Classification of Argyres-Douglas theories from M5 branes

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Classification of Argyres-Douglas theories from M5 branes

Yifan Wang[*]
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

Dan Xie[†]
Center of Mathematical Sciences and Applications, Harvard University, Cambridge, Massachusetts 02138, USA
and Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138, USA

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We obtain a large class of new four-dimensional Argyres-Douglas theories by classifying irregular punctures for the six-dimensional (2, 0) superconformal theory of ADE type on a sphere. Along the way, we identify the connection between the Hitchin system and threefold singularity descriptions of the same Argyres-Douglas theory. Other constructions such as taking degeneration limits of the irregular puncture, adding an extra regular puncture, and introducing outer-automorphism twists are also discussed. Later we investigate various features of these theories including their Coulomb branch spectrum and central charges.

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I. INTRODUCTION

Four-dimensional $\mathcal{N}=2$ superconformal field theories (SCFT) play an unusual role in the space of SCFTs across various dimensions: (1) there are various interesting IR dynamics such as the existence of a moduli space of vacua with many interesting phase structures; (2) often many aspects of the theories such as low energy effective action and Bogomol'nyi-Prasad-Sommerfield (BPS) spectrum can be solved exactly. The investigation of these theories can teach us many lessons about generic features of quantum field theory such as confinement, renormalization group flow, etc.

The space of $\mathcal{N}=2$ SCFTs has enlarged tremendously since the discovery of Seiberg-Witten (SW) solutions [1,2]. One can engineer many Lagrangian theories from $\mathcal{N}=2$ super-Yang-Mills (SYM) theories coupled to free hypermultiplets. More recently, it was found that strongly coupled matter systems such as T_N theories can also be used to construct new SCFTs [3], known as class S theories, which greatly increases the repertoire of known theories. These theories have one distinguished feature that all BPS operators on the Coulomb branch [4] have integer scaling dimensions.

There is another type of $\mathcal{N}=2$ SCFT called Argyres-Douglas (AD) theory [5]. These are *intrinsically* strongly coupled theories, the first instance of which was originally discovered at a certain point on the Coulomb branch of the pure $\mathcal{N}=2$ $SU(3)$ theory. Unlike familiar SCFTs, such as $\mathcal{N}=2$ $SU(2)$ SYM coupled to four fundamental flavors, the AD theory has some unusual properties: first, the scaling dimensions of the Coulomb branch BPS operators are fractional, and, second, there are relevant operators in the spectrum. These special features make AD theories a particularly useful class of models from which we can study generic features of conformal field theory, for example, the RG flow between various fixed points [6]. By looking at special points on the Coulomb branch of other gauge theories, new examples with an ADE classification has been found [7,8]. More recently, a further remarkable generalization of these theories which are called (G, G') theories [9] was engineered using type IIB string theory construction [10,11].

In [12], a large class of new $\mathcal{N}=2$ AD theories has been constructed using A type M5 brane construction [see [13–15] for theories engineered using A_1 (2, 0) theory]: one can engineer four-dimensional SCFTs by putting six-dimensional A_{n-1} (2, 0) theory on a punctured Riemann surface. To get a SCFT (within the construction of [12]), one must use a sphere with one irregular puncture, or a sphere with one irregular and one regular puncture. The classification of irregular punctures is very rich, and, in particular, one can also reproduce theories which are originally engineered using regular punctures on sphere [12]. These lessons suggest that the Argyres-Douglas theories actually constitute a much larger class of $\mathcal{N}=2$ theories than the usual class S theories with integral Coulomb branch spectrum, and it is interesting to further enlarge the theory space.

The main purpose of this paper is to generalize the A type construction of [12] to other types of (2, 0) theory labeled by J. The major problem in the M5 brane construction is to classify the irregular singularities [16], which we find to take the following universal form:

[*]yifanw@mit.edu
[†]dxie@cmsa.fas.harvard.edu

TABLE I. Allowed denominators for irregular singularities.

J	Allowed b
A_{N-1}	N, $N-1$
D_N	$2N-2N$
E_6	12, 9, 8
E_7	18, 14
E_8	30, 24, 20

$$\Phi = \frac{T}{z^{2+k/b}} + \cdots. \tag{1}$$

Here Φ is the holomorphic part of the Higgs field which describes the transversal deformation of M5 branes of $J = A, D, E$ type. T is a regular semisimple element of J and $k > -b$ is an arbitrary integer. The allowed set of values for b is given in Table I and the choice of T in general depends on that of b.

Given the structure of the singularity, one can obtain the SW solution by computing the spectral curve of the Hitchin system $\det(x - \Phi) = 0$. Moreover, we can directly map the corresponding $\mathcal{N} = 2$ AD theory to a threefold isolated singularity using the spectral curve (see Table II and Fig. 3).

Once the basic irregular punctures are identified, there are three variations which give rise to new theories.

(i) We can take degeneration limits for some punctures, namely, the eigenvalues of the matrices in defining the irregular singularity can become degenerate.

(ii) We can add another regular puncture, so as to obtain theories with non-Abelian flavor symmetry.

(iii) If J has a nontrivial outer-automorphism group, we can introduce twists on the punctures. This is only possible for a sphere with one irregular and one regular puncture.

Using these constructions, we can find a lot more new theories with various intriguing features.

TABLE II. Threefold singularities corresponding to our irregular punctures.

J	Singularity	b
A_{N-1}	$x_1^2 + x_2^2 + x_3^N + z^k = 0$	N
	$x_1^2 + x_2^2 + x_3^N + xz^k = 0$	$N-1$
D_N	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k = 0$	$2N-2$
	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k x_3 = 0$	N
E_6	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	12
	$x_1^2 + x_2^3 + x_3^4 + z^k x_3 = 0$	9
	$x_1^2 + x_2^3 + x_3^4 + z^k x_2 = 0$	8
E_7	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$	18
	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k x_3 = 0$	14
E_8	$x_1^2 + x_2^3 + x_3^5 + z^k = 0$	30
	$x_1^2 + x_2^3 + x_3^5 + z^k x_3 = 0$	24
	$x_1^2 + x_2^3 + x_3^5 + z^k x_2 = 0$	20

This paper is organized as follows. In Sec. II we review the basic features of AD theories and their type IIB constructions. Section III presents the classification of irregular punctures (singularities) in the M5 brane construction, as well as their relation to IIB isolated quasihomogeneous singularities. Section IV gives a detailed study of twisted irregular punctures. In Sec. V we discuss some properties such as spectrum and central charges of the AD theories that we have constructed. Finally we conclude in Sec. VI with a discussion of potential directions.

II. GENERALITY ABOUT ARGYRES-DOUGLAS THEORIES

A. Basic features

The first Argyres-Douglas type theory was found as the IR theory at a special point on the Coulomb branch of $\mathcal{N} = 2$ pure $SU(3)$ gauge theory [5]. At this point, besides the massless photons there are two extra mutually nonlocal massless dyons, and it was argued that the IR theory has to be a strongly coupled interacting SCFT [5,17]. This theory has no Higgs branch, and has a one-dimensional Coulomb branch. The Seiberg-Witten curve of this theory can be written as

$$x^2 = z^3 + u_1 z + u_2, \tag{2}$$

with Seiberg-Witten differential $\lambda = xdz$. Since the integral of λ along one cycle of the SW curve gives the mass of the BPS particles, its scaling dimension has to be 1, which implies that the scaling dimensions of x, z satisfy the condition

$$[x] + [z] = 1. \tag{3}$$

By requiring each term in the SW curve to have the same scaling dimension, we find

$$[x] = \frac{3}{5}, \qquad [z] = \frac{2}{5}, \qquad [u_1] = \frac{4}{5}, \qquad [u_2] = \frac{6}{5}. \tag{4}$$

u_1 has a scaling dimension less than 1 and therefore it is a coupling constant, while u_2 is a relevant operator parametrizing the Coulomb branch. For a $\mathcal{N} = 2$ preserving relevant deformation, the sum of scaling dimensions of coupling constant m and the relevant operator u has to be equal to 2: $[m] + [u] = 2$. The distinguished feature of AD theories among the $\mathcal{N} = 2$ SCFTs is that the Coulomb branch operators have fractional scaling dimension and they possess relevant operators.

The original method of locating AD theories on the Coulomb branch of $\mathcal{N} = 2$ gauge theory has been generalized to $SU(2)$ with various flavors in [18], and it is further generalized to $\mathcal{N} = 2$ theory with general gauge group G and fundamental matter in [7,8,17]. The AD theory from pure $SU(n+1)$ gauge theory can be labeled as (A_1, A_n)

theory, those from $SO(2n)$ gauge theory correspond to (A_1, D_n), and finally those derived from E_n gauge theory are labeled as (A_1, E_n). These labels denote the shape of the BPS quiver of the corresponding SCFTs.

Over the past decade, there have been many exciting developments in the study of these strongly coupled SCFTs, including the BPS spectrum [11,19–24], central charges and RG flows [6], Alday-Gaiotto-Tachikawa duality [25], bootstrap [26], superconformal indices [27–29], S-dualities [30,31], etc.

B. Type IIB construction

We can engineer a large class of four-dimensional $\mathcal{N}=2$ SCFTs by considering type IIB string theory on an isolated hypersurface singularity in $\mathbb{C}^4$ defined by a polynomial,

$$W(x_1, x_2, x_3, x_4) = 0, \tag{5}$$

while decoupling gravity and stringy modes.

Without loss of generality we assume the singular point is the origin and the isolated singularity condition means $dW = 0$ if and only if $x_i = 0$. The holomorphic three form on the threefold singular geometry is given by

$$\Omega = \frac{dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}{dW}. \tag{6}$$

For the resulting four-dimensional theory to be superconformal, the necessary and sufficient conditions on W are the following [10,32]:

(1) There exists a $\mathbb{C}^*$ action on W: a collection of *positive* charges $\{q_i\}$ such that $W(\lambda^{q_i} x_i) = \lambda W(x_i)$. This is related to the scaling symmetry [or $U(1)_R$ symmetry] of the resulting four-dimensional $\mathcal{N}=2$ SCFT [33].

(2) The $\mathbb{C}^*$ charges have to satisfy the condition $\sum q_i > 1$ [34].

The full Seiberg-Witten geometry of the four-dimensional SCFT can be derived from the miniversal (universal deformation with minimal base dimension) deformations of the singularity which take the form [35]

$$F(x_i, \lambda_a) = W(x_i) + \sum_{a=1}^{\mu} \lambda_a \phi_a. \tag{7}$$

Here ϕ_a is the monomial basis of the local quotient algebra,

$$\mathcal{A}_W = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{\mathcal{J}_W}, \tag{8}$$

where

$$\mathcal{J}_W = \left\langle \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \frac{\partial W}{\partial x_3}, \frac{\partial W}{\partial x_4} \right\rangle \tag{9}$$

is the Jacobian ideal.

The complex structure deformations λ_a of the singularity correspond to the parameters on the Coulomb branch of the $\mathcal{N}=2$ four-dimensional SCFT. The Milnor number $\mu \equiv \mathrm{rank}\mathcal{A}_W$ associated with the singularity captures the rank of the BPS lattice. The BPS particles correspond to D3 branes wrapping special-Lagrangian cycles in the deformed threefold. Again one can define a three form $\Omega = \frac{dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}{dF}$, whose integral over a special-Lagrangian three cycle gives the mass of the BPS particle. Demanding the integral of Ω to have mass dimension 1, we deduce the scaling dimension of the deformation parameters as

$$[\lambda_a] = \alpha(1 - q(\phi_a)), \tag{10}$$

where $q(\phi_a)$ is the $\mathbb{C}^*$ charge of monomial ϕ_a and

$$\alpha = \frac{1}{\sum_{i=1}^4 q_i - 1}. \tag{11}$$

They capture the Coulomb branch parameters of the four-dimensional $\mathcal{N}=2$ SCFT.

