)}$.

the supply condition (3.3.23) nor the intertemporal condition (3.3.21). Notice we require the sum of taxes to be zero, but taxes may be nonzero in order to satisfy the government budget constraint.

Assuming a passive fiscal policy and a given θ_t, we can solve for the terms of trade:

$$s_t = -\frac{\sigma + (\varphi(1 + \phi) - 1)\varsigma_c}{1 + (\varphi(1 + \phi) - 1)(\varsigma_c\xi + \varsigma_x\gamma)}\theta_t \tag{3.3.24}$$

and output

$$y_t = -\frac{\sigma\varsigma_x(\gamma - \xi + 1)}{1 + (\varphi(1 + \phi) - 1)(\varsigma_c\xi + \varsigma_x\gamma)}\theta_t \tag{3.3.25}$$

The terms of trade respond negatively to shocks to θ_t, but the output response is ambiguous. A sufficient condition for output to be negatively related to θ_t is $\gamma - \xi > 0$. This condition also guarantee net exports increase in response to a depreciation of the real exchange rate:[6]

$$nx_t = \varsigma_x\left[(\gamma - \xi)s_t - \theta_t\right] \tag{3.3.26}$$

This condition is a general equilibrium version of the traditional Marshall-Lerner condition. The usual condition holds in partial equilibrium and requires the sum of the export and import elasticities (in absolute value) to be greater than one. In general equilibrium, i.e., taking into account the impact of the real exchange rate on aggregate consumption, the condition boils down to $\gamma > \xi$. From now on I will maintain this assumption:

Assumption 2. *(Generalized Marshall-Lerner condition)* $\gamma > \xi$.

The reason for the effect on output to be, in principle, ambiguous is because a negative shock to θ_t represents a negative demand shock (since it decreases consumption for a given level of the terms of trade) and a positive supply shock (due to income effect on the labor supply). Fix a given import elasticity ξ. If the export elasticity γ is small, the terms of trade must fall by a lot (for a given level of y_t) after a decrease in θ_t. In this case, the equilibrium level of output will fall, as represented by point

---

[6]Since net exports can be positive or negative, I define $nx_t \equiv NX_t/\overline{Y}$, i.e., as a share of steady state output instead of log-deviations from steady state.

Figure 3-1: Effects of a shock to θ_t
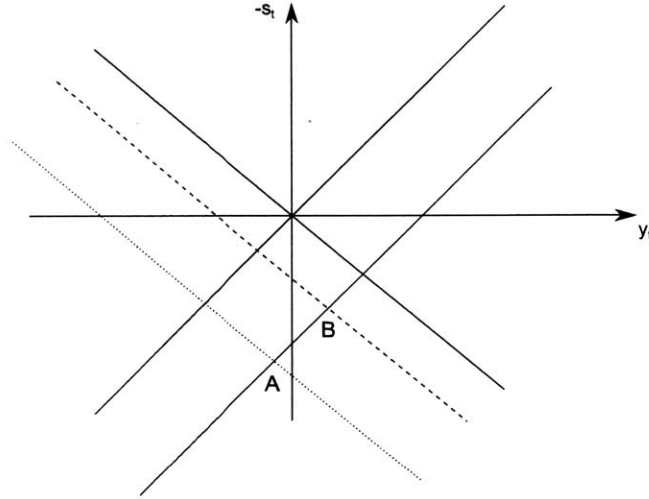
A at figure 1. If export elasticity is sufficiently high, then the shift in demand is small and the positive supply movement dominates (as in point B at the figure). The Marshall-Lerner condition guarantees the export elasticity is sufficiently high such that output indeed increases in equilibrium.

The expressions above determine output and terms of trade for a given θ_t. It remains to determine the evolution of θ_t. Consider the external solvency condition

$$0 = \int_0^\infty e^{-\rho t} \left[(\gamma - \xi) s_t - \theta_t \right] dt \Rightarrow \int_0^\infty e^{-\rho t} \theta_t dt = 0 \qquad (3.3.27)$$

where I used the fact that s_t is proportional to θ_t.

Under the passive fiscal policy, the consumption tax is constant, and θ_t evolves according to

$$\dot{\theta}_t = \frac{1}{\sigma} \psi_t \qquad (3.3.28)$$

Using the external solvency constraint, we can solve for θ_t:

$$\theta_t = \frac{1}{\sigma} \left[\int_0^t \psi_s ds - \int_0^\infty e^{-\rho s} \psi_s ds \right] \qquad (3.3.29)$$

Hence, in response to a positive risk premium shock, θ_t is negative in the short

140

run and it is positive in the long run. The idea is that the increase in interest rates caused by the risk premium shock will induce a steeper consumption profile. Since the present value of consumption must still be consistent with external solvency, consumption will fall in the short run and increase in the long run.

Proposition 13. *Suppose assumption 1 holds and $\psi_t \geq 0$ for every $t \geq 0$. Fiscal policy is passive. The short and long-run response of the economy is*

- **Equilibrium allocations:**

$$c_t = \kappa_c^P \theta_t; \qquad y_t = -\kappa_y^P \theta_t; \qquad nx_t = -\kappa_{nx}^P \theta_t; \qquad s_t = -\kappa_s^P \theta_t$$

where the coefficients κ_c^P, κ_y^P, κ_{nx}^P, and κ_s^P are all positive (see the appendix for the expressions).[7]

- **Short-run:**

$$c_0 < 0; \qquad y_0 > 0; \qquad nx_0 > 0; \qquad s_0 > 0$$

- **Long-run:**

$$\lim_{t \to \infty} c_t > 0; \qquad \lim_{t \to \infty} y_t < 0; \qquad \lim_{t \to \infty} nx_t < 0; \qquad \lim_{t \to \infty} s_t < 0$$

In response to a positive risk premium shock, consumption falls, leading to a depreciation of the real exchange rate. The drop in consumption combined with the depreciation of the terms of trade leads to an increase in net exports. Given assumption 1, output increases, since the increase in net exports more than compensate for the drop in consumption. In the long-run this pattern is reversed as the economy accumulates external assets, leading to an increase in consumption and drop in output.

Figure 2 shows the response to a exponentially decaying risk premium shock ($\psi_t = e^{-\rho_\psi t}\psi_0$). The calibration is based mostly on Gali and Monacelli (2008): $\phi = 3$, $\epsilon = 6$,

---

[7]The superscript P stands for "passive fiscal policy".

Figure 3-2: A Sudden Stop Episode: No Intervention and Optimal Policy
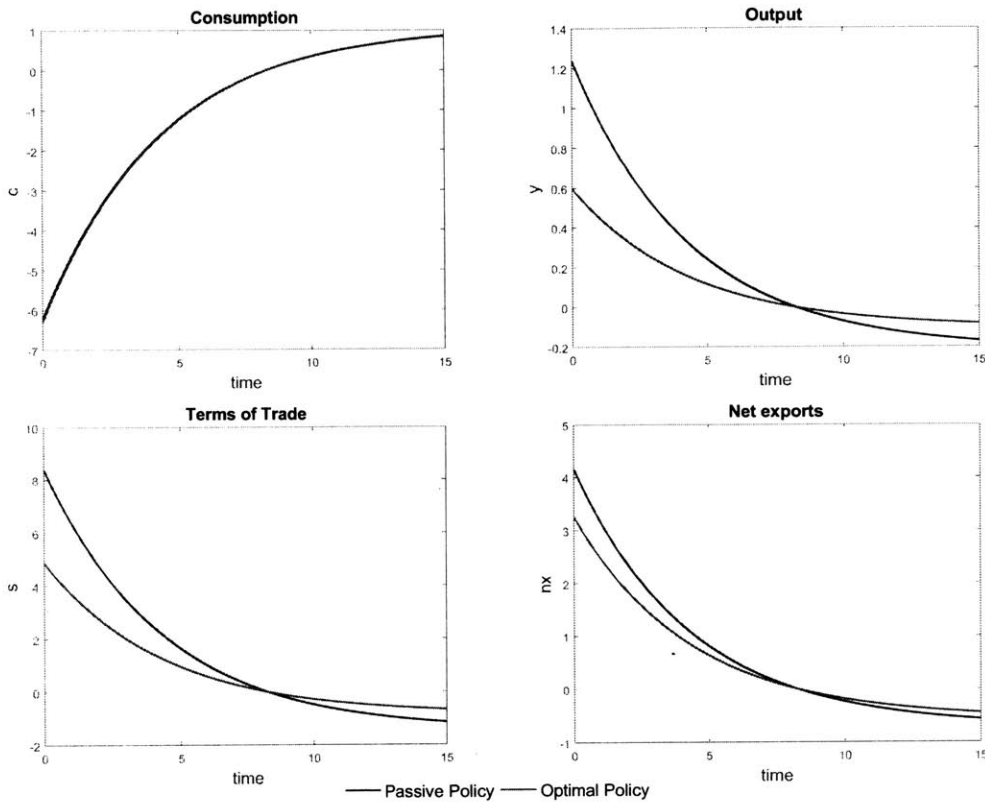
$\rho = 0.04$, $\sigma = 1.5$, $\gamma = 1.2$, and $\alpha = 0.35$. The half-life of risk premium shock is 3 years and the initial shock is 4%. Government debt is calibrated to 100% of GDP.

Even under flexible prices, the model already captures most of the salient features of a *sudden stop* episode. The main difference with the observed experience regards the behavior of output, where instead of a small boom it is typically observed a big recession (see Mendoza (2010)). However, as shown below, the behavior of output will be very different under sticky prices/wages.

3.4 The Linear Quadratic Problem

Let's consider now the problem of designing the optimal fiscal policy under flexible prices. A first order approximation of the Ramsey solution can be obtained by

142

studying a linear quadratic problem involving a second-order approximation of the objective and the linear constraints derived above.

The linear quadratic problem is

$$\min_{[\theta_t, s_t, g_t, y_t, c_t]} \left\{ \frac{1}{2} \int_0^\infty e^{-\rho t} \left[\varrho_c c_t^2 + \varrho_g g_t^2 + \varrho_y y_t^2 + \varrho_s (s_t + \varrho_{s\Psi} \hat{\Psi}_t)^2 + \varrho_\theta \left(\theta_t + \varrho_{\theta s} s_t - \rho_{\theta \Psi} \hat{\Psi}_t \right)^2 \right] dt \right\}$$

subject to

$$c_t = \theta_t + (\xi - \alpha) s_t \tag{3.4.1}$$

$$y_t = \varsigma_c \theta_t + \varsigma_g g_t + (\varsigma_c \xi + \varsigma_x \gamma) s_t \tag{3.4.2}$$

$$0 = \int_0^\infty e^{-\rho t} \left[(\gamma - \xi) s_t - \theta_t \right] dt \tag{3.4.3}$$

where the coefficients are defined in the appendix.

The first three terms on the loss function represent what someone would expect: deviations from the steady state level of consumption, government spending and output. The last two terms captures the fact that it is optimal to allow the terms of trade and the Pareto weight θ_t to react to the risk premium shock. However, it is not feasible at the same time to keep consumption and output at the steady state level and allow the terms of trade to respond to the shock. The optimal policy will balance this two objectives.

Let's start by characterizing the behavior of the Pareto weight θ_t;

Proposition 14. *Suppose assumption 1 holds. The Pareto weight θ_t evolves according to*

$$\theta_t = \theta_0 + \frac{\kappa_\theta}{\sigma} \int_0^t \psi_s ds = \kappa_\theta \theta_t^P$$

where θ_t^P is the Pareto weight under passive fiscal policy and

$$\theta_0 = -\frac{\kappa_\theta}{\sigma} \int_0^\infty e^{-\rho t} \psi_t dt = \kappa_\theta \theta_0^P \tag{3.4.4}$$

where $\kappa_\theta \in (0, 1)$ (see the appendix for the expression).

143

As in the case of passive policy, θ_t initially jumps down and then increases over time and eventually becomes positive. However, movements on θ_t are attenuated. In the presence of a positive risk premium shock, θ_0 falls but by less than under a passive policy. This imply that consumption tax must increase in response to the risk premium shock:

$$\dot{\theta}_t = \frac{1}{\sigma}\left[\psi_t - \dot{\tau}_t^c\right] \Rightarrow \dot{\tau}_t^c = (1 - \kappa_\theta)\psi_t \tag{3.4.5}$$

However, the planner does not completely undo the risk premium shock. As indicated above, even though this is a feasible policy, it is not an optimal one. The reason is that indeed the cost of transferring (foreign) consumption from the future to the present increased for the country and it is optimal to allow the price system to reflect, at least in part, this increase in cost. As we will see below, in the presence of home bias the increase in interest rate does not fully capture the economic cost of increasing consumption, generating the need for intervention.[8]

The next proposition characterizes the optimal fiscal policy:

Proposition 15. *(Optimal fiscal policy) Suppose assumption 1 holds and the risk premium shock is positive ($\psi_t \geq 0, \forall t$). Optimal fiscal policy is given by*

- *Optimal government spending:*

$$g_t = -\left(\varphi(1 + \phi) - 1\right)y_t \tag{3.4.6}$$

- *Optimal Taxation:*

$$\hat{\tau}_t^v + \hat{\tau}_t^c = -\kappa_\tau \theta_t$$

$$\dot{\hat{\tau}}_t^c = \kappa_{\tau^c}\psi_t; \qquad \hat{\tau}_0^c > 0$$

$$\dot{\hat{\tau}}_t^v = -\left(\kappa_{\tau^c} + \kappa_\tau \frac{\kappa_\theta}{\sigma}\right)\psi_t; \qquad \hat{\tau}_0^v < 0$$

where $\kappa_{\tau^c} > 0$ and $\kappa_\tau > 0$ (see the appendix for the expressions).

---

[8]A similar phenomenon appears in the literature on optimal capital controls under sticky prices. See Farhi and Werning (2012) and Farhi and Werning (2013) for a discussion and a slightly different intuition.

144

The optimal policy involves countercyclical government spending. As discussed above, when output is low, the cost of providing public goods decreases inducing the planner to increase the share of government spending.

Optimal taxation reflects the behavior of the labor wedge and intertemporal wedge. A first order approximation of the wedges gives

$$\hat{\omega}_t^l = \hat{\tau}_t^v + \hat{\tau}_t^c; \qquad \omega_t^I = -\hat{\tau}_t^c \tag{3.4.7}$$

where $\hat{\omega}_t^l \equiv \log(1 + \omega_t^l) - \log(1 + \overline{\omega}^l)$.

Notice the labor wedge increases on impact and decreases over time. As we have seen, the size of the labor wedge depends on how important exports are relative to domestic consumption. As the consumption plummets, exports becomes relatively more important in the short sun and less so as domestic consumption recovers. Therefore, in order to avoid a big depreciation, the planner decide to increase taxes in the short run and slowly decrease it over time.

Since the labor wedge is decreasing over time, the intertemporal wedge is negative. Basically, periods of high labor wedge means domestic consumption is relatively cheap, being optimal to shift consumption to the present.

Therefore, the consumption tax reduces the real interest by effectively increasing inflation, what stimulates consumption by the usual intertemporal substitution channel.

The initial level of the consumption tax is chosen to guarantee the government budget constraint is satisfied. The optimal policy then requires an initial increase in the consumption tax and a decrease in the VAT tax. Consumption tax will then increase over time while the VAT tax will be decreasing. However, in contrast to the unconventional fiscal policy in closed economy proposed by Correia et al. (2013), the VAT is not simply offsetting the increase in consumption tax. The VAT tax must decrease *faster* in order to guarantee the right level of the terms of trade.

Let's consider now the impact of the optimal policy on the government finances. First, define *total lump-sum taxation* as the sum of the present value of lump-sum

tax (if available) plus the revenue of the initial consumption tax (which is effectively lump-sum).

The next proposition shows that the optimal policy actually improves the budget situation of the government, compared to the passive policy equilibrium.

Proposition 16.

- *(Revenue-generating optimal policy): The total lump-sum taxation necessary to satisfy the government intertemporal budget constraint is smaller under the optimal policy than under the passive policy.*

- *(Fiscal Consolidation): Suppose initial government debt $\overline{D}$ is positive and $\psi_t = e^{-\rho_\psi t}\psi_0 > 0$, for $\rho_\psi > 0$. Fix a steady state level of government spending. There exist d^* such that if $\frac{\overline{D}}{\overline{P}_H \overline{Y}} \leq d^*$, then d_t is decreasing over time under the optimal policy. If $\frac{\overline{D}}{\overline{P}_H \overline{Y}} > d^*$, then D_t is increasing over time under the optimal policy.[9]*

First, notice that lump-sum taxation is not necessary, since we can use the initial consumption tax to satisfy the government budget, or introduce new taxes to achieve this goal (as a profit tax or an income tax).

