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A New Approach to the Lenard–Magri Scheme of Integrability

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Abstract: We develop a new approach to the Lenard–Magri scheme of integrability of bi-Hamiltonian PDEs, when one of the Poisson structures is a strongly skew-adjoint differential operator.

1. Introduction

There have been a number of papers on classification and study of integrable PDEs in the past few decades. As a result, the 1-component integrable PDEs have been to a large extent classified. In the 2-component case there have been only partial results, see the survey [MNW09] and references there.

In the present paper we prove integrability of the 2-component PDEs (6.1), (6.2), (6.4), (6.7). All these equations enter in the same bi-Hamiltonian hierarchy of PDEs. The corresponding two compatible Poisson structures given by (4.2) and (4.3) are local and have order 3 and 5. Equation (6.4) appeared in [MNW07], however the proof of integrability of this equation in [MNW07] uses a non-local recursion operator and it is therefore not rigorous, as such argument may lead to wrong conclusions (see [DSK13a, page 338]).

The proof of integrability (i.e., existence of integrals of motion in involution with the associated Hamiltonian vector fields of arbitrarily high order) uses the Lenard–Magri scheme. Unfortunately (or fortunately) the existing methods, developed in [Dor93, BDSK09, Wan09], do not quite work here. We therefore develop a new method based on the notion of strongly skew-adjoint differential operators, using the Lie superalgebra of variational polyvector fields.

We show that the Lenard–Magri scheme "almost" always works provided that one of the Poisson structures H_0 is a non-degenerate strongly skew-adjoint operator. Namely,

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we are able to show, by a general argument, that the Poisson brackets of conserved densities are Casimir elements for H_0, but then we need to check by simple differential order considerations that these brackets are actually zero.

Throughout the paper, unless otherwise specified, all vector spaces are considered over a field $\mathbb{F}$ of characteristic zero.

2. A Lemma on $\mathbb{Z}$-Graded Lie Superalgebras

Let $W = W_{-1} \oplus W_0 \oplus W_1 \oplus \dots$ be a $\mathbb{Z}$-graded Lie superalgebra such that elements in W_j, $j \geq -1$, have parity $j \bmod 2 \in \mathbb{Z}/2\mathbb{Z}$. Given $H \in W_1$, we define the following skew-symmetric bracket $\{\cdot\,,\,\cdot\}_H$ on W_{-1} considered as an even space:

$$\{f, g\}_H = [[H, f], g], \quad f, g \in W_{-1}. \tag{2.1}$$

Recall that if $[H, H] = 0$, then $\{\cdot\,,\,\cdot\}_H$ is actually a Lie bracket on W_{-1}, considered as an even space (see e.g. [DSK13]). We also define the space of *Casimirs* for H as

$$C_{-1}(H) := \big\{ f \in W_{-1} \,\big|\, [H, f] = 0 \big\} \subset W_{-1}. \tag{2.2}$$

Our method is based on the following lemma. Some parts of it are well known (see e.g. [Mag78, Dor93, Olv93]).

Lemma 2.1. *(a) Let $H_0, H_1 \in W_1$. Denote by $\{\cdot\,,\,\cdot\}_0$ and $\{\cdot\,,\,\cdot\}_1$ the corresponding brackets on W_{-1} given by (2.1). Let $f_0, f_1, \dots, f_{N+1} \in W_{-1}$ satisfy the equations*

$$[H_1, f_n] = [H_0, f_{n+1}] \quad \textit{for all} \;\; n = 0, \dots, N. \tag{2.3}$$

Then we have

$$\{f_m, f_n\}_0 = \{f_m, f_n\}_1 = 0 \quad \textit{for all} \;\; m, n = 0, \dots, N+1.$$

(b) Let $H_0, H_1 \in W_1$. Let $f_0, f_1, \dots, f_{N+1} \in W_{-1}$, with $f_0 \in C_{-1}(H_0)$, satisfy Eq. (2.3), and let $\{g_n\}_{n\in\mathbb{Z}_+} \subset W_{-1}$ satisfy $[H_1, g_n] = [H_0, g_{n+1}]$ for all $n \in \mathbb{Z}_+$. Then we have

$$\{f_m, g_n\}_0 = \{f_m, g_n\}_1 = 0 \quad \textit{for all } m = 0, \dots, N+1, \; n \in \mathbb{Z}_+.$$

(c) If $H \in W_1$ is such that $[H, H] = 0$, then ad H *defines a Lie algebra homomorphisms $(W_{-1}, \{\cdot\,,\,\cdot\}_H) \to (W_0, [\cdot\,,\,\cdot])$, i.e.*

$$[H, \{f, g\}_H] = [[H, f], [H, g]] \quad \textit{for all } f, g \in W_{-1}.$$

(d) Let $H_0, H_1 \in W_1$ be such that $[H_0, H_1] = 0$. Then $C_{-1}(H_0) \subset V$ is a closed with respect to the bracket $\{\cdot\,,\,\cdot\}_1$.

(e) Let $H_0, H_1 \in W_1$ be such that $[H_0, H_1] = [H_1, H_1] = 0$. Suppose that $f_0, f_1, \dots, f_{N+1} \in W_{-1}$ satisfy Eq. (2.3). Then $\{f_{n+1}, g\}_1 \in C_{-1}(H_0)$ for all $n = 0, \dots, N$ and $g \in C_{-1}(H_0)$.

Proof. First, we prove part (a). By skew-symmetry, we have $\{f_n, f_n\}_0 = \{f_n, f_n\}_1 = 0$ for every n. Assuming that $n > m$, we prove, by induction on $n - m$ that $\{f_m, f_n\}_0 = \{f_m, f_n\}_1 = 0$. We have

$$\{f_m, f_n\}_1 = [[H_1, f_m], f_n] = [[H_0, f_{m+1}], f_n] = \{f_{m+1}, f_n\}_0,$$

which is zero by inductive assumption. Similarly,

$$\{f_m, f_n\}_0 = -\{f_n, f_m\}_0 = -[[H_0, f_n], f_m] \\ = -[[H_1, f_{n-1}], f_m] = -\{f_{n-1}, f_m\}_1 = \{f_m, f_{n-1}\}_1,$$

which again is zero by induction.

Next, we prove part (b). Since, by assumption, $f_0 \in C_{-1}(H_0)$, we have $\{f_0, g_n\}_0 = [[H_0, f_0], g_n] = 0$ for all n. Furthermore,

$$\{f_0, g_n\}_1 = -\{g_n, f_0\}_1 = -[[H_1, g_n], f_0] = -[[H_0, g_{n+1}], f_0] = \{f_0, g_{n+1}\}_0,$$

which is zero by the previous case. We next prove, by induction on $m \geq 1$, that $\{f_m, g_n\}_0 = \{f_m, g_n\}_1 = 0$ for every $n \in \mathbb{Z}_+$. We have

$$\{f_m, g_n\}_0 = [[H_0, f_m], g_n] = [[H_1, f_{m-1}], g_n] = \{f_{m-1}, g_n\}_1,$$

which is zero by inductive assumption, and

$$\{f_m, g_n\}_1 = -\{g_n, f_m\}_1 = -[[H_1, g_n], f_m] = -[[H_0, g_{n+1}], f_m] \\ = [[H_0, f_m], g_{n+1}] = \{f_m, g_{n+1}\}_0,$$

which is zero by the previous case, completing the proof of part (b).

For $H \in W_1$ and $f, g \in W_{-1}$, we have, by the Jacobi identity,

$$[H, \{f, g\}_H] = [H, [[H, f], g]] = [[H, [H, f]], g] + [[H, f], [H, g]].$$

If $[H, H] = 0$, we have $[H, [H, f]] = 0$, since H is odd, proving part (c).

Next, we prove part (d). If $f, g \in C_{-1}(H_0)$, we have

$$[H_0, \{f, g\}_1] = [H_0, [[H_1, f], g]],$$

and this is zero since, by assumption, H_0 commutes with all elements H_1, f and g.

Finally, we prove part (e). Since, by assumption, H_0 commutes with both H_1 and g, we have, by the Jacobi identity,

$$[H_0, \{f_{n+1}, g\}_1] = [H_0, [[H_1, f_{n+1}], g]] = [[H_0, [H_1, f_{n+1}]], g] \\ = -[[H_1, [H_0, f_{n+1}]], g] = -[[H_1, [H_1, f_n]], g],$$

and this is zero since $(\text{ad } H_1)^2 = 0$. □

3. Application to the Theory of Hamiltonian PDEs

In the present paper we will use Lemma 2.1 in the special case when $\mathcal{W}$ is the Lie superalgebra of variational polyvector fields over an algebra of differential functions $\mathcal{V}$.