Cecotti *et al.* constructed a large class of new AD theories by putting type IIB theory on the following special class of isolated hypersurface singularities [11]:

$$f_G(x_1, x_2) + f_{G'}(x_3, x_4) = 0. \tag{12}$$

Here $f_G(x, y)$ is a polynomial of the following types:

$$\begin{aligned} f_{A_k}(x,y) &= x^2 + y^{k+1}, \\ f_{D_k}(x,y) &= x^2 y + y^{k-1}, \\ f_{E_6}(x,y) &= x^3 + y^4, \\ f_{E_7}(x,y) &= x^3 + xy^3, \\ f_{E_8}(x,y) &= x^3 + y^5. \end{aligned} \tag{13}$$

These theories are called (G, G') theories, and the scaling dimensions of various operators have a common denominator [11],

$$\begin{aligned} r &= \frac{1}{4}\frac{h_G + h_{G'}}{\gcd(h_G/2, h_{G'}/2)}, \quad \text{for } G, G' = A_1, D_{2n}, E_7, E_8 \\ r &= \frac{h_G + h_{G'}}{\gcd(h_G, h_{G'})}, \quad \text{other cases.} \end{aligned} \tag{14}$$

By looking at the defining data of these theories and reorganizing the monomials parametrizing the deformations, one notices the following obvious equivalences among these theories:

$$(G, G') \sim (G', G),$$
$$(A_1, E_6) \sim (A_2, A_3), (A_1, E_8) \sim (A_2, A_4), \ldots \quad (15)$$

The BPS quiver for these theories is given by the direct product of the G and G' type Dynkin diagram. In particular, the dimension of the charge lattice Γ is

$$\dim \Gamma = 2n_c + n_f = \mathrm{rank}(G) \times \mathrm{rank}(G'). \quad (16)$$

Here n_c is the dimension of the Coulomb branch and n_f counts the number of mass parameters.

In addition to these (G, G') theories, one can engineer a large class of new $\mathcal{N} = 2$ SCFTs by classifying the isolated hypersurface singularity with a $\mathbb{C}^*$ action satisfying the conditions $\sum q_i > 1$ [36]; see [37,38] for earlier sporadic examples. We see below that some of them can also be engineered using M5 brane constructions [39].

The connection between the IIB string theory and M5 brane constructions is most transparent in the case of (A_{k-1}, A_{n-1}) theories [11,40]. In that case, IIB string theory on the singular threefold $x^k + z^n + y^2 + w^2 = 0$ is T-dual to the IIA NS5 brane wrapped on the singular algebraic curve $x^k + z^n = 0$ at $y = w = 0$. Lifting to M-theory, we have an M5 brane wrapping the same curve. The deformations $x^i z^j$ of the curve which describe the Coulomb branch of the AD theory are identified with complex structure deformations of the threefold singularity in the IIB picture. Although such explicit duality transformation is absent in general, we can still argue by comparing the derived Coulomb branch spectrum that a special class of IIB isolated singularities is related to M5 brane configurations.

Recently, there has been an attempt in [41,42] to classify $\mathcal{N} = 2$ rank one SCFTs using Kodaira's classification of degeneration of elliptic fibrations. It would be interesting to see if we can find new rank one theories using threefold singularities.

III. CLASSIFICATION OF IRREGULAR SINGULARITIES

A. *A* type irregular singularities

One can also engineer four-dimensional $\mathcal{N} = 2$ SCFTs by putting six-dimensional A_{N-1} (2, 0) theory on a Riemann surface $\mathcal{C}$ with regular (tame) or irregular (wild) singularities [3,12,43]. The SW curve Σ of the corresponding field theory can be identified with the spectral curve of the Hitchin system defined on $\mathcal{C}$ [44,45],

$$\det(x - \Phi) = 0 \to x^N + \sum_{i=2}^{N} \epsilon_i x^{N-i} = 0. \quad (17)$$

Here $\Phi \in H^0(\mathcal{C}, \mathrm{End}(E) \otimes K_{\mathcal{C}})$ is the Higgs field transforming as a holomorphic section of the bundle $\mathrm{End}(E) \otimes K_{\mathcal{C}}$ [46], and ϵ_i is the holomorphic section of the line bundle $K^i_{\mathcal{C}}$. Moreover, the Seiberg-Witten differential is just $\lambda = x dz$.

The singularity is characterized by the singular boundary condition of the Higgs field. In particular, the regular singularity means that the Higgs field has a first order pole,

$$\Phi = \frac{T}{z} + \cdots, \quad (18)$$

where we have suppressed the regular terms and T is a nilpotent element of the Lie algebra A_{n-1}. Using the gauge invariance of the Hitchin system, the regular punctures are classified by the nilpotent orbits which can be labeled by a Young tableau $[d_1, d_2, \ldots, d_k]$ [3]. See Fig. 1 for some examples [47].

One can decorate the Riemann surface $\mathcal{C}$ with an arbitrary number of regular singularities. The Coulomb branch chiral primaries of the resulting four-dimensional $\mathcal{N} = 2$ SCFT have integer scaling dimensions and therefore there are no relevant chiral primaries in their Coulomb branch spectrum.

To get an Argyres-Douglas theory, we need to use irregular singularities. This program has been implemented in [12] for six-dimensional A_{n-1} theory (see [13–15] for construction in the A_1 case). Due to the requirement of superconformal invariance, one can have only the following two scenarios [12]: (a) a single irregular singularity on $\mathbb{P}^1$; (b) an irregular singularity and a regular singularity on $\mathbb{P}^1$. The irregular singularities have been classified in [12], and they take the following forms:

$$\Phi = \frac{T}{z^{r+2}} + \cdots, \qquad r = \frac{j}{n}, \quad \text{type I},$$
$$\Phi = \frac{T}{z^{r+2}} + \cdots, \qquad r = \frac{j}{n-1}, \quad \text{type II},$$
$$\Phi = \frac{T_\ell}{z^\ell} + \cdots + \frac{T_1}{z} + \cdots,$$
$$T_\ell \subseteq T_{\ell-1} \subseteq \cdots \subseteq T_1, \quad \text{type III},$$

where we used the usual partial ordering of Young tableau T_i via containment or more generally the partial ordering of associated nilpotent orbits.

For type I and type II theories, the SW curve can be read from the Newton polygon which captures the leading order behavior of the singularity. Assume the singularity has the following form:

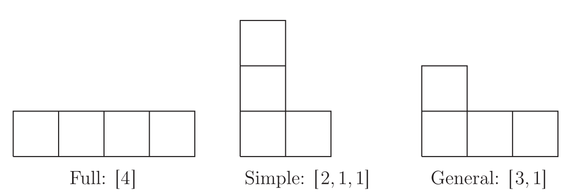

FIG. 1. Regular punctures of A_3 theories.

$$\Phi \sim \begin{pmatrix} B_1 & & \\ & \dots & \\ & & B_k \end{pmatrix}. \tag{19}$$

Here B_r are all diagonal and the order of pole satisfying the condition $r_1 < r_2 \cdots < r_k$. The size of those blocks sums up to N: $d_1 + \cdots + d_k = n$. The Newton polygon is depicted by starting with the point $(n, 0)$, and locating a point (a_i, b_i) such that the line connecting the above two points has slope r_i; next we find another point (a_{i-1}, b_{i-1}) such that the subsequent slope is r_{i-1} etc. See Fig. 2 for the Newton polygons of type I and type II singularities.

Once the Newton polygon is given, one can find out the full Seiberg-Witten curve Σ by enumerating the integral points contained in the Newton polygon, i.e. we associate to a marked point with coordinate (i, j) a monomial $x^i z^j$, and the SW curve is simply

$$\sum_{(i,j)\in S} u_{i,j} x^i z^j = 0, \tag{20}$$

where the coefficients label the parameters of the Coulomb branch of the AD theory.

One can find the scaling dimensions of these parameters by demanding that each term in (20) has the same scaling dimension and that the SW differential $\lambda = xdz$ has scaling dimension 1. Among the parameters of the physical theory, couplings are given by those with $[u_{i,j}] < 1$, Coulomb branch operators if $[u_{i,j}] > 1$ and masses if $[u_{i,j}] = 1$. For

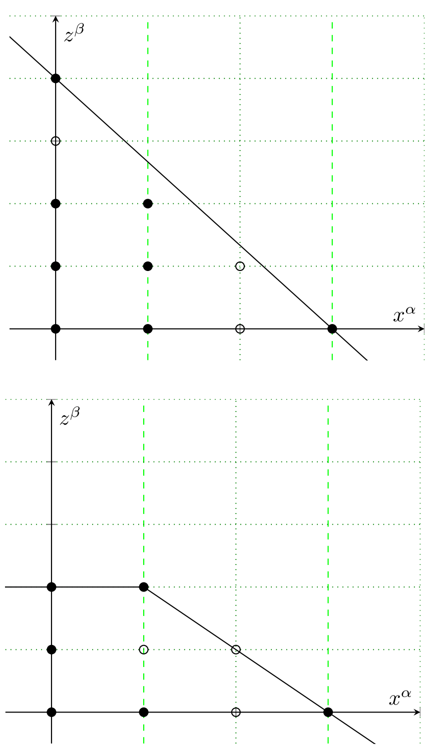

FIG. 2. Type I and II irregular singularities in A_2 theories.

type I and type II theory, some of the deformations are not allowed [50]; see Fig. 2 for the integer points labeled by the empty dots under the Newton polygon.

Now we provide a different justification for why there are only type I and type II irregular singularities. At the origin of the Coulomb branch moduli space, the SW curves for type I and type II theories are

$$\begin{aligned} x^n + z^k &= 0 \quad \text{type I}, \\ x^n + xz^k &= 0 \quad \text{type II}. \end{aligned} \tag{21}$$

The spectral curve of the A_{n-1} type Hitchin system may be written as the threefold form

$$x_1^2 + x_2^2 + x_3^n + \sum_{i=2}^{n} \epsilon_i(z) x_i^{n-i} = 0. \tag{22}$$

Here $\epsilon_i(z) \in K^i$ and is a polynomial in z. We would like to have an isolated singularity at the origin, and the only two possibilities are the following:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^n + z^k &= 0, \quad \text{type I}, \\ x_1^2 + x_2^2 + x_3^n + xz^k &= 0, \quad \text{type II}. \end{aligned} \tag{23}$$

Forgetting the first two quadratic terms which are rigid, we see that the classification of the irregular singularities in the A type Hitchin system boils down to, in the IIB perspective, two types of isolated threefold hypersurface singularities in (23) among the ones with the form of (22).

B. General case

Now we generalize the classification of irregular singularities to another type of six-dimensional (2, 0) theory labeled by a Lie algebra $J = D, E$. We still use z to denote the coordinate on $\mathbb{P}^1$ and declare that the Higgs field has the following form near $z = 0$:

$$\Phi = B + \cdots. \tag{24}$$

Here B is the singular term which can be put in the following block-diagonal form:

$$\Phi \sim \begin{pmatrix} B_1 & & \\ & \dots & \\ & & B_k \end{pmatrix} + \cdots \tag{25}$$

with the order of pole for various blocks ordered as $r_1 < r_2 < \cdots < r_k$.

For D type theory we use the fundamental representation, while for E type theory we use the adjoint representation for the polar matrices appearing in the above description. The physics should not depend on the representation we are using [51].