Figure 3 shows the evolution of taxes, spending and government debt under the assumption the initial consumption tax is chosen in order to satisfy the government budget. We see that the initial consumption tax is smaller under the optimal policy, reflecting the fact the present value of taxes necessary to close the budget is smaller in this case.

For this particular calibration, we see that government debt increases under passive and optimal policy, but it increases faster under the passive policy. As indicated in the proposition above, at lower levels of debt, it would be optimal to reduce the level of government debt, i.e., to perform a fiscal consolidation.

The next proposition characterizes the optimal allocation:

Proposition 17. *Suppose assumption 1 holds.*

---

[9]The threshold d^* may be infinite, i.e., government debt d_t is decreasing for any initial (positive) level of government debt.

146

Figure 3-3: Taxes, Spending and Debt: Passive and Optimal Policy
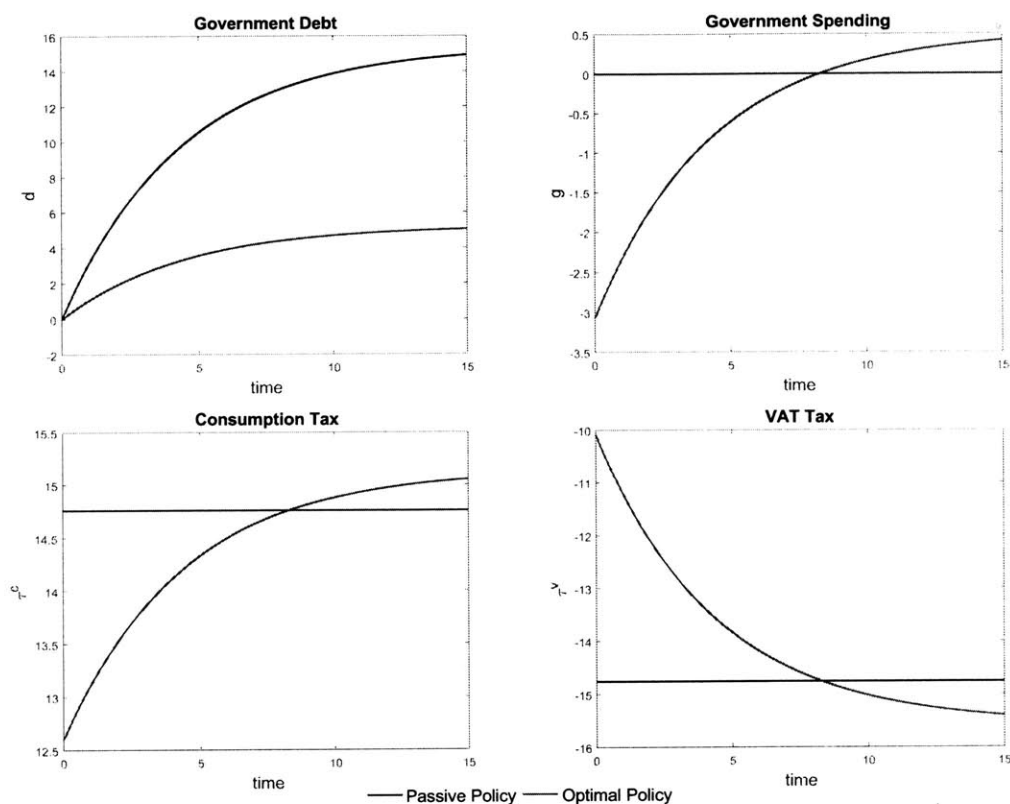
- *Optimal allocation:*

$$c_t = \kappa_c \theta_t; \qquad y_t = -\kappa_y \theta_t; \qquad nx_t = -\kappa_{nx} \theta_t; \qquad s_t = -\kappa_s \theta_t$$

where κ_c, κ_y, κ_{nx}, and κ_s are all positive constants.

- *Optimal policy vs. passive policy:*

$$\frac{s_t}{s_t^P} < 1; \qquad \frac{y_t}{y_t^P} < 1; \qquad \frac{nx_t}{nx_t^P} < 1; \qquad \frac{c_t}{c_t^P} \lessgtr 1$$

where the superscript P denotes the allocation under passive fiscal policy.

The optimal response to the shock is qualitatively similar to the one under the passive policy: consumption drops, the terms of trade depreciate, output and net ex-

ports increase. The main difference is that the optimal policy looks like an attenuated version of the passive policy allocation (see figure 2 for a comparison of the two). The main exception is consumption. Even though the drop in θ_t is attenuated, what goes in the direction of increasing consumption, the depreciation of the terms of trade is also reduced, what goes in the direction of decreasing consumption. The net effect depends on the relative strength of these two effects. In the simulation we see the net effect on consumption is indeed pretty small, and consumption under optimal fiscal policy is very close to consumption under the passive policy.

3.4.1 The Role of Openness

Let's consider now how the degree of openness of an economy interacts with the design of the fiscal policy. The next proposition characterizes both the limit of an economy who barely trades with the rest of the world ($\alpha \to 0$) and a fully open economy with no home bias ($\alpha \to 1$).

Proposition 18. *(Openness)*

- *Closed-economy limit:*

$$y_t = c_t = nx_t = \hat{\tau}_t^c + \hat{\tau}_t^v = 0; \qquad s_t = -\sigma\theta_t \qquad (3.4.8)$$

$$\dot{\theta}_t = \frac{1}{\sigma}\frac{\gamma+1}{\gamma+2}\psi_t; \qquad \dot{\hat{\tau}}_t^c = \frac{1}{\gamma+2}\psi_t; \qquad \dot{\hat{\tau}}_t^v = -\frac{1}{\gamma+2}\psi_t; \qquad (3.4.9)$$

- *Open-economy limit:*

$$c_t = \frac{\gamma}{\gamma-1}\theta_t; \qquad y_t = -\frac{\sigma(1-\varsigma_g)\gamma}{1+(\varphi(1+\phi)-1)(\varsigma_g+\gamma(1-\varsigma_g))}\theta_t; \qquad nx_t = -\left[\frac{\sigma(\gamma-1)}{1+\gamma\mathcal{G}}+1\right]\theta_t;$$

$$s_t = -\frac{\sigma}{1+\gamma\mathcal{G}}\theta_t; \qquad \dot{\theta}_t = \frac{1}{\sigma}\psi_t; \qquad \hat{\tau}_t^v + \hat{\tau}_t^c = \dot{\hat{\tau}}_t^c = \dot{\hat{\tau}}_t^v = 0 \qquad (3.4.10)$$

where $\mathcal{G} = \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}$.

Consider first the case of the closed economy limit. In this case, the economy is completely insulated from the external shock and it is optimal to keep consumption,

148

output, and net exports at the steady state level. Therefore, the optimal policy completely undo the effect of the risk premium shock.

Remember the role of the positive labor wedge was to keep the price of exports high in order to reduce the resource cost of generating a given export revenue. As the importance of exports vanish, so does the need to distort the terms of trade, leading to a zero labor wedge. Moreover, when imports are very small, the cost of increasing consumption is given entirely by the labor wedge. In this case there is a complete disconnect between the interest rate and the true economic cost of consumption, leading the planner to completely offset the risk premium shock.

Let's focus now on the open economy limit. Now both the labor wedge and the intertermporal are at their steady state levels. As we have seen, when the economy is fully open the optimal mark-up is constant and it is given by the usual monopoly rule. Therefore, taxes must add up to zero. Since the economy is fully open, the only way of increasing consumption is to increase borrowing (or reduce lending), so now the interest rate reflects the true cost of increasing consumption, and it is optimal not to offset the increase in borrowing costs.

3.5 Optimal Policy under Sticky Prices

We have considered so far the case of flexible prices. As seen above, the optimal allocation involves an immediate depreciation of the real exchange rate in response to an increase in the risk premium. Since the nominal exchange rate is fixed, this is achieved by a coordinated, simultaneous, discrete cut in prices. As in the classical argument of Friedman (1953), it may be hard to achieved in practice this coordinated immediate reduction in prices, and the actual process of adjustment may take a long time. In order to capture this process, I will assume that prices are sticky and reconsider how fiscal policy should be designed.

The next section consider the opposite extreme of flexible prices, i.e., fully rigid prices. The following section consider the intermediate case where prices are sticky, but not constant.

3.5.1 Fully Rigid Prices

The optimal policy can be obtained by solving the following linear quadratic problem:

$$\min_{[\theta_t, g_t, y_t]} \left\{ \frac{1}{2} \int_0^\infty e^{-\rho t} \left[\varrho_c \theta_t^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 + \varrho_\theta \left(\theta_t - \rho_{\theta \Psi} \hat{\Psi}_t \right)^2 \right] dt \right\} \qquad (3.5.1)$$

subject to

$$y_t = \varsigma_c \theta_t + \varsigma_g g_t \qquad (3.5.2)$$

$$0 = \int_0^\infty e^{-\rho t} \theta_t dt \qquad (3.5.3)$$

where I used the fact that $c_t = \theta_t$ and $s_t = 0$ for all $t \geq 0$.

Let's start by characterizing the optimal allocation:

Proposition 19.

- *Optimal allocation*

$$\theta_t = \frac{\kappa_\theta^R}{\sigma} \left[\int_0^t \psi_s ds - \int_0^\infty e^{-\rho s} \psi_s ds \right] \qquad (3.5.4)$$

$$y_t = \frac{\varsigma_c}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \theta_t; \qquad c_t = \theta_t; \qquad nx_t = -\varsigma_x \theta_t$$

where $\kappa_\theta^R \in (0, 1)$.

- *Optimal policy vs passive policy*

$$\frac{\theta_t}{\theta_t^P} < 1; \qquad \frac{y_t}{y_t^P} < 1; \qquad \frac{c_t}{c_t^P} < 1; \qquad \frac{nx_t}{nx_t^P} < 1 \qquad (3.5.5)$$

where the superscript P indicates the allocation under the passive fiscal policy.

As before, the increase in the real interest rate due to the risk premium shock causes consumption to drop and net exports to increase. However, in contrast with the flexible price case, output decreases, since there is no depreciation of the real exchange to compensate the drop in consumption.

150

The optimal policy is able to limit the extent of the recession and reduce the drop in consumption. As shown in the proposition below, both spending and tax policy act in the direction of smoothing the response to the risk premium shock.

Proposition 20. *(Optimal Fiscal Policy)*

- *Optimal government spending*

$$g_t = -(\varphi(1 + \phi) - 1)y_t \tag{3.5.6}$$

- *Optimal taxation*

$$\dot{\hat{\tau}}_t^c = (1 - \kappa_\theta^R)\psi_t; \qquad \hat{\tau}_0^c = -\sigma\theta_0 \tag{3.5.7}$$

- ***Revenue-generating policy:*** *The total lump-sum taxation necessary to satisfy the government intertemporal budget constraint is smaller under the optimal policy than under the passive policy.*

The expression for optimal government spending is the same as under flexible prices. The planner chooses not to deviate from the optimal provision of public goods and provide some extra stimulus (and reduce the extent of the recession, for instance). The reason is that the planner can always influence directly consumption by the choice of the consumption tax. Spending is entirely based on a static cost-benefit analysis.

The sum of the VAT and consumption tax is not determined anymore, since there is no pricing decision by the firms. For concreteness, let's assume $\hat{\tau}_t^c + \hat{\tau}_t^c = 0$. Consumption tax must be increasing in response to a positive risk premium shock, implying the VAT tax must be decreasing.

As in the case of flexible prices, optimal policy is revenue-generating, in the sense of the necessary level of taxes to satisfy the government budget is smaller under the optimal policy than under the passive policy.

By assuming prices are fully rigid, we abstracted from the behavior of the terms

of trade as well as the optimal level of taxes. The next section discuss how the terms of trade behave under sticky prices and the appropriate tax policy response.

3.5.2 Sticky Prices

The optimal policy can be obtained by solving the following linear quadratic problem:

$$\min_{[\theta_t, s_t, g_t, y_t, c_t, \pi_{H,t}]} \left\{ \frac{1}{2} \int_0^\infty e^{-\rho t} \left[\varrho_c c_t^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 + \varrho_s (s_t + \varrho_{s\Psi} \hat{\Psi}_t)^2 + \varrho_\theta \left(\theta_t + \varrho_{\theta s} s_t - \rho_{0\Psi} \hat{\Psi}_t \right)^2 + \frac{\epsilon}{\kappa} \pi_{H,t}^2 \right] dt \right\}$$
$$(3.5.8)$$

subject to

$$c_t = \theta_t + \frac{1 - \alpha}{\sigma} s_t \qquad (3.5.9)$$

$$y_t = \varsigma_c \theta_t + \varsigma_g g_t + (\varsigma_c \psi + \varsigma_x \gamma) s_t \qquad (3.5.10)$$

$$0 = \int_0^\infty e^{-\rho t} \left[(\gamma - \psi) s_t - \theta_t \right] dt \qquad (3.5.11)$$

$$\dot{s}_t = -\pi_{H,t} \qquad (3.5.12)$$

where $s_0 = 0$.

Our first result establishes that optimal government spending must be counter-cyclical:

Proposition 21. *The optimal level of government spending is given by*

$$g_t = - \left(\varphi(1 + \phi) - 1 \right) y_t$$

Notice the expression above is the same one obtained under flexible prices (see (3.6.2)). Even though it is optimal to increase government spending, there is no sense in which government spending should try to stimulate the economy. Government spending is chosen based in a pure static cost-benefit analysis.[10] The role of stimulating the economy and any dynamic consideration is left to the tax policy.

The next proposition characterizes the equilibrium allocation under sticky prices:

---

[10]See Werning (2011) for an analysis of "opportunistic" versus stimulus spending in the context of a closed economy liquidity trap model.

152

Proposition 22. *Suppose assumption 1 holds. The optimal allocation is given by*

$$\theta_t = \theta_0 - \kappa_{\theta s}^S s_t + \kappa_{\theta \Psi}^S \hat{\Psi}_t; \qquad \theta_0 < 0;$$

$$y_t = y_0 + \kappa_{ys}^S s_t + \kappa_{y\Psi}^S \hat{\Psi}_t; \qquad y_0 < 0;$$

$$nx_t = nx_0 + \kappa_{nxs}^S s_t - \kappa_{nx\Psi}^S \hat{\Psi}_t; nx_0 > 0;$$

$$c_t = c_0 + \kappa_{cs}^S s_t + \kappa_{c\Psi}^S \hat{\Psi}_t; \qquad c_0 < 0;$$

where all coefficients are positive, except for κ_{cs}^S whose sign is ambiguous (see appendix). The term $\hat{\Psi}_t$ is given by

$$\hat{\Psi}_t = \frac{1}{\sigma} \int_0^t \psi_s ds \qquad (3.5.13)$$

The proposition express the allocation in terms of the shock $\hat{\Psi}_t$ and the terms of trade s_t. The evolution of s_t is described in the appendix and it is given by a second-order differential equation, with $\hat{\Psi}_t$ as a forcing variable.

As in the flexible price case, consumption drops on impact and net exports increase due to the reduction in imports. The drop in consumption generates deflationary pressures, what further increases the real interest rate, amplifying the initial effect of the shock. In contrast to the outcome under flexible prices, output falls on impact. Hence, the economy enters into a recession in response to the risk premium shock. The reason output increases in the flexible price case is the strong depreciation of the real exchange rate and sharp increase in net exports. Given the sticky prices assumption, the terms of trade does not respond in the short run, causing the recession.

The simulation shows that the optimal policy manages to stabilize output and consumption reasonably well. Figure 4 shows the allocation under the optimal policy and compares to the allocation under a passive fiscal policy (as defined in the previous section). Instead of a major depression, the country faces a much milder recession. This is a result of the combination of the spending and tax policy. The countercyclical government spending directly stimulate the economy (regardless of the reason for the spending) and indirectly it helps to fight the deflation, reducing the real interest rate

and the drop in consumption.

The next proposition discuss the role of taxes:

Proposition 23. *(Taxes)*

- *Home-bias economy:*

$$\hat{\tau}_t^v + \hat{\tau}_t^c = (\hat{\tau}_0^v + \hat{\tau}_0^c) + \kappa_{\tau s}^S s_t + \kappa_{\tau \Psi}^S \hat{\Psi}_t; \qquad (3.5.14)$$

where the sign of $(\hat{\tau}_0^v + \hat{\tau}_0^c)$, $\kappa_{\tau s}^S$, and $\kappa_{\tau \Psi}^S$ are ambiguous.

