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Ellipsoidal bounds on state trajectories for discrete-time systems with linear fractional uncertainties

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Abstract Computation of exact ellipsoidal bounds on the state trajectories of discrete-time linear systems that have time-varying or time-invariant linear fractional parameter uncertainties and ellipsoidal uncertainty in the initial state is known to be NP-hard. This paper proposes three algorithms to compute ellipsoidal bounds on such a state trajectory set and discusses the tradeoffs between computational complexity and conservatism of the algorithms. The approach employs linear matrix inequalities to determine an initial estimate of the ellipsoid that is refined by the subsequent application of the skewed structured singular value ν. Numerical examples are used to illustrate the application of the proposed algorithms and to compare the differences between them, where small conservatism for the tightest bounds is observed.

Keywords Uncertain dynamical systems · Discrete-time systems · Bounding method · Structured singular value · Linear matrix inequalities

1 Introduction

Identifying the potential ranges for the states in an uncertain dynamical system is important in many systems engineering problems such as safety analysis (Huang

The preliminary version appears in the *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference* (Kishida and Braatz 2011).

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et al. 2002), satellite control (Rokityanski and Veres 2005), and attitude estimation of aerospace and underwater vehicles (Sanyal et al. 2008). Motivated by various applications, many papers have considered the state outer bounding problem for time-varying (Durieu et al. 2001; Polyak et al. 2004; El Ghaoui and Calafiore 1999) and time-invariant perturbations (Horak 1988; Tibken and Hofer 1995; Kishida et al. 2011), with discussing the greater difficulty for time-invariant perturbations (Puig et al. 2005). The bounding of the state vector by an ellipsoid has been deeply discussed in literature for discrete-time linear dynamical systems with unknown-but-bounded uncertainties (e.g., see Schweppe (1968), Polyak et al. (2004), and citations therein), including for additive perturbations (Durieu et al. 2001), combinations of state-space matrix and additive perturbations (Polyak et al. 2004), and linear fractional perturbations (El Ghaoui and Calafiore 1999). The relative merits of uncertainty descriptions described by ellipsoids or independent upper and lower bounds on each parameter within the context of this problem have been discussed (Chernousko 2010).

In this paper, a new approach for computing tight ellipsoidal outer bounds is presented. The approach applies to time-invariant, time-varying, and mixed parametric uncertainties, with ellipsoidal initial state uncertainties. In contrast to the vast majority of the literature that assumes that the system dynamics depend on the perturbations in a restrictive way (e.g., affine); this paper (i) treats linear fractional perturbations (as in El Ghaoui and Calafiore (1999)), which includes other dependencies such as polytopic, polynomial, and rational as special cases, and (ii) presents algorithms that propagate the uncertain state for multiple time instances, which can dramatically reduce conservatism for both time-invariant and time-varying perturbations.

The key idea of the proposed approach is to first employ linear matrix inequalities (LMIs) (Boyd et al. 1994) to estimate the orientation and ratios of axis lengths of the ellipsoid, followed by application of the skewed structured singular value ν (Smith 1990) to compute two-sided bounds on the size of the ellipsoid. The first approximation step can have either constant or exponential computational complexity depending on the users' choices, and the second step employs upper and lower bounds with polynomial computational complexity. Based on this idea, three numerical algorithms that employ various strategies to reduce the computational cost are proposed.

Section 2 presents the problem statement and some mathematical background. Section 3 presents a preliminary analysis needed for Section 4 that proposes the numerical algorithms, which are applied and compared in numerical examples in Section 5. Section 6 concludes the paper.

The following notations will be used. The maximum singular value (aka the induced 2-norm) of a matrix N is defined as $\|N\|_2 = \bar{\sigma}(N) = \sqrt{\lambda_{\max}(N^*N)}$, where N^* denotes the conjugate transpose of the matrix N and $\lambda_{\max}(N)$ denotes the maximum eigenvalue of the matrix N. The determinant of a matrix N is denoted by $|N|$.

2 Problem statement and mathematical background

This paper considers the following state trajectory bounding problem.

