Schur Weyl Duality in Complex Rank

by

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Abstract

This thesis gives an analogue to the classical Schur-Weyl duality in the setting of Deligne categories. Given a finite-dimensional unital vector space V (i.e. a vector space V with a distinguished non-zero vector $\mathbb{1}$) we give a definition of a complex tensor power of V. This is an Ind-object of the Deligne category $\underline{Rep}(S_t)$ equipped with a natural action of $\mathfrak{gl}(V)$.

This construction allows us to describe a duality between the abelian envelope of the category $\underline{Rep}(S_t)$ and a localization of the category $O_V^{\mathfrak{p}}$ (the parabolic category O for $\mathfrak{gl}(V)$ associated with the pair $(V, \mathbb{1})$).

In particular, we obtain an exact contravariant functor $\widehat{SW}_t$ from the category $\underline{Rep}^{ab}(S_t)$ (the abelian envelope of the category $\underline{Rep}(S_t)$) to a certain quotient of the category $O_V^{\mathfrak{p}}$. This quotient, denoted by $\widehat{O}_{t,V}^{\mathfrak{p}}$, is obtained by taking the full subcategory of $O_V^{\mathfrak{p}}$ consisting of modules of degree t, and localizing by the subcategory of finite dimensional modules.

It turns out that the contravariant functor $\widehat{SW}_t$ makes $\widehat{O}_{t,V}^{\mathfrak{p}}$ a Serre quotient of the category $\underline{Rep}^{ab}(S_t)^{op}$, and the kernel of $\widehat{SW}_t$ can be explicitly described.

In the second part of this thesis, we consider the case when $V = \mathbb{C}^\infty$. We define the appropriate version of the parabolic category O and its localization, and show that the latter is equivalent to a "restricted" inverse limit of categories $\widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}}$ with N tending to infinity. The Schur-Weyl functors $\widehat{SW}_{t,\mathbb{C}^N}$ then give an anti-equivalence between the category $\widehat{O}_{t,\mathbb{C}^\infty}^{\mathfrak{p}_\infty}$ and the category $\underline{Rep}^{ab}(S_t)$.

This duality provides an unexpected tensor structure on the category $\widehat{O}_{t,\mathbb{C}^\infty}^{\mathfrak{p}_\infty}$.

Thesis Supervisor: Pavel Etingof
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Contents

A mathematician is bothered that his field of research is very abstract, and resolves to switch to some practical area of mathematics. He goes to the department bulletin board to find an upcoming lecture about something practical. Luckily, a talk is scheduled that afternoon on "The Theory of Gears". Excited that he has finally found a down-to-earth area of mathematics, he arrives to hear the lecture. Then, the speaker stands up and begins: "While the theory of gears with an integer number of teeth is well understood, a gear with a complex number of teeth..."

A mathematical anecdote

Chapter 1

Introduction

1.1 Overview of the results

The study of representations in complex rank involves defining and studying families of abelian categories depending on a parameter t which are polynomial interpolations of the categories of representations of objects such as finite groups, Lie groups, Lie algebras and more. This was done by P. Deligne in [D2] for finite dimensional representations of the general linear group GL_n, the orthogonal and symplectic groups O_n, Sp_{2n} and the symmetric group S_n. Deligne defined Karoubian tensor categories $\underline{Rep}(GL_t), \underline{Rep}(OSp_t), \underline{Rep}(S_t)$, $t \in \mathbb{C}$, which at points $n = t \in \mathbb{Z}_+$ allow an essentially surjective additive functor onto the standard categories $Rep(GL_n), Rep(OSp_n), Rep(S_n)$. The category $\underline{Rep}(S_t)$ was subsequently studied by himself and others (e.g. by V. Ostrik, J. Comes in [CO], [CO2]).

This thesis gives an analogue to the classical Schur-Weyl duality in the setting of Deligne categories. In order to do this, we define the "complex tensor power" of a finite-dimensional split unital complex vector space (i.e. a vector space V with a distinguished non-zero vector $\mathbb{1}$ and a splitting $V \cong \mathbb{C}\mathbb{1} \oplus U$). This "complex tensor power" of V is an Ind-object in the category $\underline{Rep}(S_t)$, and comes with an action of $\mathfrak{gl}(V)$ on it. Furthermore, it can be shown that this object does not depend on the choice of splitting, but only on the pair $(V, \mathbb{1})$.

The "t-th tensor power" of V is defined for any $t \in \mathbb{C}$; for $n = t \in \mathbb{Z}_+$, the functor $\underline{Rep}(S_{t=n}) \to Rep(S_n)$ takes this Ind-object of $\underline{Rep}(S_{t=n})$ to the usual tensor power $V^{\otimes n}$ in $Rep(S_n)$. Moreover, the action of $\mathfrak{gl}(V)$ on the former object corresponds to the action of $\mathfrak{gl}(V)$ on $V^{\otimes n}$.

This allows us to define an additive contravariant functor, called the Schur-Weyl functor,

$$SW_{t,V} : \underline{Rep}^{ab}(S_t) \to O_V^{\mathfrak{p}}$$

Here $\underline{Rep}^{ab}(S_t)$ is the abelian envelope of the category $\underline{Rep}(S_t)$ (this envelope was described in [D2, Chapter 8], [CO2]) and the category $O_V^{\mathfrak{p}}$ is the parabolic category O for $\mathfrak{gl}(V)$ associated with the pair $(V, \mathbb{1})$.

It turns out that $SW_{t,V}$ induces an anti-equivalence of abelian categories between a Serre quotient of $\underline{Rep}^{ab}(S_t)$ and a localization of $O_V^{\mathfrak{p}}$. The latter quotient is obtained by taking the full subcategory of $O_V^{\mathfrak{p}}$ consisting of "polynomial" modules of degree t (i.e. modules on which $\mathrm{Id}_V \in \mathrm{End}(V)$ acts by the scalar t, and on which the group $GL(V/\mathbb{C}\mathbb{1})$ acts by polynomial maps), and localizing by the Serre subcategory of finite dimensional modules. This quotient is denoted by $\widehat{O}_{t,V}^{\mathfrak{p}}$.

Thus for any unital finite-dimensional space $(V, \mathbb{1})$ and for any $t \in \mathbb{C}$, the category $\widehat{O}_{t,V}^{\mathfrak{p}}$ is a Serre quotient of $\underline{Rep}^{ab}(S_t)^{op}$.

Next, we consider the categories $\widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}_N}$ for $N \in \mathbb{Z}_+$, and the corresponding Schur-Weyl functors. Defining appropriate restriction functors

$$\widehat{Res}_{t,N} : \widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}_N} \to \widehat{O}_{t,\mathbb{C}^{N-1}}^{\mathfrak{p}_{N-1}}$$

we can consider the inverse limit of the system $((\widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}_N})_{N\geq 0}, (\widehat{Res}_{t,N})_{N\geq 1})$ and a contravariant functor

$$\underline{Rep}^{ab}(S_t) \to \varprojlim_{N \in \mathbb{Z}_+} \widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}_N}$$

induced by the Schur-Weyl functors $SW_{t,\mathbb{C}^N}$.

We then define a a full subcategory of $\varprojlim_{N\in\mathbb{Z}_+} \widehat{O}_{t,\mathbb{C}^N}^{\mathfrak{p}_N}$ called "the restricted inverse limit" of the system $((\widehat{O}^{\mathfrak{p}_N}_{t,\mathbb{C}^N})_{N\geq 0}, (\widehat{Res}_{t,N})_{N\geq 1})$. Intuitively, one can describe the "the restricted inverse limit" as follows:

By definition, the objects in $\varprojlim_{N\in\mathbb{Z}_+} \widehat{O}^{\mathfrak{p}_N}_{t,\mathbb{C}^N}$ are sequences $(M_N)_{N\in\mathbb{Z}_+}$ such that $M_N \in \widehat{O}^{\mathfrak{p}_N}_{t,\mathbb{C}^N}$, together with isomorphisms $\widehat{Res}_{t,N}(M_N) \to M_{N-1}$. The objects in the restricted inverse limit are those sequences $(M_N)_{N\in\mathbb{Z}_+}$ for which the integer sequence $\{\ell(M_N)\})_{N\in\mathbb{Z}_+}$ stabilizes ($\ell(M_N)$ is the length of the $\widehat{O}^{\mathfrak{p}_N}_{t,\mathbb{C}^N}$-object M_N).

We then define the complex tensor power of the unital vector space $(\mathbb{C}^\infty, \mathbb{1} := e_1)$, and the corresponding Schur-Weyl contravariant functor $SW_{t,\mathbb{C}^\infty}$. As in the finite-dimensional case, this functor induces an exact contravariant functor $\widehat{SW}_{t,\mathbb{C}^\infty}$, and we have the following commutative diagram:

$$\begin{array}{ccc} \underline{Rep}^{ab}(S_t)^{op} & \xrightarrow{\widehat{SW}_{t,\lim}} & \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n} \\ & \searrow^{\widehat{SW}_{t,\mathbb{C}^\infty}} & \uparrow \\ & & \widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty} \end{array}$$

The contravariant functors $\widehat{SW}_{t,\mathbb{C}^\infty}$, $\widehat{SW}_{t,\lim}$ turn out to be anti-equivalences induced by the Schur-Weyl functors $SW_{t,\mathbb{C}^n}$.

The anti-equivalences $\widehat{SW}_{t,\mathbb{C}^\infty}$, $\widehat{SW}_{t,\lim}$ induce an unexpected structure of a rigid symmetric monoidal category on

$$\widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$$

We obtain an interesting corollary: the duality in this category given by the tensor structure will coincide with the one arising from the usual notion of duality in BGG category O.

The Schur-Weyl functor described above can also be used to extend other classical dualities to complex rank. Namely, one can consider categories which are constructed "on the basis of $\underline{Rep}(S_t)$". A method for constructing such categories was suggested in [Et1], and was used in [E1], [Mat] to study representations of degenerate affine Hecke algebras and of rational Cherednik algebras in complex rank. One can then try to generalize the classical Schur-Weyl dualities for these new categories: for example, one can use the notion of a complex tensor power to construct a Schur-Weyl functor between the category of representations of the degenerate affine Hecke algebra of type A of complex rank, and the category of parabolic-type representations of the Yangian $Y(\mathfrak{gl}_N)$ for $N \in \mathbb{Z}_+$. We plan to study these dualities in detail in the future.

1.2 Summary of results

Recall that the classical Schur-Weyl duality describes the relation between the actions of $\mathfrak{gl}(V)$, S_d on $V^{\otimes d}$ (here V is a finite-dimensional complex vector space, d is a non-negative integer, and S_d is the symmetric group).

In particular, it says that the actions of $\mathfrak{gl}(V)$, S_d on $V^{\otimes d}$ commute with each other, and we have a decomposition of $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-modules

$$V^{\otimes d} \cong \bigoplus_{\substack{\lambda \text{ is a Young diagram} \\ |\lambda|=d}} \lambda \otimes S^\lambda V$$

We would like to extend this duality to the Deligne category $\underline{Rep}(S_t)$, by constructing an object $V^{\underline{\otimes} t}$ in $\underline{Rep}(S_t)$, together with the action of $\mathfrak{gl}(V)$ on it, which is an analogue (a polynomial interpolation) of the module $V^{\otimes d}$ for $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$.

It turns out that this can be done in the following setting:

- The space V is required to be unital, that is, we fix a distinguished non-zero vector $\mathbb{1}$ in V. We then choose any splitting $V \cong \mathbb{C}\mathbb{1} \oplus U$. It can be shown that $V^{\underline{\otimes} t}$ does not depend on the choice of the splitting, but only on the choice of the distinguished vector $\mathbb{1}$.

  For $t \notin \mathbb{Z}_+$, one can actually give a definition without choosing a splitting, as it is done by P. Etingof in [Et1] (see Section 4.5).

- The object $V^{\underline{\otimes} t}$ is not finite-dimensional (unlike $V^{\otimes d}$), but is an Ind-object (a countable direct sum) of objects from $\underline{Rep}(S_t)$.

The intuition for working in the above setting is as follows (proposed by P. Etingof in [Et1]): let $t \in \mathbb{C}$, and let x be a formal variable. The expression x^t is not polynomial in t, and has no algebraic meaning, but if we present x as $x := 1 + y$, we can write:

$$x^t = (1+y)^t = \sum_{k \in \mathbb{Z}_+} \binom{t}{k} y^k$$

The function $\binom{t}{k}$ is polynomial in t, so the expression $\sum_{k \in \mathbb{Z}_+} \binom{t}{k} y^k$ is just a formal power series with polynomial coefficients. This explains why it is convenient to work with unital vector spaces.

Notice that for $t \notin \mathbb{Z}_+$, the above sum is infinite, which also explains why the object $V^{\underline{\otimes} t}$ can be expected to be an *Ind*-object of $\underline{Rep}(S_t)$ ($\underline{Rep}(S_t)$ has only finite direct sums).

To define the object $V^{\underline{\otimes} t}$ for a unital vector space $(V, \mathbb{1})$, we use the following notation:

Notation 1.2.0.1.

- We denote by $\mathfrak{p}_{(V, \mathbb{C}\mathbb{1})} \subset \mathfrak{gl}(V)$ the parabolic Lie subalgebra which consists of all the endomorphisms $\phi : V \to V$ for which $\phi(\mathbb{1}) \in \mathbb{C}\mathbb{1}$. We will write $\mathfrak{p} := \mathfrak{p}_{(V, \mathbb{C}\mathbb{1})}$ for short.

- $\bar{\mathfrak{P}}_{\mathbb{1}}$ denotes the mirabolic subgroup corresponding to $\mathbb{1}$, i.e. the group of automorphisms $\Phi : V \to V$ such that $\Phi(\mathbb{1}) = \mathbb{1}$, and $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}} \subset \mathfrak{p}$ denotes the algebra of endomorphisms $\phi : V \to V$ for which $\phi(\mathbb{1}) = 0$ (thus $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}} = Lie(\bar{\mathfrak{P}}_{\mathbb{1}})$).

- $\mathfrak{U}_{\mathbb{1}}$ denotes the subgroup of $\bar{\mathfrak{P}}_{\mathbb{1}}$ of automorphisms $\Phi : V \to V$ for which $Im(\Phi - \mathrm{Id}_V) \subset \mathbb{C}\mathbb{1}$, and $\mathfrak{u}_{\mathfrak{p}}^+ \subset \bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$ denotes the algebra of endomorphisms $\phi : V \to V$ for which $\mathrm{Im}\,\phi \subset \mathbb{C}\mathbb{1} \subset \mathrm{Ker}\,\phi$ (thus $\mathfrak{u}_{\mathfrak{p}}^+ = Lie(\mathfrak{U}_{\mathbb{1}})$).

Fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$.

Recall that we have a splitting $\mathfrak{gl}(V) \cong \mathfrak{p} \oplus \mathfrak{u}_{\mathfrak{p}}^-$, where $\mathfrak{u}_{\mathfrak{p}}^- \cong U$. This gives us an analogue of triangular decomposition:

$$\mathfrak{gl}(V) \cong \mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{u}_{\mathfrak{p}}^- \oplus \mathfrak{u}_{\mathfrak{p}}^+ \oplus \mathfrak{gl}(U)$$

with $\mathfrak{u}_\mathfrak{p}^+ \cong U^*$.

The definition of $V^{\underline{\otimes} t}$ is essentially an analogue of the isomorphism of $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-modules

$$V^{\otimes d} \cong \bigoplus_{k=0,...,d} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,d\}))^{S_k}$$

Here the action of $\mathfrak{gl}(V)$ on the right hand side (viewed as a $\mathbb{Z}_+$-graded space) is given as follows: Id_V acts by the scalar t, $\mathfrak{gl}(U)$ acts on each summand though its action on the spaces $U^{\otimes k}$, and $\mathfrak{u}_\mathfrak{p}^-, \mathfrak{u}_\mathfrak{p}^+$ act by operators of degrees $1, -1$ respectively.

The group S_d acts on each summand through its action on the set $Inj(\{1,...,k\},\{1,...,d\})$ of injective maps from $\{1,...,k\}$ to $\{1,...,d\}$.

In the Deligne category $\underline{Rep}(S_t)$, we have objects Δ_k which are analogues of the $S_k \times S_d$ representation $\mathbb{C}Inj(\{1,...,k\},\{1,...,d\})$. The objects Δ_k carry an action of S_k, therefore we can define a $\mathbb{Z}_+$-graded Ind-object of $\underline{Rep}(S_t)$:

$$V^{\underline{\otimes} t} := \bigoplus_{k\geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k}$$

Next, one can define the action of $\mathfrak{gl}(V)$ on $V^{\underline{\otimes} t}$ so that Id_V acts by scalar t, $\mathfrak{gl}(U)$ acts naturally on each summand $(U^{\otimes k} \otimes \Delta_k)^{S_k}$, and $\mathfrak{u}_\mathfrak{p}^-, \mathfrak{u}_\mathfrak{p}^+$ act by operators of degrees $1, -1$ respectively.

In fact, it can be shown that the object $V^{\underline{\otimes} t}$ does not depend on the choice of the splitting, but only on the choice of the distinguished vector $\mathbb{1}$.

We also show that for $t = n \in \mathbb{Z}_+$, the functor $\underline{Rep}(S_{t=n}) \to Rep(S_n)$ takes $V^{\underline{\otimes} t=n}$ to the usual tensor power $V^{\otimes n}$ in $Rep(S_n)$, and the action of $\mathfrak{gl}(V)$ on $V^{\underline{\otimes} t=n}$ corresponds to the action of $\mathfrak{gl}(V)$ on $V^{\otimes n}$.

Remark 1.2.0.2. The Hilbert series of $V^{\underline{\otimes} t}$ corresponding to the grading $gr_0(V) := \mathbb{C}\mathbb{1}$, $gr_1(V) := U$ would be $(1+y)^t$.

Remark 1.2.0.3. Given any symmetric monoidal category $\mathcal{C}$ with unit object $\mathbf{1}$ and a fixed object $X \in \mathcal{C}$, one can similarly define the object $(\mathbf{1} \oplus X)^{\underline{\otimes} t}$ of $Ind - (\underline{Rep}(S_t) \boxtimes \mathcal{C})$.

We now proceed to the second part of the Schur-Weyl duality. Recall that in the classical Schur-Weyl duality for $\mathfrak{gl}(V)$, S_d, the module $V^{\otimes d}$ over $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$ defines a contravariant functor

$$\mathtt{SW}_{d,V} : Rep(S_d) \longrightarrow Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$$
$$\mathtt{SW}_{d,V} := \mathrm{Hom}_{S_d}(\cdot, V^{\otimes d})$$

Here

- The category $Rep(S_d)$ is the semisimple abelian category of finite-dimensional representations of S_d.
- The category $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$ is the semisimple abelian category of polynomial representations of $\mathfrak{gl}(V)$ ("polynomial" meaning that these are direct summands of finite direct sums of tensor powers of V; alternatively, one can define these as finite-dimensional representations $GL(V) \to \mathrm{Aut}(W)$ which can be extended to an algebraic map $\mathrm{End}(V) \to \mathrm{End}(W)$).

This functor takes the simple representation of S_d corresponding to the Young diagram λ either to zero, or to the simple representation $S^{\lambda}V$ of $\mathfrak{gl}(V)$. Notice that the image of functor $\mathtt{SW}_{d,V}$ lies in the full additive subcategory $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly,d}$ of $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$ whose objects are $\mathfrak{gl}(V)$-modules on which Id_V acts by the scalar d.

It is then easy to see that the contravariant functor $\mathtt{SW}_{d,V} : Rep(S_d) \to Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly,d}$ is full and essentially surjective.

Considering these dualities for a fixed finite-dimensional vector space V and every $d \in \mathbb{Z}_+$, we can construct a full, essentially surjective, additive contravariant functor

$$\mathtt{SW}_V : \bigoplus_{d \in \mathbb{Z}_+} Rep(S_d) \to Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly} \cong \bigoplus_{d \in \mathbb{Z}_+} Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly,d}$$

between semisimple abelian categories.

The simple objects in $\bigoplus_{d\in\mathbb{Z}_+} Rep(S_d)$ which $\mathtt{SW}_V$ sends to zero are (up to isomorphism) exactly those parametrized by Young diagrams λ such that λ has more than $\dim V$ rows.

Thus the contravariant functor $\mathtt{SW}_V$ induces an anti-equivalence of abelian categories between a Serre quotient of the semisimple abelian category $\bigoplus_{d\in\mathbb{Z}_+} Rep(S_d)$, and the semisimple abelian category $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$.

In our case, we would like to consider the Deligne category $\underline{Rep}(S_t)$ and a category of representations of $\mathfrak{gl}(V)$ related to the unital structure of V.

Unfortunately, the Deligne category $\underline{Rep}(S_t)$ is Karoubian but not necessarily abelian, which would make it difficult to obtain an anti-equivalence of abelian categories. However, it turns out that the Karoubian tensor category $\underline{Rep}(S_t)$ is abelian semisimple whenever $t \notin \mathbb{Z}_+$. For $t = d \in \mathbb{Z}_+$, this is not the case, but then $\underline{Rep}(S_{t=d})$ can be embedded (as a Karoubian tensor category) into a larger abelian tensor category, denoted by $\underline{Rep}^{ab}(S_{t=d})$. The construction of this abelian envelope is discussed in detail in [D2, Section 8] and in [CO2]. We will denote by $\underline{Rep}^{ab}(S_t)$ the abelian envelope of $\underline{Rep}(S_t)$ for any $t \in \mathbb{C}$, with $\underline{Rep}^{ab}(S_t)$ being just $\underline{Rep}(S_t)$ whenever $t \notin \mathbb{Z}_+$.

The structure of $\underline{Rep}^{ab}(S_t)$ as an abelian category is known, and described in [CO2] and in Section 3.2.4. In particular, it is a highest weight category (with infinitely many weights), with simple objects parametrized by all Young diagrams.

It turns out that the correct categories to consider for the Schur-Weyl duality in complex rank are the abelian category $\underline{Rep}^{ab}(S_t)$ and the parabolic category O for $\mathfrak{gl}(V)$ corresponding to the pair $(V, \mathbb{1})$.

Consider the short exact sequence of groups

$$1 \to \mathfrak{U}_{\mathbb{1}} \longrightarrow \bar{\mathfrak{P}}_{\mathbb{1}} \longrightarrow GL\left({}^{V}\big/_{\mathbb{C}\mathbb{1}}\right) \to 1$$

For any irreducible finite-dimensional algebraic representation $\rho : \bar{\mathfrak{P}}_{\mathbb{1}} \to \mathrm{Aut}(E)$ of the mirabolic subgroup, $\mathfrak{U}_{\mathbb{1}}$ acts trivially on E, and thus ρ factors through $GL\,(V/\mathbb{C}\mathbb{1})$.

This allows us to say that ρ is a $GL\,(V/\mathbb{C}\mathbb{1})$*-polynomial representation of* $\bar{\mathfrak{P}}_{\mathbb{1}}$ if

$\rho : GL\,(V/\mathbb{C}\mathbb{1}) \to \mathrm{Aut}(E)$ is a polynomial representation (i.e. if ρ extends to an algebraic map $\mathrm{End}\,(V/\mathbb{C}\mathbb{1}) \to \mathrm{End}(E)$).

Now, for any finite-dimensional algebraic representation E of $\bar{\mathfrak{P}}_{\mathbb{1}}$, we say that E is $GL\,(V/\mathbb{C}\mathbb{1})$*-polynomial* if the Jordan-Holder components of E are $GL\,(V/\mathbb{C}\mathbb{1})$-polynomial representations of $\bar{\mathfrak{P}}_{\mathbb{1}}$.

This allows us to give the following definition:

Definition 1.2.0.4. The category $O^{\mathfrak{p}}_{t,V}$ is defined to be the full subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ whose objects M satisfy the following conditions:

- M is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \bar{\mathfrak{P}}_{\mathbb{1}})$, i.e. the action of the Lie subalgebra $\mathfrak{p}_{\mathbb{C}\mathbb{1}}$ on M integrates to the action of the group $\bar{\mathfrak{P}}_{\mathbb{1}}$.

  Furthermore, we require that as a representation of $\bar{\mathfrak{P}}_{\mathbb{1}}$, M be a filtered colimit of $GL\,(V/\mathbb{C}\mathbb{1})$-polynomial representations, i.e.

  $$M|_{\bar{\mathfrak{P}}_{\mathbb{1}}} \in Ind - Rep(\bar{\mathfrak{P}}_{\mathbb{1}})_{GL(V/\mathbb{C}\mathbb{1})-poly}$$

- M is a finitely generated $\mathcal{U}(\mathfrak{gl}(V))$-module.

- $\mathrm{Id}_V \in \mathfrak{gl}(V)$ acts by $t\,\mathrm{Id}_M$ on M.

Remark 1.2.0.5. For any fixed splitting $V = \mathbb{C}\mathbb{1} \oplus U$, the first requirement can be replaced by the requirement that $M|_{\mathfrak{gl}(U)}$ be a direct sum of polynomial simple $\mathcal{U}(\mathfrak{gl}(U))$-modules, and that $\mathfrak{u}_{\mathfrak{p}}^{+}$ act locally finitely on M.

The category $O_{t,V}^{\mathfrak{p}}$ is an Artinian abelian category, and is a Serre subcategory of the usual category O for $\mathfrak{gl}(V)$.

The $\mathfrak{gl}(V)$-action on the object $V^{\underline{\otimes} t}$ is a "$O_t^{\mathfrak{p}}$-type" action, which allows us to define a contravariant functor from $\underline{Rep}^{ab}(S_t)$ to $O_{t,V}^{\mathfrak{p}}$:

$$SW_{t,V} := \mathrm{Hom}_{\underline{Rep}^{ab}(S_t)}(\cdot, V^{\underline{\otimes} t})$$

This contravariant functor is linear and additive, yet only left exact. To fix this problem, we compose this functor with the quotient functor $\hat{\pi}$ from $O_{t,V}^{\mathfrak{p}}$ to the category $\widehat{O}_{t,V}^{\mathfrak{p}}$: the localization of $O_{t,V}^{\mathfrak{p}}$ by the Serre subcategory of finite-dimensional modules. We denote the newly obtained functor by $\widehat{SW}_{t,V}$.

One of the main results of this thesis is the following theorem (c.f. Theorem 5.0.0.42):

Theorem 1. *The contravariant functor $\widehat{SW}_{t,V} : \underline{Rep}^{ab}(S_t) \to \widehat{O}_{t,V}^{\mathfrak{p}}$ is exact and essentially surjective.*

Moreover, the induced contravariant functor

$$\underline{Rep}^{ab}(S_t) \Big/ Ker(\widehat{SW}_{t,V}) \to \widehat{O}_{t,V}^{\mathfrak{p}}$$

is an anti-equivalence of abelian categories, thus making $\widehat{O}_{t,V}^{\mathfrak{p}}$ a Serre quotient of $\underline{Rep}^{ab}(S_t)^{op}$.

In the course of the proof of Theorem 1 we obtain a rather explicit description of the Serre subcategory $Ker(\widehat{SW}_{t,V})$ of $\underline{Rep}(S_t)$. This description shows that as n grows large, the kernel becomes "smaller", thus allowing one to conjecture that in the "limit case" when n tends to infinity, we would be able to obtain an anti-equivalence of categories. In Chapter 7, we will show that in the correct "limit" setting, this is indeed the case. The precise statement of this result is given below.

Consider the infinite-dimensional vector space $\mathbb{C}^\infty$ with a countable basis $e_1, e_2, e_3, ...$ and $\mathbb{1} := e_1$ as the chosen vector. As before, we can construct an *Ind*-object of $\underline{Rep}(S_t)$ which is the complex tensor power $(\mathbb{C}^\infty)^{\underline{\otimes} t}$, and define an action of the Lie algebra $\mathfrak{gl}_\infty$ on it.

We define an appropriate analogue of the category O of Harish-Chandra modules for the pair $(\mathfrak{gl}_\infty, \bar{\mathfrak{P}}_{\mathbb{1}})$; this category will be denoted by $O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$.

We then define the contravariant Schur-Weyl functor

$$SW_{t,\mathbb{C}^\infty} : \underline{Rep}^{ab}(S_t) \longrightarrow O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty},\ SW_{t,\mathbb{C}^\infty} := \mathrm{Hom}_{\underline{Rep}^{ab}(S_t)}(\cdot, (\mathbb{C}^\infty)^{\underline{\otimes} t})$$

Taking the localization of $O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$ by the Serre subcategory of the polynomial modules, we obtain the category $\widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$, and a contravariant functor

$$\widehat{SW}_{t,\mathbb{C}^\infty} : \underline{Rep}^{ab}(S_t) \longrightarrow \widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$$

We claim that this functor is an anti-equivalence of categories.

In order to do so, it is convenient to use the results of Theorem 1: namely, to deduce the anti-equivalence in the infinite-dimensional case from the results obtained in the finite-dimensional case by expressing $O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$ as an inverse limit (in some sense) of the categories $O^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$.

To understand in which sense the category $O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$ is an inverse limit of the categories $O^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$, we recall the classical Schur-Weyl duality once again.

Given a sequence of categories $\{\mathcal{C}_i\}_{i\in\mathbb{Z}_+}$ and functors $\mathcal{F}_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for every $i \geq 1$, we consider (following [S], [WW]) the inverse limit category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ to be the category whose objects are pairs $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ where $C_i \in \mathcal{C}_i$ for each $i \in \mathbb{Z}_+$ and $\phi_{i-1,i} : \mathcal{F}_{i-1,i}(C_i) \xrightarrow{\sim} C_{i-1}$ for any $i \geq 1$.

A morphism in $\varprojlim_{i\in\mathbb{Z}_+}\mathcal{C}_i$ between objects $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ and $(\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$ is a set of arrows $\{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$ satisfying compatability conditions.

In the setting of the classical Schur-Weyl duality, we can consider the restriction functors

$$\mathfrak{Res}_{n-1,n} : \mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}} \to \mathit{Rep}(\mathfrak{gl}_{n-1})_{\mathit{poly}}$$

defined by $\mathfrak{Res}_{n-1,n} := (\cdot)^{E_{n,n}}$ (for each representation $\rho_W : \mathfrak{gl}_n \to \mathrm{End}(W)$, we take the space $Ker(\rho(E_{n,n}))$). Notice that $S^\lambda\mathbb{C}^n \mapsto S^\lambda\mathbb{C}^{n-1}$ for any λ.

This allows us to consider the inverse limit category $\varprojlim_{n\in\mathbb{Z}_+}\mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}}$.

In Chapter 6, we show that $\mathit{Rep}(\mathfrak{gl}_\infty)_{\mathit{poly}}$ is a full subcategory of $\varprojlim_{n\in\mathbb{Z}_+}\mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}}$, and give an intrinsic description of this subcategory, which we will now describe in brief.

Let $m \geq 1$. We will consider the Lie subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_\infty$ which consists of matrices $A = (a_{ij})_{1\leq i,j}$ for which $a_{ij} = 0$ whenever $i > m$ or $j > m$. We will also denote by $\mathfrak{gl}_m^\perp$ the Lie subalgebra of $\mathfrak{gl}_\infty$ consisting of matrices $A = (a_{ij})_{1\leq i,j}$ for which $a_{ij} = 0$ whenever $i \leq m$ or $j \leq m$.

We can then define the specialization functors

$$\Gamma_n : \mathit{Rep}(\mathfrak{gl}_\infty)_{\mathit{poly}} \longrightarrow \mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}},\ \Gamma_n := (\cdot)^{\mathfrak{gl}_n^\perp}$$

This gives a functor

$$\Gamma_{\lim} : \mathit{Rep}(\mathfrak{gl}_\infty)_{\mathit{poly}} \longrightarrow \varprojlim_{n\in\mathbb{Z}_+}\mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}}$$

One can easily see that this functor is fully faithful, and is an equivalence between the category $\mathit{Rep}(\mathfrak{gl}_\infty)_{\mathit{poly}}$ and a subcategory of $\varprojlim_{n\in\mathbb{Z}_+}\mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}}$ called the *restricted* inverse limit of the categories $\mathit{Rep}(\mathfrak{gl}_n)_{\mathit{poly}}$.

The restricted inverse limit of categories $\{\mathcal{C}_i\}_{i \in \mathbb{Z}_+}$ and functors $\mathcal{F}_i : \mathcal{C}_i \to \mathcal{C}_{i-1}$ is defined in the following setting: the categories $\mathcal{C}_i$ are required to be finite-length abelian categories, and the functors $\mathcal{F}_i$ are required to be exact. Furthermore, assume that the functors $\mathcal{F}_i$ take simple objects to either simple objects or zero (that is, the functors $\mathcal{F}_i$ do not increase the lengths of objects). Such functors are called "shortening".

Denote by $\ell_{\mathcal{C}_i}(X)$ the length of the object $X \in \mathcal{C}_i$.

The category $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ is then an abelian category as well, and we can consider its full subcategory $\varprojlim_{i \in \mathbb{Z}_+, \text{ restr}} \mathcal{C}_i$ consisting of all objects $(\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$ such that the weakly-increasing integer sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i \in \mathbb{Z}_+}$ is bounded (and thus stabilizing). This subcategory is obviously a Serre subcategory, and a finite-length abelian category.

This category is universal in the following sense: given a finite-length abelian category $\mathcal{A}$ and exact shortening functors $\mathcal{A} \to \mathcal{C}_i$ for each i, there is a functor $\mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+, \text{ restr}} \mathcal{C}_i$.

Remark 1.2.0.6. It is worth mentioning that sometimes (as it happens in our examples), there is another description of the restricted inverse limit, which is occasionally more convenient to work with. Assume that for each i, the category $\mathcal{C}_i$ "has an object-wise filtration"; namely, that it is a direct limit of a sequence of Serre subcategories $(Fil_k(\mathcal{C}_i))_{k \in \mathbb{Z}_+}$. Furthermore, assume that the functors $\mathcal{F}_{i-1,i}$ induce functors $\mathcal{F}^k_{i-1,i} : Fil_k(\mathcal{C}_{i-1}) \to Fil_k(\mathcal{C}_i)$ for any $k \in \mathbb{Z}_+$. One can then define the category

$$\varinjlim_{k \in \mathbb{Z}_+} \varprojlim_{i \in \mathbb{Z}_+} Fil_k(\mathcal{C}_i)$$

which we call the inverse limit of categories with filtrations. Under some reasonable conditions, this category coincides with the restricted inverse limit $\varprojlim_{i \in \mathbb{Z}_+, \text{ restr}} \mathcal{C}_i$.

This approach is described in detail in Chapter 6. It is used to prove Theorem 2 below.

Returning to our motivating example, the system $((Rep(\mathfrak{gl}_n)_{poly})_{n \geq 0}, (\mathfrak{Res}_{n-1,n})_{n \geq 1})$ satisfies the requirements given above, and it can be shown that the functor $\Gamma_{\lim}$ factors through the restricted inverse limit $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$, giving an equivalence

$$\Gamma_{\lim} : Rep(\mathfrak{gl}_\infty)_{poly} \xrightarrow{\sim} \varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$$

The contravariant functors

$$\mathtt{SW}_{\mathbb{C}^n} : \bigoplus_{d\in\mathbb{Z}_+} Rep(S_d) \longrightarrow Rep(\mathfrak{gl}_n)_{poly}, \quad \mathtt{SW}_{\mathbb{C}^n} := \oplus_d \mathtt{SW}_{d,\mathbb{C}^n}$$

also factor through the category $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$, and we obtain the following commutative diagram:

$$\begin{array}{ccc}
 & & Rep(\mathfrak{gl}_n)_{poly} \\
 & \nearrow^{\mathtt{SW}_{\mathbb{C}^n}} & \uparrow \mathbf{Pr}_n \\
\bigoplus_{d\in\mathbb{Z}_+} Rep(S_d)^{op} & \xrightarrow[\mathtt{SW}_{\lim}]{} & \varprojlim_{n\geq 1,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly} \\
 & \searrow_{\mathtt{SW}_{\mathbb{C}^\infty}} & \uparrow \Gamma_{\lim} \\
 & & Rep(\mathfrak{gl}_\infty)_{poly}
\end{array}$$

with the contravariant functors $\mathtt{SW}_{\lim}$, $\mathtt{SW}_{\mathbb{C}^\infty}$ being anti-equivalences.

Inspired by the classical situation described above, we define restriction functors

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n} \longrightarrow \widehat{O}^{\mathfrak{p}_{n-1}}_{t,\mathbb{C}^{n-1}}$$

for each $n \geq 1$. These functors come from exact functors

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{t,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{t,\mathbb{C}^{n-1}} \qquad \mathfrak{Res}_{n-1,n} := (\cdot)^{E_{n,n}}$$

The functors $\mathfrak{Res}_{n-1,n}$ take polynomial $\mathfrak{gl}_n$-modules to polynomial $\mathfrak{gl}_{n-1}$-modules, and therefore induce exact functors

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n} \longrightarrow \widehat{O}^{\mathfrak{p}_{n-1}}_{t,\mathbb{C}^{n-1}}$$

The Schur-Weyl contravariant functors $\widehat{SW}_{t,\mathbb{C}^n}$ turn out to be compatible with the functors $\widehat{\mathfrak{Res}}_{n-1,n}$. That is, for any $n \in \mathbb{Z}_{>0}$, there exists a natural isomorphism

$$\hat{\eta}_n : \widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{SW}_{t,\mathbb{C}^n} \longrightarrow \widehat{SW}_{t,\mathbb{C}^{n-1}}$$

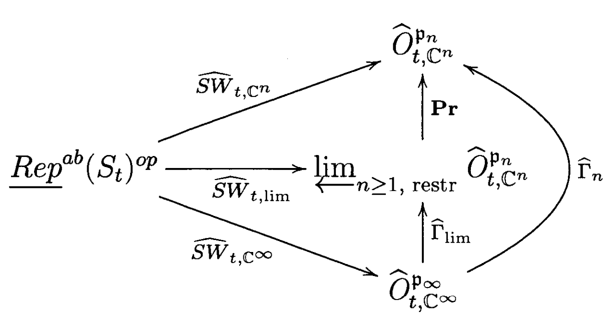
From Theorem 1, we obtain the following result:

Theorem 2. *The Schur-Weyl contravariant functors $\widehat{SW}_{t,\mathbb{C}^n}$ induce an anti-equivalence of abelian categories, given by the contravariant functor*

$$\begin{aligned}\widehat{SW}_{t,lim} : \underline{Rep}^{ab}(S_t) \longrightarrow \varprojlim_{n\in\mathbb{Z}_+,\ restr} \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n}\\ X \mapsto \left(\{\widehat{SW}_{t,\mathbb{C}^n}(X)\}_{n\in\mathbb{Z}_+}, \{\hat{\eta}_n(X)\}_{n\geq 1}\right)\\ (f: X\to Y) \mapsto \{\widehat{SW}_{t,\mathbb{C}^n}(f) : \widehat{SW}_{t,\mathbb{C}^n}(Y) \to \widehat{SW}_{t,\mathbb{C}^n}(X)\}_{n\in\mathbb{Z}_+}\end{aligned}$$

Furthermore, we prove that the category $\widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$ is equivalent the restricted inverse limit of the system categories $\left(\widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n}, \widehat{\mathfrak{Res}}_{n-1,n}\right)$ when n tends to infinity. The projection functors $\widehat{\Gamma}_n : \widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty} \to \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$ are isomorphic to the the functors induced by the invariants functors $\Gamma_n = (\cdot)^{\mathfrak{gl}_n^\perp} : O^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$.

We obtain the following commutative diagram:



Corollary 3. *The contravariant functor* $\widehat{SW}_{t,\mathbb{C}^\infty}$ *is an anti-equivalence of abelian categories.*

This anti-equivalence allows us to obtain an unexpected tensor structure on the category

$$\widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{t,\mathbb{C}^n}$$

Namely, the equivalence from Theorem 2 implies that this is a rigid symmetric monoidal category.

Finally, we show that the duality in $\underline{Rep}^{ab}(S_t)$ (given by the tensor structure) corresponds to the duality in the category $\widehat{O}^{\mathfrak{p}}_{t,V}$, i.e. that there is an isomorphism of (covariant) functors

$$\widehat{SW}_{t,V}((\cdot)^*) \longrightarrow \hat{\pi}(SW_{t,V}(\cdot)^\vee)$$

This gives a new interpretation to the notion of duality in the category $\widehat{O}^{\mathfrak{p}}_{t,V}$.

In particular, it turns out that the rigidity (duality) coming from the newly obtained tensor structure on the $\widehat{O}^{\mathfrak{p}_\infty}_{t,\mathbb{C}^\infty}$ corresponds to the a priori unrelated notion of duality in the BGG category O.

Chapter 2

Notation and definitions

The base field will be $\mathbb{C}$.

2.0.1 Finite-length categories

Let $\mathcal{C}$ be an abelian category, and C be an object of $\mathcal{C}$. A *Jordan-Holder filtration* for C is a finite sequence of subobjects of C

$$0 = C_0 \subset C_1 \subset ... \subset C_n = C$$

such that each subquotient C_{i+1}/C_i is simple.

The Jordan-Holder filtration might not be unique, but the simple factors C_{i+1}/C_i are unique (up to re-ordering and isomorphisms). Consider the multiset of the simple factors: each simple factor is considered as an isomorphism class of simple objects, and its multiplicity is the multiplicity of its isomorphism class in the Jordan-Holder filtration of C. This multiset is denoted by $JH(C)$, and its elements are called the *Jordan-Holder components* of C.

The *length* of the object C, denoted by $\ell_{\mathcal{C}}(C)$, is defined to be the size of the finite multiset $JH(C)$.

Definition 2.0.1.1. An abelian category $\mathcal{C}$ is called a *finite-length category* if every object admits a Jordan-Holder filtration.

2.0.2 Tensor categories

The following standard notation will be used thoughout the thesis:

Notation 2.0.2.1. Let $\mathcal{C}$ be a rigid symmetric monoidal category. We denote by $\mathbf{1}$ the unit object. Also, for any object M, we denote by M^* the dual of M.

2.0.3 Karoubian categories

Definition 2.0.3.1 (Karoubian category)**.** We will call a category $\mathcal{A}$ Karoubian[1] if it is an additive category, and every idempotent morphism is a projection onto a direct factor.

Definition 2.0.3.2 (Block of a Karoubian category)**.** A block in an Karoubian category is a full subcategory generated by an equivalence class of indecomposable objects, defined by the minimal equivalence relation such that any two indecomposable objects with a non-zero morphism between them are equivalent.

2.0.4 Serre subcategories and quotients

Definition 2.0.4.1 (Serre subcategory)**.** A (nonempty) full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called a *Serre subcategory* if for any exact sequence

$$0 \to M' \to M \to M'' \to 0$$

M is in $\mathcal{C}$ iff M' and M'' are in $\mathcal{C}$.

Definition 2.0.4.2 (Serre quotient)**.** Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ be a Serre subcategory.

[1]Deligne calls such categories "pseudo-abelian" (c.f. [D2, 1.9]).

We define the category $\mathcal{A}/\mathcal{C}$, called *the Serre quotient* of $\mathcal{A}$ by $\mathcal{C}$, whose objects are the objects of $\mathcal{A}$ and where the morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(X,Y) := \varinjlim_{\substack{X'\subset X, Y'\subset Y \\ X/X', Y'\in\mathcal{C}}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$$

The category $\mathcal{A}/\mathcal{C}$ comes with a quotient functor, $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$, which takes $X \in \mathcal{A}$ to $X \in \mathcal{A}/\mathcal{C}$, and $f : X \to Y$ in $\mathcal{A}$ to its image in $\varinjlim_{\substack{X'\subset X, Y'\subset Y \\ X/X', Y'\in\mathcal{C}}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y')$.

Remark 2.0.4.3. It is easy to see that the category $\mathcal{A}/\mathcal{C}$ is abelian, and the functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact.

Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ an exact functor. Then we can consider the full subcategory $Ker(\mathcal{F})$ of $\mathcal{A}$ whose objects are $X \in \mathcal{A}$ for which $\mathcal{F}(X) = 0$.

Then $Ker(\mathcal{F})$ is a Serre subcategory, and the functor $\mathcal{F}$ factors through the functor $\pi : \mathcal{A} \to \mathcal{A}/Ker(\mathcal{F})$: we have a functor

$$\bar{\mathcal{F}} : \mathcal{A}/Ker(\mathcal{F}) \to \mathcal{B} \text{ such that } \mathcal{F} = \bar{\mathcal{F}} \circ \pi$$

One can easily check that the functor $\bar{\mathcal{F}} : \mathcal{A}/Ker(\mathcal{F}) \to \mathcal{B}$ is exact and faithful.

Remark 2.0.4.4. Let $\mathcal{A}$ be an abelian category, and $\mathcal{C}$ be a Serre subcategory. Consider the quotient functor, $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{C}$. Then $Ker(\pi) = \mathcal{C}$, and any exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ which takes all the objects of $\mathcal{C}$ to zero factors through π.

2.0.5 Ind-completion of categories

Let $\mathcal{A}$ be a small category.

Definition 2.0.5.1 (Ind-completion). The Ind-completion of $\mathcal{A}$, denoted by $Ind - \mathcal{A}$, is the full subcategory of the category $Fun(\mathcal{A}^{op}, \mathbf{Set})$, whose objects are functors which are filtered colimits of representable functors $\mathcal{A}^{op} \to \mathbf{Set}$.

Remark 2.0.5.2. The Yoneda lemma gives us a fully faithful functor $\jmath : \mathcal{A} \to Fun(\mathcal{A}^{op}, \mathbf{Set})$ which restricts to a fully faithful functor $\iota : \mathcal{A} \to Ind - \mathcal{A}$.

An easy consequence of the definition is the following Lemma:

Lemma 2.0.5.3. *The objects of $\iota(\mathcal{A})$ are compact objects in $Ind - \mathcal{A}$.*

Corollary 2.0.5.4. *Given an object $A \in \mathcal{A}$ and a collection of objects $\{A_i\}_{i \in I}, A_i \in \mathcal{A}$ (here I is a discrete set), we have:*

$$\mathrm{Hom}_{Ind-\mathcal{A}}(A, \bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \mathrm{Hom}_{\mathcal{A}}(A, A_i)$$

We will also use the following property of the Ind-completion (c.f. [KS, Theorem 8.6.5, p.194]):

Theorem 2.0.5.5. *Assume the category $\mathcal{A}$ is abelian. Then the category $Ind - \mathcal{A}$ is abelian as well, and the functor ι is exact. Furthermore, the category $Ind - \mathcal{A}$ is a Grothendieck category (in the sense of [KS, Definition 8.3.24, p.186]), and thus any functor $\mathcal{F} : Ind - \mathcal{A} \to \mathcal{C}$ commuting with small colimits admits a right adjoint.*

2.0.6 Actions on tensor powers of a vector space

Let U be a vector space over $\mathbb{C}$, and let $k \geq 0$.

Notation 2.0.6.1.

1. Let $A \in \mathrm{End}(U)$. We denote the operator $\mathrm{Id}_U \otimes \mathrm{Id}_U \otimes ... \otimes A \otimes ... \otimes \mathrm{Id}_U$ on $U^{\otimes k}$ (with A acting on the i-th factor of the tensor product) by $A^{(i)}$.

   The diagonal action of A on $U^{\otimes k}$ would then be

   $$\sum_{1 \leq i \leq n} A^{(i)} = A \otimes \mathrm{Id}_U \otimes ... \otimes \mathrm{Id}_U + \mathrm{Id}_U \otimes A \otimes ... \otimes \mathrm{Id}_U + ... + \mathrm{Id}_U \otimes ... \otimes \mathrm{Id}_U \otimes A$$

   and will sometimes be denoted by $A|_{U^{\otimes k}}$.

2. Similarly, given a functional $f \in U^*$, we have an operator $f^{(l)}$ defined as

$$f^{(l)} : U^{\otimes k} \to U^{\otimes k-1}$$

$$u_1 \otimes ... \otimes u_k \mapsto f(u_l) u_1 \otimes ... \otimes u_{l-1} \otimes u_{l+1} \otimes ... \otimes u_k$$

3. Finally, given $u \in U$, we define the operator $u^{(l)}$ as

$$u^{(l)} : U^{\otimes k} \to U^{\otimes k+1}$$

$$u_1 \otimes ... \otimes u_k \mapsto u_1 \otimes ... \otimes u_{l-1} \otimes u \otimes u_l \otimes ... \otimes u_k$$

Notation 2.0.6.2. Let U be a finite-dimensional vector space, and let $f \in U^*$, $u \in U$. Denote by $T_{f,u} \in \mathrm{End}(U)$ the rank one operator $v_1 \mapsto f(v_1)u$ (i.e. the image of $f \otimes u$ under the isomorphism $U^* \otimes U \to \mathrm{End}(U)$).

Notation 2.0.6.3. Let λ be a Young diagram. Denote by S^λ the Schur functor corresponding to λ (c.f. [FH, Chapter 6]). When applied to a finite-dimensional vector space U, this is either zero (iff $l(\lambda) > \dim U$), or an irreducible finite-dimensional representation of the Lie algebra $\mathfrak{gl}(U)$, which integrates to a representation of the group $GL(U)$.

We will denote the full additive subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}(U))}$ generated by $\{S^\lambda U\}_\lambda$ (λ running over all Young diagrams) by $Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$, and call its objects polynomial representations of the Lie algebra $\mathfrak{gl}(U)$ (or the algebraic group $GL(U)$).

The category $Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$ is obviously a semisimple abelian category, and it contains all the finite-dimensional representations of $\mathfrak{gl}(U)$ which can be obtained as submodules of a direct sum of tensor powers of the tautological representation U of $\mathfrak{gl}(U)$.

Alternatively, one can describe these representations as finite-dimensional representations $\rho : GL(U) \to \mathrm{Aut}(W)$ which can be extended to an algebraic map $\mathrm{End}(U) \to \mathrm{End}(W)$.

2.0.7 Symmetric group and Young diagrams

Notation 2.0.7.1.

- S_n will denote the symmetric group ($n \in \mathbb{Z}_+$).

- The notation λ will stand for a partition (weakly decreasing sequence of non-negative integers), a Young diagram λ, and the corresponding irreducible representation of $S_{|\lambda|}$. Here $|\lambda|$ is the sum of entries of the partition, or, equivalently, the number of cells in the Young diagram λ.

- All the Young diagrams will be considered in the English notation, i.e. the lengths of the rows decrease from top to bottom.

- The length of the partition λ, i.e. the number of rows of Young diagram λ, will be denoted by $\ell(\lambda)$.

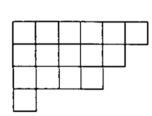
- The i-th entry of a partition λ, as well as the length of the i-th row of the corresponding Young diagram, will be denoted by λ_i (if $i > \ell(\lambda)$, then $\lambda_i := 0$).

- $\mathfrak{h}$ (in context of representations of S_n) will denote the permutation representation of S_n, i.e. the n-dimensional representation $\mathbb{C}^n$ with S_n acting by $g.e_j = e_{g(j)}$ on the standard basis $e_1, .., e_n$ of $\mathbb{C}^n$.

- For any Young diagram λ and an integer n such that $n \geq |\lambda| + \lambda_1$, we denote by $\tilde{\lambda}(n)$ the Young diagram obtained by adding a row of length $n - |\lambda|$ on top of λ.

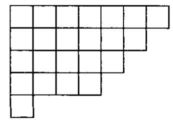
- Let $\mathcal{I}_\lambda^{m,+}$ denote the set of all Young diagrams obtained from λ by adding m boxes, no two in the same column, and $\mathcal{I}_\lambda^{m,-}$ denote the set of all Young diagrams obtained from λ by removing m boxes, no two in the same column. We will also denote: $\mathcal{I}_\lambda^+ := \biguplus_{m\geq 0}\mathcal{I}_\lambda^{m,+}$, $\mathcal{I}_\lambda^- := \biguplus_{0\leq m\leq|\lambda|}\mathcal{I}_\lambda^{m,-}$.

Example 2.0.7.2. Consider the Young diagram λ corresponding to the partition $(6, 5, 4, 1)$:



The length of λ is 4, and $|\lambda| = 16$. For $n = 23$, we have:

$$\tilde{\lambda}(n) =$$



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Chapter 3

Preliminaries

3.1 Classical Schur-Weyl duality

In this section we give a short overview of the classical Schur-Weyl duality.

Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $E := V^{\otimes d}$. Then S_d acts on E by permuting the factors of the tensor product (the action is semisimple, by Mashke's theorem):

$$\sigma.(v_1 \otimes v_2 \otimes ... \otimes v_d) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes ... \otimes v_{\sigma^{-1}(d)}$$

Denote by A the image of $\mathbb{C}[S_d]$ in $\mathrm{End}_{\mathbb{C}}(E)$.

Since $\mathbb{C}[S_d]$ is semisimple by Mashke's theorem, we have the following corollary of the Double Centralizer Theorem:

Proposition 3.1.0.3. *Let $B := \mathrm{End}_A(E)$. Then*

- *B is semisimple.*
- *$A = \mathrm{End}_B(E)$.*
- *As an $A \otimes_{\mathbb{C}} B$-module, E decomposes as*

$$E = \bigoplus_i V_i \otimes W_i$$

where V_i are all the irreducible representations of A, and W_i are all the irreducible representations of B. In particular, there is a bijection between the sets of non-isomorphic irreducible representations of A and B.

Consider the diagonal action of the Lie algebra $\mathfrak{gl}(V)$ on E (i.e. $a \in \mathfrak{gl}(V)$ acts on E by $a|_E = \sum_{1 \leq i \leq d} a^{(i)}$).

Then we have the following result, known as Schur-Weyl duality:

Theorem 3.1.0.4 (Schur-Weyl).

- *B is the image of $\mathcal{U}(\mathfrak{gl}(V))$ (the universal enveloping algebra of $\mathfrak{gl}(V)$) in $\mathrm{End}_{\mathbb{C}}(E)$, and thus E is a semisimple $\mathfrak{gl}(V)$-module.*
- *The images of $\mathbb{C}[S_d]$ and $\mathcal{U}(\mathfrak{gl}(V))$ in $\mathrm{End}_{\mathbb{C}}(E)$ are centralizers of each other.*
- *As $\mathbb{C}[S_d] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-module,*

$$E = \bigoplus_{\lambda:|\lambda|=d} \lambda \otimes S^{\lambda}V$$

We now define a contravariant functor

$$\mathtt{SW}_{d,V} : Rep(S_d) \to Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly},\ \mathtt{SW}_{d,V} := \mathrm{Hom}_{S_d}(\cdot, V^{\otimes d})$$

The contravariant functor $\mathtt{SW}_{d,V}$ is $\mathbb{C}$-linear and additive, and sends a simple module λ of S_d to $S^{\lambda}V$.

Next, consider the contravariant functor

$$\mathtt{SW}_V : \bigoplus_{d \in \mathbb{Z}_+} Rep(S_d) \to Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly},\ \mathtt{SW}_V := \oplus_d \mathtt{SW}_{d,V}$$

(the category $\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d)$ is equivalent to the category of Schur functors, and is obviously semisimple). This functor $\mathtt{SW}_V$ is clearly essentially surjective and full (this is easy to see, since $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$ is a semisimple category with simple objects $S^{\lambda}V \cong \mathtt{SW}(\lambda)$).

The kernel of the functor $\mathtt{SW}_V$ is the full additive subcategory (direct factor) of $\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d)$ generated by simple objects λ such that $\ell(\lambda) > \dim V$; taking the quotient, we see that $\mathtt{SW}_V$ defines an equivalence of categories

$$\mathtt{SW}_V : \left(\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d) \right)_{\text{length } \leq \dim V} \to Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly}$$

where $\left(\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d) \right)_{\text{length } \leq \dim V}$ is the full additive subcategory (direct factor) of $\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d)$ generated by simple objects λ such that $\ell(\lambda) \leq \dim V$.

3.2 Deligne category $\underline{Rep}(S_\nu)$

This section follows [CO, D2, Et1]. We will use the parameter ν instead of the parameter t used in Introduction.

3.2.1 General description

For any $\nu \in \mathbb{C}$, the category $\underline{Rep}(S_\nu)$ is generated, as a $\mathbb{C}$-linear Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of S_n, and any object in $\underline{Rep}(S_\nu)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $\nu \notin \mathbb{Z}_+$, $\underline{Rep}(S_\nu)$ is a semisimple abelian category.

Notation 3.2.1.1. We will denote Deligne's category for integer value $n \geq 0$ of ν as $\underline{Rep}(S_{\nu=n})$, to distinguish it from the classical category $Rep(S_n)$ of representations of the symmetric group S_n. Similarly for other categories arising in this text.