We determine what kind of combination of r_i and forms of B_i are needed to define a SCFT. This may be done using a similar method which has been used in [12] for the type A case.

However, given the correspondence between A type irregular singularities in the Hitchin system on $\mathcal{C}$ from the M5 brane perspective and isolated singularities of the form (23) in IIB geometry, we find it more convenient to generalize the classification of A type irregular singularities in the language of IIB isolated singularities.

To begin with, let us review some properties of the D type and E type Hitchin system. For D type theories, the SW curve looks like

$$x^{2n} + \sum_{i=1}^{n-1} \epsilon_{2i}(z) x^{2n-2i} + (\tilde{\epsilon}_n(z))^2 = 0. \tag{26}$$

Here $\epsilon_{2i} \in K^{2i}$ for $i = 1, \ldots, n-1$ and $\tilde{\epsilon}_n(z) \in K^n$. The novelty here compared to the type A discussion is that the term constant in x is constrained to be a perfect square. The coefficients in these differentials parametrize the Coulomb branch of the four-dimensional SCFT. The spectral curve for the E type Hitchin system is much more complicated due to constraints among the differentials. Here the important fact for us is that the independent invariant polynomials parametrizing the Coulomb branch are

$$\begin{aligned}
&E_6\colon\ \epsilon_2(z), \epsilon_5(z), \epsilon_6(z), \epsilon_8(z), \epsilon_9(z), \epsilon_{12}(z),\\
&E_7\colon\ \epsilon_2(z), \epsilon_6(z), \epsilon_8(z), \epsilon_{10}(z), \epsilon_{12}(z), \epsilon_{14}(z), \epsilon_{18}(z),\\
&E_8\colon\ \epsilon_2(z), \epsilon_8(z), \epsilon_{12}(z), \epsilon_{14}(z), \epsilon_{18}(z), \epsilon_{20}(z), \epsilon_{24}(z),\\
&\qquad \epsilon_{30}(z).
\end{aligned} \tag{27}$$

The above differentials are holomorphic sections of various line bundles $\epsilon_i(z) \in K^i$ over $\mathcal{C}$. To utilize the IIB description, one can put the SW curve in the threefold form [40],

$$\begin{aligned}
A_{n-1}\colon\ & x_1^2 + x_2^2 + x_3^n + \epsilon_2(z) x_3^{n-2} + \cdots + \epsilon_{n-1}(z) x_3 + \epsilon_n(z) = 0,\\
D_n\colon\ & x_1^2 + x_2^{n-1} + x_2 x_3^2 + \epsilon_2(z) x_2^{n-2} + \cdots + \epsilon_{2n-4}(z) x_2 + \epsilon_{2n-2}(z) + \tilde{\epsilon}_n(z) x_3 = 0,\\
E_6\colon\ & x_1^2 + x_2^3 + x_3^4 + \epsilon_2(z) x_2 x_3^2 + \epsilon_5(z) x_2 x_3 + \epsilon_6(z) x_3^2 + \epsilon_8(z) x_2 + \epsilon_9(z) x_3 + \epsilon_{12}(z) = 0,\\
E_7\colon\ & x_1^2 + x_2^3 + x_2 x_3^3 + \epsilon_2(z) x_2^2 x_3 + \epsilon_6(z) x_2^2 + \epsilon_8(z) x_2 x_3 + \epsilon_{10}(z) x_3^2\\
& + \epsilon_{12}(z) x_2 + \epsilon_{14}(z) x_3 + \epsilon_{18}(z) = 0,\\
E_8\colon\ & x_1^2 + x_2^3 + x_3^5 + \epsilon_2(z) x_2 x_3^3 + \epsilon_8(z) x_2 x_3^2 + \epsilon_{12}(z) x_3^3\\
& + \epsilon_{14}(z) x_2 x_3 + \epsilon_{18}(z) x_3^2 + \epsilon_{20}(z) x_2 + \epsilon_{24}(z) x_3 + \epsilon_{30}(z) = 0.
\end{aligned} \tag{28}$$

Now $\epsilon_i(z)$ are polynomials in z, and we would like to find out the choice of ϵ_i to turn on such that there is an isolated singularity at the origin.

Threefold hypersurface singularities of the form (28) are called compound Du Val (cDV) singularities in singularity theory [52].

It is straightforward to prove (see Appendix A for details) that the isolated quasihomogeneous threefold singularities of cDV type are precisely the ones listed in Table IV.

We find the form of the irregular singularity for the Higgs field Φ on $\mathcal{C}$ such that the leading order differential (which defines the singularity) is given by the terms listed in Table IV. In other words, we identify the Hitchin system that describes the same Coulomb branch spectrum of some $\mathcal{N} = 2$ SCFT as does the IIB singular geometry.

It is straightforward to check by explicitly comparing the Coulomb branch parameters from the spectral curve of the Hitchin system with those from the complex structure deformations of the threefold singularity (see Sec. III C 1 for an illustration in D type theories) that the Higgs field has the following singular form,

$$\Phi_z = \frac{T}{z^{2+k/b}} + \cdots, \tag{29}$$

where T is a regular semisimple element of the Lie algebra J and the possible values of b are listed in Table I which are in one to one correspondence with the degrees of the leading order differentials in Table IV [53]. Those are the irregular singularities that we focus on in this paper and we denote the resulting $\mathcal{N} = 2$ SCFT by $J^{(b)}[k]$. We summarize the connections between the AD theory and its various descriptions in Fig. 3.

When the allowed denominator b is taken to be the dual Coxeter number h of the corresponding Lie algebra J, the four-dimensional $\mathcal{N} = 2$ SCFT $J^{(h)}[k]$ engineered using this singularity corresponds to the (J, A_{k-1}) theory of [11] reviewed in the previous section. Moreover, most of the theories constructed in [37,38] using Arnold's unimodal and bimodal singularities [35] are also included in our construction [54].

Interestingly Table I also shows up in the discussion of *nondegenerate* [55] Hitchin systems with irregular

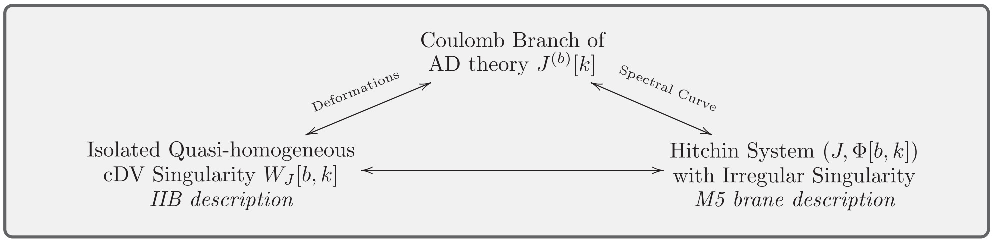

FIG. 3. Connection between AD theory, the cDV singularity, and the Hitchin system.

singularities for which the Higgs field has a leading polar matrix of the regular semisimple type (go to a covering space of the z plane if necessary) [56,57]. We start with an irregular singularity of the type

$$\Phi = \frac{T}{z^n} + \cdots, \quad n \in \mathbb{Z}, n > 1. \tag{30}$$

When T is already regular semisimple on the z plane, the story is straightforward and we do not have further constraints on T. On the other hand if T is not semisimple, for example nilpotent, it is well known that after lifting to a b-fold cover of the local patch around the singularity on the z plane by $z = u^b$, (24) is gauge equivalent to a local model with leading polar matrix T' semisimple and leading singularity of order $a + 1$ [57,58]. The local model written in terms of the original coordinate z is [59]

$$\Phi = \frac{T'}{z^{1+\frac{a}{b}}} + \cdots. \tag{31}$$

The ratio $s = a/b$ (we take b to be the minimal possible integer) is called the *slope* or Katz invariant associated with the local model of the Higgs field [57,60]. It can be shown using the relation between Higgs bundles and opers that the denominator b of s must always be a divisor of d_i for some $1 \leq i \leq \mathrm{rank}(J)$, which are degrees of the fundamental invariants of J (see Table III) [60]. Further imposing that T' is regular semisimple restricts the denominator b of the slope to take the values summarized in Table I [61]. For example, for A_n type theories, the possible slopes are simply $n + 1$ and n, which correspond to the type I and II theories described previously in [12].

TABLE III. Relevant Lie algebra data: h denotes the Coxeter number and $\{d_i\}$ are degrees of the fundamental invariants.

J	dim J	h	$\{d_i\}_{i=1,\ldots,\mathrm{rank}\,(J)}$
A_{n-1}	$n^2 - 1$	n	$2, 3, \ldots, n$
D_n	$n(2n-1)$	$2n-2$	$2, 4, \ldots, 2n-2; n$
E_6	78	12	2, 5, 6, 8, 9, 12
E_7	133	18	2, 6, 8, 10, 12, 14, 18
E_8	248	30	2, 8, 12, 14, 18, 20, 24, 30

1. Maximal irregular singularity

A special class of AD theories can be constructed from irregular singularity of the maximal type: namely the Higgs field behaves as

$$\Phi = \frac{T_\ell}{z^\ell} + \frac{T_{\ell-1}}{z^{\ell-1}} + \cdots + \frac{T_1}{z} + \cdots. \tag{32}$$

Here T_i are in regular semisimple orbits of J. This corresponds to the case where k is a positive integer multiple of b in (31). The dimension of the Coulomb branch is [56]

$$\text{dim Coulomb} = \frac{\ell(\dim J - \mathrm{rank}\, J)}{2} - \dim J. \tag{33}$$

The number of mass parameters of this theory is $n_f = \mathrm{rank}\, J$, and therefore the dimension of the BPS charge lattice is

TABLE IV. Isolated quasihomogeneous cDV singularities.

	Singularity	Leading order differential	Label
A_{n-1}	$x_1^2 + x_2^2 + x_3^n + z^k = 0$	$\epsilon_n = z^k$	$A_{n-1}^{(n)}[k]$
	$x_1^2 + x_2^2 + x_3^n + z^k x_3 = 0$	$\epsilon_{n-1} = z^k$	$A_{n-1}^{(n-1)}[k]$
D_n	$x_1^2 + x_2^{n-1} + x_2 x_3^2 + z^k = 0$	$\epsilon_{2n-2} = z^k$	$D_n^{(2n-2)}[k]$
	$x_1^2 + x_2^{n-1} + x_2 x_3^2 + z^k x_3 = 0$	$\tilde{\epsilon}_n = z^k$	$D_n^{(n)}[k]$
E_6	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	$\epsilon_{12} = z^k$	$E_6^{(12)}[k]$
	$x_1^2 + x_2^3 + x_3^4 + z^k x_3 = 0$	$\epsilon_9 = z^k$	$E_6^{(9)}[k]$
	$x_1^2 + x_2^3 + x_3^4 + z^k x_2 = 0$	$\epsilon_8 = z^k$	$E_6^{(8)}[k]$
E_7	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$	$\epsilon_{18} = z^k$	$E_7^{(18)}[k]$
	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k x_3 = 0$	$\epsilon_{14} = z^k$	$E_7^{(14)}[k]$
E_8	$x_1^2 + x_2^3 + x_3^5 + z^k = 0$	$\epsilon_{30} = z^k$	$E_8^{(30)}[k]$
	$x_1^2 + x_2^3 + x_3^5 + z^k x_3 = 0$	$\epsilon_{24} = z^k$	$E_8^{(24)}[k]$
	$x_1^2 + x_2^3 + x_3^5 + z^k x_2 = 0$	$\epsilon_{20} = z^k$	$E_8^{(20)}[k]$

$$\begin{aligned}\dim\Gamma &= 2\dim\text{Coulomb} + n_f\\ &= \ell(\dim(J) - \operatorname{rank} J) - 2\dim(J) + \operatorname{rank} J\\ &= \operatorname{rank} J[(\ell-2)h(J) - 1].\end{aligned} \tag{34}$$

This corresponds to the $(J, A_{(\ell-2)h(J)-1})$ theory of [11].

