Self-shrinkers and translating solitons of mean curvature flow

by

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Abstract

We study singularity models of mean curvature flow ("MCF") and their generalizations. In the first part, we focus on rigidity and curvature estimates for self-shrinkers. We give a rigidity theorem proving that any self-shrinker which is graphical in a large ball must be a hyperplane. This result gives a stronger version of the Bernstein type theorem for shrinkers proved by Ecker-Huisken. One key ingredient is a curvature estimate for almost stable shrinkers. By proving curvature estimates for mean convex shrinkers, we show that any shrinker which is mean convex in a large ball must be a round cylinder. This generalizes a result by Colding-Ilmanen-Minicozzi : no curvature bound assumption is needed. This part is joint work with Jonathan Zhu.

In the second part, we consider λ-hypersurfaces which can be thought of as a generalization of shrinkers. We first give various gap and rigidity theorems. We then establish the Bernstein type theorem for λ-hypersurfaces and classify λ-curves.

In the last part, we study translating solitons of MCF from four aspects: volume growth, entropy, stability, and curvature estimates. First, we show that every properly immersed translator has at least linear volume growth. Second, using Huisken's monotonicity formula, we compute the entropy of the grim reaper and the bowl solitons. Third, we estimate the spectrum of the stability operator L for translators and give a rigidity result of L-stable translators. Finally, we provide curvature estimates for L-stable translators, graphical translators and translators with small entropy.

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Chapter 1

Introduction

The main subject of this thesis is the study of singularity models of mean curvature flow ("MCF"). MCF was first studied by Brakke [Bra78] in the context of geometric measure theory. More recently, there have been many new breakthroughs starting with Huisken's work [Hui84]. Here, we will mainly focus on the rigidity, curvature estimates and uniqueness of singularities of MCF.

MCF is the negative gradient flow for the area functional. Any closed hypersurface contracts under MCF and becomes extinct in finite time. Hence, singularities are unavoidable. One of the most fundamental problems in MCF is to understand the singularities. By the combined work of Huisken [Hui90], Ilmanen [Ilm97] and White [Whi94], singularities are modeled by self-shrinkers, that is, special solutions to MCF that move by rescaling: $\Sigma_t = \sqrt{-t}\Sigma$ $(t < 0)$. Any self-shrinker Σ satisfies the equation

$$H = \frac{\langle x, \mathbf{n} \rangle}{2}. \tag{1.0.1}$$

The simplest examples in $\mathbf{R}^{n+1}$ are generalized cylinders $\mathbf{S}^k(\sqrt{2k}) \times \mathbf{R}^{n-k}$.

In [EH89], Ecker and Huisken gave the first Bernstein type theorem for self-shrinkers which states that any entire self-shrinking graph with polynomial volume growth must be a hyperplane. Later, L. Wang [Wan11a] removed the volume growth assumption. In a joint work [GZ15] with Jonathan Zhu, we proved the following rigidity theorem which gives a stronger, quantitative version of the Bernstein type theorem for self-shrinkers.

Theorem 1.0.1. *Given $n \le 6$ and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth, complete self-shrinker with entropy $\lambda(\Sigma) \le \lambda_0$ satisfying*

(†) Σ is graphical in B_R,

then Σ is a hyperplane.

It is important to note that we do not assume any curvature bounds in the theorem. Indeed, the conclusion follows readily, and in all dimensions, if one assumes even a mild curvature estimate. Our theorem represents quite a strong sense of rigidity for the hyperplane, since we recover the hyperplane by assuming only the graphical condition (†) on a large, but compact, set.

It is natural to seek such rigidity theorems for the hyperplane in the class of self-shrinkers, both since self-shrinkers model singularities of MCF and since they are minimal hypersurfaces in Euclidean space with a conformal metric. In the theory of minimal surfaces, the Bernstein Theorem is one of the most fundamental theorems, and has many important applications, such as regularity theory for minimal submanifolds.

The key ingredient in the proof of Theorem 1.0.1 is a curvature estimate for graphical shrinkers, and it is this step that is responsible for the restriction $n \le 6$.

Theorem 1.0.2. *Given $n \le 6$ and λ_0, there exists $C = C(n, \lambda_0)$ so that for any self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ with entropy $\lambda(\Sigma) \le \lambda_0$ satisfying*

($\star$) Σ is graphical in B_R for $R > 2$,

we have $\quad |A|(x) \le C(1 + |x|),$ *for all* $x \in B_{R-1} \cap \Sigma.$

Note that the constant C does not depend on the radius R. In fact, we can prove that this curvature estimate holds for a much larger class of self-shrinkers. The assumption ($\star$) can be replaced by either $H > 0$ in B_R, or Σ is L-stable in B_R. The reason is that all three conditions imply a lower bound for the lowest eigenvalue of the stability operator L. This enables us to derive stability type inequalities and then apply techniques developed by [CS85], [SS81] and [SSY75] for minimal hypersurfaces.

One central problem in the study of the singularities is the uniqueness of blowups. Namely, whether different sequences of dilations might give different blowups. For compact singularities of MCF, this uniqueness problem is better understood; see for instance [Sch14] and [Ses08]. The first uniqueness theorem for blowups at noncompact singularities was obtained by Colding-Ilmanen-Minicozzi [CIM15], who proved that if one blowup at a singularity of MCF is a multiplicity-one cylinder, then every subsequential limit is also a cylinder. The key to proving this uniqueness result is the rigidity theorem [CIM15, Theorem 0.1], which says that any self-shrinker that is mean convex and with bounded $|A|$ on a large and compact set must also be a cylinder. In a joint work [GZ16] with Jonathan Zhu, using curvature estimates for mean convex self-shrinkers, we further showed that this rigidity holds without the curvature bound.

Theorem 1.0.3. *Given $n \leq 6$ and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies*

(‡) $H \geq 0$ on $B_R \cap \Sigma$,

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $0 \leq k \leq n$.

Theorem 1.0.3 also extends Huisken [Hui93] and Colding-Minicozzi's [CM12a] classification results of mean convex shrinkers for $n \leq 6$ and gives a stronger, quantitative version of their results. If we assume any positive lower bound of mean curvature, we proved that this rigidity theorem holds for all dimensions.

There are some gap phenomena for the second fundamental form of self-shrinkers. Cao and Li [CL13] proved that any complete self-shrinker with $|A|^2 \leq \frac{1}{2}$ in arbitrary codimension is a cylinder (see also [LS11]). In [Gua14b], using the rigidity of generalized cylinders ([CIM15],[GZ16]), we showed that any self-shrinker with $|A|^2$ sufficiently close to $1/2$ must also be a cylinder. By computing the Laplacian of $|\nabla A|$, Ding and Xin [DX14] proved that self-shrinkers in $\mathbf{R}^3$ with $|A| = \textit{constant}$ are cylinders. In [Gua14b], we gave a new and simpler proof of this result by analyzing the point where $|x|$ achieves its minimum. This technique could also be used to prove results for Lagrangian self-shrinkers and λ-hypersurfaces with $|A| = constant$.

In the second part of this thesis, we study λ-hypersurfaces, which are defined by the equation

$$H - \frac{1}{2}\langle x, \mathbf{n} \rangle = \lambda, \quad \text{where } \lambda \text{ is any constant.} \tag{1.0.2}$$

Note that when $\lambda = 0$, λ-hypersurfaces are exactly self-shrinkers and thus they can be viewed as a generalization of self-shrinkers.

λ-hypersurfaces arise in the study of weighted volume-preserving MCF [CW14a] and an isoperimetric problem in a Gaussian weighted space [MR15]. It was proved that λ-hypersurfaces are critical points of this variational problem and that the only smooth stable ones are hyperplanes; see [MR15].

Our results include the following: gap and rigidity results of λ-hypersurfaces in terms of the second fundamental form; the Bernstein type theorem for λ-hypersurfaces; the classification of one dimensional λ-curves. All these results can be viewed as generalizations of results for self-shrinkers.

In the last part of this thesis, we present our results on translating solitons (or *translators* for short) which are also special solutions of MCF. Translators play an important role in the study of MCF. On one hand, every translator M gives a translating solution to MCF. On the other hand, they arise as blow-up solutions of MCF at type II singularities.

Our first result concerns the lower volume growth of translators. In [MW12], Munteanu and Wang proved that any noncompact gradient shrinking Ricci soliton has at least linear volume growth. Li and Wei [LW14] proved that every noncompact properly immersed self-shrinker also has at least linear volume growth. In [Gua15], we showed that for properly immersed translators, an analog of Li-Wei's result can be obtained.

Theorem 1.0.4. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete properly immersed translator. Then* *for any $x \in \Sigma$, there exists a constant $C > 0$ such that*

$$Vol(\Sigma \cap B_r(x)) = \int_{\Sigma \cap B_r(x)} d\mu \geq Cr \quad \text{for all } r \geq 1. \tag{1.0.3}$$

Colding-Minicozzi [CM12a] introduced the entropy of hypersurfaces. One of the key properties of the entropy is that it is non-increasing along MCF. Colding-Ilmanen-Minicozzi-White [CIMW13] showed the round sphere minimizes entropy among all closed self-shrinkers. Later, Bernstein and Wang [BW14] extended this result and proved that the round sphere minimizes entropy among all closed hypersurfaces up to dimension six. A natural problem is to determine the entropy of the grim reaper and the bowl solitons. In [Gua15], using Huisken's monotonicity formula and various estimates, we showed that the entropy of the grim reaper is 2 and the entropy of bowl soliton $\Gamma^n \subset \mathbf{R}^{n+1}$ ($n \geq 2$) is equal to the entropy of the sphere $\mathbf{S}^{n-1}$.

Translators can also be viewed as minimal hypersurfaces in Euclidean space with a conformal metric. Therefore, the method we developed in the proof of Theorem 1.0.2 can also be used to obtain curvature estimates for certain class of translators. In [GZ15], we gave uniform curvature estimates for L-stable translators.

Theorem 1.0.5. *Given $n \leq 5$ and λ_0, there exists $C = C(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is an L-stable translator satisfying $Vol(\Sigma \cap B_r(x)) \leq \lambda_0 r^n$ for all $x \in \mathbf{R}^{n+1}$ and $r > 0$, then $|A|(x) \leq C$, for all $x \in \Sigma$.*

If we consider weak solutions (more precisely, integral "boundary-less" rectifiable varifolds; see [Whi09]) of the translator equation, then these weak solutions with small entropy are indeed smooth. Combining this regularity result with Allard's compactness theorem, we were able to prove a curvature estimate for translators with small entropy in $\mathbf{R}^3$.

Chapter 2

Background

We begin by providing the necessary background material on mean curvature flow, especially the self-shrinking and translating solutions to mean curvature flow.

2.1 Mean curvature flow

Mean curvature flow ("MCF") describes the evolution of a hypersurface moving by its mean curvature vector and it corresponds to the negative gradient flow of the area functional. We say that a family of hypersurfaces $M_t \subset \mathbf{R}^{n+1}$ is flowing by mean curvature if it satisfies

$$\partial_t x = -H\mathbf{n}, \tag{2.1.1}$$

where H is the mean curvature, $\mathbf{n}$ is the unit normal and x is the position vector.

Note that minimal surfaces ($H = 0$) are stationary solutions to MCF. We also consider hypersurfaces M_t satisfying $(\partial_t x)^\perp = -H\mathbf{n}$. This equation is equivalent to (2.1.1) up to diffeomorphisms tangent to M_t.

Example 2.1.1. *Let us consider a sphere of radius R in $\mathbf{R}^{n+1}$. Since the mean curvature is everywhere n/R, (2.1.1) reduces to an ODE given by $R'(t) = -n/R(t)$. This equation can be easily solved to get $R(t) = \sqrt{R^2 - 2nt}$. So the sphere shrinks homothetically to a point at time $T = \frac{R^2}{2n}$ and the flow becomes singular.*

MCF is a nonlinear parabolic PDE for the evolving hypersurface. In particular, we have the parabolic maximum principle and there are many important consequences, including the following:

(1) If the initial hypersurface is compact and embedded, then it remains embedded during the flow.

(2) Any two smooth compact solutions of MCF which are initially disjoint stay disjoint.

For $n = 1$, MCF for curves is called *curve shortening flow.* Gage and Hamilton [GH86] showed that under the curve shortening flow any simple closed and convex curve in $\mathbf{R}^2$ remains convex and shrinks to a point in finite time. Later, Grayson [Gra87] showed that this is true for all simple closed curves, i.e., any simple closed curve eventually becomes convex and shrinks to a point under the flow. The theory of curve shortening flow is relatively complete by this nice result. However, MCF is much more complicated in higher dimensions and thus we will mainly focus on $n \geq 2$.

Huisken [Hui84] showed that convex hypersurfaces remain convex and shrink to a "round" point under MCF:

Theorem 2.1.2 ([Hui84]). *For any smooth closed and convex initial hypersurface M_0, the solution of MCF remains convex until it disappears into a point in finite time. In the process, the solution becomes asymptotically round. That is after appropriate rescaling (e.g., keeping the area fixed) it converges smoothly to a round sphere.*

By the maximum principle, any closed hypersurface contracts under MCF and becomes extinct in finite time. For convex hypersurfaces, Theorem 2.1.2 implies that the singularities are modeled by round spheres. However, for general hypersurfaces, the situation is far more complicated; see for instance Grayson's dumbbell [Gra89].

One of the most important questions in the study of MCF is to understand the singularities and the key starting point is Huisken's monotonicity formula [Hui90]. We define the function Φ on $\mathbf{R}^{n+1} \times (-\infty, 0)$ by

$$\Phi(x,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}} \tag{2.1.2}$$

and then set $\Phi_{z_0,\tau}(x,t) = \Phi(x - z_0, t - \tau)$. Huisken proved the following monotonicity formula for MCF:

Theorem 2.1.3 ([Hui90]). *If M_t is a solution to MCF and u is a C^2 function, then*

$$\frac{d}{dt}\int_{M_t} u\Phi_{(z_0,\tau)} = -\int_{M_t}\left|H\mathbf{n} - \frac{(x-z_0)^\perp}{2(\tau-t)}\right|^2 u\Phi_{(z_0,\tau)} + \int_{M_t}(u_t - \Delta u)\Phi_{(z_0,\tau)}. \quad (2.1.3)$$

When u is identically one, we get

$$\frac{d}{dt}\int_{M_t} \Phi_{(z_0,\tau)} = -\int_{M_t}\left|H\mathbf{n} - \frac{(x-z_0)^\perp}{2(\tau-t)}\right|^2 \Phi_{(z_0,\tau)}\,. \quad (2.1.4)$$

This formula leads us to do a parabolic rescaling of MCF. Namely, if M_t is a solution of MCF, then for $\lambda > 0$ and (y, T), we obtain a new MCF M_t^λ by defining

$$M_t^\lambda \equiv M_t^{(y,T),\lambda} = \lambda^{-1}(M_{\lambda^2 t+T} - y). \quad (2.1.5)$$

A singularity at time T of MCF is defined to be of *Type I* if

$$\sup_{M_t}|A| \leq \frac{C}{\sqrt{2(T-t)}}, \quad (2.1.6)$$

where C is a positive constant. Using Theorem 2.1.3, Huisken [Hui90] proved that under *Type I* hypothesis, there exists a subsequence $\lambda_i \to \infty$ such that $M_t^{\lambda_i}$ converges smoothly to a limiting flow N_t which is self-similarly shrinking and satisfies

$$N_t = \sqrt{-t}\,N_{-1}, \quad t < 0. \quad (2.1.7)$$

Here N_{-1} is the time -1 slice of N_t and satisfies the equation

$$H = \frac{\langle x, \mathbf{n}\rangle}{2}. \quad (2.1.8)$$

The flow N_t is called a *tangent flow* at (y, T). Any hypersurface satisfying (2.1.8) is called a *self-shrinker*.

Ilmanen [Ilm97] and White [Whi94] extended Huisken's result and proved that a subsequence of M_t^λ converges weakly (in the sense of varifolds) to a self-shrinking flow. Therefore, singularities of MCF are modeled by self-shrinkers.

Throughout, we will use the following notation. Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth hypersurface. Then ∇_Σ (or ∇), div, and Δ are the gradient, divergence, and Laplacian, respectively, on Σ. $\mathbf{n}$ is the outward unit normal, $H=\text{div}_\Sigma \mathbf{n}$ is the mean curvature, A is the second fundamental form, and x is the position vector. With this convection, the mean curvature H is n/r on the sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ of radius r. If e_i is an orthonormal frame for Σ, then the coefficients of the second fundamental form are defined to be $a_{ij} = \langle \nabla_{e_i} e_j, \mathbf{n} \rangle$.

2.2 Self-shrinkers of mean curvature flow

As in Section 2.1, we call a hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ a *self-shrinker*, or just *shrinker*, if it is the $t = -1$ time-slice of a MCF that evolves by shrinking homothetically to the origin $x = 0$. Such hypersurfaces are characterized by the equation

$$H = \frac{\langle x, \mathbf{n} \rangle}{2}. \tag{2.2.1}$$

The simplest examples in $\mathbf{R}^{n+1}$ are generalized cylinders $\mathbf{S}^k(\sqrt{2k}) \times \mathbf{R}^{n-k}$. Since self-shrinkers model the singularities of MCF, the study of self-shrinkers is thus vital to understanding MCF.

The first major result of the classification of self-shrinkers is due to Huisken ([Hui90, Hui93]), who showed that the only smooth complete embedded self-shrinkers in $\mathbf{R}^{n+1}$ with $H \geq 0$, polynomial volume growth and $|A|$ bounded are generalized cylinders $\mathbf{S}^k \times \mathbf{R}^{n-k}$. Later, Colding-Minicozzi [CM12a] removed the assumption of $|A|$ bounded.

Theorem 2.2.1. *([CM12a])* $\mathbf{S}^k \times \mathbf{R}^{n-k}$ *are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and* $H \geq 0$ *in* $\mathbf{R}^{n+1}$.

Without the assumption of mean convexity, numerical results show that it is impossible to give a complete classification; see [Ang92], [KM14], [Møl11] and [Ngu09] for examples. Recently, Brendle [Bre16] proved a well-known conjecture that the round sphere is the only closed embedded self-shrinker with genus 0 in $\mathbf{R}^3$.

In [CM12a], Colding-Minicozzi gave a variational characterization of self-shrinkers. For any hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$, the F-functional of Σ is defined by

$$F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma \mathrm{e}^{-\frac{|x|^2}{4}} d\mu. \tag{2.2.2}$$

Taking the first variation of the F-functional, we see that self-shrinkers are precisely the critical points of F. Therefore, any self-shrinker Σ^n can be viewed as a minimal hypersurface with respect to the Gaussian metric $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{2n}}\delta_{ij}$ on $\mathbf{R}^{n+1}$. The second variation formula of the F-functional is the following.

Lemma 2.2.2 ([CM12a]). *Suppose $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth self-shrinker. If Σ_s is a normal variation of Σ with variation vector filed $\Sigma_0' = f\mathbf{n}$, then*

$$\frac{d^2}{ds^2}\Big|_{s=0} F(\Sigma_s) = \int_\Sigma -\Big(\Delta f - \frac{1}{2}\langle x, \nabla f\rangle + |A|^2 f + \frac{1}{2} f\Big) f e^{-\frac{|x|^2}{4}} d\mu \tag{2.2.3}$$

From the second variation formula, Colding-Minicozzi introduced the following stability operator L defined by

$$L = \Delta - \frac{1}{2}\langle x, \nabla\cdot\rangle + |A|^2 + \frac{1}{2}. \tag{2.2.4}$$

We also use the drift operator $\mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla\cdot\rangle$. Note that for self-shrinkers, the mean curvature H and and the normal part $\langle v, \mathbf{n}\rangle$ of a constant vector filed v are eigenfunctions of L with $LH = H$ and $L\langle v, \mathbf{n}\rangle = \frac{1}{2}\langle v, \mathbf{n}\rangle$.

The entropy λ of a hypersurface Σ is defined as

$$\lambda(\Sigma) = \sup_{x_0, t_0} F_{x_0, t_0}(\Sigma) = \sup_{x_0, t_0} (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \mathrm{e}^{-\frac{|x-x_0|^2}{4t_0}} d\mu, \tag{2.2.5}$$

where the supremum is taking over all $t_0 > 0$ and $x_0 \in \mathbf{R}^{n+1}$. Using Huisken's monotonicity formula, we see that F-functional is not monotone under MCF. However, the entropy is nonincreasing under MCF. Moreover, it was proven in [CM12a] that for a self-shrinker, the entropy is achieved by the F-functional $F_{0,1}$, so no supremum is needed. Cheng and Zhou [CZ13] (see also [DX13]) proved the equivalence of finite entropy, polynomial volume growth, Euclidean volume growth and properness of self-shrinkers.

2.3 Translating solitons of mean curvature flow

As an evolution equation, MCF has another important special solution called *translating solitons.* A smooth hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ is called a *translating soliton*, or *translator* for short, if it satisfies the equation

$$H = -\langle y, \mathbf{n} \rangle. \tag{2.3.1}$$

Here $y \in \mathbf{R}^{n+1}$ is a constant vector.

Such a hypersurface Σ evolves under MCF by translation (in the direction y). Note that if a MCF is not developing a type I singularity, then we say that we have a *type II singularity.* Translators arise as blow-up solutions of MCF at type II singularities. For instance, Huisken and Sinestrari [HS99] proved that at type II singularity of a mean convex flow (a MCF with mean convex solution), there exists a blow-up solution which is a convex translating soliton.

For simplicity, we assume that $y = \mathbf{E}_{n+1}$, so translators satisfy the equation

$$H = -\langle \mathbf{E}_{n+1}, \mathbf{n} \rangle. \tag{2.3.2}$$

In $\mathbf{R}^{n+1}$, there is a unique (up to rigid motion) rotationally symmetric, strictly convex translator, denoted by Γ^n. For $n = 1$, the translator Γ^1 is the *grim reaper* and given as the graph of the function

$$u(x) = -\log \cos x, \quad x \in (-\pi/2, \pi/2). \tag{2.3.3}$$

The next lemma records several useful identities for translators.

Lemma 2.3.2. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a translator satisfying $H = -\langle \mathbf{E}_{n+1}, \mathbf{n}\rangle$, then*

$$LA = 0, \tag{2.3.9}$$

$$LH = 0, \tag{2.3.10}$$

$$L|A|^2 = 2|\nabla A|^2 - |A|^4, \tag{2.3.11}$$

$$\mathcal{L} x_{n+1} = 1. \tag{2.3.12}$$

Proof. Recall that for a general hypersurface, the second fundamental form A satisfies

$$\Delta A = -|A|^2 A - HA^2 - Hess_H. \tag{2.3.13}$$

We fix a point $p \in \Sigma$, and choose a local orthonormal frame e_i such that its tangential covariant derivatives vanish. By the equation of translators, we obtain that

$$\nabla_{e_j}\nabla_{e_i} H = a_{ij,k}\langle \mathbf{E}_{n+1}, e_k\rangle - a_{ik}a_{jk}H. \tag{2.3.14}$$

Note that $\langle \mathbf{E}_{n+1}, \nabla A\rangle_{ij} = a_{ij,k}\langle \mathbf{E}_{n+1}, e_k\rangle$. Combining (2.3.13) and (2.3.14) gives

$$(LA)_{ij} = (\Delta A)_{ij} + |A|^2 a_{ij} + \langle \mathbf{E}_{n+1}, \nabla A\rangle_{ij} = 0. \tag{2.3.15}$$

This is the first identity. Taking the trace gives the second identity. For the third identity, we have

$$L|A|^2 = \mathcal{L}|A|^2 + |A|^4 = 2\langle A, \mathcal{L}A\rangle + 2|\nabla A|^2 + |A|^4 = 2|\nabla A|^2 - |A|^4. \tag{2.3.16}$$

For the last identity, recall that in general we have $\Delta x = -H\mathbf{n}$. Hence,

$$\mathcal{L}x_{n+1} = \Delta x_{n+1} + \langle \mathbf{E}_{n+1}, \nabla x_{n+1}\rangle = \langle -H\mathbf{n}, \mathbf{E}_{n+1}\rangle + |\mathbf{E}_{n+1}^T|^2 = 1. \tag{2.3.17}$$

□

The next lemma records several useful identities for translators.