If ν is a non-negative integer, then the category $\underline{Rep}(S_\nu)$ has a tensor ideal $\mathfrak{I}_\nu$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \longrightarrow Y$ such that $tr(fu) = 0$ for any morphism $u : Y \longrightarrow X$). In that case, the classical category $Rep(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $\underline{Rep}(S_{\nu=n})/\mathfrak{I}_\nu$ (equivalent as Karoubian rigid symmetric monoidal categories).

The full, essentially surjective functor $\underline{Rep}(S_{\nu=n}) \to Rep(S_n)$ defining this equivalence will be denoted by $\mathcal{S}_n$.

Note that $\mathcal{S}_n$ sends $\mathfrak{h}$ to the permutation representation of S_n.

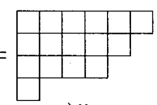
Remark 3.2.1.2. Although $\underline{Rep}(S_\nu)$ is not semisimple and not even abelian when $\nu = n \in \mathbb{Z}_+$, a weaker statement holds (see [D2, Proposition 5.1]): consider the full subcategory $\underline{Rep}(S_{\nu=n})^{(n/2)}$ of $\underline{Rep}(S_\nu)$ whose objects are directs summands of sums of $\mathfrak{h}^{\otimes m}, 0 \leq m \leq \frac{n}{2}$. This subcategory is abelian semisimple, and the restriction $\mathcal{S}_n|_{\underline{Rep}(S_{\nu=n})^{(n/2)}}$ is fully faithful.

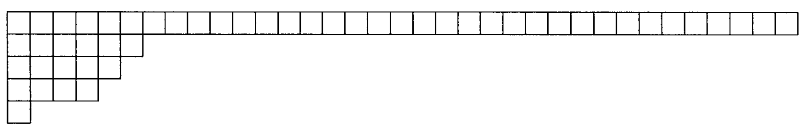
The indecomposable objects of $\underline{Rep}(S_\nu)$, regardless of the value of ν, are parametrized (up to isomorphism) by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\underline{Rep}(S_\nu)$ corresponding to the Young diagram τ by X_τ.

For non-negative integer $\nu =: n$, we have: the partitions λ for which X_λ has a non-zero image in the quotient $\underline{Rep}(S_{\nu=n})/\mathfrak{I}_{\nu=n} \cong Rep(S_n)$ are exactly the λ for which $\lambda_1 + |\lambda| \leq n$.

If $\lambda_1 + |\lambda| \leq n$, then the image of λ in $Rep(S_n)$ is the irreducible representation of S_n corresponding to the Young diagram $\tilde{\lambda}(n)$ (see notation in Chapter 2).

This allows one to intuitively treat the indecomposable objects of $\underline{Rep}(S_\nu)$ as if they were parametrized by "Young diagrams with a very long top row". The indecomposable object X_λ would be treated as if it corresponded to $\tilde{\lambda}(\nu)$, i.e. a Young diagram obtained by adding a very long top row ("of size $\nu - |\lambda|$"). This point of view is useful to understand how to extend constructions for S_n involving Young diagrams to $\underline{Rep}(S_\nu)$.

Example 3.2.1.3. The indecomposable object X_λ, where $\lambda =$  can be thought of as a Young diagram with a "very long top row of length $(\nu - 16)$":



3.2.2 Lifting objects

We start with an equivalence relation on the set of all Young diagrams, defined in [CO, Definition 5.1]:

Definition 3.2.2.1. Let λ be any Young diagram, and set

$$\mu_\lambda(\nu) = (\nu - |\lambda|, \lambda_1 - 1, \lambda_2 - 2, ...)$$

Given two Young diagrams λ, λ', denote $\mu_\lambda(\nu) =: (\mu_0, \mu_1, ...), \mu_{\lambda'}(\nu) =: (\mu'_0, \mu'_1, ...)$.

We put $\lambda \overset{\nu}{\sim} \lambda'$ if there exists a bijection $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that $\mu_i = \mu'_{f(i)}$ for any $i \geq 0$.

We will call a $\overset{\nu}{\sim}$-class *trivial* if it contains exactly one Young diagram.

The following lemma is proved in [CO, Corollary 5.6, Proposition 5.8]:

Lemma 3.2.2.2.

1. *If $\nu \notin \mathbb{Z}_+$, then any Young diagram λ lies in a trivial $\overset{\nu}{\sim}$-class.*

2. *The non-trivial $\overset{\nu}{\sim}$-classes are parametrized by all Young diagrams λ such that $\tilde{\lambda}(\nu)$ is a Young diagram (in particular, $\nu \in \mathbb{Z}_+$), and are of the form $\{\lambda^{(i)}\}_i$, with*

$$\lambda = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \ldots$$
$$\text{and } \lambda^{(i+1)} \setminus \lambda^{(i)} = \text{ strip in row } i+1 \text{ of length } \lambda_i - \lambda_{i+1} + 1 \text{ for } i > 0$$
$$\text{and } \lambda^{(1)} \setminus \lambda^{(0)} = \text{ strip in row } 1 \text{ of length } \nu - |\lambda| - \lambda_1 + 1$$

We now consider Deligne's category $\underline{Rep}(S_T)$, where T is a formal variable (c.f. [CO, Section 3.2]). This category is $\mathbb{C}((T-\nu))$-linear, but otherwise it is very similar to Deligne's category $\underline{Rep}(S_\nu)$ for generic ν. For instance, as a $\mathbb{C}((T-\nu))$-linear Karoubian tensor category, $\underline{Rep}(S_T)$ is generated by one object, again denoted by $\mathfrak{h}$.

One can show that $\underline{Rep}(S_T)$ is split semisimple and thus abelian, and its simple objects are parametrized by Young diagrams of arbitrary size.

In [CO, Section 3.2], Comes and Ostrik defined a map

$$lift_\nu : \left\{ {\text{objects in } \underline{Rep}(S_\nu) \atop \text{up to isomorphism}} \right\} \to \left\{ {\text{objects in } \underline{Rep}(S_T) \atop \text{up to isomorphism}} \right\}$$

We will not give the precise definition of this map, but will list some of its useful properties. It is defined to be additive (i.e. $lift_\nu(A \oplus B) \cong lift_\nu(A) \oplus lift_\nu(B)$ for any $A, B \in \underline{Rep}(S_\nu)$) and satisfies $lift_\nu(\mathfrak{h}) \cong \mathfrak{h}$. Moreover, we have:

Proposition 3.2.2.3. *Let A, B be two objects in $\underline{Rep}(S_\nu)$.*

1. $lift_\nu(A \otimes B) \cong lift_\nu(A) \otimes lift_\nu(B)$.

2. $\dim_{\underline{Rep}(S_\nu)} A = (\dim_{\underline{Rep}(S_T)} lift_\nu(A))|_{T=\nu}$.

3. $\dim_{\mathbb{C}} \mathrm{Hom}_{\underline{Rep}(S_\nu)}(A, B) = \dim_{Frak(\mathbb{C}[[T]])} \mathrm{Hom}_{\underline{Rep}(S_T)}(lift_\nu(A), lift_\nu(B))$.

4. The map $lift_\nu$ is injective.

5. For any λ, $lift_\nu(X_\lambda) = X_\lambda$ for all but finitely many $\nu \in \mathbb{C}$.

Proof. C.f. [CO, Proposition 3.12]. □

Remark 3.2.2.4. It was proved both in [D2, Section 7.2] and in [CO, Proposition 3.28] that the dimensions of the indecomposable objects X_λ in $\underline{Rep}(S_T)$ are polynomials in T whose coefficients depend on λ (given λ, this polynomial can be written down explicitly). Such polynomials are denoted by $P_\lambda(T)$.

Furthermore, it was proved in [CO, Proposition 5.12] that given $d \in \mathbb{Z}_+$ and a Young diagram λ, λ belongs to a trivial $\stackrel{d}{\sim}$-class iff $P_\lambda(d) = 0$.

The following result is proved in [CO, Lemma 5.20], and is a stronger version of the statement in Proposition 3.2.2.3(e):

Lemma 3.2.2.5 (Comes, Ostrik). *Consider the $\stackrel{\nu}{\sim}$-equivalence relation on Young diagrams.*

- *Whenever λ lies in a trivial $\stackrel{\nu}{\sim}$-class, $lift_\nu(X_\lambda) = X_\lambda$.*
- *For a non-trivial $\stackrel{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$,*

$$lift_\nu(X_{\lambda^{(0)}}) = X_{\lambda^{(0)}}, \qquad lift_\nu(X_{\lambda^{(i)}}) = X_{\lambda^{(i)}} \oplus X_{\lambda^{(i-1)}} \; \forall i \geq 1$$

Based on Lemmas 3.2.2.2, 3.2.2.5, Comes and Ostrik prove the following theorem (c.f. [CO, Theorem 5.3, Proposition 5.22, Theorems 6.4, 6.10], [CO2, Proposition 2.7]):

Theorem 3.2.2.6. *The indecomposable objects $X_\lambda, X_{\lambda'}$ belong to the same block of $\underline{Rep}(S_\nu)$ iff $\lambda \stackrel{\nu}{\sim} \lambda'$. The structure of the blocks of $\underline{Rep}(S_\nu)$ is described below:*

- *For a trivial* $\stackrel{\nu}{\sim}$*-class* $\{\lambda\}$*, the object* X_λ *satisfies:*

$$\dim \mathrm{End}_{\underline{Rep}(S_\nu)}(X_\lambda) = 1$$

  and the block of $\underline{Rep}(S_\nu)$ *corresponding to* $\{\lambda\}$ *is equivalent to the category* $Vect_{\mathbb{C}}$ *of finite dimensional complex vector spaces (in particular, it is a semisimple abelian category, so we will call these blocks semisimple).*

- *Let* $\{\lambda^{(i)}\}_i$ *be a non-trivial* $\stackrel{\nu}{\sim}$*-class, and let* $i \geq 1, j \geq 0$*. Then the block corresponding to* $\{\lambda^{(i)}\}_i$ *is not an abelian category (in particular, not semisimple), and the objects* $X_{\lambda^{(i)}}$ *satisfy:*

$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)}\left(X_{\lambda^{(j)}}, X_{\lambda^{(i)}}\right) = 0 \textit{ if } |j-i| \geq 2$$
$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)}\left(X_{\lambda^{(j)}}, X_{\lambda^{(i)}}\right) = 1 \textit{ if } |j-i| = 1$$
$$\dim \mathrm{End}_{\underline{Rep}(S_\nu)}\left(X_{\lambda^{(i)}}\right) = 2 \textit{ for } i \geq 1$$
$$\dim \mathrm{End}_{\underline{Rep}(S_\nu)}\left(X_{\lambda^{(0)}}\right) = 1$$

  This block has the following associated quiver:

$$X_{\lambda^{(0)}} \underset{\beta_0}{\overset{\alpha_0}{\leftrightarrows}} X_{\lambda^{(1)}} \underset{\beta_1}{\overset{\alpha_1}{\leftrightarrows}} X_{\lambda^{(2)}} \underset{\beta_2}{\overset{\alpha_2}{\leftrightarrows}} \cdots$$

  with relations $\alpha_0 \circ \beta_0 = 0$*,* $\beta_i \circ \beta_{i-1} = 0$*,* $\alpha_i \circ \alpha_{i-1} = 0$*,* $\beta_i \circ \alpha_i = \alpha_{i+1} \circ \beta_{i+1}$ *for* $i \geq 0$*.*

3.2.3 Objects Δ_k

In this subsection we define the objects Δ_k in the category $\underline{Rep}(S_\nu)$, and list some of their properties. These objects are defined for any $k \in \mathbb{Z}_+$ and any $\nu \in \mathbb{C}$.

By definition, Δ_k is the image of an idempotent $x_k \in \mathrm{End}_{\underline{Rep}(S_\nu)}(\mathfrak{h}^{\otimes k})$ (the latter is given explicitly in [CO2, Section 3.1]), and satisfies:

Lemma 3.2.3.1. $\mathcal{S}_n(\Delta_k) \cong \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\}) \cong Ind_{S_{n-k} \times S_k \times S_k}^{S_n \times S_k} \mathbb{C}$.

This is part of the definition of the functor $\mathcal{S}_n$ in [D2, Theorem 6.2].

Remark 3.2.3.2. The tensor functor $\mathcal{S}_n$ takes $\mathfrak{h}^{\otimes k}$ to $\mathbb{C}Fun(\{1,...,k\},\{1,...,n\})$ ([CO] uses this as part of the definition).

Example 3.2.3.3. $\Delta_0 \cong \mathbf{1}$ (unit object in monoidal category $\underline{Rep}(S_\nu)$), $\Delta_1 \cong \mathfrak{h}$.

Remark 3.2.3.4. Deligne in [D2] denotes the full subcategory of $\underline{Rep}(S_\nu)$ whose objects are $\{\Delta_k\}_{k\geq 0}$ by $Rep_0(S_\nu)$. This subcategory is a tensor subcategory (with respect to the tensor product in $\underline{Rep}(S_\nu)$), and it is used as the first step in defining the category $\underline{Rep}(S_\nu)$. Namely, one first describes the structure of $Rep_0(S_\nu)$ as a $\mathbb{C}$-linear rigid symmetric monoidal category (see [D2, Section 2]) and then defines $\underline{Rep}(S_\nu)$ as the Karoubi envelope of $Rep_0(S_\nu)$.

Comes and Ostrik, on the other hand, consider the full subcategory (denoted by $\underline{Rep}_0(S_\nu)$) of $\underline{Rep}(S_\nu)$ whose objects are $\{\mathfrak{h}^{\otimes k}\}_{k\geq 0}$. This is also a tensor subcategory. They start by defining the structure of $\underline{Rep}_0(S_\nu)$ as a $\mathbb{C}$-linear rigid symmetric monoidal category (see [CO, Section 2]) and then define $\underline{Rep}(S_\nu)$ as the Karoubi envelope of $\underline{Rep}_0(S_\nu)$.

In [D2, Section 8.2], Deligne showed that these two definitions are equivalent.

We now describe the Hom-spaces between the objects Δ_k. We start by introducing the following notation (see [CO, Section 2]):

Notation 3.2.3.5.

- By a *partition* π of a set S we will denote a collection $\{\pi_i\}_{i\in I}, \pi_i \subset S$, such that $\pi_i \cap \pi_j = \emptyset$ if $i \neq j$, and $\bigcup_{i\in I} \pi_i = S$. The subsets π_i will be called *parts* of π. The number of parts of π will be denoted by $l(\pi)$.

- Let $P_{r,s}$ be the set of all partitions of the set $\{1,...,r,1',...,s'\}$; $P_{0,s}$ is then the set of all partitions of $\{1',...,s'\}$, $P_{r,0}$ is the set of all partitions of $\{1,...,r\}$, $P_{0,0} :=$ $\{\text{empty partition}\}$.

- Let $\bar{P}_{r,s}$ be the subset of $P_{r,s}$ consisting of all the partitions π such that i, j do not lie in the same part of π whenever $i \neq j$, and similarly for i', j'.

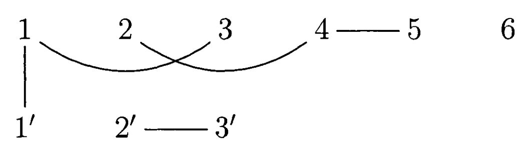
- The following diagrammatic notation will be used for elements of $P_{r,s}$ (resp. $\bar{P}_{r,s}$): let $\pi \in P_{r,s}$. We will represent π by any graph whose vertices are labeled $1, ..., r, 1', ..., s'$, and whose connected components partition the vertices into disjoint subsets corresponding to parts of π.

  For our convenience, we will always present such graphs as graphs with two rows of aligned vertices: the top row contains r vertices labeled by numbers $1, ..., r$, and the bottom row contains s vertices labeled by numbers $1', ..., s'$.

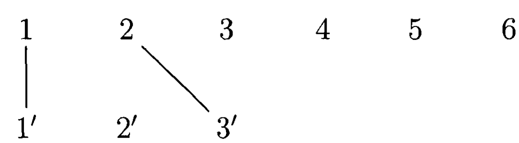
Remark 3.2.3.6. In this diagrammatic representation, partitions $\pi \in \bar{P}_{r,s}$ are exactly those which are represented by bipartite graphs with $deg(v) \leq 1$ for any vertex v. These partitions have exactly one diagram which represents them.

Example 3.2.3.7.

1. Let $\pi \in P_{6,3}, \pi := \{\{1,1',3\},\{2,4,5\},\{2',3'\},\{6\}\}$. The diagram representing π can be drawn as:



2. Let $\pi' \in P_{6,3}, \pi' := \{\{1,1'\},\{2,3'\},\{2'\},\{3\},\{4\},\{5\},\{6\}\}$. The diagram representing π' is:



   Notice that $\pi \notin \bar{P}_{6,3}$, but $\pi' \in \bar{P}_{6,3}$.

   We now describe how to "glue" two diagrams together to obtain a new diagram.

   Let $\pi \in P_{r,s}, \rho \in P_{s,t}$. We will denote the vertices in the top (resp. bottom) row of π by $1, ..., r$ (resp. $1', ..., s'$),and the vertices in the top (resp. bottom) row of ρ by $1', ..., s'$ (resp. $1'', ..., t''$).

We draw the diagram of π on top of the diagram of ρ, with the bottom row of π (vertices $1', ..., s'$) identified with the top row of ρ. We will call the diagram obtained *the gluing of* π, ρ, and will denote it by $D_{\pi,\rho}$.

We next consider the diagram induced by $D_{\pi,\rho}$ on the vertices $1, ..., r, 1'', ..., t''$ (by "induced diagram" we mean the diagram in which two vertices lie in the same connected component iff they were in the same connected component of $D_{\pi,\rho}$). This diagram (and the partition in $P_{r,t}$ it represents) will be denoted by $\rho \star \pi$.

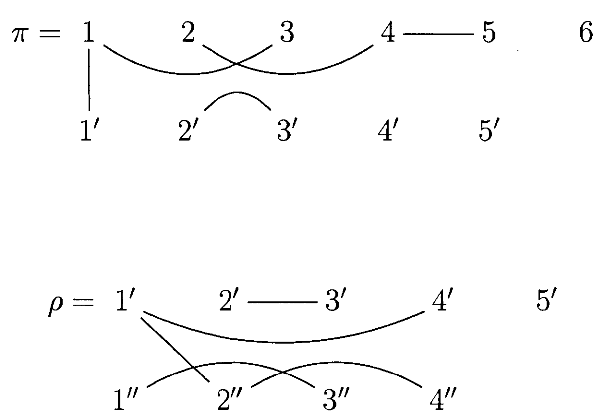
The second piece of information we want to retain from the diagram $D_{\pi,\rho}$ is the number of connected components lying entirely in the middle row. We will denote this number by $n(\rho, \pi)$. Thus

$$\sharp \text{ connected components of } D_{\pi,\rho} = l(D_{\pi,\rho}) = n(\rho,\pi) + l(\rho \star \pi)$$

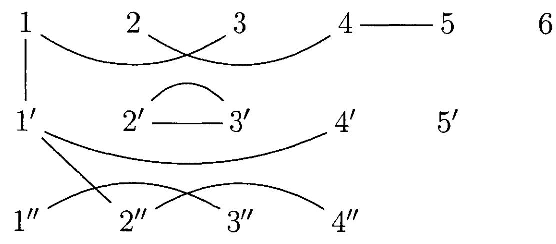
Example 3.2.3.8. Let $\pi \in P_{6,5}, \pi := \{\{1, 1', 3\}, \{2, 4, 5\}, \{2', 3'\}, \{4'\}, \{5'\}, \{6\}\}$,

$\rho \in P_{5,4}, \rho = \{\{1', 2'', 4', 4''\}, \{2', 3'\}, \{5'\}, \{1'', 3''\}\}$.

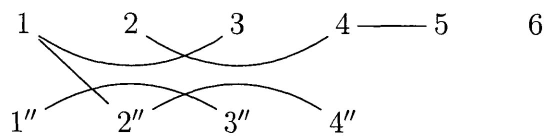
Then the diagrams of π, ρ can be drawn as:



Next, we draw the gluing of π, ρ, denoted by $D_{\pi,\rho}$:

$D_{\pi,\rho} =$ 

Then

$\rho \star \pi =$ 

i.e. $\rho \star \pi = \{\{1, 2'', 3, 4''\}, \{1'', 3''\}, \{2, 4, 5\}, \{6\}\}$ (partition of the set $\{1, ..., 6, 1'', ..., 4''\}$), and $n(\rho, \pi) = 2$.

The following statement is used by Comes and Ostrik in [CO, Definition 2.11] as the definition in the construction of $\underline{Rep}(S_\nu)$; Deligne derives it in [D2, Proposition 8.3].

Definition 3.2.3.9. Let $r, s \geq 1$. The space $\mathrm{Hom}_{\underline{Rep}(S_\nu)}(\mathfrak{h}^{\otimes r}, \mathfrak{h}^{\otimes s})$ is defined to be $\mathbb{C}P_{r,s}$, and the composition of morphisms between tensor powers of $\mathfrak{h}$ is bilinear and given by the following formula: for $\pi \in P_{r,s}, \rho \in P_{s,t}$,

$$\rho \circ \pi := \nu^{n(\rho,\pi)} \rho \star \pi \in \mathbb{C}P_{r,t}$$

The following statement is used as a definition in [D2, Definition 3.12], and can easily be derived from the definition of Δ_k (c.f. [CO2, Section 3.1]) and from Definition 3.2.3.9.

Lemma 3.2.3.10. *Let $r, s \geq 1$. The space $\mathrm{Hom}_{\underline{Rep}(S_\nu)}(\Delta_r, \Delta_s)$ is $\mathbb{C}\bar{P}_{r,s}$, and the composition of morphisms between the objects Δ_k is given by the following formula: for $\pi \in \bar{P}_{r,s}, \rho \in \bar{P}_{s,t}$,*

$$\rho \circ \pi = \sum_{\tau \in \bar{P}_{r,t} : \rho\star\pi \subset \tau} p_{\rho,\pi,\tau}(\nu) \tau \in \mathbb{C}\bar{P}_{r,t}$$

where

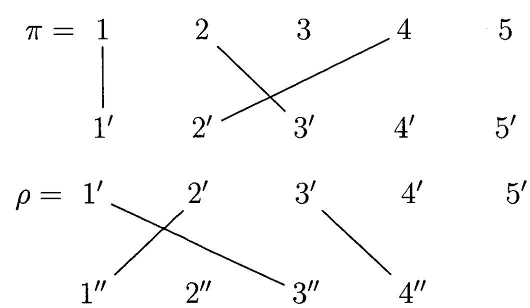
- *For* $\tau \in \bar{P}_{r,t}$, $\rho \star \pi \subset \tau$ *means that the diagram of* τ *contains the diagram of* $\rho \star \pi$ *as a subgraph (equivalently,* τ *is a coarser partition of the set* $\{1, ..., r, 1'', ..., t''\}$ *than* $\rho \star \pi$*),*

- $p_{\rho,\pi,\tau}$ *is the polynomial*

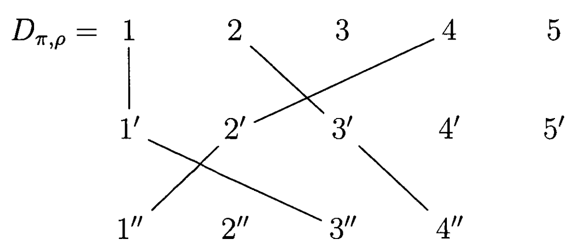
$$p_{\rho,\pi,\tau}(x) = (x - l(\tau))(x - l(\tau) - 1)...(x - l(\tau) - n(\rho, \pi) + 1)$$

Example 3.2.3.11. Let $\pi \in \bar{P}_{5,5}, \rho \in \bar{P}_{5,4}, \pi := \{\{1, 1'\}, \{2, 3'\}, \{2', 4\}, \{3\}, \{4'\}, \{5\}, \{5'\}\}$,

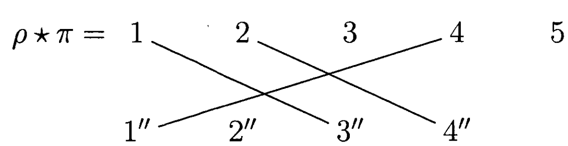
$\rho := \{\{1', 3''\}, \{1'', 2'\}, \{2''\}, \{3', 4''\}, \{4'\}, \{5'\}\}$. The diagrams representing π, ρ can be drawn as:



Gluing π and ρ together, we get:



Then

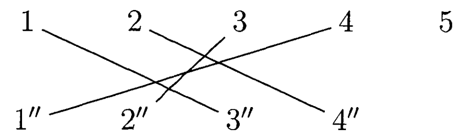


i.e. $\rho \star \pi = \{\{1, 3''\}, \{1'', 4\}, \{2''\}, \{2, 4''\}, \{3\}, \{5\}\}$ and $n(\rho, \pi) = 2$.

Next, we are looking for $\tau \in \bar{P}_{5,4}$ such that the diagram of τ contains $\rho \star \pi$ as a subgraph. There are three such partitions τ:

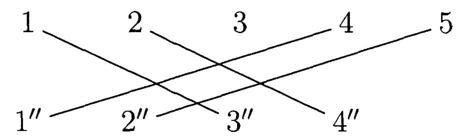
- $\tau_1 = \rho \star \pi$, in which case $p_{\rho,\pi,\tau_1}(x) = (x-6)(x-7)$.

- $\tau_2 =$



  in which case $p_{\rho,\pi,\tau_2}(x) = (x-5)(x-6)$.

- $\tau_3 =$



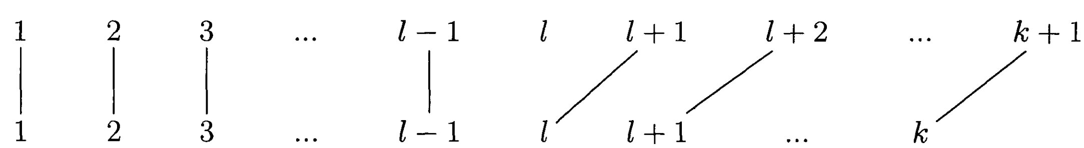
  in which case $p_{\rho,\pi,\tau_3}(x) = (x-5)(x-6)$.

Thus $\rho \circ \pi = (\nu-6)(\nu-7)\tau_1 + (\nu-5)(\nu-6)\tau_2 + (\nu-5)(\nu-6)\tau_3$.

The following morphisms between the objects Δ_k will be used frequently.

Let $r \geq 0$, $k \geq 1$, $1 \leq l \leq k$.

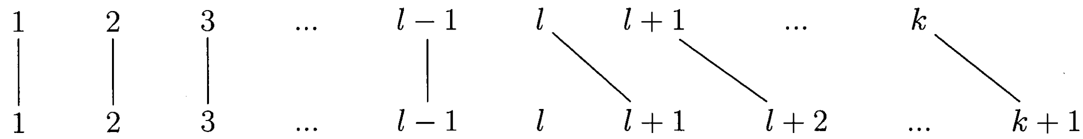
Definition 3.2.3.12. Denote by res_l the morphism $\Delta_{k+1} \to \Delta_k$ given by the diagram



By abuse of notation, we will also denote by res_l the maps $\bar{P}_{r,k+1} \to \bar{P}_{r,k}$, $\mathbb{C}\bar{P}_{r,k+1} \to \mathbb{C}\bar{P}_{r,k}$ given by $\pi \mapsto res_l \circ \pi$.

Notice that given $\pi \in \bar{P}_{r,k+1}$, a diagram describing the partition $res_l \circ \pi \in \bar{P}_{r,k}$ can be obtained by removing a vertex (labeled l') from position l of the bottom row of the diagram of π, and shifting the labels of the vertices lying to the right. If the vertex removed was connected to another vertex by an edge, then the edge is removed as well, but the second vertex stays.

Definition 3.2.3.13. Denote by res_l^* the morphism $\Delta_k \to \Delta_{k+1}$ given by the diagram



By abuse of notation, we will also denote by res_l^* the maps $\bar{P}_{r,k} \to \bar{P}_{r,k+1}$, $\mathbb{C}\bar{P}_{r,k} \to \mathbb{C}\bar{P}_{r,k+1}$ given by $\pi \mapsto res_l^* \circ \pi$.

Notice that given $\pi \in \bar{P}_{r,k}$, a diagram describing the partition $res_l^* \circ \pi \in \bar{P}_{r,k+1}$ can be obtained by inserting a solitary vertex (labeled l') in position l of the bottom row of the diagram of π, and shifting the labels of the vertices lying to the right.

Remark 3.2.3.14. Let $n \in \mathbb{Z}_+$. Fix k, l such that $1 \leq k \leq n-1, 1 \leq l \leq k$. Denote

$$\mathtt{res}_l := \mathcal{S}_n(res_l) : \mathbb{C}Inj(\{1, ..., k+1\}, \{1, ..., n\}) \to \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\})$$

$$\mathtt{res}_l^* := \mathcal{S}_n(res_l^*) : \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\}) \to \mathbb{C}Inj(\{1, ..., k+1\}, \{1, ..., n\})$$

Denote by ι_l the injection

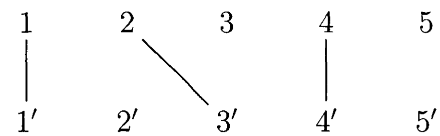
$$\{1, ..., k\} \hookrightarrow \{1, ..., k+1\},\ i \mapsto \begin{cases} i & \text{if } i < l \\ i+1 & \text{if } i \geq l \end{cases}$$

Then given $g : \{1, ..., k+1\} \hookrightarrow \{1, ..., n\}$, we have $\mathtt{res}_l(g) = g \circ \iota_l$, and given $f : \{1, ..., k+1\} \hookrightarrow \{1, ..., n\}$, we have

$$\mathtt{res}_l^*(f) = \sum_{\substack{g \in Inj(\{1,...,k+1\},\{1,...,n\}): \\ g \circ \iota_l = f}} g$$

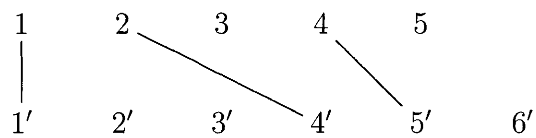
Example 3.2.3.15. Let $\pi \in \bar{P}_{5,5}, \pi := \{\{1,1'\},\{2'\},\{2,3'\},\{3\},\{4,4'\},\{5\},\{5'\}\}$, i.e.

$\pi =$



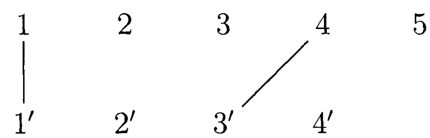
Then

$res_3^*(\pi) =$



and

$res_3(\pi) =$



Remark 3.2.3.16. One can define an endomorphism $\bar{x}_k \in \mathrm{End}_{\underline{Rep}(S_T)}(\mathfrak{h}^{\otimes k})$ similar to the idempotent x_k, so that $\mathrm{Im}(\bar{x}_k) \cong lift_\nu(\Delta_k)$ for any ν (this is a direct consequence of the definition of $lift_\nu$).

By abuse of notation, we will denote $\mathrm{Im}(\bar{x}_k)$ by Δ_k as well, and use the isomorphism $\Delta_k \cong lift_\nu(\Delta_k)$ in Lemma 5.0.0.40.

3.2.4 Abelian envelope

This section follows [CO2], [D2, Proposition 8.19].

As it was mentioned before, the category $\underline{Rep}(S_\nu)$ is defined as a Karoubian category. For $\nu \notin \mathbb{Z}_+$, it is semisimple and thus abelian, but for $\nu \in \mathbb{Z}_+$, it is not abelian. Fortunately, it has been shown that $\underline{Rep}(S_\nu)$ possesses an "abelian envelope", that is, that it can be embedded in an abelian tensor category, and this abelian tensor category has a universal mapping property.

The following result was conjectured by Deligne in [D2, 8.21.2], and proved by Comes and Ostrik in [CO2, Theorem 1.2]:

Theorem 3.2.4.1. *Let $n \in \mathbb{Z}_+$. There exists an abelian $\mathbb{C}$-linear rigid symmetric monoidal category $\underline{Rep}^{ab}(S_{\nu=n})$, and an embedding (fully faithful tensor functor)*

$\iota : \underline{Rep}(S_{\nu=n}) \to \underline{Rep}^{ab}(S_{\nu=n})$ which makes the pair $(\underline{Rep}^{ab}(S_{\nu=n}), \iota)$ the "abelian envelope" of $\underline{Rep}(S_{\nu=n})$ in the following sense:

Let $\mathcal{T}$ be an abelian $\mathbb{C}$-linear rigid symmetric monoidal category such that all Hom*-spaces are finite-dimensional and all objects have finite length; in addition, let there be a tensor functor of Karoubian categories $\mathcal{G} : \underline{Rep}(S_{\nu=n}) \to \mathcal{T}$. Then the functor $\mathcal{G}$ factors through one of the following:*

1. *The functor $\mathcal{S}_n : \underline{Rep}(S_{\nu=n}) \to Rep(S_n)$ (this happens iff $\mathcal{G}(\Delta_{n+1}) = 0$).*

2. *The functor $\iota : \underline{Rep}(S_{\nu=n}) \to \underline{Rep}^{ab}(S_{\nu=n})$ (this happens iff $\mathcal{G}(\Delta_{n+1}) \neq 0$).*

For $\nu \notin \mathbb{Z}_+$, we will put $(\underline{Rep}^{ab}(S_\nu), \iota) := (\underline{Rep}(S_\nu), \mathrm{Id}_{\underline{Rep}(S_\nu)})$.

An explicit construction of the category $\underline{Rep}^{ab}(S_{\nu=n})$ is given in [CO2]. We will only list the results which will be used in this thesis.

Remark 3.2.4.2. The category $\underline{Rep}^{ab}(S_\nu)$ is a pre-Tannakian category (c.f. [CO2, Section 2.1, Corollary 4.7]). This means, in particular, that the objects in $\underline{Rep}^{ab}(S_\nu)$ have finite length, and the Hom-spaces are finite-dimensional.

We start by introducing the category $\mathcal{C}_q$ of finite-dimensional representations of the quantum $SL(2)$. The category $\mathcal{C}_q$ is a $\mathbb{C}$-linear abelian category, and has the structure of a highest weight category (with infinitely many weights). When q is a root of unity, this category can be non-semisimple, and this is the case which will be of interest to us.

This category has a structure very similar to the structure of the category $\underline{Rep}^{ab}(S_\nu)$; moreover, the $\mathbb{C}$-linear Karoubian category $TL(q)$ of tilting modules in $\mathcal{C}_q$ (also known to be equivalent to the Temperley-Lieb category) has a structure very similar to that of $\underline{Rep}(S_\nu)$. See [CO2, Par. 1.4, 2.3, 4.3.3] for more details.

We will use the description of the structure of $\mathcal{C}_q$ given in [CO2], [A], [APW]. The main facts about $\mathcal{C}_q$ which will be used are concentrated in the following lemma.

Lemma 3.2.4.3.

1. *All the projective modules in $\mathcal{C}_q$ are injective and conversely. Thus all the projective modules of $\mathcal{C}_q$ are tilting modules, i.e. lie in the category $TL(q)$.*

2. *For $q \neq \pm 1$ being a root of unity of even order, $TL(q)$ has at least one non-semisimple block (all of the non-semisimple blocks of $TL(q)$ are equivalent as Karoubian categories); the isomorphism classes of indecomposable objects in this block can be labeled $\mathrm{Q}_0, \mathrm{Q}_1, \ldots$. Each Q_i has a unique highest weight.*

   Denote by L_i the simple module in $\mathcal{C}_q$ having the same highest weight as Q_i, and by $\mathrm{M}_i, \mathrm{M}_i^\vee, \mathrm{P}_i$ the corresponding standard, co-standard and indecomposable projective modules in $\mathcal{C}_q$. With these notations, we have:

   - *For any $i \geq 0$, there exists an injective map $\mathrm{M}_i \to \mathrm{Q}_i$. Moreover, $[\mathrm{Q}_i : \mathrm{L}_i] = 1$.*
   - *The module Q_0 is standard, co-standard and simple.*
   - *The module Q_i is a projective module iff $i \geq 1$. Furthermore, $\{\mathrm{Q}_i\}_{i \geq 1}$ is the complete set of isomorphism classes of indecomposable projective modules in the corresponding block of $\mathcal{C}_q$.*

Proof. 1. This follows from [APW, Theorem 9.12], [A, 5.7].

2. For general information on non-semisimple blocks of $TL(q)$ and the indecomposable tilting modules, c.f. [CO2, Lemma 2.11, Section 4.3.3], [A, Theorem 2.5, Corollary 2.6].

   - Q_i is a standardly filtered having the same highest weight as L_i (by definition), therefore there exists an injective map $\mathrm{M}_i \to \mathrm{Q}_i$ (c.f. [H, Proposition 3.7]). Also, from [A, Theorem 2.5] we know that the highest weight of Q_i occurs with multiplicity 1, therefore $[\mathrm{Q}_i : \mathrm{L}_i] = 1$.
   - C.f. [A, Section 4].
   - Let St_q be the Steinberg module (c.f. [A, Section 5], [CO2, Section 4.3.3]).

By [APW, Lemma 9.10], for any finite dimensional module $E \in \mathcal{C}_q$, the module $St_q \otimes E$ is projective. Furthermore, [APW, Theorem 9.12] implies that any projective module in $\mathcal{C}_q$ is a direct summand of $St_q \otimes E$ for some E.

Next, it is known that a module $M \in TL(q)$ has quantum dimension zero iff it is a direct summand of $St_q \otimes E$ for some $E \in TL(q)$ (c.f. [CO2, Section 4.3.3]). Thus a module $M \in TL(q)$ has quantum dimension zero iff it is projective.

Finally, [CO2, Lemma 2.11] tells us that Q_i has quantum dimension zero iff $i > 0$. Thus Q_i is a projective module iff $i > 0$. From Part (1) we deduce that any indecomposable projective module in the corresponding block of $\mathcal{C}_q$ is isomorphic to Q_i for some $i > 0$.

□

We can now describe the blocks of the category $\underline{Rep}^{ab}(S_\nu)$ (c.f. [CO2, Proposition 2.9, Section 4]).

Theorem 3.2.4.4. *The blocks of the category $\underline{Rep}^{ab}(S_\nu)$, just like the blocks of $\underline{Rep}(S_\nu)$, are parametrized by $\overset{\nu}{\sim}$-equivalence classes. For each $\overset{\nu}{\sim}$-equivalence class C, the block $\underline{Rep}^{ab}(S_\nu)_C$ corresponding to C contains $\iota(\underline{Rep}(S_\nu)_C)$ (the block of $\underline{Rep}(S_\nu)$ corresponding to C, namely, the indecomposable objects X_λ such that $\lambda \in C$).*

- *For a trivial $\overset{\nu}{\sim}$-class $C = \{\lambda\}$, the block $\underline{Rep}^{ab}(S_\nu)_C$ is equivalent to the category $Vect_{\mathbb{C}}$ of finite dimensional complex vector spaces (and is also equivalent to $\underline{Rep}(S_\nu)_C$).*
- *For a non-trivial $\overset{\nu}{\sim}$-class $C = \{\lambda^{(i)}\}_{i \geq 0}$, the block $\underline{Rep}^{ab}(S_\nu)_C$ is equivalent, as an abelian category, to (any) non-semisimple block of the category $\mathcal{C}_q$ (such a block exists if $q \neq \pm 1$ is a root of unity of even order). The indecomposable object $X_{\lambda^{(i)}}$ corresponds to the indecomposable tilting module Q_i (using the notation of Lemma 3.2.4.3).*

Using the theorem above, we can prove different properties of the category $\underline{Rep}^{ab}(S_\nu)$ in the following way:

1. Reduce the proof to a block-by-block check;

2. Prove the property for the semisimple blocks by checking that it holds for the category $Vect_{\mathbb{C}}$.

3. Prove the property for the non-semisimple blocks by importing the relevant result for the category $\mathcal{C}_q$.

Using this approach, we prove that $\underline{Rep}^{ab}(S_\nu)$ is a highest weight category (with infinitely many weights).

Proposition 3.2.4.5. *The category $\underline{Rep}^{ab}(S_\nu)$ is a highest weight category corresponding to the partially ordered set ({Young diagrams}, $\geq$), where*

$$\lambda \geq \mu \text{ iff } \lambda \overset{\nu}{\sim} \mu, \lambda \subset \mu$$

(namely, $\lambda^{(i)} \geq \lambda^{(j)}$ if $i \leq j$).

Proof. As it was said before, this can be proved by checking each block separately. The semisimple blocks obviously satisfy the requirement; for the non-semisimple blocks, the theorem follows from the fact that $\mathcal{C}_q$ is a highest weight category and from Theorem 3.2.2.6. □

Proposition 3.2.4.6.

1. *In the category $\underline{Rep}^{ab}(S_\nu)$ all projective objects are injective and conversely.*

2. *All projective objects of $\underline{Rep}^{ab}(S_\nu)$ lie in $\underline{Rep}(S_\nu)$.*

Proof. The statement is obvious for semisimple blocks. For non-semisimple blocks, the first part follows from [APW, Theorem 9.12].

To prove second part of the Proposition, we recall that the equivalence between non-semisimple blocks of $\underline{Rep}^{ab}(S_\nu), \mathcal{C}_q$, by definition restricts to an equivalence between non-semisimple blocks of $\underline{Rep}(S_\nu), TL(q)$ (see [CO2]). Lemma 3.2.4.3 states that the corresponding statement is true for $\mathcal{C}_q, TL(q)$, and we are done. □

We will use the following notation for simple, standard, co-standard, and indecomposable projective objects in $\underline{Rep}^{ab}(S_\nu)$:

Notation 3.2.4.7. Let λ be any Young diagram. We will denote the simple (resp. standard, co-standard, indecomposable projective) object corresponding to λ by $\mathbf{L}(\lambda)$ (resp. $\mathbf{M}(\lambda)$, $\mathbf{M}(\lambda)^*$, $\mathbf{P}(\lambda)$).

Remark 3.2.4.8. We will show in Corollary 3.2.4.12 that the co-standard object $\mathbf{M}(\lambda)^*$ is the dual (in terms of the tensor structure of $\underline{Rep}^{ab}(S_\nu)$) of the standard object $\mathbf{M}(\lambda)$. This justifies the notation $\mathbf{M}(\lambda)^*$.

Remark 3.2.4.9. Notice that if λ lies in a non-trivial $\overset{\nu}{\sim}$-class, the objects $\mathbf{L}(\lambda^{(i)})$, $\mathbf{M}(\lambda^{(i)})$, $\mathbf{M}(\lambda^{(i)})^*$, $\mathbf{P}(\lambda^{(i)})$ correspond to the modules $\mathrm{L}_i, \mathrm{M}_i, \mathrm{M}_i^\vee, \mathrm{P}_i \in \mathcal{C}_q$, respectively.

Proposition 3.2.4.10.

1. *Assume λ lies in a trivial $\overset{\nu}{\sim}$-class. Then*

$$X_\lambda \cong \mathbf{P}(\lambda) \cong \mathbf{M}(\lambda)^* \cong \mathbf{M}(\lambda) \cong \mathbf{L}(\lambda)$$

2. *Let $\{\lambda^{(i)}\}_{i\geq 0}$ be a non-trivial $\overset{\nu}{\sim}$-class, and $\mathcal{B}_\lambda$ the corresponding block of $\underline{Rep}^{ab}(S_\nu)$. Then*

   - $X_{\lambda^{(0)}} \cong \mathbf{L}(\lambda^{(0)}) \cong \mathbf{M}(\lambda^{(0)}) \cong \mathbf{M}(\lambda^{(0)})^*$.
   - *For any $i \geq 0$, $\mathbf{P}(\lambda^{(i)}) \cong X_{\lambda^{(i+1)}}$.*
   - *For any $i \geq 0$, we have short exact sequences*

$$0 \to \mathbf{M}(\lambda^{(i+1)}) \to \mathbf{P}(\lambda^{(i)}) \to \mathbf{M}(\lambda^{(i)}) \to 0$$

$$0 \to \mathbf{M}(\lambda^{(i)})^* \to \mathbf{P}(\lambda^{(i)}) \to \mathbf{M}(\lambda^{(i+1)})^* \to 0$$

   - *The socle filtration of $\mathbf{P}(\lambda^{(i)})$ has successive quotients*

$$\mathbf{L}(\lambda^{(i)}); \mathbf{L}(\lambda^{(i+1)}) \oplus \mathbf{L}(\lambda^{(i-1)}); \mathbf{L}(\lambda^{(i)})$$

if $i > 0$, and successive quotients $\mathbf{L}(\lambda^{(0)}); \mathbf{L}(\lambda^{(1)}); \mathbf{L}(\lambda^{(0)})$ if $i = 0$.

Proof.

Follows directly from Theorem 3.2.4.4, part (1).

2. From Lemma 3.2.4.3 and from the equivalence described in Theorem 3.2.4.4, we immediately conclude that

$$X_{\lambda^{(0)}} \cong \mathbf{L}(\lambda^{(0)}) \cong \mathbf{M}(\lambda^{(0)}) \cong \mathbf{M}(\lambda^{(0)})^*$$

From Lemma 3.2.4.3, we know that $\{X_{\lambda^{(i)}}\}_{i \geq 1}$ is the set of isomorphism classes of indecomposable projective objects in the block $\mathcal{B}_\lambda$. In other words, there exists a bijective map $f : \mathbb{Z}_+ \to \mathbb{Z}_{>0}$ such that $X_{\lambda^{(f(i))}} \cong \mathbf{P}(\lambda^{(i)})$ for any $i > 0$.

Lemma 3.2.4.3 also tells us that there exists an injective map $\psi_j : \mathbf{M}(\lambda^{(j)}) \hookrightarrow X_{\lambda^{(j)}}$ for any $j \geq 0$.

Composing $\psi_{f(i)}$ with the map $X_{\lambda^{(f(f(i)))}} \cong \mathbf{P}(\lambda^{(f(i))}) \twoheadrightarrow \mathbf{M}(\lambda^{(f(i))})$, we get a non-zero map $X_{\lambda^{(f(f(i)))}} \to X_{\lambda^{(f(i))}}$. Using Theorem 3.2.2.6, we see that $|f(f(i)) - f(i)| \leq 1$ for any $i \geq 0$, i.e. $|f(i) - i| \leq 1$ for any $i \geq 1$ (since f is surjective).

Notice that

$$\dim \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{P}(\lambda^{(0)}), \mathbf{M}(\lambda^{(0)})) = \dim \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(X_{\lambda^{(f(0))}}, X_{\lambda^{(0)}}) \geq 1$$

which means that $f(0) = 1$. Together with the condition: $|f(i) - i| \leq 1$ for any $i \geq 1$, this implies that $f(i) = i + 1$ for any $i \geq 1$.

Thus we proved that for any $i \geq 0$, $\mathbf{P}(\lambda^{(i)}) \cong X_{\lambda^{(i+1)}}$. We now use the BGG reciprocity for the highest weight category $\underline{Rep}^{ab}(S_\nu)$:

Sublemma 3.2.4.11.

(a) For any $M \in \underline{Rep}^{ab}(S_\nu)$*, we have:*

$$\dim \operatorname{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{P}(\lambda^{(j)}), M) = \left[M{:}\mathbf{L}(\lambda^{(j)})\right]$$

for any $j \geq 0$.

(b)

$$\left(\mathbf{P}(\lambda^{(j)}){:}\mathbf{M}(\lambda^{(i)})\right) = \left[\mathbf{M}(\lambda^{(i)}){:}\mathbf{L}(\lambda^{(j)})\right] = \left[\mathbf{M}(\lambda^{(i)})^*{:}\mathbf{L}(\lambda^{(j)})\right]$$

for any $i, j \geq 0$ *(the brackets in left hand side denote multiplicity in the standard filtration).*

Proof. The proof is standard (see e.g. [H, Theorem 3.9(c), Theorem 3.11]).

□

Applying Sublemma 3.2.4.11(a) to $M := \mathbf{P}(\lambda^{(i)}) \cong X_{\lambda^{(i+1)}}$ and using Theorem 3.2.2.6, we see that the composition factors of $\mathbf{P}(\lambda^{(i)})$ are

$$\mathbf{L}(\lambda^{(i)}), \mathbf{L}(\lambda^{(i+1)}), \mathbf{L}(\lambda^{(i-1)}), \mathbf{L}(\lambda^{(i)}) \text{ if } i > 0$$

and

$$\mathbf{L}(\lambda^{(0)}), \mathbf{L}(\lambda^{(1)}), \mathbf{L}(\lambda^{(0)}) \text{ if } i = 0$$

(notice that the cosocle of $\mathbf{P}(\lambda^{(i)})$ is necessarily $\mathbf{L}(\lambda^{(i)})$ for any $i \geq 0$).

Fix $i \geq 1$. We have a map $\mathbf{P}(\lambda^{(i)}) \twoheadrightarrow \mathbf{M}(\lambda^{(i)})$ and a map $\mathbf{M}(\lambda^{(i)}) \xhookrightarrow{\psi_i} \mathbf{P}(\lambda^{(i-1)})$.

Comparing the composition factors of $\mathbf{P}(\lambda^{(i)}), \mathbf{P}(\lambda^{(i+1)})$, we conclude that one of the following holds:

- $\mathbf{M}(\lambda^{(i)}) \cong \mathbf{L}(\lambda^{(i)})$.
- The socle filtration of $\mathbf{M}(\lambda^{(i)})$ has successive quotients $\mathbf{L}(\lambda^{(i-1)}); \mathbf{L}(\lambda^{(i)})$.

Applying Sublemma 3.2.4.11(b) to $j := i - 1$, we see that

$$\left[\mathbf{M}(\lambda^{(i)}){:}\mathbf{L}(\lambda^{(i-1)})\right] = \left(\mathbf{P}(\lambda^{(i-1)}){:}\mathbf{M}(\lambda^{(i)})\right) = 1$$

Thus we conclude that the socle filtration of $\mathbf{M}(\lambda^{(i)})$ has successive quotients $\mathbf{L}(\lambda^{(i-1)}); \mathbf{L}(\lambda^{(i)})$.

The socle filtration of the standard objects immediately implies that the following sequence is exact:

$$0 \to \mathbf{M}(\lambda^{(i+1)}) \xrightarrow{\psi_{i+1}} \mathbf{P}(\lambda^{(i)}) \longrightarrow \mathbf{M}(\lambda^{(i)}) \to 0$$

Next, the socle filtration of $\mathbf{M}(\lambda^{(i)})^*$ can be obtained from the socle filtration of $\mathbf{M}(\lambda^{(i)})$, and it has successive quotients $\mathbf{L}(\lambda^{(i)}); \mathbf{L}(\lambda^{(i-1)})$.

Since $\mathbf{P}(\lambda^{(i-1)})$ is projective, we get a map $\phi : \mathbf{P}(\lambda^{(i-1)}) \to \mathbf{M}(\lambda^{(i)})^*$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{P}(\lambda^{(i-1)}) & \xrightarrow{\phi} & \mathbf{M}(\lambda^{(i)})^* \\ \downarrow & \swarrow & \\ \mathbf{L}(\lambda^{(i-1)}) & & \end{array}$$

The socle filtration of $\mathbf{M}(\lambda^{(i)})^*$ then implies that the map ϕ is surjective, and we get an exact sequence

$$0 \to \mathbf{M}(\lambda^{(i-1)})^* \to \mathbf{P}(\lambda^{(i-1)}) \to \mathbf{M}(\lambda^{(i)})^* \to 0$$

Finally, the socle filtration of $\mathbf{P}(\lambda^{(i)})$ can be deduced from the above exact sequences and the socle filtrations of the standard and the co-standard objects.

□

Corollary 3.2.4.12. *The co-standard object* $\mathbf{M}(\lambda)^*$ *is the dual (in terms of the tensor structure of* $\underline{Rep}^{ab}(S_\nu)$*) of the standard object* $\mathbf{M}(\lambda)$.

Proof. By the construction of the Deligne category $\underline{Rep}(S_\nu)$, all the objects in $\underline{Rep}(S_\nu)$ are automatically self-dual (c.f. [D2, Section 2.16], [CO, Section 2.2]). So Proposition 3.2.4.10 immediately implies that $\mathbf{M}(\lambda)^*$ is the dual of $\mathbf{M}(\lambda)$ if whenever λ lies in a trivial $\stackrel{\nu}{\sim}$-class, or is minimal in its non-trivial $\stackrel{\nu}{\sim}$-class.

It remains to check the case when λ lies in a non-trivial $\stackrel{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\geq 0}$, and $\lambda = \lambda^{(i)}, i > 0$. Then we have an exact sequence

$$\mathbf{P}(\lambda^{(i+1)}) \xrightarrow{f} \mathbf{P}(\lambda^{(i)}) \longrightarrow \mathbf{M}(\lambda^{(i)}) \to 0$$

Since the category $\underline{Rep}^{ab}(S_\nu)$ is pre-Tannakian, the duality functor $X \mapsto X^*$ is contravariant and exact, and we conclude that the dual of $\mathbf{M}(\lambda^{(i)})$ is the kernel of the map $f^* : \mathbf{P}^*(\lambda^{(i)}) \to \mathbf{P}^*(\lambda^{(i+1)})$. The autoduality of $\mathbf{P}(\lambda^{(i+1)}), \mathbf{P}(\lambda^{(i)})$, together with Proposition 3.2.4.10 and the fact that $f^* \neq 0$, immediately implies that the dual of $\mathbf{M}(\lambda^{(i)})$ is $\mathbf{M}(\lambda^{(i)})^*$, as wanted. □

3.3 Parabolic category O

In this section, we present the results on the parabolic category O which we will use. The material of this section is mostly based on [H, Chapter 9].

We start with some definitions.

Definition 3.3.0.13. A unital vector space is a vector space V with a distinguished non-zero vector $\mathbb{1}$.

Fix a unital vector space $(V, \mathbb{1})$ with $\dim V < \infty$.

We will use Notation 1.2.0.1 for the parabolic subalgebra $\mathfrak{p}_{(V,\mathbb{C}\mathbb{1})}$ of $\mathfrak{gl}(V)$, as well as the mirabolic subgroup $\bar{\mathfrak{P}}_{\mathbb{1}}$, the mirabolic subalgebra $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$, the unipotent group $\mathfrak{U}_{\mathbb{1}}$ and the nilpotent subalgebra $\mathfrak{u}_{\mathfrak{p}}^{+}$.

Remark 3.3.0.14. Notice that $\mathfrak{p}$ has a one-dimensional center (scalar endomorphisms of V), and we have: $\mathfrak{p} \cong \mathbb{C}\,\mathrm{Id}_V \oplus \bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$.

We now want to talk about a subcategory of the category of finite-dimensional representations of the mirabolic group $\bar{\mathfrak{P}}_{\mathbb{1}}$. For this, we will use the following lemma:

Lemma 3.3.0.15. *(a) There is a short exact sequence of groups*

$$1 \to \mathfrak{U}_{\mathbb{1}} \longrightarrow \bar{\mathfrak{P}}_{\mathbb{1}} \longrightarrow GL\left({}^{V}\big/_{\mathbb{C}\mathbb{1}}\right) \to 1$$

(b) For any irreducible finite-dimensional algebraic representation $\rho : \bar{\mathfrak{P}}_{\mathbb{1}} \to \mathrm{Aut}(E)$ of the mirabolic subgroup, $\mathfrak{U}_{\mathbb{1}}$ acts trivially on E, and thus ρ factors through $GL\left(V/\mathbb{C}\mathbb{1}\right)$.

Proof. In part (a), one only needs to check that this sequence is exact at $\bar{\mathfrak{P}}_{\mathbb{1}}$. This is obvious once we choose a splitting $V = \mathbb{C}\mathbb{1} \oplus U$; with a chosen splitting, this short exact sequence splits and we get an isomorphism $\bar{\mathfrak{P}}_{\mathbb{1}} \cong GL(U) \ltimes U^*$.

In part (b), recall that the group $\mathfrak{U}_{\mathbb{1}}$ is unipotent (and even abelian), so there exists a non-zero vector $v \in E$ which is fixed by $\mathfrak{U}_{\mathbb{1}}$.

Since E is an irreducible finite-dimensional representation of $\bar{\mathfrak{P}}_{\mathbb{1}}$, E has a basis consisting of vectors of the form $\rho(x)v$ for some $x \in \bar{\mathfrak{P}}_{\mathbb{1}}$. On each such vector $\rho(x)v$, $\mathfrak{U}_{\mathbb{1}}$ acts trivially; thus $\mathfrak{U}_{\mathbb{1}}$ acts trivially on E.

□

We now consider the category $Rep(\bar{\mathfrak{P}}_{\mathbb{1}})$ of finite-dimensional algebraic representations of $\bar{\mathfrak{P}}_{\mathbb{1}}$. In this category, we define:

Definition 3.3.0.16.