- *Open-economy limit: As $\alpha \to 1$, we obtain*

$$\hat{\tau}_0^v + \hat{\tau}_0^c = \left(\frac{\gamma(1 - \varsigma_g)}{\epsilon} - 1 \right) \sigma \theta_0;$$

$$\kappa_{\tau s}^S = \left(\frac{\gamma(1 - \varsigma_g)}{\epsilon} - 1 \right) (1 + \gamma \mathcal{G})$$

where $\mathcal{G} = \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}$.

The proposition shows that, in general, whether the planner should subsidize or tax firms will depend on parameters. The fully open economy limit clarifies the role the export elasticity has in determining the level of taxes.

If the export elasticity is low, i.e., if the following condition is satisfied

$$\gamma < \frac{\epsilon}{1 - \varsigma_g} \qquad (3.5.15)$$

then taxes should increase in response to a increase in the risk premium.

If the condition is violated, the planner should subsidize firms and induce a depreciation of the real exchange rate. This result reflects the action of two opposing forces. Under sticky prices, the planner has an incentive to keep the terms of trade stable, since volatility of the real exchange rate becomes costly. The cost of price variability is captured by the parameter ϵ. On the other hand, output is inefficiently low, since the real exchange rate is too appreciated in the short run, and the more elastic exports are, the more output will respond to a depreciation. Notice that the

154

Figure 3-4: Equilibrium under sticky prices: Passive and Optimal Policy
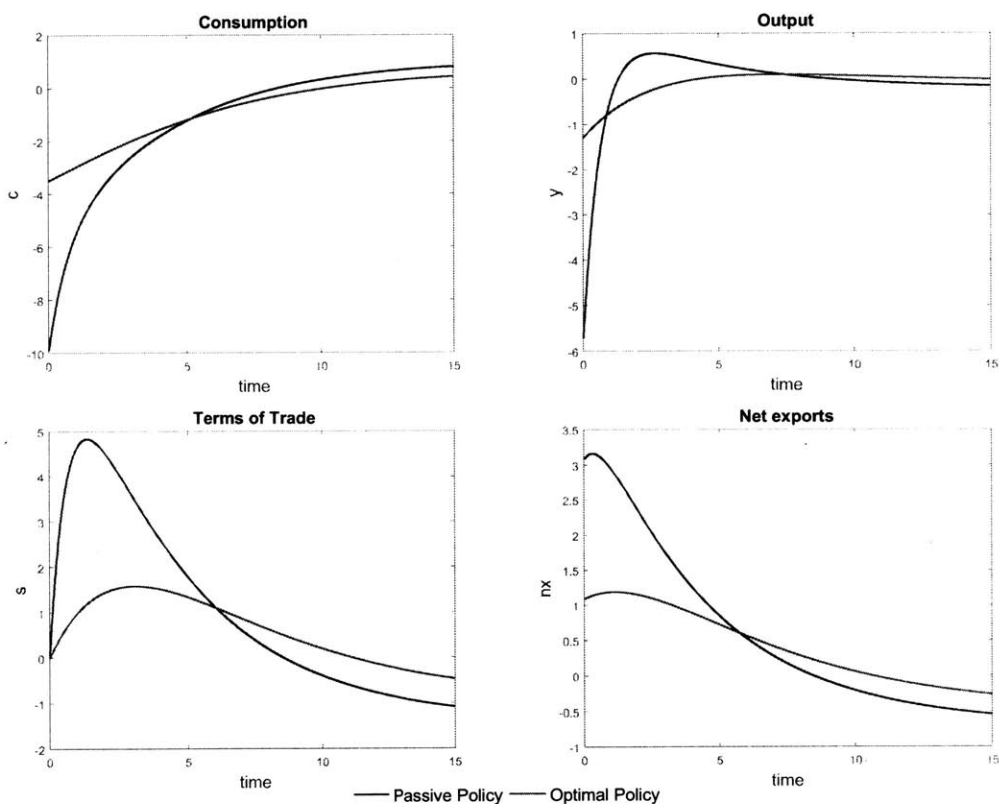
share of output that responds to the terms of trade is given by $1-\varsigma_g$, since government spending consists entirely of domestic goods. Condition above captures the two legs of the trade-off.

This imply that highly elastic countries should try to depreciate the real exchange rate through fiscal policy. However, if the export elasticity is low, then it is optimal to perform a *fiscal appreciation*, i.e., the planner should tax instead of subsidizing domestic firms.

Notice that if we had focused only on the Cole-Obstfeld case $(\psi = \gamma = 1))$, as it is typically done in the literature, we would conclude that it is never optimal to try to depreciate the real exchange rate, since this is only optimal for γ sufficiently greater than one.

Figure 5 shows the path of taxes, spending and government debt. Government

spending is a mirror image of the output path. Consumption taxes are increasing over time while the VAT is decreasing. The initial response of the VAT and consumption tax are such that the sum of taxes is positive, i.e., fiscal policy will act in the direction of appreciating the terms of trade. Noticed this will tend to reduce deflation, leading to a reduction in the real interest rate limiting the drop in consumption.

Figure 3-5: Taxes, Spending and Debt: Passive and Optimal Policy
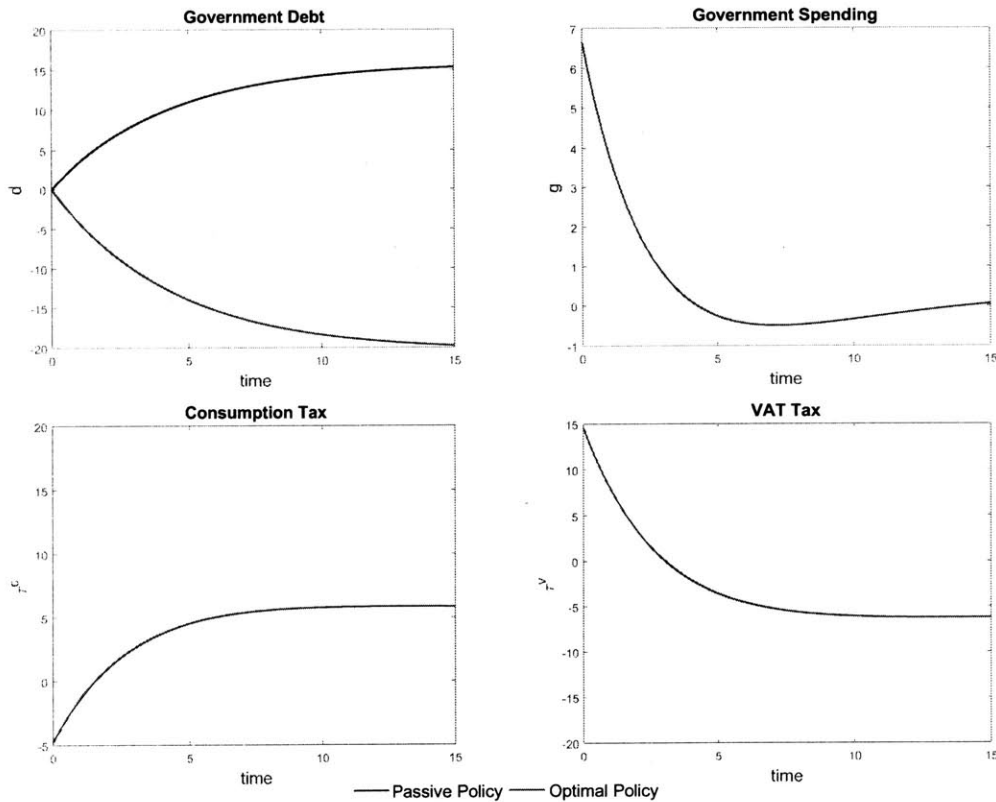
Paralleling the approach under flexible prices, I used the initial tax on consumption to satisfy the government under both the passive and optimal fiscal policy. The initial level of consumption tax is smaller under the optimal policy, indicating the optimal fiscal policy is also revenue-generating under sticky prices.

The simulation also shows that it is optimal to perform a fiscal consolidation. Government debt actually decreases under the optimal policy. As in the flexible price case, this depend on the initial level of debt. For higher levels of the initial debt,

156

government debt actually increases, but less than under the passive policy. Moreover, given the same initial level of debt, it is optimal to perform a fiscal consolidation under sticky prices, but not under flexible prices.

3.6 Downward Nominal Wage Rigidities

So far I have considered the impact of nominal rigidities coming from price stickiness, but there is an extensive empirical literature documenting *wage stickiness*, in particular for downward wage movements.[11]

I will now assume that prices are completely flexible, but nominal wages are downward rigid. This imply that in period where this constraint is binding, the nominal wage will be "'too high", meaning the labor supply will exceed the labor demand at the current wage. Therefore, downward nominal wage rigidity will cause unemployment in this economy.

Let's consider now the effect of a positive risk premium shock. For simplicity, let's assume $\gamma = \xi$ and that fiscal policy is passive. If wages are completely flexible, the nominal wage satisfy

$$w_t = \sigma \theta_t + \varphi \phi y_t = \overline{\kappa}_w^P \theta_t^P \tag{3.6.1}$$

where $\overline{\kappa}_w^P > 0$ and θ_t satisfy (3.3.29).

This imply that wages would go down on impact and stay negative while $\theta_t < 0$. If wages are downward rigid, this cannot happen and we would have unemployment instead. The next proposition characterizes what happens in the case of passive fiscal policy.

Proposition 24. *Suppose $\xi = \gamma$, fiscal policy is passive, and the risk premium shock is positive ($\psi_t \geq 0 \ \forall t$). Define T_0^P as the time period such that $\theta_t^P = 0$.*

---

[11]For empirical evidence regarding the European case, see Babecký et al. (2010). See Schmitt-Grohé and Uribe (2011) for a discussion of the impact of downward nominal rigidities under fixed exchange rates in a related model.

- **Unemployment period**: for $t \leq T_0^P$, nominal wage satisfy $w_t = 0$ and

$$c_t = \overline{\kappa}_c^P \theta_t^P; \qquad y_t = \overline{\kappa}_y^P \theta_t^P; \qquad nx_t = -\overline{\kappa}_{nx}^P \theta_t^P; \qquad s_t = -\overline{\kappa}_s^P \theta_t^P; \qquad u_t = -\overline{\kappa}_u^P \theta_t^P$$

where $\overline{\kappa}_c^P$, $\overline{\kappa}_y^P$, $\overline{\kappa}_{nx}^P$, $\overline{\kappa}_s^P$, $\overline{\kappa}_u^P$ and are all positive constants.

- **Full employment period**: for $t > T_0^P$, unemployment satisfy $u_t = 0$ and

$$c_t = \kappa_c^P \theta_t; \qquad y_t = -\kappa_y^P \theta_t; \qquad nx_t = -\kappa_{nx}^P \theta_t; \qquad s_t = -\kappa_s^P \theta_t; \qquad w_t = \kappa_w^P \theta_t$$

where κ_c, κ_y, κ_{nx}, and κ_s are all positive constants.

A key difference compared to the case of flexible prices and wages is that output responds positively to θ_t in the unemployment period. Hence, under downward nominal wage rigidity, a positive risk premium shock generates a recession. Moreover, on impact unemployment shoots up and slowly decreases as consumption and output recovers. As in the case of flexible prices, the terms of trade depreciate mitigating in part the effect of the shock.

How should fiscal policy respond to the interest rate shock under downward wage rigidity? It turns out the optimal policy can do as well as in the case of flexible prices, but the fiscal policy required to implement the optimal is different. In particular, labor income tax (or payroll tax) becomes important.

Proposition 25. *(Optimal fiscal policy) Suppose $\xi = \gamma$ and the risk premium shock is positive ($\psi_t \geq 0 \ \forall t$). Define T_0^P as the time period such that $\theta_t^P = 0$.*

- *Optimal government spending:*

$$g_t = -\left(\varphi(1+\phi) - 1\right) y_t \tag{3.6.2}$$

158

- *Optimal Taxation:*

$$\hat{\tau}_t^v + \hat{\tau}_t^c + \hat{\tau}_t^l = -\frac{\sigma\alpha}{\xi}\theta_t$$

$$\hat{\tau}_t^v = \sigma\frac{\xi - \alpha}{\xi}\theta_t$$

$$\dot{\hat{\tau}}_t^c = \kappa_{\tau^c}\psi_t$$

$$\dot{\hat{\tau}}_t^l = -\psi_t$$

where $\kappa_{\tau^c} > 0$ *(see the appendix for the expression).*

- *Revenue-generating: optimal fiscal policy is revenue-generating.*

Again we see that it is not optimal to deviate from the optimal provision of public goods, just as in the case of stick prices. The optimal allocation involves no unemployment. The VAT tax is chosen in order to guarantee the downward constraint on wages does not bind. Hence, the government provides a reduction at the VAT tax on impact in order to increase demand for labor and, consequently, nominal wages. The total sum of taxes is chosen in order to guarantee the terms of trade will not depreciate beyond the optimal level. Consumption tax is increases over time in order to partially undo the effects of the interest rate shock on consumption. The labor income tax adjust in order to guarantee the planner can simultaneously control the terms of trade, affect the real interest rate the consumer faces, and the constraint on wages is not violated.

As in the case of flexible and rigid prices, optimal policy is revenue-generating. This means the amount of taxes necessary to guarantee the government budget constraint is satisfied is smaller under the optimal policy. Hence, the optimal policy does not impose an additional burden on government finances, it actually does the opposite by reducing the financing needs of the government.

3.7 Conclusion

How should a country respond to a sudden stop? In this project I tried to answer this question in the context of a member currency union, where the fiscal policy is the main line of defense.

The first lesson is that government spending should be countercyclical, since the cost of providing public goods falls in a recession. This prediction holds regardless of the degree of openness of the economy or the degree of price flexibility. Therefore, spending should be performed to the extent it is useful for the society, any attempt to stimulate the economy occurs through taxation.

Optimal tax policy depends on the degree of price flexibility. Under flexible prices, consumption tax increases over time in order to reduce the real interest rate, and the sum of the VAT and consumption tax is positive on impact, what tends to appreciate the terms of trade. Under sticky prices, the response of taxes depend on how elastic exports are.

Both under flexible and sticky prices, we see that the present value of taxes necessary to satisfy the budget constraint of the government is smaller under the optimal fiscal policy. Moreover, depending on the initial level of debt, it may be optimal to perform a fiscal consolidation and reduce the size of government debt.

Optimal fiscal policy is then able of at the same time to stabilize the economy and limit the extent of the recession, as well as to control government finances and the size of the debt.