Lemma 2.3.2. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a translator satisfying $H = -\langle \mathbf{E}_{n+1}, \mathbf{n}\rangle$, then*

$$LA = 0, \tag{2.3.9}$$

$$LH = 0, \tag{2.3.10}$$

$$L|A|^2 = 2|\nabla A|^2 - |A|^4, \tag{2.3.11}$$

$$\mathcal{L}\, x_{n+1} = 1. \tag{2.3.12}$$

Proof. Recall that for a general hypersurface, the second fundamental form A satisfies

$$\Delta A = -|A|^2 A - HA^2 - Hess_H. \tag{2.3.13}$$

We fix a point $p \in \Sigma$, and choose a local orthonormal frame e_i such that its tangential covariant derivatives vanish. By the equation of translators, we obtain that

$$\nabla_{e_j}\nabla_{e_i} H = a_{ij,k}\langle \mathbf{E}_{n+1}, e_k\rangle - a_{ik}a_{jk}H. \tag{2.3.14}$$

Note that $\langle \mathbf{E}_{n+1}, \nabla A\rangle_{ij} = a_{ij,k}\langle \mathbf{E}_{n+1}, e_k\rangle$. Combining (2.3.13) and (2.3.14) gives

$$(LA)_{ij} = (\Delta A)_{ij} + |A|^2 a_{ij} + \langle \mathbf{E}_{n+1}, \nabla A\rangle_{ij} = 0. \tag{2.3.15}$$

This is the first identity. Taking the trace gives the second identity. For the third identity, we have

$$L|A|^2 = \mathcal{L}|A|^2 + |A|^4 = 2\langle A, \mathcal{L}A\rangle + 2|\nabla A|^2 + |A|^4 = 2|\nabla A|^2 - |A|^4. \tag{2.3.16}$$

For the last identity, recall that in general we have $\Delta x = -H\mathbf{n}$. Hence,

$$\mathcal{L}x_{n+1} = \Delta x_{n+1} + \langle \mathbf{E}_{n+1}, \nabla x_{n+1}\rangle = \langle -H\mathbf{n}, \mathbf{E}_{n+1}\rangle + |\mathbf{E}^T_{n+1}|^2 = 1. \tag{2.3.17}$$

□

Chapter 3

Rigidity and curvature estimates for self-shrinkers

In this chapter, we present the joint work with Jonathan Zhu on the rigidity and curvature estimates for self-shrinkers. The chapter begins with Section 3.1 where we prove the rigidity of the hyperplane, that is, Theorem 1.0.1. Next, in Section 3.2, we prove the curvature estimate for almost stable self-shrinkers which plays a key role in the proof of Theorem 1.0.1. Then, in Section 3.3, we give the proof of Theorem 1.0.3 by applying curvature estimates for mean convex self-shrinkers. Finally, we prove that any shrinker in $\mathbf{R}^3$ with $|A| = constant$ is a cylinder in Section 3.4. This chapter is based on [GZ15], [GZ16] and [Gua14b].

3.1 Rigidity of the hyperplane

We state our strong rigidity theorem for graphical self-shrinkers:

Theorem 3.1.1 (Theorem 1.0.1). *Given $n \leq 6$ and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth, complete self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

(†) Σ is graphical in B_R,

then Σ is a hyperplane.

Remark 3.1.2. *It is important to note that we do not assume any curvature bounds in the above theorem. Indeed, the conclusion follows readily, and in all dimensions, if one assumes even a mild curvature estimate.*

Here we take the graphical condition (†) to mean that there is a constant vector $v \in \mathbf{R}^{n+1}$ such that the hypersurface normal $\mathbf{n}$ satisfies $\langle v, \mathbf{n} \rangle > 0$ at each point of $\Sigma \cap B_R$. This is equivalent to each connected component of $\Sigma \cap B_R$ being a graph over a region in the hyperplane $\langle v, x \rangle = 0$.

In fact, we show that the hyperplane is rigid amongst a larger class of self-shrinkers, namely those that satisfy an 'almost stability' on large balls. This constitutes our main theorem, stated as follows:

Theorem 3.1.3. *Given $n \leq 6$ and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth complete self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

(‡) Σ is $\frac{1}{2}$-stable in B_R,

then Σ is a hyperplane.

We will define a more general term, δ-stability; essentially it means that the second variation operator L of the Gaussian area functional has first eigenvalue at least $-\delta$. In particular $\frac{1}{2}$-stability is a weaker assumption than L-stability.

Theorem 3.1.1 follows immediately from Theorem 3.1.3 together with the observation that a graphical self-shrinker is $\frac{1}{2}$-stable.

If we assume that $\langle v, \mathbf{n} \rangle$ has a positive lower bound for some constant unit vector v, then Theorem 3.1.1 holds in all dimensions; see Theorem 3.1.13.

3.1.1 The stability inequality

The next lemma records three useful identities from [CM12a].

Lemma 3.1.4 ([CM12a]). *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth self-shrinker, then we have*

$$LH = H, \tag{3.1.1}$$

$$\mathcal{L}|A|^2 = |A|^2 - 2|A|^4 + 2|\nabla A|^2, \tag{3.1.2}$$

and

$$L\langle v, \mathbf{n}\rangle = \frac{1}{2}\langle v, \mathbf{n}\rangle, \tag{3.1.3}$$

where $v \in \mathbf{R}^{n+1}$ is a constant vector.

Now we introduce the notion of *δ-stability* for self-shrinkers and prove a stability type inequality for graphical self-shrinkers.

Throughout, we will set $\rho = \mathrm{e}^{-|x|^2/4}$.

Definition 3.1.5. *Given δ, we say that a self-shrinker Σ is* δ-stable *in a domain Ω if*

$$\int_\Sigma (-\phi L\phi)\rho + \delta \int_\Sigma \phi^2 \rho \geq 0 \tag{3.1.4}$$

for any compactly supported function ϕ in Ω.

Note that integrating by parts gives that (3.1.4) is equivalent to

$$\int_\Sigma \Big(|A|^2 + \frac{1}{2} - \delta\Big)\phi^2\rho \leq \int_\Sigma |\nabla\phi|^2\rho. \tag{3.1.5}$$

In particular, when $\delta = 0$, our 0-stability is just the L-stability defined in [CM12b]. The next lemma shows that if a self-shrinker is graphical in B_R, then it is $\frac{1}{2}$-stable in B_R. The proof is essentially the same as Lemma 2.1 in [Wan11a]. For convenience of the reader, we also include a proof here.

Lemma 3.1.6. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a self-shrinker which is graphical in B_R, then it is $\frac{1}{2}$-stable in B_R, i.e., for any compactly supported function ϕ in B_R, we have*

$$\int_\Sigma |A|^2\phi^2\rho \leq \int_\Sigma |\nabla\phi|^2\rho. \tag{3.1.6}$$

Proof. Since Σ is graphical in B_R, we can find a constant vector $v \in \mathbf{R}^{n+1}$ such that $w(x) = \langle v, \mathbf{n}(x)\rangle$ is positive on $B_R \cap \Sigma$. Lemma 3.1.4 gives that $Lw = \frac{1}{2}w$. Hence, the function $h = \log w$ is well-defined, and it follows that h satisfies the equation

$$\mathcal{L}h = -|\nabla h|^2 - |A|^2. \tag{3.1.7}$$

For any compactly supported function ϕ in B_R, multiplying by $\phi^2\rho$ on both sides of (3.1.7) and integrating by parts, we then have

$$\int_\Sigma (|A|^2 + |\nabla h|^2)\phi^2\rho = -\int_\Sigma (\phi^2 \mathcal{L} h)\rho = \int_\Sigma 2\phi\langle\nabla\phi, \nabla h\rangle\rho. \tag{3.1.8}$$

Combining this with the inequality $2\phi\langle\nabla\phi, \nabla h\rangle \leq \phi^2|\nabla h|^2 + |\nabla\phi|^2$ gives

$$\int_\Sigma |A|^2\phi^2\rho \leq \int_\Sigma |\nabla\phi|^2\rho. \tag{3.1.9}$$

□

Remark 3.1.7. *Similar to the above and to the proof of Lemma 9.15 in [CM12a], we can prove that if Σ is a self-shrinker and ψ is a nontrivial function on Σ with $L\psi = \mu\psi$, and $\psi > 0$ on $\Sigma \cap B_R$, then Σ is μ-stable on B_R.*

In particular, a self-shrinker with $H > 0$ on $B_R \cap \Sigma$ is 1-stable in B_R.

We will often refer the inequality (3.1.6) as *the stability inequality.*

Cheng and Zhou [CZ13] (see also [DX13]) proved the equivalence of finite entropy, polynomial volume growth and properness of self-shrinkers. In particular, if Σ^n is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$, then for any p and $r > 0$, we have

$$\mathrm{Vol}(B_r(p) \cap \Sigma) \leq \mathrm{e}^{-\frac{1}{4}} \int_{B_r(p)\cap\Sigma} \mathrm{e}^{-\frac{|x-p|^2}{4r^2}} \leq \mathrm{e}^{-\frac{1}{4}} (4\pi)^{\frac{n}{2}} \lambda_0\, r^n. \tag{3.1.10}$$

In proofs we will often allow a constant C to change from line to line; nevertheless C will always depend only on the parameters as stated in the respective theorems. To emphasize particular cases where constants differ we will use primes (C', C'').

3.1.2 Integral curvature estimate and improvement

First, we use the rapid decay of the Gaussian weight $\rho = \mathrm{e}^{-|x|^2/4}$ to show that shrinkers which are $\frac{1}{2}$-stable on large balls B_R satisfy an integral curvature estimate that decays exponentially in R.

Proposition 3.1.8. *Given n and λ_0, there exists $C = C(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$, which is $\frac{1}{2}$-stable in B_R for some $R > 1$, then we have*

$$\int_{B_{R-1}\cap\Sigma} |A|^2 \leq CR^n e^{-\frac{R}{4}}. \tag{3.1.11}$$

Proof. Let $a > 0$. We choose a smooth cutoff function ϕ so that $\phi \equiv 1$ on B_{R-a}, $\phi \equiv 0$ outside B_R and $|\nabla\phi| \leq \frac{2}{a}$. From the stability inequality (3.1.6), since $|\nabla\phi|$ is supported in $B_R \setminus B_{R-a}$, we get

$$\int_{B_{R-a}\cap\Sigma} |A|^2\rho \leq \frac{4}{a^2}\mathrm{e}^{-\frac{1}{4}(R-a)^2}\,\mathrm{Vol}(B_R \cap \Sigma) \leq \frac{C}{a^2}R^n\mathrm{e}^{-\frac{1}{4}(R-a)^2}, \tag{3.1.12}$$

where we used the volume estimate (3.1.10) for the second inequality. Therefore, we obtain that

$$\int_{B_{R-2a}\cap\Sigma} |A|^2 \leq \mathrm{e}^{\frac{1}{4}(R-2a)^2}\int_{B_{R-a}\cap\Sigma} |A|^2\rho \leq \frac{C'}{a^2}R^n\mathrm{e}^{-\frac{a}{2}R}. \tag{3.1.13}$$

Taking $a = \frac{1}{2}$ gives the result. □

Using the pointwise curvature estimate in Theorem 3.2.1 together with the very tight integral estimate in Proposition 3.1.8, we are able to improve the pointwise estimate and show that the curvature is in fact uniformly small, so long as R is sufficiently large.

Theorem 3.1.9. *Given n, λ_0, C and $\delta > 0$, there exists $R_0 = R_0(n, \lambda, C, \delta)$ such that if $R \geq R_0 > 2$ and $\Sigma^n \subset \mathbb{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$, which is $\frac{1}{2}$-stable in B_R and satisfies*

$$|A|(x) \leq C(1 + |x|), \qquad \textit{for all } x \in B_{R-1} \cap \Sigma, \tag{3.1.14}$$

then in fact

$$|A|(x) \leq \delta, \qquad \textit{for all } x \in B_{R-2} \cap \Sigma. \tag{3.1.15}$$

Proof. We will use the Simons-type inequality for self-shrinkers. By Lemma 3.1.4, we get

$$\begin{aligned}\Delta|A|^2 &= \frac{1}{2}\langle x, \nabla|A|^2\rangle + 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 + 2|\nabla A|^2 \\ &\geq -\frac{1}{8}|x|^2|A|^2 - 2|\nabla|A||^2 + 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 + 2|\nabla A|^2 \\ &\geq -\frac{1}{8}|x|^2|A|^2 + |A|^2 - 2|A|^4.\end{aligned} \tag{3.1.16}$$

The assumed curvature estimate (3.1.14) allows us to estimate part of the $|A|^4$ term, turning the above into a linear differential inequality. Specifically, for $x \in B_{R-1} \cap \Sigma$ we have

$$\Delta|A|^2 \geq -\frac{R^2}{8}|A|^2 - 2CR^2|A|^2 = -C'R^2|A|^2. \tag{3.1.17}$$

This will allow us to use the mean value inequality as follows. Fix $x_0 \in B_{R-2} \cap \Sigma$ and set

$$g(s) = s^{-n} \int_{B_s(x_0)\cap\Sigma} |A|^2. \tag{3.1.18}$$

Using a mean value inequality for general hypersurfaces (see Lemma 1.18 in [CM11]), we have the following inequality:

$$g'(s) \geq \frac{1}{2s^{n+1}} \int_{B_s(x_0)\cap\Sigma} (s^2 - |x-x_0|^2)\Delta|A|^2 - \frac{1}{s^{n+1}} \int_{B_s(x_0)\cap\Sigma} |A|^2\langle x-x_0, H\mathbf{n}\rangle. \tag{3.1.19}$$

By the shrinker equation, we have $|H| \leq \frac{1}{2}|x| \leq \frac{R}{2}$, so together with (3.1.17) we then obtain

$$g'(s) \geq -C'R^2s^{1-n} \int_{B_s(x_0)\cap\Sigma} |A|^2 - \frac{R}{2}s^{-n} \int_{B_s(x_0)\cap\Sigma} |A|^2 = -C'R^2sg(s) - \frac{R}{2}g(s). \tag{3.1.20}$$

Therefore, the quantity

$$g(s)\exp\left(C'R^2s^2 + \frac{R}{2}s\right) \tag{3.1.21}$$

is nondecreasing in s. Applying this monotonicity at scale $s = R^{-1} \leq 1$ and using the integral estimate Proposition 3.1.8 gives

$$|A|^2(x_0) \leq \frac{e^{C'+\frac{1}{2}}}{\omega_n} R^n \int_{B_{\frac{1}{R}}(x_0)\cap\Sigma} |A|^2 \leq C'' R^n \int_{B_R\cap\Sigma} |A|^2 \leq C R^{2n} e^{-\frac{R}{4}}. \tag{3.1.22}$$

Here ω_n is the volume of the unit ball in $\mathbf{R}^n$.

The exponential factor decays faster than any polynomial factor, so taking R large enough gives the desired result. □

Remark 3.1.10. *Note that for $n \leq 6$, the curvature hypothesis (3.1.14) is automatically satisfied by Theorem 3.2.1.*

3.1.3 Proof of Theorem 3.1.3

Now, we will give the proof of Theorem 3.1.3 by assuming the curvature estimate Theorem 3.2.1. To finish the proof, we also need two results on shrinkers. The first result is the following smooth version of the compactness theorem for self-shrinkers in [CIM15] (see also Lemma 3.3.9).

Lemma 3.1.11 ([CIM15]). *Let $\Sigma_i \subset \mathbf{R}^{n+1}$ be a sequence of smooth self-shrinkers with $\lambda(\Sigma_i) \leq \lambda_0$ and*

$$|A| \leq C \quad \text{on } B_{R_i} \cap \Sigma_i, , \tag{3.1.23}$$

where $R_i \to \infty$. Then there exists a subsequence Σ_i' that converges smoothly and with multiplicity one to a complete embedded self-shrinker Σ with $|A| \leq C$ and

$$\lim_{i\to\infty} \lambda(\Sigma_i') = \lambda(\Sigma). \tag{3.1.24}$$

The second result is the following entropy gap theorem of Brakke.

Lemma 3.1.12 ([Bra78]). *There exists $\varepsilon(n) > 0$ such that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth complete self-shrinker with*

$$\lambda(\Sigma^n) \leq 1 + \varepsilon, \tag{3.1.25}$$

then Σ^n must be a hyperplane.

Proof of Theorem 3.1.3. Given $n \leq 6$ and λ_0, Theorem 3.2.1 gives a constant $C = C(n, \lambda_0)$ that controls the linear growth of $|A|$. Moreover, Theorem 3.1.9 enables us to make $|A|$ as small as we want. In particular, we can choose R_0 such that if Σ^n is graphical in B_R for $R \geq R_0$, then $|A| \leq 1/2$ for all $x \in B_{R-2} \cap \Sigma$.

Now we claim that there exists a constant $R \geq R_0$ such that Theorem 3.1.3 holds. Otherwise, there is a sequence of smooth, complete, non-flat self-shrinkers $\Sigma_i \subset \mathbf{R}^{n+1}$ ($\Sigma_i \neq \mathbf{R}^n$) with $\lambda(\Sigma_i) \leq \lambda_0$ and Σ_i is graphical in B_{R_i} for $R_i \to \infty$ and $R_i \geq R_0$. Theorem 3.1.9 gives that $|A| \leq 1/2$ on $B_{R_i-2} \cap \Sigma_i$. Applying the compactness of Lemma 3.1.11, there is a subsequence Σ_i' that converges smoothly and with multiplicity one to a complete embedded self-shrinker Σ with $|A| \leq 1/2$ and

$$\lim_{i\to\infty} \lambda(\Sigma_i') = \lambda(\Sigma) .$$

Recall that any smooth complete self-shrinker with $|A|^2 < 1/2$ is a hyperplane (see [CL13]), so in particular the limit Σ must be a hyperplane. By the the convergence in entropy and Lemma 3.1.12, we know that for sufficiently large i, the self-shrinker Σ_i must be a hyperplane. This provides the desired contradiction, completing the proof. □

If we assume that $\langle V, \mathbf{n}\rangle$ has a positive lower bound for some constant unit vector V, then in all dimensions we have the following rigidity theorem.

Theorem 3.1.13. *Given n, λ_0 and $\delta > 0$, there exists $R = R(n, \lambda_0, \delta)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

- *$w = \langle V, \mathbf{n}\rangle \geq \delta$ on $B_R \cap \Sigma$ for some constant unit vector V,*

then Σ is a hyperplane.

The proof of Theorem 3.1.13 follows easily from the following curvature estimate (which is a direct consequence of Ecker-Huisken [EH91, Theorem 3.1]) and some ingredients from [GZ15].

Theorem 3.1.14. *Given n and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ which satisfies*

- *$w = \langle V, \mathbf{n} \rangle \geq \delta$ on $B_R \cap \Sigma$ for some constant unit vector V and $R > 2$,*

we have

$$|A| \leq C, \quad \text{on } B_{R/2} \cap \Sigma. \tag{3.1.26}$$

3.2 Curvature estimates for almost stable shrinkers

In this section, we will prove the following curvature estimate which implies Theorem 1.0.2. We will give a self-contained proof for the cases $n \leq 5$. We will also sketch how the result follows for $n \leq 6$ from the arguments of Schoen-Simon [SS81].

Theorem 3.2.1. *Given $n \leq 6$ and λ_0, there exists $C = C(n, \lambda_0)$ so that for any self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

- *Σ is $\frac{1}{2}$-stable in B_R for $R > 2$,*

we have

$$|A|(x) \leq C(1 + |x|), \quad \text{for all } x \in B_{R-1} \cap \Sigma. \tag{3.2.1}$$

The $\frac{1}{2}$-stability here can be replaced by δ-stability for a fixed $\delta \leq 1$, in particular, this includes the strictly mean convex case $H > 0$ and the L-stability.

In [Son14] and [Wan15], the authors also showed the linear growth of the second fundamental form for properly embedded self-shrinkers with finite genus in $\mathbf{R}^3$. As a consequence, they obtained some results for two-dimensional shrinkers.

3.2.1 Small energy curvature estimates

First, we prove a small energy curvature estimate of Choi-Schoen [CS85] type for self-shrinkers, allowing us to obtain pointwise estimates on $|A|$ from suitable integral estimates. Similar estimates can also be found in [SU81] for harmonic maps.

Theorem 3.2.2. *There exists $\varepsilon = \varepsilon(n) > 0$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is self-shrinker, properly embedded in $B_{r_0}(x_0)$ for some $r_0 \leq \theta = \min\{1, |x_0|^{-1}\}$, which satisfies*

$$\int_{B_{r_0}(x_0)\cap\Sigma} |A|^n < \varepsilon, \tag{3.2.2}$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x_0) \cap \Sigma$,

$$\sigma^2|A|^2(y) \leq 1. \tag{3.2.3}$$

Proof. We will follow the Choi-Schoen type argument and argue by contradiction.

Consider the function f defined on $B_{r_0}(x_0)\cap\Sigma$ by $f(x) = (r_0-r(x))^2|A|^2(x)$, where $r(x) = |x - x_0|$. This function vanishes on $\partial B_{r_0}(x_0)$, so it achieves its maximum at some $y_0 \in B_{r_0}(x_0) \cap \Sigma$. If $f(y_0) \leq 1$, then we are done. So we assume $f(y_0) \geq 1$ and will show that this leads to a contradiction for ε sufficiently small.

Choose $\sigma > 0$ so that

$$\sigma^2|A|^2(y_0) = \frac{1}{4}. \tag{3.2.4}$$

Then $f(y_0) \geq 1$ implies that $2\sigma \leq r_0 - r(y_0)$. Using this bound for σ we see that $B_\sigma(y_0) \subset B_{r_0}(x_0)$, and the triangle inequality gives that for $x \in B_\sigma(y_0)$, we have

$$\frac{1}{2} \leq \frac{r_0 - r(x)}{r_0 - r(y_0)} \leq \frac{3}{2}. \tag{3.2.5}$$

Combining (3.2.5) with the fact that f achieves its maximum at y_0, we get that

$$(r_0 - r(y_0))^2 \sup_{B_\sigma(y_0)\cap\Sigma} |A|^2 \leq 4 \sup_{B_\sigma(y_0)\cap\Sigma} f = 4f(y_0) = 4(r_0 - r(y_0))^2|A|^2(y_0). \tag{3.2.6}$$

This gives the following estimate

$$\sup_{B_\sigma(y_0)\cap\Sigma} |A|^2 \leq 4|A|^2(y_0) = \frac{1}{\sigma^2}. \tag{3.2.7}$$

By the Simons-type inequality (3.1.16), it follows from (3.2.7) that on $B_\sigma(y_0)\cap\Sigma$,

$$\Delta|A|^2 \geq -\frac{1}{8}|x|^2|A|^2 - \frac{2}{\sigma^2}|A|^2. \tag{3.2.8}$$

A simple computation then gives

$$\Delta|A|^n \geq -\frac{n}{2}\Big(\frac{1}{8}|x|^2 + \frac{2}{\sigma^2}\Big)|A|^n. \tag{3.2.9}$$

Next, for $0 < s \leq \sigma$, we define the function $g(s)$ by

$$g(s) = \frac{1}{s^n}\int_{\Sigma_{y_0,s}} |A|^n,$$

where we used $\Sigma_{y_0,s}$ to denote $B_s(y_0)\cap\Sigma$.

Again using the mean value inequality (3.1.19), we also have

$$g'(s) \geq \frac{1}{2s^{n+1}}\int_{\Sigma_{y_0,s}} (s^2 - |x-y_0|^2)\Delta|A|^n - \frac{1}{s^{n+1}}\int_{\Sigma_{y_0,s}} |A|^n\langle x-y_0, H\mathbf{n}\rangle. \tag{3.2.10}$$

Combining this with (3.2.9) gives

$$\begin{aligned} g'(s) &\geq -\frac{1}{2s^{n-1}}\int_{\Sigma_{y_0,s}} \frac{n}{2}\Big(\frac{1}{8}|x|^2 + \frac{2}{\sigma^2}\Big)|A|^n - \frac{1}{s^n}\frac{\sqrt{n}}{\sigma}\int_{\Sigma_{y_0,s}} |A|^n \\ &\geq -\frac{n}{4}\Big(\frac{1}{8}(|x_0|+1)^2 + \frac{2}{\sigma^2}\Big)sg(s) - \frac{\sqrt{n}}{\sigma}g(s), \end{aligned} \tag{3.2.11}$$

where we used that $|x| \leq (|x_0|+1)$ and $|H| \leq \sqrt{n}|A| \leq \frac{\sqrt{n}}{\sigma}$ for $x\in B_\sigma(y_0)\cap\Sigma$.

It follows that the function

$$h(t) = g(t)\exp\Big\{\Big(\frac{n}{4}\Big(\frac{(|x_0|+1)^2}{8} + \frac{2}{\sigma^2}\Big)\frac{t^2}{2} + \frac{\sqrt{n}}{\sigma}t\Big\} \tag{3.2.12}$$

is non-decreasing for $0 < t \leq \sigma$.

Applying at $t = \sigma$, since $\sigma \leq r_0 \leq \min\{1, |x_0|^{-1}\}$, we get

$$\frac{\omega_n}{2^n\sigma^n} = \omega_n|A|^n(y_0) = h(0) \leq h(\sigma) \leq \mathrm{e}^{\frac{5n}{16}+\sqrt{n}}\frac{1}{\sigma^n}\int_{B_\sigma(y_0)\cap\Sigma} |A|^n \leq \frac{\mathrm{e}^{2n}}{\sigma^n}\varepsilon, \tag{3.2.13}$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$.