- Let $\rho : \bar{\mathfrak{P}}_{\mathbb{1}} \to \mathrm{Aut}(E)$ be an irreducible finite-dimensional algebraic representation of the mirabolic subgroup. The above lemma states that ρ factors through $GL(V/\mathbb{C}\mathbb{1})$. We say that ρ is a *$GL(V/\mathbb{C}\mathbb{1})$-polynomial representation of* $\bar{\mathfrak{P}}_{\mathbb{1}}$ if $\rho : GL(V/\mathbb{C}\mathbb{1}) \to \mathrm{Aut}(E)$ is a polynomial representation (c.f. Notation 2.0.6.1). Recall that the latter condition is equivalent to saying that ρ extends to an algebraic map $\mathrm{End}(V/\mathbb{C}\mathbb{1}) \to \mathrm{End}(E)$.

- The category of *$GL(V/\mathbb{C}\mathbb{1})$-polynomial representations of* $\bar{\mathfrak{P}}_{\mathbb{1}}$ is defined to be the Serre subcategory of $Rep(\bar{\mathfrak{P}}_{\mathbb{1}})$ generated by the irreducible $GL(V/\mathbb{C}\mathbb{1})$-polynomial representations of $\bar{\mathfrak{P}}_{\mathbb{1}}$.

  That is, a finite-dimensional algebraic representation E of $\bar{\mathfrak{P}}_{\mathbb{1}}$ is called *$GL(V/\mathbb{C}\mathbb{1})$-polynomial* if the Jordan-Holder components of E are $GL(V/\mathbb{C}\mathbb{1})$-polynomial representations of $\bar{\mathfrak{P}}_{\mathbb{1}}$.

  We denote the category of $GL(V/\mathbb{C}\mathbb{1})$-polynomial representations of $\bar{\mathfrak{P}}_{\mathbb{1}}$ by $Rep(\bar{\mathfrak{P}}_{\mathbb{1}})_{GL(V/\mathbb{C}\mathbb{1})-poly}$.

We are now ready to give a definition of the parabolic category O which we are going to consider:

Definition 3.3.0.17. We define the parabolic category O for $(\mathfrak{gl}(V), \mathfrak{p})$, denoted by $O_V^{\mathfrak{p}}$, to be the full subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ whose objects M satisfy the following conditions:

- M is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \bar{\mathfrak{P}}_{\mathbb{1}})$, i.e. the action of the Lie subalgebra $\mathfrak{p}_{\mathbb{C}\mathbb{1}}$ on M integrates to the action of the mirabolic group $\bar{\mathfrak{P}}_{\mathbb{1}}$.

Furthermore, we require that as a representation of $\bar{\mathfrak{P}}_{\mathbb{1}}$, M be a filtered colimit of $GL\,(V/\mathbb{C}\mathbb{1})$-polynomial representations, i.e.

$$M|_{\bar{\mathfrak{P}}_{\mathbb{1}}} \in Ind - Rep(\bar{\mathfrak{P}}_{\mathbb{1}})_{GL(V/\mathbb{C}\mathbb{1})-poly}$$

- M is a finitely generated $\mathcal{U}(\mathfrak{gl}(V))$-module.

We will also use the notation $Ind - O_V^{\mathfrak{p}}$ to denote the Ind-completion of $O_V^{\mathfrak{p}}$ (i.e. full subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ whose objects M satisfy the first of the above conditions).

When the space V is fixed, we will sometimes omit the subscript V and write $O^{\mathfrak{p}}$ for short.

We now fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$. The Lie algebra $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$ can then be expressed as $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}} \cong \mathfrak{gl}(U) \ltimes U^*$, and we have: $\mathfrak{u}_{\mathfrak{p}}^+ \cong U^*$, $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}} \cong \mathfrak{u}_{\mathfrak{p}}^+ \oplus \mathfrak{gl}(U)$.

Moreover, in that case we have a splitting $\mathfrak{gl}(V) \cong \mathfrak{p} \oplus \mathfrak{u}_{\mathfrak{p}}^-$, where $\mathfrak{u}_{\mathfrak{p}}^- \cong U$. This gives us an analogue of the triangular decomposition:

$$\mathfrak{gl}(V) \cong \mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{u}_{\mathfrak{p}}^- \oplus \mathfrak{u}_{\mathfrak{p}}^+ \oplus \mathfrak{gl}(U)$$

We can now rewrite the above definition of the parabolic category O (compare with the usual definition in [H, Section 9.3]):

Definition 3.3.0.18. We define the parabolic category O for $(\mathfrak{gl}(V), \mathfrak{p})$, denoted by $O_V^{\mathfrak{p}}$, to be the full subcategory of $Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ whose objects M satisfy the following conditions:

- Viewed as a $\mathcal{U}(\mathfrak{gl}(U))$-module, M is a direct sum of polynomial simple $\mathcal{U}(\mathfrak{gl}(U))$-modules (that is, M belongs to $Ind - Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$).
- M is locally finite over $\mathfrak{u}_{\mathfrak{p}}^+$.
- M is a finitely generated $\mathcal{U}(\mathfrak{gl}(V))$-module.

Remark 3.3.0.19. One can replace the requirement that $\mathfrak{u}_{\mathfrak{p}}^+$ act locally finitely on M by the requirement that $\mathcal{U}(\mathfrak{u}_{\mathfrak{p}}^+)$ act locally nilpotently on M.

The next propositions are based on [H, Section 9.3] as well:

Proposition 3.3.0.20. *The category* $O_V^{\mathfrak{p}}$ *(resp.* $Ind - O_V^{\mathfrak{p}}$*) is closed under taking direct sums, submodules, quotients and extensions in* $O_{\mathfrak{gl}(V)}$*, as well as tensoring with finite dimensional* $\mathfrak{gl}(V)$*-modules.*

Recall that in the category O we have the notion of a duality (c.f. [H, Section 3.2]): namely, given a $\mathfrak{gl}(V)$-module M with finite-dimensional weight spaces, we can consider the twisted action of $\mathfrak{gl}(V)$ on the dual space M^*, given by $A.f := f \circ A^T$, where A^T means the transpose of $A \in \mathfrak{gl}(V)$. This gives us a $\mathfrak{gl}(V)$-module M^*. We then take $M^\vee$ to be the maximal submodule of M^* lying in the category O. The module $M^\vee$ is called the dual of M in O, and we get an exact functor $(\cdot)^\vee : O \to O^{op}$.

Proposition 3.3.0.21. *The category* $O_V^{\mathfrak{p}}$ *is closed under taking duals, and the duality functor* $(\cdot)^\vee : O_V^{\mathfrak{p}} \to \left(O_V^{\mathfrak{p}}\right)^{op}$ *is an equivalence of categories.*

Definition 3.3.0.22. A module M over the Lie algebra $\mathfrak{gl}(V)$ will be said to be of degree $K \in \mathbb{C}$ if $\mathrm{Id}_V \in \mathfrak{gl}(V)$ acts by $K\,\mathrm{Id}_M$ on M.

We will denote by $O_{\nu,V}^{\mathfrak{p}}$ the full subcategory of $O_V^{\mathfrak{p}}$ whose objects are modules of degree ν.

Note that for a fixed decomposition $V = \mathbb{C}\mathbb{1} \oplus U$, for a module M of $O_V^{\mathfrak{p}}$ to be of degree ν is the same as to require that $\mathrm{Id}_{\mathbb{C}\mathbb{1}} \in \mathfrak{gl}(V)$ acts on each subspace $S^\lambda U$ of M by the scalar $\nu - |\lambda|$.

Definition 3.3.0.23. Let $\nu \in \mathbb{C}$. Define the functor $deg_\nu : Mod_{\mathcal{U}(\mathfrak{gl}(V))} \to Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ by putting $deg_\nu(E)$ to be the maximal submodule of E of degree ν (see Definition 3.3.0.22). For a morphism $f : E \to E'$ of $\mathfrak{gl}(V)$-modules, we put $deg_\nu(f) := f|_{deg_\nu(E)}$.

Let $E \in Mod_{\mathcal{U}(\mathfrak{gl}(V))}$. The maximal submodule of E of degree ν is well-defined: it is the subspace of E consisting of all vectors on which Id_V acts by the scalar ν, and it is a $\mathfrak{gl}(V)$-submodule since Id_V lies in the center of $\mathfrak{gl}(V)$.

One can show that the functor $deg_\nu : Mod_{\mathcal{U}(\mathfrak{gl}(V))} \to Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ is left-exact. Moreover, it is easy to show that the category $O^{\mathfrak{p}}_{\nu,V}$ is a direct summand of $O^{\mathfrak{p}}_V$, and the functor $deg_\nu : O^{\mathfrak{p}}_V \to O^{\mathfrak{p}}_{\nu,V}$ is exact.

3.3.1 Structure of the category $O^{\mathfrak{p}}_{\nu,V}$

In this subsection, we present some basic facts about the parabolic category O for $(\mathfrak{gl}(V), \mathfrak{p})$ of degree ν and the indecomposable objects inside it.

Fix $\nu \in \mathbb{C}$, and let $(V, \mathbb{1})$ be a finite-dimensional unital vector space with a fixed splitting $V = \mathbb{C}\mathbb{1} \oplus U$.

Definition 3.3.1.1. Let λ be a Young diagram.

$M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ is defined to be the $\mathfrak{gl}(V)$-module

$$\mathcal{U}(\mathfrak{gl}(V)) \otimes_{\mathcal{U}(\mathfrak{p})} S^\lambda U$$

where $\mathfrak{gl}(U)$ acts naturally on $S^\lambda U$, $\mathrm{Id}_V \in \mathfrak{p}$ acts on $S^\lambda U$ by scalar ν, and $\mathfrak{u}^+_{\mathfrak{p}}$ acts on $S^\lambda U$ by zero.

Thus $M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ is the parabolic Verma module for $(\mathfrak{gl}(V), \mathfrak{p})$ with highest weight $(\nu - |\lambda|, \lambda)$ iff $\dim V - 1 \geq \ell(\lambda)$, and zero otherwise.

We will sometimes refer to the parabolic Verma modules as "standard modules".

Definition 3.3.1.2. $L(\nu - |\lambda|, \lambda)$ is defined to be zero (if $\dim V - 1 < \ell(\lambda)$), or the simple module for $\mathfrak{gl}(V)$ of highest weight $(\nu - |\lambda|, \lambda)$ otherwise.

The following basic lemma will be very helpful:

Lemma 3.3.1.3. *Let λ such that $\ell(\lambda) \leq \dim V - 1$. We then have an isomorphism of $\mathfrak{gl}(U)$-modules:*

$$M_{\mathfrak{p}}(\nu - |\lambda|, \lambda) \cong SU \otimes S^\lambda U$$

Proof. Follows directly from the definition of $M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ and the PBW theorem for $\mathfrak{gl}(V)$. □

Proposition 3.3.1.4.

1. *Let μ, τ be two Young diagrams. Then*

$$\dim \mathrm{Hom}_{O^{\mathfrak{p}}_{\nu,V}}(M_{\mathfrak{p}}(\nu - |\mu|, \mu), M_{\mathfrak{p}}(\nu - |\tau|, \tau)) = 0$$

   if μ, τ lie in different $\overset{\nu}{\sim}$-classes.

2. *Fix a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}$, $\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset ....$ For any $i, j \in \mathbb{Z}_+$, we have*

$$\dim \mathrm{Hom}_{O^{\mathfrak{p}}_{\nu,V}}\left(M_{\mathfrak{p}}(\nu - \left|\lambda^{(j)}\right|, \lambda^{(j)}), M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})\right) = 0 \ \text{ if } j \neq i, i+1, \text{ and}$$
$$\dim \mathrm{Hom}_{O^{\mathfrak{p}}_{\nu,V}}\left(M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}), M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})\right) = 1 \ \text{ if } \dim V - 1 \geq \ell(\lambda^{(i+1)})$$

3. *For any Young diagram λ such that $\dim V - 1 \geq \ell(\lambda)$, we have:*

$$\dim \mathrm{End}_{O^{\mathfrak{p}}_{\nu,V}}(M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)) = 1$$

Proof. 1. Consider a $\mathfrak{gl}(V)$-morphism $M_{\mathfrak{p}}(\nu - |\mu|, \mu) \longrightarrow M_{\mathfrak{p}}(\nu - |\tau|, \tau)$, and assume it is not zero. Then the weights $(\nu - |\mu|, \mu)$, $(\nu - |\tau|, \tau)$ are W-linked, i.e. there exists an element w in the Weyl group such that $w((\nu - |\mu|, \mu) + \rho) = (\nu - |\tau|, \tau) + \rho$, where $\rho = (\dim V, \dim V - 1, \dim V - 2, ...) = \dim V(1, 1, 1, ...) - (1, 2, 3, ...)$.

This is equivalent to saying that $w(\nu - |\mu|, \mu_1 - 1, \mu_2 - 2, ...) = (\nu - |\tau|, \tau_1 - 1, \tau_2 - 2, ...)$, which means that μ, τ lie in the same $\overset{\nu}{\sim}$-class (in fact, we get that $w = (1, 2, ..., k) \in S_{\dim V} = Weyl(\mathfrak{gl}(V))$ for some $k > 1$).

2. Consider a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i, \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset ...$ (recall that this can occur only if $\nu \in \mathbb{Z}_+$). Let $i, j \geq 0$, and assume there is a non-zero $\mathfrak{gl}(V)$-morphism $M_{\mathfrak{p}}(\nu - \left|\lambda^{(j)}\right|, \lambda^{(j)}) \longrightarrow M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$.

Frobenius reciprocity then gives us a morphism of $\mathfrak{gl}(U)$-modules:

$$S^{\lambda^{(j)}} U \longrightarrow M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})|_{\mathfrak{gl}(U)}$$

By Lemma 3.3.1.3, the $\mathfrak{gl}(U)$-module $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})|_{\mathfrak{gl}(U)}$ is either zero (if $\dim(U) = \dim V - 1 < \ell(\lambda^{(i)})$), or isomorphic to

$$SU \otimes S^{\lambda^{(i)}}U \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^{\mu}U$$

We immediately conclude that $\lambda^{(i)} \subset \lambda^{(j)}$ (which means that $i \leq j$), and that $\lambda^{(j)} \in \mathcal{I}^+_{\lambda^{(i)}}$. In fact, Lemma 3.2.2.2 implies that $j = i+1$ in that case, since for $j \geq i+2$, we get: $\lambda^{(j)}_k = \lambda^{(i)}_k + 1$ for any $k = i+2, ..., j$, contradicting $\lambda^{(j)} \in \mathcal{I}^+_{\lambda^{(i)}}$.

It remains to check that

$$\dim \operatorname{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\nu,V}} \left(M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}), M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})\right) = 1 \text{ if } \dim V - 1 \geq \ell(\lambda^{(i+1)})$$

We start by noticing that the same Frobenius reciprocity argument used above guarantees us that

$$\dim \operatorname{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\nu,V}} \left(M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}), M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})\right) \leq 1$$

so we only need to check that if $\dim V - 1 \geq \ell(\lambda^{(i+1)})$, then there exists a non-zero morphism

$$M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \longrightarrow M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$$

This statement can be proved by induction on $i \geq 0$.

Base: Assume $\dim V - 1 \geq \ell(\lambda^{(0)})$. We need to check that $M_{\mathfrak{p}}(\nu - \left|\lambda^{(0)}\right|, \lambda^{(0)})$ is not simple, i.e. isn't equal to $L(\nu - \left|\lambda^{(0)}\right|, \lambda^{(0)})$. But the latter is finite-dimensional (since $(\nu - \left|\lambda^{(0)}\right|, \lambda^{(0)})$ is an integral dominant weight), while $M_{\mathfrak{p}}(\nu - \left|\lambda^{(0)}\right|, \lambda^{(0)})$ clearly isn't finite-dimensional (due to Lemma 3.3.1.3, for example).

Step: Let $i \geq 1$, and assume $\dim V - 1 \geq \ell(\lambda^{(i+1)})$. If there exists a non-zero morphism $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \longrightarrow M_{\mathfrak{p}}(\nu - \left|\lambda^{(i-1)}\right|, \lambda^{(i-1)})$, then this morphism is not injective (can be seen from Lemma 3.3.1.3), therefore $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ is not simple; so there exists a non-zero morphism $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \longrightarrow M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$, as needed.

3. This statement follows immediately from Lemma 3.3.1.3, which gives us an isomorphism of of $\mathfrak{gl}(U)$-modules:

$$M_{\mathfrak{p}}(\nu - |\lambda|, \lambda) \cong SU \otimes S^{\lambda}U \cong \bigoplus_{\mu \in \mathcal{I}^{+}_{\lambda}} S^{\mu}U$$

□

The previous proposition immediately implies:

Corollary 3.3.1.5. *Let λ lie in a trivial $\stackrel{\nu}{\sim}$-class. Then $M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ is either zero (iff $\dim V - 1 < \ell(\lambda)$), or a simple $\mathfrak{gl}(V)$-module. In particular, if $\nu \notin \mathbb{Z}_{+}$, this is true for any Young diagram λ.*

Proof. Recall that since $\nu \notin \mathbb{Z}_{+}$, each Young diagram λ lies in a trivial $\stackrel{\nu}{\sim}$-class (see Lemma 3.2.2.2). The result follows from Proposition 3.3.1.4. □

Remark 3.3.1.6. Note that Proposition 3.3.1.4 implies that the category $O^{\mathfrak{p}}_{\nu,V}$ decomposes into blocks (each of the blocks is an abelian category in its own right). To each $\stackrel{\nu}{\sim}$-class of Young diagrams corresponds a block of $O^{\mathfrak{p}}_{\nu,V}$ (if for all Young diagrams λ in this $\stackrel{\nu}{\sim}$-class, $\ell(\lambda) > \dim V - 1$, then the corresponding block is zero), and to each non-zero block of $O^{\mathfrak{p}}_{\nu,V}$ corresponds a unique $\stackrel{\nu}{\sim}$-class.

Proposition 3.3.1.4 also implies that the block corresponding to a trivial $\stackrel{\nu}{\sim}$-class is either semisimple (i.e. equivalent to the category $Vect_{\mathbb{C}}$), or zero.

Now fix a non-trivial $\stackrel{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$, and $i \geq 0$ such that $\ell(\lambda^{(i+1)}) \leq \dim V - 1$.

Proposition 3.3.1.4 implies that the maximal non-trivial submodule of $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ is $L(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)})$. We conclude that

Corollary 3.3.1.7. *Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\stackrel{\nu}{\sim}$-class, and $i \geq 0$ be such that $\ell(\lambda^{(i)}) \leq \dim V - 1$.*

Then there is a short exact sequence

$$0 \to L(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to 0$$

Corollary 3.3.1.8. *The isomorphism classes of the generalized Verma modules and the simple polynomial modules in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ form a basis for the Grothendieck group of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.*

Remark 3.3.1.9. Notice that for $i := \max\{i \geq 0 \mid \ell(\lambda^{(i)}) \leq \dim V - 1\}$, we have

$$M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \cong M^{\vee}_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \cong L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$$

We also obtain the BGG resolution in category $O^{\mathfrak{p}}_{\nu,V}$ as an immediate corollary:

Corollary 3.3.1.10. *Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\overset{\nu}{\sim}$-class. Then there is a long exact sequence of $\mathfrak{gl}(V)$-modules (BGG resolution of $L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ by parabolic Verma modules)*

$$\begin{aligned} \ldots \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+2)}\right|, \lambda^{(i+2)}) \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to \\ \to L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to 0 \end{aligned}$$

Proof. Follows immediately from Corollary 3.3.1.7. □

Remark 3.3.1.11. For $i = 0$, such a resolution is a special case of BGG resolutions in parabolic category O discussed in [H, Chapter 9, Par. 16].

We now consider the projective cover $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ of $L(\nu - |\lambda|, \lambda)$ in $O^{\mathfrak{p}}_{\nu,V}$. The existence of $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ and some of its properties are listed in the following proposition:

Proposition 3.3.1.12.

(a) Category $O^{\mathfrak{p}}_{\nu,V}$ has enough projectives; in particular, there exists a projective cover of $L(\nu - |\lambda|, \lambda)$, which will be denoted by $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$.

(b) For any Young diagram λ, the following equality holds:

$$\dim \operatorname{Hom}_{O^{\mathfrak{p}}_{\nu,V}}(P_{\mathfrak{p}}(\nu - |\lambda|, \lambda), M) = [M : L(\nu - |\lambda|, \lambda)]$$

(c) The projective module $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ is indecomposable and standardly filtered (i.e. has a filtration where all the successive quotients are parabolic Verma modules).

(d) (BGG reciprocity) The following equality holds for any Young diagrams λ, μ:

$$(P_{\mathfrak{p}}(\nu - |\lambda|, \lambda) : M_{\mathfrak{p}}(\nu - |\mu|, \mu)) = [M_{\mathfrak{p}}(\nu - |\mu|, \mu) : L(\nu - |\lambda|, \lambda)]$$

(the brackets in left hand side denote multiplicity in the standard filtration).

(e) The duality functor $(\cdot)^{\vee} : O^{\mathfrak{p}}_{\nu,V} \to \left(O^{\mathfrak{p}}_{\nu,V}\right)^{op}$ takes projective modules to injective modules and vice versa. In particular, there are enough injectives in the category $O^{\mathfrak{p}}_{\nu,V}$, and the indecomposable injective modules are exactly $P^{\vee}_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ (which is the injective hull of $L(\nu - |\lambda|, \lambda)$).

(f) Whenever λ is not the minimal Young diagram in a non-trivial $\overset{\nu}{\sim}$-class, the modules $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ are self-dual and therefore injective. In these cases the following equality holds:

$$\dim \operatorname{Hom}_{O^{\mathfrak{p}}_{\nu,V}}(M, P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)) = [M : L(\nu - |\lambda|, \lambda)]$$

Proof. The proofs of (a) - (e) can be found in [H, Chapter 9, Par. 8 and Chapter 3, Par. 9-11]; the proof of the first part of (f) is based on [H, Chapter 9, Par. 14] and on Corollary 3.3.1.7. The equality in (f) can then be proved in the same way as the equality in (b). □

We can now describe the standard filtration of the indecomposable projectives $P_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$, and their other useful properties:

Proposition 3.3.1.13.

(a) Assume λ lies in a trivial $\overset{\nu}{\sim}$-class. Then

$$P_{\mathfrak{p}}(\nu - |\lambda|, \lambda) \cong M_{\mathfrak{p}}(\nu - |\lambda|, \lambda) = L(\nu - |\lambda|, \lambda)$$

(b) Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\overset{\nu}{\sim}$-class. Then

$$P_{\mathfrak{p}}(\nu - |\lambda^{(0)}|, \lambda^{(0)}) \cong M_{\mathfrak{p}}(\nu - |\lambda^{(0)}|, \lambda^{(0)})$$

(c) Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\overset{\nu}{\sim}$-class and let $i \geq 1$. Then for i such that $\ell(\lambda^{(i)}) \leq \dim V - 1$, we have short exact sequences

$$0 \to M_{\mathfrak{p}}(\nu - |\lambda^{(i-1)}|, \lambda^{(i-1)}) \to P_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to M_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0$$

$$0 \to M_{\mathfrak{p}}^{\vee}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to P_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to M_{\mathfrak{p}}^{\vee}(\nu - |\lambda^{(i-1)}|, \lambda^{(i-1)}) \to 0$$

and the socle filtration of $P_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)})$ has successive quotients

$$L(\nu - |\lambda^{(i)}|, \lambda^{(i)}); L(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}) \oplus L(\nu - |\lambda^{(i-1)}|, \lambda^{(i-1)}); L(\nu - |\lambda^{(i)}|, \lambda^{(i)})$$

For i such that $\ell(\lambda^{(i)}) > \dim V - 1$,

$$P_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) = M_{\mathfrak{p}}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) = M_{\mathfrak{p}}^{\vee}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) = L(\nu - |\lambda^{(i)}|, \lambda^{(i)}) = 0$$

(d) Let $\{\lambda^{(i)}\}_i$ be a non-trivial $\overset{\nu}{\sim}$-class, and let $i \geq 1, j \geq 0$. Then

$$\dim \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\nu,V}}\left(P_{\mathfrak{p}}(\nu - \left|\lambda^{(j)}\right|, \lambda^{(j)}), P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})\right) = \begin{cases} 2 & \text{if } i = j, \\ & \ell(\lambda^{(i)}) \leq \dim V - 1 \\ 1 & \text{if } |i-j| = 1, \\ & \ell(\lambda^{(i)}), \ell(\lambda^{(j)}) \leq \dim V - 1 \\ 0 & \text{else} \end{cases}$$

Proof. Parts (a), (b) follow directly from the fact that $P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ is standardly filtered, the BGG reciprocity (see Proposition 3.3.1.12) and Proposition 3.3.1.4. The BGG reciprocity also implies that for a non-trivial $\overset{\nu}{\sim}$-class, denoted by $\{\lambda^{(i)}\}_i$, we have the short exact sequence

$$0 \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i-1)}\right|, \lambda^{(i-1)}) \to P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to 0$$

whenever $i \geq 1$. Taking duals in of the modules in this sequence, we obtain a short exact sequence

$$0 \to M^{\vee}_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to M^{\vee}_{\mathfrak{p}}(\nu - \left|\lambda^{(i-1)}\right|, \lambda^{(i-1)}) \to 0$$

To compute the socle filtration, notice that

$$Soc(P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})) \cong L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$$

since

$$\dim \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\nu,V}}(L(\nu - \left|\lambda^{(j)}\right|, \lambda^{(j)}), P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})) = [L(\nu - \left|\lambda^{(j)}\right|, \lambda^{(j)}) : L(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})] = \delta_{i,j}$$

(see Proposition 3.3.1.12(f)).

The short exact sequences above then imply that

$$Soc^2(P_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})) \cong L(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \oplus L(\nu - \left|\lambda^{(i-1)}\right|, \lambda^{(i-1)})$$

The socle filtration description then follows, and this proves Part (c).

The dimensions of the Hom-spaces between the indecomposable projectives can be inferred from the socle filtration and Proposition 3.3.1.12(b, f), which proves Part (d). □

Chapter 4

Complex tensor powers of a unital object

In this chapter, we fix a finite-dimensional unital vector space $(V, \mathbb{1})$. The goal of this chapter is to define an object $V^{\underline{\otimes}\nu}$ which will be an interpolation of the tensor powers $V^{\otimes n}$ for $n \in \mathbb{Z}_+$ to arbitrary $\nu \in \mathbb{C}$.

4.1 Description of the setting

In this subsection we will describe the category $Ind - (\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$, which will be used to define the complex tensor power of the vector space V.

The notation $\boxtimes$ stands for the Deligne tensor product of abelian categories (c.f. [D1, Section 5]); in this subsection, we will explain that this category can also be described as the category of Ind-objects of $\underline{Rep}^{ab}(S_\nu)$ carrying an "$O^{\mathfrak{p}}_{\nu,V}$-type" action of $\mathfrak{gl}(V)$.

Fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$.

Definition 4.1.0.14. Let $X \in Ind - \underline{Rep}^{ab}(S_\nu)$. We say that $\mathfrak{gl}(V)$ acts on X if given a homomorphism of $\mathbb{C}$-algebras

$$\rho_X : \mathcal{U}(\mathfrak{gl}(V)) \to \mathrm{End}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X)$$

We say that this action is an $O^{\mathfrak{p}}_{\nu,V}$-action if:

- $\mathfrak{gl}(U) \subset \mathfrak{gl}(V)$ acts polynomially on X, i.e. X decomposes as a direct sum $X \cong \bigoplus_{i\in I} Y_i \otimes E_i$ in $Ind - \underline{Rep}^{ab}(S_\nu)$, with

  1. $Y_i \in Ind - \underline{Rep}^{ab}(S_\nu)$,
  2. E_i being polynomial $\mathfrak{gl}(U)$-modules,

  and the following commutative diagram holds for any $a \in \mathcal{U}(\mathfrak{gl}(U))$:

$$\begin{array}{ccc} X & \longrightarrow & \bigoplus_{i\in I} Y_i \otimes E_i \\ {\scriptstyle \rho_X(a)}\downarrow & & \downarrow{\scriptstyle \oplus_i a|_{E_i}} \\ X & \longrightarrow & \bigoplus_{i\in I} Y_i \otimes E_i \end{array}$$

- $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}})$ acts locally finitely on X, i.e. for any $Y \in \underline{Rep}^{ab}(S_\nu)$, $f : Y \to X$, we have:

$$\sum_{a\in\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}})} (\rho_X(a) \circ f)(Y)$$

  belongs to $\underline{Rep}^{ab}(S_\nu)$ (i.e. is a compact object).

- Id_V acts on X by the morphism $\nu \cdot \mathrm{Id}_X$.

The category $Ind - (\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ is the category of pairs (X, ρ_X) where $X \in Ind-\underline{Rep}^{ab}(S_\nu)$ and ρ_X is an $O^{\mathfrak{p}}_{\nu,V}$-action on X. The morphisms in $Ind-(\underline{Rep}^{ab}(S_\nu)\boxtimes O^{\mathfrak{p}}_{\nu,V})$ are

$$\mathrm{Hom}((X,\rho_X),(Y,\rho_Y)) := \{f \in \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X,Y) \mid f\circ\rho_X(a) = \rho_Y(a)\circ f\, \forall a \in \mathcal{U}(\mathfrak{gl}(V))\}$$

Remark 4.1.0.15. Inside the category $Ind-(\underline{Rep}^{ab}(S_\nu)\boxtimes O^{\mathfrak{p}}_{\nu,V})$ we have the full subcategory $Ind - (\underline{Rep}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ whose objects are (X, ρ_X) where $X \in Ind - \underline{Rep}(S_\nu)$ and ρ_X is an $O^{\mathfrak{p}}_{\nu,V}$-action on X.

Now let $\nu = n \in \mathbb{Z}_+$.

By [KS, Corollary 6.3.2], there exists a unique (up to unique isomorphism) functor

$$\hat{\mathcal{S}}_n : (Ind - \underline{Rep}(S_{\nu=n})) \longrightarrow Ind - Rep(S_n)$$

which commutes with (small) filtered colimits and satisfies

$$\hat{\mathcal{S}}_n \circ \iota_{\underline{Rep}(S_\nu) \to (Ind - \underline{Rep}(S_\nu))} \cong \mathcal{S}_n$$

(the notation is the same as in Chapter 2 and Subsection 3.2.1).

Now consider the category $Ind - (Rep(S_n) \boxtimes O^{\mathfrak{p}}_{n,V})$; it is the full subcategory of the category of modules $Mod_{\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))}$ whose objects M satisfy: as a $\mathbb{C}[S_n]$-module, M is a direct sum of finite dimensional simple modules, while as a $\mathcal{U}(\mathfrak{gl}(V))$-module, $M \in O^{\mathfrak{p}}_{n,V}$.

We can define the functor

$$\hat{\mathcal{S}}_n : Ind - (\underline{Rep}(S_{\nu=n}) \boxtimes O^{\mathfrak{p}}_{n,V}) \longrightarrow Ind - (Rep(S_n) \boxtimes O^{\mathfrak{p}}_{n,V})$$

by setting $\hat{\mathcal{S}}_n(X, \rho_X) := \hat{\mathcal{S}}_n(X)$ (with action of $\mathfrak{gl}(V)$ given by $\hat{\mathcal{S}}_n(\rho_X)$). The above description of $Ind - (\underline{Rep}(S_{\nu=n}) \boxtimes O^{\mathfrak{p}}_{n,V})$ guarantees that this functor is well-defined.

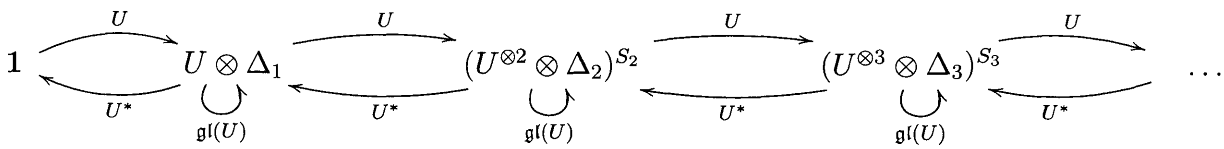
4.2 Definition of a complex tensor power: split unital vector space

Fix $\nu \in \mathbb{C}$, and fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$.

Definition 4.2.0.16 (Complex tensor power). Define the object $V^{\underline{\otimes}\nu}$ of $Ind - (\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ by setting

$$V^{\underline{\otimes}\nu} := \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k}$$

The action on $\mathfrak{gl}(V)$ on $V^{\underline{\otimes}\nu}$ is given as follows:



- $\mathrm{Id}_V \in \mathfrak{gl}(V)$ acts by scalar ν,

- $u \in U \cong \mathfrak{u}_{\mathfrak{p}}^-$ acts by operator

$$(U^{\otimes k} \otimes \Delta_k)^{S_k} \xrightarrow{F_u} (U^{\otimes k+1} \otimes \Delta_{k+1})^{S_{k+1}}$$
$$F_u := \frac{1}{k+1} \sum_{1 \le l \le k+1} u^{(l)} \otimes res_l^*$$

Here $res_l^* : \Delta_k \to \Delta_{k+1}$ as in Definition 3.2.3.13, $u^{(l)}$ as in Notation 2.0.6.1 and $k \geq 0$.

- $f \in U^* \cong \mathfrak{u}_{\mathfrak{p}}^+$ acts by operator

$$(U^{\otimes k} \otimes \Delta_k)^{S_k} \xrightarrow{E_f} (U^{\otimes k-1} \otimes \Delta_{k-1})^{S_{k-1}}$$
$$E_f := \sum_{1 \le l \le k} f^{(l)} \otimes res_l$$

Here $res_l : \Delta_k \to \Delta_{k-1}$ as in Definition 3.2.3.12, $f^{(l)}$ as in Notation 2.0.6.1 and $k \geq 1$. The action of f on $(U^{\otimes 0} \otimes \mathbf{1})^{S_0} \cong \mathbf{1}$ is zero.

- $A \in \mathfrak{gl}(U) \subset \mathfrak{gl}(V)$ acts on $(U^{\otimes k} \otimes \Delta_k)^{S_k}$ by

$$\sum_{1 \le i \le k} A^{(i)}|_{U^{\otimes k}} \otimes \mathrm{Id}_{\Delta_k} : (U^{\otimes k} \otimes \Delta_k)^{S_k} \longrightarrow (U^{\otimes k} \otimes \Delta_k)^{S_k}$$

Lemma 4.2.0.17. *The action of $\mathfrak{gl}(V)$ described in Definition 4.2.0.16 is well-defined.*

Proof. See Subsection 5.2.1. □

Remark 4.2.0.18. The actions of the elements of $\mathfrak{u}_\mathfrak{p}^+$, $\mathfrak{u}_\mathfrak{p}^-$ are in fact uniquely determined by the actions of Id_V and $\mathfrak{gl}(U)$.

To see this, note that the ideal in the Lie algebra $\mathfrak{gl}(V)$ generated by the Lie subalgebra $\mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{gl}(U_N)$ is the entire $\mathfrak{gl}(V)$. Given two $\mathfrak{gl}(V)$-modules M_1, M_2 and an isomorphism $M_1 \to M_2$ which is equivariant with respect to the Lie subalgebra $\mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{gl}(U)$, the above fact implies that this isomorphism is also $\mathfrak{gl}(V)$-equivariant.

In other words, if there exists a way to define an action of $\mathfrak{gl}(V)$ whose restriction to the the Lie subalgebra $\mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{gl}(U)$ is given by the formulas above, then such an action of $\mathfrak{gl}(V)$ is unique.

Remark 4.2.0.19. Notice that $\mathfrak{gl}(U)$ acts semisimply on $V^{\underline{\otimes}\nu}$, $U^* \cong \mathfrak{u}_\mathfrak{p}^+$ acts locally finitely on $V^{\underline{\otimes}\nu}$, thus making it an object of $Ind - (\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ (in fact, an object of the category $Ind - (\underline{Rep}(S_{\nu=n})) \boxtimes O^{\mathfrak{p}}_{n,V})$, since $\Delta_k \in \underline{Rep}(S_\nu)$ for any $k \in \mathbb{Z}_+$).

Remark 4.2.0.20. The definition of $V^{\underline{\otimes}\nu}$ makes it a $\mathbb{Z}_+$-graded object in $Ind - \underline{Rep}(S_\nu)$. This grading corresponds to the natural grading on $V^{\otimes n}$ seen as a tensor power of the graded object $V = \mathbb{C}\mathbb{1} \oplus U, gr_0(V) := \mathbb{C}\mathbb{1}, gr_1(V) := U$.

Remark 4.2.0.21. In Subsection 4.4, we will show that the object $V^{\underline{\otimes}\nu}$ of $Ind-(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ does not really depend on the splitting of V, but rather only on the pair $(V, \mathbb{1})$. In the case when $\nu \notin \mathbb{Z}_+$, one can actually give an equivalent definition of $V^{\underline{\otimes}\nu}$ without using a splitting (c.f. [Et1] and Section 4.5).

The following technical lemma will be useful to us later on. The meaning of this lemma is that the operator F_u acting on $V^{\underline{\otimes}\nu}$ is "almost injective".

Lemma 4.2.0.22. *Let $l \leq k$, and consider a non-zero morphism in $\underline{Rep}(S_\nu)$*

$$\phi : U^{\otimes l} \otimes \Delta_l \longrightarrow U^{\otimes k} \otimes \Delta_k$$

Let $u \in U \cong \mathfrak{u}_\mathfrak{p}^-, u \neq 0$. Then $F_u \circ \phi \neq 0$, where $F_u \circ \phi := \frac{1}{k+1} \sum_{1 \leq l \leq k+1} (u^{(l)} \otimes res_l^*) \circ \phi$.

Proof. Will be proved in Section 5.2.2, Lemma 5.2.2.1. □

4.3 Compatablility of the definitions of the complex and the integer tensor powers

Finally, we prove that the definition of a complex tensor power of a split unital vector space is compatible with the usual notion of a tensor power of a vector space.

We continue with a fixed splitting $V = \mathbb{C}\mathbb{1} \oplus U$. Let $V^{\underline{\otimes} n}$ be as in Definition 4.2.0.16.

Define the action of $\mathfrak{gl}(V)$ on the space $\bigoplus_{k=0,...,n}(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$ via the decomposition

$$\mathfrak{gl}(V) \cong \mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{u}_\mathfrak{p}^- \oplus \mathfrak{u}_\mathfrak{p}^+ \oplus \mathfrak{gl}(U)$$

by setting

- Id_V acts by the scalar n,

- $u \in U \cong \mathfrak{u}_\mathfrak{p}^-$ acts by operator

$$(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k} \xrightarrow{F_u} (U^{\otimes k+1} \otimes \mathbb{C}Inj(\{1,...,k+1\},\{1,...,n\}))^{S_{k+1}}$$
$$F_u = \frac{1}{k+1} \sum_{1 \leq l \leq k+1} u^{(l)} \otimes \mathtt{res}_l^*$$

  for any $k \geq 0$. Here $\mathtt{res}_l^*$ is the map defined in Remark 3.2.3.14, $u^{(l)}$ as in Notation 2.0.6.1.

- $f \in U^* \cong \mathfrak{u}_\mathfrak{p}^+$ acts by operator

$$(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k} \xrightarrow{E_f} (U^{\otimes k-1} \otimes \mathbb{C}Inj(\{1,...,k-1\},\{1,...,n\}))^{S_{k-1}}$$
$$E_f = \sum_{1 \leq l \leq k} f^{(l)} \otimes \mathtt{res}_l$$

  whenever $k \geq 1$. Here $\mathtt{res}_l$ is the map defined in Remark 3.2.3.14, $f^{(l)}$ as in Notation 2.0.6.1.

  The action of f on $(U^{\otimes 0} \otimes \mathbb{C})^{S_0} \cong \mathbb{C}$ is zero.

- $\mathfrak{gl}(U)$ acts naturally on each summand $(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$:

  $A \in \mathfrak{gl}(U)$ acts by

$$\sum_{1 \leq i \leq k} A^{(i)}|_{U^{\otimes k}} \otimes \mathrm{Id}_{\mathbb{C}Inj(\{1,...,k\},\{1,...,n\})} :$$
$$(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k} \longrightarrow (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

Notice that the space $\bigoplus_{k=0,...,n}(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$ automatically possesses a structure of a $\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-module: the group S_n acts on each summand $\mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))$ through its action on the set $\{1,...,n\}$.

Lemma 4.3.0.23. *There is an isomorphism of $\mathfrak{gl}(V)$-modules*

$$\Phi : V^{\otimes n} \xrightarrow{\sim} \bigoplus_{k=0,...,n} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

where $\Phi(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}) = 1$ (lies in degree zero of the right hand side).

Moreover, this isomorphism is an isomorphism of $\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-modules.

Proof. Will be proved in Section 5.2.3, Lemma 5.2.3.3. □

Proposition 4.3.0.24. *Consider the functor*

$$\hat{\mathcal{S}}_n : Ind - (\underline{Rep}(S_{\nu=n}) \boxtimes O^{\mathfrak{p}}_{n,V}) \longrightarrow Ind - (Rep(S_n) \boxtimes O^{\mathfrak{p}}_{n,V})$$

(c.f. Subsection 4.1). Then $\hat{\mathcal{S}}_n(V^{\underline{\otimes} n}) \cong V^{\otimes n}$.

Remark 4.3.0.25. Restricting the action of $\mathfrak{gl}(V)$ to $\mathfrak{gl}(U)$, it can be seen from the proof we get an isomorphism of $\mathbb{Z}_+$-graded $\mathfrak{gl}(U)$-modules.

Proof. Recall from Lemma 3.2.3.1 that $\mathcal{S}_n(\Delta_k) = \mathbb{C}Inj(\{1,...,k\},\{1,...,n\})$ (this is zero if $k > n$). By definition of the action of $\mathfrak{gl}(V)$ on $\bigoplus_{k=0,...,n}(U^{\otimes k} \otimes$

$\mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$ given above, we have an isomorphism of $\mathbb{C}[S_n]\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{gl}(V))$-modules

$$\Psi : \hat{\mathcal{S}}_n(V^{\underline{\otimes} n})|_{\mathfrak{gl}(U)} \to \bigoplus_{k=0,...,n} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

Using Lemma 4.3.0.23 above, we obtain the desired result.

□

Example 4.3.0.26. Let $\dim U = 1$. In that case V can be viewed as the tautological representation of $\mathfrak{gl}_2$, with standard basis v_0, v_1. The tensor power $V^{\otimes n}$ is then a span of weight vectors of the form

$$v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n},\ i_1,...,i_n \in \{0,1\}$$

(the weight of this vector is $(n - \sum_{j=1,..,n} i_j, \sum_{j=1,..,n} i_j))$. This allows us to establish an isomorphism

$$V^{\otimes n} \to \bigoplus_{k=0,...,n} \mathbb{C}\{S \subset \{1,...,n\} | \, |S| = k\} \cong \bigoplus_{k=0,...,n} \mathbb{C}Inj(\{1,...,k\},\{1,...,n\})^{S_k}$$

$$v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n} \mapsto S := \{j \in \{1,...,n\} | i_j = 1\}$$

The operators $E \in U^* \cong \mathfrak{u}_{\mathfrak{p}}^+, F \in U \cong \mathfrak{u}_{\mathfrak{p}}^-$ act on $\bigoplus_{k=0,...,n} \mathbb{C}\{S \subset \{1,...,n\} | \, |S| = k\}$ by

$$E(S) = \sum_{T:T\subset S, |T|=k-1} T,\ F(S) = \frac{1}{k+1} \sum_{T:S\subset T\subset \{1,...,n\}, |T|=k+1} T$$

where S is a subset of $\{1,...,n\}$ of size k. The operator $\mathrm{Id}_U \in \mathrm{End}(V)$ acts on $\bigoplus_{k=0,...,n} \mathbb{C}\{S \subset \{1,...,n\} | \, |S| = k\}$ by $S \mapsto |S| \cdot S$.

In particular, we immediately see that

$$\hat{\mathcal{S}}_n(V^{\underline{\otimes} n}) \cong \bigoplus_{k\geq 0} \mathcal{S}_n((\Delta_k)^{S_k}) \cong \bigoplus_{k\geq 0} \mathbb{C}Inj(\{1,...,k\},\{1,...,n\})^{S_k} \cong V^{\otimes n}$$

4.4 Independence of $V^{\underline{\otimes}\nu}$ on the choice of splitting

In this subsection, we show that our definition of $V^{\underline{\otimes}\nu}$ does not depend on the choice of splitting $V = \mathbb{C}\mathbb{1} \oplus U$ we made earlier, but rather only on the pair $(V, \mathbb{1})$.

Consider the following category **Uni** of unital vector spaces. The objects of this category will be tuples $(V, \mathbb{1}, U)$, where V is a finite-dimensional vector space, $\mathbb{1} \in V \setminus \{0\}$, and U is a subspace of V such that $V = \mathbb{C}\mathbb{1} \oplus U$.

The morphisms in this category are given by

$$Mor_{\mathbf{Uni}}\left((V, \mathbb{1}, U), (V', \mathbb{1}', U')\right) := \{\phi \in \mathrm{Hom}_{\mathbb{C}}(V, V') : \phi(\mathbb{1}) = \mathbb{1}'\}$$

Remark 4.4.0.27. This category has a natural structure of a symmetric monoidal tensor category, with

$$(V, \mathbb{1}, U) \otimes (V', \mathbb{1}', U') := (V \otimes V', \mathbb{1} \otimes \mathbb{1}', U \otimes \mathbb{C}\mathbb{1}' \oplus \mathbb{C}\mathbb{1} \otimes U' \oplus U \otimes U')$$

We now construct a functor

$$(\cdot)^{\underline{\otimes}\nu} : \mathbf{Uni} \longrightarrow Ind - \underline{Rep}(S_\nu)$$

On objects of **Uni**, this is just $(V, \mathbb{1}, U) \mapsto V^{\underline{\otimes}\nu} := \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k}$. On morphisms, this functor is

$$\phi : (V, \mathbb{1}, U) \to (V', \mathbb{1}', U') \quad \mapsto \quad \Phi : \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k} \to \bigoplus_{k \geq 0} (U'^{\otimes k} \otimes \Delta_k)^{S_k}$$

with the matrix coefficients $\Phi^{l,k} : (U^{\otimes k} \otimes \Delta_k)^{S_k} \to (U'^{\otimes l} \otimes \Delta_l)^{S_l}$ of Φ coming from maps $U^{\otimes k} \otimes \Delta_k \to U'^{\otimes l} \otimes \Delta_l$ which are defined by the formula

$$\sum_{\substack{\iota : \{1, \ldots, l\} \to \{1, \ldots, k\} \\ \text{strictly increasing}}} \phi_{U,U'}^{(Im(\iota))} \otimes \phi_{U,\mathbb{1}'}^{(\{1,\ldots,k\} \setminus Im(\iota))} \otimes \left(\Delta_k \overset{res_\iota}{\to} \Delta_l \right)$$

Here

- The map $\phi_{U,U'} : U \to U'$ is the composition $U \hookrightarrow V \xrightarrow{\phi} V' \twoheadrightarrow U'$.

  The notation $\phi_{U,U'}^{(Im(\iota))}$ means that we apply the map $\phi_{U,U'}$ only to the factors $\iota(1), \iota(2), ..., \iota(l)$ of $U^{\otimes k}$.

- The map $\phi_{U,\mathbb{1}'} : U \to \mathbb{C}$ is defined so that the composition

$$U \hookrightarrow V \xrightarrow{\phi} V' \twoheadrightarrow \mathbb{C}\mathbb{1}'$$

  is the map $u \mapsto \phi_{U,\mathbb{1}'}(u)\mathbb{1}'$. The notation $\phi_{U,\mathbb{1}'}^{(\{1,...,k\}\setminus Im(\iota))}$ means that we apply the map $\phi_{U,\mathbb{1}'}$ only to those factors i of $U^{\otimes k}$ for which $i \notin Im(\iota)$.

- The map res_ι is the map $\Delta_k \to \Delta_l$ given by the diagram $\pi \in \bar{P}_{k,l}$ with edges $\{(\iota(i), i')\}_{1 \leq i \leq l}$.

Note that Φ is upper-triangular in terms of the matrix coefficients $\Phi^{l,k}$.

Lemma 4.4.0.28. *The functor* $(\cdot)^{\underline{\otimes}\nu} : \mathbf{Uni} \longrightarrow Ind - \underline{Rep}(S_\nu)$ *is well-defined.*

Proof. We only need to check that this functor preserves composition. Indeed, let $\phi : (V, \mathbb{1}, U) \to (V', \mathbb{1}', U')$, $\psi : (V', \mathbb{1}', U') \to (V'', \mathbb{1}'', U'')$, and denote by Φ, Ψ their respective images under the functor $(\cdot)^{\underline{\otimes}\nu}$.

We have: $(\psi \circ \phi)_{U,U''} = \psi_{U',U''} \circ \phi_{U,U'}$, and $(\psi \circ \phi)_{U,\mathbb{1}''} = \psi_{U',\mathbb{1}''} \circ \phi_{U,U'} + \phi_{U,\mathbb{1}'}$.

Thus

$$(\psi \circ \phi)^{\underline{\otimes}\nu} : \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k} \to \bigoplus_{k \geq 0} (U''^{\otimes k} \otimes \Delta_k)^{S_k}$$

is a map in $Ind - \underline{Rep}(S_\nu)$, with matrix coefficients coming from the maps $U^{\otimes k} \otimes \Delta_k \to U''^{\otimes l} \otimes \Delta_l$ given by

$$\sum_{\substack{\iota:\{1,...,l\} \to \{1,...,k\} \\ \text{strictly increasing}}} (\psi_{U',U''} \circ \phi_{U,U'})^{(Im(\iota))} \otimes (\psi_{U',\mathbb{1}''} \circ \phi_{U,U'} + \phi_{U,\mathbb{1}'})^{(\{1,...,k\}\setminus Im(\iota))} \otimes \left(\Delta_k \xrightarrow{res_\iota} \Delta_l\right)$$

Next,

$$\sum_{\substack{\iota:\{1,...,l\}\to\{1,...,k\}\\ \text{strictly increasing}}} (\psi_{U',U''}\circ\phi_{U,U'})^{(Im(\iota))}\otimes(\psi_{U',\mathbb{1}''}\circ\phi_{U,U'}+\phi_{U,\mathbb{1}'})^{(\{1,...,k\}\setminus Im(\iota))}\otimes\left(\Delta_k\overset{res_\iota}{\to}\Delta_l\right)=$$

$$=\sum_{l\leq j\leq k}\sum_{\substack{\iota_2:\{1,...,l\}\to\{1,...,j\}\\ \text{str. incr.}}}\sum_{\substack{\iota_1:\{1,...,j\}\to\{1,...,k\}\\ \text{str. incr.}}}(\psi_{U',U''}\circ\phi_{U,U'})^{(Im(\iota_1\circ\iota_2))}\otimes(\psi_{U',\mathbb{1}''}\circ\phi_{U,U'})^{(Im(\iota_1)\setminus Im(\iota_1\circ\iota_2))}\otimes$$

$$\otimes\,\phi_{U,\mathbb{1}'}^{(\{1,...,k\}\setminus Im(\iota_1))}\otimes\left(\Delta_k\overset{res_{\iota_1}}{\to}\Delta_j\right)\circ\left(\Delta_j\overset{res_{\iota_2}}{\to}\Delta_l\right)$$

Thus the (l,k) matrix coefficient of $(\psi\circ\phi)^{\underline{\otimes}\nu}$ is $\sum_{l\leq j\leq k}\Psi^{l,j}\circ\Phi^{j,k}$, as wanted. □

Let $(V,\mathbb{1},U)\in\mathbf{Uni}$. Then $\mathrm{Aut}_{\mathbf{Uni}}((V,\mathbb{1},U))=\bar{\mathfrak{P}}_{V,\mathbb{1}}$ (the mirabolic subgroup of $GL(V)$ preserving $\mathbb{1}$; c.f. Definition 1.2.0.1). Given two splittings $V=\mathbb{C}\mathbb{1}\oplus U$, $V=\mathbb{C}\mathbb{1}\oplus W$, we have a map $\mathrm{Id}_V:(V,\mathbb{1},U)\to(V,\mathbb{1},W)$, and we get a commutative diagram

$$\begin{array}{ccc}\bar{\mathfrak{P}}_{V,\mathbb{1}}=\mathrm{Aut}_{\mathbf{Uni}}((V,\mathbb{1},U)) & \xrightarrow{(\cdot)^{\underline{\otimes}\nu}} & \mathrm{Aut}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k\geq0}(U^{\otimes k}\otimes\Delta_k)^{S_k})\\ \Big\downarrow{\scriptstyle Ad_{\mathrm{Id}_V}} & & \Big\downarrow{\scriptstyle Ad_{(\mathrm{Id}_V)^{\underline{\otimes}\nu}}}\\ \bar{\mathfrak{P}}_{V,\mathbb{1}}=\mathrm{Aut}_{\mathbf{Uni}}((V,\mathbb{1},W)) & \xrightarrow{(\cdot)^{\underline{\otimes}\nu}} & \mathrm{Aut}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k\geq0}(W^{\otimes k}\otimes\Delta_k)^{S_k})\end{array}$$

Consider the action

$$\rho_U:\mathfrak{gl}(V)\longrightarrow\mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k\geq0}(U^{\otimes k}\otimes\Delta_k)^{S_k})$$

given in Section 4.2.

Lemma 4.4.0.29. *The action $\rho_U|_{\mathfrak{gl}(U)\oplus U^*}$ integrates to the action of $\bar{\mathfrak{P}}_{V,\mathbb{1}}$ on $\bigoplus_{k\geq0}(U^{\otimes k}\otimes\Delta_k)^{S_k})$ given above, i.e. we have a commutative diagram*

$$\begin{array}{ccc}\bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}}\cong\mathfrak{gl}(U)\oplus U^* & \xrightarrow{\rho_U} & \mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k\geq0}(U^{\otimes k}\otimes\Delta_k)^{S_k})\\ \Big\downarrow{\scriptstyle exp} & & \Big\downarrow{\scriptstyle exp}\\ \bar{\mathfrak{P}}_{V,\mathbb{1}}=\mathrm{Aut}_{\mathbf{Uni}}((V,\mathbb{1},U)) & \xrightarrow{(\cdot)^{\underline{\otimes}\nu}} & \mathrm{Aut}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k\geq0}(U^{\otimes k}\otimes\Delta_k)^{S_k})\end{array}$$

Proof. Let $\phi \in \bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}}$, and denote by $\widetilde{\Phi}_s$ the image of $exp(s\phi)$ ($s \in \mathbb{C}$) under the functor $(\cdot)^{\underline{\otimes} \nu}$. We want to show that $\frac{d}{ds}\widetilde{\Phi}_s = \rho_U(\phi)$. Writing these expressions in terms of matrix coefficients, we want to show that for any $l, k \geq 0$, we have:

$$\widetilde{\Phi}_s^{l,k} \stackrel{?}{=} \begin{cases} s\rho_U(\phi) + o(s) & \text{if } l \neq k \\ \mathrm{Id} + s\rho_U(\phi) + o(s) & \text{if } l = k \end{cases}$$

Indeed,

$$\sum_{\substack{\iota:\{1,\ldots,l\}\to\{1,\ldots,k\} \\ \text{strictly increasing}}} exp(s\phi)_{U,U}^{Im(\iota)} \otimes exp(s\phi)_{U,\mathbb{1}}^{(\{1,\ldots,k\}\setminus Im(\iota))} \otimes \left(\Delta_k \stackrel{res_\iota}{\to} \Delta_l\right) =$$

$$= \sum_{\substack{\iota:\{1,\ldots,l\}\to\{1,\ldots,k\} \\ \text{strictly increasing}}} (\mathrm{Id}_U + s\phi_{U,U} + o(s))^{Im(\iota)} \otimes (s\phi_{U,\mathbb{1}} + o(s))^{(\{1,\ldots,k\}\setminus Im(\iota))} \otimes \left(\Delta_k \stackrel{res_\iota}{\to} \Delta_l\right) =$$

$$= \begin{cases} s\sum_{i\in\{1,\ldots,k\}} \phi_{U,\mathbb{1}}^{(i)} \otimes res_i + o(s) & \text{if } l = k-1 \\ \mathrm{Id}_{U^{\otimes k}\otimes\Delta_k} + s\sum_{i\in\{1,\ldots,k\}} \phi_{U,U}^{(i)} \otimes \mathrm{Id}_{\Delta_k} + o(s) & \text{if } l = k \\ o(s) & \text{else} \end{cases}$$

We conclude that

$$\widetilde{\Phi}_s^{l,k} = \begin{cases} s\rho_U(\phi) + o(s) & \text{if } l \neq k \\ \mathrm{Id} + s\rho_U(\phi) + o(s) & \text{if } l = k \end{cases}$$

as wanted. □

We obtained an action of the subalgebra $\bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}}$ on the *Ind*-object $V^{\underline{\otimes} \nu}$ of $\underline{Rep}(S_\nu)$, and this action does not depend on the choice of splitting, in the sense that the diagram

$$\begin{array}{ccc}
 & \mathfrak{gl}(U) \oplus U^* \longrightarrow \mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) & \\
\nearrow & & \downarrow Ad_{(\mathrm{Id}_V)^{\underline{\otimes} \nu}} \\
\bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}} & & \\
\searrow & & \\
 & \mathfrak{gl}(W) \oplus W^* \longrightarrow \mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k \geq 0}(W^{\otimes k} \otimes \Delta_k)^{S_k}) &
\end{array}$$

is commutative for any two splittings $V = \mathbb{C}\mathbb{1} \oplus U$, $V = \mathbb{C}\mathbb{1} \oplus W$.