Problem 1 Let $x_k, c_k \in \mathbb{R}^n$ denote the state and nominal state vectors at time instance $k \in \{0, 1, 2, \ldots\}$, $p \in \mathbb{R}^m$ denote a vector of uncertain real parameters, and $T : \mathbb{R}^m \to \mathbb{R}^{n\times n}$ be a rational function of its arguments. Given an uncertain value for the initial states x_0,

$$(x_0 - c_0)^T E_0 (x_0 - c_0) \leq 1, \quad E_0 > 0, \tag{1}$$

and discrete-time uncertain dynamical system

$$x_{k+1} = T(p)x_k, \; p_{\min} \leq p \leq p_{\max}, \; k = 0, 1, 2, \ldots, \tag{2}$$

determine an ellipsoidal outer bound on the state vector x_k specified by $E_k > 0$, and c_k, $k = 1, 2, 3, \ldots$, such that

$$\min \log \det E_k^{-1} \tag{3}$$

subject to

$$E_k > 0 \text{ and } (x_k - c_k)^T E_k (x_k - c_k) \leq 1,$$
$$\forall x_k \in S_k = \{x_k \text{ satisfying (1) and (2)}\}.$$

The objective (3) is to determine, for each time instance, the ellipsoid of minimum volume that outer bounds the state vector.[1] This paper proposes to approach Problem 1 through a combination of LMIs and the skewed structured singular value, which will write $T(p)$ in terms of a linear fractional transformation (LFT) that represents the uncertain real parameter p as a real structured perturbation matrix.

Definition 1 (Mixed structured perturbation (Zhou et al. 1995))
A *mixed structured perturbation* $\mathbf{\Delta}$ is a matrix with the specified structure:

$$\mathbf{\Delta} = \Big\{ \varDelta : \varDelta \in \operatorname{diag}\Big\{ \delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}}, \delta_1^c I_{k_{m_r+1}}, \ldots, \delta_{m_c}^c I_{k_{m_r+m_c}}, \varDelta_{m_r+m_c+1}^C, \ldots, \varDelta_{m_r+m_c+m_C}^C \Big\}\Big\},$$

with real scalars δ_i^r, complex scalars δ_j^c, and full complex blocks $\varDelta_q^C \in \mathbb{C}^{k_q\times k_q}$. The integers m_r, m_c, m_C, and k_i define the structure of the perturbation. A real scalar δ_i^r (or complex scalar δ_i^c) is said to be *repeated* if the integer $k_i > 1$.

Definition 2 (LFT (Zhou et al. 1995)) For any

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)}, \quad \varDelta_p \in \mathbb{C}^{q_1\times p_1},$$

such that the inverse $(I - N_{11}\varDelta_p)^{-1}$ exists, the mapping

[1] Alternative objectives, such as minimizing the trace as in El Ghaoui and Calafiore (1999), can be addressed by the algorithms in this paper by slightly modifying the first step of the LMI formulation.

$$F_u(N, \Delta_p) = N_{22} + N_{21}\Delta_p(I - N_{11}\Delta_p)^{-1}N_{12},$$

is an *(upper)* (LFT).

$I - N_{11}\Delta_p$ is not invertible for some perturbation Δ_p of interest if and only if the LFT is ill-posed. The existence of the inverse of $I - N_{11}\Delta_p$ for perturbations Δ_p within some set under consideration can be evaluated using the structured singular value (Zhou et al. 1995). To simplify the presentation, this paper assumes that this verification is carried out before applying the propsed algorithms.

To express an uncertain parameter vector $p \in \mathbb{R}^m$ defined by box constraints in terms of an LFT, let

$$\begin{aligned} p &= p_c + W_p\delta\bar{p}, \quad \|\delta\bar{p}\|_\infty \leq 1, \\ p_c &= \frac{1}{2}(p_{\max} + p_{\min}), \quad W_p = \frac{1}{2}\operatorname{diag}\{p_{\max} - p_{\min}\}, \end{aligned} \tag{4}$$

then, the uncertain system (2) can be written as

$$x_{k+1} = T(p_c + W_p\delta\bar{p})x_k \tag{5}$$

$$= F_u(N, \Delta_p)x_k \tag{6}$$

where

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad \Delta_p = \{\ \operatorname{diag}\ \{\delta\bar{p}_1 I_{k_1}, \ldots, \delta\bar{p}_m I_{k_m}\} : |\delta\bar{p}_i| \leq 1,\ i = 1, \ldots, m\},$$

where the values of k_i, $i = 1, \ldots, m$, depend on the order and structure of the map T.