We expect this class of theories to have a rich set of features among all AD theories (some of which we exhibit in Sec. V B 1). They typically have Higgs branches, large flavor symmetries and are likely to have three-dimensional mirror quiver gauge theories [12,64]. There exist exactly marginal operators in the Coulomb branch spectrum and the theory can undergo nontrivial S-duality transformations.

2. Degeneration of irregular singularities

Let us now consider a degeneration of the irregular singularity considered in the previous subsection.

$$\Phi = \frac{T_\ell}{z^\ell} + \frac{T_{\ell-1}}{z^{\ell-1}} + \cdots + \frac{T_1}{z} + \cdots, \tag{35}$$

which is specified by a sequence of semisimple elements: $\rho = \{T_1, T_2, \ldots, T_\ell\}$ of J. Previously we took all of these matrices from the regular semisimple orbits. In general, we could take them to be from other semisimple orbits, and the only constraints are [12,56]

$$T_1 \subseteq T_2 \subseteq \cdots \subseteq T_\ell. \tag{36}$$

For this type of puncture, the local contribution of the singularity ρ to the dimension of the Coulomb branch is

$$\dim_\rho \text{Coulomb} = \frac{1}{2}\sum_{i=1}^{\ell} \dim(T_i), \tag{37}$$

where $\dim(T_i)$ is the (complex) dimension of the corresponding orbit. Moreover, the number of the mass parameters in the resulting AD theory from this singularity is equal to the number of distinguished eigenvalues of T_1.

Some $\mathcal{N}=2$ SCFTs constructed using this type of singularity have exactly marginal operators and nontrivial S-duality. One example of such has been explored in [30].

C. Some explicit examples of AD theories

1. D type AD theories and the Newton polygon

The SW curve for the D type theory can be easily read off from the spectral curve of the Hitchin system,

$$x^{2n} + \sum_{i=1}^{n-1} \epsilon_{2i}(z) x^{2n-2i} + (\tilde{\epsilon}_n(z))^2 = 0. \tag{38}$$

There are two types of AD theories $D_n^{(n)}[k]$ and $D_n^{(2n-2)}[k]$ which correspond to the following singular SW curves,

$$\begin{aligned} x^{2n} + z^{2k} &= 0,\\ x^{2n} + z^k x^2 &= 0,\end{aligned} \tag{39}$$

with the SW differential $\lambda = xdz$. The Higgs field takes the following singular forms accordingly:

$$\begin{aligned}\Phi &= \frac{T}{z^{2+\frac{k}{n}}} + \cdots,\\ \Phi &= \frac{T}{z^{2+\frac{k}{2n-2}}} + \cdots.\end{aligned} \tag{40}$$

One can read off the scaling dimension of x and z by requiring that the SW differential has scaling dimension 1: $[x] + [z] = 1$. The full SW curve can be easily found from the Newton polygon; see Fig. 4.

Let us for illustration consider the example $D_4^{(6)}[5]$. We can write down the full SW curve using the Newton polygon (Fig. 4) as follows. We list monomials $x^\alpha z^\beta$ that correspond to filled dots in the Newton polygon. The half-filled dots on the x^0 axis indicate that we should only regard the square root of the corresponding monomial z^β as parametrizing independent deformations. This is due to the Pfaffian constraint. Hence we have

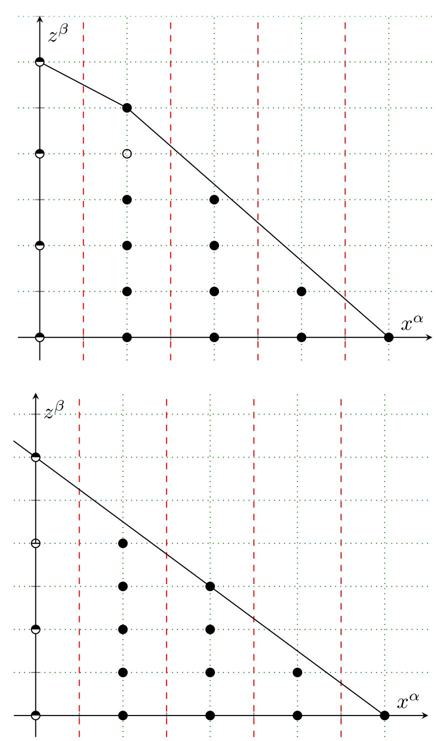

FIG. 4. Newton polygons for $D_4^{(6)}[5]$ and $D_4^{(4)}[6]$ theories.

$$\begin{aligned} &x^8 + x^6(u_{1,1}z + u_{1,0}) + x^4(u_{2,3}z^3 + u_{2,3}z^2 + u_{2,1}z + u_{2,0}) \\ &\quad + x^2(z^5 + u_{3,3}z^3 + u_{3,2}z^2 + u_{3,1}z + u_{3,0}) \\ &\quad + (\tilde{u}_3z^3 + \tilde{u}_2z^2 + \tilde{u}_1z + \tilde{u}_0)^2 = 0. \end{aligned} \tag{41}$$

From the scaling dimensions of x and z,

$$[x] = \frac{5}{11}, \qquad [z] = \frac{6}{11}, \tag{42}$$

we can read off the dimensions of the Coulomb branch parameters. In particular, there are no mass parameters and the chiral primaries have dimensions

$$\Delta_{\text{Coulomb}} = \left\{\frac{12}{11}, \frac{14}{11}, \frac{14}{11}, \frac{18}{11}, \frac{20}{11}, \frac{20}{11}, \frac{24}{11}, \frac{30}{11}\right\} \tag{43}$$

among which the relevant ones are paired with coupling constants as expected.

We can equivalently use the IIB description with the isolated hypersurface singularity

$$W(x,y,z,w) = w^2 + x^3 + xy^2 + z^5 = 0 \tag{44}$$

whose local quotient algebra is

$$\begin{aligned} \mathcal{A}_W = \{&1, x, y, z, xz, y^2, yz, z^2, xz^2, y^2z, yz^2, z^3, \\ &y^2z^2, xz^3, yz^3, y^2z^3\}. \end{aligned} \tag{45}$$

From the $\mathbb{C}^*$ charges of the coordinates

$$q_x = \frac{1}{3}, \quad q_y = \frac{1}{3}, \quad q_z = \frac{1}{5}, \quad q_w = \frac{1}{2} \tag{46}$$

and

$$\alpha = \frac{1}{\sum_{q_i} - 1} = \frac{30}{11} \tag{47}$$

we recover the same Coulomb branch spectrum as in (43).

2. An irregular singularity and a regular singularity

To build AD theories with generically non-Abelian flavor symmetries, we can consider an irregular singularity and a regular singularity on $\mathbb{P}^1$. The regular singularity is labeled by a nilpotent orbit of J [the Nahm description (or Higgs branch description) is better]. See [43] for various types of punctures. There are a variety of new theories by choosing different regular and irregular punctures.

If we take the irregular punctures with pole order denominator b given by the Coxeter number h of the Lie algebra J,

$$\Phi \sim \frac{T}{z^{2+k/h}} + \cdots, \tag{48}$$

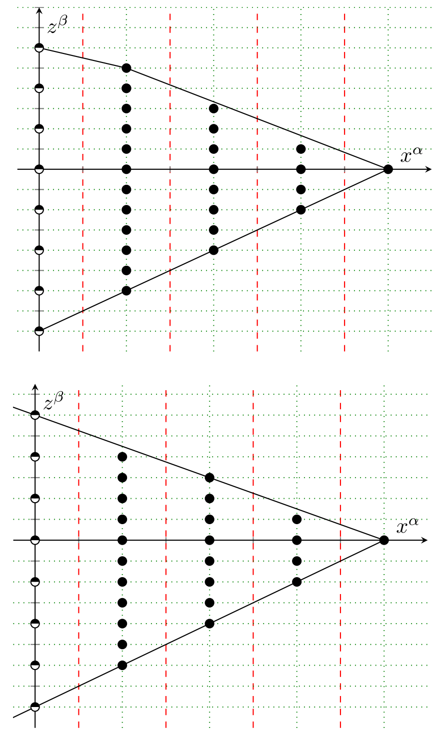

FIG. 5. Newton polygons for $(D_4^{(6)}[5], F)$ and $(D_4^{(4)}[6], F)$ theories.

and choose the full regular puncture, we can construct AD theories with non-Abelian J flavor symmetry which we denote by $(J^{(h)}[k], F)$ (see Fig. 5 for their Newton polygons). These theories are called $D_p(J)$ theories in [65,66] with $p = k + h$.

IV. TWISTED IRREGULAR SINGULARITY

If the underlying Lie algebra J has a nontrivial outer-automorphism group Out(J) (see Table V), it induces an automorphism on the Hitchin moduli space. Therefore, we may consider the projection onto Out(J) invariant configurations of the Higgs field Φ. This can be done locally at the singularities via introducing monodromy twist by an element $o \in \text{Out}(J)$,

$$\Phi(e^{2\pi i}z) = g[o(\Phi(z))]g^{-1}, \tag{49}$$

for some $g \in J/\mathfrak{g}^\vee$ (here $\mathfrak{g}^\vee$ is the invariant subalgebra of J), which we refer to as twisted singularities. Globally, the

TABLE V. Outer-automorphisms of simple Lie groups [67].

J	A_{2N}	A_{2N-1}	D_N	E_6	D_4
Automorphism Out(J)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
Invariant subalgebra $\mathfrak{g}^\vee$	B_N	C_N	B_{N-1}	F_4	G_2
Langlands dual $\mathfrak{g}$	C_N	B_N	C_{N-1}	F_4	G_2

twisted singularities must come in pairs connected by twist lines (or cuts).

A. Review of regular twisted singularities

Twisted singularities of the regular type from which one builds usual class S theories have been explored extensively in [43]. From Table V, we see a nontrivial o can have order 2 (for $J = A_{2n-1}, D_n, E_6$), or order 3 (for $J = D_4$) [68]. The Lie algebra J acquires a grading with respect to the eigenvalues of o,

$$J = \begin{cases} J_1 + J_{-1} & \text{for } |o| = 2 \\ J_1 + J_\omega + J_{\omega^2} & \text{for } |o| = 3, \end{cases} \tag{50}$$

where $\omega^3 = 1$ and we use the subscript on J to denote the eigensubspaces.

The local model for the Higgs field near the singularity takes the following form when o has order 2,

$$\Phi = \frac{T_1}{z} + \frac{U_1}{z^{1/2}} + T_0 + \cdots, \tag{51}$$

where T_1, $T_0 \in J_1 = \mathfrak{g}^\vee$ and $U_1 \in J_{-1}$. When the twist element o has order 3 with eigenvalue ω (third root of unity), we have instead

$$\Phi = \frac{T_1}{z} + \frac{U_1}{z^{2/3}} + \frac{W_1}{z^{1/3}} + T_0 + \cdots, \tag{52}$$

where T_1, $T_0 \in J_1 = \mathfrak{g}^\vee$, $U_1 \in J_\omega$ and $W_1 \in J_{\omega^2}$.