It remains to show the action of $\mathfrak{gl}(V)$ on $V^{\underline{\otimes} \nu}$ does not depend on the choice of splitting:

Lemma 4.4.0.30. *The diagram*

$$\begin{array}{ccc}
 & \bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}} \oplus U \oplus \mathbb{C}\,\mathrm{Id}_V \longrightarrow \mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) & \\
\nearrow & & \downarrow Ad_{(\mathrm{Id}_V)^{\underline{\otimes} \nu}} \\
\mathfrak{gl}(V) & & \\
\searrow & & \\
 & \bar{\mathfrak{p}}_{V,\mathbb{C}\mathbb{1}} \oplus W \oplus \mathbb{C}\,\mathrm{Id}_V \longrightarrow \mathrm{End}_{Ind-\underline{Rep}(S_\nu)}(\bigoplus_{k \geq 0}(W^{\otimes k} \otimes \Delta_k)^{S_k}) &
\end{array}$$

is commutative.

Proof. This follows directly from the definition of action of W (respectively, U) on $\bigoplus_{k \geq 0}(W^{\otimes k} \otimes \Delta_k)^{S_k}$ (respectively, $\bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}$). □

Thus we conclude that

Corollary 4.4.0.31. *The definition of the complex tensor power $V \mapsto V^{\underline{\otimes} \nu}$ as an object in $Ind-(\underline{Rep}(S_{\nu=n})) \boxtimes O^{\mathfrak{p}}_{n,V})$ depends only on the distinguished non-zero vector $\mathbb{1}$, rather than on the splitting $V = \mathbb{C}\mathbb{1} \oplus U$.*

Remark 4.4.0.32. The functor $(\cdot)^{\underline{\otimes} \nu}$ is a symmetric monoidal functor.

Indeed, let $(V, \mathbb{1}, U), (V', \mathbb{1}', U') \in \mathbf{Uni}$. The canonical isomorphism of S_n representations

$$\Upsilon_n : V^{\otimes n} \otimes V'^{\otimes n} \longrightarrow (V \otimes V')^{\otimes n}$$

and its inverse Υ_n^{-1} can be rewritten using the isomorphism in Lemma 4.3.0.23; these interpolate easily to morphisms in $Ind - \underline{Rep}(S_\nu)$:

$$\underline{\Upsilon}_\nu : V^{\underline{\otimes}\nu} \otimes V'^{\underline{\otimes}\nu} \longrightarrow (V \otimes V')^{\underline{\otimes}\nu}$$

and

$$\underline{\Upsilon}'_\nu : (V \otimes V')^{\underline{\otimes}\nu} \longrightarrow V^{\underline{\otimes}\nu} \otimes V'^{\underline{\otimes}\nu}$$

so that $\underline{\Upsilon}_\nu \circ \underline{\Upsilon}'_\nu = \mathrm{Id}$, $\underline{\Upsilon}'_\nu \circ \underline{\Upsilon}_\nu = \mathrm{Id}$.

Remark 4.4.0.33. We can now consider the category $\mathbf{Uni}'$ of finite-dimensional unital vector spaces: that is, the objects in $\mathbf{Uni}'$ are pairs $(V, \mathbb{1})$, with $\dim V < \infty$, $\mathbb{1} \in V \setminus \{0\}$, and the morphisms are

$$\mathrm{Hom}_{\mathbf{Uni}'}((V, \mathbb{1}), (V', \mathbb{1}')) := \{\phi \in \mathrm{Hom}_{\mathbb{C}}(V, V') : \phi(\mathbb{1}) = \mathbb{1}'\}$$

By definition, we have a forgetful functor $Forget : \mathbf{Uni} \to \mathbf{Uni}'$, and this functor is an equivalence of categories.

This allows us to define a functor $(\cdot)^{\underline{\otimes}\nu} : \mathbf{Uni}' \to Ind - \underline{Rep}(S_\nu)$ for each choice of functor $Forget^{-1} : \mathbf{Uni}' \to \mathbf{Uni}$; the latter is defined up to isomorphism. We do not currently have a definition of the functor $(\cdot)^{\underline{\otimes}\nu} : \mathbf{Uni}' \to Ind - \underline{Rep}(S_\nu)$ which does not involve a choice of $Forget^{-1}$.

4.5 Definition of a complex tensor power when $\nu \notin \mathbb{Z}_+$

Let $\nu \in \mathbb{C} \setminus \mathbb{Z}_+$, and let $(V, \mathbb{1})$ be a finite-dimensional unital vector space. In this section we discuss an alternative definition of the ν-th tensor power of $(V, \mathbb{1})$, which does not use a splitting of V, but is applicable only when $\nu \notin \mathbb{Z}_+$.

As before, we use Notation 1.2.0.1 for the parabolic subalgebra $\mathfrak{p}$, its "mirabolic" subalgebra $\bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$ and the nilpotent subalgebra $\mathfrak{u}_{\mathfrak{p}}^+$.

We will also denote by $\mathfrak{p}_0$ the subalgebra of $\mathfrak{p}$ consisting of all the endomorphisms $\phi : V \to V$ for which $\operatorname{Im} \phi \subset \mathbb{C}\mathbb{1}$ (notice that $\mathfrak{u}_{\mathfrak{p}}^+ = \mathfrak{p}_0 \cap \bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$).

The quotient $\mathfrak{l} := \mathfrak{p} / \mathfrak{u}_{\mathfrak{p}}^+$ is then a reductive Lie algebra which can be decomposed as a direct sum of reductive Lie algebras $\mathfrak{l} \cong \mathfrak{l}_1 \oplus \mathfrak{l}_2$, where $\mathfrak{l}_1 := \mathfrak{p}_0 / \mathfrak{u}_{\mathfrak{p}}^+$ is a one-dimensional Lie algebra, and $\mathfrak{l}_2 := \bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}} / \mathfrak{u}_{\mathfrak{p}}^+$.

Notice that the Lie algebra $\mathfrak{p}$ has a one-dimensional center ($Z(\mathfrak{p}) = \mathbb{C} \operatorname{Id}_V$), and so does $\mathfrak{l}$. In fact, we have a canonical splitting $\mathfrak{p} \cong \mathbb{C} \operatorname{Id}_V \oplus \bar{\mathfrak{p}}_{\mathbb{C}\mathbb{1}}$, and thus a canonical splitting $\mathfrak{l} \cong Z(\mathfrak{l}) \oplus \mathfrak{l}_2$.

Consider the quotient space $U := V / \mathbb{C}\mathbb{1}$. Since both V and $\mathbb{C}\mathbb{1}$ are $\mathfrak{p}$-modules, U also has the structure of a $\mathfrak{p}$-module. Moreover, $\mathfrak{p}_0$ obviously acts trivially on U, so the action of $\mathfrak{p}$ on U factors through an action of $\mathfrak{l}_2$ on U.

We now give the following definition of a parabolic Verma module for $(\mathfrak{gl}(V), \mathfrak{p})$ (this is actually the parabolic Verma module of highest weight $(\nu - |\lambda|, \lambda)$):

Definition 4.5.0.34. Let λ be a Young diagram.

If $\ell(\lambda) > \dim V - 1$, we define the parabolic Verma module $M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ to be zero.

Otherwise, consider the $\mathfrak{l}_2$-module $S^\lambda U$ (i.e. the Schur functor S^λ applied to the $\mathfrak{l}_2$-module U). We make $S^\lambda U$ a $\mathfrak{l}$-module by requiring that $Z(\mathfrak{l})$ act on $S^\lambda U$ by the scalar ν, and then lift the action of $\mathfrak{l}$ on $S^\lambda U$ to an action of $\mathfrak{p}$ on $S^\lambda U$ by requiring that $\mathfrak{u}_{\mathfrak{p}}^+$ act trivially on $S^\lambda U$.

Finally, we define

$$M_{\mathfrak{p}}(\nu - |\lambda|, \lambda) := \mathcal{U}(\mathfrak{gl}(V)) \otimes_{\mathcal{U}(\mathfrak{p})} S^{\lambda}U$$

Remark 4.5.0.35. Recall that the $\mathfrak{gl}(V)$-module $M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$ is irreducible, since $\nu \notin \mathbb{Z}_+$ (c.f. Proposition 3.3.1.4).

The following definition is proposed in [Et1] (we still assume that $\nu \notin \mathbb{Z}_+$).

Definition 4.5.0.36. Define the object $V^{\otimes^{Del'} \nu}$ of $Ind-(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ through the formula

$$V^{\otimes^{Del'} \nu} := \bigoplus_{\lambda \text{ is a Young diagram}} X_\lambda \otimes M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$$

Proposition 4.5.0.37. *Fix a splitting* $V \cong \mathbb{C}\mathbb{1} \oplus U$.

The object $V^{\otimes^{Del'} \nu}$ *defined in Definition 4.5.0.36 is isomorphic to the object* $V^{\underline{\otimes} \nu}$ *defined in Definition 4.2.0.16.*

Proof. Since we assumed $\nu \notin \mathbb{Z}_+$, the categories $\underline{Rep}(S_\nu)$, $O^{\mathfrak{p}}_{\nu,V}$ are semisimple abelian categories (see Sections 3.2, 3.3). In this case, any object A of $Ind-(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$ can be written as a direct sum with summands of the form $L(\nu - |\lambda|, \lambda) \otimes X_\mu$ (λ, μ are Young diagrams).

In the case of the object $V^{\underline{\otimes} \nu}$ of $Ind-(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}}_{\nu,V})$, we get:

$$V^{\underline{\otimes} \nu} \cong \bigoplus_{\lambda,\mu} L(\nu - |\lambda|, \lambda) \otimes X_\mu \otimes Mult_{\lambda,\mu}$$

(here $Mult_{\lambda,\mu}$ is the multiplicity space of $L(\nu - |\lambda|, \lambda) \otimes X_\mu$ in $V^{\underline{\otimes} \nu}$, not necessarily finite dimensional).

Recall from Section 3.3 that for any Young diagram λ, we have an isomorphism of $\mathfrak{gl}(V)$-modules

$$L(\nu - |\lambda|, \lambda) \cong M_{\mathfrak{p}}(\nu - |\lambda|, \lambda)$$

We now need to prove that $\dim Mult_{\lambda,\mu} = \delta_{\lambda,\mu}$, and we are done.

To do this, consider

$$V^{\underline{\otimes}\nu} \cong \bigoplus_{\lambda,\mu} L(\nu - |\lambda|, \lambda) \otimes X_\mu \otimes Mult_{\lambda,\mu}$$

as an object of $Ind - \underline{Rep}(S_\nu)$ with an action of $\mathfrak{gl}(U)$. Using Lemmas 5.0.0.40, 3.3.1.3, we get the following decompositions:

$$V^{\underline{\otimes}\nu} \cong \bigoplus_{\mu} \bigoplus_{\rho \in \mathcal{I}_\mu^+} S^\rho U \otimes X_\mu$$

and

$$\bigoplus_{\lambda,\mu} L(\nu - |\lambda|, \lambda) \otimes X_\mu \otimes Mult_{\lambda,\mu} \cong \bigoplus_{\lambda,\mu} \bigoplus_{\rho \in \mathcal{I}_\lambda^+} S^\rho U \otimes X_\mu \otimes Mult_{\lambda,\mu}$$

Thus for any Young diagram μ, we have

$$\bigoplus_{\rho \in \mathcal{I}_\mu^+} S^\rho U \cong \bigoplus_{\lambda} \bigoplus_{\rho \in \mathcal{I}_\lambda^+} S^\rho U \otimes Mult_{\lambda,\mu}$$

and we immediately conclude that $\dim Mult_{\lambda,\mu} = \delta_{\lambda,\mu}$, proving the statement of the proposition. □

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Chapter 5

Schur-Weyl duality in Deligne setting: $\underline{Rep}^{ab}(S_\nu)$ and $O^{\mathfrak{p}}_{\nu,V}$

We fix $\nu \in \mathbb{C}$, and a finite-dimensional unital vector space $(V, \mathbb{1})$ (so $V^{\underline{\otimes}\nu}$ is defined).

Definition 5.0.0.38. Define the Schur-Weyl functor

$$SW_\nu^{ind} : Ind - \underline{Rep}^{ab}(S_\nu) \longrightarrow Mod_{\mathcal{U}(\mathfrak{gl}(V))}$$

by

$$SW_\nu^{ind} := \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\cdot, V^{\underline{\otimes}\nu})$$

We will also consider the restriction of the functor SW_ν^{ind} to the category $\underline{Rep}^{ab}(S_\nu)$, which will be denoted by SW_ν.

Remark 5.0.0.39. The functor $SW_\nu^{ind} : Ind - \underline{Rep}^{ab}(S_\nu) \longrightarrow Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ (as well as its restriction SW_ν) is a contravariant $\mathbb{C}$-linear additive left-exact functor.

It turns out that the image of the functor $SW_\nu : \underline{Rep}^{ab}(S_\nu) \to Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ lies in $O^{\mathfrak{p}}_{\nu,V}$, as we will prove in Lemma 5.0.0.41.

The following technical lemma will be used to perform most of the computations:

Lemma 5.0.0.40. *Let τ be a Young diagram, $k \in \mathbb{Z}_+$ and μ a partition of k.*

- *Assume τ lies in a trivial $\overset{\nu}{\sim}$-class. Then*

$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k) = \begin{cases} 1 & \textit{if } \mu \in \mathcal{I}^+_\tau \\ 0 & \textit{else} \end{cases}$$

- *Assume τ lies in a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$, and let j be such that $\tau = \lambda^{(j)}$. Then we have:*

$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_{\lambda^{(0)}} \otimes \mu, \Delta_k) = \begin{cases} 1 & \textit{if } \mu \in \mathcal{I}^+_{\lambda^{(0)}} \\ 0 & \textit{else} \end{cases}$$

  and if $j > 0$, we have:

$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_{\lambda^{(j)}} \otimes \mu, \Delta_k) = \begin{cases} 2 & \textit{if } \mu \in \mathcal{I}^+_{\lambda^{(j)}} \cap \mathcal{I}^+_{\lambda^{(j-1)}} \\ 1 & \textit{if } \mu \in \left(\mathcal{I}^+_{\lambda^{(j)}} \setminus \mathcal{I}^+_{\lambda^{(j-1)}}\right) \cup \left(\mathcal{I}^+_{\lambda^{(j-1)}} \setminus \mathcal{I}^+_{\lambda^{(j)}}\right) \\ 0 & \textit{else} \end{cases}$$

Proof. The proof will use the $lift_\nu$ maps discussed in Subsection 3.2.2.

We know that

$$\dim \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k) = \dim \mathrm{Hom}_{\underline{Rep}(S_T)\boxtimes Rep(S_k)}(lift_\nu(X_\tau) \otimes \mu, lift_\nu(\Delta_k)) =$$
$$= \dim \mathrm{Hom}_{\underline{Rep}(S_T)\boxtimes Rep(S_k)}(lift_\nu(X_\tau) \otimes \mu, \Delta_k)$$

So it is enough to prove that

$$\dim \mathrm{Hom}_{\underline{Rep}(S_T)\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k) = \begin{cases} 1 & \text{if } \mu \in \mathcal{I}^+_\tau \\ 0 & \text{else} \end{cases}$$

and the statement of the lemma will then follow from Lemma 3.2.2.5.

To compute $\dim \operatorname{Hom}_{\underline{Rep}(S_T)\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k)$, note that

$$\dim \operatorname{Hom}_{\underline{Rep}(S_T)\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k) = \dim \operatorname{Hom}_{\underline{Rep}(S_{\nu=n})\boxtimes Rep(S_k)}(X_\tau \otimes \mu, \Delta_k) =$$
$$= \dim \operatorname{Hom}_{S_n \times S_k}(\mathcal{S}_n(X_\tau) \otimes \mu, \mathcal{S}_n(\Delta_k)) =$$
$$= \dim \operatorname{Hom}_{S_n \times S_k}(\tilde{\tau}(n) \otimes \mu, \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\}))$$

for $n >> 0, n \in \mathbb{Z}$ (the first equality follows from Proposition 3.2.2.3, while the second relies on the fact that $\mathcal{S}_n$ is fully faithful on $\underline{Rep}(S_{\nu=n})^{(n/2)}$).

But

$$\mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\}) \cong \bigoplus_{\rho:|\rho|=k} \bigoplus_{\lambda \in \mathcal{I}^+_\rho, |\lambda|=n} \lambda \otimes \rho$$

so

$$\dim \operatorname{Hom}_{S_n \times S_k}(\tilde{\tau}(n) \otimes \mu, \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\})) = \begin{cases} 1 & \text{if } \tilde{\tau}(n) \in \mathcal{I}^+_\mu \\ 0 & \text{else} \end{cases}$$

It remains to check that for $n >> 0$,

$$\tilde{\tau}(n) \in \mathcal{I}^+_\mu \Leftrightarrow \mu \in \mathcal{I}^+_\tau$$

The first condition is equivalent to saying that

$$... \leq \mu_{i+1} \leq \tilde{\tau}(n)_{i+1} \leq \mu_i \leq ... \leq \tilde{\tau}(n)_2 \leq \mu_1 \leq \tilde{\tau}(n)_1 = n - |\tau|$$

which is equivalent to

$$... \leq \mu_{i+1} \leq \tau_i \leq \mu_i \leq ... \leq \tau_1 \leq \mu_1$$

(a reformulation of the condition $\mu \in \mathcal{I}^+_\tau$) provided that $n >> 0$. □

Lemma 5.0.0.41. *The image of the functor $SW_\nu : \underline{Rep}^{ab}(S_\nu) \to Mod_{\mathcal{U}(\mathfrak{gl}(V))}$ lies in $O^{\mathfrak{p}}_{\nu,V}$.*

Proof. Fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$. We want to prove that for any $M \in \underline{Rep}^{ab}(S_\nu)$, $SW_\nu(M)$ is a Noetherian $\mathcal{U}(\mathfrak{gl}(V))$-module of degree ν on which $\mathfrak{gl}(U)$ acts polynomially (i.e. $M|_{\mathfrak{gl}(U)} \in Ind - Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$) and $\mathfrak{u}_{\mathfrak{p}}^+$ acts locally finitely.

Recall that for any $M \in \underline{Rep}^{ab}(S_\nu)$, $SW_\nu(M) = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(M, V^{\underline{\otimes}\nu})$, with the action of $\mathfrak{gl}(V)$ coming from its action on $V^{\underline{\otimes}\nu}$ (c.f. Definition 4.2.0.16). So

$$SW_\nu(M)|_{\mathfrak{gl}(U)} = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(M, V^{\underline{\otimes}\nu}|_{\mathfrak{gl}(U)}) \cong \bigoplus_{k\geq 0} \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(M, (U^{\otimes k} \otimes \Delta_k)^{S_k})$$

with $\mathfrak{gl}(U)$ acting through its action on each $(U^{\otimes k} \otimes \Delta_k)^{S_k}$. This immediately implies that $SW_\nu(M)$ has degree ν.

Next, $(U^{\otimes k} \otimes \Delta_k)^{S_k}$ is an object of $\underline{Rep}^{ab}(S_\nu) \boxtimes Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$, so the spaces $\mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(M, (U^{\otimes k} \otimes \Delta_k)^{S_k})$ are polynomial $\mathfrak{gl}(U)$-modules. Thus $M|_{\mathfrak{gl}(U)} \in Ind - Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$.

The above $\mathfrak{gl}(U)$-decomposition of $SW_\nu(M)$ gives us a $\mathbb{Z}_+$-grading on $SW_\nu(M)$, with each grade being finite-dimensional. Definition 4.2.0.16 tells us that $\mathfrak{u}_{\mathfrak{p}}^+$ acts on this space by operators of degree -1, so $\mathfrak{u}_{\mathfrak{p}}^+$ acts locally finitely on $SW_\nu(M)$.

We now prove that for any $M \in \underline{Rep}^{ab}(S_\nu)$, $SW_\nu(M)$ is a Noetherian $\mathcal{U}(\mathfrak{gl}(V))$-module. Recall that the functor SW_ν is a contravariant left-exact functor. Using this, together with the fact that the category $\underline{Rep}^{ab}(S_\nu)$ has enough projectives, it is enough to prove that for any indecomposable projective $\mathbf{P} \in \underline{Rep}^{ab}(S_\nu)$, we have: $SW_\nu(\mathbf{P})$ is a Noetherian $\mathcal{U}(\mathfrak{gl}(V))$-module.

Next, recall that any indecomposable projective in $\underline{Rep}^{ab}(S_\nu)$ is isomorphic to X_λ, where λ either lies in a trivial $\stackrel{\nu}{\sim}$-class, or is not minimal in its non-trivial $\stackrel{\nu}{\sim}$-class (c.f. Proposition 3.2.4.10).

Lemma 5.0.0.40 tells us that if λ lies in a trivial $\stackrel{\nu}{\sim}$-class, then

$$SW_\nu(X_\lambda)|_{\mathfrak{gl}(U)} \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_\lambda, \bigoplus_{k\geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) \cong$$

$$\cong \bigoplus_{k\geq 0} \bigoplus_{\mu:|\mu|=k} \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_\lambda \otimes \mu, \Delta_k) \otimes S^\mu U \cong \bigoplus_{\mu \in \mathcal{I}_\lambda^+} S^\mu U$$

If λ lies in a non-trivial $\stackrel{\nu}{\sim}$-class, $\lambda = \lambda^{(i)}, i > 0$, then

$$SW_\nu(X_{\lambda^{(i)}})|_{\mathfrak{gl}(U)} \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_{\lambda^{(i)}}, \bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) \cong$$

$$\cong \bigoplus_{k\geq 0} \bigoplus_{\mu:|\mu|=k} \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_{\lambda^{(i)}} \otimes \mu, \Delta_k) \otimes S^\mu U \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^\mu U \oplus \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i-1)}}} S^\mu U$$

In both cases, we can consider $SW_\nu(X_\lambda)|_{\mathfrak{gl}(U)}$ as a $\mathbb{Z}_+$-graded space, with grade j being the direct sum of those $S^\mu U$ for which $|\mu| = j$. Then the non-zero elements of $\mathfrak{u}_\mathfrak{p}^+$ act by operators of degree -1 (see Definition 4.2.0.16).

Next, recall that in any case, the parabolic Verma module restricted to $\mathfrak{gl}(U)$ decomposes as

$$M_\mathfrak{p}(\nu - |\lambda|, \lambda)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_\lambda} S^\mu U$$

and has $\mathcal{U}(\mathfrak{gl}(V))$-length at most 2. Using the above property of the action of $\mathfrak{u}_\mathfrak{p}^+$, we see that $SW_\nu(X_\lambda)$ has a finite filtration where each quotient is the image of a parabolic Verma module (therefore all the quotients have finite length).

□

We can now define another Schur-Weyl functor which we will consider: it is the contravariant functor $\widehat{SW}_\nu : \underline{Rep}^{ab}(S_\nu) \longrightarrow \widehat{O}^\mathfrak{p}_{\nu,V}$ where

$$\widehat{O}^\mathfrak{p}_{\nu,V} := O^\mathfrak{p}_{\nu,V} \Big/ Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly,\nu}$$

is the Serre quotient of $O^\mathfrak{p}_{\nu,V}$ by the Serre subcategory $Mod_{\mathcal{U}(\mathfrak{gl}(V)),poly,\nu}$ of polynomial $\mathfrak{gl}(V)$-modules of degree ν. We will denote the quotient functor by

$$\hat{\pi} : O^\mathfrak{p}_{\nu,V} \longrightarrow \widehat{O}^\mathfrak{p}_{\nu,V}$$

and define

$$\widehat{SW}_\nu := \hat{\pi} \circ SW_\nu$$

The main goal of this section is to prove the following theorem:

Theorem 5.0.0.42. *The contravariant functor $\widehat{SW}_\nu : \underline{Rep}^{ab}(S_\nu) \to \widehat{O}^{\mathfrak{p}}_{\nu,V}$ is exact and essentially surjective.*

Moreover, the induced contravariant functor

$$\underline{Rep}^{ab}(S_\nu) \Big/ Ker(\widehat{SW}_\nu) \to \widehat{O}^{\mathfrak{p}}_{\nu,V}$$

is an anti-equivalence of abelian categories, thus making $\widehat{O}^{\mathfrak{p}}_{\nu,V}$ a Serre quotient of $\underline{Rep}^{ab}(S_\nu)^{op}$.

The exactness of $\widehat{SW}_\nu$ will be proved in Lemma 5.0.0.48.

The rest of the proof of Theorem 5.0.0.42 will be done by considering separately semisimple and non-semisimple blocks in $\underline{Rep}^{ab}(S_\nu)$. The semisimple block case will be discussed in Subsection 5.1.1, and the non-semisimple block case will be discussed in Subsection 5.1.2 (specifically, Proposition 5.1.2.12 and Theorem 5.1.2.17).

In particular, we will obtain the following result, which will be used in Chapter 7:

Lemma 5.0.0.43. *The functor $\widehat{SW}_{\nu,\mathbb{C}^n}$ takes a simple object to either a simple object, or zero. More specifically, we have:*

- *Let λ be a Young diagram lying in a trivial $\overset{\nu}{\sim}$-class. Then*

$$\widehat{SW}_{\nu,\mathbb{C}^n}(\mathbf{L}(\lambda)) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda))$$

- *Consider a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\geq 0}$. Then*

$$\widehat{SW}_{\nu,\mathbb{C}^n}(\mathbf{L}(\lambda^{(i)})) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}))$$

  whenever $i \geq 0$.

We now introduce the following notation.

Definition 5.0.0.44. We will denote by $SW_\nu^{*,ind}$ the contravariant functor

$$Mod_{\mathcal{U}(\mathfrak{gl}(V))} \longrightarrow Ind - \underline{Rep}^{ab}(S_\nu)$$

which is right adjoint to SW_ν^{ind} (such a functor exists by Theorem 2.0.5.5, since SW_ν^{ind} obviously commutes with small colimits).

We will also denote by SW_ν^* the restriction of $SW_\nu^{*,ind}$ to $O_{\nu,V}^{\mathfrak{p}}$.

Remark 5.0.0.45. The functors $SW_\nu^{*,ind}$, SW_ν^* are contravariant, $\mathbb{C}$-linear, additive and left-exact (due to $SW_\nu^{*,ind}$ being a right-adjoint).

We will use the following notation:

Notation 5.0.0.46. The unit natural transformations corresponding to the contravariant adjoint functors $SW_\nu^{ind}, SW_\nu^{*,ind}$ will be denoted by

$$\eta : \mathrm{Id}_{Mod_{\mathcal{U}(\mathfrak{gl}(V))}} \to SW_\nu^{ind} \circ SW_\nu^{*,ind}, \ \epsilon : \mathrm{Id}_{Ind-\underline{Rep}^{ab}(S_\nu)} \to SW_\nu^{*,ind} \circ SW_\nu^{ind}$$

In particular, we have the restriction of the natural transformation ϵ:

$$\epsilon : \iota_{\underline{Rep}^{ab}(S_\nu) \to Ind-\underline{Rep}^{ab}(S_\nu)} \to SW_\nu^* \circ SW_\nu$$

These transformations satisfy the following conditions (see [MacL, Chapter 1, par. 1, Theorem 1]):

Lemma 5.0.0.47.

$$\forall E \in O_{\nu,V}^{\mathfrak{p}}, \ SW_\nu^*(\eta_E) \circ \epsilon_{SW_\nu^*(E)} = \mathrm{Id}_{SW_\nu^{*,ind}(E)}$$
$$\forall X \in \underline{Rep}^{ab}(S_\nu), \ SW_\nu(\epsilon_X) \circ \eta_{SW_\nu(X)} = \mathrm{Id}_{SW_\nu(X)}$$

We can now demonstrate the exactness of $\widehat{SW}_\nu$:

Lemma 5.0.0.48. *The functor* $\widehat{SW}_\nu : \underline{Rep}^{ab}(S_\nu) \to \widehat{O}^{\mathfrak{p}}_{\nu,V}$ *is exact.*

Proof. Let $M \in \underline{Rep}^{ab}(S_\nu)$, and let $i > 0$. We want to show that the $\mathfrak{gl}(V)$-module $\mathrm{Ext}^i(M, V^{\underline{\otimes}\nu})$ is finite dimensional.

Consider $V^{\underline{\otimes}\nu}$ as an object in $\underline{Rep}(S_\nu)$. As such, it is a direct sum $\bigoplus_\lambda X_\lambda \otimes V_\lambda$, where V_λ is the multiplicity space of X_λ (in fact, for a fixed splitting $V = \mathbb{C}\mathbb{1} \oplus U$, V_λ has the structure of a $\mathfrak{gl}(U)$-module).

We know from Proposition 3.2.4.10 that X_λ are injective objects iff $\tilde{\lambda}(\nu)$ is not a Young diagram. Furthermore, there are only finitely many Young diagrams λ such that $\tilde{\lambda}(\nu)$ is a Young diagram as well; for these λ, the space V_λ is finite dimensional and isomorphic to $S^{\tilde{\lambda}(\nu)}V$ (by Proposition 4.3.0.24).

Finally, notice that $\mathrm{Ext}^i(M, X_\lambda)$ is finite dimensional for any Young diagram λ, since all the Hom-spaces in $\underline{Rep}^{ab}(S_\nu)$ are finite-dimensional (c.f. Remark 3.2.4.2). We conclude that

$$\mathrm{Ext}^i(M, V^{\underline{\otimes}\nu}) \cong \bigoplus_{\substack{\lambda: \\ \tilde{\lambda}(\nu) \text{ is a Young diagram}}} \mathrm{Ext}^i(M, X_\lambda) \otimes S^{\tilde{\lambda}(\nu)}V$$

is finite dimensional. □

5.1 Proof of Theorem 5.0.0.42

5.1.1 Case of a semisimple block

In this subsection we consider a semisimple block in $\underline{Rep}^{ab}(S_\nu)$. We know that semisimple blocks are parametrized by Young diagrams lying in a trivial $\stackrel{\nu}{\sim}$-class. Let us denote our block by $\mathcal{B}_\lambda$, λ being the corresponding Young diagram.

The objects of such a block are finite direct sums of the simple object X_λ, so the block is equivalent to $\mathbf{Vect}_\mathbb{C}$ as an abelian category.

If $\ell(\lambda) \leq \dim V - 1$, then the block $\mathfrak{B}_\lambda$ corresponding to λ in $O^{\mathfrak{p}}_{\nu,V}$ is also semisimple, and its objects are finite direct sums of the parabolic Verma module $M_\mathfrak{p}(\nu - |\lambda|, \lambda)$ (which, in this case, is simple and coincides with $L_\mathfrak{p}(\nu - |\lambda|, \lambda)$).

Notice that $M_{\mathfrak{p}}(\nu-|\lambda|\,,\lambda)$ is infinite-dimensional and simple, so the functor $\hat{\pi}$ restricted to $\mathfrak{B}_\lambda$ is an equivalence of abelian categories.

The proof of Theorem 5.0.0.42 for $\mathcal{B}_\lambda$ is then reduced to proving following proposition:

Proposition 5.1.1.1. *Let λ be a Young diagram which lies in a trivial $\overset{\nu}{\sim}$-class. Then $SW_\nu(X_\lambda) \cong M_{\mathfrak{p}}(\nu - |\lambda|\,,\lambda)$.*

Remark 5.1.1.2. Recall that $M_{\mathfrak{p}}(\nu - |\lambda|\,,\lambda)$ is zero if $\ell(\lambda) > \dim V - 1$, see Definition 3.3.1.1.

Proof. Fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$. Based on Lemma 5.0.0.40, we see that as a $\mathfrak{gl}(U)$-module, the space

$$SW_\nu(X_\lambda) = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_\lambda, V^{\underline{\otimes}\nu})$$

is isomorphic to

$$\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_\lambda, \bigoplus_{k\geq 0}(U^{\otimes k}\otimes \Delta_k)^{S_k}) \cong \bigoplus_{k\geq 0}\bigoplus_{\mu:|\mu|=k} \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_\lambda\otimes\mu, \Delta_k)\otimes S^\mu U \cong$$

$$\cong \bigoplus_{\mu\in\mathcal{I}^+_\lambda} S^\mu U$$

Notice that this expression is zero if $\ell(\lambda) > \dim(U) = \dim V - 1$. Recall that by definition of $V^{\underline{\otimes}\nu}$, $\mathfrak{u}^+_{\mathfrak{p}}$ acts on the graded space $V^{\underline{\otimes}\nu} \cong \bigoplus_{k\geq 0}(U^{\otimes k}\otimes\Delta_k)^{S_k}$ by operators of degree -1, therefore, it acts by zero on the subspace $S^\lambda U$ of $\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_\lambda, V^{\underline{\otimes}\nu})$.

We conclude that if $\ell(\lambda) \leq \dim(U) = \dim V - 1$, then $M_{\mathfrak{p}}(\nu-|\lambda|\,,\lambda)$ maps to $SW_\nu(X_\lambda)$ inducing an identity map on the subspaces $S^\lambda U$. Now, $M_{\mathfrak{p}}(\nu - |\lambda|\,,\lambda)$ is simple, so this map is injective, and since

$$SW_\nu(X_\lambda)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu\in\mathcal{I}^+_\lambda} S^\mu U \cong M_{\mathfrak{p}}(\nu - |\lambda|\,,\lambda)|_{\mathfrak{gl}(U)}$$

as $\mathfrak{gl}(U)$-modules, we conclude that the above map from $M_{\mathfrak{p}}(\nu-|\lambda|\,,\lambda)$ to $SW_\nu(X_\lambda)$ is an isomorphism. □

Thus we proved that

Corollary 5.1.1.3. *The functor SW_ν restricted to a semisimple block $\mathcal{B}_\lambda$ of $\underline{Rep}^{ab}(S_\nu)$ is either zero (iff $\ell(\lambda) > \dim V - 1$), or is an equivalence of abelian categories between $\mathcal{B}_\lambda$ and the block $\mathfrak{B}_\lambda$ of $O^{\mathfrak{p}}_{\nu,V}$; furthermore, the functor $\widehat{SW}_\nu$ restricted to $\mathcal{B}_\lambda$ is either zero or an equivalence of abelian categories between $\mathcal{B}_\lambda$ and the block $\hat{\pi}(\mathfrak{B}_\lambda)$ of $\widehat{O}^{\mathfrak{p}}_{\nu,V}$.*

Recall from Section 3.2 that for $\nu \notin \mathbb{Z}_+$, $\underline{Rep}(S_\nu)$ is abelian semisimple and in particular $\underline{Rep}^{ab}(S_\nu) = \underline{Rep}(S_\nu)$. Denote by $\underline{Rep}(S_\nu)^{(\leq \dim V - 1)}$ the full semisimple abelian subcategory of $\underline{Rep}(S_\nu)$ generated by simple objects X_λ where λ runs over all the Young diagrams of length at most $\dim V - 1$.

Note that $\underline{Rep}(S_\nu)^{(\leq \dim V - 1)}$ is the Serre quotient of $\underline{Rep}(S_\nu)$ by the full semisimple abelian subcategory generated by simple objects X_λ where λ runs over all the Young diagrams of length at least $\dim V$.

Then we immediately get the following corollary:

Corollary 5.1.1.4. *Assume $\nu \notin \mathbb{Z}_+$. Then $SW_\nu : \underline{Rep}(S_\nu) \to O^{\mathfrak{p}}_{\nu,V}$ is a full, essentially surjective, additive $\mathbb{C}$-linear contravariant functor between semisimple abelian categories, inducing an anti-equivalence of abelian categories between $\underline{Rep}(S_\nu)^{(\leq \dim V - 1)}$ and $O^{\mathfrak{p}}_{\nu,V}$.*

Remark 5.1.1.5. If $\nu \notin \mathbb{Z}_+$, then $\widehat{O}^{\mathfrak{p}}_{\nu,V} \cong O^{\mathfrak{p}}_{\nu,V}$ with $\hat{\pi} \cong \mathrm{Id}_{O^{\mathfrak{p}}_{\nu,V}}$, so Corollary 5.1.1.4 is just Theorem 5.0.0.42 in the case $\nu \notin \mathbb{Z}_+$.

5.1.2 Case of a non-semisimple block

Throughout this subsection, we will use the results from Sections 3.2 and 3.3, and we will denote $O^{\mathfrak{p}}_{\nu,V}$ by $O^{\mathfrak{p}}_\nu$ for short.

Fix a splitting $V = \mathbb{C}\mathbb{1} \oplus U$.

In this subsection we consider a non-semisimple block in $\underline{Rep}^{ab}(S_\nu)$. Recall that such blocks occur only when $\nu \in \mathbb{Z}_+$, so we will assume that this is the case.

We know that non-semisimple blocks are parametrized by Young diagrams λ such that $\lambda_1 + |\lambda| \leq \nu$; the projective objects in such a block correspond to the elements of the (non-trivial) $\stackrel{\nu}{\sim}$-class of λ (see Proposition 3.2.4.10).

Let us denote our block by $\mathcal{B}_\lambda$.

If $\ell(\lambda) \leq \dim V - 1$, then the block $\mathfrak{B}_\lambda$ corresponding to λ in $O^{\mathfrak{p}}_\nu$ is also non-semisimple. We will continue with the blocks $\mathcal{B}_\lambda$, $\mathfrak{B}_\lambda$ fixed, and insert some notation for the convenience of the reader.

Notation 5.1.2.1. We will denote the simple objects, standard objects, co-standard and indecomposable projective objects in $\mathcal{B}_\lambda$ by $\mathbf{L}_i, \mathbf{M}_i, \mathbf{M}^*_i, \mathbf{P}_i$ $(i \in \mathbb{Z}_+)$ respectively, with $\mathbf{L}_i$ standing for $\mathbf{L}(\lambda^{(i)})$ and similarly for $\mathbf{M}_i$, $\mathbf{M}^*_i$ and $\mathbf{P}_i$. The structure of these objects is discussed in Subsection 3.2.4.

Notice that $\mathbf{M}_0 = \mathbf{M}^*_0 = \mathbf{L}_0 = X_{\lambda^{(0)}}$, $\mathbf{P}_i = X_{\lambda^{(i+1)}}$ for $i \in \mathbb{Z}_+$ (see Proposition 3.2.4.10).

Notation 5.1.2.2. We will denote the simple modules, the parabolic Verma modules, their duals (the co-standard objects in $O^{\mathfrak{p}}_\nu$) and the indecomposable parabolic projective modules in $\mathfrak{B}_\lambda$ by $L_i, M_i, M^\vee_i, P_i$ $(i \in \mathbb{Z}_+)$ respectively, with M_i standing for $M_{\mathfrak{p}}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ and similarly for $L_i, M^\vee_i$ and P_i.

The structure of the modules $L_i, M_i, M^\vee_i, P_i, (i \in \mathbb{Z}_+)$ is discussed in Section 3.3 and in [H, Chapter 9].

We put $k_\lambda := \min\{k \geq 0 \mid \ell(\lambda^{(k)}) > \dim V - 1\}$. Then $P_i = M_i = M^\vee_i = L_i = 0$ whenever $i \geq k_\lambda$.

The goal of this section is to prove Theorem 5.0.0.42 for the blocks $\mathcal{B}_\lambda$, $\mathfrak{B}_\lambda$. In order to do this, we will prove the following theorem:

Theorem 5.1.2.3. *The functor SW_ν satisfies:*

(a) $SW_\nu(\mathbf{L}_i) \cong L_{i+1}$ whenever $i \geq 1$.

(b) $SW_\nu(\mathbf{M}_i) \cong M_i$ whenever $i \geq 0$.

(c) $SW_\nu(\mathbf{M}^*_i) \cong M^\vee_i$ whenever $i \geq 2$.

(d) $SW_\nu(\mathbf{P}_i) \cong P_{i+1}$ whenever $i \geq 0$ and $i < k_\lambda - 1$ or $i \geq k_\lambda$ (recall that in the latter case $P_{i+1} = 0$); $SW_\nu(\mathbf{P}_{k_\lambda - 1}) \cong L_{k_\lambda - 1}$.

(e) $SW_\nu(\mathbf{M}_0 = \mathbf{M}_0^* = \mathbf{L}_0) \cong M_0$.

(f) $SW_\nu(\mathbf{M}_1^*) \cong Ker(P_1 \twoheadrightarrow L_1)$.

Proof. Statement (a) is proved in Proposition 5.1.2.8. Statements (b)-(d), (f) are proved in Proposition 5.1.2.11. Statement (e) is proved in Lemma 5.1.2.10. □

We start by establishing some useful properties of the functor $\hat{\pi}$ and of the category $\hat{\pi}(\mathfrak{B}_\lambda)$.

Proposition 5.1.2.4. $\hat{\pi}(P_i), i > 0$ *are indecomposable injective and projective objects in* $\widehat{O}_\nu^{\mathfrak{p}}$.

Recall that for $i > 0$, P_i is an indecomposable injective and projective module and has no finite-dimensional submodules nor quotients (c.f. Proposition 3.3.1.13). So Proposition 5.1.2.4 is a special case of the following lemma:

Lemma 5.1.2.5. *Let $\mathcal{A}$ be an abelian category where all objects have finite length, and $\mathcal{A}'$ be a Serre subcategory of $\mathcal{A}$. We consider the Serre quotient $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{A}'$.*

Let $I \in \mathcal{A}$ (respectively, $P \in \mathcal{A}$) be an injective (respectively, projective) object, such that I has no non-trivial subobject nor quotient lying in $\mathcal{A}'$.

Then $\pi(I)$ is an injective (respectively, projective) object in $\mathcal{A}/\mathcal{A}'$. Moreover, if I is indecomposable, so is $\pi(I)$.

Proof. We start by noticing that we have two functors $R_1, R_2 : \mathcal{A} \to \mathcal{A}'$ which are adjoint to the inclusion $\mathcal{A}' \to \mathcal{A}$ on different sides: the first functor, R_1, takes an object $A \in \mathcal{A}$ to its maximal subobject lying in $\mathcal{A}'$, and the second, R_2, takes A to its maximal quotient lying in $\mathcal{A}'$.

These functors are defined since for any $A \in \mathcal{A}$, we can take its maximal (in terms of length) subobject lying in $\mathcal{A}'$, and this subobject will be well-defined. Similarly for the maximal quotient of A lying in $\mathcal{A}'$.

We need to prove that $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(\cdot, \pi(I))$ is an exact functor. By definition, for any $E \in \mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(\pi(E), \pi(I)) := \varinjlim_{\substack{Y \subset E, X \subset I \\ E/Y, X \in \mathcal{A}'}} \mathrm{Hom}_{\mathcal{A}}(Y, I/X) = \varinjlim_{\substack{Y \subset E \\ E/Y \in \mathcal{A}'}} \mathrm{Hom}_{\mathcal{A}}(Y, I)$$

(since I has no non-trivial subobjects lying in $\mathcal{A}'$).

The colimit is taken with respect to the direct system

$$\{\mathrm{Hom}_{\mathcal{A}}(Y, I) : Y \subset E, E/Y \in \mathcal{A}'\},$$
$$\text{arrows } \mathrm{Hom}_{\mathcal{A}}(Y_2, I) \to \mathrm{Hom}_{\mathcal{A}}(Y_1, I) \text{ whenever } Y_1 \hookrightarrow Y_2$$

Now, I is an injective object in $\mathcal{A}$, so the arrows in this direct system are surjective:

$$\{\mathrm{Hom}_{\mathcal{A}}(Y, I) : Y \subset E, E/Y \in \mathcal{A}'\},$$
$$\text{arrows } \mathrm{Hom}_{\mathcal{A}}(Y_2, I) \twoheadrightarrow \mathrm{Hom}_{\mathcal{A}}(Y_1, I) \text{ whenever } Y_1 \hookrightarrow Y_2$$

Then one easily sees that the colimit is $\mathrm{Hom}_{\mathcal{A}}(Y_E, I)$, where $Y_E := Ker(E \twoheadrightarrow R_2(E))$. Thus

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(\pi(E), \pi(I)) := \mathrm{Hom}_{\mathcal{A}}(Y_E, I)$$

So we need to prove that given an exact sequence

$$0 \to E' \longrightarrow E \longrightarrow E'' \to 0$$

of objects in $\mathcal{A}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Y_{E'}, I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y_E, I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y_{E''}, I) \to 0$$

is also exact. Notice that since I is injective in $\mathcal{A}$, we have an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(R_2(E), I) \to \mathrm{Hom}_{\mathcal{A}}(E, I) \to \mathrm{Hom}_{\mathcal{A}}(Y_E, I) \to 0$$

and since I has no non-trivial subobjects in $\mathcal{A}'$, we get $\mathrm{Hom}_{\mathcal{A}}(E, I) \cong \mathrm{Hom}_{\mathcal{A}}(Y_E, I)$. The sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(E', I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E, I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(E'', I) \to 0$$

is exact (since I is injective in $\mathcal{A}$), so the sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Y_{E'}, I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y_E, I) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y_{E''}, I) \to 0$$

is exact as well.

Thus we proved that $\mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(\cdot, \pi(I))$ is an exact functor, so $\pi(I)$ is an injective object in $\mathcal{A}/\mathcal{A}'$.

The fact that $\pi(P)$ is a projective object in $\mathcal{A}/\mathcal{A}'$ is proved in the same way.

Now, assume $\pi(I)$ is decomposable, $\pi(I) \cong X_1 \oplus X_2$ in $\mathcal{A}/\mathcal{A}'$, $X_1, X_2 \neq 0$. Then we can find $E_1, E_2 \in \mathcal{A}$ such that E_1, E_2 have no non-trivial subobject nor quotient lying in $\mathcal{A}'$, and such that $\pi(E_i) = X_i$, $i = 1, 2$. Then one immediately sees that

$$\mathrm{Hom}_{\mathcal{A}}(I, E_i) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(\pi(I), X_i), \mathrm{Hom}_{\mathcal{A}}(E_i, I) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(X_i, \pi(I)),$$
$$\mathrm{Hom}_{\mathcal{A}}(E_i, E_j) = \mathrm{Hom}_{\mathcal{A}/\mathcal{A}'}(X_i, X_j)$$

for $i = 1, 2$. In particular, since π is exact and I, E_1, E_2 have no non-trivial subobject nor quotient lying in $\mathcal{A}'$, we see that $E_1 \oplus E_2 \cong I$. We conclude that if I is indecomposable, so is $\pi(I)$. □

The following corollary will be useful when proving that the functor $\widehat{SW}_\nu$ is full and essentially surjective:

Corollary 5.1.2.6. *The image of the category $\mathfrak{B}_\lambda$ under the functor $\hat{\pi}$ has enough injec-*

tives and enough projectives. Moreover, $\{\hat{\pi}(P_i)\}_{0<i\leq k_\lambda-1}$ *is the full set of representatives of isomorphism classes of indecomposable injective (respectively, projective) objects in* $\hat{\pi}(\mathfrak{B}_\lambda)$.

Proof. Let $E \in \mathfrak{B}_\lambda$. We know from Proposition 3.2.4.6 that the category $\mathfrak{B}_\lambda$ has enough injective and enough projective modules. This means that there exist an injective module I and a projective module P, together with an injective map $E \to I$ and a surjective map $P \to E$.

Since the functor $\hat{\pi}$ is exact, we get an injective map $\hat{\pi}(E) \to \hat{\pi}(I)$ and a surjective map $\hat{\pi}(P) \to \hat{\pi}(E)$.

Next, recall that $\{P_i, 0 \leq i \leq k_\lambda - 1\}$ (respectively, $\{P_i^\vee, 0 \leq i \leq k_\lambda - 1\}$) is the full set of representatives of isomorphism classes of indecomposable projective (respectively, injective) modules in $\mathfrak{B}_\lambda$. For $i > 0$, the object $\hat{\pi}(P_i \cong P_i^\vee)$ was proved to be injective and projective (c.f. Proposition 5.1.2.4), so it remains to check the following statement:

The object $\hat{\pi}(P_0) \cong \hat{\pi}(L_1) \cong \hat{\pi}(P_0^\vee)$ is neither injective nor projective, and has a projective cover and an injective hull in $\mathfrak{B}_\lambda$ which are direct sums of objects $\hat{\pi}(P_i), i > 0$.

To prove the latter claim, notice that the maps $L_1 \hookrightarrow P_1$, $P_1 \twoheadrightarrow L_1$ in $\mathfrak{B}_\lambda$ become maps $\hat{\pi}(L_1) \hookrightarrow \hat{\pi}(P_1)$, $\hat{\pi}(P_1) \twoheadrightarrow \hat{\pi}(L_1)$ in $\hat{\pi}(\mathfrak{B}_\lambda)$ (since the functor $\hat{\pi}$ is exact). Knowing that $\hat{\pi}(P_1)$ is an indecomposable injective and projective object, we conclude that $\hat{\pi}(L_1)$ is neither injective nor projective, and $\hat{\pi}(P_1)$ is both its projective cover and its injective hull. □

We now compute the decomposition into $\mathfrak{gl}(U)$-irreducibles of $SW_\nu(\mathbf{L}_i), SW_\nu(\mathbf{M}_i), SW_\nu(\mathbf{P}_i)$:

Lemma 5.1.2.7. *We have the following isomorphisms of $\mathfrak{gl}(U)$-modules:*

$$SW_\nu(\mathbf{L}_i)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}} \cap \mathcal{I}^+_{\lambda^{(i+1)}}} S^\mu U \ , \ SW_\nu(\mathbf{M}_i)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^\mu U \qquad \forall i \geq 1$$

$$SW_\nu(\mathbf{L}_0 = \mathbf{M}_0 = \mathbf{M}_0^* \cong X_{\lambda^{(0)}})|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(0)}}} S^\mu U$$

$$SW_\nu(\mathbf{P}_i \cong X_{\lambda^{(i+1)}})|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^\mu U \oplus \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i+1)}}} S^\mu U \qquad \forall i \geq 0$$

$$SW_\nu(\mathbf{M}_i^*)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^\mu U \qquad \forall i \geq 2$$

Proof. Consider $V^{\underline{\otimes}\nu}$ as an object in $\underline{Rep}(S_\nu)$. As such, it is a direct sum $\bigoplus_\mu X_\mu \otimes V_\mu$, where V_μ is the multiplicity space of X_μ. In fact, V_μ has the structure of $\mathbb{Z}_+$-graded $\mathfrak{gl}(U)$-module, each grade being a polynomial $\mathfrak{gl}(U)$-module.

We now consider the full subcategory $\mathcal{D}$ of $\mathcal{B}_\lambda$ whose objects are those which do not have $\mathbf{L}_0$ among their composition factors. Recall that $\mathbf{L}_i \in \mathcal{D}$ for $i > 0$, and $\mathbf{M}_i, \mathbf{M}_i^*, \mathbf{P}_i \in \mathcal{D}$ whenever $i \geq 2$.

We will denote by $\mathcal{F}$ the following functor from $\mathcal{D}$ to the category $Ind{-}Mod_{\mathcal{U}(\mathfrak{gl}(U)),poly}$:

$$\mathcal{F} := SW_\nu(\cdot)|_{\mathfrak{gl}(U)} = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\cdot, V^{\underline{\otimes}\nu})$$

Next, for any $X \in \mathcal{D}$, we have the following isomorphism of $\mathfrak{gl}(U)$-modules:

$$\mathcal{F}(X) = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X, V^{\underline{\otimes}\nu}) \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X, \bigoplus_{i>0} X_{\lambda^{(i)}} \otimes V_{\lambda^{(i)}})$$

Since we know that $X_{\lambda^{(i)}} = \mathbf{P}_{i-1}$ is injective for $i > 0$ (see Proposition 3.2.4.10), we immediately conclude that $\mathcal{F}$ is exact.

Now, one easily sees from the $\mathfrak{gl}(U)$-decomposition $V^{\underline{\otimes}\nu}|_{\mathfrak{gl}(U)} = \bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}$, together with Lemma 5.0.0.40, that for any $i \geq 0$,

$$SW_\nu(\mathbf{P}_i = X_{\lambda^{(i+1)}})|_{\mathfrak{gl}(U)} \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_{\lambda^{(i+1)}}, \bigoplus_{k\geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) \cong$$

$$\cong \bigoplus_{k\geq 0} \bigoplus_{\mu:|\mu|=k} \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_{\lambda^{(i+1)}} \otimes \mu, \Delta_k) \otimes S^\mu U \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i)}} S^\mu U \oplus \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i+1)}} S^\mu U$$

(the k-th grade of $SW_\nu(\mathbf{P}_i)$ is the direct sum of $S^\mu U$ such that $|\mu| = k$).

Fix $i \geq 1$. We can now apply $\mathcal{F}$ to the following long exact sequences in $\mathcal{D}$ (these exact sequences exist due to Proposition 3.2.4.10):

$$... \to \mathbf{P}_{i+3} \to \mathbf{P}_{i+2} \to \mathbf{P}_{i+1} \to \mathbf{M}_{i+1} \to 0$$

$$0 \to \mathbf{M}^*_{i+1} \to \mathbf{P}_{i+1} \to \mathbf{P}_{i+2} \to \mathbf{P}_{i+3} \to ...$$

$$0 \to \mathbf{L}_i \to \mathbf{M}_{i+1} \to \mathbf{M}_{i+2} \to \mathbf{M}_{i+3} \to ...$$

Using

$$\mathcal{F}(\mathbf{P}_i) \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i)} \cup \mathcal{I}^+_{\lambda(i+1)}} S^\mu U$$

and the fact that $\mathcal{F}$ is exact, we conclude that

$$\mathcal{F}(\mathbf{M}_{i+1}) \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i+1)}} S^\mu U \cong \mathcal{F}(\mathbf{M}^*_{i+1})$$

and

$$\mathcal{F}(\mathbf{L}_i) \cong V_{\lambda^{(i+1)}} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i+1)} \cap \mathcal{I}^+_{\lambda(i)}} S^\mu U$$

It remains to check the $\mathfrak{gl}(U)$-structure of $SW_\nu(\mathbf{L}_0 \cong X_{\lambda^{(0)}})|_{\mathfrak{gl}(U)}$, $SW_\nu(\mathbf{M}_1)|_{\mathfrak{gl}(U)}$.

Similarly to the decomposition of $\mathcal{F}(\mathbf{P}_i)$, we use the $\mathfrak{gl}(U)$-decomposition $V^{\underline{\otimes}\nu} = \bigoplus_{k\geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}$, together with Lemma 5.0.0.40, to get the following isomorphisms of

$\mathfrak{gl}(U)$-modules:

$$SW_\nu(X_{\lambda^{(0)}})|_{\mathfrak{gl}(U)} \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(X_{\lambda^{(0)}}, \bigoplus_{k\geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}) \cong$$

$$\cong \bigoplus_{k\geq 0} \bigoplus_{\mu:|\mu|=k} \mathrm{Hom}_{\underline{Rep}(S_\nu)\boxtimes Rep(S_k)}(X_{\lambda^{(0)}} \otimes \mu, \Delta_k) \otimes S^\mu U \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(0)}}} S^\mu U$$

In particular, we have:

$$V_{\lambda^{(0)}} \oplus V_{\lambda^{(1)}} \cong SW_\nu(X_{\lambda^{(0)}})|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(0)}}} S^\mu U$$

Recall that

$$V_{\lambda^{(0)}} \cong (S^{\tilde{\lambda^{(0)}}(\nu)}V)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu:\, \tilde{\lambda^{(0)}}(\nu)\in\mathcal{I}^+_\mu} S^\mu U$$

(c.f. Proposition 4.3.0.24). Now, for any Young diagram μ, we have (c.f. the proof of Lemma 5.0.0.40):

$$\tilde{\lambda^{(0)}}(\nu) \in \mathcal{I}^+_\mu \Leftrightarrow \left[\mu \in \mathcal{I}^+_{\lambda^{(0)}} \text{ and } \mu_1 + \left|\lambda^{(0)}\right| \leq \nu\right]$$

On the other hand, the description of non-trivial $\overset{\nu}{\sim}$-classes (c.f. Lemma 3.2.2.2) tells us that $\lambda^{(0)} \subset \lambda^{(1)}$, and $\lambda^{(1)} \setminus \lambda^{(0)}$ is a strip in row 1 of length $\nu - \left|\lambda^{(0)}\right| - \lambda^{(0)}_1 + 1$. Thus

$$\{\mu : \tilde{\lambda^{(0)}}(\nu) \in \mathcal{I}^+_\mu\} = \mathcal{I}^+_{\lambda^{(0)}} \setminus \mathcal{I}^+_{\lambda^{(1)}}$$

and so $V_{\lambda^{(1)}} \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(0)}}\cap\mathcal{I}^+_{\lambda^{(1)}}} S^\mu U$. We have already seen that $V_{\lambda^{(2)}} \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(2)}}\cap\mathcal{I}^+_{\lambda^{(1)}}} S^\mu U$, and we conclude that

$$SW_\nu(\mathbf{M}_1)|_{\mathfrak{gl}(U)} \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{M}_1, V^{\underline{\otimes}\nu}) = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{M}_1, \bigoplus_{i\geq 0} X_{\lambda^{(i)}} \otimes V_{\lambda^{(i)}}) \cong$$

$$\cong V_{\lambda^{(1)}} \oplus V_{\lambda^{(2)}} \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(0)}}\cap\mathcal{I}^+_{\lambda^{(1)}}} S^\mu U \oplus \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(2)}}\cap\mathcal{I}^+_{\lambda^{(1)}}} S^\mu U \cong \bigoplus_{\mu\in\mathcal{I}^+_{\lambda^{(1)}}} S^\mu U$$

(the last isomorphism can be inferred from Lemma 3.2.2.2).

