Nonuniqueness of infinity ground states

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Calculus of Variations

Nonuniqueness of infinity ground states

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Abstract In this paper, we construct a dumbbell domain for which the associated principal ∞-eigenvalue is not simple. This gives a negative answer to the outstanding problem posed in Juutinen et al. (Arch Ration Mech Anal 148(2):89–105, 1999; The infinity Laplacian: examples and observations, 2001). It remains a challenge to determine whether simplicity holds for convex domains.

1 Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$. According to Juutinen–Lindqvist–Manfredi [2], a continuous function $u \in C(\bar{\Omega})$ is said to be an *infinity ground state in* Ω if it is a positive viscosity solution of the following equation:

$$\begin{cases} \max\left\{\lambda_\infty - \frac{|Du|}{u},\ \Delta_\infty u\right\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here

$$\lambda_\infty = \lambda_\infty(\Omega) = \frac{1}{\max_\Omega d(x, \partial\Omega)}$$

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is the principal ∞-eigenvalue, and Δ_∞ is the infinity Laplacian operator, i.e,

$$\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}.$$

The above equation is the limit as $p \to +\infty$ of the equation

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = \lambda_p^p |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is the Euler–Lagrange equation of minimizing the nonlinear Rayleigh quotient

$$\inf_{\phi \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |D\phi|^p \, dx}{\int_\Omega |\phi|^p \, dx}$$

and λ_p is the principal eigenvalue of p-Laplacian. Precisely speaking, let u_p be a positive solution of equation (1.2) satisfying

$$\int_\Omega u_p^p \, dx = 1.$$

If u_∞ is a limiting point of $\{u_p\}$, i.e, there exists a subsequence $p_j \to +\infty$ such that

$$u_{p_j} \to u_\infty \quad \text{uniformly in } \bar{\Omega},$$

it was proved in [2] that u_∞ is a viscosity solution of the Eq. (1.1) and

$$\lim_{p \to +\infty} \lambda_p = \lambda_\infty.$$

We say that u is *a variational infinity ground state* if it is a limiting point of $\{u_p\}$.

A natural problem regarding Eq. (1.1) is to deduce whether or not infinity ground states in a given domain are unique up to a multiplicative factor; in this case, λ_∞ is said to be *simple*. The simplicity of λ_∞ has only been established for those domains where the distance function $d(x, \partial\Omega)$ is an infinity ground state ([7]). Such domains includes the ball, stadium, and torus. It has been a significant outstanding open problem to verify if simplicity holds in general domains or to exhibit an example for which simplicity fails. In this paper, we resolve this problem by constructing a planar domain where simplicity fails to hold. It is not clear to us whether variational infinity ground states are unique. Our result, however, shows that variational infinity ground states in general are not continuous with respect to domain. A somewhat similar nonuniqueness result has been proved very recently for the nonlocal infinity eigenvalue problem ([5]). Surprisingly, the nonlocal version is much simpler. Its ground states possess several interesting properties which are not true in the local case. In particular, nonlocal infinity ground states even have explicit representation formulas.

For $\delta \in (0, 1)$, denote the dumbbell

$$D_0 = B_1(\pm 5e_1) \cup R$$

for $R = (-5, 5) \times (-\delta, \delta)$ and $e_1 = (1, 0)$. Throughout this paper, $B_r(x)$ represents the open ball centered at x with radius r. The following is our main result (Fig. 1)

Theorem 1 *There exists $\delta_0 > 0$ such that when $\delta \le \delta_0$, the dumbbell D_0 possesses an infinity ground state u_∞ which satisfies $u_\infty(5, 0) = 1$ and $u_\infty(-5, 0) \le \frac{1}{2}$. In particular, u is not a variational ground state and $\lambda_\infty(D_0)$ is not simple.*

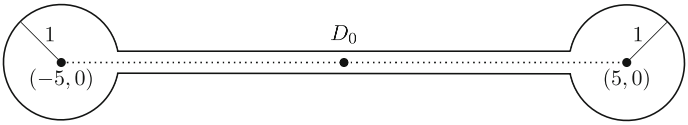

Fig. 1 The dumbbell domain D_0

We remark that the infinity ground state described in the theorem is nonvariational simply because it is not symmetric with respect to the x_2-axis, which variational ground states can be showed to be. This immediately follows from the fact that λ_p is simple, which implies any solution u_p of (1.2) on $\Omega = D_0$ must be symmetric with respect to the x_2-axis. We also remark that the number "$\frac{1}{2}$" in the above theorem is not special. By choosing a suitable δ_0, we can in fact make $u_\infty(-5, 0)$ less than any positive number.