The transformation from (5) to (6) is always possible for any well-posed rational function by using block-diagram algebra (Zhou et al. 1995), and by application of multidimensional realization algorithms (Russell et al. 1997). The LFT for any particular function is not unique, and LFTs are desired in which the dimension of Δ_p is minimal, so as to minimize the computational cost of the proposed algorithms. Multidimensional model reduction algorithms (e.g., see Russell and Braatz (1998) and references cited therein) can be applied to an LFT to reduce its dimensions before applying the proposed algorithms.

The proposed outer bounding algorithms utilize the skewed structured singular value and scaled main loop theorem, which are given below.

Definition 3 (Skewed structured singular value ν (Smith 1990)) Let the set of matrices with specified structure be represented in boldface font, $\mathbf{\Delta_p}$, and let the unit ball in the space of the mixed structured perturbation be $\mathbf{B\Delta} := \{\Delta : \Delta \in \mathbf{\Delta}, \bar{\sigma}(\Delta) \leq 1\}$. The skewed structured singular value ν of N with respect to mixed structured perturbation matrices $\mathbf{\Delta_p}$ and $\mathbf{\Delta}_C$ of appropriate dimensions is defined by

$$
\nu_{\Delta_p,\Delta_C}(N) = \begin{cases} 0, \\ \text{if } \forall\kappa < \infty,\ \nexists\Delta = \mathrm{diag}\{\Delta_p, \kappa\Delta_C\}, \Delta_i \in \mathbf{B\Delta}_i \text{ s.t. } |I - N\Delta| = 0, \\ \big(\min\{\kappa \geq 0 : \exists\Delta = \mathrm{diag}\{\Delta_p,\ \kappa\Delta_C\}, \\ \qquad\qquad\qquad\qquad \Delta_i \in \mathbf{B\Delta}_i \text{ s.t. } |I - N\Delta| = 0\}\big)^{-1}, \\ \text{otherwise.} \end{cases}
$$

Upper and lower bounds on the skewed structured singular value ν can be computed in polynomial time, with no more effort than non-skewed structured singular value calculations (Fan and Tits 1992; Ferreres 1999), by a variety of methods including power iterations and linear matrix inequalities. The next result relates the LFT (6) for the uncertain system (2) to the skewed structured singular value ν.

Theorem 1 (Scaled main loop theorem (Smith 1990; Fan and Tits 1992)) *For any well-posed LFT in the uncertain dynamical system* $x_{k+1} = F_u(N, \Delta_p)x_k$,

$$
\max_{\Delta_p \in \mathbf{B\Delta}_p} \|F_u(N, \Delta_p)\|_2 = \nu_{\Delta_p,\Delta_C}(N),
$$

where $\mathbf{\Delta}_C$ *is the set of full complex perturbation matrices of appropriate dimension.*

By defining the matrix N appropriately, this result can also be applied to compute bounds on the minimum diameter of an ellipsoid that overbounds the state vector when the ellipsoid's axis orientations and relative lengths are pre-specified. The next section gives some preliminary analysis used in the subsequent derivation of LMI-based algorithms for determining the ellipsoid's axis orientations and relative lengths.

3 Preliminary analysis: uncertainty only in the initial state

For a known nonsingular linear system, (2) simplifies to

$$
x_{k+1} = Tx_k, \quad T : \text{nonsingular}.
$$

Given a state x_k with uncertainty that can take any value within the ellipsoid,

$$
(x_k - c_k)^T E_k (x_k - c_k) \leq 1, \quad E_k > 0,
$$

an outer bounding ellipsoid parameterized by $E_{k+1} > 0$ and c_{k+1} satisfies

$$
(x_{k+1} - c_{k+1})^T E_{k+1} (x_{k+1} - c_{k+1}) \leq 1
$$

for all $x_{k+1} \in S_{k+1}$. By application of the S-procedure (Boyd and Vandenberghe 2004), the state is outer bounded by this ellipsoid if and only if there exists $\lambda \geq 0$ that satisfies the linear matrix inequality

$$\begin{bmatrix} \bar{E}_{k+1} & -\bar{c}_{k+1} & 0 \\ -\bar{c}_{k+1}^T & -1 & -\bar{c}_{k+1}^T \\ 0 & -\bar{c}_{k+1} & -\bar{E}_{k+1} \end{bmatrix} - \lambda \begin{bmatrix} E_k & -E_k c_k & 0 \\ -c_k^T E_k^T & c_k^T E_k c_k - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq 0, \qquad (7)$$