□

Proposition 5.1.2.8. *For a simple object $\mathbf{L}_i$ in $\mathcal{B}_\lambda$, $i > 0$, we have: $SW_\nu(\mathbf{L}_i) \cong L_{i+1}$ (recall that the latter is defined to be zero if $i \geq k_\lambda - 1$).*

Proof. Fix $i > 0$. By definition, $SW_\nu(\mathbf{L}_i) := \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_i, V^{\underline{\otimes}\nu})$ with $\mathfrak{gl}(V)$-action on this space induced from the action of $\mathfrak{gl}(V)$ on $V^{\underline{\otimes}\nu}$.

By Lemma 5.1.2.7, we have the following decomposition of $SW_\nu(\mathbf{L}_i)$ as a $\mathfrak{gl}(U)$-module:

$$SW_\nu(\mathbf{L}_i)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}} \cap \mathcal{I}^+_{\lambda^{(i+1)}}} S^\mu U$$

If $i \geq k_\lambda - 1$, then $\ell(\mu) > \dim V - 1$ for any $\mu \in \mathcal{I}^+_{\lambda^{(i+1)}} \cap \mathcal{I}^+_{\lambda^{(i)}}$, so we get that $SW_\nu(\mathbf{L}_i) = \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_i, V^{\underline{\otimes}\nu}) = 0 = L_{i+1}$ and we are done.

Otherwise, notice that $\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_i, V^{\underline{\otimes}\nu})$ is a $\mathbb{Z}_+$-graded $\mathfrak{gl}(U)$-module, with the grading inherited from $V^{\underline{\otimes}\nu}$: $S^\mu U$ lies in grade $|\mu|$. The minimal grade is thus $\left|\lambda^{(i+1)}\right|$, and it consists of the $\mathfrak{gl}(U)$-module $S^{\lambda^{(i+1)}}U$.

Recall that $\mathfrak{u}^+_{\mathfrak{p}}$ acts on the graded space $V^{\underline{\otimes}\nu} \cong \bigoplus_{k \geq 0}(U^{\otimes k} \otimes \Delta_k)^{S_k}$ by operators of degree -1, therefore it acts by zero on the subspace $S^{\lambda^{(i+1)}}U$ of $\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_i, V^{\underline{\otimes}\nu})$.

So there must be a non-zero map $M_{i+1} \to SW_\nu(\mathbf{L}_i)$, and its image can be either M_{i+1} itself or L_{i+1}. From the decomposition of $SW_\nu(\mathbf{L}_i)$ as a $\mathfrak{gl}(U)$-module, we see that the image is L_{i+1}, and the induced map $L_{i+1} \to SW_\nu(\mathbf{L}_i)$ is an isomorphism. □

The following lemma will be useful to us later:

Lemma 5.1.2.9.

$$\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_0, SW^*_\nu(L_0)) = 0$$

Proof. Recall that by definition of the functor SW^*_ν,

$$\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_0, SW^*_\nu(L_0)) = \mathrm{Hom}_{Ind-(\underline{Rep}^{ab}(S_\nu)\boxtimes O^{\mathfrak{p}}_{\nu,V})}(\mathbf{L}_0 \otimes L_0, V^{\underline{\otimes}\nu})$$

Recall also that the space $\mathfrak{u}_{\mathfrak{p}}^{-}$ acts on $\mathbf{L}_0 \otimes L_0$ by nilpotent operators, since L_0 is finite-dimensional and $\mathbb{Z}_+$-graded, and each non-zero element of $\mathfrak{u}_{\mathfrak{p}}^{-}$ acts by operator of degree 1. Now let

$$\phi \in \mathrm{Hom}_{Ind-(\underline{Rep}^{ab}(S_\nu)\boxtimes O^{\mathfrak{p}}_{\nu,V})}(\mathbf{L}_0 \otimes L_0, V^{\underline{\otimes}\nu})$$

The map ϕ is zero iff $\phi|_{\mathbf{L}_0\otimes S^{\lambda^{(0)}}U} = 0$.

Fix $k \in \mathbb{Z}_+$ so that we have an inclusion of $\underline{Rep}^{ab}(S_\nu)$-objects:

$$\phi(\mathbf{L}_0 \otimes v) \subset (\Delta_k \otimes U^{\otimes k})^{S_k}$$

where v is the highest weight vector in $S^{\lambda^{(0)}}U$. Then we automatically get an inclusion of $\underline{Rep}^{ab}(S_\nu)$-objects

$$\phi(\mathbf{L}_0 \otimes S^{\lambda^{(0)}}U) \subset (\Delta_k \otimes U^{\otimes k})^{S_k}$$

Let $l := \left|\lambda^{(0)}\right|$. Since $\mathbf{L}_0 = X_{\lambda^{(0)}}$, it is a summand of Δ_i iff $i \geq l$, so we immediately see that $k \geq l$.

Now, $\mathbf{L}_0 \otimes S^{\lambda^{(0)}}U$ is a direct summand of $\Delta_l \otimes U^{\otimes l}$, so one can easily find $\psi : \Delta_l \otimes U^{\otimes l} \to \Delta_k \otimes U^{\otimes k}$ such that $\phi(\mathbf{L}_0 \otimes S^{\lambda^{(0)}}U) = \mathrm{Im}(\psi)$.

As we said before, $\mathfrak{u}_{\mathfrak{p}}^{-}$ acts on $\mathbf{L}_0 \otimes L_0$ by nilpotent operators, so for any $u \in U \cong \mathfrak{u}_{\mathfrak{p}}^{-}$, $(F_u)^N \circ \phi = 0$ for $N >> 0$.

On the other hand, we know that $(F_u)^N \circ \psi \neq 0$ if $\psi \neq 0$ (by applying Lemma 4.2.0.22 iteratively to $\psi, F_u \circ \psi, ..., (F_u)^{N-1} \circ \psi$).

We conclude that

$$\mathrm{Hom}_{Ind-(\underline{Rep}^{ab}(S_\nu)\boxtimes O^{\mathfrak{p}}_{\nu,V})}(\mathbf{L}_0 \otimes L_0, V^{\underline{\otimes}\nu}) = 0$$

as needed. □

We now use Lemma 5.1.2.9 to compute the images of the "exceptional" objects in our blocks $\mathcal{B}_\lambda$, $\mathfrak{B}_\lambda$ under the functors SW_ν, SW^*_ν, respectively.

Lemma 5.1.2.10.

(a) $SW_\nu(\mathbf{L}_0 = \mathbf{M}_0 = X_{\lambda^{(0)}}) \cong M_0$.

(b) $SW^*_\nu(L_0) = 0$.

Proof. (a) We will use an agrument similar to the one in the proof of Proposition 5.1.2.8. Recall that from Lemma 5.1.2.7, we have the following isomorphism of $\mathbb{Z}_+$-graded $\mathfrak{gl}(U)$-modules:

$$SW_\nu(\mathbf{L}_0 = \mathbf{M}_0 = X_{\lambda^{(0)}}) \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(0)}}} S^\mu U$$

($S^\mu U$ lies in degree $|\mu|$). Recall also that $\mathfrak{u}^+_{\mathfrak{p}}$ acts on the right hand side by operators of degree -1.

This implies that there is a non-zero map of $\mathfrak{gl}(V)$-modules

$$\phi : M_0 = M_{\mathfrak{p}}(\nu - \left|\lambda^{(0)}\right|, \lambda^{(0)}) \to SW_\nu(\mathbf{L}_0)$$

From the $\mathfrak{gl}(U)$-decomposition of M_0 (c.f. Lemma 3.3.1.3), if this map ϕ is injective, then it is bijective as well.

So we only need to check that ϕ is injective. Indeed, assume ϕ is not injective. Since ϕ is not zero, ϕ must factor through L_0, so we have:

$$\dim \mathrm{Hom}_{O^{\mathfrak{p}}_\nu}(L_0, SW_\nu(\mathbf{L}_0)) \geq 1$$

But from adjointness of SW_ν, SW^*_ν, together with Lemma 5.1.2.9, we have:

$$\mathrm{Hom}_{O^{\mathfrak{p}}_\nu}(L_0, SW_\nu(\mathbf{L}_0)) \cong \mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}_0, SW^*_\nu(L_0)) = 0$$

We obtained a contradiction, which means that ϕ is injective.

(b) Recall that since SW_ν, SW^*_ν are adjoint, for any Young diagram μ we have

$$\mathrm{Hom}_{Ind-\underline{Rep}^{ab}(S_\nu)}(\mathbf{L}(\mu), SW^*_\nu(L_0)) \cong \mathrm{Hom}_{O^{\mathfrak{p}}_\nu}(L_0, SW_\nu(\mathbf{L}(\mu)))$$

The latter is zero by Propositions 5.1.1.1, 5.1.2.8, and Lemma 5.1.2.9. Hence $SW^*_\nu(L_0) = 0$.

□

We can now prove the following proposition:

Proposition 5.1.2.11.

(a) $SW_\nu(\mathbf{M}_i) \cong M_i$ *whenever* $i \geq 0$.

(b) $SW_\nu(\mathbf{M}^*_i) \cong M_i^\vee$ *whenever* $i \geq 2$.

(c) $SW_\nu(\mathbf{P}_i) \cong P_{i+1}$ *whenever* $i \geq 0$*, and* $i < k_\lambda - 1$ *or* $i \geq k_\lambda$ *(recall that in the latter case* $P_{i+1} = 0$*);* $SW_\nu(\mathbf{P}_{k_\lambda - 1}) \cong L_{k_\lambda - 1}$.

(d) $SW_\nu(\mathbf{M}^*_1) \cong Ker(P_1 \twoheadrightarrow L_1)$.

Proof. (a) For $i = 0$, we have already proved (in Lemma 5.1.2.10) that $SW_\nu(\mathbf{M}_0) \cong M_0$. We now fix $i \in \mathbb{Z}_{>0}$.

Recall from Lemma 5.1.2.7 that for $i > 0$, we have an isomorphism of $\mathfrak{gl}(U)$-modules:

$$SW_\nu(\mathbf{M}_i)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}} S^\mu U$$

Similarly to the argument in the proofs of Propositions 5.1.1.1 and 5.1.2.8, we have a $\mathbb{Z}_+$-grading on the space $SW_\nu(\mathbf{M}_i)$, which is inherited from the grading on $V^{\underline{\otimes}\nu}$. Grade j of $SW_\nu(\mathbf{M}_i)$ is then $\bigoplus_{\mu \in \mathcal{I}^+_{\lambda^{(i)}}, |\mu| = j} S^\mu U$. The minimal grade is thus $\left|\lambda^{(i)}\right|$, and it consists of the $\mathfrak{gl}(U)$-module $S^{\lambda^{(i)}} U$.

Recall that by definition of $V^{\underline{\otimes}\nu}$, $\mathfrak{u}^{\mathfrak{p}}_+$ acts on the graded space $V^{\underline{\otimes}\nu} \cong \bigoplus_{k \geq 0} (U^{\otimes k} \otimes \Delta_k)^{S_k}$ by operators of degree -1, therefore, it acts by zero on the subspace $S^{\lambda^{(i)}} U$ of $SW_\nu(\mathbf{M}_i) = \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{M}_i, V^{\underline{\otimes}\nu})$.

If $i \geq k_\lambda$, then $SW_\nu(\mathbf{M}_i) = 0 = M_i$, and we are done.

Otherwise, $M_i \neq 0$, and there must be a non-zero map $M_i \to SW_\nu(\mathbf{M}_i)$; its image can be either M_i itself or L_i. In the first case, we get an isomorphism $M_i \cong SW_\nu(\mathbf{M}_i)$ (from the $\mathfrak{gl}(U)$-decomposition of both), and again, we are done.

We will now assume that we are in the second case, and there is a non-zero map $L_i \to SW_\nu(\mathbf{M}_i)$. Then the $\mathfrak{gl}(U)$-decomposition of the quotient ${}^{SW_\nu(\mathbf{M}_i)}\big/_{L_i}$ means that this module is congruent to L_{i+1}. Notice that at this point we can assume that $i < k_\lambda - 1$ (otherwise $L_{i+1} = 0$, so $M_i = L_i \cong SW_\nu(\mathbf{M}_i)$). We will prove that under this assumption, we arrive to a contradiction.

Now, consider the short exact sequence

$$0 \to \mathbf{L}_{i-1} \xrightarrow{\phi} \mathbf{M}_i \xrightarrow{\psi} \mathbf{L}_i \to 0$$

Since the contravariant functor SW_ν is left-exact, we have (using Proposition 5.1.2.8 and Lemma 5.1.2.10, part (a))

$$0 \to SW_\nu(\mathbf{L}_i) \cong L_{i+1} \xrightarrow{SW_\nu(\psi)} SW_\nu(\mathbf{M}_i) \xrightarrow{SW_\nu(\phi)} SW_\nu(\mathbf{L}_{i-1}) \cong L_i$$

So $SW_\nu(\psi)$ is an insertion of a direct summand, $SW_\nu(\phi)$ is a projection onto a direct summand, and we get:

$$SW_\nu(\mathbf{M}_i) \cong L_i \oplus L_{i+1}$$

We now use the unit natural transformation ϵ described in Notation 5.0.0.46. We have a commutative diagram:

$$\begin{array}{ccccc} SW_\nu^*(SW_\nu(\mathbf{L}_{i-1})) & \xrightarrow{SW_\nu^*(SW_\nu(\phi))} & SW_\nu^*(SW_\nu(\mathbf{M}_i)) & \xrightarrow{SW_\nu^*(SW_\nu(\psi))} & SW_\nu^*(SW_\nu(\mathbf{L}_i)) \\ \epsilon_{\mathbf{L}_{i-1}} \uparrow & & \epsilon_{\mathbf{M}_i} \uparrow & & \epsilon_{\mathbf{L}_i} \uparrow \\ \mathbf{L}_{i-1} & \xrightarrow{\phi} & \mathbf{M}_i & \xrightarrow{\psi} & \mathbf{L}_i \end{array}$$

which can be rewritten as

$$\begin{array}{ccccc} SW_\nu^*(L_i) & \xrightarrow{SW_\nu^*(SW_\nu(\phi))} & SW_\nu^*(L_i) \oplus SW_\nu^*(L_{i+1}) & \xrightarrow{SW_\nu^*(SW_\nu(\psi))} & SW_\nu^*(L_{i+1}) \\ {\scriptstyle \epsilon_{\mathbf{L}_{i-1}}}\uparrow & & {\scriptstyle \epsilon_{\mathbf{M}_i}}\uparrow & & {\scriptstyle \epsilon_{\mathbf{L}_i}}\uparrow \\ \mathbf{L}_{i-1} & \xrightarrow{\phi} & \mathbf{M}_i & \xrightarrow{\psi} & \mathbf{L}_i \end{array}$$

Since the contravariant functor SW_ν^* is additive, $SW_\nu^*(SW_\nu(\phi))$ is an insertion of a direct summand, and $SW_\nu^*(SW_\nu(\psi))$ is a projection onto a direct summand.

Now, the relations in Lemma 5.0.0.47 imply that $\epsilon_{\mathbf{L}_{i-1}}, \epsilon_{\mathbf{M}_i}, \epsilon_{\mathbf{L}_i}$ are all non-zero as long as $SW_\nu(\mathbf{L}_{i-1}), SW_\nu(\mathbf{M}_i), SW_\nu(\mathbf{L}_i)$ are non-zero, which is guaranteed by the assumption $i < k_\lambda - 1$.

This means that the image of $\mathbf{L}_{i-1}$ under $SW_\nu^*(SW_\nu(\phi)) \circ \epsilon_{\mathbf{L}_{i-1}}$ is $\mathbf{L}_{i-1}$, and it lies inside the direct summand $SW_\nu^*(L_i)$ of $SW_\nu^*(SW_\nu(\mathbf{M}_i))$. We then deduce that $\epsilon_{\mathbf{M}_i}$ is injective (since it is not zero on the socle $\mathbf{L}_{i-1}$ of $\mathbf{M}_i$), and that its image lies entirely inside the direct summand $SW_\nu^*(L_i)$ of $SW_\nu^*(SW_\nu(\mathbf{M}_i))$ (since $\mathbf{M}_i$ is indecomposable).

But this clearly contradicts the right half of the above commutative diagram, since it means that

$$SW_\nu^*(SW_\nu(\psi)) \circ \epsilon_{\mathbf{M}_i} = 0$$

while we have already established that $\epsilon_{\mathbf{L}_i} \circ \psi \neq 0$.

(b) The proof for $SW_\nu(\mathbf{M}_k^*)$ is very similar to the one given for $SW_\nu(\mathbf{M}_k)$. Now fix $i \geq 2$.

Consider the short exact sequence

$$0 \to \mathbf{L}_i \xrightarrow{\phi} \mathbf{M}_i^* \xrightarrow{\psi} \mathbf{L}_{i-1} \to 0$$

Since the contravariant functor SW_ν is left-exact, we have (using Proposition 5.1.2.8)

$$0 \to SW_\nu(\mathbf{L}_{i-1}) \cong L_i \xrightarrow{SW_\nu(\psi)} SW_\nu(\mathbf{M}_i^*) \xrightarrow{SW_\nu(\phi)} SW_\nu(\mathbf{L}_i) \cong L_{i+1}$$

Furthermore, Lemma 5.1.2.7 tells us that for $i > 0$, we have an isomorphism of $\mathfrak{gl}(U)$-modules:

$$SW_\nu(\mathbf{M}_i^*)|_{\mathfrak{gl}(U)} \cong \bigoplus_{\mu \in \mathcal{I}^+_{\lambda(i)}} S^\mu U$$

This decomposition, together with the $\mathfrak{gl}(U)$-decomposition of L_i, L_{i+1}, tell us that the above exact sequence can be completed to a short exact sequence

$$0 \to SW_\nu(\mathbf{L}_{i-1}) \cong L_i \xrightarrow{SW_\nu(\psi)} SW_\nu(\mathbf{M}_i^*) \xrightarrow{SW_\nu(\phi)} SW_\nu(\mathbf{L}_i) \cong L_{i+1} \to 0$$

Applying the (exact) functor $(\cdot)^\vee : O_\nu^\mathfrak{p} \to (O_\nu^\mathfrak{p})^{op}$ to the above exact sequence, we conclude that $SW_\nu(\mathbf{M}_i^*)^\vee$ is isomorphic to either M_i or $L_i \oplus L_{i+1}$. This implies that $SW_\nu(\mathbf{M}_i^*)$ is isomorphic to either $M_i^\vee$ (which is what we want to show) or $L_i \oplus L_{i+1}$.

We will now assume that we are in the case $SW_\nu(\mathbf{M}_i^*) \cong L_i \oplus L_{i+1}$. Furthermore, we will assume that $i < k_\lambda - 1$ (otherwise $L_{i+1} = 0$, so $M_i^\vee = L_i \cong SW_\nu(\mathbf{M}_i^*)$). We will prove that under this assumption, we arrive to a contradiction. Since we assumed that $i < k_\lambda - 1$, we have: $L_i, L_{i+1} \neq 0$, which means that $SW_\nu(\psi), SW_\nu(\phi) \neq 0$ are insertion of and projection onto direct summands, respectively.

We now construct the commutative diagram

$$\begin{array}{ccccc}
SW_\nu^*(SW_\nu(\mathbf{L}_i)) & \xrightarrow{SW_\nu^*(SW_\nu(\phi))} & SW_\nu^*(SW_\nu(\mathbf{M}_i^*)) & \xrightarrow{SW_\nu^*(SW_\nu(\psi))} & SW_\nu^*(SW_\nu(\mathbf{L}_{i-1})) \\
\epsilon_{\mathbf{L}_i}\uparrow & & \epsilon_{\mathbf{M}_i^*}\uparrow & & \epsilon_{\mathbf{L}_{i-1}}\uparrow \\
\mathbf{L}_i & \xrightarrow{\phi} & \mathbf{M}_i^* & \xrightarrow{\psi} & \mathbf{L}_{i-1}
\end{array}$$

which can be rewritten as

$$\begin{array}{ccccc}
SW_\nu^*(L_{i+1}) & \xrightarrow{SW_\nu^*(SW_\nu(\phi))} & SW_\nu^*(L_{i+1}) \oplus SW_\nu^*(L_i) & \xrightarrow{SW_\nu^*(SW_\nu(\psi))} & SW_\nu^*(L_i) \\
\epsilon_{\mathbf{L}_i}\uparrow & & \epsilon_{\mathbf{M}_i^*}\uparrow & & \epsilon_{\mathbf{L}_{i-1}}\uparrow \\
\mathbf{L}_i & \xrightarrow{\phi} & \mathbf{M}_i^* & \xrightarrow{\psi} & \mathbf{L}_{i-1}
\end{array}$$

Exactly the same arguments as in part (a) now apply (we use the fact that $\mathbf{M}_i^*$ is indecomposable), and we get a contradiction.

(c) Let $i \geq 0$. Consider the exact sequence

$$0 \to \mathbf{M}_{i+1} \xrightarrow{\phi} \mathbf{P}_i \xrightarrow{\psi} \mathbf{M}_i \to 0$$

Since the contravariant functor SW_ν is left-exact, we get an exact sequence

$$0 \to SW_\nu(\mathbf{M}_i) \xrightarrow{SW_\nu(\psi)} SW_\nu(\mathbf{P}_i) \xrightarrow{SW_\nu(\phi)} SW_\nu(\mathbf{M}_{i+1})$$

and in particular (see part (a)): $M_i \cong SW_\nu(\mathbf{M}_i) \hookrightarrow SW_\nu(\mathbf{P}_i)$.

If $i \geq k_\lambda - 1$, then part (a) tells us that $SW_\nu(\mathbf{M}_{i+1}) = M_{i+1} = 0$. We conclude that $M_i \cong SW_\nu(\mathbf{M}_i) \cong SW_\nu(\mathbf{P}_i)$. In particular, $SW_\nu(\mathbf{P}_i) = 0$ if $i \geq k_\lambda$, and

$$SW_\nu(\mathbf{M}_{k_\lambda - 1}) \cong SW_\nu(\mathbf{P}_{k_\lambda - 1}) \cong M_{k_\lambda - 1} \cong L_{k_\lambda - 1}$$

From now on, we will assume that $i < k_\lambda - 1$, and thus $M_i, M_{i+1} \neq 0$.

Now, P_{i+1} is the injective hull of M_i, so there is a map $f : SW_\nu(\mathbf{P}_i) \to P_{i+1}$ such that the following diagram is commutative:

$$\begin{array}{ccccc} 0 & \longrightarrow & SW_\nu(\mathbf{M}_i) & \xrightarrow{SW_\nu(\psi)} & SW_\nu(\mathbf{P}_i) \\ & & \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & M_i & \longrightarrow & P_{i+1} \end{array}$$

From the $\mathfrak{gl}(U)$-decomposition of $SW_\nu(\mathbf{M}_i), SW_\nu(\mathbf{P}_i), SW_\nu(\mathbf{M}_{i+1})$ (see Lemma 5.1.2.7), we see that the map $SW_\nu(\phi)$ is surjective. This means that there is a non-zero map $\bar{f} : SW_\nu(\mathbf{M}_{i+1}) \to M_{i+1}$ so the diagram below is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SW_\nu(\mathbf{M}_i) & \xrightarrow{SW_\nu(\psi)} & SW_\nu(\mathbf{P}_i) & \xrightarrow{SW_\nu(\phi)} & SW_\nu(\mathbf{M}_{i+1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & M_i & \longrightarrow & P_{i+1} & \longrightarrow & M_{i+1} & \longrightarrow & 0 \end{array}$$

Since $SW_\nu(\mathbf{M}_{i+1}) \cong M_{i+1}$, we see that $\bar{f}$ is either an isomorphism, or zero. In the former case, f is an isomorphism as well, and we are done.

So it remains to prove that $\bar{f} \neq 0$.

Assume $\bar{f} = 0$. This means that the image of f is $M_i \subset P_{i+1}$, and thus $SW_\nu(\mathbf{P}_i) = M_i \oplus M_{i+1}$, with the maps $SW_\nu(\psi), SW_\nu(\phi)$ being an insertion of a direct summand and a projection onto a direct summand, respectively.

We now construct the commutative diagram

$$\begin{array}{ccccc}
SW^*_\nu(SW_\nu(\mathbf{M}_{i+1})) & \xrightarrow{SW^*_\nu(SW_\nu(\phi))} & SW^*_\nu(SW_\nu(\mathbf{P}_i)) & \xrightarrow{SW^*_\nu(SW_\nu(\psi))} & SW^*_\nu(SW_\nu(\mathbf{M}_i)) \\
\uparrow{\scriptstyle \epsilon_{\mathbf{M}_{i+1}}} & & \uparrow{\scriptstyle \epsilon_{\mathbf{P}_i}} & & \uparrow{\scriptstyle \epsilon_{\mathbf{M}_i}} \\
\mathbf{M}_{i+1} & \xrightarrow{\phi} & \mathbf{P}_i & \xrightarrow{\psi} & \mathbf{M}_i
\end{array}$$

which can be rewritten as

$$\begin{array}{ccccc}
SW^*_\nu(M_{i+1}) & \xrightarrow{SW^*_\nu(SW_\nu(\phi))} & SW^*_\nu(M_{i+1}) \oplus SW^*_\nu(M_i) & \xrightarrow{SW^*_\nu(SW_\nu(\psi))} & SW^*_\nu(M_i) \\
\uparrow{\scriptstyle \epsilon_{\mathbf{M}_{i+1}}} & & \uparrow{\scriptstyle \epsilon_{\mathbf{P}_i}} & & \uparrow{\scriptstyle \epsilon_{\mathbf{M}_i}} \\
\mathbf{M}_{i+1} & \xrightarrow{\phi} & \mathbf{P}_i & \xrightarrow{\psi} & \mathbf{M}_i
\end{array}$$

The same type of argument as in part (a) now applies (we use the fact that $\mathbf{P}_i$ is indecomposable), and we get a contradiction.

(d) Consider the short exact sequences

$$0 \to \mathbf{L}_1 \longrightarrow \mathbf{M}^*_1 \longrightarrow \mathbf{L}_0 \to 0$$

$$0 \to \mathbf{L}_0 \longrightarrow \mathbf{P}_0 \longrightarrow \mathbf{M}^*_1 \to 0$$

The contravariant functor SW_ν is left-exact, so we have (using part (a), Lemma 5.1.2.10(a) and Proposition 5.1.2.8)

$$0 \to SW_\nu(\mathbf{L}_0) \cong M_0 \longrightarrow SW_\nu(\mathbf{M}^*_1) \longrightarrow SW_\nu(\mathbf{L}_1) \cong L_2$$

$$0 \to SW_\nu(\mathbf{M}_1^*) \longrightarrow SW_\nu(\mathbf{P}_0) \cong P_1 \longrightarrow SW_\nu(\mathbf{L}_0) \cong M_0$$

The first of these two exact sequences implies that $[SW_\nu(\mathbf{M}_1^*) : L_1] = 1$, hence the map $SW_\nu(\mathbf{M}_1^*) \to P_1$ in the second sequence is not an isomorphism. The second one then means that $SW_\nu(\mathbf{M}_1^*)$ is the kernel of the unique non-zero map $P_1 \to M_0$, which factors through the canonical map $P_1 \to L_1$. Thus $SW_\nu(\mathbf{M}_1^*) \cong Ker(P_1 \twoheadrightarrow L_1)$.

□

Proposition 5.1.2.12. *The contravariant functor $\widehat{SW}_\nu : \mathcal{B}_\lambda \to \hat{\pi}(\mathfrak{B}_\lambda)$ is essentially surjective.*

Proof. We first prove a Sublemma:

Sublemma 5.1.2.13.

(a) Let I be an injective object in $\hat{\pi}(\mathfrak{B}_\lambda)$. Then there exists a projective object P in $\mathcal{B}_\lambda$ such that $\widehat{SW}_\nu(P) = I$.

(b) Consider the restriction of $\widehat{SW}_\nu$ to the full subcategory of $\mathcal{B}_\lambda$ consisting of projective objects. This restriction is a full contravariant functor to $\hat{\pi}(\mathfrak{B}_\lambda)$.

Proof. Recall from Corollary 5.1.2.6 that the set of isomorphism classes of indecomposable injective objects in $\hat{\pi}(\mathfrak{B}_\lambda)$ is $\{\hat{\pi}(P_i)\}_{0<i<k_\lambda}$. The set of isomorphism classes of indecomposable projective objects in $\mathcal{B}_\lambda$ is $\{\mathbf{P}_i\}_{i\geq 0}$ (c.f. Section 3.2.4).

We know from Proposition 5.1.2.11 that $\hat{\pi}(P_i) \cong \widehat{SW}_\nu(\mathbf{P}_{i-1})$ for any $0 < i < k_\lambda$. This immediately implies the first part of the sublemma.

We now consider the restriction of $\widehat{SW}_\nu$ to the full subcategory of $\mathcal{B}_\lambda$ consisting of projective objects.

To see that this restriction is full, we need to check that for any $i, j \geq 0$, the map

$$\widehat{SW}_{\nu,\mathbf{P}_j,\mathbf{P}_i} : \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{P}_j, \mathbf{P}_i) \to \mathrm{Hom}_{\hat{\mathcal{O}}_\nu^\mathfrak{p}}(\widehat{SW}_\nu(\mathbf{P}_i), \widehat{SW}_\nu(\mathbf{P}_j)) \tag{5.1}$$

is surjective.

We use the following observation (which follows from the definition of the Serre quotient):

Observation 5.1.2.14. Let $E, E' \in O^{\mathfrak{p}}_{\nu}$. Assume E has no finite-dimensional quotients and E' has no finite-dimensional submodules. Then

$$\mathrm{Hom}_{\hat{O}^{\mathfrak{p}}_{\nu}}(\hat{\pi}(E), \hat{\pi}(E')) = \mathrm{Hom}_{O^{\mathfrak{p}}_{\nu}}(E, E')$$

In particular, this is true for E, E' being $P_i, M_i, M_i^{\vee}, L_i$ $(i \geq 1)$.

Recall from Theorem 3.2.2.6, Propositions 3.2.4.10 and 5.1.2.11 that if $|i-j| > 1$, or if $i \geq k_{\lambda}$, or if $j \geq k_{\lambda}$, then the right hand side Hom-space in (5.1) is zero and there is nothing to prove.

If either $i = k_{\lambda} - 1$ or $j = k_{\lambda} - 1$, we only need to check the cases
$(i, j) = (k_{\lambda} - 1, k_{\lambda} - 2), (k_{\lambda} - 2, k_{\lambda} - 1), (k_{\lambda} - 1, k_{\lambda} - 1)$.

In all three cases $\mathrm{Hom}_{\hat{O}^{\mathfrak{p}}_{\nu}}(\widehat{SW}_{\nu}(\mathbf{P}_i), \widehat{SW}_{\nu}(\mathbf{P}_j))$ is one-dimensional, so we only need to check that the above map $\widehat{SW}_{\nu,\mathbf{P}_j,\mathbf{P}_i}$ is not zero. The case $i = j = k_{\lambda} - 1$ is obvious. Since $\widehat{SW}_{\nu}$ is contravariant and exact, the exact sequences

$$0 \to \mathbf{M}^*_{k_{\lambda}-2} \to \mathbf{P}_{k_{\lambda}-2} \to \mathbf{P}_{k_{\lambda}-1}$$

and

$$\mathbf{P}_{k_{\lambda}-1} \to \mathbf{P}_{k_{\lambda}-2} \to \mathbf{M}_{k_{\lambda}-2} \to 0$$

become

$$\hat{\pi}(L_{k_{\lambda}-1}) \to \hat{\pi}(P_{k_{\lambda}-1}) \to \hat{\pi}(M^{\vee}_{k_{\lambda}-2}) \to 0$$

and

$$0 \to \hat{\pi}(M_{k_{\lambda}-2}) \to \hat{\pi}(P_{k_{\lambda}-1}) \to \hat{\pi}(L_{k_{\lambda}-1})$$

which proves that $\widehat{SW}_{\nu,\mathbf{P}_j,\mathbf{P}_i}$ is not zero if $(i,j) = (k_{\lambda}-1, k_{\lambda}-2)$ or $(i,j) = (k_{\lambda}-2, k_{\lambda}-1)$.

We can now assume that $i, j < k_\lambda - 1$, and thus $\widehat{SW}_\nu(\mathbf{P}_i) \cong \hat{\pi}(P_{i+1}), \widehat{SW}_\nu(\mathbf{P}_j) \cong \hat{\pi}(P_{j+1})$.

If $|i-j| = 1$, then both Hom-spaces are at most one-dimensional and we only need to check that the above map $\widehat{SW}_{\nu, \mathbf{P}_j, \mathbf{P}_i}$ is not zero. Assume $j = i+1$ (the case $j = i-1$ is proved in a similar way). Let $\beta_{i+1} : \mathbf{P}_i \to \mathbf{P}_{i+1}$ be a non-zero morphism.

Then the kernel of β_{i+1} is $\mathbf{M}_i^*$, and since $\widehat{SW}_\nu$ is contravariant and exact, we get:

$$Coker(\widehat{SW}_\nu(\beta_{i+1})) \cong \widehat{SW}_\nu(\mathbf{M}_i^*) \cong \hat{\pi}(M_i^\vee) \ncong \hat{\pi}(P_{i+1})$$

which means that $\widehat{SW}_\nu(\beta_{i+1}) \neq 0$. Similarly, given a non-zero morphism $\alpha_{i+1} : \mathbf{P}_{i+1} \to \mathbf{P}_i$, we have:

$$Ker(\widehat{SW}_\nu(\alpha_{i+1})) \cong \widehat{SW}_\nu(Coker(\alpha_{i+1})) \cong \widehat{SW}_\nu(\mathbf{M}_i) \cong \hat{\pi}(M_i) \ncong \hat{\pi}(P_{i+1})$$

which means that $\widehat{SW}_\nu(\alpha_{i+1}) \neq 0$.

Finally, if $i = j$, then the space $\mathrm{End}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{P}_i)$ is spanned by endomorphisms $\mathrm{Id}_{\mathbf{P}_i}, \gamma_i$ of $\mathbf{P}_i$, where $Im(\gamma_i) \cong \mathbf{L}_i$ ($\gamma_i := \alpha_{i+1} \circ \beta_{i+1}$ in the above notation).

Since $\widehat{SW}_\nu$ is contravariant and exact, and (by assumption) $i < k_\lambda - 1$, we see that $\widehat{SW}_\nu(\gamma_i)$ will be a non-zero endomorphism of $\hat{\pi}(P_{i+1})$ factoring though $\hat{\pi}(L_{i+1})$. This means that $\widehat{SW}_\nu(\mathrm{Id}_{\mathbf{P}_i}), \widehat{SW}_\nu(\gamma_i)$ span $\mathrm{End}_{\hat{\mathcal{O}}_\nu^{\mathfrak{p}}}(\hat{\pi}(P_{i+1}))$.

This proves that for any $i, j \geq 0$, the map in (5.1) is surjective, and we are done. $\square$

We now show that $\widehat{SW}_\nu$ is essentially surjective. Indeed, let $E \in \hat{\pi}(\mathfrak{B}_\lambda)$. Then E has an injective resolution

$$0 \to E \to I^0 \xrightarrow{f} I^1$$

(I^0, I^1 are injective objects in $\hat{\pi}(\mathfrak{B}_\lambda)$). From the Sublemma 5.1.2.13 above, we know that there exist projective objects $P^0, P^1 \in \mathcal{B}_\lambda$ and a morphism $g : P^1 \to P^0$ such that

$$\widehat{SW}_\nu(P^0) = I^0, \widehat{SW}_\nu(P^1) = I^1, \widehat{SW}_\nu(g) = f$$

Then $E \cong Ker(f) \cong \widehat{SW}_\nu(Coker(g))$ (since $\widehat{SW}_\nu$ is exact). Thus $\widehat{SW}_\nu$ is essentially surjective. $\square$

Remark 5.1.2.15. The functor $\widehat{SW}_\nu : \mathcal{B}_\lambda \to \hat{\pi}(\mathfrak{B}_\lambda)$ is not full. For example, consider

$$\mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\mathbf{P}_{k_\lambda-1}, \mathbf{L}_{k_\lambda-2}) \xrightarrow{\widehat{SW}_\nu} \mathrm{Hom}_{\hat{\mathcal{O}}^{\mathfrak{p}}_\nu}(\widehat{SW}_\nu(\mathbf{L}_{k_\lambda-2}), \widehat{SW}_\nu(\mathbf{P}_{k_\lambda-1})) = \mathrm{End}_{\hat{\mathcal{O}}^{\mathfrak{p}}_\nu}(\hat{\pi}(L_{k_\lambda-1}))$$

The Hom-space in the left hand side is clearly zero, while the Hom-space in the right hand side is one-dimensional.

We now consider the Serre subcategory $Ker(\widehat{SW}_\nu|_{\mathcal{B}_\lambda})$ of $\mathcal{B}_\lambda$ (this is a Serre subcategory since $\widehat{SW}_\nu$ is exact). This subcategory is the Serre subcategory of $\mathcal{B}_\lambda$ generated by the simple objects $\mathbf{L}_i$, $i \geq k_\lambda - 1$.

We define the quotient of $\mathcal{B}_\lambda$ by $Ker(\widehat{SW}_\nu|_{\mathcal{B}_\lambda})$:

$$\overline{\pi} : \mathcal{B}_\lambda \longrightarrow \overline{\pi}(\mathcal{B}_\lambda)$$

By definition of $Ker(\widehat{SW}_\nu|_{\mathcal{B}_\lambda})$, the functor $\widehat{SW}_\nu$ factors through $\overline{\pi}$ and we get an exact contravariant functor

$$\widehat{\widehat{SW}}_\nu : \overline{\pi}(\mathcal{B}_\lambda) \longrightarrow \hat{\pi}(\mathfrak{B}_\lambda)$$

such that

$$\begin{array}{ccc}
\mathcal{B}_\lambda^{op} & \xrightarrow{SW_\nu} & \mathfrak{B}_\lambda \\
\downarrow{\scriptstyle \overline{\pi}^{op}} & \searrow{\scriptstyle \widehat{SW}_\nu} & \downarrow{\scriptstyle \hat{\pi}} \\
\overline{\pi}(\mathcal{B}_\lambda)^{op} & \xrightarrow{\widehat{\widehat{SW}}_\nu} & \hat{\pi}(\mathfrak{B}_\lambda)
\end{array}$$

Notice that all the functors in this commutative diagram except SW_ν are exact.

We now prove some properties of the functor $\overline{\pi}$ and the category $\overline{\pi}(\mathcal{B}_\lambda)$.

Lemma 5.1.2.16.

(a) The objects $\overline{\pi}(\mathbf{P}_i)$ are indecomposable injective (and projective) objects in $\overline{\pi}(\mathcal{B}_\lambda)$ for

any $i \leq k_\lambda - 2$.

(b) *The category* $\overline{\pi}(\mathcal{B}_\lambda)$ *has enough injectives and enough projectives.*

(c) *Moreover,* $\{\overline{\pi}(\mathbf{P}_i)\}_{0 \leq i \leq k_\lambda - 2}$ *is the full set of representatives of isomorphism classes of indecomposable injective (respectively, projective) objects in* $\overline{\pi}(\mathcal{B}_\lambda)$.

Proof. To prove the first statement, we use Lemma 5.1.2.5 and the information on the structure of $\mathbf{P}_i$ given in Proposition 3.2.4.10.

The proof of the last two statements mimics the proof of Corollary 5.1.2.6.

All we need to show is that the object $\overline{\pi}(\mathbf{P}_{k_\lambda - 1})$ is neither injective nor projective in $\overline{\pi}(\mathcal{B}_\lambda)$, but has a projective cover and an injective hull in $\overline{\pi}(\mathcal{B}_\lambda)$, both being direct sums of objects $\overline{\pi}(\mathbf{P}_i), i \leq k_\lambda - 2$.

But $\overline{\pi}(\mathbf{P}_{k_\lambda - 1}) \cong \overline{\pi}(\mathbf{L}_{k_\lambda - 2})$ (c.f. Proposition 3.2.4.10), and we have a surjective map $\overline{\pi}(\mathbf{P}_{k_\lambda - 2}) \twoheadrightarrow \overline{\pi}(\mathbf{L}_{k_\lambda - 2})$ and an injective map $\overline{\pi}(\mathbf{L}_{k_\lambda - 2}) \hookrightarrow \overline{\pi}(\mathbf{P}_{k_\lambda - 2})$. Since $\overline{\pi}(\mathbf{P}_{k_\lambda - 2})$ is an indecomposable injective and projective object in $\overline{\pi}(\mathcal{B}_\lambda)$, we conclude that $\overline{\pi}(\mathbf{P}_{k_\lambda - 1})$ is neither injective nor projective in $\overline{\pi}(\mathcal{B}_\lambda)$, but $\overline{\pi}(\mathbf{P}_{k_\lambda - 2})$ is the projective cover and the injective hull of $\overline{\pi}(\mathbf{P}_{k_\lambda - 1})$ in $\overline{\pi}(\mathcal{B}_\lambda)$. □

Theorem 5.1.2.17. *The functor* $\widehat{\overline{SW}}_\nu : \overline{\pi}(\mathcal{B}_\lambda) \to \hat{\pi}(\mathfrak{B}_\lambda)$ *is an anti-equivalence of abelian categories. That is,* $\widehat{\overline{SW}}_\nu : \overline{\pi}(\mathcal{B}_\lambda) \to \hat{\pi}(\mathfrak{B}_\lambda)$ *is an essentially surjective, fully faithful, exact contravariant functor.*

Proof.

- Proof that $\widehat{\overline{SW}}_\nu$ is faithful: by definition, if $\widehat{\overline{SW}}_\nu(X) = 0$ for some $X \in \overline{\pi}(\mathcal{B}_\lambda)$, then $X = 0$. Now, let $f : X \to Y$ in $\overline{\pi}(\mathcal{B}_\lambda)$, and assume $\widehat{\overline{SW}}_\nu(f) = 0$. Then $\widehat{\overline{SW}}_\nu(Im(f)) = 0$, i.e. $Im(f) = 0$, and thus $f = 0$.

- The fact that $\widehat{\overline{SW}}_\nu$ is essentially surjective follows directly from the fact that $\widehat{SW}_\nu$ is essentially surjective, c.f. Proposition 5.1.2.12.

- Proof that $\widehat{\overline{SW}}_\nu$ is full:

We start with the following sublemma:

Sublemma 5.1.2.18. *Let $Proj_{\overline{\pi}(\mathcal{B}_\lambda)}$ be the full subcategory of projective objects in $\overline{\pi}(\mathcal{B}_\lambda)$, and $Inj_{\hat{\pi}(\mathfrak{B}_\lambda)}$ be the full subcategory of injective objects in $\hat{\pi}(\mathfrak{B}_\lambda)$. Then $\widehat{\overline{SW}}_\nu$ induces an anti-equivalence of additive categories $Proj_{\overline{\pi}(\mathcal{B}_\lambda)} \to Inj_{\hat{\pi}(\mathfrak{B}_\lambda)}$.*

Proof. The first thing we need to check is that given a projective object in $\overline{\pi}(\mathcal{B}_\lambda)$, $\widehat{\overline{SW}}_\nu$ takes it to an injective object in $\hat{\pi}(\mathfrak{B}_\lambda)$. By Lemma 5.1.2.16, it is enough to check this for $\overline{\pi}(\mathbf{P}_i)$ for $i \leq k_\lambda - 2$, in which case this follows straight from the definition of $\widehat{\overline{SW}}_\nu$ together with Proposition 5.1.2.11 and Corollary 5.1.2.6.

Now,

$$\mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(\overline{\pi}(\mathbf{P}_i), \overline{\pi}(\mathbf{P}_j)) = \mathrm{Hom}_{\mathcal{B}_\lambda}(\mathbf{P}_i, \mathbf{P}_j),\ i, j \leq k_\lambda - 2$$

(since $\mathbf{P}_i, \mathbf{P}_j$ have no non-trivial subobjects nor quotients lying in $Ker(\widehat{SW}_\nu)$). The proof of Sublemma 5.1.2.13 then implies that the contravariant functor

$$\widehat{\overline{SW}}_\nu : Proj_{\overline{\pi}(\mathcal{B}_\lambda)} \to Inj_{\hat{\pi}(\mathfrak{B}_\lambda)}$$

is full and essentially surjective. We have already established that $\widehat{\overline{SW}}_\nu$ is faithful, which concludes the proof of the sublemma. □

Let $X \in \overline{\pi}(\mathcal{B}_\lambda)$. Since $\overline{\pi}(\mathcal{B}_\lambda)$ has enough projectives (which are also injectives), there exists an exact sequence

$$0 \to X \to I_X^0 \to I_X^1$$

in $\overline{\pi}(\mathcal{B}_\lambda)$, where I_X^0, I_X^1 are injective (and thus projective as well).

Now let $P \in \overline{\pi}(\mathcal{B}_\lambda)$ be a projective object. Sublemma 5.1.2.18 then tells us that $\widehat{\overline{SW}}_\nu(P)$ is an injective object in $\hat{\pi}(\mathfrak{B}_\lambda)$. Together with the fact that $\widehat{\overline{SW}}_\nu$ is exact, this gives us the following commutative diagram, whose rows are short exact sequences:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P, X) & \to & \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P, I_X^0) & \to & \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P, I_X^1) \\
\widehat{SW}_\nu \downarrow & & \widehat{SW}_\nu \downarrow & & \widehat{SW}_\nu \downarrow \\
\mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(P)) & \to & \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(I_X^0), \widehat{SW}_\nu(P)) & \to & \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(I_X^1), \widehat{SW}_\nu(P))
\end{array}$$

By Sublemma 5.1.2.18, the two rightmost vertical arrows are isomorphisms, which means that the arrow

$$\widehat{SW}_\nu : \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P, X) \to \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(P))$$

is an isomorphism as well.

Now, let $Y \in \overline{\pi}(\mathcal{B}_\lambda)$. There exists an exact sequence

$$P_Y^1 \to P_Y^0 \to Y \to 0$$

in $\overline{\pi}(\mathcal{B}_\lambda)$, where P_Y^0, P_Y^1 are projective. We then get the following commutative diagram, whose rows are short exact sequences:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(Y, X) & \to & \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P_Y^0, X) & \to & \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(P_Y^1, X) \\
\widehat{SW}_\nu \downarrow & & \widehat{SW}_\nu \downarrow & & \widehat{SW}_\nu \downarrow \\
\mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(Y)) & \to & \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(P_Y^0)) & \to & \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(P_Y^0))
\end{array}$$

We have already established that the two rightmost vertical arrows are isomorphisms, which means that the left vertical arrow

$$\widehat{SW}_\nu : \mathrm{Hom}_{\overline{\pi}(\mathcal{B}_\lambda)}(Y, X) \to \mathrm{Hom}_{\hat{\pi}(\mathfrak{B}_\lambda)}(\widehat{SW}_\nu(X), \widehat{SW}_\nu(Y))$$

is an isomorphism as well. Thus $\widehat{SW}_\nu$ is fully faithful.

□

5.2 Proofs of technical Lemmas

5.2.1 Action of $\mathfrak{gl}(V)$ on a complex tensor power

Lemma 5.2.1.1. *The action of* $\mathfrak{gl}(V)$ *described in Definition 4.2.0.16 is well-defined.*

Proof. Let $u, u_1, u_2 \in U \cong \mathfrak{u}_\mathfrak{p}^-, f, f_1, f_2 \in U^* \cong \mathfrak{u}_\mathfrak{p}^+, A \in \mathfrak{gl}(U)$. We have to check that the morphisms in $\underline{Rep}(S_\nu)$ by which u, f, A act are well-defined and satisfy the same commutation relations as do $u, f, A \in \mathfrak{gl}(V)$.

The first claim is obvious for the actions of f and A and one only needs to check that the image of $(U^{\otimes k} \otimes \Delta_k)^{S_k}$ under $\frac{1}{k+1}\sum_{1 \leq l \leq k+1} u^{(l)} \otimes res_l^*$ is S_{k+1}-invariant. For this, we will prove

Lemma 5.2.1.2. *Let* $\sigma \in S_{k+1}, l \in \{1, ..., k+1\}$. *Then there exists* $\rho_l(\sigma) \in S_k$ *such that*

$$\sigma \circ res_l^* = res_{\sigma(l)}^* \circ \rho_l(\sigma), \sigma \circ u^{(l)} = u^{(\sigma(l))} \circ \rho_l(\sigma)$$

Proof. We define the permutation $\rho_l(\sigma)$ to be the diagram in $\bar{P}_{k,k}$ constructed as follows.

Consider the diagram $\sigma \in \bar{P}_{k+1,k+1}$. Remove vertex l in its top row, vertex $\sigma(l)$ in the bottom row, as well as the edge connecting these vertices. The obtained diagram will lie in $\bar{P}_{k,k}$ and will have no solitary vertices; thus it represents a permutation in S_k.

The diagram obtained is the same we would get by considering the diagram for $\sigma \circ res_l^* \in \bar{P}_{k,k+1}$, and removing the unique solitary vertex $\sigma(l)$ from the bottom row of $\sigma \circ res_l^*$. From this construction we immediately get: $\sigma \circ res_l^* = res_{\sigma(l)}^* \circ \rho_l(\sigma)$. One then easily sees that $\sigma \circ u^{(l)} = u^{(\sigma(l))} \circ \rho_l(\sigma)$ holds as well. □

We now see that for any $\sigma \in S_{k+1}$,

$$\frac{1}{k+1}\sum_{1 \leq l \leq k+1} (\sigma \circ u^{(l)}) \otimes (\sigma \circ res_l^*) = \frac{1}{k+1}\sum_{1 \leq l \leq k+1} (u^{(\sigma(l))} \circ \rho_l(\sigma)) \otimes (res_{\sigma(l)}^* \circ \rho_l(\sigma))$$

Restricted to $(U^{\otimes k} \otimes \Delta_k)^{S_k}$, the latter morphism equals

$$\frac{1}{k+1} \sum_{1 \le l \le k+1} u^{(\sigma(l))} \otimes res^*_{\sigma(l)} = \frac{1}{k+1} \sum_{1 \le l \le k+1} u^{(l)} \otimes res^*_l$$

as wanted.

Moving on to the commutation relations, one only needs to check that the following commutation relations between operators on $(U^{\otimes k} \otimes \Delta_k)^{S_k}$ hold (the rest are obvious):

(a)

$$\frac{1}{(k+1)(k+2)} \sum_{\substack{1 \le l_1 \le k+1 \\ 1 \le l_2 \le k+2}} (u_2^{(l_2)} \circ u_1^{(l_1)}) \otimes (res^*_{l_2} \circ res^*_{l_1}) \stackrel{?}{=} \frac{1}{(k+1)(k+2)} \sum_{\substack{1 \le l_2 \le k+1 \\ 1 \le l_1 \le k+2}} (u_1^{(l_1)} \circ u_2^{(l_2)}) \otimes (res^*_{l_1} \circ res^*_{l_2})$$

(b)

$$\sum_{\substack{1 \le l_1 \le k \\ 1 \le l_2 \le k-1}} (f_2^{(l_2)} \circ f_1^{(l_1)}) \otimes (res_{l_2} \circ res_{l_1}) \stackrel{?}{=} \sum_{\substack{1 \le l_2 \le k \\ 1 \le l_1 \le k-1}} (f_1^{(l_1)} \circ f_2^{(l_2)}) \otimes (res_{l_1} \circ res_{l_2})$$

(c)

$$\frac{1}{(k+1)} \sum_{1 \le l_1, l_2 \le k+1} (f^{(l_2)} \circ u^{(l_1)}) \otimes (res_{l_2} \circ res^*_{l_1}) \stackrel{?}{=} \frac{1}{k} \sum_{1 \le l_1, l_2 \le k} (u^{(l_1)} \circ f^{(l_2)}) \otimes (res^*_{l_1} \circ res_{l_2}) +$$
$$+ (\nu - k) f(u) \mathrm{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} - T_{f,u}|_{(U^{\otimes k} \otimes \Delta_k)^{S_k}}$$

These identities are proved below.

(a) The claim follows immediately from the following easy computations (consequences of Lemma 3.2.3.10):

- For $1 \le l_1 < l_2 \le k+2$,

$$res^*_{l_2} \circ res^*_{l_1} = res^*_{l_1} \circ res^*_{l_2 - 1}$$

as operators on Δ_k. We also have $u_2^{(l_2)} \circ u_1^{(l_1)} = u_1^{(l_1)} \circ u_2^{(l_2-1)}$.

- For $k+1 \geq l_1 \geq l_2 \geq 1$,

$$res^*_{l_2} \circ res^*_{l_1} = res^*_{l_1+1} \circ res^*_{l_2}$$

as operators on Δ_k. We also have $u_2^{(l_2)} \circ u_1^{(l_1)} = u_1^{(l_1+1)} \circ u_2^{(l_2)}$.

(b) The claim follows immediately from the following easy computations (consequences of Lemma 3.2.3.10):

- For $1 \leq l_1 \leq l_2 \leq k-1$,

$$res_{l_2} \circ res_{l_1} = res_{l_1} \circ res_{l_2+1}$$

as operators on Δ_k. We also have $f_2^{(l_2)} \circ f_1^{(l_1)} = f_1^{(l_1)} \circ f_2^{(l_2+1)}$.

- For $k \geq l_1 > l_2 \geq 1$,

$$res_{l_2} \circ res_{l_1} = res_{l_1-1} \circ res_{l_2}$$

as operators on Δ_k. We also have $f_2^{(l_2)} \circ f_1^{(l_1)} = f_1^{(l_1-1)} \circ f_2^{(l_2)}$.

(c) We have:

- For any $1 \leq l \leq k+1$, $res_l \circ res^*_l = (\nu - k)\,\mathrm{Id}_{\Delta_k}$ and thus

$$(f^{(l)} \circ u^{(l)}) \otimes (res_l \circ res^*_l) = (\nu - k) f(u)\,\mathrm{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}}$$

- For $1 \leq l_1 < l_2 \leq k+1$,

$$res_{l_2} \circ res^*_{l_1} = res^*_{l_1} \circ res_{l_2-1} - C_{(l_1,\ldots,l_2-1)}$$

as operators on Δ_k, where $C_{(l_1,\ldots,l_2-1)} : \Delta_k \to \Delta_k$ is the action of the cycle

$C_{(l_1,\ldots,l_2-1)} \in S_k$ on Δ_k. We also have

$$f^{(l_2)} \circ u^{(l_1)} = u^{(l_1)} \circ f^{(l_2-1)} = T^{(l_1)}_{f,u} \circ C_{(l_1,\ldots,l_2-1)}$$

Thus

$$(f^{(l_2)} \circ u^{(l_1)}) \otimes (res_{l_2} \circ res^*_{l_1}) = (u^{(l_1)} \circ f^{(l_2-1)}) \otimes (res^*_{l_1} \circ res_{l_2-1}) - \\ - (T^{(l_1)}_{f,u} \circ C_{(l_1,\ldots,l_2-1)}) \otimes C_{(l_1,\ldots,l_2-1)}$$

as operators on $U^{\otimes k} \otimes \Delta_k$.

Finally, note that $res^*_{l_1} \circ res_{l_2-1} \circ C^{-1}_{(l_1,\ldots,l_2-1)} = res^*_{l_1} \circ res_{l_1}$.