For the reader's convenience, we sketch the idea of the proof. Consider the union of two disjoint balls with distinct radius $U_\epsilon = B_1(5e_1) \cup B_{1-\epsilon}(-5e_1)$ for $\epsilon \in (0, 1)$. If u is an infinity ground state of U_ϵ, the uniqueness of λ_∞ ([2]) immediately implies that $u \equiv 0$ in $B_{1-\epsilon}(-5e_1)$. A similar conclusion also holds for the principal eigenfunction of Δ_p. It is therefore natural to expect that such a degeneracy of u on the smaller ball may change very little if we add a narrow tube connecting these two balls. The key is to get uniform control of the width of the tube as $\epsilon \to 0$ for variational infinity ground states in an asymmetric perturbation D_ϵ of D_0; this is proved in Lemma 3. Lemma 3 also implies the sensitivity of principal eigenfunctions of Δ_p when p gets large. An important step is to show that, within the narrow tube, the $W^{1,p}$ norm of principal eigenfunction of Δ_p is uniformly controlled by its maximum norm (Lemma 2). We would like to point out that such a procedure as described above does not work for finite p.

2 Proof

We first prove several lemmas. Throughout this section, we write $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The following estimate follows easily from comparison with the fundamental solution of the p-Laplacian, i.e. $|x|^{\frac{p-2}{p-1}}$.

Lemma 1 *Let $R = (-1, 1) \times (-\delta, \delta)$ for $\delta \in (0, \frac{1}{2})$. Assume that $\lambda \in (0, 2)$ and $u \leq 1$ is a positive solution of*

$$\begin{cases} -\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \lambda^p u^{p-1} & \text{in } R \\ u(t, \pm\delta) = 0 & \text{for } t \in [-1, 1]. \end{cases} \tag{2.3}$$

Then for $p \geq 7$

$$u(x) \leq 6|x \pm \delta e_2|^{\frac{p-2}{p-1}}. \tag{2.4}$$

Proof Denote $w(x) = 6|x - \delta e_2|^\alpha - \frac{1}{2}|x - \delta e_2|^{2\alpha}$ for $\alpha = \frac{p-2}{p-1}$. Note that if $w = f(u)$, then

$$\Delta_p w = |f'|^{p-2} f' \Delta_p u + (p-1)|f'|^{p-2} f'' |Du|^p.$$

Since $\Delta_p|x|^\alpha = 0$ and $|x-\delta e_2| < 2$, a direct computation using the above formula shows that for $p \geq 7$,

$$-\Delta_p w = (p-1)|x-\delta e_2|^{\frac{-p}{p-1}}\alpha^p(6-|x-\delta e_2|^\alpha)^{p-2} > \frac{4^{p-3}}{2} \geq 2^p \quad \text{in } R.$$

It is straightforward to check $w > 0$ in R and

$$w(\pm 1, x_2) \geq 4 \quad \text{for } |x_2| \leq \delta.$$

Hence

$$u(x) \leq w(x) \quad \text{on } \partial R.$$

Combining with $-\Delta_p u \leq 2^p$, (2.4) follows from the comparison principal.

The following estimate may not be optimal, but is sufficient for our purposes.