3.A Appendices

3.A.1 Derivations: Section 3

The Ramsey Problem

The Ramsey problem can be written as

$$
\max_{\{\Theta_t, S_t, Y_t, G_t\}} \left\{ \int_0^\infty e^{-\rho t} \left[\frac{1}{1-\sigma} \left(\Theta_t C_t^* S_t^{\xi-\alpha} \right)^{1-\sigma} + \chi \log G_t - \frac{1}{1+\phi} \left(\frac{Y_t}{A_{H,t}} \right)^{\varphi(1+\phi)} \right] dt \right\}
$$
$$(3.A.1)$$

subject to

$$
Y_t = \left[(1-\alpha)\Theta_t S_t^\xi + \alpha \Lambda_{H,t} S_t^\gamma \right] C_t^* + G_t \tag{3.A.2}
$$

$$
\frac{E_0}{P_0^* C_0^*} = \alpha \int_0^\infty e^{-\rho t} \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} \left[\Lambda_{H,t} S_t^{\gamma-1} - \Theta_t S_t^{\xi-1} \right] ds \tag{3.A.3}
$$

The first-order conditions are

$$
\chi G_t^{-1} = \lambda_t \tag{3.A.4}
$$

$$
\varphi \left(\frac{Y_t}{A_{H,t}} \right)^{\varphi(1+\phi)} \frac{1}{Y_t} = \lambda_t \tag{3.A.5}
$$

$$
\left(\Theta_t C_t^* S_t^{\xi-\alpha} \right)^{1-\sigma} - \lambda_t (1-\alpha)\Theta_t S_t^\xi C_t^* = \alpha \Gamma \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} S_t^{\xi-1}\Theta
$$

$$
(\xi - \alpha) \left(\Theta_t C_t^* S_t^{\xi-\alpha} \right)^{1-\sigma} - \lambda_t \left[\xi(1-\alpha)\Theta_t S_t^\xi + \gamma\alpha\Lambda_{H,t} S_t^\gamma \right] C_t^* =
$$
$$
-\alpha \Gamma \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} \left[(\gamma-1)\Lambda_{H,t} S_t^{\gamma-1} - (\xi-1)S_t^{\xi-1}\Theta_t \right]
$$

Proof of proposition 1

The optimality condition for Θ_t can be written as

$$
C_t^{-\sigma} = \lambda_t (1-\alpha)S_t^\alpha + \alpha \Gamma \Psi_t^{-\sigma} \left(\frac{C_t^*}{C_0^*} \right)^{1-\sigma} \frac{S_t^{\alpha-1}}{C_t^*} \tag{3.A.6}
$$

161

Combining the optimality condition for Θ_t and S_t, we get

$$\alpha\Gamma\Psi_t^{-\sigma}\left(\frac{C_t^*}{C_0^*}\right)^{1-\sigma}\frac{S_t^{\alpha-1}}{C_t^*} = \alpha\lambda_t\frac{\left[\gamma\Lambda_{H,t}S_t^\gamma + (1-\alpha)\Theta_t S_t^\xi\right]S_t^\alpha}{(\gamma-1)\Lambda_{H,t}S_t^\gamma + (1-\alpha)S_t^\xi\Theta_t} \tag{3.A.7}$$

Combining the last two conditions, we get

$$\frac{C_t^{-\sigma}S_t^{-\alpha}}{\lambda_t} = \frac{(\gamma-1+\alpha)\Lambda_{H,t}S_t^\gamma + (1-\alpha)S_t^\xi\Theta_t}{(\gamma-1)\Lambda_{H,t}S_t^\gamma + (1-\alpha)S_t^\xi\Theta_t} = 1 + \frac{\alpha\Lambda_{H,t}S_t^\gamma}{(\gamma-1)\Lambda_{H,t}S_t^\gamma + (1-\alpha)S_t^\xi\Theta_t} \tag{3.A.8}$$

Hence, we obtain the expression in the text for the labor wedge.

Let's consider now the intertemporal wedge, which is defined by

$$e^{\int_t^{t+s}\omega_z^I dz} = \frac{e^{-\rho t}\frac{C_t^{-\sigma}}{P_t}}{e^{-\rho(t+s)}\frac{C_{t+s}^{-\sigma}}{P_{t+s}}e^{\rho s + \int_t^{t+s}\psi_z dz}} \Rightarrow \omega_t^I = \sigma\frac{\dot{C}_t}{C_t} + (1-\alpha)\frac{\dot{P}_H}{P_H} - \psi_t = \sigma\frac{\dot{\Theta}_t}{\Theta_t} - \psi_t \tag{3.A.9}$$

where I am assuming C_t^* is constant.

Notice the labor wedge can be written as

$$1 + \omega_t^l = \frac{\Theta_t^{-\sigma}}{\lambda_t(C_t^*)^\sigma S_t} \Rightarrow \frac{\dot{\omega}_t^l}{1+\omega_t^l} = -\sigma\frac{\dot{\Theta}_t}{\Theta_t} + \psi_t - \left(\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{S}_t}{S_t} + \psi_t\right) \tag{3.A.10}$$

where, for convenience, I added and subtracted ψ_t.

The optimality condition for Θ_t can be rewritten as

$$\Theta_t^{-\sigma} = (1-\alpha)(C_t^*)^\sigma\lambda_t S_t + \alpha\Gamma\Psi_t^{-\sigma}(C_0^*)^{\sigma-1} \Rightarrow -\sigma\frac{\dot{\Theta}_t}{\Theta_t} + \psi_t = \frac{1-\alpha}{1+\omega_t^l}\left(\frac{\dot{\lambda}_t}{\lambda_t} + \frac{\dot{S}_t}{S_t} + \psi_t\right) \tag{3.A.11}$$

Combining the previous three conditions, we get

$$\omega_t^I = \frac{1-\alpha}{\omega_t^l + \alpha}\frac{\dot{\omega}_t^l}{1+\omega_t^l} \tag{3.A.12}$$

Stationary Solution

Consider a stationary solution for the Ramsey problem. Assume $\overline{E} = 0$, i.e., $\overline{\Theta} = \Lambda_H \overline{S}^{\gamma-\psi}$.

162

Combining conditions (3.A.4) and (3.A.5):

$$\frac{\overline{G}}{\overline{Y}} = \frac{\chi}{\varphi} \left(\frac{\overline{Y}}{\overline{A}_H} \right)^{-\varphi(1+\phi)} \tag{3.A.13}$$

rearranging

$$\chi = \varphi \overline{N}^{1+\phi} \varsigma_g \tag{3.A.14}$$

From the optimality condition for Θ_t, we get

$$\alpha \Gamma \Lambda_H \overline{S}^{\gamma-1} = \overline{\lambda}(1-\alpha)\Lambda_H \overline{C}^* \overline{S}^\gamma - \overline{C}^{1-\sigma} \tag{3.A.15}$$

Plugging the expression above into the optimality condition for S_t:

$$\overline{C}^{1-\sigma} = \frac{\gamma}{\gamma - \alpha} \overline{\lambda} \Lambda_H \overline{C}^* \overline{S}^\gamma \tag{3.A.16}$$

Consumption can be expressed as

$$\overline{C} = \Lambda_H \overline{C}^* \overline{S}^{\gamma-\alpha} \tag{3.A.17}$$

The terms of trade can be written as

$$\overline{S} = \left[\frac{\gamma - \alpha}{\gamma} (\Lambda_H \overline{C}^*)^{-\sigma} \frac{1}{\overline{\lambda}} \right]^{\frac{1}{\sigma(\gamma-\alpha)+\alpha}} \tag{3.A.18}$$

Using the condition for $\overline{\Theta}$ on the aggregate demand equation, we obtain

$$\overline{Y} - \overline{G} = \Lambda_H \overline{C}^* \overline{S}^\gamma \tag{3.A.19}$$

Combining the conditions above and the expression for $\overline{C}^{1-\sigma}$, we get

$$\overline{C}^{1-\sigma} = \frac{\gamma \varphi}{\gamma - \alpha} \overline{N}^{1+\phi} \left(1 - \varsigma_g \right) \tag{3.A.20}$$

This expression will be useful on the derivation of the quadratic approximation below.

163

Plugging the expression for $\overline{S}^\gamma$ into (3.A.19), we get

$$1 - \frac{\chi}{\varphi}\left(\frac{\overline{Y}}{\overline{A}_{II}}\right)^{-\varphi(1+\phi)} = \left[\frac{\gamma-\alpha}{\varphi\gamma}\left(\Lambda_H \overline{C}^*\right)^{\frac{\alpha(1-\sigma)}{\gamma}} \overline{A}_H^{\frac{(\gamma-\alpha)(1-\sigma)}{\gamma}}\right]^{\frac{\gamma}{\sigma(\gamma-\alpha)+\alpha}} \left(\frac{\overline{Y}}{\overline{A}_{II}}\right)^{-\left[\frac{\gamma(\varphi(1+\phi)-1)}{\sigma(\gamma-\alpha)+\alpha}+1\right]} \tag{3.A.21}$$

Given the level of $\overline{Y}$, we can determine the value of $\overline{S}$, $\overline{G}$, $\overline{N}$, and $\overline{C}$ using the conditions above. Conversely, given the steady state level of government spending, we can determine the level of output.

$$\overline{Y} = \frac{\overline{A}_H^{\frac{\gamma\varphi(1+\phi)}{\gamma(\varphi(1+\phi)-1)+\sigma(\gamma-\alpha)+\alpha}}}{(1-\varsigma_g)^{\frac{\sigma(\gamma-\alpha)+\sigma}{\gamma(\varphi(1+\phi)-1)+\sigma(\gamma-\alpha)+\alpha}}} \left[\frac{\gamma-\alpha}{\varphi\gamma}\left(\Lambda_H \overline{C}^*\right)^{\frac{\alpha(1-\sigma)}{\gamma}}\right]^{\frac{\gamma}{\gamma(\varphi(1+\phi)-1)+\sigma(\gamma-\alpha)+\alpha}} \tag{3.A.22}$$

In order to support the level of output above, taxes must satisfy the condition:

$$\frac{(1-\overline{\tau}^v)(1-\overline{\tau}^l)}{1+\overline{\tau}^c} = \frac{\varphi\epsilon}{\epsilon-1}\overline{N}^{1+\phi}\frac{\overline{C}^\sigma \overline{S}^\alpha}{\overline{Y}} = \frac{\varphi\epsilon}{\epsilon-1}\left(\frac{\overline{Y}}{\overline{A}_{II}}\right)^{\varphi(1+\phi)-1}\frac{(\Lambda_H\overline{C})^\sigma}{\overline{A}_{II}}\overline{S}^{\sigma(\gamma-\alpha)+\alpha} \tag{3.A.23}$$

Plugging the expression for $\overline{S}$, we get

$$\frac{(1-\overline{\tau}^v)(1-\overline{\tau}^l)}{1+\overline{\tau}^c} = \frac{\epsilon}{\epsilon-1}\frac{\gamma-\alpha}{\gamma} \tag{3.A.24}$$

The wage bill can be written as

$$\frac{\overline{WN}}{\overline{P_H Y}} = \frac{1}{1-\overline{\tau}^l}\frac{\overline{N}^{1+\phi}}{\overline{Y}}\overline{C}^\sigma\overline{S}^\alpha = (1-\overline{\tau}^v)\frac{\epsilon-1}{\varphi\epsilon} \tag{3.A.25}$$

he tax rate on labor income can be written

$$\overline{\tau}^l = 1 - \frac{1}{1-\overline{\tau}^v}\frac{\epsilon}{\epsilon-1}\frac{\gamma-\alpha}{\gamma} \tag{3.A.26}$$

The revenue from the labor tax is then given by

$$\overline{\tau}^l\frac{\overline{WN}}{\overline{P_H Y}} = (1-\overline{\tau}^v)\frac{\epsilon-1}{\varphi\epsilon} - \frac{\gamma-\alpha}{\varphi\gamma} \tag{3.A.27}$$

If we assume $\sigma = 1$, we can obtain explicit solutions. Output is given by

$$\overline{Y} = \left(\frac{\gamma - \alpha + \gamma\chi}{\varphi\gamma} \right)^{\frac{1}{\varphi(1+\phi)}} \overline{A}_H \qquad (3.A.28)$$

Plugging the condition above into the optimality condition for government spending:

$$\frac{\overline{G}}{\overline{Y}} = \frac{\gamma\chi}{\gamma - \alpha + \gamma\chi} \qquad (3.A.29)$$

The value of $\overline{S}$ is given by

$$\overline{S} = \left[\frac{\gamma - \alpha}{\gamma - \alpha + \gamma\chi} \left(\frac{\gamma - \alpha + \gamma\chi}{\varphi\gamma} \right)^{\frac{1}{\varphi(1+\phi)}} \frac{\overline{A}_H}{\Lambda_H \overline{C}^*} \right]^{\frac{1}{\gamma}} \qquad (3.A.30)$$

Using the supply condition, we can determine the level of taxes:

$$\frac{(1 - \overline{\tau}^v)(1 - \overline{\tau}^l)}{1 + \overline{\tau}^c} = \frac{\epsilon}{\epsilon - 1} \frac{\gamma - \alpha}{\gamma} \qquad (3.A.31)$$

Notice we require $\gamma > \alpha$ in order for the expressions above to be meaningful. For instance, when $\alpha = 1$ the planner acts as a typical monopolist, what requires an elasticity bigger than one to guarantee the existence of the solution to the monopoly problem.

Finally, consider the government budget constraint. Let's focus in a steady state where $\overline{T} = \hat{\tau}^c = 0$.

The government constraint is then given by

$$\frac{\overline{D}}{\overline{P_H Y}} = \left(\frac{\overline{\tau}^v - \varsigma_g}{\rho} \right) + \frac{1}{\rho} \left[\overline{\tau}^p \left(1 - \overline{\tau}^v - \frac{\overline{WN}}{\overline{P_H Y}} \right) + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}} \right] \qquad (3.A.32)$$

If we set the tax on profits equal to zero, we get

$$\frac{\rho \overline{D}}{\overline{P_H Y}} = \overline{\tau}^v \left(1 - \frac{\epsilon - 1}{\varphi\epsilon} \right) - \varsigma_g + \left(\frac{\epsilon - 1}{\varphi\epsilon} - \frac{\gamma - \alpha}{\varphi\gamma} \right) \qquad (3.A.33)$$

165

Proof of proposition 2

The expressions for y_t and s_t are presented in the text. It remains to determine consumption and net exports. Plugging the expression for s_t into the Backus-Smith condition, we obtain:

$$c_t = \frac{\alpha + (\varphi(1+\phi) - 1)(\varsigma_c \alpha + \varsigma_x \gamma)}{1 + (\varphi(1+\phi) - 1)(\varsigma_c \xi + \varsigma_x \gamma)} \theta_t \qquad (3.A.34)$$

Net exports (deflated by domestic CPI) are given by

$$NX_t = \frac{P_{H,t}(Y_t - G_t) - P_t C_t}{P_{H,t}} = \alpha \overline{C}^* \overline{S}^\xi e^{c_t^* + \xi s_t} \left[\Lambda_H \overline{S}^{\gamma - \xi} e^{\lambda_{H,t} + (\gamma - \xi)s_t} - \overline{\Theta} e^{\theta_t} \right] \quad (3.A.35)$$

Let $nx_t \equiv \frac{NX_t}{\overline{Y}}$. Net exports can be written as

$$nx_t = \varsigma_x \left[(\gamma - \psi)s_t - \theta_t \right] \qquad (3.A.36)$$

Using the expression for s_t, we get

$$nx_t = -\frac{1 + \sigma(\gamma - \xi) + (\varphi(1+\phi) - 1)\gamma(\varsigma_c + \varsigma_x)}{1 + (\varphi(1+\phi) - 1)(\varsigma_c \xi + \varsigma_x \gamma)} \theta_t \qquad (3.A.37)$$

assuming the remaining disturbances are equal to zero.

The short and long run behavior of the equilibrium objects is a consequence of the evolution of θ_t (given in (3.3.29)), where $\theta_0 < 0$ and $\lim_{t \to \infty} \theta_t > 0$.

3.A.2   Derivations: Section 4

An Approximate Welfare Criterion

Utility from consumption can be written as

$$\frac{C_t^{1-\sigma}}{1-\sigma} = \frac{\overline{C}^{1-\sigma}}{1-\sigma} e^{(1-\sigma)c_t} \approx \frac{\overline{C}^{1-\sigma}}{1-\sigma} + \varphi \overline{N}^{1+\phi}(\varsigma_c + \varsigma_x) \frac{\gamma}{\gamma - \alpha} \left[c_t + \frac{1-\sigma}{2} c_t^2 \right] \quad (3.A.38)$$

where I used the fact $\overline{C}^{1-\sigma} = \varphi \overline{N}^{1+\phi}(\varsigma_c + \varsigma_x) \frac{\gamma}{\gamma - \alpha}$.

The log-linear version of the Backus-Smith condition holds exactly:

$$c_t = \theta_t + c_t^* + (1 - \alpha)s_t \tag{3.A.39}$$

The utility from government spending can be written as

$$\chi \log G_t = \chi \log \overline{G} + \varphi \overline{N}^{1+\phi} \varsigma_g g_t \tag{3.A.40}$$

where I used the fact that $\chi = \varphi \overline{N}^{1+\phi} \varsigma_g$.