Recall from [BDSK09] that an algebra of differential function $\mathcal{V}$ in the variables u_i, $i \in I = \{1, \dots, \ell\}$, is a differential algebra extension of the algebra of differential polynomials $R_\ell = \mathbb{F}[u_i^{(n)} \,|\, i \in I, n \in \mathbb{Z}_+]$, with the "total derivative" ∂ defined on generators by $\partial u_i^{(n)} = u_i^{(n+1)}$, and endowed with commuting derivations $\frac{\partial}{\partial u_i^{(n)}} : \mathcal{V} \to \mathcal{V}$ extending the usual partial derivatives on R_ℓ, such that for every $f \in \mathcal{V}$ we have $\frac{\partial f}{\partial u_i^{(n)}} = 0$ for all but finitely many values of i and n, and satisfying the commutation relation $[\frac{\partial}{\partial u_i^{(n)}}, \partial] = \frac{\partial}{\partial u_i^{(n-1)}}$ for every $i \in I, n \in \mathbb{Z}_+$ (the RHS is considered to be 0 for $n = 0$).

Recall from [DSK13] some properties of the Lie superalgebra $\mathcal{W}$ of variational polyvector fields over $\mathcal{V}$ that we will need. It is a $\mathbb{Z}$-graded Lie superalgebra $\mathcal{W} = \mathcal{W}_{-1} \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \dots$, with the parity compatible with the $\mathbb{Z}$-grading. Furthermore, $\mathcal{W}_{-1} = \mathcal{V}/\partial\mathcal{V}$ is the space of local functionals, $\mathcal{W}_0 = \mathcal{V}^\ell$ is the Lie algebra of evolutionary vector fields, i.e. derivations of $\mathcal{V}$ commuting with ∂, which have the form $X_P = \sum_{i \in I, n \in \mathbb{Z}_+} (\partial^n P_i) \frac{\partial}{\partial u_i^{(n)}}$, $P \in \mathcal{V}^\ell$. The bracket of two evolutionary vector fields is given by the formula $[X_P, X_Q] = X_{[P,Q]}$, where

$$[P, Q] = D_Q(\partial)P - D_P(\partial)Q,$$

and $D_P(\partial) \in \mathrm{Mat}_{\ell \times \ell}\, \mathcal{V}[\partial]$ is the *Frechet derivative* of P:

$$D_P(\partial)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial P_i}{\partial u_j^{(n)}} \partial^n. \tag{3.1}$$

The Lie bracket between elements $P \in W_0$ and $\int f \in W_{-1}$ is given by

$$[P, \textstyle\int f] = \int X_P(f) = \int P \cdot \delta f, \tag{3.2}$$

where $\delta f = \left(\frac{\delta f}{\delta u_i}\right)_{i \in I} \in \mathcal{V}^\ell$ denotes the vector of variational derivatives of $\int f \in \mathcal{V}/\partial\mathcal{V}$:

$$\frac{\delta f}{\delta u_i} = \sum_{n \in \mathbb{Z}_+} (-\partial)^n \frac{\partial f}{\partial u_i^{(n)}}. \tag{3.3}$$

Finally, $\mathcal{W}_1$ is the space of skew-adjoint $\ell \times \ell$ matrix differential operators over $\mathcal{V}$, the Lie bracket between $H \in \mathcal{W}_1$ and $\int f \in \mathcal{V}/\partial\mathcal{V} = \mathcal{W}_{-1}$ is given by

$$[H, \textstyle\int f] = H(\partial)\delta f, \tag{3.4}$$

and a *Poisson structure* on $\mathcal{V}$ is an element $H \in \mathcal{W}_1$ such that $[H, H] = 0$.

For $H \in \mathcal{W}_1$, the corresponding skew-symmetric bracket (2.1) on $\mathcal{W}_{-1} = \mathcal{V}/\partial\mathcal{V}$ is given by the usual formula

$$\{\textstyle\int f, \int g\}_H = \int \delta g \cdot H(\partial)\delta f, \tag{3.5}$$

and this bracket defines a Lie algebra structure on $\mathcal{V}/\partial\mathcal{V}$ if and only if H is a Poisson structure on $\mathcal{V}$. In this context, the space (2.2) of Casimir elements for H is

$$C_{-1}(H) = \left\{ \textstyle\int f \in \mathcal{V}/\partial\mathcal{V} \,\middle|\, H(\partial)\delta f = 0 \right\}. \tag{3.6}$$

Recall that the *Hamiltonian partial differential equation* for the Poisson structure $H \in \mathcal{W}_1$ and the Hamiltonian functional $\int h \in \mathcal{V}/\partial\mathcal{V}$ is the following evolution equation in the variables $u_1, \dots, u_\ell$:

$$\frac{du_i}{dt} = \sum_{j=1}^{\ell} H_{ij}(\partial) \frac{\delta h}{\delta u_j}. \tag{3.7}$$

An *integral of motion* for the Hamiltonian equation (3.7) is a local functional $\int f \in \mathcal{V}/\partial\mathcal{V}$ such that $\{\int h, \int f\}_H = 0$. Equation (3.7) is said to be *integrable* if there is an infinite sequence of linearly independent integrals of motion $\int h_0 = \int h, \int h_1, \int h_2, \dots$ in involution: $\{\int h_m, \int h_n\}_H = 0$ for all $m, n \in \mathbb{Z}_+$.

One of the main techniques for proving integrability of a Hamiltonian equation is based on the so called *Lenard–Magri scheme of integrability*. This applies when a given evolution equation has a bi-Hamiltonian form, i.e., it can be written in Hamiltonian form in two ways:

$$\frac{du}{dt} = H_1(\partial)\delta h_0 = H_0(\partial)\delta h_1, \tag{3.8}$$

where H_0, H_1 are compatible Poisson structures on $\mathcal{V}$, namely they satisfy $[H_0, H_0] = [H_0, H_1] = [H_1, H_1] = 0$. In this situation, the Lenard–Magri scheme consists in finding a sequence of local functionals $\int h_0, \int h_1, \int h_2, \dots$ satisfying the recursive conditions

$$H_1(\partial)\delta h_n = H_0(\partial)\delta h_{n+1}, \tag{3.9}$$

for all $n \in \mathbb{Z}_+$. Lemma 2.1(a) and (c) guarantees that, in this situation, all local functionals $\int h_n$, $n \in \mathbb{Z}_+$ are integrals of motion in involution with respect to both Poisson brackets $\{\cdot\,,\,\cdot\}_0$ and $\{\cdot\,,\,\cdot\}_1$, and the higher symmetries $P_n = H_0(\partial)\delta h_n$ commute. Indeed, we have the following immediate consequences of Lemma 2.1.

Corollary 3.1. *Let H_0, H_1 be Poisson structures on $\mathcal{V}$, and let $\{\cdot\,,\,\cdot\}_0$ and $\{\cdot\,,\,\cdot\}_1$ be the corresponding brackets on $\mathcal{V}/\partial\mathcal{V}$ given by (3.5). Let $\{\int h_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{V}/\partial\mathcal{V}$ be a sequence of local functionals satisfying the Lenard–Magri recursive equations (3.9). Then all elements $\int h_n$ are integrals of motion for the bi-Hamiltonian equation (3.8) in involution with respect to both Poisson brackets for H_0 and H_1: $\{\int h_m, \int h_n\}_0 = \{\int h_m, \int h_n\}_1 = 0$, and all Hamiltonian vector fields $P_n = H_1(\partial)\delta h_n$ commute: $[P_m, P_n] = 0$, for all $m, n \in \mathbb{Z}_+$.*

Proof. The first statement is a special case of Lemma 2.1(a), and the second statement follows by Lemma 2.1(c). □