Lemma 2 *Let $R_4 = (-4,4) \times (-\delta,\delta)$ for $\delta \in (0, \frac{1}{2})$. Assume that $\lambda \leq 2$ and $u \leq 1$ is a positive solution of*

$$\begin{cases} -\Delta_p u = -\mathrm{div}(|Du|^{p-2}Du) = \lambda^p u^{p-1} & \textit{in } R_4 \\ u(t, \pm\delta) = 0 & \textit{for } t \in [-4,4]. \end{cases} \tag{2.5}$$

Then, for $p \geq 7$ and $R_1 = (-1,1) \times (-\delta,\delta)$,

$$\int_{R_1} u|Du|^{p-1}\,dx + \int_{R_1} |Du|^p\,dx \leq C_0^p. \tag{2.6}$$

Here $C_0 > 1$ is a universal constant (independent of p and δ).

Proof For $i = 1,2,3,4$, we write $R_i = (-i,i) \times (-\delta,\delta)$. Throughout the proof, $C > 1$ represents various numbers which are independent of p and δ. We first prove an estimate which is a slight modification of a well known result ([4],[6]).

Suppose that $\xi \in C_0^\infty(R_4)$ and $0 \leq \xi \leq 1$. Multiplying $u^{1-p}\xi^p$ on both sides of (2.5) and using Hölder's inequality, we get

$$S \leq \frac{p}{p-1} S^{1-\frac{1}{p}} ||D\xi||_{L^p(R_2)} + 2^{p+1},$$

where $S = \int_{R_4} |\frac{Du}{u}|^p \xi^p\,dx$. If $\frac{S}{2} \geq 2^{p+1}$, then

$$\frac{S}{2} \leq \frac{p}{p-1} S^{1-\frac{1}{p}} ||D\xi||_{L^p(R_2)}.$$

Since $(\frac{p}{p-1})^p \leq 4$, we have that

$$S = \int_{R_4} \left|\frac{Du}{u}\right|^p \xi^p\,dx \leq \max\left\{2^{p+2},\ 4 \cdot 2^p \int_{R_4} |D\xi|^p\,dx\right\}. \tag{2.7}$$

Let $g_1(t) \in C_0^\infty(-4,4)$ satisfy $0 \leq g_1 \leq 1$, $|g_1'| \leq 2$ and

$$g_1(t) = 1 \quad \text{for } t \in [-3,3].$$

Also, for $m \in \mathbb{N}$, denote $\delta_m = \delta(1 - \frac{1}{m})$. Choose $h_m(t) \in C_0^\infty(-\delta, \delta)$ such that $0 \le h_m \le 1$, $|h_m^{'}| \le \frac{2m}{\delta}$ and

$$h_m(t) = 1 \quad \text{for } t \in [-\delta_m, \delta_m].$$

For $x = (x_1, x_2)$, let $\xi_m(x_1, x_2) = g_1(x_1) h_m(x_2)$. Then

$$|D\xi_{m+1}|^p \le 2^p (2^p + |h_{m+1}^{'}|^p)$$

and

$$4 \cdot 2^p \cdot \int_{R_4} |D\xi_{m+1}|^p \, dx \le 32 \cdot 8^p + 32 \cdot 8^p \cdot \left(\frac{m+1}{\delta}\right)^{p-1} \le C^p \left(\frac{m}{\delta}\right)^{p-1}.$$

Hence by (2.7)

$$\int_{[-3,3]\times[-\delta_{m+1},\delta_{m+1}]} \left|\frac{Du}{u}\right|^p dx \le C^p \left(\frac{m}{\delta}\right)^{p-1}.$$

Owing to Lemma 1 and translation, we have that for $x = (x_1, x_2) \in [-3, 3] \times (-\delta, \delta)$

$$u(x_1, x_2) \le 6 \min\{(\delta - x_2)^{\frac{p-2}{p-1}},\ (\delta + x_2)^{\frac{p-2}{p-1}}\}.$$

In particular, we have

$$u(x_1, x_2) \le 6 \left(\frac{\delta}{m}\right)^{\frac{p-2}{p-1}} \quad \text{in } A_m,$$

where $A_m = [-3, 3] \times [\delta_m, \delta_{m+1}]$. Hence

$$\int_{A_m} |Du|^p \, dx \le C^p \cdot \left(\frac{m}{\delta}\right)^{p-1} \left(\frac{m}{\delta}\right)^{\frac{p(p-2)}{p-1}} \le C^p \cdot \left(\frac{m}{\delta}\right)^{\frac{1}{p-1}};$$

again we emphasize C is independent of p and δ.