This gives a contradiction for sufficiently small ε. $\square$

3.2.2 Schoen-Simon-Yau type estimates

Theorem 3.2.2 holds in any dimension, but of course the hypotheses require an L^n bound on $|A|$, and at face value the stability inequality (3.1.6) only provides an L^2 bound. So we use the idea of Schoen-Simon-Yau [SSY75] to derive a higher dimensional version of the stability inequality. This is the next theorem which we call the Schoen-Simon-Yau type estimates for self-shrinkers.

Theorem 3.2.3. *Suppose that $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth self-shrinker which is $\frac{1}{2}$-stable in B_R, and let ϕ be a smooth function with compact support in B_R. Then for all $q \in [0, \sqrt{2/(n+1)})$, we have*

$$\int_\Sigma |A|^{4+2q}\phi^2 e^{-\frac{|x|^2}{4}} \leq C\int_\Sigma |A|^{2+2q}|\nabla\phi|^2 e^{-\frac{|x|^2}{4}} + CR^2\int_\Sigma |A|^{2+2q}\phi^2 e^{-\frac{|x|^2}{4}}, \qquad (3.2.14)$$

for some $C = C(n,q)$.

Proof. We apply the stability inequality (3.1.6) with test function $|A|^{1+q}\phi$. This gives

$$\begin{aligned}\int_\Sigma |A|^{4+2q}\phi^2\rho \leq \int_\Sigma |A|^{2+2q}|\nabla\phi|^2\rho + (1+q)^2\int_\Sigma |A|^{2q}\phi^2|\nabla|A||^2\rho \\ + 2(1+q)\int_\Sigma |A|^{1+2q}\phi\langle\nabla\phi, \nabla|A|\rangle\rho.\end{aligned} \qquad (3.2.15)$$

From the Simons-type identity for self-shrinkers, i.e., Lemma 3.1.4, we have

$$|A|\mathcal{L}|A| = \frac{1}{2}|A|^2 - |A|^4 + |\nabla A|^2 - |\nabla|A||^2. \qquad (3.2.16)$$

Now we use the following inequality for general hypersurfaces from [CM12a, Lemma 10.2]

$$\left(1 + \frac{2}{n+1}\right)|\nabla|A||^2 \leq |\nabla A|^2 + \frac{2n}{n+1}|\nabla H|^2. \qquad (3.2.17)$$

The self-shrinker equation implies that $|\nabla H| \leq \frac{1}{2}|x||A| \leq \frac{R}{2}|A|$ for all $x \in B_R(0)\cap\Sigma$.

Therefore,

$$|A|\mathcal{L}|A| \geq -|A|^4 + \frac{2}{n+1}|\nabla|A||^2 - \frac{n}{n+1}\frac{R^2}{2}|A|^2. \tag{3.2.18}$$

Multiplying by $|A|^{2q}\phi^2\rho$ and integrating by parts, we get

$$\begin{aligned} \int_\Sigma |A|^{1+2q}\phi^2\mathcal{L}|A|\rho &= -\int_\Sigma \langle \nabla|A|, \nabla(|A|^{1+2q}\phi^2)\rangle\rho \\ &= -\int_\Sigma |A|^{1+2q}\langle\nabla|A|, \nabla\phi^2\rangle\rho - (1+2q)\int_\Sigma |A|^{2q}\phi^2|\nabla|A||^2\rho \\ &\geq \int_\Sigma \left(\frac{2}{n+1}|\nabla|A||^2 - \frac{nR^2}{2(n+1)}|A|^2 - |A|^4\right)|A|^{2q}\phi^2\rho. \end{aligned} \tag{3.2.19}$$

This implies

$$\begin{aligned} \frac{2}{n+1}\int_\Sigma |\nabla|A||^2|A|^{2q}\phi^2\rho \leq& \int_\Sigma |A|^{4+2q}\phi^2\rho + \frac{nR^2}{2(n+1)}\int_\Sigma |A|^{2+2q}\phi^2\rho \\ &- 2\int_\Sigma |A|^{1+2q}\phi\langle\nabla\phi, \nabla|A|\rangle\rho - (1+2q)\int_\Sigma |A|^{2q}\phi^2|\nabla|A||^2\rho. \end{aligned} \tag{3.2.20}$$

Combining (3.2.15) and (3.2.20) gives

$$\begin{aligned} \frac{2}{n+1}\int_\Sigma |\nabla|A||^2|A|^{2q}\phi^2\rho \leq& \int_\Sigma |A|^{2+2q}|\nabla\phi|^2\rho + q^2\int_\Sigma |A|^{2q}|\nabla|A||^2\phi^2\rho \\ &+ 2q\int_\Sigma |A|^{1+2q}\phi\langle\nabla\phi, \nabla|A|\rangle\rho + \frac{nR^2}{2(n+1)}\int_\Sigma |A|^{2+2q}\phi^2\rho. \end{aligned} \tag{3.2.21}$$

Using the absorbing inequality, we obtain that $2|A|\phi\langle\nabla\phi, \nabla|A|\rangle \leq \frac{1}{a}|A|^2|\nabla\phi|^2 + a|\nabla|A||^2\phi^2$. Therefore, we get

$$\begin{aligned} \left(\frac{2}{n+1} - q^2 - aq\right)\int_\Sigma |\nabla|A||^2|A|^{2q}\phi^2\rho \leq& \left(1+\frac{q}{a}\right)\int_\Sigma |A|^{2+2q}|\nabla\phi|^2\rho \\ &+ \frac{nR^2}{2(n+1)}\int_\Sigma |A|^{2+2q}\phi^2\rho. \end{aligned} \tag{3.2.22}$$

Applying the Cauchy-Schwarz inequality to absorb the cross-term in (3.2.15) and then substituting (3.2.22) gives

$$\begin{aligned}\int_\Sigma |A|^{4+2q}\phi^2\rho \leq \left(2+\frac{2(1+q)^2(1+\frac{q}{a})}{\frac{2}{n+1}-q^2-aq}\right)\int_\Sigma |A|^{2+2q}|\nabla\phi|^2\rho \\ + \frac{\frac{n}{n+1}(1+q)^2R^2}{\frac{2}{n+1}-q^2-aq}\int_\Sigma |A|^{2+2q}\phi^2\rho.\end{aligned} \tag{3.2.23}$$

Thus so long as $q^2 < \frac{2}{n+1}$, there exists a constant C depending on n and q such that

$$\int_\Sigma |A|^{4+2q}\phi^2\rho \leq C\int_\Sigma |A|^{2+2q}|\nabla\phi|^2\rho + CR^2\int_\Sigma |A|^{2+2q}\phi^2\rho. \tag{3.2.24}$$

□

Remark 3.2.4. *In particular, if we set $q=0$, then*

$$\int_\Sigma |A|^4\phi^2\rho \leq C\int_\Sigma |A|^2|\nabla\phi|^2\rho + CR^2\int_\Sigma |A|^2\phi^2\rho. \tag{3.2.25}$$

It will also be useful to record that (for $n\leq 6$) we may set $q=1/2$ to obtain

$$\int_\Sigma |A|^5\phi^2\rho \leq C\int_\Sigma |A|^3|\nabla\phi|^2\rho + CR^2\int_\Sigma |A|^3\phi^2\rho. \tag{3.2.26}$$

3.2.3 Proof of Theorem 3.2.1

We now record a lemma that estimates a 'scale-invariant energy' $r^{p-n}\int_{B_r(x)\cap\Sigma}|A|^p$. This lemma will be convenient for obtaining our curvature estimates for low dimensions, but in fact holds for all $n\geq 2$.

Lemma 3.2.5. *Given $n\geq 2$, $\lambda_0>0$ and $2\leq p\leq 4$, there exists $C=C(n,p,\lambda_0)$ so that if $\Sigma^n\subset\mathbf{R}^{n+1}$ is a self-shrinker with $\lambda(\Sigma)\leq\lambda_0$ and Σ is $\frac{1}{2}$-stable in B_R for some $R>2$, then for all $x_0\in B_{R-1}\cap\Sigma$ and $r\leq\frac{1}{2}\theta$, where $\theta=\min\{1,|x_0|^{-1}\}$, we have*

$$\int_{B_r(x_0)\cap\Sigma}|A|^p \leq Cr^{n-p}. \tag{3.2.27}$$

Proof. Fix $x_0\in B_{R-1}\cap\Sigma$, and set $r(x)=|x-x_0|$. Also let $r_0\leq\theta$. First note that with this choice of r_0, we have

$$\frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \le \mathrm{e}. \tag{3.2.28}$$

This is clear if $|x_0| \le 1$, since in this case any $x \in B_{r_0}(x_0)$ satisfies $|x| \le |x_0| + r_0 \le 2$. On the other hand, if $|x_0| \ge 1$ then $r_0 \le |x_0|$, so $\mathrm{e}^{-\frac{1}{4}(|x_0|+r_0)^2} \le \mathrm{e}^{-\frac{1}{4}|x|^2} \le \mathrm{e}^{-\frac{1}{4}(|x_0|-r_0)^2}$ for $x \in B_{r_0}(x_0)$, and hence

$$\frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \le \mathrm{e}^{|x_0| r_0} \le \mathrm{e}. \tag{3.2.29}$$

Now we may fix a smooth cutoff function ϕ with $\phi = 1$ if $r \le r_0$, $\phi = 0$ if $r > 2r_0$, and such that $|\nabla \phi| \le \frac{2}{r_0}$.

From the stability inequality (3.1.6) and the above discussion we have

$$\begin{aligned}\int_{B_{r_0}(x_0)\cap\Sigma} |A|^2 &\le \frac{1}{\inf_{B_{r_0}(x_0)} \rho} \int_\Sigma |A|^2 \phi^2 \rho \le \frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \int_\Sigma |\nabla \phi|^2 \\ &\le \frac{4\mathrm{e}}{r_0^2} \operatorname{Vol}(B_{2r_0}(x_0) \cap \Sigma) \le C r_0^{n-2},\end{aligned} \tag{3.2.30}$$

since $r_0 \le \frac{1}{|x_0|}$. Again we have used the volume bound (3.1.10) for the last inequality. This completes the proof for $p = 2$.

Now further suppose $r_0 \le \frac{1}{2}\theta$. Arguing as above using Remark 3.2.4 ($q = 0$), we obtain

$$\int_{B_{r_0}(x_0)\cap\Sigma} |A|^4 \le C\left(\frac{4}{r_0^2} + (|x_0|+1)^2\right) \int_{B_{2r_0}(x_0)\cap\Sigma} |A|^2. \tag{3.2.31}$$

We may now apply (3.2.30) at radius $2r_0$ to conclude that

$$\int_{B_{r_0}(x_0)\cap\Sigma} |A|^4 \le C\left(\frac{4}{r_0^2} + (|x_0|+1)^2\right) r_0^{n-2} \le C' r_0^{n-4}, \tag{3.2.32}$$

where we have again used that $r_0 \le \min(1, |x_0|^{-1})$. This completes the proof for $p = 4$.

For $2 < p < 4$, we obtain the desired result by interpolating (3.2.30) and (3.2.31) using Hölder's inequality. □

We are now ready to give, in the case $n \leq 5$, the proof of the curvature estimate Theorem 3.2.1.

Proof of Theorem 3.2.1 for $n \leq 5$. Fix $p \in B_{R-1} \cap \Sigma$, and set $r(x) = |x - p|$. Also let $r_0 \leq \frac{1}{4} \min\{1, |p|^{-1}\}$. We have

$$\frac{\sup_{B_{r_0}(p)} \rho}{\inf_{B_{r_0}(p)} \rho} \leq \mathrm{e}. \tag{3.2.33}$$

The goal is to show that Σ has small energy at a uniform scale δr_0, in the sense that

$$\int_{B_{\delta r_0}(p) \cap \Sigma} |A|^n < \varepsilon, \tag{3.2.34}$$

where ε is as in Theorem 3.2.2 and δ depends only on n and λ_0. From Theorem 3.2.2 we would then conclude that

$$|A|(p) \leq \frac{1}{\delta r_0} \leq C(1 + |p|) \tag{3.2.35}$$

as claimed.

To achieve this, we will use a logarithmic cutoff function. In particular, we fix a large integer k to be determined later, and define a cutoff function η by

$$\eta = \begin{cases} 1 & \text{if } r \leq \mathrm{e}^{-k} r_0, \\ \frac{\log(r_0) - \log(r)}{k} & \text{if } \mathrm{e}^{-k} r_0 < r \leq r_0, \\ 0 & \text{if } r > r_0. \end{cases} \tag{3.2.36}$$

Note that $|\nabla \eta| \leq \frac{1}{kr}$ and $|\nabla \eta|$ is supported in the annulus between $\mathrm{e}^{-k} r_0$ and r_0. Using (3.2.33) as in the proof of Lemma 3.2.5, we obtain from the stability inequality (3.1.6) that

$$\int_{B_{\mathrm{e}^{-k} r_0}(p) \cap \Sigma} |A|^2 \eta^2 \leq \frac{\sup_{B_{r_0}(p)} \rho}{\inf_{B_{r_0}(p)} \rho} \int_\Sigma |\nabla \eta|^2 \leq \frac{\mathrm{e}}{k^2} \int_{(B_{r_0}(p) \setminus B_{\mathrm{e}^{-k} r_0}(p)) \cap \Sigma} \frac{1}{r^2}. \tag{3.2.37}$$

Since $n \geq 2$ we can use the usual trick as well as the volume estimate (3.1.10) to bound the integral:

$$\int_{(B_{r_0}(p)\setminus B_{e^{-k}r_0}(p))\cap\Sigma}\frac{1}{r^2}=\sum_{l=0}^{k-1}\int_{(B_{e^{-l}r_0}(p)\setminus B_{e^{-l-1}r_0}(p))\cap\Sigma}\frac{1}{r^2}\leq C\sum_{l=0}^{k-1}e^{2(l+1)}r_0^{-2}e^{-nl}r_0^n$$
$$\leq C'r_0^{n-2}\sum_{l=0}^{k-1}e^{-(n-2)l}\leq C'kr_0^{n-2}. \tag{3.2.38}$$

Thus

$$\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^2\leq\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^2\eta^2\leq\frac{C}{k}r_0^{n-2}. \tag{3.2.39}$$

This completes the proof for $n=2$, by taking k sufficiently large.

Now using the Cauchy-Schwarz inequality and that $B_{e^{-k}r_0}(p)\subset B_{r_0}(p)$, we have

$$\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^3\eta^2\leq\left(\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^2\eta^2\right)^{\frac{1}{2}}\left(\int_{B_{r_0}(p)\cap\Sigma}|A|^4\eta^2\right)^{\frac{1}{2}}. \tag{3.2.40}$$

Using (3.2.37) for the $|A|^2$ factor and Lemma 3.2.5 for the $|A|^4$ factor we conclude that

$$\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^3\leq\int_{B_{e^{-k}r_0}(p)\cap\Sigma}|A|^3\eta^2\leq\frac{C}{k}r_0^{n-3}. \tag{3.2.41}$$

Thus, again by taking k sufficiently large, the result follows for $n=3$.

Consider the remaining cases $n=4,5$. Using (3.2.33) to estimate the weight ρ as before, and applying Remark 3.2.4, we obtain

$$\int_\Sigma|A|^n\eta^2\leq C\left(\int_\Sigma|A|^{n-2}|\nabla\eta|^2+(|p|+1)^2\int_\Sigma|A|^{n-2}\eta^2\right). \tag{3.2.42}$$

By (3.2.38) or (3.2.41) respectively, the second term can be estimated using that $r_0\leq\min(1,|p|^{-1})$,

$$(|p|+1)^2\int_\Sigma|A|^{n-2}\eta^2\leq\frac{C}{k}(|p|+1)^2r_0^{n-(n-2)}\leq\frac{C'}{k}. \tag{3.2.43}$$

To handle the first term we use the logarithmic trick again:

$$\begin{aligned}\int_\Sigma |A|^{n-2}|\nabla\eta|^2 &\le \frac{1}{k^2}\sum_{l=0}^{k-1}\int_{B_{\mathrm{e}^{-l}r_0(p)}\setminus B_{\mathrm{e}^{-l-1}r_0(p)}} |A|^{n-2}\frac{1}{r^2} \\ &\le \frac{C}{k^2}\sum_{l=0}^{k-1} \mathrm{e}^{2(l+1)}r_0^{-2}\mathrm{e}^{(n-(n-2))l}r_0^{n-(n-2)} = \frac{C'}{k}.\end{aligned} \tag{3.2.44}$$

Here, we used Lemma 3.2.5 to get to the second line. Thus

$$\int_{B_{\mathrm{e}^{-k}r_0}(p)\cap\Sigma} |A|^n \le \frac{C}{k} \tag{3.2.45}$$

in the remaining cases $n = 4, 5$, and the proof is complete. □

To obtain the desired curvature estimate Theorem 3.2.1 in the case $n = 6$, we need to apply the Schoen-Simon theory [SS81], with a few minor modifications that we will outline here. Of course, this argument will indeed apply for all $2 \le n \le 6$.

The key point is to apply the theory at the optimal scale $\theta = \min(1, |x_0|^{-1})$ (see for instance Section 12 of [CM12a]). At this scale, the conformal metric $g_{ij} = \mathrm{e}^{-\frac{|x|^2}{2n}}\delta_{ij}$ is uniformly Euclidean in the sense that the volume form $\rho = \mathrm{e}^{-\frac{|x|^2}{4}}$ satisfies the familiar estimate

$$\frac{\sup_{B_\theta(x_0)}\rho}{\inf_{B_\theta(x_0)}\rho} \le \mathrm{e}, \tag{3.2.46}$$

with similar uniform estimates for its derivatives.

In particular, consider a smooth self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$, which has entropy $\lambda(\Sigma) \le \lambda_0$ and is $\frac{1}{2}$-stable in B_R, $R > 1$. For $x_0 \in B_{R-1}\cap\Sigma$, we wish to apply the Schoen-Simon theory to the (renormalised) F-functional

$$\mathrm{e}^{\frac{1}{4}|x_0|^2}\int_\Sigma \mathrm{e}^{-\frac{1}{4}|x|^2}. \tag{3.2.47}$$

By the uniform estimates mentioned above, it may be verified that this functional satisfies all the conditions in Section 1 of [SS81] with parameters depending only on n. Also, the entropy bound gives a required density bound (3.1.10). The main issue with this approach is that Σ may not be stable for the F functional. At the scale θ, however, the $\frac{1}{2}$-stability inequality (3.1.6) gives that

$$\int_\Sigma |A|^2\phi^2 \le \frac{\sup_{B_\theta(x_0)}\rho}{\inf_{B_\theta(x_0)}\rho}\int_\Sigma |\nabla\phi|^2 \le \mathrm{e}\int_\Sigma |\nabla\phi|^2, \tag{3.2.48}$$

for compactly supported functions ϕ in $B_\theta(x_0)$. All the arguments of [SS81] then go through with this slightly weaker stability inequality, at the cost of carrying around the universal constant e.

In particular, the conclusions of Theorem 3 of [SS81] hold, so since Σ is smooth we conclude that

$$|A|(x_0) \le C\theta^{-1} \le C(1+|x_0|), \tag{3.2.49}$$

where $C = C(n, \lambda_0)$ as desired.

Remark 3.2.6. *The Schoen-Simon argument goes through assuming only a δ-stability inequality (3.1.5) at the optimal scale $\theta = \min(1, |x_0|^{-1})$, for δ depending only on n. (That is, assuming a uniform lower bound for the least eigenvalue of L on balls $B_\theta(x_0)$.)*

In particular, the curvature estimate also applies to self-shrinkers with bounded entropy and positive mean curvature $H > 0$. In this strictly mean convex setting, the curvature estimate was already known to Colding-Ilmanen-Minicozzi. This is the next lemma.

Lemma 3.2.7. *Given $n \le 6$ and $\alpha > 0$, there exists $C = C(n,\alpha)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a shrinker with $\lambda(\Sigma) \le \alpha$ and $H > 0$ on $B_R \cap \Sigma$ for some $R > 2$, then on $B_{R-1}\cap\Sigma$ we have*

$$|A| \le C(1+|x|). \tag{3.2.50}$$

3.2.4 L-stable self-shrinkers in $\mathbf{R}^3$

In this subsection, we will show that when $n = 2$, the curvature estimate for L-stable shrinkers does not depend on the area growth. Indeed, it can be shown that L-stable shrinkers in $\mathbf{R}^3$ have (local) quadratic area growth. The result also holds for $1/2$-stable shrinkers. The main result is the following:

Theorem 3.2.8. *Let Σ^2 be a smooth properly embedded self-shrinker in $\mathbf{R}^3$. If Σ^2 is L-stable (0-stable) in B_R for $R > 2$, then for all $x \in B_{R-1} \cap \Sigma$,*

$$|A|(x) \leq C(1 + |x|), \tag{3.2.51}$$

where C is a numerical constant.

We will first establish a uniform area bound for L-stable shrinkers. We need the following lemmas relating area growth and total curvature of geodesic balls.

The first lemma is due to Shiohama and Tanaka ([ST89],[ST93]; see also [ER11]).

Lemma 3.2.9 ([ST93]). *If $\Sigma^2 \subset \mathbf{R}^3$ is a smooth surface, and $B_r^{\Sigma}(x) \subset \Sigma^2$ is the geodesic ball centered at x of radius r. We assume that $B_r^{\Sigma}(x) \cap \partial\Sigma = \emptyset$. Let $l(r)$ denote the length of the boundary $\partial B_r^{\Sigma}(x)$, then we have*

(1) for almost every r,

$$l'(r) \leq 2\pi\chi(B_r^{\Sigma}(x)) - \int_{B_r^{\Sigma}(x)} K_{\Sigma}, \tag{3.2.52}$$

(2) for all $0 \leq s < t$,

$$l(t) - l(s) \leq \int_s^t l'(r). \tag{3.2.53}$$

Here, $'$ denotes the derivative with respect to r and $\chi(B_r^{\Sigma}(x))$ is the Euler characteristic of $B_r^{\Sigma}(x)$.

Remark that $l(r)$ is not even continuous in general; see [Har64] and [ST89] for more discussion.

The second lemma is taken from Corollary 2.7 in [CM11], we will also include a proof here for the reader's convenience.

Lemma 3.2.10 ([CM11]). *If $\Sigma^2 \subset \mathbf{R}^3$ is a smooth surface, $B_{r_0}^{\Sigma}(x) \subset \Sigma^2$ is the geodesic ball centered at x of radius r_0, and $B_{r_0}^{\Sigma}(x) \cap \partial\Sigma = \emptyset$, then*

$$\int_0^{r_0} \int_0^t \int_{B_s^{\Sigma}(x)} K_{\Sigma} = \int_{B_{r_0}^{\Sigma}(x)} K_{\Sigma} \frac{(r_0 - r)^2}{2} = \int_{B_{r_0}^{\Sigma}(x)} (H^2 - |A|^2) \frac{(r_0 - r)^2}{4}, \tag{3.2.54}$$

where r is the intrinsic distance function.

Proof. If we set $f(t) = \int_0^t \int_{B_s^\Sigma(x)} K_\Sigma$ and $g(t) = (r_0 - t)^2/2$, then integration by parts twice gives that

$$\int_0^{r_0} f(t)g''(t)dt = \int_0^{r_0} f''(t)g(t)dt. \tag{3.2.55}$$

Combining this with the coarea formula gives the first equality

$$\int_0^{r_0} \int_0^t \int_{B_s^\Sigma(x)} K_\Sigma = \int_0^{r_0} \int_{\partial B_t^\Sigma(x)} K_\Sigma \frac{(r_0 - t)^2}{2} = \int_{B_{r_0}^\Sigma(x)} K_\Sigma \frac{(r_0 - r)^2}{2}. \tag{3.2.56}$$

The second equality follows from the fact that $2K_\Sigma = H^2 - |A|^2$. □

Combining these two lemmas with the coarea formula, we obtain the following corollary.

Corollary 3.2.11. *If $\Sigma^2 \subset \mathbf{R}^3$ is a smooth surface, $B_{r_0}^\Sigma(x) \subset \Sigma^2$ is the geodesic ball centered at x of radius r_0, and $B_{r_0}^\Sigma(x) \cap \partial\Sigma = \emptyset$, then*

$$Area(B_{r_0}^\Sigma(x)) - \pi {r_0}^2 \leq \int_{B_{r_0}^\Sigma(x)} |A|^2 \frac{(r_0 - r)^2}{4}. \tag{3.2.57}$$

Proof. For $0 < r \leq r_0$, by Lemma 3.2.9,

$$l'(r) \leq 2\pi\chi(B_r^\Sigma(x)) - \int_{B_r^\Sigma(x)} K_\Sigma \leq 2\pi - \int_{B_r^\Sigma(x)} K_\Sigma. \tag{3.2.58}$$

Integrating (3.2.58) gives

$$\text{Area}(B_{r_0}^\Sigma(x)) - \pi {r_0}^2 \leq - \int_0^{r_0} \int_0^t \int_{B_s^\Sigma(x)} K_\Sigma. \tag{3.2.59}$$

Now the corollary follows easily from Lemma 3.2.10. □

We will now specialize Corollary 3.2.11 to the case where Σ is a L-stable self-shrinker and this will give a uniform area bound for L-stable self-shrinkers.