The Coulomb branch of the resulting four-dimensional SCFT receives local contributions from not only the Spaltenstein-dual nilpotent orbit $d(\mathcal{O}_\mathbf{p})$ in $\mathfrak{g}^\vee$ associated with the leading polar matrix T_1 [70], but also components of the subleading polar matrices U_1, W_1. Altogether the local contribution to the Coulomb branch has dimension [43]

$$\dim_\mathbf{p} \text{Coulomb} = \frac{1}{2}\dim_\mathbb{C} d(\mathcal{O}_\mathbf{p}) + \frac{1}{2}\dim J/\mathfrak{g}^\vee. \tag{53}$$

To obtain the total Coulomb branch dimension, we add up the local contributions from the singularities and the global contribution [71],

$$\dim \text{Coulomb} = \sum_i \dim_{\mathbf{p}_i} \text{Coulomb} + (g-1)\dim J, \tag{54}$$

where g is the genus of the Riemann surface $\mathcal{C}$.

On the other hand, the local contribution to the Higgs branch has quaternionic dimension [43],

$$\dim_\mathbf{p} \text{Higgs} = \frac{1}{2}(\dim \mathfrak{g} - \text{rank}\, \mathfrak{g} - \dim_\mathbb{C} \mathcal{O}_\mathbf{p}). \tag{55}$$

The total quaternionic dimension of the Higgs branch is given by

$$\dim \text{Higgs} = \sum_i \dim_{\mathbf{p}_i} \text{Higgs} + \text{rank}\, \mathfrak{g}^\vee. \tag{56}$$

B. Maximal twisted irregular singularities

We now extend the twisted singularities to the irregular type which can be achieved by decorating the irregular singularities considered previously with appropriate local monodromy twist $o \in \text{Out}(J)$. As explained before, demanding conformal invariance and o-invariance, we specialize to the case of one irregular twisted singularity and one regular twisted singularity on $\mathbb{P}^1$. Unlike the untwisted case, we do not have a classification for these twisted irregular singularities at the moment. Nonetheless we see a number of subclasses can already be constructed easily and have interesting features. We leave the general classification of twisted singularities that give rise to AD theories to a future publication. Since the IIB description for these twisted singularities is not known, it would also be interesting to figure out the corresponding IIB threefold singular geometry.

First let us consider the case where the irregular singularity is of the maximal type discussed in Sec. III B 1 with a $\mathbb{Z}_2$ twist. The local structure of irregular singularity is

$$\Phi = \frac{T_\ell}{z^\ell} + \frac{U_\ell}{z^{\ell-1/2}} + \frac{T_{\ell-1}}{z^{\ell-1}} + \frac{U_{\ell-1}}{z^{\ell-3/2}} + \cdots + \frac{T_1}{z} + \cdots, \tag{57}$$

where T_i are regular semisimple elements of $J_1 = \mathfrak{g}^\vee$ which is even under the $\mathbb{Z}_2$ twist and $U_i \in J_{-1}$ is odd. We denote the data defining the twisted irregular singularity collectively by $\tilde{\rho} = \{T_i, U_j | 1 \le i \le \ell, 2 \le j \le \ell\}$.

The local contribution to the Coulomb branch dimension can be obtained by studying the pole structure of the differentials ϵ_{d_i}. Expanded in z, if the leading singular term in ϵ_{d_i} that is not completely determined by the singular part of Φ (i.e. T_m for $1 \le m \le \ell$ and U_m for $2 \le m \le \ell$) has pole order p_{d_i}, the irregular singularity contributes $\sum_{i=1}^{\text{rank}\, J} p_{d_i}$ to the Coulomb branch dimension. Taking into account the local contribution from the regular twisted singularity (53), and the global contribution (now with $g = 0$ this is $-\dim J$), we obtain the total Coulomb branch dimension.

The number of distinctive eigenvalues of T_1 corresponds to the number of mass parameters contributed by the twisted irregular singularity, which is the rank of $\mathfrak{g}^\vee$ for the maximal case we consider here. We expect the local contribution to the Higgs branch also has quaternionic dimension rank $\mathfrak{g}^\vee$.

For example consider a $\mathbb{Z}_2$ twist of the ADE type Hitchin system with regular semisimple polar matrices T_m. The set of fundamental invariants splits under the action o,

$$\begin{aligned} A_{2n-1}o\colon\ & \epsilon_k \to (-1)^k \epsilon_k \quad \text{for } 2 \le k \le 2n \\ D_{n\neq 4}o\colon\ & \epsilon_{2i} \to \epsilon_{2i} \quad \text{for } 1 \le i \le n-1, \text{ and } \tilde{\epsilon}_n \to -\tilde{\epsilon}_n \\ E_6 o\colon\ & \epsilon_i \to \epsilon_k \quad \text{for } k = 2,6,8,12, \text{ and } \epsilon_k \to -\epsilon_k \quad \text{for } k = 5,9. \end{aligned} \tag{58}$$

We denote the invariant subset by s_1 and its complement by s_2. The local contribution from the irregular twisted singularity to the Coulomb branch dimension is

$$\begin{aligned} \dim_{\tilde{\rho}}\text{Coulomb} &= \sum_{i\in s_1} (d_i - 1)\ell + \sum_{i\in s_2} ((d_i - 1)\ell + 1/2) \\ &= \frac{\ell h(J)\text{rank}(J)}{2} + \frac{|s_2|}{2}, \end{aligned} \tag{59}$$

where $|s_2|$ is the size of the set s_2 and the ith summand gives the order of the highest pole in the differential ϵ_i whose coefficient is not purely determined by the polar matrices T_m and U_n from $\tilde{\rho}$ [56].

As for degenerations of the maximal twisted irregular singularity, we have the following *conjectured* formula, in analogy to the untwisted case, for counting the local contribution to the Coulomb branch dimension in terms of semisimple orbits of T_i in $\mathfrak{g}^\vee$ [72]:

$$\boxed{\dim_{\tilde{\rho}}\text{Coulomb} = \frac{1}{2}\left(\sum_{i=1}^{\ell} \dim T_i + \sum_{j=2}^{\ell} (\dim J/\mathfrak{g}^\vee - |s_2|) + \dim J/\mathfrak{g}^\vee\right).} \tag{60}$$

For T_i regular semisimple, we have $\dim T_m = (\dim \mathfrak{g}^\vee - \text{rank}\, \mathfrak{g}^\vee)/2$ and the above reduces to (59).

Therefore the total Coulomb branch dimension of the AD theory, constructed from a maximal twisted irregular singularity and a principal regular singularity, is

$$\begin{aligned} \dim \text{Coulomb} &= \frac{(\ell+1)h(J)\text{rank}(J)}{2} + |s_2| - \dim J \\ &= \frac{\text{rank}(J)((\ell-1)h(J) - 2)}{2} + |s_2|. \end{aligned} \tag{61}$$

On the other hand, the total Higgs branch quarternionic dimension of this AD theory is given by

$$\dim \text{Higgs} = \frac{1}{2}(\dim \mathfrak{g} - \text{rank}\, \mathfrak{g}) + \text{rank}\, \mathfrak{g}^\vee. \tag{62}$$

The detailed Coulomb branch spectrum can be obtained from the SW curve as before. Below we take $J = D_n$ for illustration. The singular SW curve, up to transformations that fix xdz, takes the form [73]

$$x^{2n} + x^2 z^{2(n-1)(\ell-2)} = 0, \tag{63}$$

which fixes the scaling dimensions

$$[x] = \frac{\ell-2}{\ell-1}, \qquad [z] = \frac{1}{\ell-1}. \tag{64}$$

The crucial difference from the untwisted cases in the previous sections is that among the deformations of the singular SW curve, the Pfaffian $\tilde{\epsilon}_n$ is constrained to have half integer powers of z. With this in mind, we can quickly enumerate the Coulomb branch operators, in particular, the total number of them is, from ϵ_{2i} and $\tilde{\epsilon}_n$,

$$\begin{aligned} \dim \text{Coulomb} &= \sum_{i=1}^{n-1} (2i(\ell-1) - \ell) + (n-1)(\ell-1) \\ &= (n-1)(n(\ell-1) - 1), \end{aligned} \tag{65}$$

which agrees with (61). Furthermore, we have $2n-2$ mass parameters: $n-1$ of them correspond to the Casmirs of $\text{USP}(2n-2)$ flavor symmetry and the other $n-1$ of them come from the irregular singularity. From (62), the total Higgs branch quarternionic dimension is

$$\begin{aligned} \dim \text{Higgs} &= \frac{1}{2}(\dim C_{n-1} - \text{rank}\, C_{n-1}) + n - 1 \\ &= n(n-1). \end{aligned} \tag{66}$$

We discuss some details about the $\mathbb{Z}_2$ twisted regular singularities and the resulting AD theories in Appendix B. We leave the generalizations to the $\mathbb{Z}_2$ twist for A_{2n} theories and the $\mathbb{Z}_3$ twist for D_4 theories to the interested readers.

C. *D* type twisted irregular singularities

The generalization to cases with leading polar matrix *nilpotent* is straightforward for D type theories since the $\mathbb{Z}_2$ outer-automorphism of $SO(2n)$ can be identified with $O(2n)/SO(2n)$. In other words, the condition (49) becomes

$$\Phi(ze^{2\pi i}) = \tilde{g}\Phi(z)\tilde{g}^{-1}, \tag{67}$$

with $\tilde{g} \in O(2n)$ and $\det \tilde{g} = -1$.

The twisted version of D_n singularity with slope denominator $b = n$ takes the following form with k odd [74],

$$\Phi = \frac{1}{z^{2+\frac{k}{2n}}}\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & \omega^k & & & \\ & & -\omega^k & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \omega^{k(n-1)} \\ & & & & & -\omega^{k(n-1)} & 0 \end{pmatrix} + \cdots,$$
$$\omega^{2n} = 1 \tag{68}$$

with $O(2n)$ gauge transformation

$$\tilde{g} = \left(\begin{array}{c|c} 0 & I_{2n-2} \\ J_2 & 0 \end{array}\right), \tag{69}$$

where we defined I_m as the $m \times m$ identity matrix and $J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Alternatively we can also twist the irregular singularities with $b = 2n-2$, and obtain for arbitrary $k \in \mathbb{Z}$

$$\Phi = \frac{1}{z^{2+\frac{k}{2n-2}}}\begin{pmatrix} 0 & 0 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \omega^{k(n-2)} \\ & & & & & -\omega^{k(n-2)} & 0 \end{pmatrix},$$
$$\omega^{2n-2} = 1 \tag{70}$$

with $O(2n)$ gauge transformation

$$\tilde{g} = \left(\begin{array}{c|c|c} I_2 & 0 & 0 \\ \hline 0 & 0 & I_{2n-4} \\ \hline 0 & J_2 & 0 \end{array}\right) \tag{71}$$

for k odd and

$$\tilde{g} = \left(\begin{array}{c|c|c} J_2 & 0 & 0 \\ \hline 0 & 0 & I_{2n-4} \\ \hline 0 & I_2 & 0 \end{array}\right) \tag{72}$$

for k even.

For example consider the AD theory constructed from a twisted D_n singularity of the form (70) with $k = 1$ and a simple regular puncture whose pole structure is $\{1, \ldots, 1, 1/2\}$. The singular SW curve is

$$x^{2n} + x^2 z = 0, \tag{73}$$

which fixes

$$[x] = \frac{1}{2n-1}, \qquad [z] = \frac{2n-2}{2n-1}. \tag{74}$$

The AD theory has $n-1$ Coulomb branch operators with dimensions $2i/(2n-1)$ with $n \le i \le 2n-2$ and one mass parameter [75].