Accordingly,

$$\int_{[-3,3]\times[0,\delta]} u^2 |Du|^p \, dx = \sum_{m=1}^{\infty} \int_{A_m} u^2 |Du|^p \, dx \le 36 \cdot C^p \sum_{m=1}^{\infty} \frac{1}{m^{\frac{3}{2}}} \le C^p.$$

Similarly, we can prove that

$$\int_{[-3,3]\times[-\delta,0]} u^2 |Du|^p \, dx \le C^p,$$

and therefore

$$\int_{R_3} u^2 |Du|^p \, dx \le C^p.$$

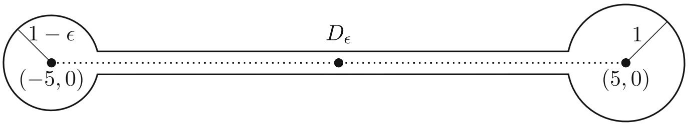

Fig. 2 The asymetric dumbbell domain D_ϵ

Using Hölder's inequality and the assumption that $u \leq 1$, we also have that

$$\begin{aligned}\int_{R_3} u^2|Du|^{p-1}\,dx &\leq 6^{\frac{1}{p}} \cdot \left(\int_{R_3} u^{\frac{2p}{p-1}}|Du|^p\,dx\right)^{\frac{p-1}{p}} \\ &\leq 2 \cdot \left(\int_{R_3} u^2|Du|^p\,dx\right)^{\frac{p-1}{p}} \\ &\leq C^p.\end{aligned}$$

Choose $g_2(t) \in C_0^\infty(-3,3)$ such that $0 \leq g_2 \leq 1$, $|g_2^{'}| \leq 2$ and

$$g_2(t) = 1 \quad \text{for } t \in [-2, 2].$$

Multiplying $w(x) = u^2 \cdot g_2(x_1)$ on both sides of (2.5) leads to

$$\int_{R_2} u|Du|^p\,dx \leq C^p.$$

Again, by Hölder's inequality, we have that

$$\int_{R_2} u|Du|^{p-1}\,dx \leq C^p.$$

Finally, select $g_3(t) \in C_0^\infty(-2,2)$ satisfying $0 \leq g_3 \leq 1$, $|g_3^{'}| \leq 2$ and

$$g_3(t) = 1 \quad \text{for } t \in [-1, 1].$$

Multiplying $w(x) = u \cdot g_3(x_1)$ on both sides of (2.5) leads to

$$\int_{R_1} |Du|^p\,dx \leq C^p.$$

Consequently, (2.6) holds, as desired. □

Remark 1 Combining Lemma 1 and 2, it is easy to see that we can refine (2.6) to be

$$\int_{R_1} u|Du|^{p-1}\,dx + \int_{R_1} |Du|^p\,dx \leq \tilde{C}_0^p \cdot \delta^{\frac{p(p-2)}{p-1}}$$

for a universal positive constant $\tilde{C}_0$ independent of p and δ.

Now let

$$\delta_0 = \frac{1}{16C_0} < \frac{1}{16}.$$

Here $C_0 > 1$ is the same number in Lemma 2. For $\epsilon \in (0, \frac{1}{2})$, write

$$D_\epsilon = B_{1-\epsilon}(-5e_1) \cup R \cup B_1(5e_1)$$

and $R = (-5, 5) \times (-\delta, \delta)$. Note that D_ϵ is not symmetric with respect to the x_2 axis and $\max_{D_\epsilon} d(x, \partial D_\epsilon) = 1$ (Fig. 2).

The following lemma says that the principal eigenfunction of p-Laplacian, although unique up to multiplicative factor, is actually very sensitive to the domain when p gets large. Especially, it implies that variational infinity ground states are not continuous with respect to domain.