The disutility from labor is given by

$$\frac{N_t^{1+\phi}}{1+\phi} = \frac{\overline{N}^{1+\phi}}{1+\phi} e^{\varphi(1+\phi)(y_t - a_{H,t})} \approx \frac{\overline{N}^{1+\phi}}{1+\phi} + \varphi \overline{N}^{1+\phi} \left[(y_t - a_{H,t}) + \frac{\varphi(1+\phi)}{2}(y_t - a_{H,t})^2 \right] \tag{3.A.41}$$

The aggregate demand condition can be expressed as:

$$e^{y_t} = (1 - \varsigma_g) \exp \left(\log \frac{(1 - \alpha)\overline{\Theta S}^\xi e^{\theta_t + \xi s_t} + \alpha \Lambda_H \overline{S}^\gamma e^{\lambda_{H,t} + \gamma s_t}}{(1 - \alpha)\overline{\Theta S}^\xi + \alpha \Lambda_H \overline{S}^\gamma} + c_t^* \right) + \varsigma_g e^{g_t} \tag{3.A.42}$$

up to second-order, we get

$$y_t + \frac{1}{2}y_t^2 = \varsigma_c \theta_t + \varsigma_x \lambda_{H,t} + \varsigma_g g_t + (\varsigma_c + \varsigma_x)c_t^* + (\gamma \varsigma_x + \xi \varsigma_c) s_t + $$
$$\frac{1}{2} \left[\varsigma_c(\theta_t + c_t^* + \xi s_t)^2 + \varsigma_x(\lambda_{H,t} + c_t^* + \gamma s_t)^2 + \varsigma_g g_t^2 \right]$$

Let $u_t \equiv \frac{C_t^{1-\sigma}}{1-\sigma} + \chi \log G_t - \frac{1}{1+\phi} N_t^{1+\phi}$, $\overline{u} \equiv \frac{\overline{C}^{1-\sigma}}{1-\sigma} + \chi \log \overline{G} - \frac{1}{1+\phi} \overline{N}^{1+\phi}$, and $\hat{u}_t \equiv \frac{u_t - \overline{u}}{\varphi \overline{N}^{1+\phi}}$.
The term $\hat{u}_t$ can be written as

$$\hat{u}_t = \frac{\gamma(\varsigma_c + \varsigma_x)}{\gamma - \alpha} \left[c_t + \frac{1 - \sigma}{2} c_t^2 \right] + \varsigma_g g_t - \left[(y_t - a_{H,t}) + \frac{\varphi(1 + \phi)}{2}(y_t - a_{H,t})^2 \right]$$
$$= -\frac{1}{2} \left[\varsigma_c(\theta_t + c_t^* + \xi s_t)^2 + \varsigma_x(\lambda_{H,t} + c_t^* + \gamma s_t)^2 + \varsigma_g g_t^2 - y_t^2 + \varphi(1 + \phi)(y_t - a_{H,t})^2 \right]$$
$$- \frac{1}{2} \frac{\gamma(\sigma - 1)(\varsigma_c + \varsigma_x)}{\gamma - \alpha} c_t^2 + \frac{\gamma \varsigma_x + \alpha \varsigma_c}{\gamma - \alpha} \left[\theta_t + (\xi - \gamma)s_t \right]$$

where I am ignoring terms independent of policy.

We can use the external solvency constraint to eliminate the linear terms:

$$0 = \rho \int_0^\infty e^{-\int_0^t (\rho + \psi_s) ds} e^{(1-\sigma)c_t^* + (\xi-1)s_t} \overline{\Theta S}^{\xi-1} \left[e^{\lambda_{H,t} + (\gamma - \xi)s_t} - e^{\theta_t} \right] dt \qquad (3.A.43)$$

where I used the fact that $E_0 = 0$.

Taking a second order approximation, we get

$$\int_0^\infty e^{-\rho t} \left[\theta + (\xi - \gamma)s_t - \lambda_{H,t} \right] dt =$$

$$-\frac{1}{2} \int_0^\infty e^{-\rho t} \left[(\theta_t + (1 - \sigma)c_t^* + (\xi - 1)s_t - \sigma\hat{\Psi}_t)^2 - \left(\lambda_{H,t} + (\gamma - 1)s_t + (1 - \sigma)c_t^* - \sigma\hat{\Psi}_t \right)^2 \right] dt$$

Let $\mathbb{U} = \int e^{-\rho t} \hat{u}_t dt$ be our measure of welfare. This can be written as

$$\mathbb{U} = -\int_0^\infty \frac{e^{-\rho t}}{2} \left[\varrho_\theta (\theta_t + (1 - \sigma)c_t^* + (\xi - 1)s_t - \sigma\hat{\Psi}_t)^2 - \varrho_\theta \left(\lambda_{H,t} + (1 - \sigma)c_t^* + (\gamma - 1)s_t - \sigma\hat{\Psi}_t \right)^2 \right.$$

$$\left. + \varrho_c c_t^2 + \varsigma_c (\theta_t + c_t^* + \xi s_t)^2 + \varsigma_x (\lambda_{H,t} + c_t^* + \gamma s_t)^2 + \varsigma_g g_t^2 - y_t^2 + \varphi(1 + \phi)(y_t - a_{H,t})^2 \right] dt + t.i.p.$$

where $\varrho_\theta \equiv \frac{\gamma \varsigma_x + \alpha \varsigma_c}{\gamma - \alpha}$ and $\varrho_c \equiv \frac{\gamma(\sigma-1)}{\gamma - \alpha}(1 - \varsigma_g)$.

In the case there is only a risk premium shock, we get

$$\mathbb{U}_t = \varrho_\theta (\theta_t + (\xi - 1)s_t - \sigma\hat{\Psi}_t)^2 - \varrho_\theta \left((\gamma - 1)s_t - \sigma\hat{\Psi}_t \right)^2 + \varsigma_c (\theta_t + \xi s_t)^2 + \varsigma_x \gamma^2 s_t^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 + \varrho_c c_t^2$$

where $\varrho_y \equiv \varphi(1 + \phi) - 1$ and $\mathbb{U} = \int_0^\infty e^{-\rho t} \mathbb{U}_t dt$.[12]

We can simplify the expression above even further by combining the terms involving s_t. Define ϱ_s as

$$\varrho_s \equiv \varsigma_x \gamma^2 - \varrho_\theta(\gamma - 1)^2 = \frac{\alpha(1 - \varsigma_g)}{\gamma - \alpha} \left[\gamma^2 + (1 - 2\alpha)\gamma - (1 - \alpha) \right] \qquad (3.A.44)$$

·Utility at period t can then be written as

$$\mathbb{U}_t = \varrho_\theta (\theta_t + (\xi - 1)s_t - \sigma\hat{\Psi}_t)^2 + \varrho_s \left(s_t + \varrho_{s\Psi}\hat{\Psi}_t \right)^2 + \varsigma_c (\theta_t + \xi s_t)^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 + \varrho_c (\theta_t + (\xi - \alpha)s_t)^2$$

We can also rewrite the problem as

$$\mathbb{U}_t = \tilde{\varrho}_\theta (\theta_t + \varrho_{\theta s} s_t - \varrho_{\theta\Psi}\hat{\Psi}_t)^2 + \tilde{\varrho}_s \left(s_t + \tilde{\varrho}_{s\Psi}\hat{\Psi}_t \right)^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 + \varrho_c (\theta_t + (\xi - \alpha)s_t)^2$$

---

[12]In order to have a concave problem, we need to assume parameter as such $\varrho_s > 0$.

where

$$\tilde{\varrho}_\theta \equiv \varrho_\theta + \varsigma_c$$

$$\varrho_{\theta s} \equiv \xi - \frac{\varrho_\theta}{\varrho_\theta + \varsigma_c}$$

$$\varrho_{\theta \Psi} \equiv \frac{\varrho_\theta \sigma}{\varrho_\theta + \varsigma_c}$$

$$\tilde{\varrho}_s \equiv \varsigma_x \left[\frac{(\gamma - 1)(\gamma + 1 - \alpha) + (1 - \alpha)\gamma}{\gamma - \alpha} + (1 - \alpha)\frac{\gamma + 1 - \alpha}{\gamma} \right]$$

$$\tilde{\varrho}_{s\Psi} \equiv \frac{\varrho_\theta \sigma}{\tilde{\varrho}_s} \frac{\gamma(\gamma - 1) + (\gamma - \alpha)(1 - \alpha)}{\gamma}$$

This is the form presented in the text (where I omitted the tilde over the coefficients).

Linear-quadratic problem

The first-order conditions for the linear quadratic problem are

$$g_t = \lambda_t$$

$$(\varphi(1 + \phi) - 1) y_t = -\lambda_t$$

$$[\varrho_\theta + \varsigma_c + \varrho_c]\theta_t + [\varrho_\theta(\psi - 1) + \varsigma_c\psi + \varrho_c(\psi - \alpha)]s_t - \varrho_\theta\sigma\hat{\Psi}_t + \Gamma = \varsigma_c\lambda_t$$

$$[\varrho_\theta(\psi - 1) + \varsigma_c\psi + \varrho_c(\psi - \alpha)]\theta_t + [\varrho_\theta(\psi - 1)^2 + \varrho_s + \varsigma_c\psi^2 + \varrho_c(\psi - \alpha)^2]s_t$$

$$- (\varrho_\theta\sigma(\psi - 1) - \varrho_s\varrho_{s\Psi})\hat{\Psi}_t - (\gamma - \psi)\Gamma = (\varsigma_c\psi + \varsigma_x\gamma)\lambda_t$$

Combining the first two conditions, we get that government spending is countercyclical

$$g_t = -(\varphi(1 + \phi) - 1) y_t \tag{3.A.45}$$

Plugging the expression above into the aggregate demand condition

$$y_t = \frac{\varsigma_c\theta_t + (\varsigma_c\psi + \varsigma_x\gamma)s_t}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \tag{3.A.46}$$

169

The optimality condition for θ_t can be written as

$$\frac{\gamma(1-\varsigma_g)\sigma}{\gamma-\alpha}\theta_t + \varsigma_c s_t - \varrho_\theta\sigma\hat{\Psi}_t + \Gamma = \varsigma_c\lambda_t \qquad (3.A.47)$$

The optimality condition for s_t can be written as

$$\varsigma_c\theta_t + \left[\varrho_\theta(\psi-1)^2 + \varrho_s + \varsigma_c\psi^2 + \varrho_c(\psi-\alpha)^2\right]s_t + (\gamma-\psi)\varrho_\theta\sigma\hat{\Psi}_t - (\gamma-\psi)\Gamma = (\varsigma_c\psi + \varsigma_x\gamma)\lambda_t$$

Combining the optimality condition for θ_t and s_t, we get

$$\frac{\varsigma_c\left[(\gamma-\alpha)^2 + \alpha(\gamma-\xi)\right]}{(\gamma-\alpha)(\xi-\alpha)}\theta_t + \frac{1-\varsigma_g}{\gamma-\alpha}\left[\gamma(\gamma-\alpha) + \alpha(1-\alpha)(\gamma-\xi)\right]s_t = \gamma(\varsigma_c + \varsigma_x)\lambda_t \qquad (3.A.48)$$

using the expression for λ_t, we get

$$s_t = -\kappa_s\theta_t \qquad (3.A.49)$$

where

$$\kappa_s = \sigma\frac{(\gamma-\alpha)^2 + \alpha(\gamma-\xi) + \gamma(\gamma-\alpha)(\xi-\alpha)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{\gamma(\gamma-\alpha) + \alpha(1-\alpha)(\gamma-\xi) + \gamma(\gamma-\alpha)((1-\alpha)\xi + \alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}} \qquad (3.A.50)$$

A sufficient condition for the expression above to be positive is $\gamma \geq \xi$. The derivation above will now be used to show the results discussed in the text.

Proof of proposition 3

Consider the dynamics for θ_t. The optimality condition for θ_t can be written as

$$\omega_\theta\theta_t = \varrho_\theta\sigma\hat{\Psi}_t - \Gamma \qquad (3.A.51)$$

170

where

$$\omega_\theta \equiv \frac{\gamma(\gamma+1-\alpha)+(1-\alpha)\left[(\gamma-\alpha)+\alpha\frac{\gamma-\xi}{\gamma-\alpha}\right]+\gamma[(\gamma+1-\alpha)(\alpha\gamma+(1-\alpha)\xi)+(1-\alpha)(\xi-\alpha)]\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{[\alpha(1-\varsigma_g)\sigma]^{-1}\left[\gamma(\gamma-\alpha)+\alpha(1-\alpha)(\gamma-\xi)+\gamma(\gamma-\alpha)((1-\alpha)\xi+\alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}\right]}$$

$$(3.A.52)$$

Differentiating with respect to time:

$$\dot{\theta}_t = \frac{\kappa_\theta}{\sigma}\psi_t \qquad (3.A.53)$$

where $\kappa_\theta \equiv \varrho_\theta/\omega_\theta$, or more explicitly.

$$\kappa_\theta = \frac{\frac{\gamma+1-\alpha}{\gamma-\alpha}\left[\gamma(\gamma-\alpha)+\alpha(1-\alpha)(\gamma-\xi)+\gamma(\gamma-\alpha)((1-\alpha)\xi+\alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}\right]}{\gamma(\gamma+1-\alpha)+(1-\alpha)\left[(\gamma-\alpha)+\alpha\frac{\gamma-\xi}{\gamma-\alpha}\right]+\gamma[(\gamma+1-\alpha)(\alpha\gamma+(1-\alpha)\xi)+(1-\alpha)(\xi-\alpha)]\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}$$

$$(3.A.54)$$

The level of θ_t is given by

$$\theta_t = \theta_0 + \frac{\kappa_\theta}{\sigma}\int_0^t \psi_s ds \qquad (3.A.55)$$

The value of θ_0 is determined by the external solvency constraint:

$$0 = \int_0^\infty e^{-\rho t}\left[(\gamma-\xi)s_t - \theta_t\right]dt \Rightarrow 0 = \int_0^\infty e^{-\rho t}\theta_t dt \qquad (3.A.56)$$

Plugging the value for θ_t into the expression above, we get

$$\theta_0 = -\frac{\kappa_\theta}{\sigma}\int_0^\infty e^{-\rho t}\psi_t dt = \kappa_\theta\theta_0^P \qquad (3.A.57)$$

It only remains to show that $\kappa_\theta \in (0,1)$. The generalized Marshall-Lerner is a sufficient condition to guarantee the expression is positive. Computing $1 - \kappa_\theta$, we obtain

$$1 - \kappa_\theta \equiv \frac{(1-\alpha)[(1-\alpha)\gamma+\alpha\xi-\alpha]+\gamma(1-\alpha)(\xi-\alpha)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{\gamma(\gamma+1-\alpha)+(1-\alpha)\left[(\gamma-\alpha)+\alpha\frac{\gamma-\xi}{\gamma-\alpha}\right]+\gamma[(\gamma+1-\alpha)(\alpha\gamma+(1-\alpha)\xi)+(1-\alpha)(\xi-\alpha)]\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}$$

$$(3.A.58)$$

Again the condition $\gamma > \xi$ guarantees the coefficient is positive.

Proof of propositions 4 and 5

The optimality condition for government spending were derived above. Consider now the evolution of consumption taxes:

$$\dot{\theta}_t = \frac{1}{\sigma}\left[\psi - \hat{\tau}_t^c\right] \Rightarrow \dot{\hat{\tau}}_t^c = \psi_t - \sigma\dot{\theta}_t = (1 - \kappa_\theta)\psi_t = \kappa_{\tau^c}\psi_t \tag{3.A.59}$$

where $\kappa_{\tau^c} \equiv 1 - \kappa_\theta$.

Taxes are given by the supply condition (3.3.23):

$$(\hat{\tau}_t^v + \hat{\tau}_t^c) = (\kappa_s - \sigma)\theta_t - (\varphi(1 + \phi) - 1)y_t = -\kappa_\tau \theta_t$$

where I imposed $\hat{\tau}_t^l = 0$ and

$$\kappa_\tau \equiv \frac{\sigma\alpha\left[(\alpha\xi + (1 - \alpha)\gamma - \alpha) + \gamma(\xi - \alpha)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}\right]}{\gamma(\gamma - \alpha) + \alpha(1 - \alpha)(\gamma - \xi) + \gamma(\gamma - \alpha)((1 - \alpha)\xi + \alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}} \tag{3.A.60}$$

since $\xi > \alpha$ and $\gamma > \alpha$ (by assumption), the expression above is non-negative.