The main problem in applying the Lenard–Magri scheme of integrability is to show that at each step n the recursive equation (3.9) can be solved for $\int h_{n+1} \in \mathcal{V}/\partial\mathcal{V}$. This problem is split in three parts. First, under the assumption that $\mathcal{V}$ is a domain and the Poisson structure H_0 is non-degenerate (cf. Definition 3.2 below), Theorem 3.3 below guarantees that, if an element $F \in \mathcal{V}^\ell$ exists such that $H_1(\partial)\delta h_n = H_0(\partial)F$, then F is *closed*, i.e. it has self-adjoint Frechet derivative: $D_F(\partial)^* = D_F(\partial)$. Next, Theorem 3.5 below shows that, if $F \in \mathcal{V}^\ell$ is closed, then it is *exact* in a normal extension $\widetilde{\mathcal{V}}$ of the algebra of differential algebra function $\mathcal{V}$: $F = \delta\int h$ for some $h \in \widetilde{\mathcal{V}}$. Hence, we reduced our problem to proving that $H_1(\partial)\delta h_n \in H_0(\partial)\mathcal{V}^\ell$. There is no universal technique to solve this problem, but there are various approaches which work in specific examples (see e.g. [BDSK09,DSK13,DSK12,Dor93,Olv93,Wan09]). In Proposition 3.8 below we propose an ansatz for solving this problem, under the assumption that the Poisson structure H_0 is strongly skew-adjoint (cf. Definition 3.6 below), and that the given finite Lenard–Magri sequence $\int h_0, \dots, \int h_n$ starts with a Casimir element for H_0: $H_0(\partial)\delta(\int h_0) = 0$. In the following sections we will be able to apply successfully this ansatz to prove integrability of the compatible bi-Hamiltonian PDE's in two variables (6.1), (6.2), (6.4) and (6.7).

Definition 3.2. *Assume that $\mathcal{V}$ is a domain. A matrix differential operator $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ is* non-degenerate *if it is not a left (or right) zero divisor in $\mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ (equivalently, if its Dieudonné determinant in non-zero).*

Theorem 3.3 (see e.g. [BDSK09, Thm. 2.7]). *Let $H_0,\ H_1 \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ be compatible Poisson structures on the algebra of differential functions $\mathcal{V}$, which is assumed to be a domain, and suppose that H_0 is non-degenerate. If $\int h_0, \int h_1 \in \mathcal{V}/\partial\mathcal{V}$ and $F \in \mathcal{V}^\ell$ are such that $H_1(\partial)\delta h_0 = H_0(\partial)\delta h_1$ and $H_1(\partial)\delta h_1 = H_0(\partial)F$, then F is closed: $D_F(\partial) = D_F(\partial)^*$.*

Consider the following filtration of the algebra of differential functions $\mathcal{V}$:

$$\mathcal{V}_{m,i} = \left\{ f \in \mathcal{V} \,\middle|\, \frac{\partial f}{\partial u_j^{(n)}} = 0 \text{ for all } (n, j) > (m, i) \right\},$$

where $>$ denotes lexicographic order. By definition, $\frac{\partial}{\partial u_j^{(n)}}(\mathcal{V}_{m,i})$ is zero for $(n, j) > (m, i)$, and it is contained in $\mathcal{V}_{m,i}$ for $(n, j) \leq (m, i)$.

Definition 3.4. *The algebra of differential functions $\mathcal{V}$ is called* normal *if $\frac{\partial}{\partial u_i^{(m)}}(\mathcal{V}_{m,i}) = \mathcal{V}_{m,i}$ for all $i \in I, m \in \mathbb{Z}_+$.*

Note that any algebra of differential function can be extended to a normal one (see [DSK13a]).

Theorem 3.5 ([BDSK09, Prop. 1.9]) *If $F \in \mathcal{V}^\ell$ is exact, i.e. $F = \delta f$ for some $\int f \in \mathcal{V}/\partial\mathcal{V}$, then it is closed, i.e. $D_F(\partial) = D_F(\partial)^*$. Conversely, if $\mathcal{V}$ is a normal algebra of differential functions and $F \in \mathcal{V}^\ell$ is closed, then it is exact.*

Note that if $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ is a skew-adjoint operator, then $H(\partial)\mathcal{V}^\ell \perp \mathrm{Ker}\, H(\partial)$, where the orthogonal complement is with respect to the pairing $\mathcal{V}^\ell \times \mathcal{V}^\ell \to \mathcal{V}/\partial\mathcal{V}$ given by $(F, P) \mapsto \int F \cdot P$.

Definition 3.6. *A skew-adjoint operator $H \in \mathrm{Mat}_{\ell\times\ell}\,\mathcal{V}[\partial]$ is called* strongly skew-adjoint *if the following conditions hold:*

(i) $\mathrm{Ker}\, H(\partial) \subset \delta(\mathcal{V}/\partial\mathcal{V})$,
(ii) $\left(\mathrm{Ker}\, H(\partial)\right)^\perp = H(\partial)\mathcal{V}^\ell$.

Let $\int h_0 \in C_{-1}(H_0)$, and let $\int h_0, \dots, \int h_N \in \mathcal{V}/\partial\mathcal{V}$ be a finite sequence satisfying the Lenard–Magri recursive equations (3.9). Lemma 2.1(d) and (e), in this context, imply the following result:

Corollary 3.7. *If H_0 and H_1 are compatible Poisson structures on $\mathcal{V}$, then $H_0(\partial)\delta\{\int h_n, \int g\}_1 = 0$ for all $n = 0, \dots, N$ and for all $\int g \in C_{-1}(H_0)$.*

If H_0 is non-degenerate, its kernel in $\mathcal{V}^\ell$ is finite-dimensional. Therefore it is reasonable to hope that, by computing explicitly $\mathrm{Ker}\, H_0$ and $C_{-1}(H_0)$, and carefully looking at the recursive equations (3.9) one can prove that, in fact, $\{\int h_n, \int g\}_1 = 0$ for all $\int g \in C_{-1}(H_0)$. In this case, assuming that H_0 is strongly skew-adjoint, the following proposition guarantees that we can successfully apply the Lenard–Magri scheme.

Proposition 3.8. *Suppose that* H_0 *is a strongly skew-adjoint operator and that* $\{\int h, \int g\}_1 = 0$ *for all* $\int g \in C_{-1}(H_0)$*, Then* $H_1(\partial)\delta h \in H_0(\partial)\mathcal{V}^\ell$.

Proof. By assumption, we have $\int \delta g \cdot H_1(\partial)\delta h = 0$ for all $\int g \in C_{-1}(H_0)$, i.e. $H_1(\partial)\delta h \perp \delta C_{-1}(H_0)$. By condition (i) of the strong skew-adjointness assumption on H_0 this implies that $H_1(\partial)\delta h \perp \operatorname{Ker} H_0(\partial)$, and therefore, by the condition (ii), we conclude that $H_1(\partial)\delta h \in H_0(\partial)\mathcal{V}^\ell$, proving the claim. □

To conclude the section, we state the following result, which will be used in the following sections.

Corollary 3.9. *Let* H_0, H_1 *be Poisson structures on* $\mathcal{V}$*. Let* $\{\int f_n\}_{n=0}^N \subset \mathcal{V}/\partial\mathcal{V}$ *be a finite sequence satisfying the Lenard–Magri recursive equations* (3.9)*, with* $\int f_0 \in C_{-1}(H_0)$*, and let* $\{\int g_n\}_{n=0}^\infty \subset \mathcal{V}/\partial\mathcal{V}$ *be an infinite sequence also satisfying the Lenard–Magri recursive equations* (3.9)*. Then the two Lenard–Magri sequences are compatible, in the sense that* $\{\int f_m, \int g_n\}_0 = \{\int f_m, \int g_n\}_1 = 0$ *for all* $m = 0, \dots, N$, $n \in \mathbb{Z}_+$.

Proof. It is a special case of Lemma 2.1(b). □

4. The Bi-Poisson Structure (H_0, H_1)

Consider the following algebra of differential functions in two variables u, v:

$$\mathcal{V} = \mathbb{F}[u, v^{\pm 1}, u', v', u'', v'', \dots]. \tag{4.1}$$

It is contained in the normal extension $\widetilde{\mathcal{V}} = \mathcal{V}[\log v]$, see [DSK13a, Ex. 4.5].