Theorem 3.2.12. *If $\Sigma^2 \subset \mathbf{R}^3$ is a L-stable and 2-sided self-shrinker and $B^{\Sigma}_{r_0}(x_0)$ is a geodesic ball for $r_0 \leq \rho_0 = \min\{1, \frac{1}{|x_0|}\}$, then*

$$Area(B^{\Sigma}_{r_0}(x_0)) \leq \frac{4\pi}{4-e} r_0^2. \tag{3.2.60}$$

Proof. Combining Corollary 3.2.11 and the stability inequality gives

$$\begin{aligned} 4(\text{Area}(B^{\Sigma}_{r_0}(x_0)) - \pi {r_0}^2) &\leq \int_{B^{\Sigma}_{r_0}(x_0)} |A|^2 (r_0 - r)^2 \\ &\leq \mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \int_{B^{\Sigma}_{r_0}(x_0)} \left(|\nabla(r_0 - r)|^2 - \frac{1}{2}(r_0 - r)^2\right) \mathrm{e}^{-\frac{|x|^2}{4}} \\ &\leq \mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \int_{B^{\Sigma}_{r_0}(x_0)} \mathrm{e}^{-\frac{|x|^2}{4}}, \end{aligned} \tag{3.2.61}$$

where the last inequality used that $|\nabla r| = 1$.

Now we consider two cases $|x_0| \leq 1$ and $|x_0| \geq 1$.

If $|x_0| \leq 1$, then $r_0 \leq \rho_0 = \min\{1, \frac{1}{|x_0|}\} = 1$ and $\mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \leq \mathrm{e}$. By (3.2.61), we get

$$4(\text{Area}(B^{\Sigma}_{r_0}(x_0)) - \pi {r_0}^2) \leq \mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \int_{B^{\Sigma}_{r_0}(x_0)} \mathrm{e}^{-\frac{|x|^2}{4}} \leq \mathrm{e}\,\text{Area}(B^{\Sigma}_{r_0}(x_0)). \tag{3.2.62}$$

This gives the desired estimate (3.2.60).

If $|x_0| \geq 1$, then $r_0 \leq \rho_0 = \min\{1, \frac{1}{|x_0|}\} = \frac{1}{|x_0|} \leq |x_0|$. Note that $\mathrm{e}^{-\frac{|x|^2}{4}} \leq \mathrm{e}^{-\frac{(|x_0|-r_0)^2}{4}}$ on $B^{\Sigma}_{r_0}(x_0)$, then (3.2.61) implies that

$$\begin{aligned} 4(\text{Area}(B^{\Sigma}_{r_0}(x_0)) - \pi {r_0}^2) &\leq \mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \int_{B^{\Sigma}_{r_0}(x_0)} \mathrm{e}^{-\frac{|x|^2}{4}} \\ &\leq \mathrm{e}^{\frac{(|x_0|+r_0)^2}{4}} \int_{B^{\Sigma}_{r_0}(x_0)} \mathrm{e}^{-\frac{(|x_0|-r_0)^2}{4}} \\ &= \mathrm{e}^{|x_0| r_0} \text{Area}(B^{\Sigma}_{r_0}(x_0)) \leq \mathrm{e}\,\text{Area}(B^{\Sigma}_{r_0}(x_0)). \end{aligned} \tag{3.2.63}$$

This gives (3.2.60) and completes the proof. □

We also have an intrinsic version of Theorem 3.2.2 and the proof is essentially the same as Theorem 3.2.2.

Theorem 3.2.13. *There exists $\varepsilon > 0$ so that if $x_0 \in \Sigma^2 \subset \mathbf{R}^3$ is a smooth self-shrinker for $r_0 \leq \rho_0 = \min\{1, \frac{1}{|x_0|}\}$ which satisfies*

$$\int_{B^{\Sigma}_{r_0}(x_0)} |A|^2 < \varepsilon, \tag{3.2.64}$$

then for all $0 < \sigma \leq r_0$ and $y \in B^{\Sigma}_{r_0-\sigma}(x_0)$,

$$\sigma^2 |A|^2(y) \leq 1. \tag{3.2.65}$$

Combining Theorem 3.2.12, Theorem 3.2.13, and suitable cut-off functions gives Theorem 3.2.8.

3.3 Uniqueness theorem for generalized cylinders

The aim of this section is to prove the following uniqueness theorem:

Theorem 3.3.1 (Theorem 1.0.3). *Given $n \leq 6$ and λ_0, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies*

(‡) $H \geq 0$ on $B_R \cap \Sigma$,

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $0 \leq k \leq n$.

If we assume a positive lower bound of the mean curvature, then we obtain the following rigidity theorem that holds in all dimensions.

Theorem 3.3.2. *Given n, λ_0 and $\delta > 0$, there exists $R = R(n, \lambda_0, \delta)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies*

(‡) $H \geq \delta$ on $B_R \cap \Sigma$,

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $1 \leq k \leq n$.

3.3.1 Local curvature estimates for mean convex shrinkers

One of the key ingredients in the proof of Theorem 3.3.1 is the following local curvature estimate for mean convex shrinkers in all dimensions.

Theorem 3.3.3. *Given n and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ which satisfies*

($\star$) *$H \geq \delta$ on $B_R \cap \Sigma$ for $R > 2$,*

we have

$$|A|(x) \leq \frac{CR}{R - |x|} H(x), \quad \text{for all } x \in B_{R-1} \cap \Sigma. \tag{3.3.1}$$

The proof of Theorem 3.3.3 is inspired by the interior curvature estimates of Ecker-Huisken [EH91], which give a local curvature estimate for MCF with bounded gradient. Similar type arguments can also be found in [AG92] and [CNS88].

Next, we give the proof of Theorem 3.3.3. Fix n and $\delta > 0$. Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a shrinker which satisfies

($\star$) $H \geq \delta$ on $B_R \cap \Sigma$ for $R > 2$.

Set

$$v = \frac{1}{H} \quad \text{and} \quad v_0 = \frac{1}{\delta}. \tag{3.3.2}$$

We have $v \leq v_0$ on $B_R \cap \Sigma$. Lemma 3.1.4 gives that $LH = H$. Hence, v satisfies the equation

$$\Delta v = 2\frac{|\nabla H|^2}{H^3} - \frac{\Delta H}{H^2} = \frac{1}{2}\langle x, \nabla v\rangle + 2\frac{|\nabla v|^2}{v} + \Big(|A|^2 - \frac{1}{2}\Big)v. \tag{3.3.3}$$

Let $f = |A|^2 h(v^2)$. $h > 0$ will be determined later. Then we have

$$\Delta f = h(v^2)\Delta|A|^2 + |A|^2\Delta h(v^2) + 2\langle\nabla|A|^2, \nabla h(v^2)\rangle. \tag{3.3.4}$$

Note that

$$\nabla h(v^2) = h'\nabla v^2 = 2h'v\nabla v, \tag{3.3.5}$$

and

$$\Delta h(v^2) = h'\Delta v^2 + h''|\nabla v^2|^2. \tag{3.3.6}$$

Combining this with Lemma 3.1.4 gives that

$$\begin{aligned}\Delta f = &h\Big(2|\nabla A|^2 + (1-2|A|^2)|A|^2 + \frac{1}{2}\langle x, \nabla|A|^2\rangle\Big) + |A|^2\Big(h''|\nabla v^2|^2 + h'\Delta v^2\Big)\\ &+ 2\langle\nabla|A|^2, \nabla h(v^2)\rangle.\end{aligned} \tag{3.3.7}$$

Now we estimate the right hand side of the equation (3.3.7). First, we have

$$2\langle\nabla|A|^2, \nabla h(v^2)\rangle = \frac{\langle\nabla h, \nabla f\rangle}{h} - |A|^2\frac{|\nabla h|^2}{h} + 4h'|A|v\langle\nabla|A|, \nabla v\rangle. \tag{3.3.8}$$

Using absorbing inequality gives that

$$4h'|A|v\langle\nabla|A|, \nabla v\rangle \leq \frac{2(h')^2|A|^2v^2|\nabla v|^2}{h} + 2h|\nabla|A||^2. \tag{3.3.9}$$

This implies

$$2\langle\nabla|A|^2, \nabla h(v^2)\rangle \geq \frac{\langle\nabla h, \nabla f\rangle}{h} - 6\frac{(h')^2|A|^2v^2|\nabla v|^2}{h} - 2h|\nabla|A||^2. \tag{3.3.10}$$

We also have that

$$\begin{aligned}h''|\nabla v^2|^2 + h'\Delta v^2 =&h'\Big[v\langle x, \nabla v\rangle + 4|\nabla v|^2 + (2|A|^2-1)v^2 + 2|\nabla v|^2\Big]\\ &+ 4h''v^2|\nabla v|^2\end{aligned} \tag{3.3.11}$$

and

$$\frac{1}{2}\langle x, \nabla f\rangle = \frac{h}{2}\langle x, \nabla|A|^2\rangle + \frac{|A|^2}{2}\langle x, \nabla h\rangle = \frac{h}{2}\langle x, \nabla|A|^2\rangle + |A|^2h'v\langle x, \nabla v\rangle. \tag{3.3.12}$$

Therefore, we obtain that

$$\Delta f \geq \frac{\langle \nabla h, \nabla f\rangle}{h} + \frac{1}{2}\langle x, \nabla f\rangle + (1-2|A|^2)|A|^2 h + 2h' v^2|A|^4 - h' v^2 |A|^2 + \Big[4h'' v^2 + 6\Big(h' - \frac{(h')^2 v^2}{h}\Big)\Big]|A|^2|\nabla v|^2. \tag{3.3.13}$$

Now we choose

$$h(y) = \frac{y}{1-ky}, \tag{3.3.14}$$

where $k = (2v_0^2)^{-1}$. Simple computations give that

$$h'(y) = \frac{1}{(1-ky)^2}, \tag{3.3.15}$$

and

$$h''(y) = \frac{2k}{(1-ky)^3}. \tag{3.3.16}$$

Set $y = v^2$ gives the following identities

$$h(v^2) - h'(v^2)v^2 = -kh^2, \tag{3.3.17}$$

and

$$4h'' v^2 + 6\Big(h' - \frac{(h')^2 v^2}{h}\Big) = \frac{2k}{(1-kv^2)^2} h(v^2). \tag{3.3.18}$$

Inserting these inequalities into (3.3.13) implies that

$$\Delta f \geq \frac{\langle \nabla h, \nabla f\rangle}{h} + \frac{1}{2}\langle x, \nabla f\rangle - f + 2kf^2 + \frac{2k|\nabla v|^2}{(1-kv^2)^2} f. \tag{3.3.19}$$

Here, we used that $h' v^2 |A|^2 \leq 2f$.

We will set

$$a = \frac{\nabla h(v^2)}{h} \quad \text{and} \quad d = \frac{2k|\nabla v|^2}{(1-kv^2)^2}. \tag{3.3.20}$$

Lemma 3.3.4. *Let $\phi(x) = ((\rho^2 - |x - x_0|^2)_+)^3$ and $\mu(x) = \rho^2 - |x - x_0|^2$, where $x_0 \in \mathbf{R}^{n+1}$ and $\rho > 0$. If Σ^n is a shrinker, then on $B_\rho(x_0) \cap \Sigma$ we have*

$$\Delta \phi = 24\mu|(x - x_0)^T|^2 - 6n\mu^2 + 6\mu^2 H\langle x - x_0, \mathbf{n}\rangle. \tag{3.3.21}$$

In particular, we have the estimate

$$|\Delta\phi(x)| \leq 24\mu\rho^2 + 6n\mu^2 + 3\mu^2\rho|x| \leq (24+6n)\rho^4 + 3\rho^3|x|. \tag{3.3.22}$$

Proof. Since $\nabla\phi = -3(\rho^2 - |x-x_0|^2)^2\nabla|x-x_0|^2 = -6\mu^2(x-x_0)^T$, we have

$$\begin{aligned}\Delta\phi &= -6\,\mathrm{div}(\mu^2(x-x_0)^T)\\ &= -6\Big[2\mu\langle\nabla\mu,(x-x_0)^T\rangle + \mu^2\Big(n - \langle x-x_0,\mathbf{n}\rangle H\Big)\Big]\\ &= 24\mu|(x-x_0)^T|^2 - 6n\mu^2 + 6\mu^2 H\langle x-x_0,\mathbf{n}\rangle.\end{aligned} \tag{3.3.23}$$

The second claim follows from the shrinker equation and the fact that $\mu \leq \rho^2$. □

Proof of Theorem 3.3.3. Now fix a point $x_0 \in B_{R-1}\cap\Sigma$ and set $\rho = R - |x_0|$. Let ϕ be the function defined in Lemma 3.3.4. We will work on $B_\rho(x_0)\cap\Sigma$. Using (3.3.19) gives that

$$\begin{aligned}\Delta(\phi f) &= \phi\Delta f + f\Delta\phi + 2\langle\nabla\phi,\nabla f\rangle\\ &\geq \phi\Big[\langle a + \frac{x}{2},\nabla f\rangle - f + 2kf^2 + df\Big] + f\Delta\phi + 2\langle\nabla\phi,\nabla f\rangle.\end{aligned} \tag{3.3.24}$$

Note that

$$\langle a,\nabla(\phi f)\rangle = \phi\langle a,\nabla f\rangle + f\langle a,\nabla\phi\rangle \tag{3.3.25}$$

and

$$\langle\nabla\phi,\nabla(f\phi)\rangle = f|\nabla\phi|^2 + \phi\langle\nabla\phi,\nabla f\rangle. \tag{3.3.26}$$

This implies

$$\begin{aligned}\Delta(\phi f) \geq\ & \langle a + \frac{x}{2},\nabla(\phi f)\rangle - \langle a + \frac{x}{2},\nabla\phi\rangle f + \phi\Big[(d-1)f + 2kf^2\Big]\\ &+ f\Delta\phi + \frac{2}{\phi}\langle\nabla\phi,\nabla(f\phi)\rangle - 2\frac{|\nabla\phi|^2}{\phi}f.\end{aligned} \tag{3.3.27}$$

Now we set $F(x) = \phi(x)f(x)$ and consider its maximum on $B_\rho(x_0)\cap\Sigma$. Since F vanishes on $\partial B_\rho(x_0)\cap\Sigma$, F achieves its maximum at some interior point $y_0 \in B_\rho(x_0)\cap$

Σ. At the point y_0, we have

$$\nabla F(y_0) = 0 \text{ and } \Delta F(y_0) \leq 0. \tag{3.3.28}$$

In the following, we will work at the point y_0. By (3.3.27) and $f(y_0) > 0$, we have

$$\langle a + \frac{y_0}{2}, \nabla\phi\rangle + 2\frac{|\nabla\phi|^2}{\phi} \geq \phi(d-1) + 2k\phi f + \Delta\phi. \tag{3.3.29}$$

Note that

$$|a|^2 = 4\Big(\frac{h'}{h}\Big)^2 v^2|\nabla v|^2 = \frac{2}{kv^2}d \leq \frac{|y_0|^2}{2k}d. \tag{3.3.30}$$

This yields

$$\langle a, \nabla\phi\rangle \leq (d+1)\phi + \frac{|a|^2}{4(d+1)}\frac{|\nabla\phi|^2}{\phi} \leq (d+1)\phi + \frac{|y_0|^2}{8k}\frac{|\nabla\phi|^2}{\phi}. \tag{3.3.31}$$

Combining (3.3.31) with (3.3.29) gives that

$$2k\phi f \leq -\Delta\phi + 2\phi + \Big(2 + \frac{|y_0|^2}{8k}\Big)\frac{|\nabla\phi|^2}{\phi} + \frac{|y_0|}{2}|\nabla\phi|. \tag{3.3.32}$$

By the definition of ϕ, we have

$$\phi \leq \rho^6, \quad |\nabla\phi| \leq 6\rho^5 \quad \text{and} \quad \frac{|\nabla\phi|^2}{\phi} \leq 36\rho^4. \tag{3.3.33}$$

Combining this with Lemma 3.3.4, $|y_0| \leq R$ and (3.3.32) yields that

$$F(y_0) = \phi(y_0)f(y_0) \leq C(\rho^6 + R\rho^5 + R^2\rho^4), \tag{3.3.34}$$

where C is a constant depending on n and δ.

Since F achieves its maximum at y_0, we have $F(x_0) \leq F(y_0)$. This implies

$$\frac{\rho^6|A|^2(x_0)}{H^2(x_0) - k} = F(x_0) \leq F(y_0) \leq C(\rho^6 + R\rho^5 + R^2\rho^4). \tag{3.3.35}$$

In particular, we have

$$|A|(x_0) \leq C\Big(1 + \frac{R}{\rho}\Big)H(x_0). \tag{3.3.36}$$

Since x_0 is an arbitrary point in $B_{R-1} \cap \Sigma$, we complete the proof of Theorem 3.3.3. □

A direct corollary of Theorem 3.3.3 is the following result.

Theorem 3.3.5. *Given n and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded self-shrinker $\Sigma^n \subset \mathbf{R}^{n+1}$ which satisfies*

- *$H \geq \delta$ on $B_R \cap \Sigma$ for $R > 2$,*

we have

$$|A| \leq CH, \quad \textit{for all } x \in B_{R/2} \cap \Sigma. \tag{3.3.37}$$

3.3.2 Proof of Theorem 3.3.1

In this section we prove Theorem 3.3.1 and Theorem 3.3.2 by adapting the iteration and improvement scheme used to prove [CIM15, Theorem 0.1]. We will briefly outline this scheme here; recall that the two key ingredients are the so-called iterative step [CIM15, Proposition 2.1] and the improvement step [CIM15, Proposition 2.2] (compare Proposition 3.3.7 below). In the iterative step, it is shown that if a self-shrinker is almost cylindrical (quantified by H and $|A|$) on a large scale, then it is still close to a cylinder on a larger scale, albeit with some loss in the estimates. It is important here that the scale extends by a fixed multiplicative factor.

Proposition 3.3.6. *(Iteration; [CIM15, Proposition 2.1]) Given $\lambda_0 < 2$ and n, there exist positive constants R_0, δ_0, C_0 and θ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a shrinker with $\lambda(\Sigma) \leq \lambda_0$, $R \geq R_0$, and*

- *$B_R \cap \Sigma$ is smooth with $H \geq 1/4$ and $|A| \leq 2$,*

then $B_{(1+\theta)R} \cap \Sigma$ is smooth with $H \geq \delta_0$ and $|A| \leq C_0$.

On the other hand, in the improvement step, it is shown that if a shrinker is close to a cylinder on some scale, then the estimates can be improved so long as we decrease the scale by a fixed amount. We will show that the initial closeness in the improvement step only needs to be quantified by H — using our curvature estimate Theorem 3.3.3 and a compactness result of shrinkers, we can show that the bounded curvature assumption in the improvement step (Proposition 2.2 of [CIM15]) can be removed, which in turn implies Theorem 3.3.1. Our improvement step is stated as follows:

Proposition 3.3.7. *(Improvement) Given n and λ_0, let $\delta_0 \in (0, 1/4)$ be given by Proposition 3.3.6. Then there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a shrinker with $\lambda(\Sigma) \leq \lambda_0$ and*

- *$H \geq \delta_0$ on $B_R \cap \Sigma$,*

then $H \geq 1/4$ and $|A| \leq 2$ on $B_{R-4} \cap \Sigma$.

The main argument in the improvement step is to control the derivatives of the tensor $\tau = A/H$. These estimates are shown to decay exponentially as $R^{\alpha}\mathrm{e}^{-R/4}$ for some α, allowing one to extend good cylindrical estimates from a fixed scale $5\sqrt{2n}$ to almost the whole ball of radius R. For us, instead of assuming $|A| \leq C$ for some constant C as in [CIM15], our curvature estimates give that $|A| \leq CR$ for shrinkers with positive mean curvature H in B_R. In the proof of Proposition 3.3.7, we show that this is still enough to control the derivatives of τ, possibly with a worse exponent α of R. But the exponential factor still decays much faster than any polynomial factor, so the polynomial factor can be eventually absorbed into the exponential factor as long as we choose R sufficiently large. The remaining details of our proof will be deferred to Section 3.3.3.

To complete the iteration and improvement scheme, we first apply Proposition 3.3.7, then apply Proposition 3.3.6 and repeat the process. The multiplicative factor extends the scale by more than the fixed decrease if R is large enough, so we get strict mean convexity on all of Σ, which must therefore be a cylinder by the classification of mean convex shrinkers (Theorem 2.2.1). Thus we have:

Proposition 3.3.8. *Given n and $\lambda_0 < 2$, let $\delta_0 \in (0, 1/4)$ be given by Proposition 3.3.6. Then there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies*

- *$H \geq \delta_0$ on $B_R \cap \Sigma$,*

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $1 \leq k \leq n$.

We also need the following compactness theorem for self-shrinkers which plays an important role in our argument:

Lemma 3.3.9 (Compactness). *Let $\Sigma_i \subset \mathbf{R}^{n+1}$ be a sequence of shrinkers with $\lambda(\Sigma_i) \leq \lambda_0$ and*

$$|A|(x) \leq C(1 + |x|) \quad on \quad B_i \cap \Sigma_i. \tag{3.3.38}$$

Then there exists a subsequence Σ_i' that converges smoothly and with multiplicity one to a complete embedded shrinker Σ with

$$|A|(x) \leq C(1 + |x|) \quad and \quad \lim_{i \to \infty} \lambda(\Sigma_i') = \lambda(\Sigma)\,. \tag{3.3.39}$$

Proof. The key is that the a priori bound on $|A|$ is uniform on compact subsets. Thus, as in Lemma 2.7 in [CIM15], for any R we may obtain smooth convergence in B_R by covering with a finite number of balls. Passing to a diagonal argument gives the overall smooth convergence to a smooth, complete, embedded shrinker Σ with $\lambda(\Sigma) \leq \lambda_0$. Again arguing as in [CIM15], if multiplicity is greater than one then the limit Σ must be L-stable. But there are no such shrinkers with polynomial volume growth (see Theorem 0.5 in [CM12b]), so the multiplicity must be one. □

Now we first give the proof of Theorem 3.3.2.

Proof of Theorem 3.3.2. Since we assumed $H \geq \delta$ on $B_R \cap \Sigma$, the curvature estimate Theorem 3.3.3 gives in particular that $|A| \leq CH \leq \frac{C}{2}|x|$ on $B_{R/2} \cap \Sigma$. Applying the compactness Lemma 3.3.9 we get that Σ is smoothly close to $\mathbf{S}^k \times \mathbf{R}^{n-k}$ in $B_{R/2}$ for R sufficiently large. We may assume that $\lambda(\Sigma) \leq \lambda_0 < 2$, and $H \geq \delta_0$ on $B_{R/2} \cap \Sigma$. The result then follows from Proposition 3.3.8. □

Now we give the proof of Theorem 3.3.1 via Proposition 3.3.8.

Proof of Theorem 3.3.1 using Proposition 3.3.8. First, the Harnack inequality gives that either $H \equiv 0$ or $H > 0$. If $H \equiv 0$ in B_R, then Σ is a hyperplane in B_R. Thus by the rigidity of the hyperplane (Theorem 3.1.1), Σ must be a hyperplane if R is sufficiently large.

Next, we assume $H > 0$ in B_R. Lemma 3.2.7 then gives a curvature estimate on $B_{R-1} \cap \Sigma$. By the compactness of Lemma 3.3.9, we can assume that Σ is smoothly close to $\mathbf{S}^k \times \mathbf{R}^{n-k}$ in B_{R_1} for some $k \geq 0$, where R_1 can be taken as large as we wish. If $k = 0$, then again the rigidity of the hyperplane means that Σ must be a hyperplane, although this is a contradiction since in this case we assume $H > 0$ on $B_R \cap \Sigma$. So $k \geq 1$, and consequently H is approximately $\sqrt{k/2}$ on $B_{R_1} \cap \Sigma$, then Theorem 3.3.1 follows directly from Proposition 3.3.8. □

3.3.3 Proof of the improvement step

In this subsection, we prove Proposition 3.3.7 by sketching the necessary modifications of the proof of Proposition 2.2 in [CIM15].

As discussed earlier, the central argument is the very tight estimate on the tensor $\tau = A/H$, that decays exponentially in R. Thus, our main modification is the following lemma, which removes the curvature bound of Corollary 4.12 in [CIM15] by accepting a slightly larger power of R, although we still have the exponential decay.