Similarly, we can start with a twisted D_n singularity of the form (68) with $k = 1$ and a simple regular puncture. The singular SW curve is [76]

$$x^{2n} + z = 0, \tag{75}$$

which fixes

$$[x] = \frac{1}{2n+1}, \qquad [z] = \frac{2n}{2n+1}. \tag{76}$$

The AD theory has $n-1$ Coulomb branch operators with dimensions $2i/(2n+1)$ with $n+1 \le i \le 2n-1$ and no mass parameters [77].

It is also straightforward to construct AD theories from the above twisted irregular singularities in the presence of twisted full regular singular singularities. See Fig. 6 for examples of the Newton polygons for these theories. Note that in contrast to those of the untwisted theories in Fig. 5, the x-independent monomials that would contribute to the SW curve now correspond to half-filled dots that have been shifted below by one unit because the Pfaffian invariant $\tilde{\epsilon}_n$ is odd under a $\mathbb{Z}_2$ twist.

V. MORE PROPERTIES OF AD THEORIES

A. Coulomb branch spectrum

As we have discussed in the previous sections, for D type theory it is straightforward to read off the spectrum of the AD theory from an irregular singularity and possibly an additional regular singularity using the Newton polygon. For E type theories, the spectral curve representation of the spectrum is rather redundant, in which case it is more convenient to use the IIB threefold singularity description whose complex structure deformations constitute the full Coulomb branch spectrum with no redundancy [78]. The scaling dimensions of the Coulomb branch parameters which correspond to *certain* coefficients in $\epsilon_i(z)$ can be read off easily following the procedure in Sec. II B. However, some of the deformation parameters have negative four-dimensional scaling dimension and correspond

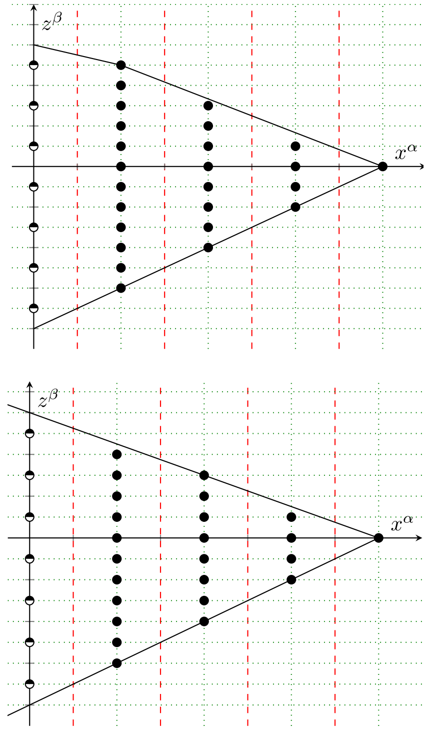

FIG. 6. Twisted D_4 theories with twisted full regular singularities.

to irrelevant couplings which must be removed from the list of physical Coulomb branch parameters.

To incorporate an additional regular singularity, we can simply allow $\epsilon_i(z)$ to have a pole in z according to the pole structure $\{p_{d_i}\}$ with $1 \leq i \leq \operatorname{rank} J$ associated to the regular singularity,

$$\epsilon_i(z) = \cdots + \frac{u_{i,1}}{z} + \frac{u_{i,2}}{z^2} \cdots + \frac{u_{i,p_{d_i}}}{z^{p_{d_i}}}. \tag{77}$$

The $u_{i,j}$'s are unconstrained Coulomb branch parameters whose scaling dimensions can again be easily fixed (all positive). Together with the parameters associated with the isolated threefold singularity in the absence of poles in $\epsilon_i(z)$, they make up the entire Coulomb branch spectrum.

B. Central charges

There are a number of ways to compute the central charges of the AD theories that we have constructed. Some of them are more useful than others depending on the input.

When there exists a weak coupling description, the central charges a, c are determined by [79]

$$2a - c = \frac{1}{4}\sum_i (2[u_i] - 1), \qquad a - c = \frac{1}{24}(n_v - n_h), \tag{78}$$

where $[u_i]$ denotes the scaling dimension of the Coulomb branch operator u_i, n_v counts the number of vector multiplets and n_h counts the number of hypermultiplets. More generally, when the theory has a Higgs branch and is completely Higgsed, we can rewrite the second equation as [6,12]

$$a - c = -\frac{\dim_{\mathbb{H}} \text{Higgs}}{24}. \tag{79}$$

For AD theories constructed from type J (2, 0) SCFTs on $\mathbb{P}^1$ with the integral pole at the irregular singularity, we expect $\dim_{\mathbb{H}} \text{Higgs} = \operatorname{rank} J$.

For strongly coupled theories, there is a formula for the central charges from topological field theories [79],

$$a = \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5r}{24} + \frac{h}{24}, \qquad c = \frac{R(B)}{3} + \frac{r}{6} + \frac{h}{12}, \tag{80}$$

where $R(A)$, $R(B)$ are the R-charges of path integral measure factors and r, h are the number of free vector multiplets and hypermultiplets at *generic* points of the Coulomb branch. For the theories we consider, r coincides with the rank of the Coulomb branch and h is 0. Moreover $R(A)$ can be expressed in terms of the scaling dimensions of the Coulomb branch operators,

$$R(A) = \sum_i ([u_i] - 1). \tag{81}$$

For generic strongly coupled $\mathcal{N} = 2$ SCFTs, it is difficult to compute $R(B)$. However we have the following formula for (G, G') theories from [66]:

$$R(B) = \frac{1}{4}\frac{r(G)r(G')h(G)h(G')}{h(G) + h(G')}. \tag{82}$$

This formula has an elegant extension for general isolated hypersurface singularities in the IIB description [36]. Given a IIB singular threefold defined by $W(x_1, x_2, x_3, x_4) = 0$ in $\mathbb{C}^3$ which has a $\mathbb{C}^*$ action with positive charges $\{q_i\}$, we can compute $R(B)$ by [36]

$$R(B) = \frac{\mu\alpha}{4}, \tag{83}$$

where μ is the Milnor number for W and

$$\alpha = \frac{1}{\sum_{i=4}^{r} q_i - 1} \tag{84}$$

is the scaling dimension of the constant deformation.

Since all of the AD theories from untwisted irregular singularity considered here have IIB description in terms of isolated hypersurface singularities, we can extract their a and c central charges from (83) and (80).

TABLE VI. Milnor numbers (BPS lattice dimension) for $J^{(b)}[k]$ theories.

J	$\mu(b_1)$	$\mu(b_2)$	$\mu(b_3)$
A_n	$n(k-1)$	$(n+1)k-n$	
D_n	$n(k-1)$	$(2n-2)k-n$	
E_6	$6(k-1)$	$8k-6$	$9k-6$
E_7	$7(k-1)$	$9k-7$	
E_8	$8(k-1)$	$10k-8$	$12k-8$

TABLE VII. Coulomb branch dimension r, number of marginal operators r_{marg} and relevant operators r_{rel} for $J^{(b)}[bm]$ theories.

J	r	r_{marg}	r_{rel}	n_f
A_n	$\frac{1}{2}n(m(n+1)-2)$	$n-1$ $(m>1)$ $n-2$ $(m=1)$	$mn-1$	n
$D_{n>3}$	$n(m(n-1)-1)$	$n-1$	$mn-1$	n
E_6	$6(6m-1)$	5	$6m-1$	6
E_7	$7(9m-1)$	6	$7m-1$	7
E_8	$8(15m-1)$	7	$8m-1$	8

As for the twisted theories, generally only the $2a-c$ anomaly can be obtained from the Coulomb branch spectrum and more techniques need to be developed in order to compute a and c separately, for example, three-dimensional mirror symmetry for the S^1 reduction of these AD theories would be a useful tool. Below we focus on the central charges for the untwisted theories (see Appendix B for examples in the twisted case).

It is known for cDV singularities of index n (i.e. cA_n, cD_n, cE_n) with the additional coordinate z that its Milnor number is given by [35]

$$\mu = n\left(\frac{1}{q(z)}-1\right), \tag{85}$$

where $q(z)$ is the $\mathbb{C}^*$ charge of z. For example, for $b=h$ the Coxeter number, the Milnor number is simply

$$\mu = n(k-1). \tag{86}$$

In general we have $q(z)=\frac{b}{hk}$ and $\alpha=\frac{hk}{b+k}$; thus

$$\mu = n\left(\frac{hk}{b}-1\right), \tag{87}$$

(See Table VI for a list) which leads to

$$R(B)=\frac{nhk(hk-b)}{4b(b+k)} \tag{88}$$

using (83).

1. Examples from the maximal irregular singularities

In general it is straightforward to compute $R(A)$ using the Coulomb branch spectrum obtained from either the spectral curve or the associated threefold singularity. Although a closed form expression of $R(A)$ for general $J^{(b)}[k]$ is not available at present, for the special subclass of theories $J^{(b)}[bm]\cong(J,A_{hm-1})$ with $m\in\mathbb{Z}^+$ which originate from maximal irregular singularities introduced in Sec. III B 1, the problem is vastly simplified.

Since $J^{(b)}[bm]$ does not depend on the (allowed) choice of b up to marginal deformations, its Coulomb branch spectrum is captured uniformly by a single spectral curve or its corresponding threefold cDV singularity for given J and positive integer m. In particular, following the procedure outlined in the previous section, it is easy to see that the Coulomb branch of $J^{(b)}[bm]$ has dimension $n(mh/2-1)$ among which there are $n-1$ marginal operators [80] and $mn-1$ relevant operators (see Table VII). Moreover, from the Coulomb branch spectrum one can compute a and c central charges which we record in Table VIII.

2. Limits of the central charges

Here we consider various large parameter limits of the $J^{(b)}[k]$ theories and obtain the asymptotic behaviors for

TABLE VIII. Central charges a and c for $J^{(b)}[bm]$ theories.

J	a	c
A_n	$\frac{n(m(2m(n+1)(n+2)-5)-5)}{24(m+1)}$	$\frac{n(m(m(n+1)(n+2)-2)-2)}{12(m+1)}$
D_n	$\frac{n(m(4m(n-1)(2n-1)-5)-5)}{24(m+1)}$	$\frac{n(m(m(n-1)(2n-1)-1)-1)}{6(m+1)}$
E_6	$\frac{312m^2-5m-5}{4(m+1)}$	$\frac{78m^2-m-1}{m+1}$
E_7	$\frac{7(684m^2-5m-5)}{24(m+1)}$	$\frac{7(171m^2-m-1)}{6(m+1)}$
E_8	$\frac{5(372m^2-m-1)}{3(m+1)}$	$\frac{4(465m^2-m-1)}{3(m+1)}$

central charges a and c. In particular, we see that $a = c$ in these limits.