Theorem 4.1. *The following is a compatible pair of Poisson structures* $(H_0, H_1) \in \operatorname{Mat}_{2\times 2}\mathcal{V}[\partial]$*:*

$$H_0(\partial) = \begin{pmatrix} \partial^3 + \partial \circ u + u\partial & v\partial \\ \partial \circ v & 0 \end{pmatrix}, \tag{4.2}$$

and

$$H_1(\partial) = \begin{pmatrix} 0 & \partial \circ \frac{1}{v^2} \\ \frac{1}{v^2}\partial & -\frac{1}{v^2} Q(\partial) \circ \frac{1}{v^2} \end{pmatrix}, \tag{4.3}$$

where

$$Q(\partial) = \partial^5 + 3\partial \circ (\partial \circ u + u\partial)\partial + 2(\partial^3 \circ u + u\partial^3) + 8(\partial \circ u^2 + u^2\partial).$$

Proof. H_0 is a well known Poisson structure, see e.g. [Ito82, Dor93]. A simple proof of this fact can be found in [BDSK09]. H_1 is obviously skew-adjoint. The proof that H_1 satisfies Jacobi identity and is compatible with H_0 is a rather lengthy computation (one has to verify equation (1.49) from [BDSK09]). This has been checked with the use of the computer. □

5. Casimirs for H_0 and H_1

In this paper we will apply the Lenard–Magri scheme for the bi-Poisson structure (H_0, H_1) to find integrable hierarchies of bi-Hamiltonian equations. As explained in Sect. 1, in order to do so, it is convenient to find the Casimir elements for H_0 and H_1.

Proposition 5.1. *(a) The kernel of $H_0(\partial)$ is spanned by*

$$\xi^{0,0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi^{0,1} = \begin{pmatrix} \frac{1}{v} \\ -\frac{u}{v^2} - \frac{3}{2}\frac{(v')^2}{v^4} + \frac{v''}{v^3} \end{pmatrix}. \tag{5.1}$$

(b) We have $\xi^{0,0} = \delta(\int h^{0,0})$ and $\xi^{0,1} = \delta(\int h^{0,1})$, where

$$\int h^{0,0} = \int v \ \ \text{and} \ \ \int h^{0,1} = \int \Big(\frac{u}{v} - \frac{1}{2}\frac{(v')^2}{v^3}\Big).$$

(c) The matrix differential operator $H_0(\partial)$ is strongly skew-adjoint.
(d) The kernel of $H_1(\partial)$ is spanned by

$$\xi^{1,0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^{1,1} = \begin{pmatrix} u'' + 4u^2 \\ \frac{v^2}{2} \end{pmatrix}. \tag{5.2}$$

(e) We have $\xi^{1,0} = \delta(\int h^{1,0})$ and $\xi^{1,1} = \delta(\int h^{1,1})$, where

$$\int h^{1,0} = \int u \ \ \text{and} \ \ \int h^{1,1} = \int \Big(\frac{1}{2}uu'' + \frac{4}{3}u^3 + \frac{1}{6}v^3\Big).$$

(f) The matrix differential operator $H_1(\partial)$ is strongly skew-adjoint.

Proof. Recall that the dimension (over the field of constants) of the kernel of a non-degenerate matrix differential operator is at most the degree of its Dieudonné determinant, see e.g. [DSK13]. Clearly, the Dieudonné determinants of both H_0 and H_1 have degree 2, therefore their kernels have dimensions (over $\mathbb{F}$) at most 2. On the other hand, obviously $\xi^{0,0} \in \operatorname{Ker} H_0(\partial)$, and $\xi^{1,0} \in \operatorname{Ker} H_1(\partial)$. Moreover, the following straightforward identities

$$\begin{gathered} (\partial^3 + \partial \circ u + u\partial)\frac{1}{v} + v\partial\Big(-\frac{u}{v^2} - \frac{3}{2}\frac{(v')^2}{v^4} + \frac{v''}{v^3}\Big) = 0, \\ \frac{1}{v^2}\partial(u'' + 4u^2) - \frac{1}{v^2}Q(\partial)\frac{1}{2} = 0, \end{gathered} \tag{5.3}$$

imply respectively that $\xi^{0,1} \in \operatorname{Ker} H_0(\partial)$, and $\xi^{1,1} \in \operatorname{Ker} H_1(\partial)$, proving parts (a) and (d). Parts (b) and (e) follow by straightforward computations. Note that parts (b) and (e) exactly say, respectively, that the operators H_0 and H_1 satisfy condition (i) of Definition 3.6 of strong skew-adjointness. We are left to prove condition (ii) for both H_0 and H_1. Take $P_0 = \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \perp \operatorname{Ker} H_0$. Since $\int P_0 \cdot \xi^{0,0} = 0$ (and since v is invertible in $\mathcal{V}$), we have that $q_0 = (v\alpha_0)'$, for some $\alpha_0 \in \mathcal{V}$. Denote $r_0 = \frac{1}{v}(p_0 - \alpha_0''' - (u\alpha_0)' - u\alpha_0') \in \mathcal{V}$, so that $p_0 = (\partial^3 + \partial \circ u + u\partial)\alpha_0 + vr_0$. The condition $\int P_0 \cdot \xi^{0,1} = 0$ then reads

$$\int \Big(r_0 + \frac{1}{v}(\partial^3 + \partial \circ u + u\partial)\alpha_0 + \Big(-\frac{u}{v^2} - \frac{3}{2}\frac{(v')^2}{v^4} + \frac{v''}{v^3}\Big)(v\alpha_0)'\Big) = 0.$$

After integration by parts, the last two terms under integration cancel by the first identity in (5.3), hence the above equation implies that $r_0 = \beta_0' \in \partial\mathcal{V}$. In conclusion, $P_0 = H_0(\partial)\begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix}$, proving condition (ii) for H_0. Similarly, take $P_1 = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \perp \operatorname{Ker} H_1$.

Since $\int P_1 \cdot \xi^{1,0} = 0$ (and since v is invertible in $\mathcal{V}$), we have that $p_1 = \left(\frac{\beta_1}{v^2}\right)'$, for some $\beta_1 \in \mathcal{V}$. Denote $r_1 = v^2 q_1 + Q(\partial)\frac{\beta_1}{v^2} \in \mathcal{V}$, so that $q_1 = \frac{r_1}{v^2} - \frac{1}{v^2}Q(\partial)\frac{\beta_1}{v^2}$. The condition $\int P_1 \cdot \xi^{1,1} = 0$ then reads

$$\int\Big(\big(\frac{\beta}{v^2}\big)'(u'' + 4u^2) - \frac{1}{2}Q(\partial)\frac{\beta_1}{v^2} + \frac{r_1}{2}\Big) = 0.$$

After integration by parts, the first two terms under integration cancel by the second identity in (5.3), hence the above equation implies that $r_1 = \alpha_1' \in \partial\mathcal{V}$. In conclusion, $P_1 = H_1(\partial)\begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$, proving condition (ii) for H_1. □

By Proposition 5.1(b)–(e), the space of Casimir elements for H_0 and H_1 are respectively

$$C_{-1}(H_0) = \mathrm{Span}_{\mathbb{F}}\big\{\textstyle\int h^{0,0}, \int h^{0,1}\big\} \quad \text{and} \quad C_{-1}(H_1) = \mathrm{Span}_{\mathbb{F}}\big\{\textstyle\int h^{1,0}, \int h^{1,1}\big\}.$$

Let, as before, $\{\cdot\,,\,\cdot\}_0$ and $\{\cdot\,,\,\cdot\}_1$ be the Poisson brackets (3.5) associated to the Poisson structures H_0 and H_1 respectively.