We can then determine the evolution of the VAT tax

$$\dot{\hat{\tau}}_t^v = -\left(\frac{\kappa_\tau\kappa_\theta}{\sigma} + \kappa_{\tau^c}\right)\psi_t \tag{3.A.61}$$

Consider now the government solvency constraint:

$$-\frac{(\sigma\theta_0 + \hat{\tau}_0^c)}{\overline{D}^{-1}\overline{P_H Y}} = \int_0^\infty e^{-\rho t}\left[\frac{\rho\overline{D}\varphi(1 + \phi)}{\overline{P_H Y}}y_t + (1 - \varsigma_g)(\hat{\tau}_t^v + \hat{\tau}_t^c + \hat{\tau}_t^l) - \varsigma_g(g_t - y_t) - \tilde{T}_t + \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}}\right)\hat{\tau}_t^l\right]dt \tag{3.A.62}$$

using the fact the present value of output and of the sum of taxes is equal to zero, we get

$$-(\sigma\theta_0 + \hat{\tau}_0^c)\frac{\overline{D}}{\overline{P_H Y}} = \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}}\right)\int_0^\infty e^{-\rho t}\hat{\tau}_t^l dt \tag{3.A.63}$$

If we assume a zero labor tax, initial consumption must satisfy

$$\hat{\tau}_0^c = -\sigma \theta_0 \tag{3.A.64}$$

Notice that if we were to use lump sum taxes to satisfy the budget, the present-value of the lump-sum taxes would be $-\sigma \theta_0 \frac{\overline{D}}{\overline{P_H Y}}$. A similar calculation would show that the same expression holds for the case of passive policy, but with θ_0^P instead of θ_0. Since $\theta_0 = \kappa_\theta \theta_0^P$, the required amount of taxes required to satisfy the budget under the optimal policy is smaller than under the passive policy.

The initial VAT is given by

$$\hat{\tau}_0^v = \sigma \left(1 - \frac{\kappa_\tau}{\sigma} \right) \theta_0 = \sigma \frac{(\gamma - \alpha)^2 + \alpha(\gamma - \psi) + \gamma \left[(\gamma - \alpha)(\alpha\gamma + (1-\alpha)\psi - \alpha) + \alpha(\gamma - \psi) \right] \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{\gamma(\gamma - \alpha) + \alpha(1-\alpha)(\gamma - \psi) + \gamma(\gamma - \alpha)((1-\alpha)\psi + \alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}} \theta_0 \tag{3.A.65}$$

where the coefficient multiplying θ_0 is positive.

The value of debt (in utility terms) at period t is given by

$$\hat{d}_t^g \frac{\overline{D}}{\overline{P_H Y}} = -\left[\left(\frac{\rho \overline{D}}{\overline{P_H Y}} + \varsigma_g \right) \varphi(1+\phi)\kappa_y + (1 - \varsigma_g)\kappa_\tau \right] \int_t^\infty e^{-\rho(s-t)} \theta_s ds \tag{3.A.66}$$

where I used the fact that $y_t = -\kappa_y \theta_t$ shown below.

The value of debt in monetary terms is

$$\hat{D}_t^g = (\sigma \theta_t + \hat{\tau}_t^c) + \hat{d}_t^g \tag{3.A.67}$$

The first term is given by

$$\sigma\theta_t + \hat{\tau}_t^c = \sigma(\theta_t - \theta_0) + \kappa_{\tau^c} \int_0^t \psi_s ds = (\sigma\kappa_\theta + \kappa_{\tau^c}) \int_0^t \psi_s ds = \int_0^t \psi_s ds = \frac{1 - e^{-\rho_\psi t}}{\rho_\psi} \psi_0 \tag{3.A.68}$$

where the last equality uses the assumption $\psi_t = e^{-\rho_\psi t} \psi_0$.

Similarly, using the decaying risk premium shock, we get

$$\int_t^\infty e^{-\rho(s-t)} \theta_s ds = \frac{\theta_t}{\rho} + \kappa_\theta \frac{\int_t^\infty e^{-\rho(s-t)} \psi_s ds}{\rho} = \kappa_\theta \frac{1 - e^{-\rho_\psi t}}{\rho_\psi(\rho + \rho_\psi)} \psi_0 \tag{3.A.69}$$

Combining the expressions above, we get

$$\hat{D}_t^g = \frac{1 - e^{-\rho_\psi t}}{\rho_\psi} \psi_0 \left[1 - \left(\frac{\overline{D}}{\overline{P}_H \overline{Y}} \right)^{-1} \left[\left(\frac{\rho \overline{D}}{\overline{P}_H \overline{Y}} + \varsigma_g \right) \varphi(1 + \phi)\kappa_y + (1 - \varsigma_g)\kappa_\tau \right] \frac{\kappa_\theta}{\rho + \rho_\psi} \right] \tag{3.A.70}$$

rearranging

$$\hat{D}_t^g = \frac{1 - e^{-\rho_\psi t}}{\rho_\psi} \psi_0 \left(\frac{\overline{D}}{\overline{P}_H \overline{Y}} \right)^{-1} \left[\frac{\overline{D}}{\overline{P}_H \overline{Y}} \left(1 - \frac{\rho}{\rho + \rho_\psi} \varphi(1 + \phi)\kappa_y \kappa_\theta \right) - [\varsigma_g \varphi(1 + \phi)\kappa_y + (1 - \varsigma_g)\kappa_\tau] \frac{\kappa_\theta}{\rho + \rho_\psi} \right] \tag{3.A.71}$$

If the term multiplying $\frac{\overline{D}}{\overline{P}_H \overline{Y}}$ is negative, then government debt is necessarily decreasing over time. If the term multiplying $\frac{\overline{D}}{\overline{P}_H \overline{Y}}$ is positive, there is a threshold $d^* > 0$ such that government debt is increasing for $\frac{\overline{D}}{\overline{P}_H \overline{Y}} > d^*$ and decreasing otherwise.

Proof of proposition 6

Output is given by

$$y_t = -k_y \theta_t \tag{3.A.72}$$

where

$$\kappa_y \equiv \frac{(\gamma - \psi)\alpha\sigma\gamma(\gamma + 1 - \alpha)(1 - \varsigma_g)(1 + (\varphi(1 + \phi) - 1)\varsigma_g)^{-1}}{\gamma(\gamma - \alpha) + \alpha(1 - \alpha)(\gamma - \psi) + \gamma(\gamma - \alpha)((1 - \alpha)\xi + \alpha\gamma)\frac{(\varphi(1 + \phi) - 1)(1 - \varsigma_g)}{1 + (\varphi(1 + \phi) - 1)\varsigma_g}} \tag{3.A.73}$$

Therefore, output will respond negatively to θ_t if the (generalized) Marshall-Lerner condition is satisfied ($\gamma > \xi$). In the special case $\xi = \gamma$, which encompass the Cole-Obstfeld case, output should be zero.

Consumption is given by

$$c_t = \theta_t + (\xi - \alpha)s_t = (1 - (\xi - \alpha)\kappa_s)\theta_t = \kappa_c\theta_t \tag{3.A.74}$$

where

$$\kappa_c = \frac{\alpha\gamma(\gamma + 1 - \alpha)\left[1 + \gamma \frac{(\varphi(1 + \phi) - 1)(1 - \varsigma_g)}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \right]}{\gamma(\gamma - \alpha) + \alpha(1 - \alpha)(\gamma - \xi) + \gamma(\gamma - \alpha)((1 - \alpha)\xi + \alpha\gamma)\frac{(\varphi(1 + \phi) - 1)(1 - \varsigma_g)}{1 + (\varphi(1 + \phi) - 1)\varsigma_g}} \tag{3.A.75}$$

174

Therefore, consumption responds positively to θ_t. Consider now the behavior of net exports:

$$nx_t = (\gamma - \xi)s_t - \theta_t = - \left[(\gamma - \xi)\kappa_s + 1\right]\theta_t = -\kappa_{nx}\theta_t \qquad (3.A.76)$$

where

$$\kappa_{nx} \equiv \frac{\gamma(\gamma-\alpha)[\alpha\xi+(1-\alpha)\gamma-\alpha]+\gamma(\gamma-\alpha)\left[(1-\alpha)(\gamma-\xi)((1-\alpha)\xi+\alpha\gamma)+(\xi-\alpha)^2\right]\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{(\xi-\alpha)\left[\gamma(\gamma-\alpha)+\alpha(1-\alpha)(\gamma-\xi)+\gamma(\gamma-\alpha)((1-\alpha)\xi+\alpha\gamma)\frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}\right]}$$

$$(3.A.77)$$

3.A.3 Derivations: Section 5

Fully Rigid Prices

The optimal policy can be obtained by solving the following linear quadratic problem:

$$\min_{[\theta_t, g_t, y_t]} \left\{ \frac{1}{2} \int_0^\infty e^{-\rho t} \left[\varrho_\theta(\theta_t - \sigma\hat{\Psi}_t)^2 + (\varsigma_c + \varrho_c)\theta_t^2 + \varsigma_g g_t^2 + \varrho_y y_t^2 \right] dt \right\} \qquad (3.A.78)$$

subject to

$$y_t = \varsigma_c \theta_t + \varsigma_g g_t \qquad (3.A.79)$$

$$0 = \int_0^\infty e^{-\rho t}\theta_t dt \qquad (3.A.80)$$

The first-order conditions are

$$g_t = \lambda_t$$

$$-(\varphi(1+\phi)-1)y_t = \lambda_t$$

$$(\varrho_c + \varsigma_c)\theta_t + \varrho_\theta\left(\theta_t - \sigma\hat{\Psi}_t\right) = \varsigma_c\lambda_t + \Gamma$$

This imply the usual condition for government spending:

$$g_t = -(\varphi(1+\phi)-1)y_t \qquad (3.A.81)$$

175

Output is then given by

$$y_t = \frac{\varsigma_c}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \theta_t \tag{3.A.82}$$

The Lagrange multiplier is then given by

$$\lambda_t = -\frac{(\varphi(1 + \phi) - 1)\varsigma_c}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \theta_t \tag{3.A.83}$$

Differentiating the condition for θ_t, we get

$$\dot{\theta}_t = \frac{\alpha(\gamma + 1 - \alpha)}{\gamma\sigma + (\gamma - \alpha)(1 - \alpha)^2 \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}} \psi_t \equiv \kappa_\theta^R \psi_t \tag{3.A.84}$$

Plugging into the external solvency condition,

$$\theta_0 = -\kappa_\theta^R \int_0^\infty e^{-\rho t} \psi_t dt \tag{3.A.85}$$

Therefore, θ_t is given by

$$\theta_t = \kappa_\theta^R \left[\int_0^t \psi_s ds - \int_0^\infty e^{-\rho s} \psi_s ds \right] \tag{3.A.86}$$

The government debt expression is

$$-(\sigma\theta_0 + \hat{\tau}_0^c)\frac{\overline{D}}{\overline{P_H Y}} = \int_0^\infty e^{-\rho t} \left[-\sigma c_t \left[\overline{\tau}^v - \varsigma_g\right] + (\overline{\tau}^v - \varsigma_g)y_t - \varsigma_g(g_t - y_t) + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}}(\varphi(1 + \phi)y_t) \right] dt$$

Since the present value of θ_t is equal to zero (and output is proportional to θ_t), we conclude that initial consumption tax satisfy:

$$\hat{\tau}_0^c = -\sigma\theta_0 \tag{3.A.87}$$

The evolution of the consumption tax is given by

$$\dot{\theta}_t = \frac{1}{\sigma}\left[\psi_t - \dot{\hat{\tau}}_t^c\right] \Rightarrow \dot{\hat{\tau}}_t^c = \left(1 - \sigma\kappa_\theta^R\right)\psi_t \equiv \kappa_{\tau^c}^R \psi_t \tag{3.A.88}$$

where

$$\kappa_{\tau^c}^R \equiv 1 - \sigma \frac{\alpha(\gamma + 1 - \alpha)}{\gamma\sigma + (\gamma - \alpha)(1 - \alpha)^2 \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}} = \frac{\sigma(\gamma - \alpha)(1 - \alpha) + (\gamma - \alpha)(1 - \alpha)^2 \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}{\gamma\sigma + (\gamma - \alpha)(1 - \alpha)^2 \frac{(\varphi(1+\phi)-1)(1-\varsigma_g)}{1+(\varphi(1+\phi)-1)\varsigma_g}}$$

$$(3.A.89)$$

The VAT tax is given by

$$\hat{\tau}_t^v = -\hat{\tau}_t^c \tag{3.A.90}$$

The government debt at period t is given by

$$\hat{d}_t^g \frac{\overline{D}}{\overline{P_H Y}} = \left[-\sigma \left[\overline{\tau}^v - \varsigma_g \right] + \left[(\overline{\tau}^v - \varsigma_g) + \left(\varsigma_g + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}} \right) \varphi(1 + \phi) \right] \frac{\varsigma_c}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \right] \int_t^\infty e^{-\rho(s-t)} \theta_s ds$$

$$(3.A.91)$$

where

$$\int_t^\infty e^{-\rho(s-t)} \theta_s ds = \frac{\theta_t}{\rho} + \frac{\kappa_\theta^R}{\rho} \int_t^\infty e^{-\rho(s-t)} \psi_s ds = \frac{\kappa_\theta^R}{\rho_\psi} \frac{1 - e^{-\rho_\psi t}}{\rho + \rho_\psi} \psi_0 \tag{3.A.92}$$

The level of debt is then given by

$$\hat{D}_t = \sigma\theta_t + \hat{\tau}_t^c + \omega_d \frac{\kappa_\theta^R}{\rho_\psi} \frac{1 - e^{-\rho_\psi t}}{\rho + \rho_\psi} \psi_0 = \left[1 + \omega_d \frac{\kappa_\theta^R}{\rho + \rho_\psi} \right] \frac{1 - e^{-\rho_\psi t}}{\rho_\psi} \psi_0 \tag{3.A.93}$$

where

$$\omega_d = \left(\frac{\overline{D}}{\overline{P_H Y}} \right)^{-1} \left[-\sigma \left[\overline{\tau}^v - \varsigma_g \right] + \left[(\overline{\tau}^v - \varsigma_g) + \left(\varsigma_g + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}} \right) \varphi(1 + \phi) \right] \frac{\varsigma_c}{1 + (\varphi(1 + \phi) - 1)\varsigma_g} \right]$$

$$(3.A.94)$$

Notice that

$$1 + \omega_d \frac{\kappa_\theta^R}{\rho + \rho_\psi} = \left(\frac{\overline{D}}{\overline{P_H Y}} \right)^{-1} \left[\frac{\overline{\tau}^v - \varsigma_g}{\rho} \left(1 - \frac{\sigma\kappa_\theta^R \rho}{\rho + \rho_\psi} \right) + \frac{\overline{\tau}^l}{\rho} \frac{\overline{WN}}{\overline{P_H Y}} + \frac{\varsigma_c \left[(\overline{\tau}^v - \varsigma_g) + \left(\varsigma_g + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}} \right) \varphi(1 + \phi) \right] \kappa_0^R}{(1 + (\varphi(1 + \phi) - 1)\varsigma_g)(\rho + \rho_\psi)} \right]$$

$$(3.A.95)$$

Consider now the allocation under the passive fiscal policy:

$$\theta_t^P = \frac{1}{\sigma} \left[\int_0^t \psi_s ds - \int_0^\infty e^{-\rho s} \psi_s ds \right] \tag{3.A.96}$$

177

and

$$y_t^P = \varsigma_c \theta_t^P; \qquad c_t^P = \theta_t^P; \qquad nx_t^P = -\varsigma_x \theta_t^P; \qquad \hat{\tau}_t^c = -\sigma\theta_0^P = -\hat{\tau}_t^v \qquad (3.A.97)$$

The government debt expression is

$$\hat{d}_t^g \frac{\overline{D}}{\overline{P_H Y}} = \left[-\sigma\left[\overline{\tau}^v - \varsigma_g\right] + \left[\overline{\tau}^v + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}}\varphi(1+\phi)\right]\varsigma_c \right] \int_t^\infty e^{-\rho(s-t)}\theta_s^P \, ds$$

where

$$\int_t^\infty e^{-\rho(s-t)}\theta_s^P \, ds = \frac{\theta_t^P}{\rho} + \frac{1}{\sigma\rho}\int_t^\infty e^{-\rho(s-t)}\psi_s \, ds = \frac{1}{\sigma\rho_\psi}\frac{1-e^{-\rho_\psi t}}{\rho+\rho_\psi}\psi_0 \qquad (3.A.98)$$