- For $k+1 \geq l_1 > l_2 \geq 1$, $res_{l_2} \circ res^*_{l_1} = res^*_{l_1-1} \circ res_{l_2} - C^{-1}_{(l_2,\ldots,l_1-1)}$ as operators on Δ_k, where $C_{(l_2,\ldots,l_1-1)} : \Delta_k \to \Delta_k$ is the action of the cycle $C_{(l_2,\ldots,l_1-1)} \in S_k$ on Δ_k. We also have

$$f^{(l_2)} \circ u^{(l_1)} = u^{(l_1-1)} \circ f^{(l_2)} = T_{f,u} \circ C^{-1}_{(l_2,\ldots,l_1-1)}$$

Thus

$$(f^{(l_2)} \circ u^{(l_1)}) \otimes (res_{l_2} \circ res^*_{l_1}) = (u^{(l_1-1)} \circ f^{(l_2)}) \otimes (res^*_{l_1-1} \circ res_{l_2}) - \\ - (T^{(l_1-1)}_{f,u} \circ C^{-1}_{(l_2,\ldots,l_1-1)}) \otimes C^{-1}_{(l_2,\ldots,l_1-1)}$$

as operators on $U^{\otimes k} \otimes \Delta_k$.

Finally, note that $res^*_{l_1-1} \circ res_{l_2} \circ C_{(l_1,\ldots,l_2-1)} = res^*_{l_1-1} \circ res_{l_1-1}$.

Together these imply the following identities of operators on $(U^{\otimes k} \otimes \Delta_k)^{S_k}$:

$$\frac{1}{(k+1)} \sum_{1 \leq l_1, l_2 \leq k+1} (f^{(l_2)} \circ u^{(l_1)}) \otimes (res_{l_2} \circ res_{l_1}^*) = (\nu - k) f(u) \operatorname{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} +$$
$$+ \frac{1}{(k+1)} \sum_{1 \leq l_1 < l_2 \leq k+1} (u^{(l_1)} \circ f^{(l_2-1)}) \otimes (res_{l_1}^* \circ res_{l_2-1}) - (T_{f,u}^{(l_1)} \circ C_{(l_1,\dots,l_2-1)}) \otimes C_{(l_1,\dots,l_2-1)} +$$
$$+ \frac{1}{(k+1)} \sum_{1 \leq l_2 < l_1 \leq k+1} (u^{(l_1-1)} \circ f^{(l_2)}) \otimes (res_{l_1-1}^* \circ res_{l_2}) - (T_{f,u}^{(l_1-1)} \circ C_{(l_2,\dots,l_1-1)}^{-1}) \otimes C_{(l_2,\dots,l_1-1)}^{-1} =$$
$$= (\nu - k) f(u) \operatorname{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} - T_{f,u}|_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} +$$
$$+ \frac{1}{(k+1)} \sum_{1 \leq l_1, l_2 \leq k} (u^{(l_1)} \circ f^{(l_2)}) \otimes (res_{l_1}^* \circ res_{l_2}) + \frac{1}{(k+1)} \sum_{1 \leq l_1 \leq k+1} (u^{(l_1)} \circ f^{(l_1)}) \otimes (res_{l_1}^* \circ res_{l_1}) =$$
$$= (\nu - k) f(u) \operatorname{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} - T_{f,u}|_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} +$$
$$+ \frac{1}{(k+1)} \sum_{1 \leq l_1, l_2 \leq k} (u^{(l_1)} \circ f^{(l_2)}) \otimes (res_{l_1}^* \circ res_{l_2}) + \frac{1}{k(k+1)} \sum_{1 \leq l_1, l_2 \leq k} (u^{(l_1)} \circ f^{(l_2)}) \otimes (res_{l_1}^* \circ res_{l_2}) =$$
$$= \frac{1}{k} \sum_{1 \leq l_1, l_2 \leq k} (u^{(l_1)} \circ f^{(l_2)}) \otimes (res_{l_1}^* \circ res_{l_2}) + (\nu - k) f(u) \operatorname{Id}_{(U^{\otimes k} \otimes \Delta_k)^{S_k}} - T_{f,u}|_{(U^{\otimes k} \otimes \Delta_k)^{S_k}}$$

□

5.2.2 Proof of Lemma 4.2.0.22

Lemma 5.2.2.1. *Let $l \leq k$, and consider a non-zero morphism in $\underline{Rep}(S_\nu)$*

$$\phi : U^{\otimes l} \otimes \Delta_l \longrightarrow U^{\otimes k} \otimes \Delta_k$$

Let $u \in U \cong \mathfrak{u}_\mathfrak{p}^-, u \neq 0$. Then $F_u \circ \phi \neq 0$, where $F_u \circ \phi := \frac{1}{k+1} \sum_{1 \leq l \leq k+1} (u^{(l)} \otimes res_l^*) \circ \phi$.

Proof. Recall from Lemma 3.2.3.10 that

$$\operatorname{Hom}_{\underline{Rep}(S_\nu)}(\Delta_l, \Delta_k) := \mathbb{C}\bar{P}_{l,k}$$

where $\bar{P}_{l,k}$ is the set of partitions of $\{1, .., l, 1', ..., k'\}$ into disjoint subsets such that i, j do not lie in the same subset, and neither do i', j', for any $i \neq j, i' \neq j'$.

So

$$\mathrm{Hom}_{\underline{Rep}(S_\nu)}(U^{\otimes l} \otimes \Delta_l, U^{\otimes k} \otimes \Delta_k) \cong \mathbb{C}\bar{P}_{l,k} \otimes U^{\otimes k} \otimes U^{*\otimes l}$$

We now study the map

$$F_u \circ (\cdot) : \mathbb{C}\bar{P}_{l,k} \otimes U^{\otimes k} \otimes U^{*\otimes l} \to \mathbb{C}\bar{P}_{l,k+1} \otimes U^{\otimes k+1} \otimes U^{*\otimes l}$$

By definition of F_u, we know that

$$F_u \circ (x \otimes u_1 \otimes ... \otimes u_k \otimes f_1 \otimes ... \otimes f_l) = \frac{1}{k+1} \sum_{1 \leq s \leq k+1} res^*_s(x) \otimes u_1 \otimes ... \otimes u_{s-1} \otimes u \otimes u_s \otimes ... \otimes u_k \otimes f_1 \otimes ... \otimes f_l$$

where $x \in \bar{P}_{l,k}, u_1, ..., u_k \in U, f_1, ..., f_l \in U^*$.

As we said before, we can consider ϕ as an element of $\mathbb{C}\bar{P}_{l,k} \otimes U^{\otimes k} \otimes U^{*\otimes l}$.

Let $N := \dim U$, and choose a basis $u_1, ..., u_N$ of U such that $u_1 = u$. Then we can write

$$\phi = \sum_{\substack{x \in \bar{P}_{l,k}, \\ i_1,...,i_k \in \{1,...,N\}}} \alpha_{x,I} x \otimes u_{i_1} \otimes ... \otimes u_{i_k}$$

where I denotes the sequence $(i_1, ..., i_k)$ and $\alpha_{x,I} \in U^{*\otimes l}$.

Now assume $F_u \circ \phi = 0$, i.e.

$$\sum_{1 \leq s \leq k+1} \sum_{\substack{x \in \bar{P}_{l,k}, \\ i_1,...,i_k \in \{1,...,N\}}} \alpha_{x,I} res^*_s(x) \otimes u_{i_1} \otimes ... \otimes u_{i_{s-1}} \otimes u \otimes u_{i_s} \otimes ... \otimes u_{i_k} = 0$$

So for any $y \in \bar{P}_{l,k+1}$, and any sequence $J = (j_1, ..., j_{k+1})$ (here $j_1, .., j_{k+1} \in \{1, ..., N\}$), we have

$$\sum_{\substack{\text{triples } (x,I,s): \\ 1 \leq s \leq k+1,\, res^*_s(x)=y,\, j_s=1,\, I=(j_1,...,j_{s-1},j_{s+1},...,j_{k+1})}} \alpha_{x,I} = 0$$

We will now show that this implies that $\alpha_{x,I} = 0$ for any $x \in \bar{P}_{l,k}, I = (i_1, ..., i_k), i_1, ..., i_k \in \{1, ..., N\}$, which will mean that $\phi = 0$ and thus lead to a contradiction.

For our convenience, let us denote by $Ins_s(I)$ the sequence $(i_1, ..., i_{s-1}, 1, i_s, ..., i_k)$ (1 inserted in the s-th place). We will also use the following notation:

- For $x \in \bar{P}_{l,k}$, consider the longest sequence of consecutive solitary vertices in the bottom row of the diagram of x (if there are several such sequences of maximal length, choose the first one).

  Denote the length of this sequence by $m(x)$. Let j_x be such that $j_x + 1$ is the first element of this sequence (if this sequence is empty, then put $j_x := 1$).

  So this sequence of solitary vertices in x is $\{j_x + 1, j_x + 2, ..., j_x + m(x)\}$.

- Let $x \in \bar{P}_{l,k}, I = (i_1, ..., i_k), i_1, ..., i_k \in \{1, ..., N\}$. Consider the sequence $(i_{j_x+1}, i_{j_x+2}, ..., i_{j_x+m(x)})$ and inside it the longest segment of consecutive occurrences of 1 (if there are several such segments of maximal length, choose the first one).

  Denote the length of this segment by $M(I, x)$. Let $j_{I,x}$ be such that $j_{I,x} + 1$ is the position of the first element of this segment (if this segment is empty, i.e. $M(I, x) = 0$, then put $j_{I,x} := j_x$).

We now rewrite the equality we obtained above: for any triple x, I, s where $x \in \bar{P}_{l,k}$, I is a sequence of length k with entries in $\{1, ..., N\}$, and $1 \leq s \leq k+1$, we have:

$$\sum_{\substack{\text{triples } (x',I',s'): \\ 1 \leq s' \leq k+1,\, res^*_{s'}(x') = res^*_s(x),\, Ins_{s'}(I') = Ins_s(I)}} \alpha_{x',I'} = 0$$

We will now use two-fold descending induction on the values $m(x), M(I, x)$ to prove that $\alpha_{x,I} = 0$ for any $x \in \bar{P}_{l,k}$, and any sequence I of length k with entries in $\{1, ..., N\}$.

Base: Let x, I such that $m(x) = k, M(I, x) = k$. Then the bottom row of x consists of solitary vertices, and I consists only of 1's. Now choose any $s \in \{1, ..., k+1\}$. Then, by definition, the bottom row of $res^*_s(x)$ consists of solitary vertices, and $Ins_s(I)$ consists only of 1's.

Then for any triple (x', I', s') which satisfies $res^*_{s'}(x') = res^*_s(x), Ins_{s'}(I') = Ins_s(I)$, we will have $x' = x$, $I' = I$. The above equality then implies that $\alpha_{x,I} = 0$.

Step: Let $0 \leq M, m \leq k$, and $M + m < 2k$. Assume $\alpha_{x,I} = 0$ for any x, I such that either $m(x) > m$, or $m(x) = m, M(I, x) > M$.

Let x, I be such that $m(x) = m, M(I, x) = M$. Set $s := j_{I,x} + 1$.

All we have to do is prove the following Sublemma, and we are done.

Sublemma 5.2.2.2. *Let (x', I', s') be a triple which satisfies $res^*_{s'}(x') = res^*_s(x), Ins_{s'}(I') = Ins_s(I)$. Then one of the following statements holds:*

- $m(x') > m(x)$,
- $x' = x, M(I', x) > M(I, x)$,
- $x' = x, I' = I$.

Proof. By definition, $res^*_s(x)$ has a sequence of $m(x)+1$ consecutive solitary vertices. We assumed that $res^*_{s'}(x') = res^*_s(x)$, so x' is obtained by removal of the s'-th vertex from the bottom row of $res^*_s(x)$. So either x' has a sequence of $m(x) + 1$ consecutive solitary vertices, i.e. $m(x') = m(x) + 1$, or we are removing one of the $m(x) + 1$ consecutive solitary vertices of $res^*_s(x)$, which means that $x' = x$.

Now, assume $x' = x$, and use a similar argument for I, I'. By definition, the sequence $Ins_s(I)$ has a segment of $M(I, x) + 1$ consecutive occurrences of 1. Again, we assumed that $Ins_{s'}(I') = Ins_s(I)$, so I' is obtained by removal of the s'-th element of the sequence $Ins_s(I)$. So either I' has a segment of $M(I, x) + 1$ consecutive occurrences of 1, i.e. $M(I', x) = M(I, x) + 1$, or $I' = I$. □

□

5.2.3 Proof of Lemma 4.3.0.23

Let $V \cong \mathbb{C}\mathbb{1} \oplus U$ be a split unital finite-dimensional vector space. We will use the same notations as in Section 4.3.

The following two technical lemmas will be used to prove Lemma 4.3.0.23.

Lemma 5.2.3.1. *Let $k \in \{0, ..., n-1\}, \{j_1 < j_2 < ... < j_k\} \subset \{1, ..., n\}$, $u \in U$, $v_{j_1}, v_{j_2}, ..., v_{j_k} \in U$, and let $f_{j_1<j_2<...<j_k}$ be the map $\{1, ..., k\} \to \{1, ..., n\}$ taking i to j_i. Then*

$$e_{S_{k+1}}\left(\sum_{1\le l\le k+1}\ \sum_{\substack{g\in Inj(\{1,...,k+1\},\{1,...,n\}):\\ g\circ\iota_l = f_{j_1<j_2<...<j_k}\\ g \text{ monotone increasing}}} u^{(l)}.(v_{j_1}\otimes v_{j_2}\otimes ... \otimes v_{j_k})\otimes g\right) =$$
$$= \frac{1}{(k+1)!}\sum_{1\le l\le k+1}\sum_{\sigma\in S_k}(u^{(l)}\circ\sigma)(v_{j_1}\otimes v_{j_2}\otimes ... \otimes v_{j_k})\otimes(\mathtt{res}_l^*\circ\sigma)(f_{j_1<j_2<...<j_k})$$

Proof. We rewrite both sides of the identity we want to prove: the left hand side becomes

$$\frac{1}{(k+1)!}\sum_{\substack{\sigma\in S_{k+1},\\ 1\le l\le k+1}}(\sigma\circ u^{(l)}).(v_{j_1}\otimes v_{j_2}\otimes ... \otimes v_{j_k})\otimes\left(\sum_{\substack{g\in Inj(\{1,...,k+1\},\{1,...,n\}):\\ g\circ\iota_l = f_{j_1<j_2<...<j_k}\\ g \text{ monotone increasing}}} g\circ\sigma^{-1}\right)$$

and the right hand side becomes

$$\frac{1}{(k+1)!}\sum_{1\le l'\le k+1}\sum_{\sigma'\in S_k}(u^{(l')}\circ\sigma').(v_{j_1}\otimes v_{j_2}\otimes ... \otimes v_{j_k})\otimes\left(\sum_{\substack{g'\in Inj(\{1,...,k+1\},\{1,...,n\}):\\ g'\circ\iota_{l'} = f_{j_1<j_2<...<j_k}\circ\sigma'^{-1}}} g'\right)$$

We now define the following map:

$$\mu : S_{k+1}\times\{1, ..., k+1\} \longrightarrow S_k\times\{1, ..., k+1\}$$
$$(\sigma, l)\mapsto(\rho_l(\sigma), \sigma(l))$$

where $\rho_l(\sigma)$ is defined in Lemma 5.2.1.2.

Then it is enough to check that for every $(\sigma', l')\in S_k\times\{1, ..., k+1\}$, the following identity holds:

$$\sum_{(\sigma,l)\in\mu^{-1}(\sigma',l')} (\sigma \circ u^{(l)}).(v_{j_1} \otimes v_{j_2} \otimes \ldots \otimes v_{j_k}) \otimes \left(\sum_{\substack{g\in Inj(\{1,\ldots,k+1\},\{1,\ldots,n\}):\\ g\circ\iota_l = f_{j_1<j_2<\ldots<j_k}\\ g \text{ monotone increasing}}} g \circ \sigma^{-1} \right) =$$

$$= (u^{(l')} \circ \sigma').(v_{j_1} \otimes v_{j_2} \otimes \ldots \otimes v_{j_k}) \otimes \left(\sum_{\substack{g'\in Inj(\{1,\ldots,k+1\},\{1,\ldots,n\}):\\ g'\circ\iota_{l'} = f_{j_1<j_2<\ldots<j_k}\circ\sigma'^{-1}}} g' \right)$$

From Lemma 5.2.1.2, we know that $\sigma \circ u^{(l)} = u^{(l')} \circ \sigma'$ for any $(\sigma, l) \in \mu^{-1}(\sigma', l')$. So we need to check that

$$\sum_{(\sigma,l)\in\mu^{-1}(\sigma',l')} \sum_{\substack{g\in Inj(\{1,\ldots,k+1\},\{1,\ldots,n\}):\\ g\circ\iota_l = f_{j_1<j_2<\ldots<j_k}\\ g \text{ monotone increasing}}} (g \circ \sigma^{-1}) = \sum_{\substack{g'\in Inj(\{1,\ldots,k+1\},\{1,\ldots,n\}):\\ g'\circ\iota_{l'} = f_{j_1<j_2<\ldots<j_k}\circ\sigma'^{-1}}} g'$$

Notice that by definition of μ, $\sigma \circ \iota_l = \iota_{\sigma(l)} \circ \rho_l(\sigma)$, i.e.

$$\sigma \circ \iota_l = \iota_{l'} \circ \sigma'$$

Thus for any monotone increasing map $g : \{1, \ldots, k+1\} \to \{1, \ldots, n\}$ such that $g \circ \iota_l = f_{j_1<j_2<\ldots<j_k}$, the map $g' := g \circ \sigma^{-1}$ is an injective map $\{1, \ldots, k+1\} \to \{1, \ldots, n\}$ satisfying: $g' \circ \iota_{l'} = f_{j_1<j_2<\ldots<j_k} \circ \sigma'^{-1}$.

It remains to check that the summands in the left hand side are pairwise different and that both sides have the same number of summands.

The first of these statements is proved as follows: let $g_1 \circ \sigma^{-1}$, $g_2 \circ \tau^{-1}$ be two summands in the left hand side. Assume they are equal. This means that g_1, g_2 have the same image, and since they both are monotone increasing, we conclude that $g_1 = g_2$, which of course means that $\sigma = \tau$, and we are done.

The second statement is proved as follows: the number of summands in the right hand side is obviously $n-k$. To show that this is also the number of summands in the left hand side, we only need to check that the projection $S_{k+1}\times\{1,...,k+1\}\to\{1,...,k+1\}$ maps $\mu^{-1}(\sigma',l')$ bijectively to $\{1,...,k+1\}$.

By the definition of μ and the construction described in Lemma 5.2.1.2, we see that for every $l\in\{1,...,k+1\}$, we can (uniquely) reconstruct σ from the data (σ',l',l) so that $\mu(\sigma,l)=(\sigma',l')$: we consider the diagram of $\sigma'\in\bar{P}_{k,k}$, insert a vertex in the l-th position in the top row, a vertex in the l'-th position in the bottom row and an edge connecting the two. The obtained diagram will be σ. This completes the proof of the lemma. □

Lemma 5.2.3.2. *Let* $k\in\{0,...,n-1\},\{j_1<j_2<...<j_{k+1}\}\subset\{1,...,n\}$, $\lambda\in U^*$, $v_{j_1},v_{j_2},...,v_{j_{k+1}}\in U$, *and let* $f_{j_1<j_2<...<j_{k+1}}$ *be the map* $\{1,...,k+1\}\to\{1,...,n\}$ *taking* i *to* j_i. *Then*

$$e_{S_k}\left(\sum_{1\leq l\leq k+1}\lambda^{(l)}.(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_{k+1}})\otimes\mathtt{res}_l(f_{j_1<j_2<...<j_{k+1}})\right)=$$
$$=\frac{1}{(k+1)!}\sum_{1\leq l\leq k+1}\sum_{\sigma\in S_{k+1}}(\lambda^{(l)}\circ\sigma)(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_{k+1}})\otimes(\mathtt{res}_l\circ\sigma)(f_{j_1<j_2<...<j_{k+1}})$$

Proof. We rewrite the left hand of the identity we want to prove, and it becomes

$$\frac{1}{k!}\sum_{\substack{\sigma'\in S_k,\\ 1\leq l'\leq k+1}}(\sigma'\circ\lambda^{(l')}).(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_{k+1}})\otimes(\sigma'\circ\mathtt{res}_{l'})\,(f_{j_1<j_2<...<j_{k+1}})$$

We use the definition of the map μ from the proof of Lemma 5.2.3.1, and define the map

$$\tilde{\mu}:S_{k+1}\times\{1,...,k+1\}\longrightarrow S_k\times\{1,...,k+1\}$$
$$(\sigma,l)\mapsto\mu(\sigma^{-1},l)$$

Then it is enough to check that for every $(\sigma', l') \in S_k \times \{1, ..., k+1\}$,

$$(\sigma' \circ \lambda^{(l')}).(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_{k+1}}) \otimes (\sigma' \circ \mathtt{res}_{l'}) (f_{j_1 < j_2 < ... < j_{k+1}}) =$$
$$= \frac{1}{k+1} \sum_{(\sigma, l) \in \tilde{\mu}^{-1}(\sigma'^{-1}, l')} (\lambda^{(l)} \circ \sigma)(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_{k+1}}) \otimes (\mathtt{res}_l \circ \sigma)(f_{j_1 < j_2 < ... < j_{k+1}})$$

By definition of $\tilde{\mu}$, for every $(\sigma, l) \in \tilde{\mu}^{-1}(\sigma'^{-1}, l')$, we have: $\mu(\sigma^{-1}, l) = (\sigma'^{-1}, l')$, which means that $\sigma^{-1} \circ \iota_l = \iota_{l'} \circ \sigma'^{-1}$ (see the proof of Lemma 5.2.3.1), and so

$$\sigma' \circ \mathtt{res}_{l'} = \mathtt{res}_l \circ \sigma$$

and similarly

$$\sigma' \circ \lambda^{(l')} = \lambda^{(l)} \circ \sigma$$

Thus it only remains to check that the right hand side has $k+1$ summands, i.e. that $\tilde{\mu}^{-1}(\sigma'^{-1}, l')$ has $k+1$ elements. The latter can be easily deduced from the arguments in the proof of Lemma 5.2.3.1. □

Lemma 5.2.3.3. *There is an isomorphism of $\mathfrak{gl}(V)$-modules*

$$\Phi : V^{\otimes n} \xrightarrow{\sim} \bigoplus_{k=0,...,n} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1, ..., k\}, \{1, ..., n\}))^{S_k}$$

where $\Phi(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}) = 1$ (lies in degree zero of the right hand side).

Moreover, this isomorphism is an isomorphism of $\mathbb{C}[S_n] \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{gl}(V))$-modules.

Proof. Fix a dual basis vector $\mathbb{1}^* \in (\mathbb{C}\mathbb{1})^*$ such that $\mathbb{1}^*(\mathbb{1}) = 1$.

Given a subset $J = \{j_1 < j_2 < ... < j_k\} \subset \{1, ..., n\}$, let

$$U^{\otimes J} = \mathbb{C}\mathbb{1}^{\otimes j_1 - 1} \otimes U \otimes \mathbb{C}\mathbb{1}^{\otimes j_2 - j_1 - 1} \otimes U \otimes ... \otimes \mathbb{C}\mathbb{1}^{\otimes j_k - j_{k-1} - 1} \otimes U \otimes \mathbb{C}\mathbb{1}^{\otimes n - j_k}$$

(that is, the factors j_1, j_2, etc. are U, and the rest are $\mathbb{C}\mathbb{1}$). Then $V^{\otimes n} = \bigoplus_{J \subset \{1,..,n\}} U^{\otimes J}$.

Let

$$\Phi_J : U^{\otimes\{j_1<j_2<\ldots<j_k\}} \longrightarrow (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$
$$v_1 \otimes ... \otimes v_n \mapsto e_{S_k}(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k} \otimes f_{j_1<j_2<\ldots<j_k}) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i)$$

Here $f := f_{j_1<j_2<\ldots<j_k} \in Inj(\{1,...,k\},\{1,...,n\})$ is given by $f(s) := j_s$ and e_{S_k} is the projection

$$U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}) \to (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

Finally, set the map

$$\Phi : V^{\otimes n} = \bigoplus_{J\subset\{1,..,n\}} U^J \longrightarrow \bigoplus_{k=0,...,n} (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

to be $\sum_{J\subset\{1,..,n\}} \Phi_J$.

Notice that $\Phi(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}) = \Phi_\emptyset(\mathbb{1} \otimes \mathbb{1} \otimes ... \otimes \mathbb{1}) = 1$.

We claim that Φ is a map of $\mathfrak{gl}(V)$-modules. Again, we consider the decomposition

$$\mathfrak{gl}(V) \cong \mathbb{C}\,\mathrm{Id}_V \oplus \mathfrak{u}_\mathfrak{p}^- \oplus \mathfrak{u}_\mathfrak{p}^+ \oplus \mathfrak{gl}(U)$$

- Id_V acts by the scalar n on both sides.

- Let $u \in U \cong \mathfrak{u}_\mathfrak{p}^-$, and let $v_1 \otimes ... \otimes v_n \in U^{\otimes\{j_1<j_2<\ldots<j_k\}}$

Then u acts on $V^{\otimes n}$ by operator F_u which satisfies:

$$F_u.(v_1 \otimes ... \otimes v_n) = \sum_{i \notin J} v_1 \otimes ... \otimes \mathbb{1}^*(v_i)u \otimes ... \otimes v_n$$

and thus

$$\Phi(F_u.(v_1\otimes...\otimes v_n)) = e_{S_{k+1}}\left(\sum_{1\le l\le k+1}\sum_{\substack{g\in Inj(\{1,...,k+1\},\{1,...,n\}):\\ g\circ\iota_l=f_{j_1<j_2<...<j_k}\\ f(l-1)<g(l)<f(l)}} u^{(l)}.(v_{j_1}\otimes v_{j_2}\otimes ...\otimes v_{j_k})\otimes g\right)\cdot\prod_{i\notin J}\mathbb{1}^*(v_i)$$

Here ι_l is the injection

$$\{1,...,k\}\hookrightarrow\{1,...,k+1\},\ i\mapsto\begin{cases} i & \text{if } i<l\\ i+1 & \text{if } i\ge l\end{cases}$$

Now,

$$F_u.\Phi(v_1\otimes...\otimes v_n) = F_u.\left(e_{S_k}(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_k}\otimes f_{j_1<j_2<...<j_k})\cdot\prod_{i\notin J}\mathbb{1}^*(v_i)\right) =$$

$$= \frac{1}{(k+1)!}\prod_{i\notin J}\mathbb{1}^*(v_i)\left(\sum_{1\le l\le k+1}\sum_{\sigma\in S_k}(u^{(l)}\circ\sigma)(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_k})\otimes(\mathtt{res}_l^*\circ\sigma)(f_{j_1<j_2<...<j_k})\right)$$

We now use Lemma 5.2.3.1 to conclude that Φ is a map of $\mathfrak{u}_{\mathfrak{p}}^-$-modules.

- Let $\lambda\in U^*\cong\mathfrak{u}_{\mathfrak{p}}^+$, $k\ge 1$, and $v_1\otimes...\otimes v_n\in U^{\otimes\{j_1<j_2<...<j_k\}}$.

Then λ acts on $V^{\otimes n}$ by operator E_λ which satisfies:

$$E_\lambda.(v_1\otimes...\otimes v_n) = \sum_{j\in J} v_1\otimes...\otimes\lambda(v_j)\otimes...\otimes v_n$$

and thus

$$\Phi(E_\lambda.(v_1\otimes...\otimes v_n)) = e_{S_{k-1}}\left(\sum_{1\le l\le k}\lambda^{(l)}.(v_{j_1}\otimes v_{j_2}\otimes...\otimes v_{j_k})\otimes\mathtt{res}_l(f_{j_1<j_2<...<j_k})\right)\cdot\prod_{i\notin J}\mathbb{1}^*(v_i)$$

Now,

$$E_\lambda.\Phi(v_1 \otimes ... \otimes v_n) = E_\lambda.\left(e_{S_k}(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k} \otimes f_{j_1<j_2<...<j_k}) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i)\right) =$$

$$= \frac{1}{k!} \prod_{i \notin J} \mathbb{1}^*(v_i) \left(\sum_{1 \leq l \leq k} \sum_{\sigma \in S_k} (\lambda^{(l)} \circ \sigma)(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k}) \otimes (\mathtt{res}_l \circ \sigma)(f_{j_1<j_2<...<j_k}) \right)$$

We now use Lemma 5.2.3.2 to conclude that Φ is a map of $\mathfrak{u}_{\mathfrak{p}}^+$-modules (note that the action of λ on $(U^{\otimes 0} \otimes \mathbb{C})^{S_0} \cong \mathbb{C}$ is zero, as is on $\mathbb{C}\mathbb{1} \cong U^{\otimes \emptyset}$). This Lemma is proved at the end of this section.

- $\mathfrak{gl}(U)$ acts naturally on each summand $U^{\otimes\{j_1<j_2<...<j_k\}}$ on the left and on each summand $(U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$ on the right, and this action gives us isomorphisms of $\mathfrak{gl}(U)$-modules:

$$\Phi_J : U^{\otimes\{j_1<j_2<...<j_k\}} \to U^{\otimes k} \otimes \mathbb{C}f_{j_1<j_2<...<j_k}$$

and

$$\bigoplus_{\{j_1<j_2<...<j_k\} \subset \{1,...,n\}} U^{\otimes k} \otimes \mathbb{C}f_{j_1<j_2<...<j_k} \cong (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

Note that the last argument also shows that Φ is an isomorphism.

It remains to check that Φ is also a morphism of S_n-modules. Fix $k \in \{0,...,n\}$. It is enough to check that

$$\bigoplus_{\substack{J \subset \{1,...,n\}, \\ |J|=k}} \Phi_J : \bigoplus_{\substack{J \subset \{1,...,n\}, \\ |J|=k}} U^{\otimes\{j_1<j_2<...<j_k\}} \longrightarrow (U^{\otimes k} \otimes \mathbb{C}Inj(\{1,...,k\},\{1,...,n\}))^{S_k}$$

$$v_1 \otimes ... \otimes v_n \mapsto e_{S_k}(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k} \otimes f_{j_1<j_2<...<j_k}) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i)$$

is a morphism of S_n-modules (here $J = \{j_1 < j_2 < ... < j_k\}$).

Fix $\sigma \in S_n$, and fix $J = \{j_1 < j_2 < ... < j_k\} \subset \{1, ..., n\}$. Let $\tau \in S_k$ be such that $\sigma(j_{\tau^{-1}(1)}), \sigma(j_{\tau^{-1}(2)}), ..., \sigma(j_{\tau^{-1}(k)})$ is a monotone increasing sequence. We will denote this sequence by $\sigma(J)$.

We have:

$$\sigma(\Phi_J(v_1 \otimes ... \otimes v_n)) = e_{S_k}(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k} \otimes (\sigma \circ f_{j_1 < j_2 < ... < j_k})) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i)$$

On the other hand,

$$\begin{aligned}
&\Phi_{\sigma(J)}(\sigma(v_1 \otimes ... \otimes v_n)) = \Phi_{\sigma(J)}(v_{\sigma^{-1}(1)} \otimes ... \otimes v_{\sigma^{-1}(n)}) = \\
&= e_{S_k}\left(v_{j_{\tau^{-1}(1)}} \otimes v_{j_{\tau^{-1}(2)}} \otimes ... \otimes v_{j_{\tau^{-1}(k)}} \otimes f_{j_{\sigma(\tau^{-1}(1))} < \sigma(j_{\tau^{-1}(2)}) < ... < \sigma(j_{\tau^{-1}(k)})}\right) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i) = \\
&= e_{S_k}\left(\tau.(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k}) \otimes (\sigma \circ f_{j_1 < j_2 < ... < j_k} \circ \tau^{-1})\right) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i) = \\
&= e_{S_k}(v_{j_1} \otimes v_{j_2} \otimes ... \otimes v_{j_k} \otimes (\sigma \circ f_{j_1 < j_2 < ... < j_k})) \cdot \prod_{i \notin J} \mathbb{1}^*(v_i)
\end{aligned}$$

Thus

$$\sigma \circ \left(\bigoplus_{J \subset \{1, ..., n\}, |J| = k} \Phi_J \right) = \left(\bigoplus_{J \subset \{1, ..., n\}, |J| = k} \Phi_J \right) \circ \sigma$$

and we are done.

□

Chapter 6

Restricted inverse limits of categories

6.1 Overview of restricted inverse limits

In this chapter, we discuss the notion of an inverse limit of an inverse sequence of categories and functors.

Given a system of categories $\mathcal{C}_i$ (with i running through the set $\mathbb{Z}_+$) and functors $\mathcal{F}_{i-1,i} : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for each $i \geq 1$, we define the inverse limit category $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ to be the following category:

- The objects are pairs $(\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$ where $C_i \in \mathcal{C}_i$ for each $i \in \mathbb{Z}_+$ and $\phi_{i-1,i} : \mathcal{F}_{i-1,i}(C_i) \stackrel{\sim}{\to} C_{i-1}$ for any $i \geq 1$.

- A morphism f between two objects $(\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$, $(\{D_i\}_{i \in \mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i \geq 1})$ is a set of arrows $\{f_i : C_i \to D_i\}_{i \in \mathbb{Z}_+}$ satisfying some compatability conditions.

This category is an inverse limit of the system $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ in the $(2,1)$-category of categories with functors and natural isomorphisms. It is easily seen (see Section 6.2) that if the original categories $\mathcal{C}_i$ were pre-additive (resp. additive, abelian), and the functors $\mathcal{F}_{i-1,i}$ were linear (resp. additive, exact), then the inverse limit is again pre-additive (resp. additive, abelian).

One can also show that if the original categories $\mathcal{C}_i$ were monoidal (resp. symmetric monoidal, rigid symmetric monoidal) categories, and the functors $\mathcal{F}_{i-1,i}$ were, monoidal (resp. symmetric monoidal functors), then the inverse limit is again a monoidal (resp. symmetric monoidal, rigid symmetric monoidal) category.

6.1.1 Motivating example: rings

We now consider the motivating example.

First of all, consider the inverse system of rings of symmetric polynomials

$$\dots \to \mathbb{Z}[x_1, \dots, x_n]^{S_n} \to \mathbb{Z}[x_1, \dots, x_{n-1}]^{S_{n-1}} \to \dots \to \mathbb{Z}[x_1] \to \mathbb{Z}$$

with the homomorphisms given by $p(x_1, \dots, x_n) \mapsto p(x_1, \dots, x_{n-1}, 0)$.

We also consider the ring $\Lambda_{\mathbb{Z}}$ of symmetric functions in infinitely many variables. This ring is defined as follows: first, consider the ring $\mathbb{Z}[x_1, x_2, \dots]^{\cup_{n \geq 0} S_n}$ of all power series with integer coefficients in infinitely many indeterminates $x_1, x_2, \dots$ which are invariant under any permutation of indeterminates. The ring $\Lambda_{\mathbb{Z}}$ is defined to be the subring of all the power series such that the degrees of all its monomials are bounded.

We would like to describe the ring $\Lambda_{\mathbb{Z}}$ as an inverse limit of the former inverse system.

1-st approach: The following construction is described in [Macd, Chapter I]. Take the inverse limit $\varprojlim_{n \geq 0} \mathbb{Z}[x_1, \dots, x_n]^{S_n}$ (this is, of course, a ring, isomorphic to $\mathbb{Z}[x_1, x_2, \dots]^{\cup_{n \geq 0} S_n}$), and consider only those elements $(p_n)_{n \geq 0}$ for which $deg(p_n)$ is a bounded sequence. These elements form a subring of $\varprojlim_{n \geq 0} \mathbb{Z}[x_1, \dots, x_n]^{S_n}$ which is isomorphic to the ring of symmetric functions in infinitely many variables.

2-nd approach: Note that the notion of degree gives a $\mathbb{Z}_+$-grading on each ring $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$, and on the ring $\Lambda_{\mathbb{Z}}$. The morphisms $\mathbb{Z}[x_1, \dots, x_n]^{S_n} \to \mathbb{Z}[x_1, \dots, x_{n-1}]^{S_{n-1}}$ respect this grading; furthermore, they do not send to zero any polynomial of degree $n-1$ or less, so they define an isomorphism between the i-th grades of $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$ and $\mathbb{Z}[x_1, \dots, x_{n-1}]^{S_{n-1}}$ for any $i < n$. One can then see that $\Lambda_{\mathbb{Z}}$ is an inverse limit of the rings $\mathbb{Z}[x_1, ..., x_n]^{S_n}$ in the category of $\mathbb{Z}_+$-graded rings, and its n-th grade is isomorphic to the n-th grade of $\mathbb{Z}[x_1, ..., x_n]^{S_n}$.

6.1.2 Motivating example: categories

We now move on to the categorical version of the same result.

Let $GL_n(\mathbb{C})$ (denoted by GL_n for short) be the general linear group over $\mathbb{C}$. We have an inclusion $GL_n \subset GL_{n+1}$ with the matrix $A \in GL_n$ corresponding to a block matrix $A' \in GL_{n+1}$ which has A as the upper left $n \times n$-block, and 1 in the lower right corner (the rest of the entries are zero). One can consider a similar inclusion of Lie algebras $\mathfrak{gl}_n \subset \mathfrak{gl}_{n+1}$.

Next, we consider the polynomial representations of the algebraic group GL_n (alternatively, the Lie algebra $\mathfrak{gl}_n$): these are the representations $\rho : GL_n \to \mathrm{Aut}(V)$ which can be extended to an algebraic map $Mat_{n\times n}(\mathbb{C}) \to \mathrm{End}(V)$. These representations are direct summands of finite sums of tensor powers of the tautological representation $\mathbb{C}^n$ of GL_n.

The category of polynomial representations of GL_n, denoted by $Rep(\mathfrak{gl}_n)_{poly}$, is a semisimple symmetric monoidal category, with simple objects indexed by integer partitions with at most n parts. The Grothendieck ring of this category is isomorphic to $\mathbb{Z}[x_1, ..., x_n]^{S_n}$.

We also have functors

$$\mathfrak{Res}_{n-1,n} = (\cdot)^{E_{n,n}} : Rep(\mathfrak{gl}_n)_{poly} \to Rep(\mathfrak{gl}_{n-1})_{poly}$$

On the Grothendieck rings, these functors induce the homomorphisms

$$\mathbb{Z}[x_1, ..., x_n]^{S_n} \to \mathbb{Z}[x_1, ..., x_{n-1}]^{S_{n-1}} \quad p(x_1, ..., x_n) \mapsto p(x_1, ..., x_{n-1}, 0)$$

discussed above.

Finally, we consider the infinite-dimensional group $GL_\infty = \bigcup_{n\geq 0} GL_n$, and its Lie algebra $\mathfrak{gl}_\infty = \bigcup_{n\geq 0} \mathfrak{gl}_n$. The category of polynomial representations of this group (resp. Lie algebra) is denoted by $Rep(\mathfrak{gl}_\infty)_{poly}$, and it is the free Karoubian symmetric monoidal category generated by one object (the tautological representation $\mathbb{C}^\infty$ of GL_∞). It is also known that this category is equivalent to the category of strict polynomial functors of finite degree (c.f. [HY]), it is semisimple, and its Grothendieck ring is isomorphic to the ring $\Lambda_\mathbb{Z}$.

The category $Rep(\mathfrak{gl}_\infty)_{poly}$ possesses symmetric monoidal functors

$$\Gamma_n : Rep(\mathfrak{gl}_\infty)_{poly} \to Rep(\mathfrak{gl}_n)_{poly}$$

with the tautological representation of $\mathfrak{gl}_\infty$ being sent to tautological representation of $\mathfrak{gl}_n$. These functors are compatible with the functors $\mathfrak{Res}_{n-1,n}$ (i.e. $\Gamma_{n-1} \cong \mathfrak{Res}_{n-1,n} \circ \Gamma_n$), and the functor Γ_n induces the homomorphism

$$\Lambda_\mathbb{Z} \to \mathbb{Z}[x_1, ..., x_n]^{S_n} \quad p(x_1, ..., x_n, x_{n+1}, ...) \mapsto p(x_1, ..., x_n, 0, 0, ...)$$

This gives us a fully faithful functor $\Gamma_{\lim} : Rep(\mathfrak{gl}_\infty)_{poly} \to \varprojlim_{n\geq 0} Rep(\mathfrak{gl}_n)_{poly}$.

Finding a description of the image of the functor $\Gamma_{\lim}$ inspires the following two frameworks for "special" inverse limits, which turn out to be useful in other cases as well.

6.1.3 Restricted inverse limit of categories

To define the restricted inverse limit, we work with categories $\mathcal{C}_i$ which are finite-length categories; namely, abelian categories where each object has a (finite) Jordan-Holder filtration. We require that the functors $\mathcal{F}_{i-1,i}$ be "shortening": this means that these are exact functors such that given an object $C \in \mathcal{C}_i$, we have

$$\ell_{\mathcal{C}_{i-1}}(\mathcal{F}_{i-1,i}(C)) \leq \ell_{\mathcal{C}_i}(C)$$

In that case, it makes sense to consider the full subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, with $\{\ell_{\mathcal{C}_n}(C_n)\}_{n\geq 0}$ being a bounded sequence (the condition on the functors implies that this sequence is weakly increasing).

This subcategory will be called the "restricted" inverse limit of categories $\mathcal{C}_i$ and will be denoted by $\varprojlim_{i\in\mathbb{Z}_+, \text{ restr}} \mathcal{C}_i$. It is the inverse limit of the categories $\mathcal{C}_i$ in the $(2,1)$-category of finite-length categories and shortening functors.

Considering the restricted inverse limit of the categories $Rep(\mathfrak{gl}_n)_{poly}$, we obtain a functor

$$\Gamma_{\lim} : Rep(\mathfrak{gl}_\infty)_{poly} \to \varprojlim_{n\geq 0, \text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$$

It is easy to see that $\Gamma_{\lim}$ is an equivalence. Note that in terms of Grothendieck rings, this construction corresponds to the first approach described in Subsection 6.1.1.

6.1.4 Inverse limit of categories with filtrations

Another construction of the inverse limit is as follows: let K be a filtered poset, and assume that our categories $\mathcal{C}_i$ have a K-filtration on objects; that is, we assume that for each $k \in K$, there is a full subcategory $Fil_k(\mathcal{C}_i)$, and the functors $\mathcal{F}_{i-1,i}$ respect this filtration (note that if we consider abelian categories and exact functors, we should require that the subcategories be Serre subcategories).

We can then define a full subcategory $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ such that there exists $k \in K$ for which $C_i \in Fil_k(\mathcal{C}_i)$ for any $i \geq 0$.

The category $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is automatically a category with a K-filtration on objects. It is the inverse limit of the categories $\mathcal{C}_i$ in the $(2,1)$-category of categories with K-filtrations on objects, and functors respecting these filtrations.

Remark 6.1.4.1. A more general way to describe this setting would be the following.

Assume that for each i, the category $\mathcal{C}_i$ is a direct limit of a system

$$\left((\mathcal{C}_i^k)_{k\in\mathbb{Z}_+}, \left(\mathcal{G}_i^{k-1,k} : \mathcal{C}_i^{k-1} \to \mathcal{C}_i^k\right)\right)$$

Furthermore, assume that the functors $\mathcal{F}_{i-1,i}$ induce functors $\mathcal{F}^k_{i-1,i} : \mathcal{C}^k_{i-1} \to \mathcal{C}^k_i$ for any $k \in \mathbb{Z}_+$, and that the latter are compatible with the functors $\mathcal{G}_i^{k-1,k}$. One can then define the category

$$\varinjlim_{k \in K} \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}^k_i$$

which will be the "directed" inverse limit of the system. When $\mathcal{C}^k_i := Fil_k(\mathcal{C}_i)$ and $\mathcal{G}_i^{k-1,k}$ are inclusion functors, the directed inverse limit coincides with $\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$.

All the statements in this thesis concerning inverse limits of categories with filtrations can be translated to the language of directed inverse limits.

Considering appropriate $\mathbb{Z}_+$-filtrations on the objects of the categories $Rep(\mathfrak{gl}_n)_{poly}$, we obtain a functor

$$\Gamma_{\lim} : Rep(\mathfrak{gl}_\infty)_{poly} \to \varprojlim_{n \geq 0, \mathbb{Z}_+ - filtr} Rep(\mathfrak{gl}_n)_{poly}$$

One can show that this is an equivalence. Note that in terms of Grothendieck rings, this construction corresponds to the second approach described in Subsection 6.1.1 (in fact, in this particular case one can use a grading instead of a filtration; however, this is not the case in Chapter 7).

These two "special" inverse limits may coincide, as it happens in the case of the categories $Rep(\mathfrak{gl}_n)_{poly}$, and in Chapter 7. We give a sufficient condition for this to happen. In such case, each approach has its own advantages.

The restricted inverse limit approach does not involve defining additional structures on the categories, and shows that the constructed inverse limit category does not depend on the choice of filtration, as long as the filtration satisfies some relatively mild conditions.

Yet the object-filtered inverse limit approach is sometimes more convenient to work with, as it happens in Chapter 7.

6.2 Inverse limits of categories

In this section we discuss the notion of an inverse limit of categories, based on [WW, Definition 1], [S, Section 5]. This is the inverse limit in the $(2,1)$-category of categories with functors and natural isomorphisms.

6.2.1 Definition of inverse limits of categories

Consider the partially ordered set $(\mathbb{Z}_+, \leq)$. We consider the following data ("system"):

1. Categories $\mathcal{C}_i$ for each $i \in \mathbb{Z}_+$.

2. Functors $\mathcal{F}_{i-1,i} : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for each $i \geq 1$.

Definition 6.2.1.1. Given the above data, we define the inverse limit category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ to be the following category:

- The objects are pairs $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ where $C_i \in \mathcal{C}_i$ for each $i \in \mathbb{Z}_+$ and $\phi_{i-1,i} : \mathcal{F}_{i-1,i}(C_i) \xrightarrow{\sim} C_{i-1}$ for any $i \geq 1$.

- A morphism f between two objects $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, $(\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$ is a set of arrows $\{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$ such that for any $i \geq 1$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}_{i-1,i}(C_i) & \xrightarrow{\phi_{i-1,i}} & C_{i-1} \\ {\scriptstyle \mathcal{F}_{i-1,i}(f_i)}\downarrow & & {\scriptstyle f_{i-1}}\downarrow \\ \mathcal{F}_{i-1,i}(D_i) & \xrightarrow{\psi_{i-1,i}} & D_{i-1} \end{array}$$

  Composition of morphisms is component-wise.

The definition of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ implies that for each $i \in \mathbb{Z}_+$, we can define functors

$$\mathbf{Pr}_i : \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i \to \mathcal{C}_i$$
$$C = (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})) \mapsto C_i$$
$$f = \{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+} \mapsto f_i$$

which satisfy the following property (this property follows directly from the definition of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$):

Lemma 6.2.1.2. *For any $i \geq 1$, $\mathcal{F}_{i-1,i} \circ \mathbf{Pr}_i \cong \mathbf{Pr}_{i-1}$, with a natural isomorphism given by:*

$$(\mathcal{F}_{i-1,i} \circ \mathbf{Pr}_i)(C) \overset{\phi_{i-1,i}}{\rightarrow} \mathbf{Pr}_{i-1}(C)$$

(here $C = (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1}))$).

Let $\mathcal{A}$ be a category, together with a set of functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ which satisfy: for any $i \geq 1$, there exists a natural isomorphism

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}$$

Then $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ is universal among such categories; that is, we have a functor

$$\begin{aligned} \mathcal{G} : \mathcal{A} &\to \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i \\ A &\mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1}) \\ (f : A_1 \to A_2) &\mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+} \end{aligned}$$

and $\mathcal{G}_i \cong \mathbf{Pr}_i \circ \mathcal{G}$ for every $i \in \mathbb{Z}_+$.

Finally, we give the following simple lemma:

Lemma 6.2.1.3. *Let $N \in \mathbb{Z}_+$, and assume that for any $i \geq N$, $\mathcal{F}_{i-1,i}$ is an equivalence. Then $\mathbf{Pr}_i : \varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j \to \mathcal{C}_i$ is an equivalence for any $i \geq N$.*

Proof. Set $\mathcal{F}_{ij} := \mathcal{F}_{i,i+1} \circ ... \circ \mathcal{F}_{j-1,j}$ for any $i \leq j$ (in particular, $\mathcal{F}_{ii} := \mathrm{Id}_{\mathcal{C}_i}$).

Fix $i \geq N$. Let $j \geq i$; then $\mathcal{F}_{ij}$ is an equivalence, i.e. we can find a functor

$$\mathcal{G}_j : \mathcal{C}_i \to \mathcal{C}_j$$

such that $\mathcal{F}_{ij} \circ \mathcal{G}_j \cong \mathrm{Id}_{\mathcal{C}_i}$, and $\mathcal{G}_j \circ \mathcal{F}_{ij} \cong \mathrm{Id}_{\mathcal{C}_j}$ (for $j := i$, we put $\mathcal{G}_i := \mathrm{Id}_{\mathcal{C}_i}$).

For any $j > i$, fix natural transformations

$$\eta_{j-1,j} : \mathcal{F}_{j-1,j} \circ \mathcal{G}_j \xrightarrow{\sim} \mathcal{G}_{j-1}$$

For any $j \leq i$, put $\mathcal{G}_j := \mathcal{F}_{ji}$, and $\eta_{j-1,j} := \mathrm{Id}$.

Then the universal property of $\varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j$ implies that there exists a functor

$$\mathcal{G} : \mathcal{C}_i \to \varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j$$

such that $\mathbf{Pr}_j \circ \mathcal{G} \cong \mathcal{G}_j$ for any j. The functor $\mathcal{G}$ is given by

$$\begin{aligned} \mathcal{G} : \mathcal{C}_i &\to \varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j \\ C &\mapsto (\{\mathcal{G}_j(C)\}_{j\in\mathbb{Z}_+}, \{\eta_{j-1,j}\}_{j\geq 1}) \\ f : C \to C' &\mapsto \{f_j := \mathcal{G}_j(f)\}_{j\in\mathbb{Z}_+} \end{aligned}$$

In particular, we have: $\mathbf{Pr}_i \circ \mathcal{G} \cong \mathrm{Id}_{\mathcal{C}_i}$. It remains to show that $\mathcal{G} \circ \mathbf{Pr}_i \cong \mathrm{Id}_{\varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j}$, and this will prove that $\mathbf{Pr}_i$ is an equivalence of categories.

For any $C \in \varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j$, $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$, and for any $l \leq j$ we define isomorphisms $\phi_{lj} : \mathcal{F}_{lj}(C_j) \to C_l$ given by

$$\phi_{lj} := \phi_{l,l+1} \circ \mathcal{F}_{l,l+1}(\phi_{l+1,l+2} \circ \mathcal{F}_{l+1,l+2}(\phi_{l+2,l+3} \circ \ldots \circ \mathcal{F}_{j-2,j-1}(\phi_{j-1,j})\ldots))$$

Define $\theta(C) := \{\theta(C)_j : C_j \to \mathbf{Pr}_j(\mathcal{G}(C_i)) \cong \mathcal{G}_j(C_i)\}_{j\in I}$ by setting

$$\theta(C)_j := \begin{cases} \phi_{ji}^{-1} & \text{if } j \leq i \\ \mathcal{G}_j(\phi_{ij}) & \text{if } j > i \end{cases}$$

Now, let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$, $D := (\{D_j\}_{j\in\mathbb{Z}_+}, \{\psi_{j-1,j}\}_{j\geq 1})$ be objects in $\varprojlim_{j\in\mathbb{Z}_+} \mathcal{C}_j$, together with a morphism $f : C \to D$, $f := \{f_j : C_j \to D_j\}_{j\in\mathbb{Z}_+}$.

Then the diagram

$$\begin{array}{ccc} C & \xrightarrow{\theta(C)} & (\mathcal{G}\circ \mathbf{Pr}_i)(C) \\ f\downarrow & & \downarrow(\mathcal{G}\circ\mathbf{Pr}_i)(f) \\ D & \xrightarrow{\theta(D)} & (\mathcal{G}\circ \mathbf{Pr}_i)(D) \end{array}$$

is commutative, since for $j \leq i$, the diagrams

$$\begin{array}{ccc} C_j & \xrightarrow{\phi_{ji}^{-1}} & \mathbf{Pr}_j(\mathcal{G}(C_i)) \cong \mathcal{G}_j(C_i) \\ f_j\downarrow & & \downarrow\mathcal{G}_j(f_i) \\ D_j & \xrightarrow{\psi_{ji}^{-1}} & \mathbf{Pr}_j(\mathcal{G}(D_i)) \cong \mathcal{G}_j(D_i) \end{array}$$

are commutative, and for $j > i$, the diagrams

$$\begin{array}{ccc} C_j & \xrightarrow{\mathcal{G}_j(\phi_{ij})} & \mathbf{Pr}_j(\mathcal{G}(C_i)) \cong \mathcal{G}_j(C_i) \\ f_j\downarrow & & \downarrow\mathcal{G}_j(f_i) \\ D_j & \xrightarrow{\mathcal{G}_j(\psi_{ij})} & \mathbf{Pr}_j(\mathcal{G}(D_i)) \cong \mathcal{G}_j(D_i) \end{array}$$

are commutative.

□

6.2.2 Inverse limits of pre-additive, additive and abelian categories

In this subsection, we give some more or less trivial properties of the inverse limit corresponding to the system $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ depending on the properties of the categories $\mathcal{C}_i$ and the functors $\mathcal{F}_{i-1,i}$.

Lemma 6.2.2.1. *Assume the categories $\mathcal{C}_i$ are $\mathbb{C}$-linear pre-additive categories (i.e. the* Hom*-spaces in each $\mathcal{C}_i$ are complex vector spaces), and the functors $\mathcal{F}_{i-1,i}$ are $\mathbb{C}$-linear. Then the category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ is automatically a $\mathbb{C}$-linear pre-additive category:*

given $f, g : C \to D$ in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, where $C = (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, $D =$

$(\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$, $f = \{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}, g = \{g_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$, *we have:*

$$\alpha f + \beta g := \{(\alpha f_i + \beta g_i) : C_i \to D_i\}_{i\in\mathbb{Z}_+}$$

where $\alpha, \beta \in \mathbb{C}$.

The functors $\mathbf{Pr}_i$ *are then* $\mathbb{C}$*-linear.*

Lemma 6.2.2.2. *Assume the categories* $\mathcal{C}_i$ *are additive categories (i.e. each* $\mathcal{C}_i$ *is pre-additive and has biproducts), and the functors* $\mathcal{F}_{i-1,i}$ *are additive. Then the category* $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ *is automatically a additive category:*

- *The zero object in* $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ *is* $(\{0_{\mathcal{C}_i}\}_{i\in\mathbb{Z}_+}, \{0\}_{i\geq 1})$.
- *Given* C, D *in* $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, *where* $C = (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, $D = (\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$, *we have:*

  $$C \oplus D := (\{(C_i \oplus D_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i} \oplus \psi_{i-1,i}\}_{i\geq 1})$$

  with obvious inclusion and projection maps.

The functors $\mathbf{Pr}_i$ *are then additive.*

Proof. Let $X, Y \in \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, $X = (\{X_i\}_{i\in\mathbb{Z}_+}, \{\mu_{i-1,i}\}_{i\geq 1})$, $Y = (\{Y_i\}_{i\in\mathbb{Z}_+}, \{\rho_{i-1,i}\}_{i\geq 1})$, and let $f_C : X \to C$, $f_D : X \to D$, $g_C : C \to Y$, $g_D : D \to Y$ (we denote the components of the map f_C by f_{C_i}, of the map f_D by f_{D_i}, etc.).

Denote by $\iota_{C_i}, \iota_{D_i}, \pi_{C_i}, \pi_{D_i}$ the inclusion and projection maps between C_i, D_i and $C_i \oplus D_i$. By definition, $\iota_C := \{\iota_{C_i}\}_{i\in\mathbb{Z}_+}, \iota_D := \{\iota_{D_i}\}_{i\in\mathbb{Z}_+}, \pi_C := \{\pi_{C_i}\}_{i\in\mathbb{Z}_+}, \pi_D := \{\pi_{D_i}\}_{i\in\mathbb{Z}_+}$ are the inclusion and projection maps between C, D and $C \oplus D$.

For each i, there exists a unique map $f_i : X_i \to C_i \oplus D_i$ and a unique map $g_i : C_i \oplus D_i \to Y_i$ such that

$$\pi_{C_i} \circ f_i = f_{C_i}, \pi_{D_i} \circ f_i = f_{D_i}, g_i \circ \iota_{C_i} = g_{C_i}, g_i \circ \iota_{D_i} = g_{D_i}$$

for any $i \in \mathbb{Z}_+$.