Let us start with the limit $k \to \infty$ with n being finite. In this limit we have

$$\alpha = \frac{hk}{b+k} \sim h, \qquad r \sim \frac{\mu}{2} = \frac{nhk}{2b}, \tag{89}$$

which gives

$$R(B) = \frac{\mu\alpha}{4} \sim \frac{nh^2 k}{4b}. \tag{90}$$

The Coulomb branch spectrum (in this limit $[x] \sim 1$, $[z] \sim b/k$) from the spectral curve is given by

$$\left\{ d_i - \frac{bj}{k} | 1 \leq i \leq n, j \geq 1 \text{ such that } d_i - \frac{bj}{k} > 1 \right\} \tag{91}$$

where d_i's are the degrees of fundamental invariants in Table III. From (91) we can derive

$$R(A) = \sum_\alpha ([u_\alpha] - 1) \sim \sum_{i=1}^{n} \left((d_i - 1)(b_i + 1) - \frac{b_i(b_i+1)}{2}\frac{b}{k} \right), \tag{92}$$

where b_i counts the number of Coulomb branch operators from the invariant differential ϵ_i. In the limit $k \to \infty$, we have

$$b_i \sim (d_i - 1)\frac{k}{b}, \tag{93}$$

which implies

$$R(A) \sim \frac{k}{2b} \sum_{i=1}^{n} (d_i - 1)^2 = \frac{knh(2h-1)}{12b} \tag{94}$$

where we have used the Lie algebra identities

$$\sum_{i=1}^{n} d_i = \frac{1}{2} n(h+2), \qquad \sum_{i=1}^{n} d_i^2 = \frac{1}{6}(2h^2 + 5h + 6)n. \tag{95}$$

Therefore the central charges a and c are determined to be

$$a = c = \frac{knh(h+1)}{12b}. \tag{96}$$

In particular, for the maximal slope $b = h$, we have

$$a = c = \frac{kn(h+1)}{12}. \tag{97}$$

Next let us inspect the $n \to \infty$ limit with k being finite in the A and D type theories. For $D_n^{2n-2}(k)$ and $D_n^n(k)$ theories, by studying the deformations of the spectral curves, we have in this limit

$$\begin{aligned} \text{for } D_n^{(2n-2)}[k]_{n\to\infty} \quad & a = c = \frac{(k^2-1)n}{12}, \\ \text{for } D_n^{(n)}[k]_{n\to\infty} \quad & a = c = \frac{(4k^2-1)n}{12}. \end{aligned} \tag{98}$$

Similarly for $A_n^{n+1}(k)$ and $A_n^n(k)$ theories, we have [81]

$$\begin{aligned} \text{for } A_n^{(n+1)}[k]_{n\to\infty} \quad & a = c = \frac{(k^2-1)n}{12}, \\ \text{for } A_n^{(n)}[k]_{n\to\infty} \quad & a = c = \frac{(k^2-1)n}{12}. \end{aligned} \tag{99}$$

Finally we consider theories $A_n^{(b)}[bm]$ and $D_n^{(b)}[bm]$ in the $n \to \infty$ limit (thus $b \to \infty$) with m being finite [which corresponds to taking k and $\mathrm{rank}(J)$ large with their ratio fixed]. The central charges in this limit can be easily read off from Table VIII,

$$\begin{aligned} \text{for } A_n^{(b)}[bm]_{n\to\infty} \quad & a = c = \frac{m^2 n^3}{12(m+1)}, \\ \text{for } D_n^{(b)}[bm]_{n\to\infty} \quad & a = c = \frac{m^2 n^3}{3(m+1)}. \end{aligned} \tag{100}$$

VI. CONCLUSION AND DISCUSSIONS

Using M5 branes, we have constructed a large class of new $\mathcal{N} = 2$ SCFTs by classifying the irregular punctures. We have also given the corresponding threefold hypersurface singularities in the IIB description. Along the way, we have established a map between the irregular singularities of the Hitchin system, Argyres-Douglas theories, and isolated hypersurface singularities (see Fig. 3). The main purpose of this paper is to give a classification of the possible theories within this construction, and there are many other interesting questions about these theories that one can study.

Some of the theories (e.g. from maximal irregular singularities) constructed here have exact marginal deformations, and one question is to identify the corresponding duality group. It is expected that one can find many weakly coupled gauge theory descriptions, and it is interesting to study them systematically (see [30] for some examples).

A special subclass of our theories, labeled by $(J^{(b)}[k], F)$ with $k = -b + 1$, is rigid matter in the sense that it does not contain any Coulomb branch moduli but has full flavor symmetry J [82]. By gauging a diagonal subgroup of the flavor symmetries of two such rigid matter systems, one may generate $\mathcal{N} = 2$ asymptotically free gauge theories.

With the outer-automorphism twist, we can construct $\mathcal{N}=2$ asymptotically free gauge theories for arbitrary gauge group G this way [83].

The RG flows between the AD theories have been explored in [6] for A type theories by considering various deformations of the singular SW curve. In terms of the IIB isolated threefold singularities, these flows are captured by the so-called adjacency relations [84] between different singular varieties. In particular, some of the adjacency relations among Arnold's simple singularities (which correspond to cDV singularities labeled by $J^{(h)}[2]$) were realized explicitly by RG flows in [6]. We expect a similar relation between adjacency relations among the threefold singularities considered in this paper (Table II) and the RG flows among the corresponding AD theories.

It is interesting to study various partition functions of these theories, and we expect that our M5 brane construction will be quite useful. In particular, we expect that the two point function (with insertions of operators corresponding to our irregular punctures) of the two-dimensional Toda theory will give the S^4 partition function [85]. Similarly, the two point function of the q-deformed Yang-Mills theory would give the superconformal (Schur) index [27] (see [28,29] for recent results on A_1 type AD theory). Once the index is obtained, it is interesting to find the corresponding chiral algebra [28,29].

For A type theory whose irregular singularity has integer order poles, one can compactify the theory on a circle to get a three-dimensional $\mathcal{N}=4$ SCFT. The mirror for these theories has been written down in [12,64,86] and they are all Lagrangian quiver gauge theories. We expect that the new theories engineered here using integer order pole irregular singularity (the maximal irregular singularities and their degenerations) will also have three-dimensional mirrors, and it would be interesting to develop a systematic identification.

For our theories $J^{(b)}[k]$, their central charges satisfy the condition $a=c$ in the large k or large $\mathrm{rank}(J)$ limit, and this indicates that the theories may have supergravity duals [87]. It would be very interesting to derive the supergravity dual explicitly.

As we have briefly mentioned, for the same threefold isolated quasihomogeneous singularity, had we kept the string scale finite while decoupling gravity, we would have ended up with a four-dimensional nongravitational string theory [known as little string theory (LST)] whose low energy limit gives the four-dimensional $\mathcal{N}=2$ SCFT [88–91]. There we have another holographic picture in terms of type II string theory on linear dilaton background with an $\mathcal{N}=2$ Landau-Ginzburg (LG) sector which is well defined even at finite k and $\mathrm{rank}(J)$. It would be very interesting to understand what kind of dynamics in the four-dimensional $\mathcal{N}=2$ AD theory (as a low energy sector of the full LST) we can learn from the bulk string theory description, possibly in a double-scaled limit (to cap off the dilaton throat) along the lines of [92–96].

Finally, the irregular singularities of the Hitchin system considered here are codimension-two half-BPS defects of the six-dimensional (2, 0) theory. Upon compactification on T^2 longitudinal to the defect, we obtain a half-BPS surface operator in four-dimensional $\mathcal{N}=4$ SYM [43]. It would be interesting to study the surface operators obtained this way from our irregular singularities, especially the ones with outer-automorphism twist, in relation to the geometric Langlands program [56,57].

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APPENDIX A: ISOLATED QUASIHOMOGENEOUS CDV SINGULARITIES

In this section, we prove that the isolated quasihomogeneous cDV singularities defined by

$$W_J(x_1,x_2,x_3,z)=f_J(x_1,x_2,x_3)+zg(x_1,x_2,x_3,z), \tag{A1}$$

with $J=A$, D, E, are precisely those listed in Table IV.

First, it is easy to check that the quasihomogeneous singularities in Table IV all have finite Milnor numbers thus isolated.

A necessary condition for a general quasihomogeneous singularity $W(x_i)=0$ to be isolated is that for any axis x_i there must be at least one monomial $\prod_j x_j^{k_j}$ in $W(x_i)$ such that $\sum_j k_j - k_i \le 1$; otherwise there will be a singular locus along the x_i axis [35].

Now given a quasihomogeneous cDV singularity defined by W_J in (A1) [in particular, $q(x_i)$, $q(z)>0$], $W_J(x_i,z)$ must contain a monomial(s) from the set $L=\{z^k, z^k x_1, z^k x_2, z^k x_3\}$ with $k\ge 1$ to avoid a singular locus along the z axis.

(1) For $W_J(x_i,z)$ of cA_n, cE_6 and cE_8 types, up to coordinate redefinitions, such a $W_J(x_i,z)$ is always captured by the normal forms (or their marginal deformations) in Table IV.

(2) For $W_J(x_i,z)$ of cD_n type, if $W_J(x_i,z)$ contains any of the three monomials z^k, $z^k x_1$, $z^k x_3$, up to a coordinate transformation, such a $W_J(x_i,z)$ is

captured by the two normal forms (or their marginal deformations) in Table IV. However if $W_J(x_i, z)$ only contains the $z^k x_2$ monomial from the set L, then up to a coordinate transformation, we may assume that the z dependent monomials of $W_J(x_i, z)$ are all of the form $z^i x_2^j$ with $i,\ j \geq 1$. Consequently, we have a singular locus along $x_3^2 + z^k = x_2 = x_1 = 0$.

(3) For $W_J(x_i, z)$ of cE_7 type, if $W_J(x_i, z)$ contains any of the three monomials z^k, $z^k x_1$, $z^k x_3$, up to a coordinate transformation, such a $W_J(x_i, z)$ is captured by the two normal forms (or their marginal deformations) in Table IV. However if $W_J(x_i, z)$ only contains the $z^k x_2$ monomial from the set L, then up to a coordinate transformation, we may assume that the z dependent monomials of $W_J(x_i, z)$ are all of the form $z^i x_2^j$, $z^i x_2^j x_3$ or $z^i x_3^2$ with $i,\ j \geq 1$. If $W_J(x_i, z)$ does not contain monomials of the form $z^i x_3^2$, we again end up with a singular locus along $x_3^3 + z^k = x_2 = x_1 = 0$; otherwise the $\mathbb{C}^*$ charges $q(x_2) = 1/3$ and $q(x_3) = 2/9$ demand a term of the form $t^{6m} x_2 + z^{5m} x_3^2$ in $W_J(x_i, z)$ for some $m \in \mathbb{Z}^+$ (i.e. $k \in 6\mathbb{Z}$), in which case $W_J(x_i, z)$ is simply a marginal deformation of $f_{E_7}(x_i, z) + z^{9m}$.

Hence we have completed the proof.