Proposition 5.2. *The spaces $C_{-1}(H_0)$ and $C_{-1}(H_1)$ are abelian subalgebras with respect to both Poisson brackets $\{\cdot\,,\,\cdot\}_0$ and $\{\cdot\,,\,\cdot\}_1$.*

Proof. By definition of Casimir elements, the space $C_{-1}(H_0)$ is in the kernel of the bracket $\{\cdot\,,\,\cdot\}_0$, and $C_{-1}(H_1)$ is in the kernel of the bracket $\{\cdot\,,\,\cdot\}_1$. Hence, we only need to prove that $C_{-1}(H_0)$ is an abelian subalgebra w.r.t. $\{\cdot\,,\,\cdot\}_1$, namely, by (3.5) and Proposition (5.1)(b), that

$$\int \xi^{0,0} \cdot H_1(\partial)\xi^{0,1} = 0, \tag{5.4}$$

and that $C_{-1}(H_1)$ is an abelian subalgebra w.r.t. $\{\cdot\,,\,\cdot\}_0$, namely

$$\int \xi^{1,0} \cdot H_0(\partial)\xi^{1,1} = 0. \tag{5.5}$$

We have

$$\begin{aligned}\xi^{1,0} \cdot H_0(\partial)\xi^{1,1} &= (\partial^3 + \partial \circ u + u\partial)(u'' + 4u^2) + \frac{1}{2}v\partial v^2\\ &= (u'' + 4u^2)''' + \Big(2uu'' - \frac{1}{2}(u')^2 + \frac{20}{3}u^3\Big)' + \frac{1}{3}(v^3)',\end{aligned}$$

hence (5.5) holds. A similar, but longer computation, shows that (5.4) holds as well. □

6. The Four Integrable Bi-Hamiltonian Equations of Lowest Order

Let us start computing the first Hamiltonian equations associated to each element in the kernel of H_0 and H_1. Starting with $\xi^{0,0} \in \mathrm{Ker}(H_0)$ we get the Hamiltonian vector field $P^{0,0} = H_1(\partial)\xi^{0,0}$, and the corresponding Hamiltonian equation (of order 5):

$$\begin{aligned}\frac{du}{dt} &= \left(\frac{1}{v^2}\right)'\\ \frac{dv}{dt} &= -\frac{1}{v^2}\left(\frac{1}{v^2}\right)^{(5)} + 3\frac{1}{v^2}\left(u\left(\frac{1}{v^2}\right)'\right)'' + 3\frac{1}{v^2}\left(u\left(\frac{1}{v^2}\right)''\right)'\\ &\quad +2\frac{1}{v^2}\left(\frac{u}{v^2}\right)''' + 2\frac{u}{v^2}\left(\frac{1}{v^2}\right)''' + 8\frac{1}{v^2}\left(\frac{u^2}{v^2}\right)' + 8\frac{u^2}{v^2}\left(\frac{1}{v^2}\right)'.\end{aligned} \tag{6.1}$$

Starting with $\xi^{0,1} \in \mathrm{Ker}(H_0)$ we get the Hamiltonian vector field $P^{0,1} = H_1(\partial)\xi^{0,1}$, and the corresponding Hamiltonian equation (of order 7):

$$\begin{aligned} \frac{du}{dt} &= \Big(-\frac{u}{v^4} + \frac{v''}{v^5} - \frac{3}{2}\frac{(v')^2}{v^6} \Big)' \\ \frac{dv}{dt} &= -\frac{v'}{v^4} - \frac{1}{v^2}\big(\partial^5 + 3\partial^2 \circ u\partial + 3\partial \circ u\partial^2 + 2\partial^3 \circ u + 2u\partial^3 \\ &\quad +8\partial \circ u^2 + 8u^2\partial\big)\Big(-\frac{u}{v^4} + \frac{v''}{v^5} - \frac{3}{2}\frac{(v')^2}{v^6} \Big). \end{aligned} \tag{6.2}$$

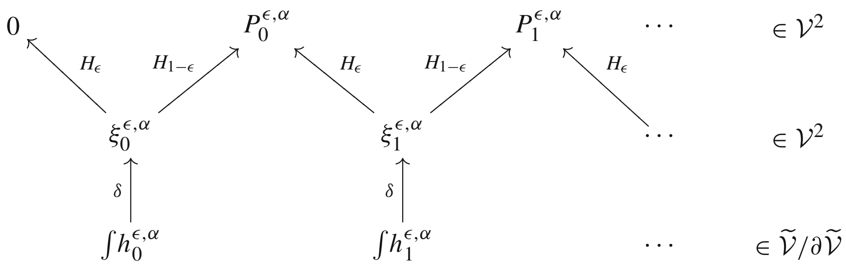
Starting with $\xi^{1,0} \in \mathrm{Ker}(H_1)$ we get the Hamiltonian vector field $P^{1,0} = H_0(\partial)\xi^{1,0}$, and the corresponding Hamiltonian equation:

$$\begin{aligned} \frac{du}{dt} &= u' \\ \frac{dv}{dt} &= v'. \end{aligned} \tag{6.3}$$

Finally, starting with $\xi^{1,1} \in \mathrm{Ker}(H_0)$ we get the Hamiltonian vector field $P^{1,1} = H_0(\partial)\xi^{1,1}$, and the corresponding Hamiltonian equation (of order 5):

$$\begin{aligned} \frac{du}{dt} &= u^{(5)} + 10uu''' + 25u'u'' + 20u^2u' + v^2v' \\ \frac{dv}{dt} &= u'''v + u''v' + 8uu'v + 4u^2v'. \end{aligned} \tag{6.4}$$

We want to prove that, for $\epsilon = 0, 1$, each element $\xi^{\epsilon,\alpha}$, $\alpha = 0, 1$ in the kernel of H_ϵ produces an infinite Lenard–Magri scheme starting with $\xi_0^{\epsilon,\alpha} = \xi^{\epsilon,\alpha}$:



(6.5)

This will easily imply that all by-Hamiltonian equations (6.1), (6.2), (6.4) and (6.7) are integrable, and they are compatible with each other. Namely, we will prove the following result.

Theorem 6.1. *(a) For $\epsilon, \alpha \in \{0, 1\}$ there is a sequence $\{\int h_n^{\epsilon,\alpha}\}_{n\in\mathbb{Z}_+} \subset \widetilde{\mathcal{V}}/\partial\widetilde{\mathcal{V}}$, such that $\delta(\int h_0^{\epsilon,\alpha}) = \xi^{\epsilon,\alpha}$ and $\delta(\int h_n^{\epsilon,\alpha}) \in \mathcal{V}^2$ for every $n \in \mathbb{Z}_+$, satisfying the Lenard–Magri recurrence relations (6.5), i.e.*

$$H_{1-\epsilon}(\partial)\delta(\textstyle\int h_n^{\epsilon,\alpha}) = H_\epsilon(\partial)\delta(\int h_{n+1}^{\epsilon,\alpha}) =: P_n^{\epsilon,\alpha}. \tag{6.6}$$

(b) The four Lenard–Magri schemes are compatible, in the sense that

$$\begin{aligned} \big\{\textstyle\int h_m^{\epsilon,\alpha}, \int h_n^{\delta,\beta}\big\}_\zeta &= 0 \quad \textit{for all } \epsilon, \delta, \zeta, \alpha, \beta = 0, 1,\ m, n \in \mathbb{Z}_+, \\ [P_m^{\epsilon,\alpha}, P_n^{\delta,\beta}] &= 0 \quad \textit{for all } \epsilon, \delta, \alpha, \beta = 0, 1,\ m, n \in \mathbb{Z}_+. \end{aligned}$$

(c) *The differential orders of the higher symmetries* $P_n^{\epsilon,\alpha} = H_\epsilon(\partial)\delta(\int h_n^{\epsilon,\alpha})$ *tend to infinity as* $n \to \infty$.

Remark 6.2. Proposition 5.1 gives the integrals of motion $\int h_n^{\epsilon,\alpha}$ for $n = 0$ and arbitrary $\epsilon, \alpha = 0, 1$. One can use the Lenard–Magri relations (6.6) to find recursively all other integrals of motions $\int h_n^{\epsilon,\alpha}$ for arbitrary $n \in \mathbb{Z}$. For example, we have

$$\int h_1^{1,0} = \int\Big(\frac{1}{3}uv^3 + \frac{8}{3}u^4 + \frac{1}{2}uu^{(4)} - 6u(u')^2\Big).$$

The next higher symmetry is $P_1^{1,0} = H_0(\partial)\delta(\int h_1^{1,0})$, and the corresponding Hamiltonian equation (of order 7) is:

$$\begin{aligned} \frac{du}{dt} &= (\partial^3 + 2u\partial + u')(u^{(4)} + 12uu'' + 6(u')^2 + \tfrac{32}{3}u^3 + \tfrac{1}{3}v^3) + v\partial(uv^2) \\ \frac{dv}{dt} &= \partial(vu^{(4)} + 12vuu'' + 6v(u')^2 + \tfrac{32}{3}vu^3 + \tfrac{1}{3}v^4). \end{aligned} \tag{6.7}$$

7. Proof of Theorem 6.1

Let $\epsilon, \alpha \in \{0, 1\}$ and let

$$\left\{\xi_n^{\epsilon,\alpha} = \begin{pmatrix} f_n^{\epsilon,\alpha} \\ g_n^{\epsilon,\alpha} \end{pmatrix}\right\}_{n=0}^{N} \subset \mathcal{V}^2 \tag{7.1}$$

be a finite sequence satisfying the Lenard–Magri recursive relations

$$H_{1-\epsilon}(\partial)\xi_{n-1}^{\epsilon,\alpha} = H_\epsilon(\partial)\xi_n^{\epsilon,\alpha} \tag{7.2}$$

for every $n = 0, \dots, N$, where $\xi_{-1}^{\epsilon,\alpha} = 0$ and $\xi_0^{\epsilon,\alpha} = \xi^{\epsilon,\alpha}$. The main task in the proof of Theorem 6.1 is to show that, when $\alpha = 1$, this sequence can be extended by one step. This is the content of Corollary 7.6 below, which is a consequence of the following five lemmas.