Lemma 3.3.10. *Given n, λ_0 and $\delta > 0$, there exists a constant $C_\tau > 0$ such that if $\lambda(\Sigma) \leq \lambda_0$, $R \geq 2$, and*

- *$B_{R+1} \cap \Sigma$ is smooth with $H \geq \delta > 0$,*

then

$$\sup_{B_{R-2}\cap\Sigma} |\nabla \tau|^2 + R^{-4} \left|\nabla^2 \tau\right|^2 \leq C_\tau \, R^{3n+4} \, e^{-R/4} \,. \tag{3.3.40}$$

Proof. First, Theorem 3.3.3 gives there exists a constant $C = C(n,\delta)$ such that $|A| \leq CRH$ in B_R. Hence, Proposition 4.8 in [CIM15] with $s = 1/2$ implies that

$$\int_{B_{R-1/2}\cap\Sigma} |\nabla\tau|^2 \, \mathrm{e}^{-|x|^2/4} \leq C\, R^{n+4}\, \mathrm{e}^{-(R-1/2)^2/4}. \tag{3.3.41}$$

Since $\mathrm{e}^{-|x|^2/4} \geq \mathrm{e}^{-\frac{R^2-2R+1}{4}}$ on B_{R-1}, it follows that

$$\int_{B_{R-1}\cap\Sigma} |\nabla\tau|^2 \leq C\, R^{n+4}\, \mathrm{e}^{-\frac{R}{4}}. \tag{3.3.42}$$

This gives the desired integral decay on $\nabla\tau$. We will combine this with elliptic theory to get the pointwise bounds. The key is that τ satisfies the elliptic equation $\mathcal{L}_{H^2}\tau = 0$ (see Proposition 4.5 in [CIM15]), that is,

$$\Delta\tau - \frac{1}{2}\langle x, \nabla\tau\rangle + \langle\nabla \log H^2, \nabla\tau\rangle = 0. \tag{3.3.43}$$

Note that we have

$$|\nabla \log H^2| = \frac{2|\nabla H|}{H} \leq \frac{|A||x|}{H} \leq CR|x|, \tag{3.3.44}$$

where we used that $|\nabla H| \leq \frac{1}{2}|A||x|$ and $|A| \leq CRH$.

Therefore, the two first order terms in the equation (3.3.43) come from x^T in $\mathcal{L}$ and $\nabla \log H^2$; both grow at most quadratically. Now we can apply elliptic theory on balls of radius $1/R^2$ to get for any $p \in B_{R-2}\cap\Sigma$ that

$$\left(|\nabla\tau|^2 + R^{-4}\,|\nabla^2\tau|^2\right)(p) \leq C\, R^{2n} \int_{B_{\frac{1}{R^2}}(p)\cap\Sigma} |\nabla\tau|^2. \tag{3.3.45}$$

Combining this with the integral bounds (3.3.42) gives the lemma. □

Now we sketch the proof of Proposition 3.3.7.

Fix n, $\lambda_0 > 0$, and $\delta_0 > 0$. Let $R > 0$ and assume that Σ is a self-shrinker in $\mathbf{R}^{n+1}$, $\lambda(\Sigma) \leq \lambda_0$ and $H \geq \delta_0$ on $\Sigma\cap B_R$. By Lemma 3.3.10, the tensor $\tau = A/H$

satisfies

$$|\nabla\tau| + |\nabla^2\tau| \leq \varepsilon_\tau \quad \text{on } B_{R-2}\cap\Sigma, \tag{3.3.46}$$

where

$$\varepsilon_\tau^2 := C\,R^{3n+8}\,\mathrm{e}^{-R/4}$$

and the constant C depends only on n, δ_0 and λ_0. As in [CIM15], the key point is that ε_τ can still be made small for large R, due to the decaying exponential factor.

Now fix small $\varepsilon_0 > 0$, to be chosen as needed, but depending only on n. Combining the compactness of Lemma 3.3.9 with the classification of mean convex shrinkers [CM12a], there exists a constant $R_1 = R_1(n, \lambda_0, \delta_0, \varepsilon_0)$ so that if $R \geq R_1$, then $B_{5\sqrt{2n}} \cap \Sigma$ is C^2 ε_0-close to a cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $1 \leq k \leq n$. The remainder of Proposition 3.3.7 follows from the proof of Proposition 2.2 in [CIM15].

3.4 Self-shrinkers with constant $|A|$

We first state the following conjecture:

Conjecture 3.4.1. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth complete embedded self-shrinker with polynomial volume growth. If $|A| =$ constant, then Σ is a generalized cylinder.*

The case $n = 1$ follows from a more general result by Abresch and Langer [AL86], which says that the only smooth complete and embedded self-shrinkers in $\mathbf{R}^2$ are the lines and a round circle. In $\mathbf{R}^3$, i.e., $n = 2$, the above conjecture was proved by Ding and Xin [DX14] using the following identity

$$\frac{1}{2}\mathcal{L}|\nabla A|^2 = |\nabla^2 A|^2 + (1 - |A|^2)|\nabla A|^2 - 3\Xi - \frac{3}{2}|\nabla|A|^2|^2, \tag{3.4.1}$$

where the operator $\mathcal{L} = \Delta - \langle \frac{x}{2}, \nabla\cdot\rangle$, h_{ij} is the second fundamental form and

$$\Xi = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml} - 2\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{im}h_{jl}. \tag{3.4.2}$$

In this section, we give a new and simpler proof of the above result without heavy computation. More precisely, we prove the following theorem.

Theorem 3.4.2. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a smooth complete embedded self-shrinker with polynomial volume growth. If the second fundamental form of Σ^2 is of constant length, i.e., $|A| = constant$, then Σ^2 is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{2-k}$ for some $k \leq 2$.*

The key idea in the proof is to analyze the point where $|x|$ achieves its minimum. Since Σ has polynomial volume growth and, thus, Σ is proper (see [CZ13] and [DX13]), such a point always exists. At this point, we have that $\nabla H = 0$. Combining this with $|A| = constant$ implies that $|A|^2 \leq 1/2$. Therefore, the conclusion follows directly from the fact that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq 1/2$ must be a generalized cylinder. We remark that our method does not apply to higher dimensions to prove Conjecture 3.4.1.

Theorem 3.4.3 ([LS11],[CL13]). *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth self-shrinker. If $|A|^2 \leq \frac{1}{2}$, then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k \leq n$. Moreover, if $|A|^2 < \frac{1}{2}$, then Σ is a hyperplane.*

Proof of Theorem 3.4.2. By Theorem 3.4.3, if $|A|^2 < \frac{1}{2}$, then Σ is a hyperplane in $\mathbf{R}^3$. Therefore, in the following we only consider the case when $|A|^2 \geq \frac{1}{2}$.

For any point $p \in \Sigma$, we can choose a local orthonormal frame $\{e_1, e_2\}$ such that the coefficients of the second fundamental form $h_{ij} = \lambda_i \delta_{ij}$ for $i, j = 1, 2$. By definition, we have

$$|\nabla H|^2 = (h_{111} + h_{221})^2 + (h_{112} + h_{222})^2. \tag{3.4.3}$$

Since $|A| = constant$, Lemma 3.1.4 gives that

$$h_{11}h_{111} + h_{22}h_{221} = h_{11}h_{112} + h_{22}h_{222} = 0 \tag{3.4.4}$$

and

$$|\nabla A|^2 = h_{111}^2 + h_{222}^2 + 3h_{112}^2 + 3h_{221}^2 = |A|^2\Big(|A|^2 - \frac{1}{2}\Big). \tag{3.4.5}$$

First, we prove that $|x| > 0$ on Σ. We argue by contradiction. Suppose Σ goes through the origin, then at the origin, we have $H = |\nabla H| = 0$. Therefore,

$$h_{11} + h_{22} = h_{111} + h_{221} = h_{112} + h_{222} = 0.$$

Combining this with (3.4.4) yields that

$$h_{111} = h_{222} = h_{112} = h_{221} = 0.$$

This implies that $|\nabla A|^2 = |A|^2(|A|^2 - \frac{1}{2}) = 0$ and, thus, $|A|^2 = \frac{1}{2}$. By Theorem 3.4.3, we conclude that Σ is $\mathbf{S}^2$ or $\mathbf{S}^1 \times \mathbf{R}$. However, this contradicts the assumption that Σ goes through the origin.

Note that Σ has polynomial volume growth implies that Σ is proper (see Theorem 4.1 in [CZ13]). By the maximum principle, Σ must intersect $\mathbf{S}^2(2)$. Hence, there exists a point $p \in \Sigma$ which minimizes $|x|$.

Now, at point p, we have $|x| > 0$ and $x^T = 0$, where x^T is the tangential projection of x. This implies that

$$4H^2(p) = |x|^2(p) \quad \text{and} \quad \nabla H(p) = 0. \tag{3.4.6}$$

Thus, we have

$$h_{111} + h_{221} = h_{112} + h_{222} = 0.$$

By (3.4.4), we obtain that

$$h_{111}(h_{11} - h_{22}) = h_{222}(h_{11} - h_{22}) = 0. \tag{3.4.7}$$

If $h_{111} = h_{222} = 0$, then we see that $|\nabla A|^2 = 0$ and $|A|^2 = \frac{1}{2}$. By Theorem 3.4.3, we conclude that Σ is a generalized cylinder.

If $h_{11} = h_{22}$, then we have

$$|A|^2 = 2h_{11}^2 = \frac{H^2(p)}{2} = \frac{|x|^2(p)}{8}. \tag{3.4.8}$$

Since every smooth complete self-shrinker must intersect the sphere $\mathbf{S}^2(2)$, we conclude that $|x|(p) \leq 2$. By (3.4.8), this gives

$$|A|^2 = \frac{|x|^2(p)}{8} \leq \frac{1}{2}. \tag{3.4.9}$$

The theorem follows immediately from Theorem 3.4.3. □

For self-shrinkers, there are some gap phenomena for the norm of the second fundamental form. Cao and Li [CL13] proved that any smooth complete self-shrinker with polynomial volume growth and $|A|^2 \leq 1/2$ in arbitrary codimension is a generalized cylinder (see also [LS11]). Colding, Ilmanen and Minicozzi [CIM15] showed that the generalized cylinders are rigid in a strong sense that any self-shrinker which is sufficiently close to one of the generalized cylinders on a large and compact set must itself be a generalized cylinder. Using this result, we prove that any self-shrinker with $|A|^2$ sufficiently close to $1/2$ must also be a generalized cylinder.

Theorem 3.4.4. *Given n and λ_0, there exists $\delta = \delta(n, \lambda_0) > 0$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying*

(†) $|A|^2 \leq \frac{1}{2} + \delta$,

then Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k \leq n$.

Chapter 4

Gap and rigidity results of λ-hypersurfaces

In this chapter, we study λ-hypersurfaces that are critical points of a Gaussian weighted area functional $\int_{\Sigma} e^{-\frac{|x|^2}{4}} dA$ for compact variations that preserve weighted volume. First, we prove various gap and rigidity theorems for λ-hypersurfaces in terms of the norm of the second fundamental form $|A|$. Second, we show that in one dimension, the only smooth complete and embedded λ-curves in $\mathbf{R}^2$ with $\lambda \geq 0$ are lines and round circles. Moreover, we establish a Bernstein type theorem for λ-hypersurfaces. All the results can be viewed as generalizations of results for self-shrinkers. This chapter is based on [Gua14a].

4.1 Introduction

We call a hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ a λ-hypersurface if it satisfies

$$H - \frac{\langle x, \mathbf{n} \rangle}{2} = \lambda, \tag{4.1.1}$$

where λ is any constant, H is the mean curvature, $\mathbf{n}$ is the outward pointing unit normal and x is the position vector.

Example 4.1.1. *We give three examples of λ-hypersurfaces in $\mathbf{R}^3$:*

(1) The sphere $\mathbf{S}^2(r)$ *with radius* $r = \sqrt{\lambda^2+4} - \lambda$.

(2) The cylinder $\mathbf{S}^1(r) \times \mathbf{R}$, *where* $\mathbf{S}^1(r)$ *has radius* $\sqrt{\lambda^2+2} - \lambda$.

(3) The hyperplane in $\mathbf{R}^3$.

λ-hypersurfaces were first studied by McGonagle and Ross in [MR15], where they investigate the following isoperimetric type problem in a Gaussian weighted Euclidean space:

Let $\mu(\Sigma)$ be the weighted area functional defined by $\mu(\Sigma) = \int_\Sigma e^{-\frac{|x|^2}{4}} dA$ for any hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$. Consider the variational problem of minimizing $\mu(\Sigma)$ among all Σ enclosing a fixed Gaussian weighted volume. Note that the variational problem is not to consider Σ enclosing a specific fixed weighted volume, but to consider variations that preserve the weighted volume.

It turns out that critical points of this variational problem are λ-hypersurfaces and the only smooth stable ones are hyperplanes; see [MR15].

In [CW14a], Cheng and Wei introduced the notation of λ-hypersurfaces by studying the weighted volume-preserving MCF. They proved that λ-hypersurfaces are critical points of the weighted area functional for the weighted volume-preserving variations. Moreover, they defined a F-functional of λ-hypersurfaces and studied F-stability, which extended a result of Colding-Minicozzi [CM12a].

Note that when $\lambda = 0$, λ-hypersurfaces are just self-shrinkers and they can be viewed as a generalization of self-shrinkers.

Self-shrinkers play a key role in the study of mean curvature flow, since they describe the singularity models of the MCF. In one dimension, smooth complete embedded self-shrinking curves are totally understood and they are just lines and round circles by the work of Abresch and Langer [AL86]. In higher dimensions, self-shrinkers are more complicated and there are only few examples; see [Ang92], [KM14], [Møl11] and [Ngu09]. There are some classification and rigidity results of self-shrinkers under certain assumptions. Ecker and Huisken [EH89] proved that if a self-shrinker is an entire graph with polynomial volume growth, then it is a hyperplane. Later, Wang [Wan11a] removed the condition of polynomial volume growth.

In this chapter, we study λ-hypersurfaces from three aspects: gap and rigidity results, one-dimensional case and entire graphic case.

The first main result is the following gap theorem for λ-hypersurfaces in terms of the norm of the second fundamental form $|A|$.

Theorem 4.1.2. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth complete embedded λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$ with polynomial volume growth, which satisfies*

$$|A| \leq \frac{\sqrt{\lambda^2+2}-|\lambda|}{2}, \tag{4.1.2}$$

then Σ is one of the following:

(1) a round sphere $\mathbf{S}^n$,

(2) a cylinder $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for $1 \leq k \leq n-1$,

(3) a hyperplane in $\mathbf{R}^{n+1}$.

Remark 4.1.3. *Note that when $\lambda = 0$, then Σ is a self-shrinker satisfying $|A|^2 \leq 1/2$. So Theorem 4.1.2 implies the gap theorem of Cao and Li [CL13] in codimension one case. Cheng, Ogata and Wei [COW14] obtained a gap theorem for λ-hypersurfaces in terms of $|A|$ and H, which also generalized Cao and Li's result.*

Partially motivated by the work of Chern, do Carmo and Kobayashi [CdCK70] on minimal submanifolds of a sphere with the second fundamental form of constant length, we consider smooth closed embedded λ-hypersurfaces $\Sigma^2 \subset \mathbf{R}^3$ with $|A| = constant$ and $\lambda \geq 0$. We prove that they are just round spheres. It can be thought of as a generalization of the result that any smooth self-shrinker in $\mathbf{R}^3$ with $|A| = constant$ is a generalized cylinder; see [DX14] and [Gua14b].

Theorem 4.1.4. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a smooth closed and embedded λ-hypersurface with $\lambda \geq 0$. If the second fundamental form of Σ^2 is of constant length, i.e., $|A| = constant$, then Σ^2 is a round sphere.*

The proof of this theorem contains two ingredients. The first ingredient is to consider the point where the norm of the position vector $|x|$ achieves its minimum. This will give that the genus is 0. The second ingredient is an interesting result from [HW54] that any smooth closed special W-surface of genus 0 is a round sphere.

Next, we turn to the one-dimensional case. Following the argument in [Man11], we show that just as self-shrinkers in $\mathbf{R}^2$, the only smooth complete and embedded λ-hypersurfaces (λ-curves) in $\mathbf{R}^2$ with $\lambda \geq 0$ are lines and round circles.

Theorem 4.1.5. *Any smooth complete embedded λ-hypersurface (λ-curve) γ in $\mathbf{R}^2$ satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$ with $\lambda \geq 0$ must either be a line or a round circle.*

In contrast to embedded self-shrinking curves, the dynamical pictures suggest that there exist some embedded λ-curves with $\lambda < 0$ which are not round circles. There also exist Abresch-Langer type curves for immersed λ-curves; see [Cha14].

In the last part, we give a Bernstein type theorem for λ-hypersurfaces, which generalizes Ecker and Huisken's result [EH89].

Theorem 4.1.6. *If a λ-hypersurface $\Sigma^n \subset \mathbf{R}^{n+1}$ is an entire graph with polynomial volume growth satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$, then Σ is a hyperplane.*

Remark 4.1.7. *A similar result is also obtained later by Cheng and Wei [CW14b] under the assumption of properness instead of polynomial volume growth. Note that they proved properness of λ-hypersurfaces implies polynomial volume growth; see Theorem 9.1 in [CW14a].*

4.2 Background

In this section, we recall some background and collect several useful formulas for λ-hypersurfaces. Throughout, we always assume hypersurfaces to be smooth complete embedded, without boundary and with polynomial volume growth.

4.2.1 Simons type identities

We will derive a Simons type identity for λ-hypersurfaces which plays a key role in our proof of Theorem 4.1.2. First, recall the operators $\mathcal{L}$ and L from [CM12a] defined by

$$\mathcal{L} = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle, \tag{4.2.1}$$

$$L = \Delta - \frac{1}{2}\langle x, \nabla \cdot \rangle + |A|^2 + \frac{1}{2}. \tag{4.2.2}$$

Lemma 4.2.1. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$, then*

$$LA = A - \lambda A^2, \tag{4.2.3}$$

$$LH = H + \lambda |A|^2, \tag{4.2.4}$$

$$\mathcal{L}|A|^2 = 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2. \tag{4.2.5}$$

Remark 4.2.2. *More general results of above formulas were already obtained by Colding-Minicozzi; see Proposition 1.2 in [CM15]. For completeness we also include a proof here. Note that when $\lambda = 0$, these formulas are just Simons identities for self-shrinkers in [CM12a].*

Proof of Lemma 4.2.1. Recall that for a general hypersurface, the second fundamental form A satisfies

$$\Delta A = -|A|^2 A - HA^2 - Hess_H. \tag{4.2.6}$$

Now we fix a point $p \in \Sigma$, and choose a local orthonormal frame e_i for Σ such that its tangential covariant derivatives vanish. So at this point, we have $\nabla_{e_i} e_j = a_{ij}\mathbf{n}$. Thus,

$$\begin{aligned} 2Hess_H(e_i, e_j) &= \nabla_{e_j}\nabla_{e_i}\langle x, \mathbf{n}\rangle = \langle x, -a_{ik}e_k\rangle_j \\ &= -a_{ikj}\langle x, e_k\rangle - a_{ij} - a_{ik}a_{jk}\langle x, \mathbf{n}\rangle \\ &= -(\nabla_{x^T} A)(e_i, e_j) - A(e_i, e_j) - \langle x, \mathbf{n}\rangle A^2(e_i, e_j). \end{aligned} \tag{4.2.7}$$

Combining (4.2.6) with (4.2.7) gives

$$LA = \Delta A - \frac{1}{2}\nabla_{x^T} A + \Big(\frac{1}{2} + |A|^2\Big)A = A - \Big(H - \frac{\langle x, \mathbf{n}\rangle}{2}\Big)A^2 = A - \lambda A^2. \quad (4.2.8)$$

This gives (4.2.3) and taking the trace gives (4.2.4). For (4.2.5), we have that

$$\begin{aligned} \mathcal{L}|A|^2 &= 2\langle \mathcal{L}A, A\rangle + 2|\nabla A|^2 \\ &= 2|A|^2 - 2\lambda\langle A^2, A\rangle - 2\Big(\frac{1}{2} + |A|^2\Big)|A|^2 + 2|\nabla A|^2 \\ &= 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2. \end{aligned} \quad (4.2.9)$$

This completes the proof. □

4.2.2 Weighted integral estimates for $|A|$

In this subsection, we prove a result which will justify our integration when hypersurfaces are non-compact and with bounded $|A|$.

Proposition 4.2.3. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a complete λ-hypersurface with polynomial volume growth satisfying $|A| \leq C_0$, then*

$$\int_\Sigma |\nabla A|^2 e^{-\frac{|x|^2}{4}} < \infty. \quad (4.2.10)$$

The proof of Proposition 4.2.3 relies on the following two lemmas from [CM12a] which show that the linear operator $\mathcal{L}$ is self-adjoint in a weighted L^2 space.

Lemma 4.2.4 ([CM12a]). *If $\Sigma \subset \mathbf{R}^{n+1}$ is a hypersurface, u is a C^1 function with compact support, and v is a C^2 function, then*

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} = -\int_\Sigma \langle \nabla u, \nabla v\rangle e^{-\frac{|x|^2}{4}}. \quad (4.2.11)$$

Lemma 4.2.5 ([CM12a]). *Suppose that $\Sigma \subset \mathbf{R}^{n+1}$ is a complete hypersurface without* *boundary. If u, v are C^2 functions with*

$$\int_\Sigma \big(|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|\big)e^{-\frac{|x|^2}{4}} < \infty, \tag{4.2.12}$$

then we get

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|x|^2}{4}} = -\int_\Sigma \langle \nabla u, \nabla v\rangle e^{-\frac{|x|^2}{4}}. \tag{4.2.13}$$

Proof of Proposition 4.2.3. By Lemma 4.2.1 and $|A| \leq C_0$, we have

$$\begin{aligned}\mathcal{L}|A|^2 &= 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2\lambda\langle A^2, A\rangle + 2|\nabla A|^2 \\ &\geq 2\Big(\frac{1}{2} - |A|^2\Big)|A|^2 - 2|\lambda||A|^3 + 2|\nabla A|^2 \geq 2|\nabla A|^2 - C,\end{aligned} \tag{4.2.14}$$

where C is a positive constant depending only on λ and C_0. We allow C to change from line to line. For any smooth function ϕ with compact support, we integrate (4.2.14) against $\frac{1}{2}\phi^2$.

By Lemma 4.2.4, we obtain that

$$-2\int_\Sigma \phi|A|\langle\nabla\phi, \nabla|A|\rangle e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}. \tag{4.2.15}$$

Using the absorbing inequality $\epsilon a^2 + \frac{b^2}{\epsilon} \geq 2ab$ gives

$$\int_\Sigma (\epsilon\phi^2|\nabla|A||^2 + \frac{1}{\epsilon}|A|^2|\nabla\phi|^2)e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2(|\nabla A|^2 - C)e^{-\frac{|x|^2}{4}}. \tag{4.2.16}$$

Now we choose $|\phi| \leq 1$, $|\nabla\phi| \leq 1$ and $\epsilon = 1/2$. Combining this with $|\nabla A| \geq |\nabla|A||$, we see that (4.2.16) gives

$$\int_\Sigma (4|A|^2 + C)e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \phi^2|\nabla A|^2 e^{-\frac{|x|^2}{4}}. \tag{4.2.17}$$

The conclusion follows from the monotone convergence theorem and the fact that Σ has polynomial volume growth.

□

A direct consequence of Proposition 4.2.3 and Lemma 4.2.5 is the following corollary.

Corollary 4.2.6. *If $\Sigma^n \subset \mathbb{R}^{n+1}$ is a complete λ-hypersurface with polynomial volume growth satisfying $|A| \leq C_0$, then*

$$\int_\Sigma \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} = 0. \tag{4.2.18}$$

4.3 Gap theorem for λ-hypersurfaces

4.3.1 Proof of Theorem 4.1.2

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. By Lemma 4.2.1, we have

$$\begin{aligned} \frac{1}{2}\mathcal{L}|A|^2 &= \Big(\frac{1}{2} - |A|^2\Big)|A|^2 - \lambda\langle A^2, A\rangle + |\nabla A|^2 \\ &\geq \Big(\frac{1}{2} - |A|^2\Big)|A|^2 - |\lambda||A|^3 + |\nabla A|^2. \end{aligned} \tag{4.3.1}$$

Proposition 4.2.3 and Corollary 4.2.6 give

$$0 = \int_\Sigma \mathcal{L}|A|^2 e^{-\frac{|x|^2}{4}} \geq \int_\Sigma \Big(\frac{1}{2} - |A|^2 - |\lambda||A|\Big)|A|^2 e^{-\frac{|x|^2}{4}} + \int_\Sigma |\nabla A|^2 e^{-\frac{|x|^2}{4}}. \tag{4.3.2}$$

Note that when $|A| \leq \frac{\sqrt{\lambda^2+2}-|\lambda|}{2}$, we have

$$\frac{1}{2} - |A|^2 - |\lambda||A| \geq 0.$$

This implies the first term of (4.3.2) on the right hand side is nonnegative. Therefore, (4.3.2) implies that all inequalities are equalities. Moreover, we have

$$|\nabla A| = \Big(\frac{1}{2} - |A|^2 - |\lambda||A|\Big)|A|^2 = 0.$$

By Theorem 4 of Laswon [Law69] that every smooth hypersurface with $\nabla A = 0$ splits isometrically as a product of a sphere and a linear space, we finish the proof.