Lemma 3 *Assume* $0 < \epsilon < \frac{1}{2}$. *If* $\delta \le \delta_0$ *and* u_∞ *is a variational infinity ground state of* D_ϵ *satisfying* $u_\infty(5, 0) = 1$, *then*

$$u_\infty(-5, 0) < \frac{1}{2}.$$

Note that δ_0 *is independent of* ϵ.

Proof We argue by contradiction and assume that $u_\infty(-5, 0) \ge \frac{1}{2}$. Now fix δ and ϵ. It is easy to see that ([3]) $\max_{D_\epsilon} u_\infty = u_\infty(5, 0) = 1$ and

$$0 < u_\infty(x) \le d(x, \partial D_\epsilon) \quad \text{for } x \in D_\epsilon.$$

Hence

$$u_\infty \le \delta \le \delta_0 \quad \text{in } [-4, 4] \times [-\delta, \delta].$$

For $p > 2$, let u_p be the principal eigenfunction of Δ_p in D_ϵ satisfying $\max_{D_\epsilon} u_p = 1$ and

$$-\Delta_p u_p = -\text{div}(|Du_p|^{p-2} Du_p) = \lambda_{\epsilon,p}^p u_p^{p-1} \quad \text{in } D_\epsilon. \tag{2.8}$$

Here $\lambda_{\epsilon,p}$ is the principal eigenvalue of Δ_p associated with D_ϵ (Fig. 2).

Passing to a subsequence if necessary, we may assume that

$$\lim_{p \to +\infty} u_p = u_\infty \quad \text{uniformly in } D_\epsilon.$$

Hence, when p is large enough,

$$u_p \le 2\delta_0 \quad \text{in } [-4, 4] \times [-\delta, \delta]. \tag{2.9}$$

Since $\lim_{p \to +\infty} \lambda_{\epsilon,p} = \lambda_{\epsilon,\infty} = 1$, we may assume that $\lambda_{\epsilon,p} \le 2$.

Now, define $g(t)$ by

$$\begin{cases} g(t) = 1 & \text{for } t \le -1 \\ g(t) = \frac{1}{2}(1 - t) & \text{for } -1 \le t \le 1 \\ g(t) = 0 & \text{for } t \ge 1. \end{cases}$$

Let

$$w(x) = u_p \cdot g(x_1).$$

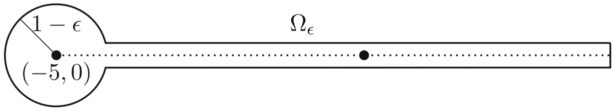

Fig. 3 The domain Ω_ϵ

and, for $\tilde{R} = (-5, 4) \times (-\delta, \delta)$, let

$$\Omega_\epsilon = B_{1-\epsilon}(-5e_1) \cup \tilde{R}.$$

Note that $\{w \neq 0\} \subset \Omega_\epsilon$ and therefore

$$\Lambda_{\epsilon,p}^p \leq \frac{\int_{\Omega_\epsilon} |Dw|^p \, dx}{\int_{\Omega_\epsilon} |w|^p \, dx} = \frac{\int_{D_\epsilon} |Dw|^p \, dx}{\int_{D_\epsilon} |w|^p \, dx}, \tag{2.10}$$

where $\Lambda_{\epsilon,p}$ is the principal eigenvalue of Δ_p associated with Ω_ϵ (Fig. 3).

Since u_p is uniformly Hölder continuous and $\lim_{p\to+\infty} u_p(-5e_1) = u_\infty(-5e_1)$, there exists $\tau \in (0, 1)$ such that

$$u_p(x) \geq \frac{1}{3} \quad \text{in } B_\tau(-5e_1), \tag{2.11}$$

for sufficiently large p.