with variables $\bar{E}_{k+1} = T^T E_{k+1} T$, $\bar{c}_{k+1} = T^T E_{k+1} c_{k+1}$, and λ. The matrix E_{k+1} and vector c_{k+1} can be computed from $\bar{E}_{k+1}$ and $\bar{c}_{k+1}$.[2] A solution to this LMI for invertible T that gives an ellipsoid of minimum volume is (see Section 4.3 of Schweppe (1973))

$$E_{k+1} = T^{-T} E_k T^{-1}, \quad c_{k+1} = T c_k,$$

which occurs for $\lambda = 1$. From the Loewner–Behrend Theorem (Berger 1979; Pronzato and Walter 1994), this minimum-volume ellipsoid is unique.

Repeating the above procedure from the uncertain initial condition implies that the minimum-volume ellipsoids for all time instances k are given by

$$E_k = (T^{-T})^k E_0 (T^{-1})^k, \quad c_k = T^k c_0. \qquad (8)$$

These ellipsoidal covers on the states are exact, which can be observed by checking the map between the boundary of the ellipsoid at time instance k and the boundary of the ellipsoid of time instance $k+1$.

Remark 1 The above analysis can be generalized to singular T. If T is singular, then there exists a nonzero $x_1 \in S_k$ that maps to the origin, and x_1 and all subsequent x_k lie in a lower dimensional space. This lower dimensional space and its covering ellipsoid are of dimension $n - m$, where n is the dimension of the matrix T and m is the number of zero eigenvalues of T.

The next section addresses uncertain state matrices.

4 Proposed algorithms: uncertain systems

To focus on delivering the main concept of the proposed approach, this section describes the simple case of $c_k = 0$ for all k (i.e., the ellipsoid on the initial uncertain state is centered at the origin; the analysis is similar for $c_k \neq 0$). By letting $c_0 = 0$ in (7), E_{k+1} specifies an outer bounding ellipsoid for x_{k+1} for fixed T if

$$T^T E_{k+1} T \leq E_k,$$

with the E_k specifying the minimum-volume ellipsoid when the inequality is an equality (8). The proposed algorithms employ this LMI to address uncertainties in the state matrix. The three algorithms discussed in this section are:

– Algorithm I: one step ahead
  This simplest algorithm is based on the bounds on the state at the previous time instance; given E_k, compute E_{k+1} at each time instance.

[2] This derivation is simpler than an equivalent LMI derived elsewhere (El Ghaoui and Calafiore 1999).

- Algorithm II: compound
  This algorithm propagates the state instead of the state bounds as in Algorithm I between time instances; given initial uncertainty E_0, compute E_{k+1} at each time instance.
- Algorithm III: receding horizon
  This algorithm combines the update strategies in Algorithms I and II by introducing a moving horizon. s: at each time step k of $k<s$, this algorithm coincides with Algorithms II and for each time step k of $k \geq s$, given E_{k-s+1}, compute E_{k+1}.

4.1 Algorithm I: one step ahead

Recall that the uncertain system (2) is written as

$$x_{k+1} = F_u(N, \Delta_p)x_k, \quad \Delta_p \in \mathbf{B\Delta}_p, \tag{9}$$

where $x_k^T E_k x_k \leq 1$ and $E_k > 0$ are given. The minimum-volume ellipsoid is described by

$$\min \log \det E_{k+1}^{-1} \tag{10}$$

$$\begin{aligned} \text{subject to} \quad & \\ & E_{k+1} > 0, \\ & F_u(N, \Delta_p)^T E_{k+1} F_u(N, \Delta_p) - E_k \leq 0, \\ & \Delta_p \in \mathbf{B\Delta}_p. \end{aligned} \tag{11}$$

For general LFTs, it is straightforward to apply the proof technique in (Braatz et al. 1994) to show that this nonconvex optimization is NP-hard. An approximate solution Y for E_{k+1} can be obtained by replacing (11) by the

- nominal system: $F_u(N, 0)^T Y F_u(N, 0) - E_k \leq 0$ (constant computational complexity), or
- average: $\bar{F}_u(N, \Delta_p)^T Y \bar{F}_u(N, \Delta_p) - E_k \leq 0$, where $\bar{F}(N, \Delta_p)$ is an elementwise averaged matrix over multiple sampled Δ_p within the uncertainty set $\mathbf{B\Delta}_p$ (constant computational complexity), or
- extreme uncertainties: $F_u(N, \Delta_p^i)^T Y F_u(N, \Delta_p^i) - E_k \leq 0$, $i = 1, \ldots, 2^m$, where m is the dimension of parameter p and Δ_p^i taken from a set defined by diagonal matrices with all combinations of -1 and 1 as diagonal elements (exponential computational complexity).