Government debt is then given by

$$\hat{D}_t^P = \sigma\theta_t + \hat{\tau}_t^c + \omega_d^P \frac{1}{\sigma\rho_\psi}\frac{1-e^{-\rho_\psi t}}{\rho+\rho_\psi}\psi_0 = \left[1 + \omega_d^P\frac{1}{\sigma(\rho+\rho_\psi)}\right]\frac{1-e^{-\rho_\psi t}}{\rho_\psi}\psi_0 \quad (3.A.99)$$

where

$$1+\omega_d^P\frac{1}{\sigma(\rho+\rho_\psi)} = \left(\frac{\overline{D}}{\overline{P_H Y}}\right)^{-1}\left[\frac{\overline{\tau}^v - \varsigma_g}{\rho}\frac{\rho_\psi}{\rho+\rho_\psi} + \frac{\overline{\tau}^l}{\rho}\frac{\overline{WN}}{\overline{P_H Y}} + \left[\overline{\tau}^v + \overline{\tau}^l\frac{\overline{WN}}{\overline{P_H Y}}\varphi(1+\phi)\right]\frac{\varsigma_c}{\sigma(\rho+\rho_\psi)}\right]$$
$$(3.A.100)$$

$$1+\omega_d\frac{\kappa_\theta^R}{\rho+\rho_\psi} = \left(\frac{\overline{D}}{\overline{P_H Y}}\right)^{-1}\left[\frac{\overline{\tau}^v - \varsigma_g}{\rho}\left(1 - \frac{\sigma\kappa_\theta^R\rho}{\rho+\rho_\psi}\right) + \frac{\overline{\tau}^l}{\rho}\frac{\overline{WN}}{\overline{P_H Y}} + \frac{\varsigma_c\left[(\overline{\tau}^v - \varsigma_g) + \left(\varsigma_g + \overline{\tau}^l\frac{\overline{WN}}{\overline{P_H Y}}\right)\varphi(1+\phi)\right]\kappa_\theta^R}{(1+(\varphi(1+\phi)-1)\varsigma_g)(\rho+\rho_\psi)}\right]$$
$$(3.A.101)$$

Government Budget

Let's consider now the government solvency condition:

$$\frac{C_t^{-\sigma}D_t^g}{P_t(1+\tau_t^c)} = \int_t^\infty e^{-\rho(s-t)}\frac{C_s^{-\sigma}}{1+\tau_s^c}\left[\frac{Q_s}{S_s}\left((\tau_s^v + \tau_s^P(1-\tau_s^v))Y_s - G_s - T_s\right) + \tau_s^c C_s + (\tau_s^l - \tau_s^P)\frac{W_s}{P_s}N_s\right]ds$$
$$(3.A.102)$$

Define real government debt (in utility terms) as $d_t^g \equiv C_t^{-\sigma}D_t^g/(P_t(1+\tau_t^c))$. We

can then write the constraint as:

$$\dot{d}_t^g = \rho d_t^g - \frac{C_t^{-\sigma}}{1+\tau_t^c}\left[\frac{Q_t}{S_t}\left((\tau_t^v + \tau_t^P(1-\tau_t^v))\,Y_t - G_t - T_t\right) + \tau_t^c C_t + (\tau_t^l - \tau_t^P)\frac{W_t N_t}{P_t}\right]$$

(3.A.103)

$$d_t^g = \Theta_t^{-\sigma}(C_t^*)^{-\sigma}\frac{D_t^g}{P_t^*(1+\tau_t^c)}$$

(3.A.104)

$$d_0^g = \int_0^\infty e^{-\rho t}\frac{C_t^{-\sigma}}{1+\tau_t^c}\left[\frac{Q_t}{S_t}\left((\tau_t^v + \tau_t^P(1-\tau_t^v))\,Y_t - G_t - T_t\right) + \tau_t^c C_t + (\tau_t^l - \tau_t^P)\frac{W_t N_t}{P_t}\right]dt$$

(3.A.105)

$$e^{-\sigma\theta_0 - \hat{\tau}_0^c}\frac{\overline{D}}{\overline{P_H Y}} = \int_0^\infty e^{-\rho t - \sigma c_t - \hat{\tau}_t^c}\left[e^{q_t - s_t}\left[\left(\left(1-(1-\overline{\tau}^v)e^{-\hat{\tau}_t^v}\right)+\overline{\tau}^P(1-\overline{\tau}^v)e^{-\hat{\tau}_t^v}\right)e^{y_t} - \varsigma_g e^{g_t} - \frac{T_t}{\overline{Y}}\right]\right. +$$

$$\left. + \frac{\overline{PC}}{\overline{P_H Y}}\left(e^{\hat{\tau}_t^c} - 1\right)e^{c_t} + \left(\left(1-\overline{\tau}^p\right) - (1-\overline{\tau}^l)e^{-\hat{\tau}_t^l}\right)\frac{\overline{WN}}{\overline{P_H Y}}e^{w_t + n_t - p_t}\right]dt$$

Taking a first order approximation, we get

$$-(\sigma\theta_0 + \hat{\tau}_0^c)\frac{\overline{D}}{\overline{P_H Y}} = \rho\frac{\overline{D}}{\overline{P_H Y}}\int_0^\infty e^{-\rho t}\left[-\sigma c_t - \hat{\tau}_t^c\right]dt +$$

$$\int_0^\infty e^{-\rho t}\left[-\alpha s_t\left[\overline{\tau}^v + \overline{\tau}^P(1-\overline{\tau}^v) - \varsigma_g\right] + (1-\overline{\tau}^p)(1-\overline{\tau}^v)\hat{\tau}_t^v + \left(\overline{\tau}^v + \overline{\tau}^P(1-\overline{\tau}^v)\right)y_t - \varsigma_g g_t - \tilde{T}_t\right]dt$$

$$+ \int_0^\infty e^{-\rho t}\left[\frac{\overline{PC}}{\overline{P_H Y}}\hat{\tau}_t^c + (1-\overline{\tau}^l)\frac{\overline{WN}}{\overline{P_H Y}}\hat{\tau}_t^l + \left(\overline{\tau}^l - \overline{\tau}^p\right)\frac{\overline{WN}}{\overline{P_H Y}}(w_t + n_t - p_t)\right]dt$$

where $\tilde{T}_t \equiv T_t/\overline{Y}$.

Note that the real wage bill can be written as

$$w_t - p_t + n_t = \sigma c_t + \varphi(1+\phi)y_t + \hat{\tau}_t^l + \hat{\tau}_t^c$$

(3.A.106)

If the steady state profit tax is equal to zero and the risk premium is the only

179

non-zero disturbance, then we have

$$- (\sigma\theta_0 + \hat{\tau}_0^c) \frac{\overline{D}}{\overline{P_H Y}} = \int_0^\infty e^{-\rho l} \left[-(\sigma c_t + \alpha s_t + \hat{\tau}_t^c + \hat{\tau}_t^v + \hat{\tau}_t^l) \left[\overline{\tau}^v - \varsigma_g \right] + (1 - \varsigma_g)(\hat{\tau}_t^v + \hat{\tau}_t^c + \hat{\tau}_t^l) \right] dt$$

$$+ \int_0^\infty e^{-\rho t} \left[(\overline{\tau}^v - \varsigma_g) y_t - \varsigma_g(g_t - y_t) - \tilde{T}_t + \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}} \right) \hat{\tau}_t^l + \overline{\tau}^l \frac{\overline{WN}}{\overline{P_H Y}} (\varphi(1 + \phi) y_t) \right] dt$$

From the supply condition, we get

$$- \frac{(\sigma\theta_0 + \hat{\tau}_0^c)}{\overline{D}^{-1} \overline{P_H Y}} = \int_0^\infty e^{-\rho t} \left[\frac{\rho \overline{D} \varphi(1 + \phi)}{\overline{P_H Y}} y_t + (1 - \varsigma_g)(\hat{\tau}_t^v + \hat{\tau}_t^c + \hat{\tau}_t^l) - \varsigma_g(g_t - y_t) - \tilde{T}_t + \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}} \right) \hat{\tau}_t^l \right] dt$$

$$\tag{3.A.107}$$

Suppose now that the risk premium is the only shock and that tax and government spending are equal to zero:

$$- \sigma\theta_0 \frac{\overline{D}}{\overline{P_H Y}} = \int_0^\infty e^{-\rho t} \left(\varphi(1 + \phi) \frac{\rho \overline{D}}{\overline{P_H Y}} + \varsigma_g \right) y_t dt + \overline{T}^{pv} = \overline{T}^{pv} \tag{3.A.108}$$

where $\overline{T}^{pv} \equiv - \int_0^\infty e^{-\rho l} \tilde{T}_t dl$ is the present value of lump-sum taxes required to balance the budget.

Notice that I used the fact that the output is proportional to θ_t and the present value of θ_t is zero to conclude the present value of output is zero.

The budget can be balanced using the lump-sum tax, if available, or it can be financed by a permanent increase in the consumption tax offset by a permanent decrease in the VAT tax, such that $\hat{\tau}_t^c + \hat{\tau}_t^v = 0$.

Similar expressions determine the debt dynamics:

$$\frac{\hat{d}_t^g \overline{D}}{\overline{P_H Y}} = \int_t^\infty e^{-\rho(s-t)} \left[\frac{\rho \overline{D}}{\overline{P_H Y}} \varphi(1 + \phi) y_s + (1 - \varsigma_g)(\hat{\tau}_s^v + \hat{\tau}_s^c + \hat{\tau}_s^l) - \varsigma_g(g_s - y_s) - \tilde{T}_s + \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}} \right) \hat{\tau}_s^l \right] ds$$

$$\tag{3.A.109}$$

where

$$\hat{d}_t^g = -\sigma\theta_t - \hat{\tau}_t^c + \hat{D}_t^g \tag{3.A.110}$$

For the case where the budget is financed with offsetting consumption and VAT taxes, we get

$$\hat{d}_t^g \frac{\overline{D}}{\overline{P_H Y}} = \left(\frac{\rho \overline{D}}{\overline{P_H Y}} \varphi(1 + \phi) + \varsigma_g \right) \int_t^\infty e^{-\rho(s-t)} y_s ds \tag{3.A.111}$$

Let's compute the following integral

$$\int_t^\infty e^{-\rho(s-t)}\theta_s ds = \frac{\theta_t}{\rho} + \frac{1}{\sigma}\frac{\int_t^\infty e^{-\rho(s-t)}\psi_s ds}{\rho} \tag{3.A.112}$$

where I used the fact $\theta_s = \theta_t + (1/\sigma)\int_t^s \psi_u du$.

Debt level at period t can be written as

$$\hat{D}_t^g = \sigma(\theta_t - \theta_0) + \kappa_D\left(\theta_t + \frac{1}{\sigma}\int_t^\infty e^{-\rho(s-t)}\psi_s ds\right) \tag{3.A.113}$$

where

$$\kappa_D \equiv \left(\frac{\overline{D}}{\overline{P_H Y}}\right)^{-1}\left(\frac{\rho\overline{D}}{\overline{P_H Y}}\varphi(1+\phi) + \varsigma_g\right)\frac{\kappa_y}{\rho} \tag{3.A.114}$$

and

$$\kappa_y = -\alpha(1-\varsigma_g)\sigma\frac{\gamma-\xi+1}{1+((1-\alpha)\xi+\alpha\gamma)(\varphi(1+\phi)-1)(1-\varsigma_g)} \tag{3.A.115}$$

Note that κ_D is negative when the Marshall-Lerner condition is satisfied, implying debt can increase or decrease over time.

3.A.4 Derivations: Section 6

The wage and pricing equations are given by

$$w_t - p_t = \sigma c_t + \varphi\phi y_t + \hat{\tau}_t^c + \hat{\tau}_t^l \tag{3.A.116}$$

$$p_{H,t} = (\varphi - 1)y_t + w_t + \hat{\tau}_t^v \tag{3.A.117}$$

Suppose now that wages are downward rigid, i.e., wages are allowed to go up, but not to go down. If the constraint is binding, then there will be unemployment, i.e., labor supply exceed labor supply.

181

We can rewrite the equations above as a labor demand and labor supply equations:

$$n_t^s = \frac{1}{\phi}\left[w_t + (1-\alpha)s_t - \sigma c_t - \hat{\tau}_t^c - \hat{\tau}_t^l\right]$$

$$n_t^d = \frac{\varphi}{\varphi-1}\left[-w_t - s_t - \hat{\tau}_t^v\right]$$

Unemployment is then given by

$$u_t = n_t^s - n_t^d = \frac{1}{\phi}\left[-\sigma\theta_t - \hat{\tau}_t^c - \hat{\tau}_t^l\right] - \frac{\varphi}{\varphi-1}\left[-s_t - \hat{\tau}_t^v\right] \tag{3.A.118}$$

where I used the fact that $w_t = 0$.

Consider first the equilibrium under passive policy. Let's conjecture the economy starts with positive unemployment given a positive risk premium shock. Equilibrium is then determined by

$$y_t = \varsigma_c\theta_t + (\varsigma_c\xi + \varsigma_x\gamma)s_t$$

$$-s_t = (\varphi - 1)y_t$$

solving the system

$$y_t = \frac{\varsigma_c\theta_t}{1 + (\varsigma_c\xi + \varsigma_x\gamma)(\varphi - 1)}; \qquad s_t = -\frac{\varsigma_c(\varphi - 1)\theta_t}{1 + (\varsigma_c\xi + \varsigma_x\gamma)(\varphi - 1)} \tag{3.A.119}$$

Unemployment is given by

$$u_t = -\left[\frac{\sigma}{\phi} + \frac{\varphi\varsigma_c}{1 + (\varsigma_c\xi + \varsigma_x\gamma)(\varphi - 1)}\right]\theta_t \tag{3.A.120}$$

For simplicity, assume $\xi = \gamma$. This imply that θ_t is given by

$$\theta_t = \frac{1}{\sigma}\left[\int_0^t \psi_s ds - \int_0^\infty e^{-\rho s}\psi_s ds\right] \tag{3.A.121}$$

Hence, on impact output will decrease, unemployment will increase, and the terms of trade will depreciate (if technology presents decreasing returns).

182

Define T_0 as the period where $\theta_{T_0} = 0$. For $t \geq T_0$, equilibrium will coincide with the flexible price allocation:

$$y_t = -\frac{\varsigma_x \sigma(\gamma - \xi + 1)}{1 + (\varphi(1 + \phi) - 1)(\varsigma_c \xi + \varsigma_x \gamma)}\theta_t \qquad (3.A.122)$$

$$s_t = -\frac{\sigma + (\varphi(1 + \phi) - 1)\varsigma_c}{1 + (\varphi(1 + \phi) - 1)(\varsigma_c \xi + \varsigma_x \gamma)}\theta_t \qquad (3.A.123)$$

The nominal wage is given by

$$w_t = \sigma\theta_t + \varphi\phi y_t = \sigma\left[1 - \frac{\varsigma_x \varphi\phi}{1 + (\varphi(1 + \phi) - 1)(\varsigma_c + \varsigma_x)\xi}\right]\theta_t \qquad (3.A.124)$$

The coefficient above is positive. Hence, the wage is positive and increasing for $t > T_0$. Notice also that for $t < T_0$, the wage would be below the steady state level, contradicting the downward wage rigidity.

Consider now the optimal policy. Given enough instruments, the downward nominal wage rigidity does not impose any additional restriction compared to the problem under flexible prices. Hence, the solution will be

$$y_t = 0; \qquad s_t = -\frac{1 - \alpha}{\xi} \qquad (3.A.125)$$

The implementation, however, is different. Unless the government artificially increase the labor supply, there is no unemployment under the optimal policy. This imply the sum of taxes must satisfy the condition:

$$\hat{\tau}_t^c + \hat{\tau}_t^v + \hat{\tau}_t^l = -\frac{\sigma\alpha}{\xi}\theta_t \qquad (3.A.126)$$

under the assumption the wage is always non-negative.