Lemma 7.1. (a) *For* $\epsilon = 0$, *the Lenard–Magri recursive relations* (7.2) *translate in the following identities for the entries of* $\xi_n^{0,\alpha}$ *for* $1 \le n \le N$:

$$\begin{aligned} (vf_n^{0,\alpha})' &= \frac{1}{v^2}(f_{n-1}^{0,\alpha})' - \frac{1}{v^2}Q(\partial)\frac{g_{n-1}^{0,\alpha}}{v^2}, \\ v(g_n^{0,\alpha})' &= \Big(\frac{g_{n-1}^{0,\alpha}}{v^2}\Big)' - (f_n^{0,\alpha})''' - (uf_n^{0,\alpha})' - u(f_n^{0,\alpha})'. \end{aligned} \tag{7.3}$$

(b) *For* $\epsilon = 1$, *Eq.* (7.2) *translate in the following identities for the entries of* $\xi_n^{1,\alpha}$ *for* $1 \le n \le N$:

$$\begin{aligned} \Big(\frac{g_n^{1,\alpha}}{v^2}\Big)' &= (f_{n-1}^{1,\alpha})''' + (uf_{n-1}^{1,\alpha})' + u(f_{n-1}^{1,\alpha})' + v(g_{n-1}^{1,\alpha})', \\ (f_n^{1,\alpha})' &= v^2(vf_{n-1}^{1,\alpha})' + Q(\partial)\Big(\frac{g_n^{1,\alpha}}{v^2}\Big). \end{aligned} \tag{7.4}$$

Proof. It follows immediately from the definitions (4.2)–(4.3) of the operators H_0 and H_1. □

The algebra of differential functions $\mathcal{V}$, defined by (4.1), admits the following two differential subalgebras:

$$\mathcal{V}^+ = \mathcal{R}_2 = \mathbb{F}[u, v, u', v', u'', v'', \dots], \quad \mathcal{V}^- = \mathbb{F}\big[u, \frac{1}{v}, u', v', u'', v'', \dots\big], \tag{7.5}$$

whose intersection is the differential subalgebra

$$\mathcal{V}^0 = \mathbb{F}[u, u', v', u'', v'', \dots]. \tag{7.6}$$

Lemma 7.2. *(a) We have $\partial\mathcal{V} \cap \mathcal{V}^+ = \partial\mathcal{V}^+$.*
(b) We have $\partial\mathcal{V} \cap \mathcal{V}^- = \partial(\mathbb{F}v \oplus \mathcal{V}^-)$,
(c) For every $k \geq 1$, we have $\partial\mathcal{V} \cap \big(\mathbb{F} \oplus \frac{1}{v^k}\mathcal{V}^-\big) = \partial\big(\mathbb{F}\frac{1}{v^{k-1}} \oplus \frac{1}{v^k}\mathcal{V}^-\big)$.

Proof. Any element $f \in \mathcal{V}$ admits a unique decomposition $f = \sum_{j=M}^N v^j c_j$, where $M \leq N \in \mathbb{Z}$, $c_j \in \mathcal{V}^0$ for all j and $c_M, c_N \neq 0$. Its derivative is then

$$f' = \sum_{j=M-1}^{N-1} (j+1)v^j c_{j+1} v' + \sum_{j=M}^{N} v^j c'_j.$$

If $f' \in \mathcal{V}^+$, then M must be non-negative, i.e. $f \in \mathcal{V}^+$, proving part (a). Suppose next that $f' \in \mathcal{V}^-$. If $N \geq 2$, then $c'_N = 0$, $c'_{N-1} + Nc_N v' = 0$, namely $0 \neq c_N \in \mathbb{F}$ and $c_{N-1} + Nc_N v \in \mathbb{F}$, which is impossible since, by assumption, $c_{N-1} \in \mathcal{V}^0$. If $N = 1$, then $c'_1 = 0$, namely $c_1 \in \mathbb{F}$, and therefore $f \in \mathbb{F}v \oplus \mathcal{V}^-$. Finally, if $N \leq 0$, then $f \in \mathcal{V}^-$. This completes the proof of part (b). A similar argument can be used to prove part (c). Indeed, let $k \geq 1$ and assume that $f' \in \mathbb{F} \oplus \frac{1}{v^k}\mathcal{V}^-$. If $N \geq -k+2$, then $c'_N = 0$, $c'_{N-1} + Nc_N v' = 0$ (when $N = 0$ or 1 we are using the fact that $\mathbb{F} \cap \partial\mathcal{V} = 0$), namely $0 \neq c_N \in \mathbb{F}$ and $c_{N-1} + Nc_N v \in \mathbb{F}$. This is impossible since, by assumption, $c_{N-1} \in \mathcal{V}^0$. If $N = -k+1$, then $c'_{-k+1} = 0$, namely $c_{-k+1} \in \mathbb{F}$, and therefore $f \in \mathbb{F}\frac{1}{v^{k-1}} \oplus \frac{1}{v^k}\mathcal{V}^-$. Finally, if $N \leq k$, then $f \in \frac{1}{v^k}\mathcal{V}^-$. □

Lemma 7.3. *(a) If $\epsilon = 0$, we have $f_n^{0,\alpha} \in \frac{1}{v}\mathcal{V}^-$ and $g_n^{0,\alpha} \in \mathbb{F} \oplus \frac{1}{v}\mathcal{V}^-$ for every $n = 0, \dots, N$.*
(b) If $\epsilon = 1$, we have $f_n^{1,\alpha} \in \mathcal{V}^+$ and $g_n^{1,\alpha} \in v^2\mathcal{V}^+$ for every $n = 0, \dots, N$.

Proof. First, let us prove part (a) by induction on $n \geq 0$. The claim clearly holds for $n = 0$ by (5.1). The inductive assumption, together with the first equation in (7.3), implies that $(vf_n^{0,\alpha})' \in \frac{1}{v}\mathcal{V}^-$. Therefore $f_n^{0,\alpha} \in \frac{1}{v}\mathcal{V}^-$ thanks to Lemma 7.2(c) for $k = 1$. Furthermore, the second equation in (7.3) implies that $(g_n^{0,\alpha})' \in \frac{1}{v}\mathcal{V}^-$, so that, by Lemma 7.2(c) with $k = 1$, we get that $g_n^{0,\alpha} \in \mathbb{F} \oplus \frac{1}{v}\mathcal{V}^-$. Similarly, we prove part (b) again by induction on $n \geq 0$. For $n = 0$ the claim holds by (5.2). The first equation in (7.4) and Lemma 7.2(a) imply that $g_n^{1,\alpha} \in v^2\mathcal{V}^+$, while the second equation in (7.4) and Lemma 7.2(a) imply that $f_n^{1,\alpha} \in \mathcal{V}^+$. □

Lemma 7.4. *For every $n = 0, \dots, N$ we have*

$$\textstyle\int \xi^{\epsilon,1-\alpha} \cdot H_{1-\epsilon}(\partial)\xi_n^{\epsilon,\alpha} \in C_{-1}(H_\epsilon). \tag{7.7}$$