An improved solution to (10) can be obtained by combining one of the approximations for (11) with the application of ν to determine an optimal scaling of the ellipsoid. Remember that the approximate solution was used to fix the shape of the ellipsoid, and does not mean approximate covering of the states. For specificity, the steps are described for the case when extreme uncertainties are used, with similar steps for the other cases.

Step 1: Solve

$$\begin{aligned} &\min \log\det Y_k^{-1} \\ &\text{subject to} \\ &Y_k > 0, \\ &F_u(N, \Delta_p^i)^T Y_k F_u(N, \Delta_p^i) - E_k \leq 0, \;\; i = 1, \ldots, 2^m, \end{aligned} \tag{12}$$

where each Δ_p^i has ± 1 as its diagonal elements.

Step 2: Set

$$\begin{aligned} M_{k+1,11} &= N_{11}, \quad M_{k+1,12} = N_{12} E_k^{-1/2}, \\ M_{k+1,21} &= Y_k^{1/2} N_{21}, \quad M_{k+1,22} = Y_k^{1/2} N_{22} E_k^{-1/2}, \end{aligned} \tag{13}$$

and compute upper and lower bounds on $\nu_{\Delta_p,\Delta_C}(M_{k+1})$.

Step 3: The ellipsoidal bound on the state

$$\begin{aligned} &\frac{1}{\nu^2_{\Delta_p,\Delta_C}(M_{k+1})} x_{k+1}^T Y_k x_{k+1} \leq 1, \text{or} \\ &x_{k+1}^T E_{k+1} x_{k+1} \leq 1, \quad E_{k+1} = \frac{1}{\nu^2_{\Delta_p,\Delta_C}(M_{k+1})} Y_k, \end{aligned} \tag{14}$$

is the ellipsoid of minimum volume with rotation and relative magnitude of axes defined by Y_k. Replacing ν with its upper bound in (14) results in an ellipsoid that is guaranteed to cover the state x_{k+1} for all perturbations within the uncertainty description.

4.2 Algorithm II: compound

In this algorithm, the uncertain state equation corresponding to (9) is

$$x_{k+1} = F_u(N_k, \Delta_{p,k}) x_0, \quad \Delta_{p,k} \in \mathbf{B\Delta}_{p,k}, \tag{15}$$

where $x_0^T E_0 x_0 \leq 1$ and $E_0 > 0$ are given. Therefore, the algorithm takes the following steps.

Step 1 (use extreme uncertainties): Solve

$$\begin{aligned} &\min \log\det Y_k^{-1} \\ &\text{subject to} \\ &Y_k > 0, \\ &F_u(N_k, \Delta_{p,k}^i)^T Y_k F_u(N_k, \Delta_{p,k}^i) - E_0 \leq 0, \;\; i = 1, \ldots, 2^m, \end{aligned} \tag{16}$$

where each $\Delta_{p,k}^i$ has ± 1 as its diagonal elements.

Step 2: Set

$$\begin{aligned} M_{k+1,11} &= N_{k,11}, \quad M_{k+1,12} = N_{k,12} E_0^{-1/2}, \\ M_{k+1,21} &= Y_k^{1/2} N_{k,21}, \quad M_{k+1,22} = Y_k^{1/2} N_{k,22} E_0^{-1/2}, \end{aligned} \tag{17}$$

and compute upper and lower bounds on $v_{\Delta_{p,k},\Delta_C}(M_{k+1})$.

Step 3: The ellipsoidal bound on the state

$$\begin{aligned} &\frac{1}{v^2_{\Delta_{p,k},\Delta_C}(M_{k+1})} x^T_{k+1} Y_k x_{k+1} \leq 1, \text{or} \\ &x^T_{k+1} E_{k+1} x_{k+1} \leq 1, \quad E_{k+1} = \frac{1}{v^2_{\Delta_{p,k},\Delta_C}(M_{k+1})} Y_k, \end{aligned} \tag{18}$$

is the ellipsoid of minimum volume with rotation and relative magnitude of axes defined by Y_k. Note that if the size of matrix N of Algorithm I is n, then the size of the matrix N_k in Algorithm II is bounded by nk for the same problem.