The nominal wage is given by

$$w_t = \sigma\theta_t + \hat{\tau}_t^c + \hat{\tau}_t^l = \sigma\frac{\xi - \alpha}{\xi}\theta_t - \hat{\tau}_t^v \geq 0 \qquad (3.A.127)$$

183

One possibility is to choose

$$\hat{\tau}_t^v = \sigma \frac{\xi - \alpha}{\xi} \theta_t \tag{3.A.128}$$

This imply that the labor income tax is

$$\hat{\tau}_t^l = -\left(\sigma \theta_t + \hat{\tau}_t^c\right) = -\int_0^t \psi_s ds - \left(\sigma \theta_0 + \hat{\tau}_0^c\right) \tag{3.A.129}$$

Consider now the government solvency constraint:

$$-\frac{(\sigma \theta_0 + \hat{\tau}_0^c)}{\overline{D}^{-1} \overline{P_H Y}} = \int_0^\infty e^{-\rho t} \left[\frac{\rho \overline{D} \varphi (1 + \phi)}{\overline{P_H Y}} y_t + (1 - \varsigma_g)(\hat{\tau}_t^v + \hat{\tau}_t^c + \hat{\tau}_t^l) - \varsigma_g(g_t - y_t) - \tilde{T}_t + \left(\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}} \right) \hat{\tau}_t^l \right] dt \tag{3.A.130}$$

where

$$\overline{\tau}^v - 1 + \frac{\overline{WN}}{\overline{P_H Y}} = -(1 - \overline{\tau}^v) \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right] \tag{3.A.131}$$

The budget constraint can be rewritten as

$$-\left[1 + \left(\frac{\overline{D}}{\overline{P_H Y}} \right)^{-1} (1 - \overline{\tau}^v) \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right] \right] (\sigma \theta_0 + \hat{\tau}_0^c) = \frac{1 - \overline{\tau}^v}{\rho} \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right] \int_0^\infty e^{-\rho t} \psi_t dt \tag{3.A.132}$$

rearranging

$$\hat{\tau}_0^c = \left[\sigma \kappa_\theta - \frac{\frac{1 - \overline{\tau}^v}{\rho} \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right]}{\left[1 + \left(\frac{\overline{D}}{\overline{P_H Y}} \right)^{-1} (1 - \overline{\tau}^v) \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right] \right]} \right] \int_0^\infty e^{-\rho t} \psi_t dt \tag{3.A.133}$$

and

$$\hat{\tau}_0^l = \frac{\frac{1 - \overline{\tau}^v}{\rho} \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right]}{\left[1 + \left(\frac{\overline{D}}{\overline{P_H Y}} \right)^{-1} (1 - \overline{\tau}^v) \left[1 - \frac{\epsilon - 1}{\epsilon \varphi} \right] \right]} \int_0^\infty e^{-\rho t} \psi_t dt \tag{3.A.134}$$

Basically, the labor income tax increases, the VAT tax decreases and the consumption tax has an ambiguous sign.

184

Bibliography

Tobias Adrian, Hyun Song Shin, et al. Financial intermediaries and monetary economics. *Handbook of Monetary Economics*, 3:601–650, 2010.

Franklin Allen and Douglas Gale. Limited market participation and volatility of asset prices. *The American Economic Review*, pages 933–955, 1994.

Fernando Alvarez, Andrew Atkeson, and Patrick J Kehoe. Money, interest rates, and exchange rates with endogenously segmented markets. *Journal of Political Economy*, 110(1), 2002.

Javier Andrés, J David López-Salido, and Edward Nelson. Tobin's imperfect asset substitution in optimizing general equilibrium. *Journal of Money, Credit & Banking (Ohio State University Press)*, 36(4), 2004.

Aloísio Araújo, Susan Schommer, and Michael Woodford. Conventional and unconventional monetary policy with endogenous collateral constraints. *American Economic Journal: Macroeconomics*, 7(1):1–43, 2015.

Jan Babecký, Philip Du Caju, Theodora Kosma, Martina Lawless, Julián Messina, and Tairi Rõõm. Downward nominal and real wage rigidity: Survey evidence from european firms*. *The Scandinavian Journal of Economics*, 112(4):884–910, 2010.

Robert J Barro and Andrew Mollerus. Safe assets. Technical report, National Bureau of Economic Research, 2014.

Suleyman Basak and Domenico Cuoco. An equilibrium model with restricted stock market participation. *Review of Financial Studies*, 11(2):309–341, 1998.

Juliane Begenau, Monika Piazzesi, and Martin Schneider. Banks' risk exposures. 2013.

Ben S Bernanke, Mark Gertler, and Simon Gilchrist. The financial accelerator in a quantitative business cycle framework. *Handbook of macroeconomics*, 1:1341–1393, 1999.

Tiago C Berriel and Arthur Mendes. Central bank balance sheet, liquidity trap, and quantitative easing. Technical report, working paper, 2015.

Saroj Bhatarai, Bulat Gafarov, and Gauti Eggertsson. Time consistency and the duration of government debt: A signalling theory of quantitative easing. In *2014 Meeting Papers*, number 1292. Society for Economic Dynamics, 2014.

Zvi Bodie, Robert C Merton, and William F Samuelson. Labor supply flexibility and portfolio choice in a life cycle model. *Journal of economic dynamics and control*, 16(3):427–449, 1992.

Alon Brav, George M Constantinides, and Christopher C Geczy. Asset pricing with heterogeneous consumers and limited participation: Empirical evidence. *Journal of Political Economy*, 110(4), 2002.

Karl Brunner and Allan H Meltzer. Mr. hicks and the" monetarists". *Economica*, pages 44–59, 1973.

Markus Brunnermeier and Yuliy Sannikov. A macroeconomic model with a financial sector. *American Economic Review*, 2014.

Markus K. Brunnermeier and Yuliy Sannikov. The i theory of money. 2015.

Ricardo J Caballero and Emmanuel Farhi. The safety trap. Technical report, National Bureau of Economic Research, 2014.

Yeung Lewis Chan and Leonid Kogan. Catching up with the joneses: Heterogeneous preferences and the dynamics of asset prices. *Journal of Political Economy*, 110(6): 1255–1285, 2002.

Han Chen, Vasco Cúrdia, and Andrea Ferrero. The macroeconomic effects of large-scale asset purchase programmes*. *The economic journal*, 122(564):F289–F315, 2012.

Yili Chien, Harold Cole, and Hanno Lustig. Is the volatility of the market price of risk due to intermittent portfolio rebalancing? *The American Economic Review*, 102(6):2859–2896, 2012.

John H Cochrane. Presidential address: Discount rates. *The Journal of Finance*, 66 (4):1047–1108, 2011.

George M Constantinides. Intertemporal asset pricing with heterogeneous consumers and without demand aggregation. *Journal of Business*, pages 253–267, 1982.

Isabel Correia, Emmanuel Farhi, Juan Pablo Nicolini, and Pedro Teles. Unconventional fiscal policy at the zero bound. *American Economic Review*, 103(4):1172–1211, 2013.

Arnaud Costinot, Guido Lorenzoni, and Iván Werning. A theory of capital controls as dynamic terms-of-trade manipulation. Technical report, National Bureau of Economic Research, 2011.

186

Vasco Curdia and Michael Woodford. The central-bank balance sheet as an instrument of monetarypolicy. *Journal of Monetary Economics*, 58(1):54–79, 2011.

Stefania D'Amico and Thomas B King. Flow and stock effects of large-scale treasury purchases: Evidence on the importance of local supply. *Journal of financial economics*, 108(2):425–448, 2013.

Marco Del Negro, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki. The great escape? a quantitative evaluation of the fed's liquidity facilities. *Federal Reserve Bank of New York Staff Report*, (520), 2011.

Peter M DeMarzo and Yuliy Sannikov. Optimal security design and dynamic capital structure in a continuous-time agency model. *The Journal of Finance*, 61(6):2681–2724, 2006.

Sebastian Di Tella. Uncertainty shocks and balance sheet recessions. Technical report, mimeo, MIT, 2012.

Douglas W Diamond and Raghuram G Rajan. Illiquid banks, financial stability, and interest rate policy. *Journal of Political Economy*, 120(3):552–591, 2012.

Peter A Diamond. National debt in a neoclassical growth model. *The American Economic Review*, 55(5):1126–1150, 1965.

Itamar Drechsler, Alexi Savov, and Philipp Schnabl. A model of monetary policy and risk premia. Technical report, National Bureau of Economic Research, 2014.

Darrell Duffie and Larry G Epstein. Stochastic differential utility. *Econometrica: Journal of the Econometric Society*, pages 353–394, 1992.

Bernard Dumas. Two-person dynamic equilibrium in the capital market. *Review of Financial Studies*, 2(2):157–188, 1989.

Gauti B Eggertsson and Michael Woodford. Zero bound on interest rates and optimal monetary policy. *Brookings Papers on Economic Activity*, 2003(1):139–233, 2003.

Emmanuel Farhi and Ivan Werning. Dealing with the trilemma: Optimal capital controls with fixed exchange rates. Technical report, National Bureau of Economic Research, 2012.

Emmanuel Farhi and Ivan Werning. Dilemma not trilemma? capital controls and exchange rates with volatile capital flows. In *14th Jacques Polak Annual Research Conference, November*, 2013.

Brett W Fawley and Christopher J Neely. Four stories of quantitative easing. *Federal Reserve Bank of St. Louis Review*, 95(January/February 2013), 2013.

Andrea Ferrero. Fiscal and monetary rules for a currency union. *Journal of International Economics*, 77(1):1–10, 2009.

Milton Friedman. The case for flexible exchange rates. 1953.

Xavier Gabaix. A sparsity-based model of bounded rationality. *The Quarterly Journal of Economics*, 129(4):1661–1710, 2014.

Joseph Gagnon, Matthew Raskin, Julie Remache, and Brian Sack. The financial market effects of the federal reserve's large-scale asset purchases. *International Journal of Central Banking*, 7(1):3–43, 2011.

Jordi Gali and Tommaso Monacelli. Monetary policy and exchange rate volatility in a small open economy. *The Review of Economic Studies*, 72(3):707–734, 2005.

Jordi Gali and Tommaso Monacelli. Optimal monetary and fiscal policy in a currency union. *Journal of International Economics*, 76(1):116–132, 2008.

Nicolae Gârleanu and Stavros Panageas. Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. *Journal of Political Economy*, 123(3):670–685, 2015.

Nicolae Gârleanu and Lasse Heje Pedersen. Margin-based asset pricing and deviations from the law of one price. *Review of Financial Studies*, page hhr027, 2011.

Nicola Gennaioli, Andrei Shleifer, and Robert Vishny. Neglected risks, financial innovation, and financial fragility. *Journal of Financial Economics*, 104(3):452–468, 2012.

Mark Gertler and Peter Karadi. A model of unconventional monetary policy. *Journal of monetary Economics*, 58(1):17–34, 2011.

Mark Gertler and Peter Karadi. Qe 1 vs. 2 vs. 3...: A framework for analyzing large-scale asset purchases as a monetary policy tool. *International Journal of Central Banking*, 9(1):5–53, 2013.

Mark Gertler and Nobuhiro Kiyotaki. Financial intermediation and credit policy in business cycle analysis. *Handbook of monetary economics*, 3(3):547–599, 2010.

Mark Gertler, Nobuhiro Kiyotaki, and Andrea Prespitino. Wholesale banking and bank runs in macroeconomic modelling of financial crises. 2015.

Francisco Gomes and Alexander Michaelides. Optimal life-cycle asset allocation: Understanding the empirical evidence. *The Journal of Finance*, 60(2):869–904, 2005.

Robin Greenwood and Dimitri Vayanos. Bond supply and excess bond returns. *Review of Financial Studies*, 27(3):663–713, 2014.

Sanford Grossman and Laurence Weiss. A transactions-based model of the monetary transmission mechanism. *The American Economic Review*, 73(5):871–880, 1983.

Jonathan Gruber. A tax-based estimate of the elasticity of intertemporal substitution. *Quarterly Journal of Finance*, 3(1):1 – 20, 2013. ISSN 20101392.

John G Gurley, Edward Stone Shaw, and Alain C Enthoven. *Money in a Theory of Finance*. Brookings Institution Washington, DC, 1960.

Fatih Guvenen. A parsimonious macroeconomic model for asset pricing. *Econometrica*, 77(6):1711–1750, 2009.

Robert E. Hall and Ricardo Reis. Maintaining central-bank solvency under new-style central banking. Hoover Institution, Stanford University, and Columbia University, February 2013.

Fumio Hayashi. Tobin's marginal q and average q: A neoclassical interpretation. *Econometrica: Journal of the Econometric Society*, pages 213–224, 1982.

Zhigu He and Arvind Krishnamurthy. A model of capital and crises. *The Review of Economic Studies*, 79(2):735–777, 2012.

Zhiguo He and Arvind Krishnamurthy. Intermediary asset pricing. *The American Economic Review*, 103(2):732–70, 2013.

John Heaton and Deborah Lucas. Market frictions, savings behavior, and portfolio choice. *Macroeconomic Dynamics*, 1(01):76–101, 1997.

Bengt Holmstrom and Jean Tirole. Financial intermediation, loanable funds, and the real sector. *the Quarterly Journal of economics*, pages 663–691, 1997.

Hyeok Jeong and Robert M Townsend. Sources of tfp growth: occupational choice and financial deepening. *Economic Theory*, 32(1):179–221, 2007.

Michael Joyce, David Miles, Andrew Scott, and Dimitri Vayanos. Quantitative easing and unconventional monetary policy–an introduction*. *The Economic Journal*, 122 (564):F271–F288, 2012.

Alejandro Justiniano, Giorgio E Primiceri, and Andrea Tambalotti. Investment shocks and business cycles. *Journal of Monetary Economics*, 57(2):132–145, 2010.

Mudit Kapoor and Shamika Ravi. Elasticity of intertemporal substitution in consumption: Empirical evidence from a natural experiment. *SSRN Working Paper Series*, 2010.

Nobuhiro Kiyotaki and John Moore. Credit cycles. *The Journal of Political Economy*, 105(2):211–248, 1997.

Arvind Krishnamurthy and Annette Vissing-Jorgensen. The effects of quantitative easing on interest rates: Channels and implications for policy. *Brookings Papers on Economic Activity*, 2011(2):215–287, 2011.

Arvind Krishnamurthy and Annette Vissing-Jorgensen. The aggregate demand for treasury debt. *Journal of Political Economy*, 120(2):233–267, 2012.

Francis A Longstaff and Jiang Wang. Asset pricing and the credit market. *Review of Financial Studies*, 25(11):3169–3215, 2012.

N Gregory Mankiw and Stephen P Zeldes. The consumption of stockholders and nonstockholders. *Journal of Financial Economics*, 29(1):97–112, 1991.

Enrique G Mendoza. Sudden stops, financial crises, and leverage. *The American Economic Review*, 100(5):1941–1966, 2010.

Ezra Oberfield and Devesh Raval. Micro data and macro technology. Technical report, National Bureau of Economic Research, 2014.

Ricardo Reis. Different types of central bank insolvency and the central role of seignorage. *Journal of Monetary Economics*, 73:20 – 25, 2015. ISSN 0304-3932. doi: http://dx.doi.org/10.1016/j.jmoneco.2015.06.001. URL http://www.sciencedirect.com/science/article/pii/S0304393215000628. Carnegie-Rochester-NYU Conference Series on Public Policy "Monetary Policy: An Unprecedented Predicament" held at the Tepper School of Business, Carnegie Mellon University, November 14-15, 2014.

Julio J Rotemberg. A monetary equilibrium model with transactions costs. *The Journal of Political Economy*, 92(1):40–58, 1984.

Krislert Samphantharak and Robert Townsend. Risk and return in village economies. Technical report, National Bureau of Economic Research, 2013.

Paul A Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *The journal of political economy*, pages 467–482, 1958.

Yuliy Sannikov. A continuous-time version of the principal-agent problem. *The Review of Economic Studies*, 75(3):957–984, 2008.

WE Schiesser. Pde boundary conditions from minimum reduction of the pde. *Applied numerical mathematics*, 20(1):171–179, 1996.

Stephanie Schmitt-Grohé and Martin Uribe. Pegs and pain. Technical report, National Bureau of Economic Research, 2011.

Dejanir Silva. Optimal unconventional monetary policy: A two-period example. Technical report, mimeo, MIT, 2015.

Christopher A Sims. Implications of rational inattention. *Journal of monetary Economics*, 50(3):665–690, 2003.

James Tobin. A general equilibrium approach to monetary theory. *Journal of money, credit and banking*, 1(1):15–29, 1969.

James Tobin and William C Brainard. Financial intermediaries and the effectiveness of monetary controls. *The American Economic Review*, pages 383–400, 1963.

Dimitri Vayanos and Jean-Luc Vila. A preferred-habitat model of the term structure of interest rates. Technical report, National Bureau of Economic Research, 2009.

Luis M Viceira. Optimal portfolio choice for long-horizon investors with nontradable labor income. *The Journal of Finance*, 56(2):433–470, 2001.

Annette Vissing-Jørgensen and Orazio P Attanasio. Stock-market participation, intertemporal substitution, and risk-aversion. *American Economic Review*, pages 383–391, 2003.

Neil Wallace. A modigliani-miller theorem for open-market operations. *The American Economic Review*, pages 267–274, 1981.

Jiang Wang. The term structure of interest rates in a pure exchange economy with heterogeneous investors. *Journal of Financial Economics*, 41(1):75–110, 1996.

Ivan Werning. Managing a liquidity trap: Monetary and fiscal policy. Technical report, National Bureau of Economic Research, 2011.

Stephen D Williamson. Liquidity, monetary policy, and the financial crisis: A new monetarist approach. *The American Economic Review*, pages 2570–2605, 2012.