□

By the proof of Theorem 4.1.2, we have the following gap result.

Corollary 4.3.1. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth complete embedded λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n}\rangle}{2} = \lambda$ with polynomial volume growth, which satisfies*

$$|A| < \frac{\sqrt{\lambda^2+2} - |\lambda|}{2}, \tag{4.3.3}$$

then Σ is a hyperplane in $\mathbf{R}^{n+1}$.

Remark 4.3.2. *Note that in Theorem 4.1.2, when Σ^n is a round sphere, this forces $\lambda = 0$. we will address this issue in the next subsection and prove a gap theorem for closed λ-hypersurfaces with arbitrary $\lambda \geq 0$.*

4.3.2 Gap Theorems for closed λ-hypersurfaces

We consider closed λ-hypersurfaces with $\lambda \geq 0$ in this subsection.

Lemma 4.3.3. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a smooth λ-hypersurface, then*

$$\mathcal{L}|x|^2 = 2n - |x|^2 - 2\lambda\langle x, \mathbf{n}\rangle. \tag{4.3.4}$$

Proof. Recall that for any hypersurface, we have $\Delta x = -H\mathbf{n}$. Therefore,

$$\begin{aligned} \mathcal{L}|x|^2 = \Delta|x|^2 - \frac{1}{2}\langle x, \nabla|x|^2\rangle &= 2\langle \Delta x, x\rangle + 2|\nabla x|^2 - |x^T|^2 \\ &= -2H\langle x, \mathbf{n}\rangle + 2n - |x^T|^2 \\ &= 2n - |x|^2 - 2\lambda\langle x, \mathbf{n}\rangle. \end{aligned} \tag{4.3.5}$$

□

A simple application of Lemma 4.3.3 and the maximum principle give the following corollary.

Corollary 4.3.4. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth closed λ-hypersurface with $\lambda \geq 0$. If $|x| \leq \sqrt{\lambda^2 + 2n} - \lambda$, then Σ is a round sphere.*

Now we are ready to address the issue mentioned in Remark 4.3.2 and prove the gap theorem for closed λ-hypersurfaces in terms of $|A|$.

Theorem 4.3.5. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth closed λ-hypersurface with $\lambda \geq 0$. If Σ satisfies*

$$|A|^2 \leq \frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n}, \tag{4.3.6}$$

then Σ is a round sphere with radius $\sqrt{\lambda^2 + 2n} - \lambda$.

Proof. Since Σ is closed, we consider the point p where $|x|$ achieves its maximum. At point p, x and $\mathbf{n}$ are in the same direction. This implies $2H(p) = 2\lambda + |x|(p)$.

By (4.3.6), we have

$$\left(\lambda + \frac{|x|(p)}{2}\right)^2 = H^2(p) \leq n|A|^2 \leq n\left(\frac{1}{2} + \frac{\lambda(\lambda + \sqrt{\lambda^2 + 2n})}{2n}\right). \tag{4.3.7}$$

This gives

$$\max_{\Sigma} |x| \leq |x|(p) \leq \sqrt{\lambda^2 + 2n} - \lambda. \tag{4.3.8}$$

By Corollary 4.3.4, we conclude that Σ is a round sphere. $\square$

We also include another weak gap theorem for closed λ-hypersurfaces in $\mathbf{R}^3$ in that the proof is interesting and using Gauss-Bonnet Formula, Minkowski Integral Formulas and the Willmore's inequality.

Theorem 4.3.6. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a closed λ-hypersurface satisfying $H - \frac{\langle x, \mathbf{n} \rangle}{2} = \lambda$ with $\lambda \geq 0$. If Σ satisfies $|A|^2 \leq \frac{1+\lambda^2}{2}$, then Σ is a round sphere.*

Proof. First, by the Gauss-Bonnet Theorem, we have the following identity:

$$\int_\Sigma H^2 = \int_\Sigma |A|^2 + 8\pi(1 - g), \tag{4.3.9}$$

where g is the genus of Σ.

Next, by Minkowski Integral Formulas and Stokes' theorem, we have

$$\int_\Sigma H\langle x, \mathbf{n} \rangle = 2\text{Area}(\Sigma), \tag{4.3.10}$$

$$\int_\Sigma \langle x, \mathbf{n}\rangle = 3\text{Volume}(\Omega). \tag{4.3.11}$$

Here, Ω is the region enclosed by Σ.

By (4.3.10), (4.3.11) and the λ-hypersurface equation, we get that

$$\int_\Sigma H \geq \lambda\text{Area}(\Sigma) = \lambda\Big(\int_\Sigma H^2 - \lambda\int_\Sigma H\Big). \tag{4.3.12}$$

This implies

$$\int_\Sigma H \geq \frac{\lambda}{1+\lambda^2}\int_\Sigma H^2. \tag{4.3.13}$$

Using (4.3.9), (4.3.10) and the $|A|$ bound, we see that

$$\begin{aligned}\int_\Sigma H^2 &\leq \Big(\frac{1+\lambda^2}{2}\Big)\text{Area}(\Sigma) + 8\pi(1-g)\\ &\leq \Big(\frac{1+\lambda^2}{2}\Big)\Big(\int_\Sigma H^2 - \lambda\int_\Sigma H\Big) + 8\pi(1-g).\end{aligned} \tag{4.3.14}$$

Combining this with (4.3.13) gives

$$\int_\Sigma H^2 \leq 16\pi(1-g). \tag{4.3.15}$$

Recall that for any smooth closed surface M in $\mathbf{R}^3$, the Willmore energy satisfies

$$\int_M H^2 \geq 16\pi, \tag{4.3.16}$$

with the equality holds if and only M is a round sphere. Therefore, by (4.3.15) we conclude that Σ is a round sphere. Actually this case implies that $\lambda = 0$.

□

4.3.3 Closed λ-hypersurfaces in $\mathbf{R}^3$ with constant $|A|$

Let $\Sigma^2 \subset \mathbf{R}^3$ be a smooth complete embedded self-shrinker. If $|A|$ is a constant, then one can show that Σ is a generalized cylinder $\mathbf{S}^k \times \mathbf{R}^{2-k}$ for some $k \leq 2$; see [DX14] and [Gua14b]. One way to prove this is to consider the point where the norm of position vector $|x|$ achieves its minimum.

For λ-hypersurfaces, we will use a similar idea and an important result from [HW54] to show that any smooth closed λ-hypersurface in $\mathbf{R}^3$ with $\lambda \geq 0$ and $|A| = constant$ is a round sphere, i.e., Theorem 4.1.4.

First of all, we recall the following ingredients from [HW54].

Definition 4.3.7. *A hypersurface in $\mathbf{R}^3$ is called a special Weingarten surface (special W-surface) if its Gauss curvature and mean curvature*[1]*, K and H, are connected by an identity*

$$F(K, H) = 0 \tag{4.3.17}$$

in which F satisfies the following condition:

- *The function $F(K, H)$ is defined and of class C^2 on the portion $4K \leq H^2$ of the $(K, H) - plane$ and satisfies*

$$F_H + HF_K \neq 0 \quad \textit{when} \quad 4K = H^2. \tag{4.3.18}$$

In [HW54], Hartman and Wintner proved the following theorem for special W-surfaces.

Theorem 4.3.8 ([HW54]). *If a closed orientable surface S of genus 0 is a special W-surface of class C^2, then S is a round sphere.*

One may easily check that a closed surface with $|A| = constant$ is a special W-surface, so by Theorem 4.3.8, we have the following corollary.

Corollary 4.3.9. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a smooth closed embedded surface of genus 0. If $|A| = constant$, then Σ is a round sphere.*

By Corollary 4.3.9, in order to prove Theorem 4.1.4, all we need to show is that any closed λ-hypersurface with constant $|A|$ has genus 0.

[1]In [HW54], they use the average rather than the sum of the principal curvatures.

Proof of Theorem 4.1.4. First, by Gauss-Bonnet Formula, Minkowski Integral Formulas and Stokes' theorem, we have

$$\int_\Sigma H^2 = \int_\Sigma |A|^2 + 8\pi(1-g), \tag{4.3.19}$$

$$\int_\Sigma H\langle x, \mathbf{n}\rangle = 2\text{Area}(\Sigma), \tag{4.3.20}$$

and

$$\int_\Sigma \langle x, \mathbf{n}\rangle = 3\text{Volume}(\Omega), \tag{4.3.21}$$

where g is the genus of Σ and Ω is the region enclosed by Σ.

Combining above identities, we deduce that

$$\int_\Sigma H^2 \geq (\lambda^2+1)\int_\Sigma = (\lambda^2+1)\text{Area}(\Sigma). \tag{4.3.22}$$

Next, we consider the point $p \in \Sigma$ where $|x|$ achieves its minimum. By Lemma 4.3.3, at point p, we have

$$H^2(p) \leq \frac{2+\lambda^2+\lambda\sqrt{\lambda^2+4}}{2}. \tag{4.3.23}$$

At point p, we can choose a local orthonormal frame $\{e_1, e_2\}$ such that the second fundamental form

$$a_{ij} = \lambda_i \delta_{ij} \quad \text{for} \quad i,j = 1,2. \tag{4.3.24}$$

Thus, we have

$$|\nabla H|^2 = (a_{111}+a_{221})^2 + (a_{112}+a_{222})^2. \tag{4.3.25}$$

Since $|A|^2 = constant$, we see that

$$a_{11}a_{111} + a_{22}a_{221} = a_{11}a_{112} + a_{22}a_{222} = 0. \tag{4.3.26}$$

Note that at point p, $|\nabla H| = 0$. This implies

$$a_{111} + a_{221} = a_{112} + a_{222} = 0.$$

Combining this with (4.3.25) and (4.3.26), we get

$$a_{111}(a_{11} - a_{22}) = a_{222}(a_{11} - a_{22}) = 0. \tag{4.3.27}$$

If $a_{11} = a_{22}$, then by (4.3.23), we have

$$|A|^2 = \frac{H^2}{2} \leq \frac{2 + \lambda^2 + \lambda\sqrt{\lambda^2 + 4}}{4}. \tag{4.3.28}$$

By Theorem 4.3.5, this implies Σ is a round sphere.

If $a_{111} = a_{222} = 0$, then $|\nabla A|^2 = 0$. Hence,

$$\Big(\frac{1}{2} - |A|^2\Big)|A|^2 = \lambda\langle A^2, A\rangle. \tag{4.3.29}$$

Thus, we have

$$\Big(|A|^2 - \frac{1}{2}\Big)|A|^2 = -\lambda\langle A^2, A\rangle \leq \lambda|A|^3. \tag{4.3.30}$$

Therefore,

$$|A|^2 \leq \frac{1 + \lambda^2 + \lambda\sqrt{\lambda^2 + 2}}{2}. \tag{4.3.31}$$

Combining this with (4.3.19) and (4.3.22) gives

$$(\lambda^2 + 1)\text{Area}(\Sigma) \leq \int_\Sigma H^2 \leq \frac{1 + \lambda^2 + \lambda\sqrt{\lambda^2 + 2}}{2}\text{Area}(\Sigma) + 8\pi(1 - g). \tag{4.3.32}$$

Observe that

$$\lambda^2 + 1 > \frac{1 + \lambda^2 + \lambda\sqrt{\lambda^2 + 2}}{2}, \tag{4.3.33}$$

and this implies the genus $g = 0$. By Corollary 4.3.9, we conclude that Σ is a round sphere. This completes the proof. □

Remark 4.3.10. *Note that our method does not apply to higher dimensions. We conjecture that any λ-hypersurface $\Sigma^2 \subset \mathbf{R}^3$ with $|A|$ = constant is a generalized cylinder.*

4.4 Embedded λ-curves in $\mathbf{R}^2$

In this section, we follow the argument in [Man11] to show that any λ-hypersurface (λ-curve) in $\mathbf{R}^2$ with $\lambda \geq 0$ must either be a line or a round circle, i.e., Theorem 4.1.5.

Proof of Theorem 4.1.5. Suppose s is an arclength parameter of γ, then the curvature is $H = -\langle \nabla_{\gamma'}\gamma', \mathbf{n}\rangle$. Note that $\nabla_{\gamma'}\mathbf{n} = H\gamma'$, so we have

$$2H' = \nabla_{\gamma'}\langle x, \mathbf{n}\rangle = H\langle x, \gamma'\rangle. \tag{4.4.1}$$

If at some point $H = 0$, then $H' = 0$. By the uniqueness theorem of ODE, we conclude that $H \equiv 0$, and, thus, γ is just a line. Therefore, we may assume that H is always nonzero and possibly reversing the orientation of the curve to make $H > 0$, i.e., γ is strictly convex.

Differentiating $|x|^2$ gives

$$(|x|^2)' = 2\langle x, \gamma'\rangle = 4\frac{H'}{H}. \tag{4.4.2}$$

Thus $H = Ce^{\frac{|x|^2}{4}}$ for some constant $C > 0$.

Since the curve is strictly convex, we introduce a variable θ by $\theta = \arccos\langle \mathbf{E}_1, n\rangle$. Differentiating with respect to the arclength parameter gives

$$\partial_s\theta = -H, \tag{4.4.3}$$

$$H_\theta = -\frac{H'}{H} = -\frac{\langle x, \gamma'\rangle}{2}, \tag{4.4.4}$$

and

$$H_{\theta\theta} = \frac{\partial_s H_\theta}{-H} = \frac{1 - 2H(H-\lambda)}{2H} = \frac{1}{2H} - H + \lambda. \tag{4.4.5}$$

Multiplying both sides of the above equation by $2H_\theta$, we get

$$\partial_\theta(H_\theta^2 + H^2 - \log H - 2\lambda H) = 0. \tag{4.4.6}$$

Therefore, the quantity

$$E = H_\theta^2 + H^2 - \log H - 2\lambda H \tag{4.4.7}$$

is a constant.

Consider the function $f(t) = t^2 - \log t - 2\lambda t$, $t > 0$. It is easy to verify that

$$f(t) \geq f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2}). \tag{4.4.8}$$

Hence,

$$E \geq f(\frac{\lambda + \sqrt{\lambda^2 + 2}}{2}). \tag{4.4.9}$$

If $E = f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$, then H is constant and γ must be a round circle.

Now we assume that $E > f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$. Note that $H = Ce^{\frac{|x|^2}{4}}$ and $H \leq |x/2| + |\lambda|$. Then H has an upper bound and $|x|$ is bounded. By the embeddedness and completeness of γ, we conclude that γ must be closed, simple and strictly convex.

If γ is not a round circle, then we consider the critical points of the curvature H. By our assumption that $E > f(\frac{\lambda+\sqrt{\lambda^2+2}}{2})$, when $H_\theta = 0$, we have $H_{\theta\theta} = \frac{1}{2H} - H + \lambda \neq 0$. So the critical points are not degenerate. By the compactness of the curve, they are finite and isolated.

Without loss of generality, we may assume $H(0) = H_{max}$ and $H(\bar{\theta})$ is the first subsequent critical point of H for $\bar{\theta} > 0$. Combining the fact that the curvature is strictly decreasing in the interval $[0, \bar{\theta}]$ with the second-order ODE of the function H is symmetric with respect to $\theta = 0$ and $\theta = \bar{\theta}$, we conclude that $H(\bar{\theta})$ must be the minimum of the curvature.

By the four-vertex theorem, we know that γ has at least four pieces like the one described above. Since our curve is closed and embedded, the curvature H is periodic with period $T < \pi$ and $\frac{T}{2} = \bar{\theta}$.

Next, we will evaluate an integral to produce a contradiction.

Since $H_{\theta\theta} = \frac{1}{2H} - H + \lambda$, we have

$$(H^2)_{\theta\theta\theta} + 4(H^2)_\theta = \frac{2H_\theta}{H} + 6\lambda H_\theta. \tag{4.4.10}$$

Now we consider the following integral

$$2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = \int_0^{\frac{T}{2}} \sin 2\theta \Big[(H^2)_{\theta\theta\theta} + 4(H^2)_\theta - 6\lambda H_\theta\Big] d\theta. \tag{4.4.11}$$

Integration by parts gives

$$\begin{aligned} 2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta &= \sin 2\theta (H^2)_{\theta\theta}\Big|_0^{\frac{T}{2}} - 2\int_0^{\frac{T}{2}} \cos 2\theta (H^2)_{\theta\theta} d\theta \\ &+ 4\int_0^{\frac{T}{2}} \sin 2\theta (H^2)_\theta d\theta - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta \\ &= 2\sin T\Big[H_\theta^2(\frac{T}{2}) + H(\frac{T}{2})H_{\theta\theta}(\frac{T}{2})\Big] - 2\cos 2\theta (H^2)_\theta\Big|_0^{\frac{T}{2}} \\ &- 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta \\ &= 2\sin T H(\frac{T}{2})H_{\theta\theta}(\frac{T}{2}) - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta. \end{aligned} \tag{4.4.12}$$

By (4.4.5) and $H_\theta(0) = H_\theta(\frac{T}{2}) = 0$, we get

$$2\int_0^{\frac{T}{2}} \sin 2\theta \frac{H_\theta}{H} d\theta = 2\sin T\Big[\frac{1}{2} - H^2(\frac{T}{2}) + \lambda H(\frac{T}{2})\Big] - 6\lambda \int_0^{\frac{T}{2}} \sin 2\theta H_\theta d\theta. \tag{4.4.13}$$

Since H is decreasing from 0 to $\frac{T}{2}$ and $\sin 2\theta$ is nonnegative, the left-hand side of (4.4.13) is nonpositive. For the right-hand side, the first term is nonnegative since $H(\frac{T}{2})$ is a minimum, and $\lambda \geq 0$ implies the second term is nonpositive. So the right-hand side of (4.4.13) is nonnegative, and this gives a contradiction. Therefore, we conclude that γ is a round circle. □

Remark 4.4.1. *For the noncompact case, we do not need the condition $\lambda \geq 0$ to prove it is a line, and we do need $\lambda \geq 0$ for the closed case. When $\lambda < 0$, there exist some embedded λ-curves which are not round circles.*

4.5 A Bernstein type theorem for λ-hypersurfaces

The aim of this section is to prove Theorem 4.1.6 which generalizes Ecker and Huisken's result [EH89]. The key ingredient is that for a λ-hypersurface Σ, the function $\langle v, \mathbf{n}\rangle$ is an eigenfunction of the operator L with eigenvalue $1/2$, where $v \in \mathbf{R}^{n+1}$ is any constant vector. Note that the result is also true for self-shrinkers. This eigenvalue result was also obtained by McGonagle and Ross [MR15].

Lemma 4.5.1. *If $\Sigma \subset \mathbf{R}^{n+1}$ is a λ-hypersurface, then for any constant vector $v \in \mathbf{R}^{n+1}$, we have*

$$L\langle v, \mathbf{n}\rangle = \frac{1}{2}\langle v, \mathbf{n}\rangle.$$

Proof. Set $f = \langle v, \mathbf{n}\rangle$. Working at a fixed point p and choosing e_i to be a local orthonormal frame, we have

$$\nabla_{e_i} f = \langle v, \nabla_{e_i}\mathbf{n}\rangle = -a_{ij}\langle v, e_j\rangle.$$

Differentiating again and using Codazzi equation gives that

$$\nabla_{e_k}\nabla_{e_i} = -a_{ijk}\langle v, e_j\rangle - a_{ij}a_{jk}\langle v, \mathbf{n}\rangle.$$

Therefore,

$$\Delta f = \langle v, \nabla H\rangle - |A|^2 f. \tag{4.5.1}$$

Using the equation of λ-hypersurfaces, we have

$$\langle v, \nabla H\rangle = \langle v, -\frac{1}{2}a_{ij}\langle x, e_j\rangle e_i\rangle = \frac{1}{2}\langle x, \nabla f\rangle. \tag{4.5.2}$$

Combining (4.5.1) and (4.5.2), we obtain that

$$Lf = \Delta f - \frac{1}{2}\langle x, \nabla f\rangle + \Big(\frac{1}{2} + |A|^2\Big) f = \frac{1}{2} f. \tag{4.5.3}$$

□

We are now in the position to prove Theorem 4.1.6.

Proof of Theorem 4.1.6. Since Σ is an entire graph, we can find a constant vector v such that $f = \langle v, \mathbf{n} \rangle > 0$. Let $u = 1/f$. Then we have

$$\nabla u = -\frac{\nabla f}{f^2} \text{ and } \Delta u = -\frac{\Delta f}{f^2} + \frac{2|\nabla f|^2}{f^3}. \tag{4.5.4}$$

By Lemma 4.5.1, we can easily get

$$\mathcal{L}u = |A|^2 u + \frac{2|\nabla u|^2}{u}. \tag{4.5.5}$$

Since Σ has polynomial volume growth, we get

$$\int_\Sigma \left(|A|^2 u + \frac{2|\nabla u|^2}{u} \right) e^{-\frac{|x|^2}{4}} = 0. \tag{4.5.6}$$

Therefore, $|A| = 0$ and Σ is a hyperplane in $\mathbf{R}^{n+1}$. □

Chapter 5

Analysis of translators of mean curvature flow

In this chapter, we study the translating solitons to mean curvature flow. First, we prove that every complete properly immersed translator has at least linear volume growth. Next, by using Huisken's monotonicity formula, we compute the entropy of the grim reaper and the bowl solitons. Then, we estimate the spectrum of the stability operator L for translators and give a rigidity result of L-stable translators. Finally, we give three types of curvature estimates for translators. This chapter is based on [Gua15], [GZ15] and [GZ16].

5.1 Volume growth

In this section, we consider the volume growth for translators and show that every properly immersed translator has at least linear volume growth. More precisely, we prove the following theorem.

Theorem 5.1.1. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete properly immersed translator. Then for any $x \in \Sigma$, there exists a constant $C > 0$ such that*

$$Vol(\Sigma \cap B_r(x)) = \int_{\Sigma \cap B_r(x)} d\mu \geq Cr \quad \text{for all } r \geq 1. \tag{5.1.1}$$

Remark 5.1.2. *Unlike self-shrinkers, translators do not necessarily have Euclidean volume growth; see the example constructed in [Ngu15]. In a recent work of Xin [Xin15], the author considered the volume growth of the intrinsic balls of translators in a conformal metric.*

Suppose $\Sigma^n \subset \mathbf{R}^{n+1}$ is a properly immersed translator. For any $x_0 \in \Sigma$, let $B_r(x_0)$ be the extrinsic ball in $\mathbf{R}^{n+1}$, and denote the volume and the weighed volume of $\Sigma \cap B_r(x_0)$ by

$$V(r) = \mathrm{Vol}(\Sigma \cap B_r(x_0)) = \int_{\Sigma \cap B_r(x_0)} d\mu, \tag{5.1.2}$$

and

$$\tilde{V}(r) = \int_{\Sigma \cap B_r(x_0)} \mathrm{e}^{x_{n+1}} d\mu. \tag{5.1.3}$$

We will first show that the weighed volume has at least exponential growth, and this relies on the following key ingredient.

Lemma 5.1.3. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a translator satisfying $H = -\langle \mathbf{E}_{n+1}, \mathbf{n} \rangle$, then*

$$\Delta e^{x_{n+1}} = e^{x_{n+1}}. \tag{5.1.4}$$

Proof. First, we have

$$\Delta \mathrm{e}^{x_{n+1}} = \mathrm{div}_\Sigma(\mathrm{e}^{x_{n+1}} \nabla_\Sigma x_{n+1}) = \mathrm{e}^{x_{n+1}} |E_{n+1}^T|^2 + \mathrm{e}^{x_{n+1}} \Delta x_{n+1}. \tag{5.1.5}$$

Hence, the identity follows from Lemma 2.3.2 and the fact that $|E_{n+1}^T|^2 = 1 - H^2$. □

For simplicity, we may assume $x_0 = 0$. By the co-area formula, we have

$$\tilde{V}(r) = \int_0^r \int_{\partial B_s \cap \Sigma} \mathrm{e}^{x_{n+1}} \frac{1}{|\nabla_\Sigma |x||} ds. \tag{5.1.6}$$

Note that $\nabla_\Sigma |x|^2 = 2x^T = 2|x| \nabla_\Sigma |x|$, so we get

$$\tilde{V}'(r) = \int_{\partial B_r \cap \Sigma} \mathrm{e}^{x_{n+1}} \frac{|x|}{|x^T|}. \tag{5.1.7}$$

On the other hand, by Lemma 5.1.3, we obtain that

$$\begin{aligned} \tilde{V}(r) = \int_{\Sigma\cap B_r} \Delta \mathrm{e}^{x_{n+1}} d\mu &= \int_{\partial B_r\cap\Sigma} \langle \nabla_\Sigma \mathrm{e}^{x_{n+1}}, \frac{x^T}{|x^T|} \rangle \\ &= \int_{\partial B_r\cap\Sigma} \langle \nabla_\Sigma x_{n+1}, \frac{x^T}{|x^T|} \rangle \mathrm{e}^{x_{n+1}}. \end{aligned} \tag{5.1.8}$$

Combining (5.1.7) and (5.1.8), we conclude that for any $r > 0$,

$$\tilde{V}(r) \leq \tilde{V}'(r). \tag{5.1.9}$$

This implies the quantity $\tilde{V}(r)\mathrm{e}^{-r}$ is monotone non-decreasing. We summarize this result in the following proposition.