4.3 Algorithm III: receding horizon

In this algorithm, the uncertain state equation corresponds to (9) is

$$x_{k+1} = \begin{cases} F_u(N_k, \Delta_{p,k}) x_0, & \Delta_{p,k} \in \mathbf{B\Delta}_{p,k}, \quad \text{for } k < s, \\ F_u(N_s, \Delta_{p,s}) x_{k-s+1}, & \Delta_{p,s} \in \mathbf{B\Delta}_{p,s}, \quad \text{for } k \geq s. \end{cases} \tag{19}$$

Therefore, the algorithm takes the following steps for $k \geq s$ (for $k < s$, the algorithm is the same as Algorithm II).

Step 1 (use extreme uncertainties): Solve

$$\begin{gathered} \min \log \det Y_k^{-1} \\ \text{subject to} \\ Y_k > 0, \\ F_u(N_s, \Delta^i_{p,s})^T Y_k F_u(N_s, \Delta^i_{p,s}) - E_{k-s+1} \leq 0, \ i = 1, \ldots, 2^m, \end{gathered} \tag{20}$$

where each $\Delta^i_{p,s}$ has ± 1 as its diagonal elements.

Step 2: Set

$$\begin{aligned} &M_{k+1,11} = N_{s,11}, \quad M_{k+1,12} = N_{s,12} E^{-1/2}_{k-s+1}, \\ &M_{k+1,21} = Y_k^{1/2} N_{s,21}, \quad M_{k+1,22} = Y_k^{1/2} N_{s,22} E^{-1/2}_{k-s+1}, \end{aligned} \tag{21}$$

and compute upper and lower bounds on $v_{\Delta_{p,s},\Delta_C}(M_{k+1})$.

Step 3: The ellipsoidal bound on the state

$$\begin{aligned} &\frac{1}{v^2_{\Delta_{p,s},\Delta_C}(M_{k+1})} x^T_{k+1} Y_k x_{k+1} \leq 1, \text{or} \\ &x^T_{k+1} E_{k+1} x_{k+1} \leq 1, \quad E_{k+1} = \frac{1}{v^2_{\Delta_{p,s},\Delta_C}(M_{k+1})} Y_k, \end{aligned} \tag{22}$$

is the ellipsoid of minimum volume with rotation and relative magnitude of axes defined by Y_k. Note that if the size of matrix N of Algorithm I is n, then the size of the matrix N_s in Algorithm III is bounded by ns for the same problem.

4.4 Comparison of Algorithms I–III

Figure 1 and Table 1 summarize the properties and dependencies of Algorithms I–III for computing the bounds on state at one time instance. The maximum computational cost at each time step of Algorithms I and III is independent of k and the computation of the bounds on ν are polynomial-time in $\dim\{N\}$, which implies that Step 2 of Algorithms I and III are polynomial-time. Numerical studies for dense matrices have observed that a computational cost for the upper and lower bounds on ν is approximately cubic as a function of the row dimension of N (Young et al. 1995); the computational cost for the bounding algorithms when sparse-matrix algebra is used to exploit the sparseness of N.

Algorithm I applies to time-varying perturbations because it propagates the ellipsoidal bound on the state at each time instance k (i.e., E_0 is used to compute E_1, E_1 is used to compute E_2, etc.), with approximately constant computational cost per time instance.

Algorithm II can treat each real parametric uncertainty as being time-invariant or time-varying and propagates the state instead of the state bounds between time instances, which requires the use of a new LFT for each time instance. The only uncertainties at each time instance are in the initial state x_0 and the uncertain parameters. The structure of the $\Delta_{p,k}$ in the ν computations depends on the time dependency of each uncertain parameter. At each time instance k, with the given initial uncertainty E_0, E_k is computed. For the special case of all parameters being time-invariant, the matrix N and structure of the Δ_p are constructed from $F_u(N, \Delta_p)^k$. Analytical expressions for these LFTs are available (Zhou et al. 1995).