To simplify notation, we now drop the p dependence and write $u_p = u$. Multiplying $ug^p(x_1)$ on both sides of (2.8), we have that

$$\frac{\int_{D_\epsilon} |Du|^p g^p \, dx}{\int_{D_\epsilon} |w|^p \, dx} \leq \lambda_{\epsilon,p}^p + \frac{p \int_{[-1,1]\times[-\delta,\delta]} u|Du|^{p-1} \, dx}{\int_{D_\epsilon} |w|^p}.$$

Due to Lemma 2 and (2.9)

$$\int_{[-1,1]\times[-\delta,\delta]} u|Du|^{p-1} \, dx \leq (2\delta_0 C_0)^p < \frac{1}{4^p}.$$

Therefore owing to (2.11),

$$\frac{p \int_{[-1,1]\times[-\delta,\delta]} u|Du|^{p-1} \, dx}{\int_{D_\epsilon} |w|^p} \leq \left(\frac{3}{4}\right)^p \frac{p}{\pi\tau^2}.$$

Since $Dw = gDu + uDg$ and $(a + b)^p \leq 2^p(a^p + b^p)$, we have that

$$\begin{aligned}
\int_{D_\epsilon} |Dw|^p \, dx &\leq \int_{D_\epsilon} |Du|^p g^p \, dx + 2^p \int_{[-1,1]\times[-\delta,\delta]} (|Du|^p g^p + \frac{u^p}{2^p}) \, dx \\
&\leq \int_{D_\epsilon} |Du|^p g^p \, dx + (\delta_0 4C_0)^p + (2\delta_0)^p \\
&\leq \int_{D_\epsilon} |Du|^p g^p \, dx + 2 \cdot \frac{1}{4^p}.
\end{aligned}$$

The first inequality is also due to the fact that

$$Dw = gDu \quad \text{in } D_\epsilon \backslash [-1, 1] \times [-\delta, \delta] .$$

Therefore by (2.11) when p is large enough

$$\frac{\int_{D_\epsilon} |Dw|^p \, dx}{\int_{D_\epsilon} |w|^p \, dx} \leq \lambda_{\epsilon,p}^p + 3 \cdot \left(\frac{3}{4}\right)^p \frac{p}{\pi \tau^2} \leq \lambda_{\epsilon,p}^p + 1. \tag{2.12}$$

Since $\max_{D_\epsilon} d(x, \partial D_\epsilon) = 1$ and $\max_{\Omega_\epsilon} d(x, \partial \Omega_\epsilon) = 1 - \epsilon$, we have $\Lambda_{\epsilon,p} \to (1-\epsilon)^{-1}$ and $\lambda_{\epsilon,p} \to 1$ as $p \to \infty$.

Thus, for sufficiently large p, we have

$$\Lambda_{\epsilon,p} \geq \frac{1}{1 - \frac{1}{2}\epsilon} \quad \text{and} \quad \lambda_{\epsilon,p} \leq \frac{1}{1 - \frac{1}{4}\epsilon}.$$

Owing to (2.10) and (2.12), we have

$$\left(\frac{2}{2-\epsilon}\right)^p \leq \left(\frac{4}{4-\epsilon}\right)^p + 1,$$

for all large enough p. Since this is a contradiction, the lemma follows.

Proof of Theorem 1: For $\epsilon \in (0, \frac{1}{2})$, let $u_{\epsilon,\infty}$ be a variational infinity ground state of D_ϵ satisfying $u_{\epsilon,\infty}(5, 0) = 1$. Since $\Delta_\infty u_{\epsilon,\infty} \leq 0$, according to [1], the sequence $\{u_{\epsilon,\infty}\}_{\epsilon>0}$ is uniformly Lipschitz continuous within any compact subset of D_0 when ϵ is small. The sequence is also controlled by $0 \leq u_{\epsilon,\infty} \leq d(x, \partial D_\epsilon)$ near the boundary. Upon a subsequence if necessary, we may assume that

$$\lim_{\epsilon \to 0} u_{\epsilon,\infty} = u_\infty.$$

Then according to Lemma 3 and stability of viscosity solutions, u_∞ is an infinity ground state of D_0 satisfying

$$u_\infty(-5, 0) \leq \frac{1}{2} \quad \text{and} \quad u_\infty(5, 0) = 1.$$

As u_∞ is not symmetric about the x_2-axis, it cannot be a variational infinity ground state associated to D_0. As there exists at least one variational ground state [2], it follows that $\lambda_\infty(D_0)$ is not simple.

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