Proposition 5.1.4. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete properly immersed translator. Then for any $x \in \Sigma$, there exists a constant $C > 0$ such that*

$$\tilde{V}(r) = \int_{\Sigma\cap B_r(x)} e^{x_{n+1}} d\mu \geq C e^r, \quad \textit{for all } r \geq 1. \tag{5.1.10}$$

With the help of Proposition 5.1.4, we can now estimate the volume growth. For any $R > 1$, we have

$$\tilde{V}'(R) \leq \mathrm{e}^R \int_{\partial B_R\cap\Sigma} \frac{|x|}{|x^T|} \leq \mathrm{e}^R V'(R). \tag{5.1.11}$$

Combining this with Proposition 5.1.4 and (5.1.9) gives that $V(R)$ has at least linear growth.

5.2 Entropy of the grim reaper and the bowl solitons

We first recall the main ingredient Huisken's monotonicity formula [Hui90].

Theorem 5.2.1 ([Hui90]). *If M_t is a solution to MCF, then we have*

$$\frac{d}{dt}\int_{M_t} \Phi_{(z_0,\tau)} = -\int_{M_t} \left| H\mathbf{n} - \frac{(x-z_0)^\perp}{2(\tau-t)} \right|^2 \Phi_{(z_0,\tau)}, \tag{5.2.1}$$

where the function Φ *on* $\mathbf{R}^{n+1} \times (-\infty, 0)$ *is given by*

$$\Phi(x,t) = (-4\pi t)^{-\frac{n}{2}} e^{\frac{|x|^2}{4t}} \tag{5.2.2}$$

and $\Phi_{z_0,\tau}(x,t) = \Phi(x - z_0, t - \tau)$.

The F-functional can be expressed as the following:

$$F_{z_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int_\Sigma \mathrm{e}^{-\frac{|x-z_0|^2}{4t_0}} = \int_\Sigma \Phi_{z_0,t_0}(x,0). \tag{5.2.3}$$

Moreover, for any z_0 in $\mathbf{R}^{n+1}$ and $t_0 > 0$, if M_t gives a MCF and suppose $t > s$, then by Huisken's monotonicity formula (5.2.1), we have

$$F_{z_0,t_0}(M_t) \leq F_{z_0,t_0+(t-s)}(M_s). \tag{5.2.4}$$

A direct consequence of (5.2.4) is that the entropy is non-increasing under MCF.

Now we apply Huisken's monotonicity formula to translating solitons.

The grim reaper or the bowl solition Γ gives a solution of MCF $\Gamma_t = \{(x, t+f(x)) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n\}$ (for $n = 1$, $x \in (-\pi/2, \pi/2)$) for all $t \in \mathbf{R}$.

For any point $(x_0, y_0) \in \mathbf{R}^{n+1}$ ($x_0 \in \mathbf{R}^n, y_0 \in \mathbf{R}$) and $t_0 > 0$, we have the following identity:

$$F_{(x_0,y_0),\,t_0}(\Gamma_t) = F_{(x_0,y_0-t),\,t_0}(\Gamma). \tag{5.2.5}$$

By (5.2.4), for any $t > s$, we get

$$F_{(x_0,y_0),\,t_0}(\Gamma_t) \leq F_{(x_0,y_0),\,t_0+(t-s)}(\Gamma_s). \tag{5.2.6}$$

Combining this with (5.2.5) gives

$$F_{(x_0,y_0-t),\,t_0}(\Gamma) \leq F_{(x_0,y_0-s),\,t_0+(t-s)}(\Gamma). \tag{5.2.7}$$

Therefore, we obtain the following crucial lemma.

Lemma 5.2.2. *For any point* $((x_0, y_0), t_0) \in \mathbf{R}^{n+1} \times (0, \infty)$ *and arbitrary* $N > 0$, *we have*

$$F_{(x_0,y_0),\,t_0}(\Gamma) \le F_{(x_0,y_0+N),\,t_0+N}(\Gamma). \tag{5.2.8}$$

5.2.1 Entropy of the grim reaper

Theorem 5.2.3. *The entropy of the grim reaper* $\Gamma^1 \subset \mathbf{R}^2$ *is* 2.

Recall that the entropy is taking the supremum of the F-functional, so we will first consider the upper bound of the F-functional $F_{(x_0,y_0+N),\,t_0+N}(\Gamma)$ by taking $N \to +\infty$, and this will yield that the entropy of the grim reaper is less than or equal to 2.

For any fixed point $(x_0, y_0) \in \mathbf{R}^2$ and $t_0 > 0$, by Lemma 5.2.2, we may choose N sufficiently large such that

$$F_{(x_0,y_0),t_0}(\Gamma) \le F_{(x_0,N+\delta),\,N}(\Gamma), \tag{5.2.9}$$

where $\delta = y_0 - t_0$.

By the equation of the grim reaper, we have

$$F_{(x_0,N+\delta),\,N}(\Gamma) = (4\pi N)^{-\frac{1}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-\frac{(x-x_0)^2+(\log\cos x+N+\delta)^2}{4N}} \frac{1}{\cos x}\, dx. \tag{5.2.10}$$

Note that

$$F_{(x_0,N+\delta),\,N}(\Gamma) \le (\pi N)^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{(\log\cos x+N+\delta)^2}{4N}} \frac{1}{\cos x}\, dx. \tag{5.2.11}$$

Now we define a function $g(N)$ by

$$g(N) = (\pi N)^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} \mathrm{e}^{-\frac{(\log\cos x+N+\delta)^2}{4N}} \frac{1}{\cos x}\, dx. \tag{5.2.12}$$

We make a change of variable and let $u = -\log\cos x$ and $t = u/N$. This gives

$$\begin{aligned} g(N^2) &= \frac{1}{N\sqrt{\pi}} \int_0^{\infty} \mathrm{e}^{-\frac{(u-N^2-\delta)^2}{4N^2}} \frac{\mathrm{e}^u}{\sqrt{\mathrm{e}^{2u}-1}} du \\ &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \mathrm{e}^{-\frac{(N^2+\delta)^2}{4N^2}} \mathrm{e}^{-\frac{t^2}{4}+(\frac{3N}{2}+\frac{\delta}{2N})t} \frac{1}{\sqrt{\mathrm{e}^{2Nt}-1}}\, dt. \end{aligned} \tag{5.2.13}$$

Next we estimate the value of $g(N^2)$ when N goes to infinity by splitting it into two parts, A and B.

For the first part,

$$\begin{aligned} A &= \frac{1}{\sqrt{\pi}} \int_0^{\frac{N}{6}} \mathrm{e}^{-\frac{(N^2+\delta)^2}{4N^2}} \mathrm{e}^{-\frac{t^2}{4}+(\frac{3N}{2}+\frac{\delta}{2N})t} \frac{1}{\sqrt{\mathrm{e}^{2Nt}-1}}\, dt \\ &\le \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\frac{(N^2+\delta)^2}{4N^2}+\frac{N^2}{4}+\frac{\delta}{12}} \int_0^{\frac{N}{6}} \frac{1}{\sqrt{\mathrm{e}^{2Nt}-1}}\, dt \\ &\le \frac{1}{2N\sqrt{\pi}} \mathrm{e}^{-\frac{\delta^2}{4N^2}-\frac{5}{12}\delta} \int_0^{\frac{N^2}{3}} \frac{1}{\sqrt{\mathrm{e}^t-1}}\, dt \le \frac{\pi}{2N\sqrt{\pi}} \mathrm{e}^{-\frac{\delta^2}{4N^2}-\frac{5}{12}\delta}. \end{aligned} \tag{5.2.14}$$

Here we use the fact that

$$\int_0^\infty \frac{1}{\sqrt{\mathrm{e}^t-1}}\, dt = \pi.$$

For the second part,

$$\begin{aligned} B &= \frac{1}{\sqrt{\pi}} \int_{\frac{N}{6}}^{\infty} \mathrm{e}^{-\frac{(N^2+\delta)^2}{4N^2}} \mathrm{e}^{-\frac{t^2}{4}+(\frac{3N}{2}+\frac{\delta}{2N})t} \frac{1}{\sqrt{\mathrm{e}^{2Nt}-1}}\, dt \\ &\le \frac{1}{\sqrt{\pi}} \big(1+2\,\mathrm{e}^{-\frac{N^2}{6}}\big) \int_{\frac{N}{6}}^{\infty} \mathrm{e}^{-\frac{1}{4}(t-N-\frac{\delta}{N})^2}\, dt \\ &\le \frac{1}{\sqrt{\pi}} \big(1+2\,\mathrm{e}^{-\frac{N^2}{6}}\big) \int_{-\frac{5N}{6}-\frac{\delta}{N}}^{\infty} \mathrm{e}^{-\frac{t^2}{4}}\, dt \le 2\big(1+2\,\mathrm{e}^{-\frac{N^2}{6}}\big). \end{aligned} \tag{5.2.15}$$

The last inequality uses

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{4}}\, dt = 2. \tag{5.2.16}$$

Taking $N \to +\infty$, by Lemma 5.2.2, (5.2.14) and (5.2.15), we obtain the following result.

Lemma 5.2.4. *For any point* $((x_0, y_0),\, t_0) \in \mathbf{R}^2 \times (0,\infty)$, *we have*

$$F_{(x_0,y_0),\,t_0}(\Gamma) \le 2, \quad .i.e., \quad \lambda(\Gamma) \le 2. \tag{5.2.17}$$

Using the similar method as above, it is easy to prove that

$$\lim_{N\to\infty} F_{(0,N),\,N}(\Gamma) = 2. \tag{5.2.18}$$

Finally, combining Lemma 5.2.4 and (5.2.18), we conclude that $\lambda(\Gamma) = 2$, which completes the proof of Theorem 5.2.3.

5.2.2 Entropy of the bowl solitons

In this section, we deal with the case when $n \geq 2$ and prove the next theorem.

Theorem 5.2.5. *The entropy of the rotationally symmetric convex translating soliton $\Gamma^n \subset \mathbf{R}^{n+1}$ $(n \geq 2)$ is equal to the entropy of the sphere $\mathbf{S}^{n-1}$, i.e.,*

$$\lambda(\Gamma^n) = \lambda(\mathbf{S}^{n-1}) = n\alpha(n)\Big(\frac{n-1}{2\pi e}\Big)^{\frac{n-1}{2}}, \tag{5.2.19}$$

where $\alpha(n)$ is the volume of unit ball in $\mathbf{R}^n$.

The idea of the proof of Theorem 5.2.5 is similar to the one dimensional case. The only difference is, unlike the grim reaper, we do not have explicit expression for the function f in (2.3.4). Therefore, we need to recall some important properties of the function f (see also [CSS07] and [Wan11b]).

By the property of rotationally symmetry, we have $f(x) = f(r)$ with $r = |x|$. The equation (2.3.4) gives the following ODE

$$f_{rr} = (1 + f_r^2)\Big(1 - \frac{(n-1)f_r}{r}\Big) \tag{5.2.20}$$

with $f(0) = \lim_{r\to 0} f'(r) = 0$ for $f : \mathbf{R}^+ \to \mathbf{R}$.

Proposition 5.2.6. *The function f in the ODE (5.2.20) satisfies the following properties:*

(1) $f'(r) < \frac{r}{n-1}$ *and* $f(r) \leq \frac{r^2}{2(n-1)}$.

(2) For any $\varepsilon > 0$, $f'(r) > (\frac{1}{n} - \varepsilon)r$, *especially,* $f'(r) > \frac{r}{2n}$.

(3) For any $\varepsilon > 0$, *there exists* $r_0 = r_0(\varepsilon) > 0$ *such that* $f'(r) > (1-\varepsilon)\frac{r}{n-1}$ *for* $r \geq r_0$.

(4) For any $\varepsilon > 0$*, there exists a constant* $M = M(\varepsilon) > 0$ *such that* $f(r) > \frac{1-\varepsilon}{2(n-1)}r^2 - M$.

Proof. The proof of the first and the second properties are straightforward. For the third one, we may argue by contradiction. If this is not true, then $f''(r) > (1+(f'(r))^2)\varepsilon$ for some r going to infinity. However, this is a contradiction, since our solution exists for all $r > 0$. The last property follows directly from the third property. $\square$

Next we show that the entropy of Γ is less than or equal to the entropy of the sphere S^{n-1}. Just as in the proof of Theorem 5.2.3, we consider the upper bound of the F-functional $F_{(x_0, y_0+N),\, t_0+N}(\Gamma)$ by taking $N \to +\infty$.

For any fixed point $(x_0, y_0) \in \mathbf{R}^{n+1}$ ($x_0 \in \mathbf{R}^n, y_0 \in \mathbf{R}$) and $t_0 > 0$, by Lemma 5.2.2, we may choose N large enough so that

$$F_{(x_0,y_0),\, t_0}(\Gamma) \le F_{(x_0, N+\delta),\, N}(\Gamma), \tag{5.2.21}$$

where $\delta = y_0 - t_0$. Then we have

$$\begin{aligned} F_{(x_0,N+\delta),\, N}(\Gamma) &= (4\pi N)^{-\frac{n}{2}} \int_{\mathbf{R}^n} \mathrm{e}^{-\frac{|x-x_0|^2+(f(x)-N-\delta)^2}{4N}} \sqrt{1+|\nabla f|^2}\, dx \\ &= (4\pi N)^{-\frac{n}{2}} \int_0^\infty \int_{\partial B_r} \mathrm{e}^{-\frac{|x-x_0|^2+(f(x)-N-\delta)^2}{4N}} \sqrt{1+|\nabla f|^2}\, dS_r\, dr. \end{aligned} \tag{5.2.22}$$

Now we fix an arbitrary small constant $\varepsilon > 0$. By Proposition 5.2.6, there exists a constant $M = M(\varepsilon) > 0$ such that $f(r) > \frac{1-\varepsilon}{2(n-1)}r^2 - M$.

By the absorbing inequality, we have $2\langle x, x_0\rangle \le \varepsilon|x|^2 + \varepsilon^{-1}|x_0|^2$. Therefore, we obtain that $|x-x_0|^2 \ge (1-\varepsilon)|x|^2 + (1-\varepsilon^{-1})|x_0|^2$. This gives

$$\begin{aligned} F_{(x_0,N+\delta),\, N}(\Gamma) &\le (4\pi N)^{-\frac{n}{2}} \int_0^\infty \int_{\partial B_r} \mathrm{e}^{-\frac{(1-\varepsilon)|x|^2+(1-\varepsilon^{-1})|x_0|^2+(f(x)-N-\delta)^2}{4N}} \sqrt{1+|\nabla f|^2}\, dS_r\, dr \\ &= \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \mathrm{e}^{-\frac{(1-\varepsilon^{-1})}{4N}|x_0|^2} \int_0^\infty r^{n-1} \mathrm{e}^{-\frac{(1-\varepsilon)r^2+(f(r)-N-\delta)^2}{4N}} \sqrt{1+(f'(r))^2}\, dr. \end{aligned} \tag{5.2.23}$$

Since $e^{-\frac{(1-\varepsilon^{-1})}{4N}|x_0|^2}$ goes to 1 when $N \to +\infty$, we only need to consider the integral:

$$g(N) = \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \int_0^\infty r^{n-1} e^{-\frac{(1-\varepsilon)r^2 + (f(r)-N-\delta)^2}{4N}} \sqrt{1+(f'(r))^2}\, dr. \tag{5.2.24}$$

Note that $f(r)$ is monotone and strictly increasing, we can make a change of variable and set $u = \frac{f(r)-N-\delta}{\sqrt{N}}$. Assume h is the inverse function of $f(r)$, then we have $h(\sqrt{N}u + N + \delta) = r$.

By Proposition 5.2.6, we get that

$$r^2 \geq 2(n-1)f(r) = 2(n-1)(\sqrt{N}u + N + \delta), \tag{5.2.25}$$

$$\sqrt{N}u + N + \delta = f(r) > \frac{1-\varepsilon}{2(n-1)} r^2 - M. \tag{5.2.26}$$

Combining this with $f'(r) > \frac{r}{2n}$ implies that

$$\begin{aligned} g(N) &\leq \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \int_0^\infty e^{-\frac{(1-\varepsilon)r^2 + (f(r)-N-\delta)^2}{4N}} (r^{n-1} + r^{n-1} f'(r))\, dr \\ &= \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \int_{\frac{-N-\delta}{\sqrt{N}}}^\infty e^{-\frac{u^2}{4}} e^{-\frac{1-\varepsilon}{4N}h^2} (h^{n-1} + h^{n-1}h')\sqrt{N}\, du \\ &\leq \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \int_{\frac{-N-\delta}{\sqrt{N}}}^\infty e^{-\frac{(1-\varepsilon)(n-1)}{2N}(\sqrt{N}u+N+\delta)} \Big[\Big(\frac{2(n-1)(M+\sqrt{N}u+N+\delta)}{1-\varepsilon} \Big)^{\frac{n-1}{2}} \\ &+ 2n \Big(\frac{2(n-1)(M+\sqrt{N}u+N+\delta)}{1-\varepsilon} \Big)^{\frac{n-2}{2}} \Big] e^{-\frac{u^2}{4}} \sqrt{N}\, du, \end{aligned} \tag{5.2.27}$$

where $h = h(\sqrt{N}u + N + \delta)$.

Set

$$\beta = e^{-\frac{u^2}{4}} e^{-\frac{(1-\varepsilon)(n-1)}{2N}(\sqrt{N}u+N+\delta)}. \tag{5.2.28}$$

Next, we split the above integral into two parts, A and B.

For part A, we have

$$A = \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}} \int_{\frac{-N-\delta}{\sqrt{N}}}^\infty \beta \Big(\frac{2(n-1)(M+\sqrt{N}u+N+\delta)}{1-\varepsilon} \Big)^{\frac{n-1}{2}} \sqrt{N}\, du. \tag{5.2.29}$$

It is easy to see that

$$\lim_{N\to+\infty} A = n\alpha(n)\Big(\frac{n-1}{2\pi(1-\varepsilon)\mathrm{e}^{1-\varepsilon}}\Big)^{\frac{n-1}{2}}. \tag{5.2.30}$$

For part B, we get

$$B = \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}}\int_{\frac{-N-\delta}{\sqrt{N}}}^{\infty} 2n\beta\Big(\frac{2(n-1)(M+\sqrt{N}u+N+\delta)}{1-\varepsilon}\Big)^{\frac{n-2}{2}}\sqrt{N}\,du, \tag{5.2.31}$$

and when $N\to+\infty$, B converges to 0.

Combining all the results above, we obtain that

$$F_{(x_0,y_0),\,t_0}(\Gamma) \le n\alpha(n)\Big(\frac{n-1}{2\pi(1-\varepsilon)\mathrm{e}^{1-\varepsilon}}\Big)^{\frac{n-1}{2}}. \tag{5.2.32}$$

Since ε is an arbitrary small number, we get an upper bound of the entropy.

Lemma 5.2.7. *For any point $(x_0,y_0)\in\mathbf{R}^{n+1}$ $(x_0\in\mathbf{R}^n, y_0\in\mathbf{R})$ and $t_0>0$, we have*

$$F_{(x_0,y_0),\,t_0}(\Gamma) \le n\alpha(n)\Big(\frac{n-1}{2\pi e}\Big)^{\frac{n-1}{2}}, \ \ .i.e.,\ \lambda(\Gamma)\le n\alpha(n)\Big(\frac{n-1}{2\pi e}\Big)^{\frac{n-1}{2}} \tag{5.2.33}$$

The next lemma shows that this upper bound can be achieved.

Lemma 5.2.8.

$$\lim_{N\to+\infty} F_{(0,N),\,N}(\Gamma) = n\alpha(n)\Big(\frac{n-1}{2\pi e}\Big)^{\frac{n-1}{2}}. \tag{5.2.34}$$

Proof. By definition, we get

$$\begin{aligned} F_{(0,N),\,N}(\Gamma) &= (4\pi N)^{-\frac{n}{2}}\int_{\mathbf{R}^n} \mathrm{e}^{-\frac{|x|^2+(f(x)-N)^2}{4N}}\sqrt{1+|\nabla f|^2}\,dx \\ &= \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}}\int_0^{\infty} r^{n-1}\mathrm{e}^{-\frac{r^2+(f(r)-N)^2}{4N}}\sqrt{1+(f'(r))^2}\,dr. \end{aligned} \tag{5.2.35}$$

Following the same method and using the same notations as in the proof of Lemma

5.2.7, by (5.2.25) and (5.2.26), we have

$$\begin{aligned} F_{(0,N),N}(\Gamma) &= \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}}\int_{-\sqrt{N}}^{\infty} \mathrm{e}^{-\frac{u^2}{4}}\,\mathrm{e}^{-\frac{h^2}{4N}}\left(h^{n-1}+h^{n-1}h'\right)\sqrt{N}\,dr \\ &\geq \frac{n\alpha(n)}{(4\pi N)^{\frac{n}{2}}}\int_{-\sqrt{N}}^{\infty} \mathrm{e}^{-\frac{u^2}{4}}\,\mathrm{e}^{-\frac{2(n-1)}{4(1-\varepsilon)N}(\sqrt{N}u+N+M)}\Big[2(n-1)(\sqrt{N}u \\ &\quad + N)\Big]^{\frac{n-1}{2}}\sqrt{N}\,du. \end{aligned} \tag{5.2.36}$$

Taking $N \to +\infty$ gives

$$\lim_{N\to+\infty} F_{(0,N),N}(\Gamma) \geq n\alpha(n)\Big(\frac{n-1}{2\pi}\mathrm{e}^{-\frac{1}{1-\varepsilon}}\Big)^{\frac{n-1}{2}}. \tag{5.2.37}$$

Since ε is arbitrary, the claim easily follows from Lemma 5.2.7. □

Finally, Lemma 5.2.8 and Lemma 5.2.7 complete the proof of Theorem 5.2.5.

5.3 L-stability and rigidity results

In this section, we first consider the spectrum of the stability operator L for translators and compute the bottom of the spectrum for grim reaper hyperplanes. Then, by using a cut off argument and a uniqueness lemma, we give a rigidity result for translators in terms of the weighed L^2 norm of the second fundamental form.

5.3.1 The spectrum of L for translators

Recall the stability operator L for translator defined by

$$L = \Delta + \langle \mathbf{E}_{n+1}, \nabla\cdot\rangle + |A|^2. \tag{5.3.1}$$

Definition 5.3.1. *We say that a translator Σ is L-stable, if for any compactly supported function f, we have*

$$\int_\Sigma (-fLf)\, e^{x_{n+1}}\, d\mu \geq 0. \tag{5.3.2}$$

It was proved by Shahriyari [Sha15] that all translating graphs are L-stable. Therefore, the grim reaper, the bowl solitons and hyperplanes are all L-stable translators.

We need the following lemma which shows that the operator $\mathcal{L}$ is self-adjoint in a weighted L^2 space.

Lemma 5.3.2. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a translator satisfying $H = -\langle \mathbf{E}_{n+1}, \mathbf{n}\rangle$, u is a C^2 function with compact support, and v is a C^2 function, then*

$$-\int_\Sigma u(\mathcal{L}v)e^{x_{n+1}} = \int_\Sigma \langle \nabla u, \nabla v\rangle e^{x_{n+1}}. \tag{5.3.3}$$

Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a translator. Note that all translators are noncompact, so there may not be a lowest eigenvalue for the operator L. However, we can still define the bottom of the spectrum μ_1 by

$$\mu_1 = \inf_f \frac{\int_\Sigma \left(|\nabla f|^2 - |A|^2 f^2\right)\mathrm{e}^{x_{n+1}}}{\int_\Sigma f^2 \mathrm{e}^{x_{n+1}}} = \inf_f \frac{-\int_\Sigma f(Lf)\mathrm{e}^{x_{n+1}}}{\int_\Sigma f^2\mathrm{e}^{x_{n+1}}}, \tag{5.3.4}$$

where the infimum is taken over all smooth functions with compact support. It is possible that $\mu_1 = -\infty$ since all translators are noncompact.