Algorithm III combines the update strategies in Algorithms I and II, so as to be more computationally efficient than Algorithm II, but with the introduction of potential conservatism. Algorithm III employs a moving horizon s: at each time instance $k < s$, this algorithm coincides with Algorithm II. As each time instance $k \geq s$, E_{k+1} is computed from E_{k-s+1} by using $F_u(N_s, \Delta_{p,s}) = F_u(N, \Delta_p)^s$ for time-invariant parameters.

Algorithms II and III can be extended to mixed time-varying and time-invariant parameter uncertainties by specifying appropriate structures for the Δ_p. For example, consider a scalar state equation $x_{k+1} = (\delta p + \delta q)x_k$ being δp time-invariant and δq time-varying whose absolute values are bounded by 1, then the state at time 2 can be expressed as

$$x_2 = F_u(N, \Delta_p)x_0$$

$$\text{where } N = \begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} & 0 \end{bmatrix}, \quad \Delta_p = \operatorname{diag}\{\delta p I_2, \delta q(0), \delta q(1)\}$$

where $\delta q(k)$ indicates the value of the uncertain parameter at time k. Regardless of the fact that δp and δq appears in the same way in the state equation, they appear differently in Δ_p due to having different time dependencies.

4.5 Remarks on the perturbation and complex ellipsoid

All of the steps also apply to complex structured perturbation matrices, in which the ellipsoid is defined over a complex space and the states can be complex. Such a system could arise when applying Fourier transforms to PDEs.

5 Numerical examples

This section gives two examples to illustrate the effectiveness of the three proposed algorithms as well as to compare their differences. The first simple example also details the usage of the proposed algorithms. In each example, uncertain parameters are bounded by known upper and lower bounds. The maximum and minimum bounds on ν were computed by using YALMIP (Löfberg 2004) and the Skew Mu Toolbox (SMT) (Ferreres and Biannic 2009).

In all figures, unless otherwise stated, the curves are the boundaries of the ellipsoids, Alg. I is red (ν lower bound) and orange (ν upper), Alg. II is purple (ν lower) and magenta (ν upper), Alg. III is green (ν lower) and blue (ν upper).

5.1 Coordinate transformation

The first numerical example uses the state-space equation:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \alpha \begin{bmatrix} 1 & p \\ -p & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}, \tag{23}$$

where $x_{1,k}$ and $x_{2,k}$ are the states at time instance k, p is a parameter, and α is a scaling constant. Equation (23) describes state vectors that rotate in 2 dimensions if $\alpha = 1/\sqrt{1+p^2}$ is used.

This simple example is a discrete-time variation of a continuous-time problem used to evaluate the accuracy of state-bounding algorithms in handling rotations in the state vector (Moore 1996). The example of a coordinate transformation that rotates and scales the state vector is a useful model problem because the application of many state-bounding algorithms, including those based on interval analysis, to such systems can produce conservatism approaching ∞ as $k \to \infty$ (Moore 1996).

The associated LFTs for each algorithms are summarized below.

$$\begin{aligned} \begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} &= F_u(N, \Delta_p) \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix}, \text{(Alg. I)} \\ &= F_u(N_k, \Delta_{p,k}) \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}, \text{(Alg. II)} \\ &= F_u(N_s, \Delta_{p,s}) \begin{bmatrix} x_{1,s-k+1} \\ x_{2,s-k+1} \end{bmatrix}, \text{(Alg. III)} \end{aligned} \tag{24}$$