APPENDIX B: EXAMPLES OF AD THEORIES FROM TWISTED IRREGULAR SINGULARITIES

For the AD theory engineered using a D type maximal twisted singularity and another full regular twisted singularity considered in Sec. IV B with singular SW curve (63), the dimensions of the Coulomb branch operators are

$$\left\{2i - \frac{k}{\ell - 1} | 1 \leq i \leq n-1, k \geq 1 \right.$$
$$\left. \text{such that}\, 2i - \frac{k}{\ell - 1} > 1 \right\} \tag{B1}$$

from ϵ_{2i} and

$$\left\{ n - \frac{2k+1}{2(\ell - 1)} | k \geq 0 \,\text{such that}\, n - \frac{2k+1}{2(\ell-1)} > 1 \right\} \tag{B2}$$

from $\tilde{\epsilon}_n$. Hence we have from (78)

$$\begin{aligned} 2a - c &= \frac{1}{4} \sum_j (2[u_j] - 1) \\ &= \frac{1}{12}(n-1)n(4(\ell-1)n - 2\ell - 1) \end{aligned} \tag{B3}$$

and from (62) and (79)

$$a - c = -\frac{n(n-1)}{24}, \tag{B4}$$

allowing us to determine a and c for this class of AD theories,

$$\begin{aligned} a &= \frac{1}{24}(n-1)n(8(\ell-1)n - 4\ell - 1), \\ c &= \frac{1}{6}(n-1)n(2(\ell-1)n - \ell). \end{aligned} \tag{B5}$$

It is straightforward to repeat the above analysis for twisted A_{2n-1} and E_6 theories from an irregular twisted singularity with regular semisimple polar matrices T_i (in $\mathfrak{g}^\vee$) and a regular twisted puncture of general type. Suppose the pole structure associated with the regular twisted singularity is denoted by $\{p_{d_i}\}$ with $1 \leq i \leq \text{rank}(J)$; then the Coulomb branch spectrum is given by

$$\left\{ p_{d_i} + 1 - \frac{k}{\ell - 1} | d_i \in s_1,\ k \geq 1 \ \text{such that}\ p_{d_i} - \frac{k}{\ell - 1} > 0 \right\} \tag{B6}$$

from the $\mathbb{Z}_2$ invariant differentials and

$$\left\{ p_{d_i} + \frac{1}{2} - \frac{2k+1}{2(\ell-1)} | d_i \in s_2, k \geq 0 \right.$$
$$\left. \text{such that}\, p_{d_i} + \frac{1}{2} - \frac{2k+1}{2(\ell-1)} > 1 \right\} \tag{B7}$$

from the $\mathbb{Z}_2$ odd differentials. In addition, we have $2|s_1| = 2(r - |s_2|)$ mass parameters, half of which correspond to the Casmirs of either $SO(2n+1)$ or F_4 flavor symmetry (the other half are associated with the twisted irregular singularity). The Higgs branch quarternionic dimensions are conjectured to be $n(n+1)$ for the twisted A_{2n-1} theory and $(52-4)/2 + 4 = 28$ for twisted E_6 theory.

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[34] If in the decoupling limit, we send the string coupling $g_s \to 0$ while keeping the string scale ℓ_s fixed, we end up with a nongravitational four-dimensional LST whose holographic dual is described by type II string theory in the background $\mathbb{R}^{3,1}\times\mathbb{R}_\phi\times(S^1\times LG(W))/\Gamma$ with a suitable Gliozzi-Scherk-Olive projection Γ that acts as an orbifold to ensure four-dimensional $\mathcal{N}=2$ spacetime supersymmetry [88–90]. Here $\mathbb{R}_\phi$ denotes the $\mathcal{N}=1$ linear dilaton SCFT with dilaton profile $\varphi=-\frac{Q}{2}\phi$ and $LG(W)$ the $\mathcal{N}=2$ LG theory with four chiral superfields x_i and superpotential $W(x_i)$. In particular, the $\mathbb{C}^*$ action on x_i is identified with the $U(1)_R$ symmetry in the LG model. In the low energy limit ($\ell_s\to 0$), we recover the four-dimensional SCFT from the LST. Now the $\mathcal{N}=1$ linear dilaton theory has central charge $3(1/2+Q^2)$, whereas the $\mathcal{N}=2$ LG model has central charge $3\sum_i(1-2q_i)$. Consistency of the type II string theory on this background requires the worldsheet theory to have a total central charge of 15 which implies $Q^2=2(\sum_i q_i-1)$; thus $\sum_i q_i-1>0$.
[35] V. I. Arnold, A. Varchenko, and S. Gusein-Zade, *Singularities of Differentiable Maps: Volume I: The Classification of Critical Points Caustics and Wave Fronts* (Springer Science & Business Media, New York, 2012), Vol. 82.
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[37] S. Cecotti and M. Del Zotto, J. High Energy Phys. 10 (2011) 099.
[38] M. Del Zotto, J. High Energy Phys. 11 (2011) 115.
[39] In fact, as we see, most of the theories in [37,38] turn out to have M5 brane constructions.
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[44] N. J. Hitchin, Proc. London Math. Soc. **s3-55**, 59 (1987).
[45] N. J. Hitchin, Duke Math. J. **54**, 91 (1987).
[46] The Higgs field Φ has origin in the five-dimensional $\mathcal{N}=2$ SYM from S^1 reduction of the six-dimensional (2, 0) type J theory. Upon twist compactification of the five-dimensional maximal super-Yang-Mills on $\mathcal{C}$, a natural principal bundle E arises on $\mathcal{C}$ with a structure group whose Lie algebra is J. Moreover two of the five five-dimensional scalars combine and become the holomorphic section of the bundle $\mathrm{End}(E)\times K_{\mathcal{C}}$ which is just Φ.
[47] We are following the convention of [43,48] here. The rows of the Young tableau correspond to Hitchin partitions which are related to the Nahm partitions via the Spaltenstein map [43,49].
[48] O. Chacaltana, J. Distler, and A. Trimm, J. High Energy Phys. 04 (2015) 173.

[49] D. H. Collingwood and W. M. McGovern, *Nilpotent Orbits in Semisimple Lie Algebra: An Introduction* (CRC Press, 1993).

[50] This happens when it is a redundant deformation due to coordinate redefinition(s) keeping $\lambda = xdz$ fixed, or because of trace relations for the differentials in the spectral curve.

[51] One needs to be careful about the constraints among the differentials appearing in the spectral curve when working with a general representation.

[52] The cDV singularities are a special class of threefold singularities defined by

$$W_J(x_1, x_2, x_3, z) = f_J(x_1, x_2, x_3) + zg(x_1, x_2, x_3, z) = 0,$$

where f_J is the usual Du Val singularity of $J = A, D, E$ type and $g(x_1, x_2, x_3, z)$ is an arbitrary polynomial. In the case in which $g(x_1, x_2, x_3, z) = z$, the cDV singularities reduce to threefold Du Val singularities. Demanding that the $\mathbb{C}^*$ charge of z satisfy $q(z) > 0$, using coordinate redefinitions, we can put any cDV singularity into the form of (28).

[53] Up to conjugation, J has infinitely many regular semisimple elements. Here the choice of T is constrained by b to ensure gauge invariance across the branch cuts on the z plane.

[54] The only exceptions are Z_{12}, S_{12} in the unimodal case, and Z_{18}, Q_{17}, S_{16} for the bimodal case.

[55] This notion of nondegeneracy is consistent with the requirement of isolated singularities that gives rise to SCFTs, which restricts to the case of a single block in (25).

[56] E. Witten, arXiv:0710.0631.

[57] E. Frenkel and B. Gross, arXiv:0901.2163 [Ann. Math. (to be published)].

[58] As an example [57], let us fix a Borel subalgebra of J and denote the Lie algebra element corresponding to the root α by X_α. We then define nilpotent elements $N = \sum_{\alpha\in\Delta} X_\alpha$ and $E = X_{-\theta}$ where Δ is the set of simple roots and θ is the longest (positive) root. The following local model for the Higgs field,

$$\Phi \sim \frac{E}{z^2} + \frac{N}{z} + \cdots,$$

is gauge equivalent to (after projecting down to the z plane)

$$\Phi \sim h\frac{E + N}{z^{1+1/h}} + \cdots,$$

where $E + N$ is regular semisimple and h is the Coxeter number.

[59] The presence of branch cuts on the z plane indicates nothing but a wrong choice of complex structure. Recall that the Hitchin equations are also invariant under nonholomorphic gauge transformations which we can make use of to remove the branch cuts [56].

[60] T.-H. Chen and M. Kamgarpour, arXiv:1404.0598.

[61] This follows from a classification result in the work of Springer [62] and later Kac *et al.* [63]. More explicitly, the nonsingle-valuedness of the Higgs field around the origin on the z plane demands a nontrivial gauge transformation g across the branch cuts,

$$\Phi(ze^{2\pi i}) = g\Phi(z)g^{-1} = \omega^a\Phi(z),$$

where ω is the bth root of unity. In general g can be identified with an element w of the Weyl group $W(J)$ and T' with an eigenvector of g (or w) which lies on a plane in the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ fixed by w. Requiring T' to be regular semisimple implies that w is a regular element in the sense of [62] which was classified and their orders correspond to the allowed b's (we omit the b's which are divisors of other ones). Related to this, T' is also what is called a regular semisimple cyclic element in [63], which has also been classified and the list again coincides with that of Table I.

[62] T. A. Springer, Inventiones Mathematicae **25**, 159 (1974).

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[65] S. Cecotti and M. Del Zotto, J. High Energy Phys. 01 (2013) 191.

[66] S. Cecotti, M. Del Zotto, and S. Giacomelli, J. High Energy Phys. 04 (2013) 153.

[67] W. Fulton and J. Harris, *Representation Theory: A First Course* (Springer, New York, 1991).

[68] As was pointed out in [43,69], the case $J = A_{2n}$ is subtle due to a discrete theta angle in the five-dimensional maximal SYM from the compactification of the six-dimensional (2, 0) type A_{2n} theory on S^1 with the $\mathbb{Z}_2$ twist. However it is unclear to us whether this subtlety affects the local irregular singularities discussed here.

[69] Y. Tachikawa, J. High Energy Phys. 11 (2011) 123.

[70] Here $\mathbf{p}$ labels the Nahm pole [43].

[71] By the Riemann-Roch theorem, the moduli associated with a jth holomorphic differential on a genus g Riemann surface $\mathcal{C}$ have dimension $H^0(K_\mathcal{C}^j) = \deg(K_\mathcal{C}^j) - (g-1) = (2j-1)(g-1)$. Including contributions from each fundamental invariant differentials and using the Lie-algebraic formula $\dim(J) = 2\sum_{i=1}^r d_i - r$, we end up with the total global contribution $(g-1)\dim J$.

[72] Heuristically the rhs of (60) counts the generalized monodromy data [56] in the presence of the $\mathbb{Z}_2$ twist. The appearances of $|s_2|$ which counts the $\mathbb{Z}_2$ odd Cartan elements of J can be understood as follows: as components of U_j with $2 \le j \le \ell$, they are parameters of the singularity rather than moduli.

[73] Note that unlike the case with a single untwisted irregular singularity, the singular AD curve with all deformations turned off cannot specify the full theory. However, one may still use the singular curve to read off the scaling dimensions of x and z.

[74] The singular boundary condition for the Higgs field with k even must be accompanied by a gauge transformation $\tilde{g} \in SO(2n)$ which leads to the untwisted irregular singularity considered before.

[75] The Coulomb branch spectrum of this twisted theory is identical to that of the (A_1, D_{2n-1}) theory but we believe they have different BPS charge lattices and thus are different $\mathcal{N} = 2$ SCFTs.

[76] Let us emphasize again that the singular SW curve here is only used to fix the scaling dimensions of x and z.

[77] As in the previous example, the Coulomb branch spectrum of this twisted theory is identical to that of the (A_1, A_{2n-2}) theory but we believe they have different BPS charge lattices and thus are different $\mathcal{N} = 2$ SCFTs.
[78] In the spectral curve, we need to remove redundant deformations due to coordinate redefinitions that leave $\lambda = xdz$ invariant and also trace relations among the invariant differentials.
[79] A. D. Shapere and Y. Tachikawa, J. High Energy Phys. 09 (2008) 109.
[80] The case $A_n^{(b)}[b]$ is an exception, which has $n-2$ marginal operators on the Coulomb branch.
[81] Note that the two theories $D_n^{(2n-2)}[2k]$ and $D_n^{(n)}[k]$ are approximately equivalent in the limit $k \to \infty$. This is most easily seen from the associated cD_n singularities. The same statement applies to $A_n^{(n+1)}[k]$ and $A_n^{(n)}[k]$.
[82] This was the rigid local model studied in [57].
[83] The case with $b = h(J)$ gives rise to pure SYM theories, whereas the other values of b lead to SYM theories coupled to matter.
[84] It can be thought of as an inclusion relation between the deformation space of the singularities.
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