Proof. By Proposition 5.1(b) and (e) and by Theorem 3.3, all elements $\xi_n^{\epsilon,\alpha}$ are closed, and therefore, by Theorem 3.5, they are exact in $\widetilde{\mathcal{V}}$, i.e. there exist $\int h_n^{\epsilon,\alpha} \in \widetilde{\mathcal{V}}/\partial\widetilde{\mathcal{V}}$ such that $\xi_n^{\epsilon,\alpha} = \delta(\int h_n^{\epsilon,\alpha})$ for all $n = 0, \dots, N$. In the Lie superalgebra $\mathcal{W}$ of variational polyvector fields we have (cf. Eqs. (3.2) and (3.4)):

$$\textstyle\int \xi^{\epsilon,1-\alpha} \cdot H_{1-\epsilon}(\partial)\xi_n^{\epsilon,\alpha} = [[H_{1-\epsilon}, \int h_n^{\epsilon,\alpha}], \int h^{\epsilon,1-\alpha}],$$

and the assumption (7.2) reads $[H_\epsilon, \int h_{n-1}^{\epsilon,\alpha}] = [H_{1-\epsilon}, \int h_n^{\epsilon,\alpha}]$, for all $n = 0, \dots, N$ (where we let $\int h_{-1}^{\epsilon,\alpha} = 0$). Condition (7.7) then holds by Lemma 2.1(e). □

Lemma 7.5. *If* $\epsilon \in \{0, 1\}$ *and* $\alpha = 1$, *we have, for every* $n = 0, \dots, N$,

$$\textstyle\int \xi^{\epsilon,0} \cdot H_{1-\epsilon}(\partial)\xi_n^{\epsilon,1} = 0. \tag{7.8}$$

Proof. For $n = 0$ the claim holds by Proposition 5.2. For arbitrary $n \geq 1$ we will prove that Eq. (7.8) holds separately for $\epsilon = 0$ and $\epsilon = 1$.

Let us consider first the case $\epsilon = 0$. By Lemma 7.4, we have that

$$\textstyle\delta \int \xi^{0,0} \cdot H_1(\partial)\xi_n^{0,1} \ \in \ \mathrm{Ker}(H_0(\partial)). \tag{7.9}$$

Recalling Proposition 5.1(a), condition (7.9) says that there exist $\alpha, \beta \in \mathbb{F}$ such that:

$$\textstyle\delta \int \xi^{0,0} \cdot H_1(\partial)\xi_n^{0,1} = \alpha\xi^{0,0} + \beta\xi^{0,1}. \tag{7.10}$$

In other words, recalling the definition (4.3) of H_1, we have

$$\begin{aligned} &\frac{\delta}{\delta u}\Big(\frac{1}{v^2}(f_n^{0,1})' - \frac{1}{v^2}Q(\partial)\frac{g_n^{0,1}}{v^2}\Big) = \beta\frac{1}{v}, \\ &\frac{\delta}{\delta v}\Big(\frac{1}{v^2}(f_n^{0,1})' - \frac{1}{v^2}Q(\partial)\frac{g_n^{0,1}}{v^2}\Big) = \alpha + \beta\Big(-\frac{u}{v^2} + \frac{v''}{v^3} - \frac{3}{2}\frac{(v')^2}{v^4}\Big). \end{aligned} \tag{7.11}$$

By Lemma 7.3(a) both elements $f_n^{0,1}$ and $g_n^{0,1}$ lie in the differential subalgebra $\mathcal{V}^-$. Note that the space $\frac{1}{v^2}\mathcal{V}^-$ is preserved by both $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$. It follows that the LHS's of both Eqs. (7.11) lie in $\frac{1}{v^2}\mathcal{V}^-$. It immediately follows that, necessarily, $\beta = 0$ (looking at the first equation), and $\alpha = 0$ (looking at the second equation). We thus have so far, by Eq. (7.10), that

$$\textstyle\delta \int \xi^{0,0} \cdot H_1(\partial)\xi_n^{0,1} = 0.$$

Recalling that $\mathrm{Ker}\,\delta = \int \mathbb{F}$, this means that

$$\textstyle\int \xi^{0,0} \cdot H_1(\partial)\xi_n^{0,1} = \int \gamma, \tag{7.12}$$

for some $\gamma \in \mathbb{F}$, and we need to prove that $\gamma = 0$. By writing explicitly Eq. (7.12), we have that

$$\frac{1}{v^2}(f_n^{0,1})' - \frac{1}{v^2}Q(\partial)\frac{g_n^{0,1}}{v^2} - \gamma \in \partial\mathcal{V}. \tag{7.13}$$

Lemma 7.2(c) with $k = 2$ then implies that the LHS of (7.13) lies in $\partial(\mathbb{F}\frac{1}{v} \oplus \frac{1}{v^2}\mathcal{V}^-) \subset \frac{1}{v^2}\mathcal{V}^-$, and therefore $\gamma = 0$.

For the case $\epsilon = 1$ we will use a similar argument. By Lemma 7.4, we have

$$\textstyle\delta \int \xi^{1,0} \cdot H_0(\partial)\xi_n^{1,1} \ \in \ \mathrm{Ker}(H_1(\partial)). \tag{7.14}$$

Recalling Proposition 5.1(d), condition (7.14) is saying that there exist $\alpha, \beta \in \mathbb{F}$ such that:

$$\textstyle\delta \int \xi^{1,0} \cdot H_0(\partial)\xi_n^{1,1} = \alpha\xi^{1,0} + \beta\xi^{1,1}. \tag{7.15}$$

In other words, recalling the definition (4.2) of H_0, we have

$$\begin{aligned} \frac{\delta}{\delta u}\big(u(f_n^{1,1})' + v(g_n^{1,1})'\big) &= \alpha + \beta(u'' + 4u^2), \\ \frac{\delta}{\delta v}\big(u(f_n^{1,1})' + v(g_n^{1,1})'\big) &= \frac{1}{2}\beta v^2. \end{aligned} \tag{7.16}$$

For $h \in \mathcal{V}$, denote by $D_{h,1}(\partial) = \sum_{n\in\mathbb{Z}_+} \frac{\partial h}{\partial u^{(n)}}\partial^n$, $D_{h,2}(\partial) = \sum_{n\in\mathbb{Z}_+} \frac{\partial h}{\partial v^{(n)}}\partial^n$, its Frechet derivatives, and by $D_{h,1}(\partial)^*$ and $D_{h,2}(\partial)^*$ the corresponding adjoint operators. Recalling the definition of the variational derivatives, Eq. (7.16) can be equivalently rewritten as follows:

$$\begin{aligned} (f_n^{1,1})' - D^*_{f_n^{1,1},1}(\partial)u' - D^*_{g_n^{1,1},1}(\partial)v' &= \alpha + \beta(u'' + 4u^2) \\ (g_n^{1,1})' - D^*_{f_n^{1,1},2}(\partial)u' - D^*_{g_n^{1,1},2}(\partial)v' &= \frac{1}{2}\beta v^2. \end{aligned} \tag{7.17}$$

Note that $\xi_n^{1,1}$ is a closed element of $\mathcal{V}^2$, namely it has self-adjoint Frechet derivative. This means that $D^*_{f_n^{1,1},1}(\partial) = D_{f_n^{1,1},1}(\partial)$, $D^*_{g_n^{1,1},2}(\partial) = D_{g_n^{1,1},2}(\partial)$, and $D^*_{f_n^{1,1},2}(\partial) = D_{g_n^{1,1},1}(\partial)$. Hence, Eq. (7.17) give

$$\begin{aligned} (f_n^{1,1})' - D_{f_n^{1,1},1}(\partial)u' - D_{f_n^{1,1},2}(\partial)v' &= \alpha + \beta(u'' + 4u^2), \\ (g_n^{1,1})' - D_{g_n^{1,1},1}(\partial)u' - D_{g_n^{1,1},2}(\partial)v' &= \frac{1}{2}\beta v^2. \end{aligned} \tag{7.18}$$

The polynomial ring $\mathcal{V}^+ = \mathbb{F}[u, v, u', v', u'', v'', \dots]$ admits the polynomial degree decomposition $\mathcal{V}^+ = \bigoplus \mathcal{V}^+[k]$, where $\mathcal{V}^+$ consists of homogeneous polynomials of degree k. For $h \in \mathcal{V}^+$, denote by $h = \sum_{k\in\mathbb{Z}_+} h[k]$ its decomposition in homogeneous components. Note that $\mathcal{V}^+[0] = \mathbb{F}$ and $\partial\mathcal{V} \cap \mathbb{F} = 0$. It follows, by looking at the homogenous components of degree 0 in both sides of the first equation of (7.18), that $\alpha = 0$. Furthermore, by looking at the homogenous components of degree 2 in both sides of the second equation of (7.18), we get

$$(g_n^{1,1}[2])' - D_{g_n^{1,1}[2],1}(\partial)u' - D_{g_n^{1,1}[2],2}(\partial)v' = \frac{1}{2}\beta v^2. \tag{7.19}$$