This means that we have a unique map $f : X \to C \oplus D$ and a unique map $g : C \oplus D \to Y$ such that

$$\pi_C \circ f = f_C, \pi_D \circ f = f_D, g \circ \iota_C = g_C, g \circ \iota_D = g_D$$

(these are the maps $f = \{f_i\}_i, g = \{g_i\}_i$).

□

Lemma 6.2.2.3. *Let $f : C \to D$ in $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$, where $C = (\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$, $D = (\{D_i\}_{i \in \mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i \geq 1})$, $f = \{f_i : C_i \to D_i\}_{i \in \mathbb{Z}_+}$.*

Assume f_i are isomorphisms for each i. Then f is an isomorphism.

Proof. Let $g_i := f_i^{-1}$ for each $i \in \mathbb{Z}_+$ (this morphism exists since f_i is an isomorphism, and is unique). All we need is to show that $g := \{g_i : D_i \to C_i\}_i$ is a morphism from D to C in $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$, i.e. that the following diagram is commutative for any $i \geq 1$:

$$\begin{array}{ccc} \mathcal{F}_{i-1,i}(C_i) & \xrightarrow{\phi_{i-1,i}} & C_{i-1} \\ {\scriptstyle \mathcal{F}_{i-1,i}(g_i)}\uparrow & & {\scriptstyle g_{i-1}}\uparrow \\ \mathcal{F}_{i-1,i}(D_i) & \xrightarrow{\psi_{i-1,i}} & D_{i-1} \end{array}$$

The morphism $g_{i-1} \circ \psi_{i-1,i}$ is inverse to $\psi_{i-1,i}^{-1} \circ f_{i-1}$, and $\phi_{i-1,i} \circ \mathcal{F}_{i-1,i}(g_j i)$ is inverse to $\mathcal{F}_{i-1,i}(f_i) \circ \phi_{i-1,i}^{-1}$.

But $\psi_{i-1,i}^{-1} \circ f_{i-1} = \mathcal{F}_{i-1,i}(f_i) \circ \phi_{i-1,i}^{-1}$, since $f = \{f_i : C_i \to D_i\}_{i \in \mathbb{Z}_+}$ is a morphism from C to D in $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$. The uniqueness of the inverse morphism then implies that $g_{i-1} \circ \psi_{i-1,i} = \phi_{i-1,i} \circ \mathcal{F}_{i-1,i}(g_i)$, and we are done. □

Proposition 6.2.2.4. *Assume the categories $\mathcal{C}_i$ are abelian, and the functors $\mathcal{F}_{i-1,i}$ are exact. Then the category $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ is automatically abelian:*

- *Given $f : C \to D$ in $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$, where $C = (\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$, $D = (\{D_i\}_{i \in \mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i \geq 1})$, $f = \{f_i : C_i \to D_i\}_{i \in \mathbb{Z}_+}$, f has a kernel and a cokernel:*

  $$Ker(f) := (\{Ker(f_i)\}_{i \in \mathbb{Z}_+}, \{\rho_{i-1,i}\}_{i \geq 1}), Coker(f) := (\{Coker(f_i)\}_{i \in \mathbb{Z}_+}, \{\mu_{i-1,i}\}_{i \geq 1})$$

where $\rho_{i-1,i}, \mu_{i-1,i}$ are the unique maps making the following diagram commutative:

$$\begin{array}{ccc}
Ker(\mathcal{F}_{i-1,i}(f_i)) \cong \mathcal{F}_{i-1,i}(Ker(f_i)) & \xrightarrow{\rho_{i-1,i}} & Ker(f_{i-1}) \\
\downarrow & & \downarrow \\
\mathcal{F}_{i-1,i}(C_i) & \xrightarrow{\phi_{i-1,i}} & C_{i-1} \\
{\scriptstyle \mathcal{F}_{ij}(f_i)}\downarrow & & {\scriptstyle f_{i-1}}\downarrow \\
\mathcal{F}_{i-1,i}(D_i) & \xrightarrow{\psi_{i-1,i}} & D_{i-1} \\
\downarrow & & \downarrow \\
Coker(\mathcal{F}_{i-1,i}(f_i)) \cong \mathcal{F}_{i-1,i}(Coker(f_i)) & \xrightarrow{\mu_{i-1,i}} & Coker(f_{i-1})
\end{array}$$

- *Given $f : C \to D$ in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, we have: $Im(f) := Ker(Coker(f)) \cong Coker(Ker(f)) =: Coim(f)$.*

Proof. The universal properties of $Ker(f), Coker(f)$ hold automatically, as a consequence of the universal properties of $Ker(f_i), Coker(f_i)$.

Now, let $f : C \to D$ in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, where $C = (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, $D = (\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$, $f = \{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$.

Consider the objects $Im(f) := Ker(Coker(f)), Coim(f) := Coker(Ker(f))$ in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. We have a canonical map $\bar{f} : Coim(f) \to Im(f)$, such that $f : C \to D$ is the composition

$$C \twoheadrightarrow Coim(f) \xrightarrow{\bar{f}} Im(f) \hookrightarrow D$$

Consider the maps $\bar{f}_i$ for each $i \in \mathbb{Z}_+$, where $\bar{f}_i$ is the canonical map such that $f_i : C_i \to D_i$ is the composition

$$C_i \twoheadrightarrow Coim(f_i) \xrightarrow{\bar{f}_i} Im(f_i) \hookrightarrow D_i$$

One then immediately sees that $\bar{f} = \{\bar{f}_i : Coim(f_i) \to Im(f_i)\}_i$.

Since the category $\mathcal{C}_i$ is abelian for each $i \in \mathbb{Z}_+$, the map $\bar{f}_i$ is an isomorphism. Lemma 6.2.2.3 then implies that $\bar{f}$ is an isomorphism as well. □

The following is a trivial corollary of the previous proposition:

Corollary 6.2.2.5. *The functors* $\mathbf{Pr}_i$ *are exact.*

This corollary, in turn, immediately implies the following statement:

Corollary 6.2.2.6. *Let* $(\mathcal{C}_i, \mathcal{F}_{ij})$ *be a system of pre-additive (respectively, additive, abelian) categories, and linear (respectively, additive, exact) functors.*

Let $\mathcal{A}$ *be a pre-additive (respectively, additive, abelian) category, together with a set of linear (respectively, additive, exact) functors* $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ *which satisfy: for any* $i \geq 1$, *there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}$$

Then $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ *is universal among such categories; that is, we have a linear (respectively, additive, exact) functor*

$$\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$$
$$A \mapsto (\{\mathcal{G}_i(A)\}_{i \in \mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i \in \mathbb{Z}_+})$$
$$f : A_1 \to A_2 \mapsto \{f_i := \mathcal{G}_i(f)\}_{i \in \mathbb{Z}_+}$$

and $\mathcal{G}_i \cong \mathbf{Pr}_i \circ \mathcal{G}$ *for every* $i \in \mathbb{Z}_+$.

6.3 Restricted inverse limit of finite-length categories

We consider the case when the categories $\mathcal{C}_i$ are finite-length. We would like to give a notion of an inverse limit of the system $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ which would be a finite-length category as well. In order to do this, we will define the notion of a "shortening" functor, and define a "stable" inverse limit of a system of finite-length categories and shortening functors.

Definition 6.3.0.7. Let $\mathcal{A}_1, \mathcal{A}_2$ be finite-length categories. An exact functor $\mathcal{F} : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ will be called *shortening* if for any object $A \in \mathcal{A}_1$, we have:

$$\ell_{\mathcal{A}_1}(A) \geq \ell_{\mathcal{A}_2}(\mathcal{F}(A))$$

Since $\mathcal{F}$ is exact, this is equivalent to requiring that for any simple object $L \in \mathcal{A}_1$, the object $\mathcal{F}(L)$ is either simple or zero.

Definition 6.3.0.8. Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a system of finite-length categories and shortening functors. We will denote by $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$ the full subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ whose objects $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ satisfy: the integer sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ stabilizes.

Note that the since the functors $\mathcal{F}_{i-1,i}$ are shortening, the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is weakly increasing. Therefor, this sequence stabilizes iff it is bounded from above.

We now show that $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$ is a finite-length category.

Lemma 6.3.0.9. *The category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\ \mathit{restr}} \mathcal{C}_i$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, and its objects have finite length.*

Moreover, given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have:

$$\ell_{\mathcal{C}}(C) \leq \max\{\ell_{\mathcal{C}_i}(C_i) | i \geq 0\}$$

Proof. Let

$$C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1}),\ C' := (\{C'_j\}_{j\in\mathbb{Z}_+}, \{\phi'_{j-1,j}\}_{j\geq 1}),\ C'' := (\{C''_j\}_{j\in\mathbb{Z}_+}, \{\phi''_{j-1,j}\}_{j\geq 1})$$

be objects in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, together with morphisms $f : C' \to C$, $g : C \to C''$ such that the sequence

$$0 \to C' \xrightarrow{f} C \xrightarrow{g} C'' \to 0$$

is exact.

If C lies in the subcategory $\mathcal{C}$, then the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is bounded from above, and stabilizes. Denote its maximum by N. For each i, the sequence

$$0 \to C'_i \xrightarrow{f_i} C_i \xrightarrow{g} C''_i \to 0$$

is exact. Therefore, $\ell_{\mathcal{C}_i}(C'_i), \ell_{\mathcal{C}_i}(C''_i) \leq N$ for each i, and thus C', C'' lie in $\mathcal{C}$ as well.

Vice versa, assuming C', C'' lie in $\mathcal{C}$, denote by N', N'' the maximums of the sequences $\{\ell_{\mathcal{C}_i}(C'_i)\}_i, \{\ell_{\mathcal{C}_i}(C''_i)\}_i$ respectively. Then $\ell_{\mathcal{C}_i}(C_i) \leq N' + N''$ for any $i \geq 0$, and so C lies in the subcategory $\mathcal{C}$ as well.

Thus $\mathcal{C}$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$.

Next, let C lie in $\mathcal{C}$. We would like to say that C has finite length. Denote by N the maximum of the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$. It is easy to see that C has length at most N; indeed, if $\{C', C'', ..., C^{(n)}\}$ is a subset of $JH_{\mathcal{C}}(C)$, then for some $i >> 0$, we have: $\mathbf{Pr}_i(C^{(k)}) \neq 0$ for any $k = 1, 2, ..., n$. $\mathbf{Pr}_i(C^{(k)})$ are distinct Jordan Holder components of C_i, so $n \leq \ell_{\mathcal{C}_i}(C_i) \leq N$. In particular, we see that

$$\ell_{\mathcal{C}}(C) \leq N = \max\{\ell_{\mathcal{C}_i}(C_i) | i \geq 0\}$$

□

Notation 6.3.0.10. Denote by $Irr(\mathcal{C}_i)$ the set of isomorphism classes of irreducible objects in $\mathcal{C}_i$, and define the pointed set

$$Irr_*(\mathcal{C}_i) := Irr(\mathcal{C}_i) \sqcup \{0\}$$

The shortening functors $\mathcal{F}_{i-1,i}$ then define maps of pointed sets

$$f_{i-1,i} : Irr_*(\mathcal{C}_i) \longrightarrow Irr_*(\mathcal{C}_{i-1})$$

Similarly, we define $Irr\left(\varprojlim_{i\in\mathbb{Z}_+, \text{ restr}} \mathcal{C}\right)$ to be the set of isomorphism classes of irreducible objects in $\mathcal{C}$, and define the pointed set

$$Irr_*(\mathcal{C}) := Irr(\mathcal{C}) \sqcup \{0\}$$

Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\mathcal{C}$. We denote by $JH(C_j)$ the multiset of the Jordan-Holder components of C_j, and let

$$JH_*(C_j) := JH(C_j) \sqcup \{0\}$$

The corresponding set lies in $Irr_*(\mathcal{C}_j)$, and we have maps of (pointed) multisets

$$f_{j-1,j} : JH_*(C_j) \to JH_*(C_{j-1})$$

Denote by $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$ the inverse limit of the system $(\{Irr_*(\mathcal{C}_i\}_{i\geq 0}, \{f_{i-1,i}\}_{i\geq 1})$. We will also denote by $pr_j : \varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i) \to Irr_*(\mathcal{C}_j)$ the projection maps.

The elements of the set $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$ are just sequences $(L_i)_{i\geq 0}$ such that $L_i \in Irr_*(\mathcal{C}_i)$, and $f_{i-1,i}(L_i) \cong L_{i-1}$.

The following lemma describes the simple objects in the category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\ \text{restr}} \mathcal{C}_i$.

Lemma 6.3.0.11. *Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\ restr} \mathcal{C}_i$. Then*

$$C \in Irr_*(\mathcal{C}) \iff \mathbf{Pr}_j(C) = C_j \in Irr_*(\mathcal{C}_j)\ \forall j$$

In other words, C is a simple object (that is, C has exactly two distinct subobjects: zero and itself) iff $C \neq 0$, and for any $j \geq 0$, the component C_j is either a simple object in $\mathcal{C}_j$, or zero.

Proof. The direction $\Leftarrow$ is obvious, so we will only prove the direction $\Rightarrow$.

Let n_0 be a position in which the maximum of the weakly-increasing integer sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is obtained. By definition of n_0, for $j > n_0$, the functors $\mathcal{F}_{j-1,j}$ do not kill any Jordan-Holder components of C_j.

Now, consider the socles of the objects C_j for $j \geq n_0$. For any $j > 0$, we have:

$$\mathcal{F}_{j-1,j}(\ socle(C_j)\ ) \overset{\phi_{j-1,j}}{\hookrightarrow} \ socle(C_{j-1})$$

and thus for $j > n_0$, we have

$$\ell_{\mathcal{C}_j}(\ socle(C_j)\ ) = \ell_{\mathcal{C}_{j-1}}(\mathcal{F}_{j-1,j}(\ socle(C_j)\ )) \leq \ell_{\mathcal{C}_{j-1}}(\ socle(C_{j-1})\ )$$

Thus the sequence

$$\{\ell_{\mathcal{C}_j}(\ socle(C_j)\ )\}_{j \geq n_0}$$

is a weakly decreasing sequence, and stabilizes. Denote its stable value by N. We conclude that there exists $n_1 \geq n_0$ so that

$$\mathcal{F}_{j-1,j}(\ socle(C_j)\ ) \overset{\phi_{j-1,j}}{\longrightarrow} \ socle(C_{j-1})$$

is an isomorphism for every $j > n_1$.

Now, denote:

$$D_j := \begin{cases} \mathcal{F}_{j,n_1}(\ socle(C_{n_1})\ ) & \text{if } j < n_1 \\ socle(C_j) & \text{if } j \geq n_1 \end{cases}$$

and we put: $D := ((D_j)_{j \geq 0}, (\phi_{j-1,j})_{j \geq 1})$ (this is a subobject of C in the category $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$). Of course, $\ell_{\mathcal{C}_j}(D_j) \leq N$ for any j, so D is an object in the full subcategory $\mathcal{C}$ of $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$.

Furthermore, since $C \neq 0$, we have: for $j >> 0$, $socle(C_j) \neq 0$, and thus $0 \neq D \subset C$.

D is a semisimple object $\mathcal{C}$, with simple summands corresponding to the elements of the inverse limit of the multisets $\varprojlim_{j \in \mathbb{Z}_+} JH_*(D_j)$.

We conclude that $D = C$, and that $socle(C_j) = C_j$ has length at most one for any $j \geq 0$.

Remark 6.3.0.12. Note that the latter multiset is equivalent to the inverse limit of multisets

$JH_*(\ socle(C_j)\ )$, so D is, in fact, the socle of C.

□

Corollary 6.3.0.13. *The set of isomorphism classes of simple objects in $\varprojlim_{i\in\mathbb{Z}_+,\ restr} \mathcal{C}_i$ is in bijection with the set $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)\setminus\{0\}$. That is, we have a natural bijection*

$$Irr_*(\mathcal{C}) \cong \varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$$

In particular, given an object $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ in $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$, we have: $JH_*(C) = \varprojlim_{i\in\mathbb{Z}_+} JH_*(C_i)$ (an inverse limit of the system of multisets $JH_*(C_j)$ and maps $f_{j-1,j}$).

It is now obvious that the projection functors $\mathbf{Pr}_i$ are shortening as well:

Corollary 6.3.0.14. *The projection functors $\mathbf{Pr}_i$ are shortening, and define the maps*

$$pr_i : Irr_*(\mathcal{C}) \longrightarrow Irr_*(\mathcal{C}_i) \qquad .$$

Lemma 6.3.0.9 and Corollary 6.3.0.14 give us:

Corollary 6.3.0.15. *Given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have:*

$$\ell_{\mathcal{C}}(C) = \max\{\ell_{\mathcal{C}_i}(C_i) | i \geq 0\}$$

It is now easy to see that the restricted inverse limit has the following universal property:

Proposition 6.3.0.16. *Let $\mathcal{A}$ be a finite-length category, together with a set of shortening functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ which satisfy: for any $i \geq 1$, there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}$$

Then $\varprojlim_{i\in\mathbb{Z}_+,\ restr} \mathcal{C}_i$ is universal among such categories; that is, we have a shortening *functor*

$$\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+,\ restr} \mathcal{C}_i$$
$$A \mapsto (\{\mathcal{G}_i(A)\}_{i \in \mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i \geq 1})$$
$$f : A_1 \to A_2 \mapsto \{f_i := \mathcal{G}_i(f)\}_{i \in \mathbb{Z}_+}$$

and $\mathcal{G}_i \cong \mathbf{Pr}_i \circ \mathcal{G}$ *for every* $i \in \mathbb{Z}_+$.

Proof. Consider the functor $\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ induced by the functors $\mathcal{G}_i$. We would like to say that for any $A \in \mathcal{A}$, the object $\mathcal{G}(A)$ lies in the subcategory $\varprojlim_{i \in \mathbb{Z}_+,\ \text{restr}} \mathcal{C}_i$, i.e. that the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above.

Indeed, since $\mathcal{G}_i$ are shortening functors, we have: $\ell_{\mathcal{C}_i}(\mathcal{G}_i(A)) \leq \ell_{\mathcal{A}}(A)$. Thus the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above by $\ell_{\mathcal{A}}(A)$.

Now, using Corollary 6.3.0.15, we obtain:

$$\ell_{\mathcal{C}}(\mathcal{G}(A)) = \max_{i \geq 0}\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\} \leq \ell_{\mathcal{A}}(A)$$

and we conclude that $\mathcal{G}$ is a shortening functor. □

6.4 Inverse limit of categories with a filtration

We now consider the case when the categories $\mathcal{C}_i$ have a filtration on the objects (we will call these "filtered categories"), and the functors $\mathcal{F}_{i-1,i}$ respect this filtration. We will then define a subcategory of the category $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ which will be denoted by $\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$ and will be called the "inverse limit of filtered categories $\mathcal{C}_i$".

Fix a directed partially ordered set $(K, \leq)$ ("directed", means that for any $k_1, k_2 \in K$, there exists $k \in K$ such that $k_1, k_2 \leq k$).

Definition 6.4.0.17 (K-filtered categories). We say that a category $\mathcal{A}$ is a *K-filtered* category if for each $k \in K$ we have a full subcategory $\mathcal{A}^k$ of $\mathcal{A}$, and these subcategories satisfy the following conditions:

1. $\mathcal{A}^k \subset \mathcal{A}^l$ whenever $k \leq l$.
2. $\mathcal{A}$ is the union of $\mathcal{A}^k, k \in K$: that is, for any $A \in \mathcal{A}$, there exists $k \in K$ such that $A \in \mathcal{A}^k$.

A functor $\mathcal{F} : \mathcal{A}_1 \to \mathcal{A}_2$ between K-filtered categories $\mathcal{A}_1, \mathcal{A}_2$ is called a *K-filtered functor* if for any $k \in K$, $\mathcal{F}(\mathcal{A}_1^k)$ is a subcategory of $\mathcal{A}_2^k$.

Remark 6.4.0.18. Let $\mathcal{F} : \mathcal{A}_1 \to \mathcal{A}_2$ be a K-filtered functor between K-filtered categories $\mathcal{A}_1, \mathcal{A}_2$. Assume the restriction of $\mathcal{F}$ to each filtration component k is an equivalence of categories $\mathcal{A}_1^k \to \mathcal{A}_2^k$. Then $\mathcal{F}$ is obviously an equivalence of (K-filtered) categories.

Remark 6.4.0.19. The definition of a K-filtration on the objects of a category $\mathcal{A}$ clearly makes $\mathcal{A}$ a direct limit of the subcategories $\mathcal{A}^k$.

Definition 6.4.0.20. We say that the system $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ is *K-filtered* if for each $i \in \mathbb{Z}_+$, $\mathcal{C}_i$ is a category with a K-filtration, and the functors $\mathcal{F}_{i-1,i}$ are K-filtered functors.

Definition 6.4.0.21. Let $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ be a K-filtered system. We define the inverse limit of this $\mathbb{Z}_+$-filtered system (denoted by $\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$) to be the full subcategory of $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ whose objects C satisfy: there exists $k_C \in K$ such that $\mathbf{Pr}_i(C) \in \mathcal{C}_i^{k_C}$ for any $i \in \mathbb{Z}_+$.

The following lemma is obvious:

Lemma 6.4.0.22. *The category $\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is automatically K-filtered: the filtration component $Fil_k(\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i)$ can be defined to be the full subcategory of $\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$ of objects C such that $\mathbf{Pr}_i(C) \in \mathcal{C}_i^k$ for any $i \in \mathbb{Z}_+$.*

This also makes the functors $\mathbf{Pr}_i : \varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i \to \mathcal{C}_i$ K-filtered functors.

Remark 6.4.0.23. Note that by definition, for any $k \in K$

$$Fil_k\left(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i\right) \cong \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i^k$$

where the inverse limit is taken over the system $((\mathcal{C}_i^k)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i}|_{\mathcal{C}_i^k})_{i\geq 1})$. Thus

$$\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i := \varinjlim_{k\in K} \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i^k$$

Lemma 6.4.0.24. *Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a K-filtered system.*

1. *Assume the categories $\mathcal{C}_i$ are additive, the functors $\mathcal{F}_{i-1,i}$ are additive, and for any $k \in K$, $\mathcal{C}_i^k$ is an additive subcategory of $\mathcal{C}_i$.*

   Then the category $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is an additive subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, and all its filtration components are additive subcategories.

2. *Assume the categories $\mathcal{C}_i$ are abelian, the functors $\mathcal{F}_{i-1,i}$ are exact, and for any $k \in K$, $\mathcal{C}_i^k$ is a Serre subcategory of $\mathcal{C}_i$.*

   Then the category $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is abelian (and a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$), and all its filtration components are Serre subcategories.

Proof. To prove the first part of the statement, we only need to check that $Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$ is an additive subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. This follows directly from the construction of direct sums in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$: let $C, D \in Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i) \subset \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. Then $\mathbf{Pr}_i(C) \in \mathcal{C}_i^k$, $\mathbf{Pr}_i(D) \in \mathcal{C}_i^k$ for any $i \in \mathbb{Z}_+$. Since $\mathcal{C}_i^k$ is an additive subcategory of $\mathcal{C}_i$, we get: $\mathbf{Pr}_i(C \oplus D) \in \mathcal{C}_i^k$ for any $i \in \mathbb{Z}_+$ (the direct sum $C \oplus D$ is taken in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$).

Thus $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is an additive subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, and all its filtration components are additive subcategories as well.

To prove the second part of the statement, it is again enough to check that $Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$.

Indeed, let

$$0 \to C' \to C \to C'' \to 0$$

be a short exact sequence in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. We want to show that $C \in Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$ iff $C', C'' \in Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$.

The functors $\mathbf{Pr}_i$ are exact, so the sequence

$$0 \to C_i' \to C_i \to C_i'' \to 0$$

is exact for any $i \in \mathbb{Z}_+$.

Since $\mathcal{C}_i^k$ is a Serre subcategory of $\mathcal{C}_i$, we have: $C_i \in \mathcal{C}_i^k$ iff $C'_i, C''_i \in \mathcal{C}_i^k$, and we are done. □

We now have the following universal property, whose proof is straight-forward:

Proposition 6.4.0.25. *Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a K-filtered system, and let $\mathcal{A}$ be a category with a K-filtration, together with a set of K-filtered functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ which satisfy: for any $i \geq 1$, there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}$$

Then $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is universal among such categories; that is, we have a functor

$$\mathcal{G} : \mathcal{A} \to \varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$$
$$A \mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1})$$
$$f : A_1 \to A_2 \mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+}$$

which is obviously K-filtered, and satisfies: $\mathcal{G}_i \cong \mathbf{Pr}_i \circ \mathcal{G}$ for every $i \in \mathbb{Z}_+$.

Next, consider the case when $\mathcal{A}$, $\{\mathcal{G}_i\}_{i\in\mathbb{Z}_+}$ satisfy the following "stabilization" condition:

Condition 6.4.0.26. For every $k \in K$, there exists $i_k \in \mathbb{Z}_+$ such that $\mathcal{G}_j : \mathcal{A}^k \to \mathcal{C}_j^k$ is an equivalence of categories for any $j \geq i_k$.

In this setting, the following proposition holds:

Proposition 6.4.0.27. *The functor $\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i$ is an equivalence of (K-filtered) categories.*

Proof. To prove that $\mathcal{G}$ is an equivalence of (K-filtered) categories, we neeed to show that

$$\mathcal{G} : \mathcal{A}^k \to Fil_k \left(\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i \right)$$

is an equivalence of categories for any $k \in K$. Recall that

$$Fil_k \left(\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i \right) \cong \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i^k$$

By Condition 6.4.0.26, for any $i > i_k$ we have a commutative diagram where all arrows are equivalences:

$$\begin{array}{ccc} \mathcal{A}^k & \xrightarrow{\mathcal{G}_i} & \mathcal{C}_i^k \\ {\scriptstyle \mathcal{G}_{i_k}} \downarrow & \swarrow {\scriptstyle \mathcal{F}_{i-1,i}} & \\ \mathcal{C}_{i-1}^k & & \end{array}$$

By Lemma 6.2.1.3, we then have: $\mathbf{Pr}_i : \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i^k \longrightarrow \mathcal{C}_i^k$ is an equivalence of categories for any $i > i_k$, and thus $\mathcal{G} : \mathcal{A}^k \to Fil_k \left(\varprojlim_{i \in \mathbb{Z}_+, K-filtr} \mathcal{C}_i \right)$ is an equivalence of categories. □

6.5 Restricted inverse limit and inverse limit of categories with a K-filtration

Let $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ be a system of finite-length categories with K-filtrations and shortening K-filtered functors, whose the filtration components are Serre subcategories.

We would like to give a sufficient condition on the K-filtration for the inverse limit of K-filtered categories to coincide with the restricted inverse limit of these categories.

Recall that since the functors $\mathcal{F}_{i-1,i}$ are shortening, we have maps

$$f_{i-1,i} : Irr_*(\mathcal{C}_i) \longrightarrow Irr_*(\mathcal{C}_{i-1})$$

and we can consider the inverse limit $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$ of the sequence of sets $Irr_*(\mathcal{C}_i)$ and maps $f_{i-1,i}$; we will denote by $pr_j : \varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i) \to Irr_*(\mathcal{C}_j)$ the projection maps.

Notice that the sets $Irr_*(\mathcal{C}_i)$ have a natural K-filtration, and the maps $f_{i-1,i}$ are K-filtered maps.

Proposition 6.5.0.28. *Assume the following conditions hold:*

1. *There exists a K-filtration on the set $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$. That is, we require:*

   For each L in $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$, there exists $k \in K$ so that $pr_i(L) \in Fil_k(Irr_*(\mathcal{C}_i))$ for any $i \geq 0$.

   We would then say that such an object L belongs in the k-th filtration component of $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$.

2. *"Stabilization condition": for any $k \in K$, there exists $N_k \geq 0$ such that the map $f_{i-1,i} : Fil_k(Irr_*(\mathcal{C}_i)) \to Fil_k(Irr_*(\mathcal{C}_{i-1}))$ be an injection for any $i \geq N_k$.*

   That is, for any $k \in K$ there exists $N_k \in \mathbb{Z}_+$ such that the (exact) functor $\mathcal{F}_{i-1,i}$ is faithful for any $i \geq N_k$.

Then the two full subcategories $\varprojlim_{i\in\mathbb{Z}_+,\ restr} \mathcal{C}_i$, $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$ of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ coincide.

Proof. Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$. As before, we denote by $JH(C_j)$ the multiset of Jordan-Holder components of C_j, and let

$JH_*(C_j) := JH(C_j) \sqcup \{0\}$.

The first condition is natural: giving a K-filtration on the objects of $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$ is equivalent to giving a K-filtration on the simple objects of $\varprojlim_{i\in\mathbb{Z}_+,\ \mathrm{restr}} \mathcal{C}_i$, i.e. on the set $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$.

Assume $C \in \varprojlim_{i\in\mathbb{Z}_+,\text{ restr}} \mathcal{C}_i$. Let $n_0 \geq 0$ be such that $\ell_{\mathcal{C}_j}(C_j)$ is constant for $j \geq n_0$. Recall that we have (Corollary 6.3.0.13):

$$JH_*(C) = \varprojlim_{i\in\mathbb{Z}_+} JH_*(C_j)$$

Choose k such that all the elements of $JH_*(C)$ lie in the k-th filtration component of $\varprojlim_{i\in\mathbb{Z}_+} Irr_*(\mathcal{C}_i)$. This is possible due to the first condition.

Then for any $L_j \in JH(C_j)$, we have: $L_j = pr_j(L)$ for some $L \in JH_*(C)$, and thus $L_j \in Fil_k(Irr_*(\mathcal{C}_j))$. We conclude that $C \in Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$.

Thus we proved that the first condition of the Theorem holds iff $\varprojlim_{i\in\mathbb{Z}_+,\text{ restr}} \mathcal{C}_i$ is a full subcategory of $\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$.

Now, let $C \in \varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i$, and let $k \in K$ be such that $C \in Fil_k(\varprojlim_{i\in\mathbb{Z}_+, K-filtr} \mathcal{C}_i)$. We would like to show that $\ell_{\mathcal{C}_i}(C_i)$ is constant starting from some i.

Indeed, the second condition of the Theorem tells us that there exists $N_k \geq 0$ such that the map

$$f_{i-1,i} : Fil_k(Irr_*(\mathcal{C}_i)) \to Fil_k(Irr_*(\mathcal{C}_{i-1}))$$

is an injection for any $i \geq N_k$.

We claim that for $i \geq N_k$, $\ell_{\mathcal{C}_i}(C_i)$ is constant. Indeed, if it weren't, then there would be some $i \geq N_k+1$ and some $L_i \in JH(C_i)$ such that $f_{i-1,i}(L_i) = 0$. But this is impossible, due to the requirement above.

□

6.6 $\mathfrak{gl}_\infty$ and the restricted inverse limit of representations of $\mathfrak{gl}_n$

In this section, we give a nice example of a restricted inverse limit of categories; namely, we will show that the category of polynomial representations of the Lie algebra $\mathfrak{gl}_\infty$ is a restricted inverse limit of the categories of polynomial representations of $\mathfrak{gl}_n$ for $n \geq 0$.

The representations of the Lie algebra $\mathfrak{gl}_\infty$ (or the group GL_∞) are discussed in detail in [PS], [DPS], as well as [SS, Section 3].

6.6.1 The Lie algebra $\mathfrak{gl}_\infty$

Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $\{e_1, e_2, e_3, ...\}$.

Consider the Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices $A = (a_{ij})_{i,j\geq 1}$ with finitely many non-zero entries. We have a natural action of $\mathfrak{gl}_\infty$ on $\mathbb{C}^\infty$, with $\mathfrak{gl}_\infty \cong \mathbb{C}^\infty \otimes \mathbb{C}^\infty_*$. Here $\mathbb{C}^\infty_* = span_{\mathbb{C}}(e_1^*, e_2^*, e_3^*, ...)$, where e_i^* is the linear functional dual to e_i: $e_i^*(e_j) = \delta_{ij}$.

We now insert more notation. Let $N \in \mathbb{Z}_+ \cup \{\infty\}$, and let $m \geq 1$. We will consider the Lie subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_N$ consisting of matrices $A = (a_{ij})_{1\leq i,j \leq N}$ for which $a_{ij} = 0$ whenever $i > m$ or $j > m$. We will also denote by $\mathfrak{gl}_m^\perp$ the Lie subalgebra of $\mathfrak{gl}_N$ consisting of matrices $A = (a_{ij})_{1\leq i,j\leq N}$ for which $a_{ij} = 0$ whenever $i \leq m$ or $j \leq m$.

Remark 6.6.1.1. Note that $\mathfrak{gl}_n^\perp \cong \mathfrak{gl}_{N-m}$ for any N, m.

6.6.2 Categories of polynomial representations

In this subsection, $N \in \mathbb{Z}_+ \cup \{\infty\}$.

We will consider the symmetric monoidal category $Rep(\mathfrak{gl}_N)_{poly}$ of polynomial representations of $\mathfrak{gl}_N$.

As a tensor category, it is generated by the tautological representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$. Namely, this is the category of $\mathfrak{gl}_N$-modules which are direct summands in finite direct sums of tensor powers of $\mathbb{C}^N$, and $\mathfrak{gl}_N$-equivariant morphisms between them.

This category is discussed in detail in [SS, Section 2.2].

It is easy to see that this is a semisimple abelian category, whose simple objects are parametrized (up to isomorphism) by all Young diagrams of arbitrary sizes: the simple object corresponding to λ is $L_\lambda^N = S^\lambda \mathbb{C}^N$.

Remark 6.6.2.1. Note that $Rep(\mathfrak{gl}_\infty)_{poly}$ is the free abelian symmetric monoidal category generated by one object (c.f. [SS, (2.2.11)]). It has a equivalent definition as the category of polynomial functors of bounded degree, which can be found in [HY], [Macd, Chapter I], [SS].

Remark 6.6.2.2. For $N \in \mathbb{Z}_+$, one can describe these representations as finite-dimensional representations $\rho : GL_N \to \mathrm{Aut}(W)$ which can be extended to an algebraic map $\mathrm{End}(GL_N) \to \mathrm{End}(W)$.

6.6.3 Specialization functors

We now define specialization functors from the category of representations of $\mathfrak{gl}_\infty$ to the categories of representations of $\mathfrak{gl}_n$ (c.f. [SS, Section 3]):

Definition 6.6.3.1.

$$\Gamma_n : Rep(\mathfrak{gl}_\infty)_{poly} \to Rep(\mathfrak{gl}_n)_{poly},\ \Gamma_n := (\cdot)^{\mathfrak{gl}_n^\perp}$$

Lemma 6.6.3.2. *The functor* Γ_n *is well-defined.*

Proof. First of all, notice that the subalgebras $\mathfrak{gl}_n, \mathfrak{gl}_n^\perp \subset \mathfrak{gl}_\infty$ commute, and therefore the subspace of $\mathfrak{gl}_n^\perp$-invariants of a $\mathfrak{gl}_\infty$-module automatically carries an action of $\mathfrak{gl}_n$.

We need to check that given a polynomial $\mathfrak{gl}_\infty$-representation M of $\mathfrak{gl}_n$, the $\mathfrak{gl}_n^\perp$-invariants of M form a polynomial respresentation of $\mathfrak{gl}_n$. It is enough to check that this is true when $M = (\mathbb{C}^\infty)^{\otimes r}$.

The latter statement is checked explicitly on basis elements of the form $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_r}$. The subspace of $\mathfrak{gl}_n^\perp$-invariants is spanned by the basis elements $e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_r}$ for which $i_1, ..., i_r \leq n$. Thus the $\mathfrak{gl}_n^\perp$-invariants of $(\mathbb{C}^\infty)^{\otimes r}$ form the $\mathfrak{gl}_n$-representation $(\mathbb{C}^n)^{\otimes r}$.

□

In particular, one proves in the same way that the $\mathfrak{gl}_n^\perp$-invariants of $(\mathbb{C}^\infty)^{\otimes r}$ form the $\mathfrak{gl}_n$-representation $(\mathbb{C}^n)^{\otimes r}$.

The following Lemmas are proved in [PS], [SS, Section 3]:

Lemma 6.6.3.3. *The functors Γ_n are symmetric monoidal functors.*

The functors $\Gamma_n : Rep(\mathfrak{gl}_\infty)_{poly} \to Rep(\mathfrak{gl}_n)_{poly}$ are additive functors between semisimple categories, and their effect on simple objects is given by the following Lemma (a direct consequence of Lemma 6.6.3.3):

Lemma 6.6.3.4. *For any Young diagram λ, $\Gamma_n(S^\lambda \mathbb{C}^\infty) \cong S^\lambda \mathbb{C}^n$.*

6.6.4 Restriction functors

Definition 6.6.4.1. Let $n \geq 1$. We define the functor

$$\mathfrak{Res}_{n-1,n} : Rep(\mathfrak{gl}_n)_{poly} \to Rep(\mathfrak{gl}_{n-1})_{poly}, \; \mathfrak{Res}_{n-1,n} := (\cdot)^{\mathfrak{gl}_{n-1}^\perp}$$

The proof that this functor is well-defined is exactly the same as that of Lemma 6.6.3.2.

Remark 6.6.4.2. Here is an alternative definition of the functors $\mathfrak{Res}_{n-1,n}$.

We say that a $\mathfrak{gl}_n$-module M is of *degree d* if $\mathrm{Id}_{\mathbb{C}^n} \in \mathfrak{gl}_n$ acts by $d\,\mathrm{Id}_M$ on M. Also, given any $\mathfrak{gl}_n$-module M, we may consider the maximal submodule of M of degree d, and denote it by $deg_d(M)$. This defines an endo-functor deg_d of $Rep(\mathfrak{gl}_n)_{poly}$.

Note that a simple module $S^\lambda \mathbb{C}^n$ is of degree $|\lambda|$.

The notion of degree gives a decomposition

$$Rep(\mathfrak{gl}_n)_{poly} \cong \bigoplus_{d \in \mathbb{Z}_+} Rep(\mathfrak{gl}_n)_{poly,d}$$

where $Rep(\mathfrak{gl}_n)_{poly,d}$ is the full subcategory of $Rep(\mathfrak{gl}_n)_{poly}$ consisting of all polynomial $\mathfrak{gl}_n$-modules of degree d.

Then

$$\mathfrak{Res}_{n-1,n} = \oplus_{d \in \mathbb{Z}_+} \mathfrak{Res}_{d,n-1,n} : Rep(\mathfrak{gl}_n)_{poly} \to Rep(\mathfrak{gl}_{n-1})_{poly}$$

where

$$\mathfrak{Res}_{d,n-1,n} : Rep(\mathfrak{gl}_n)_{poly,d} \to Rep(\mathfrak{gl}_{n-1})_{poly,d},\ \mathfrak{Res}_{d,n-1,n} := deg_d \circ \mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$$

where $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ is the usual restriction functor for the pair $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$.

Again, $\mathfrak{Res}_{n-1,n}$ are additive functors between semisimple categories, so we are interested in checking the effect of these functors on simple modules:

Lemma 6.6.4.3. $\mathfrak{Res}_{n-1,n}(S^\lambda \mathbb{C}^n) \cong S^\lambda \mathbb{C}^{n-1}$ *for any Young diagram* λ.

Proof. This is a simple corollary of the branching rues for $\mathfrak{gl}_n, \mathfrak{gl}_{n-1}$. □

Next, we notice that these functors are compatible with the functors Γ_n defined before.

Lemma 6.6.4.4. *For any* $n \geq 1$*, we have a commutative diagram:*

$$\begin{array}{ccc} Rep(\mathfrak{gl}_\infty)_{poly} & \xrightarrow{\Gamma_n} & Rep(\mathfrak{gl}_n)_{poly} \\ & \searrow^{\Gamma_{n-1}} & \downarrow \mathfrak{Res}_{n-1,n} \\ & & Rep(\mathfrak{gl}_{n-1})_{poly} \end{array}$$

That is, there is a natural isomorphism $\Gamma_{n-1} \cong \mathfrak{Res}_{n-1,n} \circ \Gamma_n$.

Proof. By definition of the functors $\Gamma_{n-1}, \mathfrak{Res}_{n-1,n}, \Gamma_n$, we have a natural transformation $\theta : \Gamma_{n-1} \to \mathfrak{Res}_{n-1,n} \circ \Gamma_n$ which is given by the injection $\theta_M : \Gamma_{n-1}(M) \hookrightarrow (\mathfrak{Res}_{n-1,n} \circ \Gamma_n)(M)$ for any $M \in Rep(\mathfrak{gl}_\infty)_{poly}$. We would like to say that θ_M are isomorphisms.

The categories in question are semisimple, so it is enough to check what happens to the simple objects. Lemmas 6.6.3.4 and 6.6.4.3 then tell us that $\theta_{S^\lambda \mathbb{C}^\infty}$ is an isomorphism for any Young diagram λ, and we are done. □

Lemma 6.6.4.5. *The functors* $\mathfrak{Res}_{n-1,n} : Rep(\mathfrak{gl}_n)_{poly} \to Rep(\mathfrak{gl}_{n-1})_{poly}$ *are symmetric monoidal functors.*

Proof. The functor Γ_n is full and essentially surjective, as well as a tensor functor. The natural isomorphism from Lemma 6.6.4.4 then provides a monoidal structure on the functor $\mathfrak{Res}_{n-1,n}$, and we can immediately see that it is symmetric as well. $\square$

6.6.5 The restricted inverse limit of categories $Rep(\mathfrak{gl}_n)_{poly}$

This subsection describes the category $Rep(\mathfrak{gl}_\infty)_{poly}$ as a "stable" inverse limit of categories $Rep(\mathfrak{gl}_n)_{poly}$.

We now define a $\mathbb{Z}_+$-filtration on $Rep(\mathfrak{gl}_n)_{poly}$ for each $n \in \mathbb{Z}_+$.

Notation 6.6.5.1. For each $k \in \mathbb{Z}_+$, let $Rep(\mathfrak{gl}_n)_{poly,\text{ length }\leq k}$ be the full additive subcategory of $Rep(\mathfrak{gl}_n)_{poly}$ generated by $S^\lambda \mathbb{C}^n$ such that $\ell(\lambda) \leq k$.

Clearly the subcategories $Rep(\mathfrak{gl}_n)_{poly,\text{ length }\leq k}$ give us a $\mathbb{Z}_+$-filtration of the category $Rep(\mathfrak{gl}_n)_{poly}$, and by Lemma 6.6.4.3, the functors $\mathfrak{Res}_{n-1,n}$ are $\mathbb{Z}_+$-filtered functors (see Section 6.4).

This allows us to consider the inverse limit

$$\varprojlim_{n\in\mathbb{Z}_+, \mathbb{Z}_+-filtr} Rep(\mathfrak{gl}_n)_{poly}$$

of $\mathbb{Z}_+$-filtered categories $Rep(\mathfrak{gl}_n)_{poly}$. This inverse limit is an abelian category with a $\mathbb{Z}_+$-filtration (by Lemma 6.4.0.24).

Note that by Lemma 6.6.4.3, the functors $\mathfrak{Res}_{n-1,n}$ are shortening functors (see Definition 6.3.0.7); futhermore, the system $((Rep(\mathfrak{gl}_n)_{poly})_{n\in\mathbb{Z}_+}, (\mathfrak{Res}_{n-1,n})_{n\geq 1})$ satisfies the conditions in Proposition 6.5.0.28, and therefore the inverse limit of this $\mathbb{Z}_+$-filtered system is also its restricted inverse limit (see Section 6.3).

Of course, since the functors $\mathfrak{Res}_{n-1,n}$ are symmetric monoidal functors, the above restricted inverse limit is a symmetric monoidal category.

Proposition 6.6.5.2. *We have an equivalence of symmetric monoidal abelian categories*

$$\Gamma_{lim} : Rep(\mathfrak{gl}_\infty)_{poly} \longrightarrow \varprojlim_{n\in\mathbb{Z}_+,\ restr} Rep(\mathfrak{gl}_n)_{poly}$$

induced by the symmetric monoidal functors

$$\Gamma_n = (\cdot)^{\mathfrak{gl}_n^{\perp}} : Rep(\mathfrak{gl}_\infty)_{poly} \longrightarrow Rep(\mathfrak{gl}_n)_{poly}$$

Proof. Define a $\mathbb{Z}_+$-filtration on the semisimple category $Rep(\mathfrak{gl}_\infty)_{poly}$ by requiring the simple object $S^\lambda\mathbb{C}^\infty$ to lie in filtra $\ell(\lambda)$. Lemma 6.6.3.4 then tells us that for any $k \in \mathbb{Z}_+$ and any $n \geq k$, the functor

$$\Gamma_n : Fil_k(Rep(\mathfrak{gl}_\infty)_{poly}) \longrightarrow Fil_k(Rep(\mathfrak{gl}_n)_{poly}) := Rep(\mathfrak{gl}_n)_{poly,\text{ length } \leq k}$$

is an equivalence. Proposition 6.4.0.27 completes the proof. □

Remark 6.6.5.3. The same result has been proved in [HY]; the approach used there is equivalent to that of inverse limits of $\mathbb{Z}_+$-filtered categories - namely, the authors give a $\mathbb{Z}_+$-grading on the objects of each category $Rep(\mathfrak{gl}_n)_{poly}$, with $S^\lambda\mathbb{C}^n$ lying in grade $|\lambda|$. The "stable" inverse limit of these graded categories, as defined in [HY], is just the inverse limit of the $\mathbb{Z}_+$-filtered categories $Rep(\mathfrak{gl}_n)_{poly}$ with the appropriate filtrations. Note that by Proposition 6.5.0.28, this construction is equivalent to our construction of a $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$.

In this case, this is also equivalent to taking the compact subobjects inside $\varprojlim_{n\in\mathbb{Z}_+} Rep(\mathfrak{gl}_n)_{poly}$.

Remark 6.6.5.4. The adjoint (on both sides) to functor $\Gamma_{\lim}$ is the functor

$$\Gamma^*_{\lim} : \varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly} \longrightarrow Rep(\mathfrak{gl}_\infty)_{poly}$$

defined below.

For any object $((M_n)_{n\geq 0}, (\phi_{n-1,n})_{n\geq 1})$ of $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$, the $\mathfrak{gl}_{n-1}$-module M_{n-1} is isomorphic (via $\phi_{n-1,n}$) to a $\mathfrak{gl}_{n-1}$-submodule of M_n.

This allows us to cosider a vector space M which is the direct limit of the vector spaces M_n and the inclusions $\phi_{n-1,n}$. On this vector space M we have a natural action of $\mathfrak{gl}_\infty$:

given $A \in \mathfrak{gl}_n \subset \mathfrak{gl}_\infty$ and $m \in M$, we have $m \in M_N$ for $N >> 0$. In particular, we can choose $N \geq n$, and then A acts on m through its action on M_N.

We can easily check that the $\mathfrak{gl}_\infty$-module M is polynomial: indeed, due to the equivalence in Proposition 6.6.5.2, there exists a polynomial $\mathfrak{gl}_\infty$-module M' such that $M_n \cong \Gamma_n(M')$ for every n, and $\phi_{n-1,n}$ are induced by the inclusions $\Gamma_{n-1}(M') \subset \Gamma_n(M')$. By definition of M, we have a $\mathfrak{gl}_\infty$- equivariant map $M \to M'$, and it is easy to check that it is an isomorphism.

We put $\Gamma^*_{\lim}((M_n)_{n\geq 0}, (\phi_{n-1,n})_{n\geq 1}) := M$, and require that the functor $\Gamma^*_{\lim}$ act on morphisms accordingly. The above construction then gives us a natural isomorphism

$$\Gamma^*_{\lim} \circ \Gamma_{\lim} \xrightarrow{\sim} \mathrm{Id}_{Rep(\mathfrak{gl}_\infty)_{poly}}.$$

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Chapter 7

Schur-Weyl duality in the limit case

7.1 Parabolic category O in infinite rank

In this section, we give a uniform definition for both the parabolic category O for $\mathfrak{gl}_n$, and for $\mathfrak{gl}_\infty$ which we will use. This will be a slight modification of the original definition to accomodate the case $N = \infty$.

Let $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.

Consider a unital vector space $(\mathbb{C}^N, \mathbb{1})$, where $\mathbb{1} := e_1$. Put $U_N := span_{\mathbb{C}}(e_2, e_3, ...) \subset \mathbb{C}^N$, so that we have a splitting $\mathbb{C}^N = \mathbb{C}e_1 \oplus U_N$. We will also denote $U_{N,*} := span(e_2^*, e_3^*, ...)$ (so $U_{N,*} = U_N^*$ whenever $N \in \mathbb{Z}$).

The following notation will be used in this chapter:

Notation 7.1.0.5.

- We denote by $\mathfrak{p}_N \subset \mathfrak{gl}_N$ the parabolic Lie subalgebra which consists of all the endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\phi(\mathbb{1}) \in \mathbb{C}\mathbb{1}$. In terms of matrices this is $span\{E_{1,1}, E_{i,j} | j > 1\}$.

- $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{p}_N$ denotes the algebra of endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\operatorname{Im} \phi \subset \mathbb{C}\mathbb{1} \subset \operatorname{Ker} \phi$. In terms of matrices, $\mathfrak{u}^+_{\mathfrak{p}_N} = span\{E_{1,j} | j > 1\}$.

We have a decomposition

$$\mathfrak{gl}_N \cong \mathfrak{gl}(U_N) \oplus \mathfrak{gl}_1 \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N}$$

Of course, for any N, $\mathfrak{u}^-_{\mathfrak{p}_N} \cong U_N$; moreover, $\mathfrak{u}^+_{\mathfrak{p}_N} \cong U_{N,*}$.

We will also use the isomorphisms $\mathfrak{gl}(U_N) \cong \mathfrak{gl}_1^{\perp} \cong \mathfrak{gl}_{N-1}$.

Definition 7.1.0.6.

- Define the category $Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$ to be the category of $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ lies in $Ind - Rep(\mathfrak{gl}_{U_N})_{poly}$; that is, $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ is a (perhaps infinite) direct sum of Schur functors applied to U_N.

  The morphisms would be $\mathfrak{gl}_N$-equivariant maps.

- We say that an object $M \in Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$ is of *degree* ν ($\nu \in \mathbb{C}$) if on every summand $S^{\lambda} U_N \subset M$, the element $E_{1,1} \in \mathfrak{gl}_N$ acts by $(\nu - |\lambda|) \operatorname{Id}_{S^{\lambda} U_N}$.

- Let $M \in Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$. We have a commutative algebra $Sym(U_N) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^-_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a structure of a $Sym(U_N)$-module.

  We say that M is *finitely generated* over $Sym(U_N)$ if M is a quotient of a "free finitely-generated $Sym(U_N)$-module"; that is, as a $Sym(U_N)$-module, M is a quotient (in $Ind-Rep(\mathfrak{gl}_N)_{poly}$) of $Sym(U_N) \otimes E$ for some $E \in Rep(\mathfrak{gl}(U_N))_{poly}$.

- Let $M \in Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$. We have a commutative algebra $Sym(U_{N,*}) \cong \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a structure of a $Sym(U_{N,*})$-module.

We say that M is *locally nilpotent* over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ if for any $v \in M$, there exists $m \geq 0$ such that for any $A \in Sym^m(U_{N,*})$ we have: $A.v = 0$.

Recall the natural $\mathbb{Z}_+$-grading on the object of $Ind - Rep(\mathfrak{gl}_N)_{poly}$.

For each $M \in Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$, the above definition implies: $\mathfrak{gl}(U_N)$ acts by operators act by operators of degree zero, $U_{N,*}$ acts by operators of degree 1. We now define the parabolic category O for $\mathfrak{gl}_N$ which we will use in this chapter:

Definition 7.1.0.7. We define the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ to be the full subcategory of $Mod_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)-poly}$ whose objects M satisfy the following requirements:

- M is of degree ν.
- M is finitely generated over $Sym(U_N)$.
- M is locally nilpotent over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$.

Of course, for a positive integer N, this is just the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ we defined in the beginning of this section.

We will also consider the localization of the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by its Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν; such modules exist iff $\nu \in \mathbb{Z}_+$. This localization will be denoted by

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \longrightarrow \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$$

and will play an important role when we consider the Schur-Weyl duality in complex rank.

7.2 Restricted inverse limit of parabolic categories O

7.2.1 Restriction functors

Definition 7.2.1.1. Let $n \geq 1$. Define the functor

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \;\; \mathfrak{Res}_{n-1,n} := (\cdot)^{\mathfrak{gl}^{\perp}_{n-1}}$$

Again, the subalgebras $\mathfrak{gl}_{n-1}, \mathfrak{gl}_{n-1}^{\perp} \subset \mathfrak{gl}_n$ commute, and therefore the subspace of $\mathfrak{gl}_{n-1}^{\perp}$-invariants of a $\mathfrak{gl}_n$-module automatically carries an action of $\mathfrak{gl}_{n-1}$.

We need to check that this functor is well-defined. In order to do so, consider the functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow Mod_{\mathcal{U}(\mathfrak{gl}_{n-1})}$. This functor is well-defined, and we will show that the objects in the image lie in the full subcategory $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ of $Mod_{\mathcal{U}(\mathfrak{gl}_{n-1})}$.

Note that the functor $\mathfrak{Res}_{n-1,n}$ can alternatively be defined as follows: for a module M in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, we restrict the action of $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$, and then only take the vectors in M attached to specific central characters. More specifically, we have:

Lemma 7.2.1.2. *The functor $\mathfrak{Res}_{n-1,n}$ is naturally isomorphic to the composition $deg_\nu \circ$ $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ (the functor deg_ν was defined in Definition 3.3.0.23).*

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. For any vector $m \in M$, we know that $\mathrm{Id}_{\mathbb{C}^n} .m = (E_{1,1} + E_{2,2} + \ldots + E_{n,n}).m = \nu m$. Then the requirement that $\mathrm{Id}_{\mathbb{C}^{n-1}} .m = (E_{1,1} + E_{2,2} + \ldots + E_{n-1,n-1}).m = \nu m$ is equivalent to requiring that $E_{n,n}.m = 0$, namely that $m \in M^{\mathfrak{gl}_{n-1}^{\perp}}$. ☐

We will now use this information to prove the following lemma:

Lemma 7.2.1.3. *The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is well-defined.*

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, and consider the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$. By definition, this is a module of degree ν. We will show that it lies in $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$.

First of all, consider the inclusion $\mathfrak{gl}(U_{n-1})^{\perp} \oplus \mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$. This inclusion gives us the restriction functor (see Definition 6.6.4.1)

$$\mathfrak{Res}_{U_{n-1},U_n} : Rep(\mathfrak{gl}(U_n))_{poly} \longrightarrow Rep(\mathfrak{gl}(U_{n-1}))_{poly}, \quad \mathfrak{Res}_{U_{n-1},U_n} := (\cdot)^{\mathfrak{gl}(U_{n-1})^{\perp}}$$

The latter is an additive functor between semisimple categories, and takes polynomial representations of $\mathfrak{gl}(U_n)$ to polynomial representations of $\mathfrak{gl}(U_{n-1})$.

Now, the restriction to $\mathfrak{gl}(U_{n-1})$ of the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$ is isomorphic to $\mathfrak{Res}_{U_{n-1},U_n}(M|_{\mathfrak{gl}(U_n)})$, and thus is a polynomial representation of $\mathfrak{gl}(U_{n-1})$.

Secondly, $\mathfrak{Res}_{n-1,n}(M)$ is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}})$, since M is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$ and $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}}) \subset \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$.

It remains to check that given $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, the module $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $Sym(U_{n-1})$. Indeed, we know that there exists a polynomial $\mathfrak{gl}(U_n)$-module E and a surjective $\mathfrak{gl}(U_n)$-equivariant morphism of $Sym(U_n)$-modules $Sym(U_n) \otimes E \twoheadrightarrow M$. Taking the $\mathfrak{gl}(U_{n-1})^{\perp}$-invariants and using Lemma 6.6.4.5, we conclude that there is a surjective $\mathfrak{gl}(U_{n-1})$-equivariant morphism of $Sym(U_{n-1})$-modules

$$Sym(U_{n-1}) \otimes E^{\mathfrak{gl}(U_{n-1})^{\perp}} \twoheadrightarrow \mathfrak{Res}_{n-1,n}(M)$$

Thus $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $Sym(U_{n-1})$.

□

Lemma 7.2.1.4. *The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is exact.*

Proof. We use Lemma 7.2.1.2. The functor $deg_\nu : O^{\mathfrak{p}_{n-1}}_{\mathbb{C}^{n-1}} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is exact, so the functor $\mathrm{Res}_{n-1,n}$ is obviously exact as well.

□

Lemma 7.2.1.5. *The functor $\mathfrak{Res}_{n-1,n}$ takes parabolic Verma modules to either parabolic Verma modules or to zero:*

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda)$$

(recall that the latter is a parabolic Verma module for $\mathfrak{gl}_{n-1}$ iff $\ell(\lambda) \leq n-2$, and zero otherwise).