By the definition of μ_1 and the minimal surface theory (see [FCS80]), we have the following characterization of L-stability for translators.

Lemma 5.3.3. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a translator, then the following are equivalent:*

(1) Σ is L-stable,

(2) $\mu_1(\Sigma) \geq 0$,

(3) there exists a positive function u satisfying $Lu = 0$.

Next, we show that if the mean curvature H of a translator has at most quadratic weighted L^2 growth, then the bottom of the spectrum μ_1 is nonpostive.

Lemma 5.3.4. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete properly embedded translator. If the mean curvature H satisfies the following weighted L^2 growth*

$$\int_{\Sigma\cap B_R} H^2 e^{x_{n+1}} \leq C_0 R^\alpha, \quad \textit{for any } R > 1, \tag{5.3.5}$$

where C_0 is a positive constant and $0 \leq \alpha < 2$, then we have $\mu_1(\Sigma) \leq 0$.

Proof. Given any fixed $\delta > 0$, if we can construct a compactly supported function u such that

$$-\int_\Sigma u(Lu)\mathrm{e}^{x_{n+1}} < \delta \int_\Sigma u^2 \mathrm{e}^{x_{n+1}}, \tag{5.3.6}$$

then this implies $\mu_1 \leq 0$.

We will use the mean curvature H and the fact that $LH = 0$ to construct our test function. Suppose η is a function with compact support and let $u = \eta H$. Then

$$L(\eta H) = \eta LH + H\mathcal{L}\eta + 2\langle\nabla\eta, \nabla H\rangle = H\mathcal{L}\eta + 2\langle\nabla\eta, \nabla H\rangle. \tag{5.3.7}$$

It follows that

$$-\int_\Sigma \eta H L(\eta H)\mathrm{e}^{x_{n+1}} = -\int_\Sigma \Big[\eta H^2\mathcal{L}\eta + 2\eta H\langle\nabla\eta, \nabla H\rangle\Big]\mathrm{e}^{x_{n+1}}. \tag{5.3.8}$$

Applying Lemma 5.3.2 gives

$$-\int_\Sigma H^2\mathcal{L}(\eta^2)\mathrm{e}^{x_{n+1}} = \int_\Sigma \langle\nabla H^2, \nabla\eta^2\rangle\mathrm{e}^{x_{n+1}}. \tag{5.3.9}$$

Note that $\mathcal{L}\eta^2 = 2\eta\mathcal{L}\eta + 2|\nabla\eta|^2$. Combining this with (5.3.8) and (5.3.9) yields that

$$-\int_\Sigma \eta H L(\eta H)\mathrm{e}^{x_{n+1}} = \int_\Sigma H^2|\nabla\eta|^2\mathrm{e}^{x_{n+1}}. \tag{5.3.10}$$

If we choose η to be one on B_R and cut off linearly to zero on $B_{2R}\backslash B_R$, then (5.3.10) gives

$$-\int_\Sigma (\eta H L(\eta H) + \delta\eta^2 H^2)\mathrm{e}^{x_{n+1}} \leq \frac{1}{R^2}\int_{\Sigma\cap B_{2R}} H^2\mathrm{e}^{x_{n+1}} - \delta\int_{\Sigma\cap B_R} H^2\mathrm{e}^{x_{n+1}}. \tag{5.3.11}$$

By the assumption (5.3.5), the right-hand side of (5.3.11) must be negative for sufficiently large R. Therefore, when R is large, the function $u = \eta H$ satisfies (5.3.6), and this completes the proof. $\square$

Lemma 5.3.3 and Lemma 5.3.4 imply that any complete L-stable translator satisfying (5.3.5) has $\mu_1 = 0$. In particular, we obtain the following theorem.

Theorem 5.3.5. *For all grim reaper hyperplanes $\Gamma \times \mathbf{R}^{n-1}$, we have $\mu_1(\Gamma \times \mathbf{R}^{n-1}) = 0$.*

Remark 5.3.6. *If $n \geq 3$, Theorem 5.3.5 does not follow directly from Lemma 5.3.4, but we can slightly modify the argument in the proof of Lemma 5.3.4 to get the result.*

Using a similar argument as in the proof of [CM12a, Lemma 9.25], we have the following characterization of μ_1 for translators.

Lemma 5.3.7. *If $\Sigma^n \subset \mathbf{R}^{n+1}$ is a complete properly embedded translator and $\mu_1(\Sigma) \neq -\infty$, then there is a positive function u on Σ such that $Lu = -\mu_1 u$.*

5.3.2 Uniqueness lemma and rigidity results

In [IR14], Impera and Rimoldi proved a rigidity theorem that if a L-stable translator satisfies a weighted L^2-condition on the norm of the second fundamental form $|A|$, then it must be a hyperplane. By relaxing the condition on the weighted L^2 growth of $|A|$, we obtain the following rigidity result, improving Theorem A in [IR14].

Theorem 5.3.8. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete L-stable translator satisfying $H = -\langle \mathbf{E}_{n+1}, \mathbf{n} \rangle$. If the norm of the second fundamental form satisfies the following weighted L^2 growth*

$$\int_{\Sigma \cap B_R} |A|^2 e^{x_{n+1}} \leq C_0 R^\alpha, \quad \text{for any } R > 1, \tag{5.3.12}$$

where C_0 is a positive constant and $0 \leq \alpha < 2$, then Σ is one of the following

(1) a hyperplane,

(2) the grim reaper Γ, when $n = 1$,

(3) a grim reaper hyperplane, i.e., $\Gamma \times \mathbf{R}$*, when* $n = 2$ *and* $1 \leq \alpha < 2$.

The proof of Theorem 5.3.8 relies on the following standard uniqueness lemma for general hypersurfaces.

Lemma 5.3.9. *Let* $\Sigma^n \subset \mathbf{R}^{n+1}$ *be a smooth complete hypersurface. If* $g > 0$ *and* h *are two functions on* Σ *which satisfy*

$$\Delta g + \langle \nabla f, \nabla g \rangle + Vg = 0, \tag{5.3.13}$$

and

$$\Delta h + \langle \nabla f, \nabla h \rangle + Vh = 0, \tag{5.3.14}$$

where f *and* V *are smooth functions on* Σ*. If* Σ *is closed or* h *satisfies the weighted* L^2 *growth*

$$\int_{\Sigma \cap B_R} h^2 e^f \leq C_0 R^\alpha, \quad \text{for any } R > 1, \tag{5.3.15}$$

where C_0 *is a positive constant and* $0 \leq \alpha < 2$*, then* $h = Cg$ *for some constant* C.

Proof. For convenience, we let $\mathcal{L} = \Delta + \langle \nabla f, \nabla \cdot \rangle$. Thus,

$$\mathcal{L}g + Vg = 0 \text{ and } \mathcal{L}h + Vh = 0. \tag{5.3.16}$$

Let $w = \frac{h}{g}$. Then by (5.3.13) and (5.3.14), we have

$$\mathcal{L}w = \frac{g\mathcal{L}h - h\mathcal{L}g}{g^2} - 2\langle \nabla w, \frac{\nabla g}{g} \rangle = -2\langle \nabla w, \frac{\nabla g}{g} \rangle. \tag{5.3.17}$$

We define a drifted operator $\mathcal{L}_g$ by

$$\mathcal{L}_g = \frac{\mathrm{e}^{-f}}{g^2}\mathrm{div}(g^2 \mathrm{e}^f \, \nabla \cdot) = \mathcal{L} + 2\langle \nabla \cdot, \frac{\nabla g}{g} \rangle. \tag{5.3.18}$$

Then (5.3.17) gives

$$\mathcal{L}_g w^2 = 2w\mathcal{L}_g w + 2|\nabla w|^2 = 2|\nabla w|^2 \geq 0. \tag{5.3.19}$$

Now, if Σ is closed, then integrating (5.3.19) finishes the proof. If Σ is noncompact, we choose a cut off function ϕ. By (5.3.18) and (5.3.19)

$$\frac{\mathrm{e}^{-f}}{g^2}\mathrm{div}(\phi^2 g^2 \mathrm{e}^f \nabla w^2) = \langle \nabla\phi^2, \nabla w^2\rangle + \phi^2 \mathcal{L}_g w^2 = \langle \nabla\phi^2, \nabla w^2\rangle + 2\phi^2|\nabla w|^2. \quad (5.3.20)$$

Using the Stokes' theorem, (5.3.20) gives

$$0 = \int_\Sigma \left[\langle \nabla\phi^2, \nabla w^2\rangle + \phi^2\mathcal{L}_g w^2\right] g^2 \mathrm{e}^f. \quad (5.3.21)$$

Applying the absorbing inequality, (5.3.21) gives

$$0 \geq \int_\Sigma \Big[-\phi^2|\nabla w|^2 - 4w^2|\nabla\phi|^2 + 2\phi^2|\nabla w|^2\Big] g^2\mathrm{e}^f. \quad (5.3.22)$$

This is equivalent to

$$\int_\Sigma \phi^2|\nabla w|^2 g^2 \mathrm{e}^f \leq 4\int_\Sigma w^2|\nabla\phi|^2 g^2\mathrm{e}^f. \quad (5.3.23)$$

If we choose ϕ to be identically one on B_R and cuts off linearly to zero from ∂B_R to ∂B_{2R}, then $|\nabla\phi| \leq 1/R$. Combining the weighted growth of h, i.e., (5.3.15) and taking $R \to \infty$, we conclude that $|\nabla w| = 0$. This completes the proof.

□

Remark 5.3.10. *We can slightly modify the proof of Lemma 5.3.9 to show that it still holds if we assume $\Delta h + \langle \nabla f, \nabla h\rangle + Vh \geq 0$ and $h \geq 0$, which is a special case of [Rim14, Theorem 8].*

As an application of Lemma 5.3.9 to L-stable translators, we have the following lemma.

Lemma 5.3.11. *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a complete L-stable translator. If the mean curvature H satisfies the weighed L^2 growth*

$$\int_{\Sigma\cap B_R} H^2 e^{x_{n+1}} \leq C_0 R^\alpha, \quad \textit{for any } R > 1, \quad (5.3.24)$$

where C_0 is a positive constant and $0 \leq \alpha < 2$, then $H \equiv 0$ or H does not change sign.

Proof. This follows from Lemma 5.3.3, Lemma 5.3.9 and the fact that $LH = 0$. □

We are now ready to prove Theorem 5.3.8.

Proof of Theorem 5.3.8. First by Lemma 5.3.11, we conclude that $H \equiv 0$ or H does not change sign. If $H \equiv 0$, then Σ is just a hyperplane, which is the first case. Next, we assume H does not change sign and $H > 0$. Then $|A|$ does not vanish.

By Lemma 2.3.2, we have

$$L|A| = \frac{|\nabla A|^2 - |\nabla |A||^2}{|A|} \geq 0. \tag{5.3.25}$$

Since $LH = 0$ and $L|A| \geq 0$, applying Remark 5.3.10 and Lemma 5.3.9 with $g = H$ and $h = |A|$, we conclude that there exists a constant C such that

$$|A| = CH. \tag{5.3.26}$$

It follows that Σ is the grim reaper or a grim reaper hyperplane $\Gamma \times \mathbf{R}^{n-1}$ (see Theorem A in [MSHS15] or [IR14]).

Note that if $n \geq 3$, the norm of the second fundamental form $|A|$ of a grim reaper hyperplane $\Gamma \times \mathbf{R}^{n-1}$ has at least quadratic weighted L^2 growth. Therefore, by our assumption, we conclude that $n \leq 2$. If $n = 1$, then Σ is the grim reaper Γ, which is the second case. If $n = 2$, then Σ is a grim reaper hyperplane $\Gamma \times \mathbf{R}$ and $1 \leq \alpha < 2$, which is the last case.

□

5.4 Curvature estimates

This section provides three types of curvature estimates for translators.

5.4.1 Curvature estimates for translators with small entropy

Theorem 5.4.1. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a smooth complete translator with $\lambda(\Sigma) \leq \alpha < 2$. Then there exists a constant $C = C(\alpha) > 0$ such that $|A|^2 \leq C$.*

In order to get a curvature estimate for translators in $\mathbf{R}^3$ with small entropy, we will use Allard's compactness theorem [All72] for integral rectifiable varifolds with locally bounded first variation. Moreover, we will restrict to "boundary-less varifolds" [Whi09] where the rectifiable varifold is mod two equivalent to an integral current without boundary. Following the notation in [Whi09, Definition 4.1], we say that an integral rectifiable varifold V is cyclic mod 2 (boundary-less) provided $\partial[V] = 0$, where $[V]$ is the rectifiable mod 2 flat chain associated to V. Note that a varifold which consists of unions of an odd number of multiplicity-one half-planes meeting along a common line is not cyclic mod 2. Under the assumptions of small entropy and the "boundary-less" condition, the following regularity result holds which is implicit in section 5 in [CIMW13] (see also section 4.1 in [BW14]). For convenience of the reader, we also include a proof here.

Lemma 5.4.2. *Let $\Sigma^2 \subset \mathbf{R}^3$ be a boundary-less (cyclic mod 2) integral rectifiable varifold with $\lambda(\Sigma) < 2$. If Σ is a weak solution of either the self-shrinker equation or the translator equation, then Σ is smooth.*

Proof. We will analyze tangent cones to prove this lemma. By a standard result from [All72] (see also section 42 in [Sim83]), any integral n-rectifiable varifold has stationary integral rectifiable tangent cones as long as the generalized mean curvature H is locally in L_p for some $p > n$. In our case, both equations guarantee that H is locally bounded. This gives the existence of stationary tangent cones at every point. Moreover, it follows from [Whi09, Theorem 1.1] that those tangent cones must also be boundary-less.

Next, we will show that any such tangent cone V is a multiplicity-one hyperplane. If $y \in \mathrm{Sing}(V) \setminus \{0\}$ ($\mathrm{Sing}(V)$ denotes the singular set of V), then a dimension reduction argument [CIMW13, Lemma 5.8] gives that every tangent cone to V at

y is of the form $V' \times \mathbf{R}_y$, where $\mathbf{R}_y$ is the line in the direction y and V' is a one-dimensional integral stationary cone in $\mathbf{R}^2$. By the lower semi-continuity of entropy, we have $\lambda(V') \leq \lambda(V) < 2$. Since any one-dimensional cone is the union of rays, the small entropy condition implies that V' consists of at most three rays. Moreover, "boundary-less" property rules out three rays. Therefore, the only such configuration that is stationary is when there are two rays to form a multiplicity-one line and this implies that $\mathrm{Sing}(V) \subset \{\mathbf{0}\}$. By [AA76], we conclude that the intersection of V with unit sphere $\mathbf{S}^2$ is a smooth closed geodesic, i.e., a great circle, and this gives that V is a multiplicity-one hyperplane.

Now combining the fact that H is locally bounded, Allard's regularity theorem [All72] (see also Theorem 24.2 in [Sim83]) gives that Σ is a $C^{1,\beta}$ manifold for some $\beta > 0$. Then elliptic theory for the self-shrinker or translator equation gives estimates on higher derivatives, and this implies that Σ is smooth. □

Proof of Theorem 5.4.1. We will argue by contradiction. Suppose there is a sequence $\Sigma_i \subset \mathbf{R}^3$ of smooth translators with $\lambda(\Sigma_i) \leq \alpha$ and points $x_i \in \Sigma_i$ with

$$|A|(x_i) > i. \tag{5.4.1}$$

Then we translate Σ_i to $\tilde{\Sigma}_i$ such that x_i is the origin. Note that at any point, we have density bounds for $\tilde{\Sigma}_i$ coming from the entropy bound. Since each $\tilde{\Sigma}_i$ has bounded mean curvature, Allard's compactness theorem [All72] gives a subsequence of the $\tilde{\Sigma}_i$'s that converges to an integral rectifiable varifold $\tilde{\Sigma}$ which weakly satisfies the translator equation and has

$$\lambda(\tilde{\Sigma}) \leq \alpha < 2. \tag{5.4.2}$$

The small entropy condition implies that the convergence is multiplicity one. As $\tilde{\Sigma}_i$ is smooth complete embedded, the Brakke flow associated to $\tilde{\Sigma}_i$ is cyclic mod 2, i.e., boundary-less, in the sense of [Whi09, Definition 4.1]. Therefore, by [Whi09, Theorem 4.2], the limit $\tilde{\Sigma}$ is also boundary-less. Thus, Lemma 5.4.2 gives that $\tilde{\Sigma}$ is smooth. Finally, Allard's regularity theorem [All72] implies that the convergence is also smooth, which contradicts (5.4.1). ☐

5.4.2 Curvature estimates for L-stable translators

The methods we have developed in Section 3.2 can also be used to to prove uniform curvature estimates for L-stable translators. We will only outline the proofs, since they are very similar to the shrinker cases. Our main theorem is the following:

Theorem 5.4.3. *(Theorem 1.0.5) Given $n \leq 5$ and λ_0, there exists $C = C(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is an L-stable translator satisfying $Vol(\Sigma \cap B_r(x)) \leq \lambda_0 r^n$ for all $x \in \mathbf{R}^{n+1}$ and $r > 0$, then*

$$|A|(x) \leq C \tag{5.4.3}$$

for all $x \in \Sigma$.

Remark 5.4.4. *As in the shrinker case (see Section 3.2.4), it can be shown that L-stable translators in $\mathbf{R}^3$ have quadratic area growth, and hence for $n = 2$ the constant C above need not depend on λ_0. More precisely, there exists a constant $C > 0$ such that the curvature of any L-stable translator in $\mathbf{R}^3$ is bounded by C, i.e., $|A| \leq C$. The translating graph case of this result was proved by Shahriyari [Sha15, Theorem 3.2]. Such uniform curvature estimates can be used to prove topology results for translators; see [Sha15].*

Recall that a translator Σ is L-stable, if for any compactly supported function ϕ, we have

$$\int_\Sigma (-\phi L\phi)\, \mathrm{e}^{x_{n+1}}\, d\mu \geq 0. \tag{5.4.4}$$

Note that (5.4.4) is equivalent to

$$\int_\Sigma |A|^2\phi^2 \mathrm{e}^{x_{n+1}} \leq \int_\Sigma |\nabla\phi|^2 \mathrm{e}^{x_{n+1}}. \tag{5.4.5}$$

Outline of the proof of Theorem 5.4.3. In order to prove Theorem 5.4.3, we also need to establish a Choi-Schoen type estimate for translators, which gives a pointwise estimate for $|A|$ as long as its L^n norm is small. When $n = 2$, such an L^2 bound can be obtained from the inequality (5.4.4). For $3 \leq n \leq 5$, we can again adapt the techniques of Schoen-Simon-Yau to improve the control on $|A|$. The corresponding theorems are as follows:

Theorem 5.4.5. *There exists $\varepsilon = \varepsilon(n) > 0$ so that if $\Sigma^n \subset \mathbf{R}^{n+1}$ is a properly embedded translator in $B_{r_0}(x_0)$ for $r_0 \leq 1$ which satisfies*

$$\int_{B_{r_0}(x_0)\cap\Sigma} |A|^n < \varepsilon, \tag{5.4.6}$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x_0)$,

$$\sigma^2|A|^2(y) \leq 1. \tag{5.4.7}$$

Theorem 5.4.6. *Suppose that $\Sigma^n \subset \mathbf{R}^{n+1}$ is a properly embedded L-stable translator. Then for all $q \in [0, \sqrt{2/(n+1)})$, we have*

$$\int_\Sigma |A|^{4+2q}\phi^2 e^{x_{n+1}} \leq C\Big[\int_\Sigma |A|^{2+2q}|\nabla\phi|^2 e^{x_{n+1}} + \int_\Sigma |A|^{2+2q}\phi^2 e^{x_{n+1}}\Big], \tag{5.4.8}$$

where $C = C(n,q)$ and ϕ is a smooth function with compact support.

The proofs of Theorem 5.4.5 and Theorem 5.4.6 are similar to the shrinker cases. Note that translators also satisfy a Simons-type identity (see Lemma 2.3.2):

$$L|A|^2 = 2|\nabla A|^2 - |A|^4. \tag{5.4.9}$$

This gives a Simons-type inequality

$$\begin{aligned} \Delta|A|^2 &\geq -\frac{1}{2}|A|^2 - 2|\nabla|A||^2 + 2|\nabla A|^2 - 2|A|^4 \\ &\geq -\frac{1}{2}|A|^2 - 2|A|^4. \end{aligned} \tag{5.4.10}$$

Using this inequality and arguing as in the proof of Theorem 3.2.2 give Theorem

5.4.5. For Theorem 5.4.6, combining (5.4.9) with (3.2.17) gives

$$|A|\Big(\Delta|A| + \langle \mathbf{E}_{n+1}, \nabla|A|\rangle\Big) \geq -|A|^4 + \frac{2}{n+1}|\nabla|A||^2 - \frac{2n}{n+1}|A|^2. \tag{5.4.11}$$

Here we use that $|\nabla H| \leq |A|$ for translators. The inequality (5.4.11) is the essential inequality for the proof of Theorem 5.4.6.

Finally, applying suitable cutoff functions as in the proof of Theorem 3.2.1 leads to the proof of Theorem 5.4.3. □

5.4.3 Curvature estimates for graphical translators

We can also use the methods in Section 3.3 to prove a curvature estimate for graphical translators. As an immediate corollary, we obtain the Bernstein type theorem for translators which was proved by Bao and Shi [BS14] by using different methods.

Theorem 5.4.7. *Given n and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded translator $\Sigma^n \subset \mathbf{R}^{n+1}$ which satisfies*

- *$w = \langle V, \mathbf{n}\rangle \geq \delta$ on $B_R(x_0) \cap \Sigma$ for some constant unit vector V and $x_0 \in \mathbf{R}^{n+1}$,*

we have

$$|A|^2(x) \leq C\Big(\frac{1}{R} + \frac{1}{R^2}\Big)w^2(x), \quad \text{for all } x \in B_{R/2}(x_0) \cap \Sigma. \tag{5.4.12}$$

Proof. Since the proof is very similar to Theorem 3.3.3, we will only sketch the argument.

The stability operator L for translators is defined by $L = \Delta + \langle \mathbf{E}_{n+1}, \nabla \cdot\rangle + |A|^2$. We have the following identities (see for instance [IR14])

$$L\langle V, \mathbf{n}\rangle = 0 \quad \text{and} \quad L|A|^2 = 2|\nabla A|^2 - |A|^2. \tag{5.4.13}$$

Set $v = 1/w$, $v_0 = 1/\delta$ and $f = |A|^2 h(v^2)$. Similar computations and estimates as in the proof of Theorem 3.3.3 give that

$$\begin{aligned}\Delta f \geq \frac{\langle \nabla h, \nabla f\rangle}{h} - \langle e_{n+1}, \nabla f\rangle - 2h|A|^4 + 2h'v^2|A|^4 \\ + \Big[4h''v^2 + 6\Big(h' - \frac{(h')^2v^2}{h}\Big)\Big]|A|^2|\nabla v|^2.\end{aligned} \tag{5.4.14}$$

Choosing $h(y) = \frac{y}{1-ky}$, where $k = (2v_0^2)^{-1}$. We then obtain that

$$\Delta f \geq \frac{\langle \nabla h, \nabla f\rangle}{h} - \langle e_{n+1}, \nabla f\rangle + 2kf^2 + \frac{2k|\nabla v|^2}{(1-kv^2)^2}f. \tag{5.4.15}$$

Let $\phi(x) = ((R^2 - |x-x_0|^2)_+)^3$. We set $F(x) = \phi(x)f(x)$ and consider its maximum on $B_R(x_0) \cap \Sigma$. Assume F achieves its maximum at some point $y_0 \in B_R(x_0) \cap \Sigma$.

Using $\nabla F(y_0) = 0$, $\Delta F(y_0) \leq 0$ and some estimates of ϕ, we have

$$F(y_0) = \phi(y_0)f(y_0) \leq C(R^4 + R^5), \tag{5.4.16}$$

where C is a constant depending on n and δ. Since F achieves its maximum at y_0, we have $F(x) \leq F(y_0)$ for all $x \in B_{R/2}(x_0) \cap \Sigma$. This implies for any $x \in B_{R/2}(x_0) \cap \Sigma$

$$\Big(\frac{R}{2}\Big)^6 \frac{|A|^2(x)}{w^2(x) - k} \leq F(x) \leq F(y_0) \leq C(R^4 + R^5). \tag{5.4.17}$$

Now the theorem follows directly. □

By taking R goes to infinity in Theorem 5.4.7, we obtain the Bernstein type theorem for translators in [BS14].

Theorem 5.4.8 ([BS14]). *Let $\Sigma^n \subset \mathbf{R}^{n+1}$ be a smooth complete translator. If there exists a positive constant δ such that*

- *$w = \langle V, \mathbf{n}\rangle \geq \delta$ for some constant unit vector V,*

then Σ must be a hyperplane.

Recently, Kunikawa [Kun15] generalized Theorem 5.4.8 to arbitrary codimension.

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