On the other hand, by looking at the homogenous components of degree 0 in both sides of the first equation of (7.4), we get that $g_n^{1,\alpha}[2] \in \mathbb{F}v^2$. But then the LHS of Eq. (7.19) is equal to zero, and therefore $\beta = 0$. We thus have so far, by Eq. (7.15), that

$$\textstyle\int \xi^{1,0} \cdot H_0(\partial)\xi_n^{1,1} = \int \gamma, \tag{7.20}$$

for some $\gamma \in \mathbb{F}$, and we need to prove that $\gamma = 0$. By writing explicitly Eq. (7.20), we get

$$\gamma = u(f_n^{1,1})' + v(g_n^{1,1})' + r', \tag{7.21}$$

for some $r \in \mathcal{V}$. By Lemma 7.3 we have that $u(f_n^{1,1})' + v(g_n^{1,1})' \in \bigoplus_{k\geq 1} \mathcal{V}^+[k]$. Moreover, by Lemma 7.2(a) we can assume $r \in \mathcal{V}^+$, and therefore $r' \in \bigoplus_{k\geq 1} \mathcal{V}^+[k]$. It follows by Eq. (7.21) that $\gamma = 0$. □

Corollary 7.6. *If* $\alpha = 1$*, there exists* $\xi_{N+1}^{\epsilon,1} \in \mathcal{V}^2$ *solving the equation*

$$H_{1-\epsilon}(\partial)\xi_N^{\epsilon,1} = H_\epsilon(\partial)\xi_{N+1}^{\epsilon,1}.$$

Proof. By the usual inductive argument, based on the recursive relations (7.2) and the fact that H_0 and H_1 are skew-adjoint (as in the proof of Lemma 2.1(a)), we know that $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,1} \perp \xi_0^{\epsilon,1}$. Moreover, Lemma 7.5 says that $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,1} \perp \xi_0^{\epsilon,0}$. Therefore, $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,1} \perp \mathrm{Ker}(H_\epsilon)$, which, by the strong skew-adjointness property of H_ϵ (cf. Proposition 5.1(c) and (f)), implies that $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,1}$ lies in the image of $H_\epsilon(\partial)$, proving the claim. □

Proof of Theorem 6.1. For $\epsilon \in \{0,1\}$, by Corollary 7.6 there exists an infinite sequence $\{\xi_n^{\epsilon,1}\}_{n\in\mathbb{Z}_+}$ starting with $\xi_0^{\epsilon,1} = \xi^{\epsilon,1}$ and satisfying the Lenard–Magri recursive relations $H_{1-\epsilon}(\partial)\xi_{n-1}^{\epsilon,1} = H_\epsilon(\partial)\xi_n^{\epsilon,1}$ for all $n \in \mathbb{Z}_+$ (where $\xi_{-1}^{\epsilon,1} = 0$). Next, we want to prove that the statement of Corollary 7.6 also holds for $\alpha = 0$. Let $\{\xi_n^{\epsilon,0}\}_{n=0}^N$ be a finite sequence starting with $\xi_0^{\epsilon,0} = \xi^{\epsilon,0}$ and satisfying the Lenard–Magri recursive relations $H_{1-\epsilon}(\partial)\xi_{n-1}^{\epsilon,0} = H_\epsilon(\partial)\xi_n^{\epsilon,0}$ for all $n = 0,\dots,N$ (where $\xi_{-1}^{\epsilon,\alpha} = 0$). By the usual inductive argument we know that $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,0} \perp \xi_0^{\epsilon,0}$. Moreover, we have, by the Lenard–Magri relations and skew-adjointness of H_0 and H_1,

$$\begin{aligned}&\textstyle\int \xi_0^{\epsilon,1}\cdot H_{1-\epsilon}(\partial)\xi_N^{\epsilon,0} = -\int \xi_N^{\epsilon,0}\cdot H_{1-\epsilon}(\partial)\xi_0^{\epsilon,1} = -\int \xi_N^{\epsilon,0}\cdot H_\epsilon(\partial)\xi_1^{\epsilon,1}\\ &\quad = \textstyle\int \xi_1^{\epsilon,1}\cdot H_\epsilon(\partial)\xi_N^{\epsilon,0} = \int \xi_1^{\epsilon,1}\cdot H_{1-\epsilon}(\partial)\xi_{N-1}^{\epsilon,0} = \dots = \int \xi_{N+1}^{\epsilon,1}\cdot H_{1-\epsilon}(\partial)\xi_{-1}^{\epsilon,0} = 0.\end{aligned}$$

Hence, $H_{1-\epsilon}(\partial)\xi_{n-1}^{\epsilon,0} \perp \mathrm{Ker}(H_\epsilon)$, and therefore, by the same argument as in the proof of Corollary 7.6, there exists $\xi_{N+1}^{\epsilon,0} \in \mathcal{V}^2$ solving the equation $H_{1-\epsilon}(\partial)\xi_N^{\epsilon,0} = H_\epsilon(\partial)\xi_{N+1}^{\epsilon,0}$. Therefore, the given finite sequence can be extended to an infinite sequence $\{\xi_n^{\epsilon,0}\}_{n\in\mathbb{Z}_+}$ satisfying the Lenard–Magri recursive relations.

So far, for each $\epsilon,\alpha \in \{0,1\}$, we have an infinite sequence $\{\xi_n^{\epsilon,\alpha}\}_{n\in\mathbb{Z}_+}$ satisfying the Lenard–Magri recursive relations (7.2). By Theorems 3.3 and 3.5 we know that all the elements of these sequences are exact in $\widetilde{\mathcal{V}}$, i.e. there are elements $\int h_n^{\epsilon,\alpha} \in \widetilde{\mathcal{V}}/\partial\widetilde{\mathcal{V}}$ such that $\delta h_n^{\epsilon,\alpha} = \xi_n^{\epsilon,\alpha}$ for all $\epsilon,\alpha \in \{0,1\}$, $n \in \mathbb{Z}_+$. and the relations (7.2) on the elements $\xi_n^{\epsilon,\alpha}$'s translate to the Eq. (6.6) on the elements $\int h_n^{\epsilon,\alpha}$'s. Part (a) of the theorem is then proved, and part (b) is an immediate consequence of Lemma 2.1(a), (b) and (c).

We are left to prove part (c). By looking at the recursive equations (7.3) it is not hard to compute the differential orders $|f_n^{\epsilon,\alpha}|$ and $|g_n^{\epsilon,\alpha}|$ of all the entries of each element of the four sequences. We have for every $n \geq 1$:

$$\begin{aligned}&|f_n^{0,0}| = 6n-2,\ |g_n^{0,0}| = 6n,\ |f_n^{0,1}| = 6n,\ |g_n^{0,1}| = 6n+2,\\ &|f_n^{1,0}| = 6n-2,\ |g_n^{1,0}| = 6(n-1),\ |f_n^{1,1}| = 6n+2,\ |g_n^{1,1}| = 6n-2.\end{aligned}$$

Hence, the differential orders of the higher symmetries are $|P_n^{0,0}| = 6n+5$, $|P_n^{0,1}| = 6n+7$, $|P_n^{1,0}| = 6n+1$, $|P_n^{1,1}| = 6n+5$. Claim (c) follows. □

Remark 7.7. Since the algebra of differential polynomials $\mathcal{V}^+$ is normal, it follows from Lemma 7.3(b) that all integrals of motion $\int h_n^{1,\alpha}$ lie in $\mathcal{V}^+/\partial\mathcal{V}^+$.

Remark 7.8. From Eq. (7.3) and Lemma 7.2(c) it is easy to see that, for every $\alpha \in \{0,1\}$ and $n \geq 0$, we have $f_n^{0,\alpha} \in \mathbb{F}\frac{1}{v} \oplus \frac{1}{v^2}\mathcal{V}^-$, $g_n^{0,\alpha} \in \frac{1}{v^2}\mathcal{V}^-$. It follows from the arguments in the proof of [BDSK09, Thm.3.2] that all integrals of motion $\int h_n^{0,\alpha}$ lie in $\mathcal{V}^-/\partial\mathcal{V}^-$.

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