Proof. Consider the parabolic Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$, where the Young diagram λ has length at most $n-1$.

By definition of the parabolic Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$, we have:

$$M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda) = \mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} S^{\lambda} U_n$$

The branching rule for $\mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$ tells us that

$$(S^\lambda U_n)|_{\mathfrak{gl}(U_{n-1})} \cong \bigoplus_{\lambda'} S^{\lambda'} U_{n-1}$$

the sum taken over the set of all Young diagrams obtained from λ by removing several boxes, no two in the same column.

So

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong \left(\bigoplus_{\lambda' \subset \lambda} M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda')\right) \otimes \mathcal{U}\left(\mathfrak{u}^-_{\mathfrak{p}_n} \Big/ \mathfrak{u}^-_{\mathfrak{p}_{n-1}}\right)$$

Here

- $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda')$ is either a parabolic Verma module for $\mathfrak{gl}_{n-1}$ of highest weight $(\nu - |\lambda|, \lambda')$ (note that it is of degree $\nu - |\lambda| + |\lambda'|$) or zero.
- $\mathfrak{gl}(U_{n-1})$ acts trivially on the space $\mathcal{U}\left(\mathfrak{u}^-_{\mathfrak{p}_n} \Big/ \mathfrak{u}^-_{\mathfrak{p}_{n-1}}\right)$. This space is isomorphic, as a $\mathbb{Z}_+$-graded vector space, to $\mathbb{C}[t]$ (t standing for $E_{n,1} \in \mathfrak{gl}_n$) and $E_{1,1}$ acts on it by derivations $-t\frac{d}{dt}$.

Thus $\mathrm{Id}_{\mathbb{C}^{n-1}} \in \mathfrak{gl}_n$ acts on the subspace $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda') \otimes t^k \subset M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ by the scalar $\nu - |\lambda| + |\lambda'| - k$.

We now apply the functor deg_ν to the module $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda))$.

To see which subspaces $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda'|, \lambda') \otimes t^k$ of $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ will survive after applying deg_ν, we require that $|\lambda| - |\lambda'| + k = 0$. But we are only considering Young diagrams λ' such that $\lambda' \subset \lambda$, and non-negative integers k, which means that the only relevant case is $\lambda' = \lambda$, $k = 0$.

We conclude that

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_N}(\nu - |\lambda|, \lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda)$$

□

Lemma 7.2.1.6. *Given a simple $\mathfrak{gl}_n$-module $L_n(\nu - |\lambda|, \lambda)$,*

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$$

(recall that the latter is a simple $\mathfrak{gl}_{n-1}$-module iff $\ell(\lambda) \leq n-2$, and zero otherwise).

Proof. Note that the statement follows immediately from Lemma 7.2.1.5 when λ lies in a trivial $\overset{\nu}{\sim}$-class; for a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$, we have short exact sequences (see Corollary 3.3.1.7):

$$0 \to L_n(\nu - \left|\lambda^{(i+1)}\right|, \lambda^{(i+1)}) \to M_{\mathfrak{p}_n}(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to L_n(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)}) \to 0$$

Using the exactness of $\mathfrak{Res}_{n-1,n}$, we can prove the required statement for $L_n(\nu - \left|\lambda^{(i)}\right|, \lambda^{(i)})$ by induction on i, provided the statement is true for $i = 0$.

So it remains to check that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$$

for the minimal Young diagram λ in any non-trivial $\overset{\nu}{\sim}$-class.

Recall that in that case, $L_n(\nu - |\lambda|, \lambda) = S^{\tilde{\lambda}(\nu)}\mathbb{C}^n$ is a finite-dimensional simple representation of $\mathfrak{gl}_n$.

The branching rule for $\mathfrak{gl}_n, \mathfrak{gl}_{n-1}$ implies that

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(S^{\tilde{\lambda}(\nu)}\mathbb{C}^n) \cong \bigoplus_{\mu} S^{\mu}\mathbb{C}^{n-1}$$

the sum taken over the set of all Young diagrams obtained from $\tilde{\lambda}(\nu)$ by removing several boxes, no two in the same column.

Considering only the summands of degree ν, we see that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong S^{\tilde{\lambda}(\nu)}\mathbb{C}^{n-1} \cong L_{n-1}(\nu - |\lambda|, \lambda)$$

□

The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ clearly takes polynomial modules to polynomial modules; together with Lemma 7.2.1.4, this means that $\mathfrak{Res}_{n-1,n}$ factors through an exact functor

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

i.e. we have a commutative diagram

$$\begin{array}{ccc} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} & \xrightarrow{\mathfrak{Res}_{n-1,n}} & O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \\ {\scriptstyle \hat{\pi}_n}\downarrow & & \downarrow{\scriptstyle \hat{\pi}_{n-1}} \\ \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} & \xrightarrow{\widehat{\mathfrak{Res}}_{n-1,n}} & \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \end{array}$$

(see Section 5 for the definition of the localizations $\hat{\pi}_n$).

7.2.2 Specialization functors

Definition 7.2.2.1. Let $n \geq 1$. Define the functor

$$\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \ \ \Gamma_n := (\cdot)^{\mathfrak{gl}_n^\perp}$$

As before, the subalgebras $\mathfrak{gl}_n, \mathfrak{gl}_n^\perp \subset \mathfrak{gl}_\infty$ commute, and therefore the subspace of $\mathfrak{gl}_n^\perp$-invariants of a $\mathfrak{gl}_\infty$-module automatically carries an action of $\mathfrak{gl}_n$.

Lemma 7.2.2.2. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is well-defined.*

Proof. The proof is essentially the same as in Lemma 7.2.1.3.

□

Next, we check that the functor Γ_n is exact:

Lemma 7.2.2.3. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is exact.*

Proof. The definition of Γ_n immediately implies that this functor is left-exact. Consider the inclusion $\mathfrak{gl}(U_n) \oplus \mathfrak{gl}(U_n)^\perp \subset \mathfrak{gl}(U_\infty)$. We then have an isomorphism of $\mathfrak{gl}(U_n)$-modules

$$\left(M|_{\mathfrak{gl}(U_\infty)}\right)^{\mathfrak{gl}(U_n)^\perp} \cong \left(M^{\mathfrak{gl}_n^\perp}\right)|_{\mathfrak{gl}(U_n)}$$

The exactness of Γ_n then follows from the additivity of the functor $(\cdot)^{\mathfrak{gl}(U_n)^\perp} : Rep(\mathfrak{gl}(U_\infty))_{poly} \to Rep(\mathfrak{gl}(U_n))_{poly}$, which is an additive functor between semisimple categories. □

The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ clearly takes polynomial $\mathfrak{gl}_\infty$-modules to polynomial $\mathfrak{gl}_n$-modules; together with Lemma 7.2.2.3, this means that Γ_n factors through an exact functor

$$\widehat{\Gamma}_n : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

i.e. we have a commutative diagram

$$\begin{array}{ccc} O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} & \xrightarrow{\Gamma_n} & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ {\scriptstyle \hat{\pi}_\infty}\downarrow & & {\scriptstyle \hat{\pi}_n}\downarrow \\ \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} & \xrightarrow{\widehat{\Gamma}_n} & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \end{array}$$

7.2.3 Restricted inverse limit of categories $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ and the category $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$

The restriction functors

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \quad \mathfrak{Res}_{n-1,n} := (\cdot)^{\mathfrak{gl}_{n-1}^\perp}$$

descibed in Subsection 7.2.1 allow us to consider the inverse limit of the system $((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2})$.

Similarly, we can consider the inverse limit of the system $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2})$.

Let $n \geq 1$.

Notation 7.2.3.1. For each $k \in \mathbb{Z}_+$, let $Fil_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ (respectively, $Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$) be the Serre subcategory of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (respectively, $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$) generated by simple modules $L_n(\nu - |\lambda|, \lambda)$ (respectively, $\hat{\pi}_n(L_n(\nu - |\lambda|, \lambda))$), with $\ell(\lambda) \leq k$.

This defines $\mathbb{Z}_+$-filtrations on the objects of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, i.e.

$$O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k \in \mathbb{Z}_+} Fil_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}), \quad \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k \in \mathbb{Z}_+} Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

Lemma 7.2.3.2. *Let $n \geq 1$. The functors*

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

and

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

are both shortening and $\mathbb{Z}_+$-filtered functors between finite-length categories with $\mathbb{Z}_+$-filtrations on objects (see Chapter 6 for the relevant definitions).

Proof. These statements follow directly from Lemma 7.2.1.6, which tells us that $\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$. □

We can now consider the inverse limits of the $\mathbb{Z}_+$-filtered systems $((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2})$ and $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2})$. By Section 6.5, these limits are equivalent to the respective restricted invverse limits

$$\varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \quad \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

The functors Γ_n described above induce exact functors

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow \varprojlim_{n\geq 1} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{\Gamma}_{\lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow \varprojlim_{n\geq 1} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

We would like to show that this functor is an equivalence of categories:

Proposition 7.2.3.3. *The functors Γ_n induce an equivalence*

$$\Gamma_{lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow \varprojlim_{n\geq 1,\ restr} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

Proof. First of all, we need to check that this functor is well-defined. Namely, we need to show that for any $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$, the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ stabilizes. In fact, it is enough to show that this sequence is bounded (since it is obviously increasing).

Recall that we have a surjective map of $Sym(\mathfrak{u}^-_{\mathfrak{p}_\infty})$-modules $Sym(\mathfrak{u}^-_{\mathfrak{p}_\infty}) \otimes E \twoheadrightarrow M$ for some $E \in Rep(\mathfrak{gl}(U_\infty))_{poly}$. Since Γ_{n+1} is exact, it gives us a surjective map $Sym(\mathfrak{u}^-_{\mathfrak{p}_{n+1}}) \otimes \Gamma_{n+1}(E) \twoheadrightarrow \Gamma_{n+1}(M)$ for any $n \geq 0$, with $\Gamma_{n+1}(E)$ being a polynomial $\mathfrak{gl}(U_{n+1})$-module.

Now,

$$\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_{n+1}})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E))$$

The sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E))\}_{n\geq 0}$ is bounded by Proposition 6.6.5.2, and thus the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ is bounded as well.

We now show that $\Gamma_{\lim}$ is an equivalence.

A construction similar to the one appearing in Subsection 6.6.5 gives a left-adjoint to the functor $\Gamma_{\lim}$; that is, we will define a functor

$$\Gamma^*_{\lim} : \varprojlim_{n\geq 1,\ \mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \longrightarrow O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$$

Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\ \mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

The isomorphisms $\phi_{n-1,n} : \mathfrak{Res}_{n-1,n}(M_n) \xrightarrow{\sim} M_{n-1}$ define $\mathfrak{gl}_{n-1}$-equivariant inclusions

$M_{n-1} \hookrightarrow M_n$. Consider the vector space

$$M := \bigcup_{n \geq 1} M_n$$

which has a natural action of $\mathfrak{gl}_\infty$ on it.

It is easy to see that the obtained $\mathfrak{gl}_\infty$-module M is a direct sum of polynomial $\mathfrak{gl}(U_\infty)$-modules, and is locally nilpotent over the algebra

$$\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_\infty}) \cong Sym(U_{\infty,*}) \cong \bigcup_{n \geq 1} Sym(U_n^*)$$

We now prove the following sublemma:

Sublemma 7.2.3.4. *Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\ restr} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. Then $M := \bigcup_{n\geq 1} M_n$ is a finitely generated module over $Sym(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.*

Proof. Recall from Section 6.3 that all the objects in the abelian category $\varprojlim_{n\geq 1,\ \text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ have finite length, and that the simple objects in this category are exactly those of the form $((L_n(\nu - |\lambda|, \lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ for a fixed Young diagram λ. So we only need to check that applying the above construction to these simple objects gives rise to finitely generated modules over $Sym(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

Using Corollary 3.3.1.8 we now reduce the proof of the sublemma to proving the following two statements:

- Let λ be a fixed Young diagram and let $((L_n(\nu - |\lambda|, \lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be a simple object in $\varprojlim_{n\geq 1,\ \text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ in which $L_n(\nu - |\lambda|, \lambda)$ is polynomial for every n (i.e. λ is minimal in its non-trivial $\overset{\nu}{\sim}$-class).

  Then $L := \bigcup_{n\geq 1} L_n(\nu - |\lambda|, \lambda)$ is a polynomial $\mathfrak{gl}_\infty$-module (in particular, a finitely generated module over $Sym(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$).

- Let λ be a fixed Young diagram and let $((M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\ \text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (this is a sequence of "compatible" parabolic Verma modules). Then

$$M := \bigcup_n M_{\mathfrak{p}_n}(\nu - |\lambda|\,, \lambda)$$

is a finitely generated module over $Sym(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

The first statement follows immediately from Proposition 6.6.5.2 (cf. Subsection 6.6.5).

To prove the second statement, recall that

$$M_{\mathfrak{p}_n}(\nu - |\lambda|\,, \lambda) \cong Sym(U_n) \otimes S^\lambda U_n$$

(Lemma 3.3.1.3). So

$$M := \bigcup_n M_{\mathfrak{p}_n}(\nu - |\lambda|\,, \lambda) \cong \bigcup_n Sym(U_n) \otimes S^\lambda U_n \cong Sym(U_\infty) \otimes S^\lambda U_\infty$$

which is clearly a finitely generated module over $Sym(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

□

This allows us to define the functor $\Gamma^*_{\lim}$ by setting

$$\Gamma^*_{\lim}(\,(M_n)_{n\geq 1},\,(\phi_{n-1,n})_{n\geq 2}\,) := \bigcup_{n\geq 1} M_n$$

and requiring that it act on morphisms accordingly.

The definition of $\Gamma^*_{\lim}$ gives us a natural transformation

$$\Gamma^*_{\lim} \circ \Gamma_{\lim} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}}$$

Restricting the action of $\mathfrak{gl}_\infty$ to $\mathfrak{gl}(U_\infty)$ and using Proposition 6.6.5.2, we conclude that this natural transformation is an isomorphism.

Notice that the definition of $\Gamma^*_{\lim}$ implies that this functor is faithful. Thus we conclude that the functor $\Gamma^*_{\lim}$ is an equivalence of categories, and so is $\Gamma_{\lim}$.

□

Proposition 7.2.3.5. *The functors $\widehat{\Gamma}_n$ induce an equivalence*

$$\widehat{\Gamma}_{lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\ restr} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

Proof. Let $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. First of all, we need to check that the functor $\widehat{\Gamma}_{\lim}$ is well-defined; that is, we need to show that the sequence $\{\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M)))\}_{n\geq 1}$ is bounded from above.

Indeed,

$$\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M))) \leq \ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M))$$

But the sequence $\{\ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M))\}_{n\geq 1}$ is bounded from above by Lemma 7.2.3.3, so the original sequence is bound from above as well.

Thus we obtain a commutative diagram

$$\begin{array}{ccccc}
Rep(\mathfrak{gl}_\infty)_{poly,\nu} & \longrightarrow & O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} & \xrightarrow{\hat{\pi}_\infty} & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \\
\downarrow{\scriptstyle \Gamma_{\lim}} & & \downarrow{\scriptstyle \Gamma_{\lim}} & & \downarrow{\scriptstyle \widehat{\Gamma}_{\lim}} \\
\varprojlim_{n\geq 1,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly,\nu} & \longrightarrow & \varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} & \xrightarrow{\hat{\pi}_{\lim}=\varprojlim_n \hat{\pi}_n} & \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}
\end{array}$$

where $Rep(\mathfrak{gl}_N)_{poly,\nu}$ is the Serre subcategory of $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ consisting of all polynomial modules of degree ν. The rows of this commutative diagram are "exact" (in the sense that $\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ is the Serre quotient of the category $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ by the Serre subcategory $Rep(\mathfrak{gl}_\infty)_{poly,\nu}$, and similarly for the bottom row).

The functors

$$\Gamma_{\lim} : Rep(\mathfrak{gl}_\infty)_{poly,\nu} \longrightarrow \varprojlim_{n\geq 1,\text{ restr}} Rep(\mathfrak{gl}_n)_{poly,\nu}$$

and

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \longrightarrow \varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

are equivalences of categories (by Propositions 6.6.5.2 and 7.2.3.3), and thus the functor $\widehat{\Gamma}_{\lim}$ is an equivalence as well.

□

7.3 Complex tensor powers of a unital vector space

Fix $N \in \mathbb{Z}_+ \cup \{\infty\}$. In this section we give a uniform construction of a complex tensor power of the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$. This definition coincides with the Definition 4.2.0.16 whenever $N < \infty$.

Again, we denote $U_N := span\{e_2, e_3, ...\}$, and $U_{N*} := span\{e_2^*, e_3^*, ...\} \subset \mathbb{C}_*^N$. As before, we have a decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}\, \mathrm{Id}_{\mathbb{C}^N} \oplus \mathfrak{u}_{\mathfrak{p}_N}^- \oplus \mathfrak{u}_{\mathfrak{p}_N}^+ \oplus \mathfrak{gl}(U_N)$$

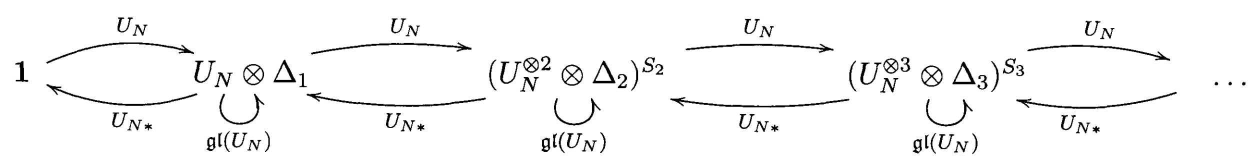
such that $U_N \cong \mathfrak{u}_{\mathfrak{p}_N}^-$, $U_{N*} \cong \mathfrak{u}_{\mathfrak{p}_N}^+$, and if N is finite, we have $U_N^* \cong U_{N*}$.

Fix $\nu \in \mathbb{C}$.

Definition 7.3.0.6 (Complex tensor power). Define the object $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ of $Ind - (\underline{Rep}^{ab}(S_\nu) \boxtimes O_{\nu,\mathbb{C}^N}^{\mathfrak{p}_N})$ by setting

$$(\mathbb{C}^N)^{\underline{\otimes}\nu} := \bigoplus_{k \geq 0} (U_N^{\otimes k} \otimes \Delta_k)^{S_k}$$

The action on $\mathfrak{gl}_N$ on $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ is given as follows:



- $E_{1,1} \in \mathfrak{gl}_N$ acts by scalar $\nu - k$ on each summand $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$.

- $A \in \mathfrak{gl}(U_N) \subset \mathfrak{gl}_N$ acts on $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$ by

$$\sum_{1\leq i\leq k} A^{(i)}|_{U_N^{\otimes k}} \otimes \mathrm{Id}_{\Delta_k} : (U_N^{\otimes k} \otimes \Delta_k)^{S_k} \longrightarrow (U_N^{\otimes k} \otimes \Delta_k)^{S_k}$$

- $u \in U_N \cong \mathfrak{u}^-_{\mathfrak{p}_N}$ acts by morphisms of degree 1, which are given explicitly in Section 4.2.
- $f \in U_{N*} \cong \mathfrak{u}^+_{\mathfrak{p}_N}$ acts by morphisms of degree -1, which are given explicitly in Section 4.2.

Remark 7.3.0.7. The proof that the object $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ lies in the category $Ind - (\underline{Rep}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ is the same as in Section 4.3. In particular, it means that the action of the mirabolic subalgebra Lie$\bar{\mathfrak{P}}_{\mathbb{1}}$ on the complex tensor power $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ integrates to an action of the mirabolic subgroup $\bar{\mathfrak{P}}_{\mathbb{1}}$, thus making $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ a Harish-Chandra module in $Ind - \underline{Rep}^{ab}(S_\nu)$ for the pair $(\mathfrak{gl}_N, \bar{\mathfrak{P}}_{\mathbb{1}})$.

The definition of the complex tensor power is compatible with the usual notion of a tensor power of a unital vector space (see Section 4.3):

Proposition 7.3.0.8. *Let $d \in \mathbb{Z}_+$. Consider the functor*

$$\hat{\mathcal{S}}_d : Ind - (\underline{Rep}(S_{\nu=d}) \boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N}) \longrightarrow Ind - (Rep(S_d) \boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N})$$

induced by the functor

$$\mathcal{S}_d : \underline{Rep}(S_{\nu=d}) \longrightarrow Rep(S_n)$$

described in Subsection 3.2.1. Then $\hat{\mathcal{S}}_d((\mathbb{C}^N)^{\underline{\otimes}d}) \cong (\mathbb{C}^N)^{\otimes d}$.

The construction of the complex tensor power is also compatible with the functors $\mathfrak{Res}_{n,n+1}$ and Γ_n defined in Definitions 7.2.1.1, 7.2.2.1. These properties can be seen as special cases of the following statement (when $N = n+1$ and $N = \infty$, respectively):

Proposition 7.3.0.9. *Let $n \geq 1$, and let $N \geq n$, $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Recall that we have an inclusion $\mathfrak{gl}_n \oplus \mathfrak{gl}_n^{\perp} \subset \mathfrak{gl}_N$, and consider the functor*

$$(\cdot)^{\mathfrak{gl}_n^{\perp}} : Ind - \left(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\right) \longrightarrow Ind - \left(\underline{Rep}^{ab}(S_\nu) \boxtimes O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\right)$$

induced by the functor $(\cdot)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The functor $(\cdot)^{\mathfrak{gl}_n^{\perp}}$ then takes $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ to $(\mathbb{C}^n)^{\underline{\otimes}\nu}$.

Proof. The functor $(\cdot)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ induces an endofunctor of $Ind{-}\underline{Rep}^{ab}(S_\nu)$. We would like to say that we have an isomorphism of $Ind{-}\underline{Rep}^{ab}(S_\nu)$-objects

$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \overset{?}{\cong} (\mathbb{C}^n)^{\underline{\otimes}\nu}$, and that the action of $\mathfrak{gl}_n \subset \mathfrak{gl}_N$ on $((\mathbb{C}^N)^{\underline{\otimes}\nu})$ corresponds to the action of $\mathfrak{gl}_n$ on $(\mathbb{C}^n)^{\underline{\otimes}\nu}$.

In order to do this, we first consider $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ as an object in $Ind - \underline{Rep}^{ab}(S_\nu)$ with an action of $\mathfrak{gl}(U_N)$:

$$(\mathbb{C}^N)^{\underline{\otimes}\nu} \cong \bigoplus_{k\geq 0}(\Delta_k \otimes U_N^{\otimes k})^{S_k}$$

If we consider only the actions of $\mathfrak{gl}(U_N), \mathfrak{gl}(U_n)$, the functor Γ_n is induced by the additive monoidal functor $(\cdot)^{\mathfrak{gl}(U_n)^{\perp}} : Ind - Rep(\mathfrak{gl}(U_N))_{poly} \to Ind - Rep(\mathfrak{gl}(U_N))_{poly}$. This shows that we have an isomorphism of $Ind{-}\underline{Rep}^{ab}(S_\nu)$-objects

$$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \cong \bigoplus_{k\geq 0}(\Delta_k \otimes U_n^{\otimes k})^{S_k} \cong (\mathbb{C}^n)^{\underline{\otimes}\nu}$$

and the actions of $\mathfrak{gl}(U_n)$ on both sides are compatible. From the definition of the complex tensor power (Definition 7.3.0.6) one immediately sees that the actions of $E_{1,1}$ on both sides are compatible as well. Remark 4.2.0.18 now completes the proof.

□

7.4 Schur-Weyl functor for the Deligne category and the Lie algebra $\mathfrak{gl}_\infty$

The definition of the Schur-Weyl contravariant functor $SW_{\nu,V}$ given in Section 5 can be naturally extended to the case when $(V, \mathbb{1}) = (\mathbb{C}^\infty, e_1)$: Fix $\nu \in \mathbb{C}$, and $N \in \mathbb{Z}_+ \cup \{\infty\}$. Again, we consider the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$ and the complement $U_N := span\{e_2, e_3, ...\}$.

Definition 7.4.0.10. Define the Schur-Weyl contravariant functor

$$SW_\nu : \underline{Rep}^{ab}(S_\nu) \longrightarrow Ind - Mod_{\mathcal{U}(\mathfrak{gl}_N)}$$

by

$$SW_\nu := \mathrm{Hom}_{\underline{Rep}^{ab}(S_\nu)}(\,\cdot\,, (\mathbb{C}^N)^{\underline{\otimes}\nu})$$

As before, the functor SW_ν is a contravariant $\mathbb{C}$-linear additive left-exact functor, and its image lies in $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ (cf. Remark 7.3.0.7).

We can now define another Schur-Weyl functor which we will consider: it is the contravariant functor $\widehat{SW}_{\nu,\mathbb{C}^N} : \underline{Rep}^{ab}(S_\nu) \longrightarrow \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$. Recall from Section 7.1 that

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \longrightarrow \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} := O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \Big/ Rep(\mathfrak{gl}_N)_{poly,\nu}$$

is the Serre quotient of $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by the Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν. We then define

$$\widehat{SW}_{\nu,\mathbb{C}^N} := \hat{\pi}_N \circ SW_{\nu,\mathbb{C}^N}$$

7.5 Classical Schur-Weyl duality and the restricted inverse limit

7.5.1 Classical Schur-Weyl duality: inverse limit

In this subsection, we prove that the classical Schur-Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$ give a duality (anti-equivalence) between the category $\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d)$ and the category

$$Rep(\mathfrak{gl}_\infty)_{poly} \cong \varprojlim_{n \in \mathbb{Z}_+, \text{ restr}} Rep(\mathfrak{gl}_n)_{poly}$$

The contravariant functor $\mathtt{SW}_{\mathbb{C}^N}$ sends the Young diagram λ to the $\mathfrak{gl}_N$-module $S^\lambda \mathbb{C}^N$.

Let $n \in \mathbb{Z}_+$. We start by noticing that the functors $\mathfrak{Res}_{n,n+1}$ and the functors Γ_n (defined in Subsection 6.6.3) are compatible with the classical Schur-Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$:

Lemma 7.5.1.1. *We have natural isomorphisms*

$$\mathfrak{Res}_{n,n+1} \circ \mathtt{SW}_{\mathbb{C}^{n+1}} \cong \mathtt{SW}_{\mathbb{C}^n}$$

and

$$\Gamma_n \circ \mathtt{SW}_{\mathbb{C}^\infty} \cong \mathtt{SW}_{\mathbb{C}^n}$$

for any $n \geq 0$.

Proof. It is enough to check this on simple objects in $\bigoplus_{d \in \mathbb{Z}_+} Rep(S_d)$, in which case the statement follows directly from the definitions of $\mathfrak{Res}_{n,n+1}$, Γ_n together with the fact that $\mathtt{SW}_{\mathbb{C}^N}(\lambda) \cong S^\lambda \mathbb{C}^N$ for any $N \in \mathbb{Z}_+ \cup \{\infty\}$. □

The above Lemma implies that we have a commutative diagram

$$\begin{array}{ccc} & & Rep(\mathfrak{gl}_n)_{poly} \\ & \overset{\mathtt{SW}_{\mathbb{C}^n}}{\nearrow} & \uparrow \mathbf{Pr}_n \\ \bigoplus_{d\in\mathbb{Z}_+} Rep(S_d)^{op} & \xrightarrow[\mathtt{SW}_{\lim}]{} & \varprojlim_{n\geq 1,\ \mathrm{restr}} Rep(\mathfrak{gl}_n)_{poly} \\ & \underset{\mathtt{SW}_{\mathbb{C}^\infty}}{\searrow} & \uparrow \Gamma_{\lim} \\ & & Rep(\mathfrak{gl}_\infty)_{poly} \end{array} \qquad \Gamma_n : Rep(\mathfrak{gl}_\infty)_{poly} \to Rep(\mathfrak{gl}_n)_{poly}$$

the functor $\Gamma_{\lim}$ being an equivalence of categories (by Proposition 6.6.5.2), and $\mathbf{Pr}_n$ being the canonical projection functor.

Proposition 7.5.1.2. *The contravariant functors*

$$\mathtt{SW}_\infty : \bigoplus_{d\in\mathbb{Z}_+} Rep(S_d) \longrightarrow Rep(\mathfrak{gl}_\infty)_{poly}$$

and

$$\mathtt{SW}_{lim} : \bigoplus_{d\in\mathbb{Z}_+} Rep(S_d) \to \varprojlim_{n\in\mathbb{Z}_+,\ restr} Rep(\mathfrak{gl}_n)_{poly}$$

are anti-equivalences of semisimple categories.

Proof. As it was said in Subsection 3.1, the functor $\mathtt{SW}_N$ is full and essentially surjective for any N. In this case, the functor $\mathtt{SW}_\infty$ is also faithful, since the simple object λ in $\bigoplus_{d\in\mathbb{Z}_+} Rep(S_d)$ is taken by the functor $\mathtt{SW}_\infty$ to the simple object $S^\lambda\mathbb{C}^\infty \neq 0$. This proves that the contravariant functor $\mathtt{SW}_\infty$ is an anti-equivalence of categories. The commutative diagram above then implies that the contravariant functor $\mathtt{SW}_{\lim}$ is an anti-equivalence as well. $\square$

7.6 $\underline{Rep}^{ab}(S_\nu)$ and the inverse limit of categories $\widehat{\mathcal{O}}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$

In this section we are going to prove that the Schur-Weyl functors defined in Section 5 give us an equivalence of categories between $\underline{Rep}^{ab}(S_\nu)$ and the restricted inverse limit

$\varprojlim_{N \in \mathbb{Z}_+, \text{ restr}} \widehat{O}^{\mathfrak{p}_N}_{\nu, \mathbb{C}^N}$.

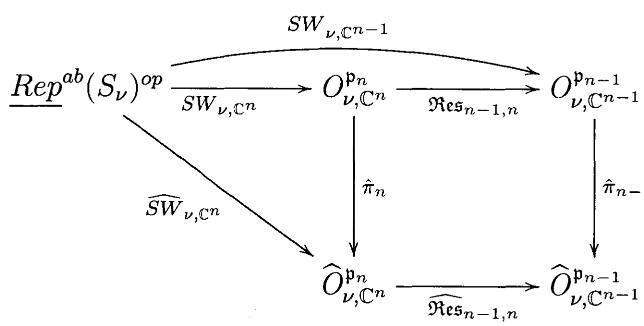
We fix $\nu \in \mathbb{C}$.

Proposition 7.6.0.3. *The functor $\mathfrak{Res}_{n-1,n}$ satisfies: $\mathfrak{Res}_{n-1,n} \circ SW_{\nu,\mathbb{C}^n} \cong SW_{\nu,\mathbb{C}^{n-1}}$, i.e. there exists a natural isomorphism $\eta_n : \mathfrak{Res}_{n-1,n} \circ SW_{\nu,\mathbb{C}^n} \to SW_{\nu,\mathbb{C}^{n-1}}$.*

Proof. Follows directly from Proposition 7.3.0.9. □

Corollary 7.6.0.4. *We have $\widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{SW}_{\nu,\mathbb{C}^n} \cong \widehat{SW}_{\nu,\mathbb{C}^{n-1}}$, i.e. there exists a natural isomorphism $\hat{\eta}_n : \widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{SW}_{\nu,\mathbb{C}^n} \to \widehat{SW}_{\nu,\mathbb{C}^{n-1}}$.*

Proof. By definition of $\widehat{\mathfrak{Res}}_{n-1,n}, \widehat{SW}_{\nu,\mathbb{C}^n}$, together with Proposition 7.6.0.3, we have a commutative diagram



Since $\hat{\pi}_{n-1} \circ SW_{\nu,\mathbb{C}^{n-1}} =: \widehat{SW}_{\nu,\mathbb{C}^{n-1}}$, we get $\widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{SW}_{\nu,\mathbb{C}^n} \cong \widehat{SW}_{\nu,\mathbb{C}^{n-1}}$. □

Notation 7.6.0.5. For each $k \in \mathbb{Z}_+$, $Fil_k(\underline{Rep}^{ab}(S_\nu))$ is defined to be the Serre subcategory of $\underline{Rep}^{ab}(S_\nu)$ generated by the simple objects $\mathbf{L}(\lambda)$ such that the Young diagram λ satisfies either of the following conditions:

- λ belongs to a trivial $\overset{\nu}{\sim}$-class, and $\ell(\lambda) \leq k$.
- λ belongs to a non-trivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i \geq 0}$, $\lambda = \lambda^{(i)}$, and $\ell(\lambda^{(i+1)}) \leq k$.

This defines a $\mathbb{Z}_+$-filtration on the objects of the category $\underline{Rep}^{ab}(S_\nu)$. That is, we have:

$$\underline{Rep}^{ab}(S_\nu) \cong \varinjlim_{k \in \mathbb{Z}_+} Fil_k(\underline{Rep}^{ab}(S_\nu))$$

Lemma 7.6.0.6. *The functors $\widehat{SW}_{\nu,\mathbb{C}^n}$ are $\mathbb{Z}_+$-filtered shortening functors (see Chapter 6 for the relevant definitions) .*

Proof. Follows from the fact that $\widehat{SW}_{\nu,\mathbb{C}^n}$ are exact, together with Lemma 5.0.0.43. □

This Lemma, together with Corollary 7.6.0.4, gives us a contravariant ($\mathbb{Z}_+$-filtered shortening) functor

$$\widehat{SW}_{\nu,\mathrm{lim}} : \underline{Rep}^{ab}(S_\nu) \longrightarrow \varprojlim_{n\geq 1,\ \mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$
$$X \mapsto \left(\{\widehat{SW}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}, \{\hat{\eta}_n(X)\}_{n\geq 2}\right)$$
$$(f : X \to Y) \mapsto \{\widehat{SW}_{\nu,\mathbb{C}^n}(f) : \widehat{SW}_{\nu,\mathbb{C}^n}(Y) \to \widehat{SW}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}$$

This functor is given by the universal property of the restricted inverse limit described in Chapter 6, and makes the diagram below commutative:

$$\begin{array}{ccc} & & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ & \overset{\widehat{SW}_{\nu,\mathbb{C}^n}}{\nearrow} & \uparrow \mathbf{Pr}_n \\ \underline{Rep}^{ab}(S_\nu)^{op} & \xrightarrow[\widehat{SW}_{\nu,\mathrm{lim}}]{} & \varprojlim_{n\geq 1,\ \mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \end{array}$$

(here $\mathbf{Pr}_n$ is the canonical projection functor).

We now show that there is an equivalence of categories $\underline{Rep}^{ab}(S_\nu)^{op}$ and $\varprojlim_{n\geq 1,\ \mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Theorem 7.6.0.7. *The Schur-Weyl contravariant functors $\widehat{SW}_{\nu,\mathbb{C}^n}$ induce an anti-equivalence of abelian categories, given by the (exact) contravariant functor*

$$\widehat{SW}_{\nu,lim} : \underline{Rep}^{ab}(S_\nu) \longrightarrow \varprojlim_{n\geq 1,\ restr} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

Proof. The functors $\widehat{SW}_{\nu,\mathbb{C}^n}$ are exact for each $n \geq 1$, which means that the functor $\widehat{SW}_{\nu,\mathrm{lim}}$ is exact as well (see Subsection 6.2.2).

To see that it is an anti-equivalence, we will use Proposition 6.4.0.27. All we need to check is that the functors $\widehat{SW}_{\nu,\mathbb{C}^n}$ satisfy the "stabilization condition" (Condition 6.4.0.26): that is, for each $k \in \mathbb{Z}_+$, there exists $n_k \in \mathbb{Z}_+$ such that

$$\widehat{SW}_{\nu,\mathbb{C}^n} : Fil_k(\underline{Rep}^{ab}(S_\nu)) \to Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an anti-equivalence of categories for any $n \geq n_k$.

Indeed, let $k \in \mathbb{Z}_+$, and let $n \geq k+1$.

The category $Fil_k(\underline{Rep}^{ab}(S_\nu))$ decomposes into blocks (corresponding to the blocks of $\underline{Rep}^{ab}(S_\nu)$), and the category $Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ decomposes into blocks corresponding to the blocks of $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

The requirement $n \geq k+1$ together with Lemma 5.0.0.43 means that for any semisimple block of $Fil_k(\underline{Rep}^{ab}(S_\nu))$, the simple object $\mathbf{L}(\lambda)$ corresponding to this block is not sent to zero under $\widehat{SW}_{\nu,\mathbb{C}^n}$. This, in turn, implies that $\widehat{SW}_{\nu,\mathbb{C}^n}$ induces an anti-equivalence between each semisimple block of $Fil_k(\underline{Rep}^{ab}(S_\nu))$ and the corresponding semisimple block of $Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$.

Now, fix a non-semisimple block $\mathcal{B}_\lambda$ of $\underline{Rep}^{ab}(S_\nu)$, and denote by $Fil_k(\mathcal{B}_\lambda)$ the corresponding non-semisimple block of $Fil_k(\underline{Rep}^{ab}(S_\nu))$. We denote by $\mathfrak{B}_{\lambda,n}$ the corresponding block in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The corresponding block of $Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ will then be $\hat{\pi}(Fil_k(\mathfrak{B}_{\lambda,n}))$.

We now check that the contravariant functor

$$\widehat{SW}_{\nu,\mathbb{C}^n}|_{Fil_k(\mathcal{B}_\lambda)} : Fil_k(\mathcal{B}_\lambda) \to \hat{\pi}(Fil_k(\mathfrak{B}_{\lambda,n}))$$

is an anti-equivalence of categories when $n \geq k+1$.

Since $n \geq k+1$, the Serre subcategories $Fil_k(\mathcal{B}_\lambda)$ and $Ker(\widehat{SW}_{\nu,\mathbb{C}^n})$ of $\underline{Rep}^{ab}(S_\nu)$ have trivial intersection (see Lemma 5.0.0.43), which means that the restriction of $\widehat{SW}_{\nu,\mathbb{C}^n}$ to the Serre subcategory $Fil_k(\mathcal{B}_\lambda)$ is both faithful and full (the latter follows from Theorem 5.0.0.42).

It remains to establish that the functor $\widehat{SW}_{\nu,\mathbb{C}^n}|_{Fil_k(\mathcal{B}_\lambda)}$ is essentially surjective when

$n \geq k+1$. This can be done by checking that this functor induces a bijection between the sets of isomorphism classes of indecomposable projective objects in $Fil_k(\mathcal{B}_\lambda), \hat{\pi}(Fil_k(\mathfrak{B}_{\lambda,n}))$ respectively. The latter fact follows from the proof of Theorem 5.1.2.3.

Thus $\widehat{SW}_{\nu,\mathbb{C}^n} : Fil_k(\mathcal{B}_\lambda) \to Fil_k(\hat{\pi}(\mathfrak{B}_{\lambda,n}))$ is an anti-equivalence of categories for $n \geq k+1$, and

$$\widehat{SW}_{\nu,\mathbb{C}^n} : Fil_k(\underline{Rep}^{ab}(S_\nu)) \to Fil_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an anti-equivalence of categories for $n \geq k+1$, which completes the proof.

□

7.7 Schur-Weyl duality for $\underline{Rep}^{ab}(S_\nu)$ and $\mathfrak{gl}_\infty$

Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \ldots$. Fix $\mathbb{1} := e_1$ and $U_\infty := span_{\mathbb{C}}(e_2, e_3, \ldots)$.

Lemma 7.7.0.8. *We have a commutative diagram*

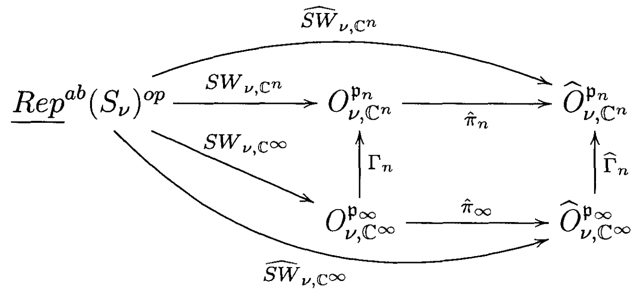
$$\begin{array}{ccc} \underline{Rep}^{ab}(S_\nu)^{op} & \xrightarrow[\widehat{SW}_{\nu,lim}]{} & \varprojlim_{n\geq 1,\ restr} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ & \searrow^{\widehat{SW}_{\nu,\mathbb{C}^\infty}} & \uparrow \widehat{\Gamma}_{lim} \\ & & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \end{array}$$

Namely, there is a natural isomorphism $\hat{\eta} : \widehat{\Gamma}_{lim} \circ \widehat{SW}_{\nu,\mathbb{C}^\infty} \to \widehat{SW}_{\nu,lim}$.

Proof. In order to prove this statement, we will show that for any $n \geq 1$, the following diagram is commutative:

$$\begin{array}{ccc} \underline{Rep}^{ab}(S_\nu)^{op} & \xrightarrow[\widehat{SW}_{\nu,\mathbb{C}^n}]{} & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ & \searrow^{\widehat{SW}_{\nu,\mathbb{C}^\infty}} & \uparrow \widehat{\Gamma}_n \\ & & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \end{array}$$

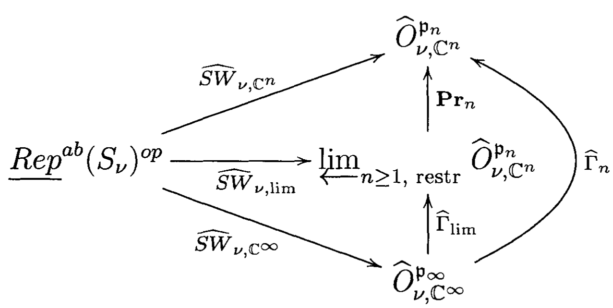
In fact, we will show that the diagram below is commutative



which will prove the required statement. The commutativity of this diagram follows from the existence of a natural isomorphism $\Gamma_n \circ SW_{\nu,\mathbb{C}^\infty} \xrightarrow{\sim} SW_{\nu,\mathbb{C}^n}$ (due to Proposition 7.3.0.9) and a natural isomorphism $\widehat{\Gamma}_n \circ \hat{\pi}_\infty \cong \hat{\pi}_n \circ \Gamma_n$ (see proof of Proposition 7.2.3.5).

□

Thus we obtain a commutative diagram



Theorem 7.7.0.9. *The contravariant functor $\widehat{SW}_{\nu,\mathbb{C}^\infty} : \underline{Rep}^{ab}(S_\nu) \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ is an anti-equivalence of abelian categories.*

Proof. The functor $\widehat{\Gamma}_{\lim}$ is an equivalence of categories (see Lemma 7.2.3.5), and the functor $\widehat{SW}_{\nu,\lim}$ is an anti-equivalence of categories (see Theorem 7.6.0.7). The commutative diagram above implies that the contravariant functor $\widehat{SW}_{\nu,\mathbb{C}^\infty}$ is an anti-equivalence of categories as well. □

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Chapter 8

Schur-Weyl functors and duality structures

In this section, we discuss the relation given by the Schur-Weyl functors between the duality structures in the category $\underline{Rep}^{ab}(S_\nu)$ (which is a rigid symmetric monoidal category) and in the category $O^{\mathfrak{p}_\infty}_{\mathbb{C}^\infty}$. The latter is a subcategory of the BGG category O and therefore inherits a duality functor.

8.1 Duality in category O

A construction similar to the duality functor described in Section 3.3 can be made for $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. All modules M in $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ are weight modules with respect to the subalgebra of diagonal matrices in $\mathfrak{gl}_\infty$, and the weight spaces are finite-dimensional (due to the polynomiality condition in the definition of $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$). This allows one to construct the restricted twisted dual $M^\vee$ in the same way as before, and obtain an exact functor

$$(\cdot)^\vee : \left(O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}\right)^{op} \longrightarrow O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$$

Remark 8.1.0.10. It is obvious that for $n \in \mathbb{Z}_+$, the functor $(\cdot)^\vee : \left(O^{\mathfrak{p}_n}_{\mathbb{C}^n}\right)^{op} \to O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ takes finite-dimensional (polynomial) modules to finite-dimensional (polynomial) modules.

In fact, one can easily check that the functor $(\cdot)^\vee : \left(O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}\right)^{op} \longrightarrow O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ takes polynomial modules to polynomial modules as well.

We now describe the above functor in terms of the restricted inverse limit of categories $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Let $n \in \mathbb{Z}_+$.

The contravariant duality functors

$$(\cdot)^\vee_n : \left(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\right)^{op} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

takes polynomial modules to polynomial modules, and therefore descends to a contravariant duality functor

$$\widehat{(\cdot)}^\vee_n : \left(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\right)^{op} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

It is a straightforward consequence of the definition of a dual of a module, that the duality functors commute with the restriction functors $\mathfrak{Res}_{n-1,n}$:

Lemma 8.1.0.11. *For any $n \geq 2$, we have:*

$$(\cdot)^\vee_{n-1} \circ \mathfrak{Res}^{op}_{n-1,n} \cong \mathfrak{Res}^{op}_{n-1,n} \circ (\cdot)^\vee_n$$

This allows us to define duality functors

$$(\cdot)^\vee_{\lim} : \left(\varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\right)^{op} \to \varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{(\cdot)}^\vee_{\lim} : \left(\varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\right)^{op} \to \varprojlim_{n\geq 1,\text{ restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

Under the equivalence $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\text{ restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ established in Subsection 7.2.3, the functor $(\cdot)^\vee_{\lim}$ corresponds to the duality functor

$$(\cdot)^\vee_\infty : \left(O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}\right)^{op} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$$

discussed in Subsection 8.1.

This functor takes polynomial $\mathfrak{gl}_\infty$-modules to polynomial $\mathfrak{gl}_\infty$-modules, and therefore descends to a contravariant duality functor

$$\widehat{(\cdot)}^\vee_\infty : \left(\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}\right)^{op} \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$$

8.2 The Schur-Weyl functor and dualities in categories $\underline{Rep}^{ab}(S_\nu)$, $O^{\mathfrak{p}}_{\nu,V}$

As a consequence of Theorem 5.0.0.42, we establish a connection between the notions of duality in the Deligne category $\underline{Rep}^{ab}(S_\nu)$ and the duality in the category $O^{\mathfrak{p}}_{\nu,V}$.

Consider the contravariant functors

$$(\cdot)^* : \underline{Rep}^{ab}(S_\nu) \to \underline{Rep}^{ab}(S_\nu) \quad \text{and} \quad (\cdot)^\vee : O^{\mathfrak{p}}_{\nu,V} \to O^{\mathfrak{p}}_{\nu,V}$$

where $(\cdot)^*$ is the duality functor on $\underline{Rep}^{ab}(S_\nu)$ (with respect to the tensor structure of $\underline{Rep}^{ab}(S_\nu)$), and $(\cdot)^\vee$ is the usual duality in the category O (c.f. Section 3.3, or [H, Section 3.2]). The Schur-Weyl functor $SW_{\nu,V}$ relates these two duality notions:

Proposition 8.2.0.12. *For any $\nu \in \mathbb{C}$, there is an isomorphism of (covariant) functors*

$$\widehat{SW}_{\nu,V}((\cdot)^*) \longrightarrow \hat{\pi}(SW_{\nu,V}(\cdot)^\vee)$$

Proof. First of all, notice that both sides are exact functors. Indeed, the duality functor on any abelian rigid monoidal category is exact, and $\widehat{SW}_\nu$ is a (contravariant) exact functor by Lemma 5.0.0.48, which implies that $\widehat{SW}_\nu((\cdot)^*)$ is exact.

On the other hand, $(\cdot)^\vee$ is exact (c.f. [H, Section 3.2]), so an argument similar to the proof of Lemma 5.0.0.48 shows that $\hat{\pi}(SW_\nu(\cdot)^\vee)$ is exact as well.

Since any object in $\underline{Rep}^{ab}(S_\nu)$ has a projective resolution, it remains to establish a natural isomorphism between the two functors when we restrict ourselves to the full subcategory of projective objects in $\underline{Rep}^{ab}(S_\nu)$.

We now use the fact that all projective objects in $\underline{Rep}^{ab}(S_\nu)$ are self-dual, since they lie in $\underline{Rep}(S_\nu)$ (c.f. Proposition 3.2.4.6). This allows us to construct the isomorphism between the two functors block-by-block.

Fix a block $\mathcal{B}_\lambda$ of $\underline{Rep}^{ab}(S_\nu)$. If this block is semisimple, then by Proposition 5.1.1.1, there is nothing to prove.

So we assume that the block $\mathcal{B}_\lambda$ is not semisimple, and use the same notation as in Subsection 5.1.2 for simple, standard, co-standard and projective objects in both $\mathcal{B}_\lambda$ and the corresponding block of $O^{\mathfrak{p}}_{\nu,V}$. We will also denote by $Proj_\lambda$ the full subcategory of projective objects in $\mathcal{B}_\lambda$.

For each $i \geq 1$, fix a non-zero morphism $\beta_i : \mathbf{P}_{i-1} \to \mathbf{P}_i$; Proposition 3.2.4.10 tells us that we have an exact sequence

$$0 \to \mathbf{M}^*_{i-1} \to \mathbf{P}_{i-1} \xrightarrow{\beta_i} \mathbf{P}_i$$

Recall from Theorem 3.2.2.6 that such a morphism β_i is unique up to a non-zero scalar, and that the morphisms $\{\beta_i, \beta_i^*\}_{i\geq 1}$ generate all the morphisms in $Proj_\lambda$.

We construct isomorphisms

$$\theta_i : SW_\nu(\mathbf{P}^*_i) \longrightarrow SW_\nu(\mathbf{P}_i)^\vee$$

iteratively (recall that such isomorphisms exist by Theorem 5.1.2.3).

We start by choosing any isomorphism $\theta_0 : SW_\nu(\mathbf{P}^*_0) \to SW_\nu(\mathbf{P}_0)^\vee$; at the i-th step, we have already constucted $\theta_0, ..., \theta_{i-1}$, and we choose an isomorphism θ_i so that the diagram below is commutative:

$$\begin{array}{ccc} SW_\nu(\mathbf{P}_i^*) & \xrightarrow{\theta_i} & SW_\nu(\mathbf{P}_i)^\vee \\ {\scriptstyle SW_\nu(\beta_i^*)}\big\uparrow & & {\scriptstyle SW_\nu(\beta_i)^\vee}\big\uparrow \\ SW_\nu(\mathbf{P}_{i-1}^*) & \xrightarrow{\theta_{i-1}} & SW_\nu(\mathbf{P}_{i-1})^\vee \end{array}$$

We now explain why it is possible to make such a choice of θ_i.

Applying the left-exact (covariant) functors $SW_\nu(\cdot)^\vee, SW_\nu((\cdot)^*)$ to the exact sequence

$$0 \to \mathbf{M}_{i-1}^* \to \mathbf{P}_{i-1} \xrightarrow{\beta_i} \mathbf{P}_i$$

and using Theorem 5.1.2.3, we see that the maps $SW_\nu(\beta_i^*), SW_\nu(\beta_i)$ are either simultaneously zero or simultaneously not zero. Since the space

$$\mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}_{\nu,V}}(SW_\nu(\mathbf{P}_{i-1}^*), SW_\nu(\mathbf{P}_i)^\vee)$$

is at most one-dimensional (c.f. Theorem 5.1.2.3 and Proposition 3.3.1.13), we can take θ_i to be any isomorphism $SW_\nu(\mathbf{P}_i^*) \to SW_\nu(\mathbf{P}_i)^\vee$, and then multiply it by a non-zero scalar to make the above diagram commutative.

We now claim that the isomorphisms θ_i define a natural transformation. Since the morphisms $\{\beta_i, \beta_i^*\}_{i\geq 1}$ generate all the morphisms in $Proj_\lambda$, we only need to check that for any $i \geq 1$, the following diagram is commutative:

$$\begin{array}{ccc} SW_\nu(\mathbf{P}_i^*) & \xrightarrow{\theta_i} & SW_\nu(\mathbf{P}_i)^\vee \\ {\scriptstyle SW_\nu(\beta_i)}\big\downarrow & & {\scriptstyle SW_\nu(\beta_i^*)^\vee}\big\downarrow \\ SW_\nu(\mathbf{P}_{i-1}^*) & \xrightarrow{\theta_{i-1}} & SW_\nu(\mathbf{P}_{i-1})^\vee \end{array}$$

The latter follows easily from the construction of θ_i, together with the fact that $\mathbf{P}_i = \mathbf{P}_i^*$ (for any $i \geq 0$) and $\theta_i = \theta_i^\vee$. □

The above construction allows us to extend this connection to the infinite-dimensional case. Namely, the anti-equivalences in Theorems 7.6.0.7 and 7.7.0.9 imply the following statement:

Corollary 8.2.0.13. *Let $N \in \mathbb{Z}_+ \cup \{\infty\}$. For any $\nu \in \mathbb{C}$, there is an isomorphism of (covariant) functors*

$$\widehat{SW}_{\nu,\mathbb{C}^N} \circ (\cdot)^* \longrightarrow \widehat{(\cdot)}_N^{\vee} \circ SW_{\nu,\mathbb{C}^N}$$

Bibliography

[A] H. H. Andersen, *Tensor products of quantized tilting modules*, Comm. Math. Phys. 149 (1992), no. 1, 149-159.

[APW] H. H. Andersen, P. Polo, K. Wen, *Representations of quantum algebras*, Invent. Math. 104 (1991), p. 1-59.

[CO] J. Comes, V. Ostrik, *On blocks of Deligne's category* $\underline{Rep}(S_t)$, Advances in Mathematics 226 (2011), no. 2, 1331-1377; arXiv:0910.5695 [math.RT].

[CO2] J. Comes, V. Ostrik, *On Deligne's category* $Rep^{ab}(S_d)$, to appear in Algebra and Number Theory; arXiv:math/1304.3491v1 [math.RT].

[DPS] E. Dan-Cohen, I. Penkov, V. Serganova, *A Koszul category of representations of finitary Lie algebras*; arxiv:1105.3407 [math.RT].

[D1] P. Deligne, *Categories tannakiennes*, The Grothendieck Festschrift, Birkhauser, Boston (1990), p. 111 - 195.

[D2] P. Deligne, *La categorie des representations du groupe symetrique* S_t*, lorsque t n'est pas un entier naturel*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., p. 209-273, Mumbai (2007); http://www.math.ias.edu/ phares/deligne/preprints.html.

[Et1] P. Etingof, *Representation theory in complex rank, I*, Transformation Groups, Volume 19, Issue 2, pp. 359-381, Springer US (2014); arXiv:1401.6321 [math.RT].

[E1] I. Entova Aizenbud, *On representations of rational Cherednik algebras in complex rank*, arXiv:1301.0120 [math.RT].

[E2] I. Entova-Aizenbud, *Schur-Weyl duality for Deligne categories*, to appear in Int. Math. Res. Not., arXiv:1403.5509 [math.RT].

[E3] I. Entova-Aizenbud, *Notes on restricted inverse limits of categories*, arXiv:1504.01121 [math.RT].

[E4] I. Entova-Aizenbud, *Deligne categories and reduced Kronecker coefficients*, arXiv:1407.1506 [math.RT].

[FH] W. Fulton, J. Harris, *Representation theory: a first course*, Volume 129 of Graduate Texts in Mathematics, Springer (1991).

[HY] J. Hong, O. Yacobi, *Polynomial Representations and Categorifications of Fock Space*, Algebras and Representation Theory, Volume 16, Issue 5, Springer Netherlands (2012), pp. 1273-1311; arXiv:1101.2456 [math.RT].

[H] J. E. Humphreys, *Representations of semisimple Lie algebras in the BGG category O*, Volume 94 of Graduate Studies in Mathematics, American Mathematical Soc. (2008).

[KS] M. Kashiwara, P. Schapira, *Categories and sheaves*, Volume 332 of Grundlehren der mathematischen Wissenschaften, Springer (2006).

[Macd] I. G. Macdonald, *Symmetric Functions and Hall polynomials*, Oxford Mathematical Monographs, 1995.

[MacL] S. MacLane, *Categories for the working mathematician*, Volume 5 of Graduate Texts in Mathematics, Springer (1998).

[Mat] A. Mathew, *Categories parameterized by schemes and representation theory in complex rank*, Journal of Algebra 381 (2013), p. 140-163; arxiv:1006.1381 [math.RT].

[PS] I. Penkov, K. Styrkas, *Tensor representations of classical locally finite Lie algebras*, Developments and trends in infinite-dimensional Lie theory, Progr. Math. 288, Birkhauser Boston, (2011), pp. 127Ũ150; arXiv:0709.1525.

[S] O. M. Schnürer, *Equivariant sheaves on flag varieties*, Math. Zeitschrift 267 (2011), no. 1-2, 27-80; arXiv:0809.4785 [math.RT].

[SS] S. V. Sam, A. Snowden, *Stability patterns in representation theory*; arXiv:1302.5859 [math.RT].

[WW] B. Webster, G. Williamson, *The bounded below equivariant derived category*, `http://people.virginia.edu/~btw4e/equivariant.pdf`.