where

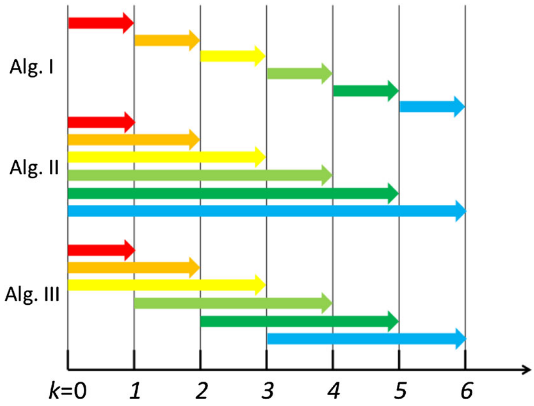


Fig. 1 The propagation of information in Algorithms I–III

Table 1 Comparison of algorithms*

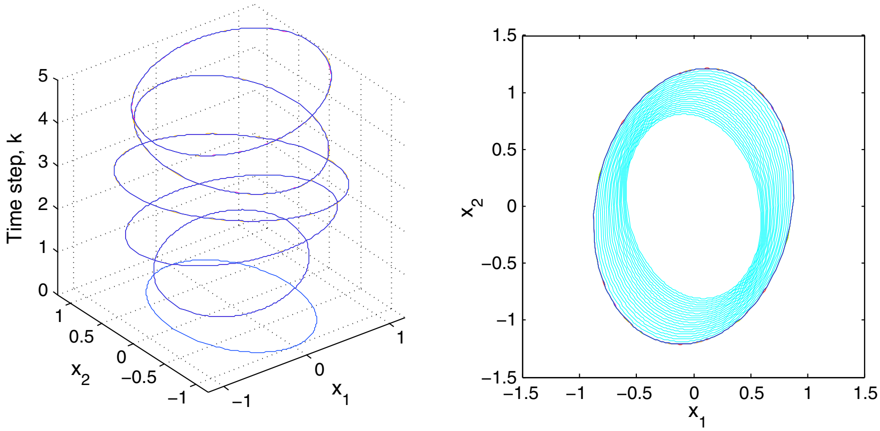
At the kth step	I	II	III
Computational cost ratio of Step 2	1	k	$\min\{k, s\}$
Bounds	Possibly loose	Tight	Moderate
Dependency	x_{k-1}	x_0	x_{k-1-s} to x_{k-1}

* Same for both time-invariant and time-varying

$$N = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \alpha \begin{bmatrix} 0 & w_p \\ -w_p & 0 \end{bmatrix} & \alpha \begin{bmatrix} 1 & p_c \\ -p_c & 1 \end{bmatrix} \end{bmatrix}, \quad \Delta_p \in \{\delta\bar{p} I_2 : -1 \le \delta\bar{p} \le 1\}, \qquad (25)$$

and N_k, N_s, $\Delta_{p,k}$, and $\Delta_{p,s}$ are given by the expressions for multiplication of LFTs (Zhou et al. 1995). The time dependency of the parameters appears in the structures of $\Delta_{p,k}$ and $\Delta_{p,s}$.

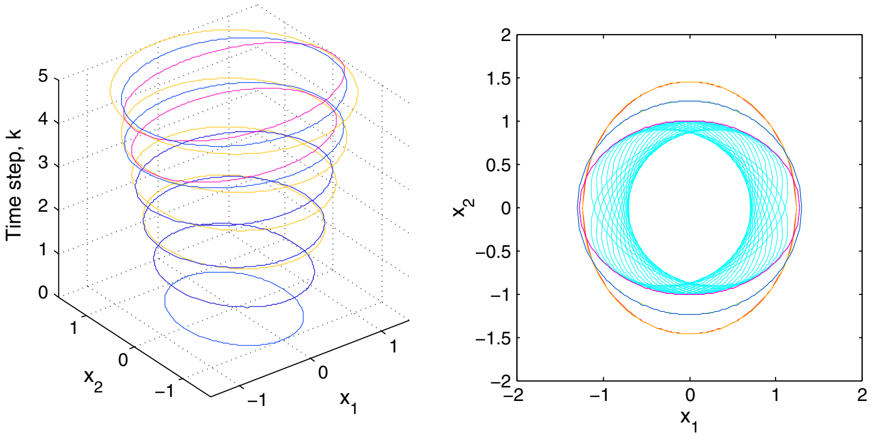
For the uncertain parameter $p \in [0.9, 1.1]$ and $\alpha = 1/\sqrt{2}$, the ellipsoidal state outer bounds obtained by Algorithms I–III are indistinguishable each other (Fig. 2a) and very tight (Fig. 2b). On the other hand, for $p \in [-0.3, 0.3]$ and $\alpha = 1$, the outer bounding ellipsoids produced by Algorithms I–III are different; with Algorithm II with time-invariant p having the smallest conservatism and Algorithm I with time-varying p having the largest, as expected from the theoretical derivations in Section 4 (see Fig. 3). None of the three algorithms have the large conservatism as obtained by interval analysis (Moore 1996). Depending on the system, Algorithm I, which treats the parameter as being time-varying, ranged from producing the same tight outer ellipsoids as Algorithms II–III to producing larger ellipsoids (compare Figs. 2 and 3).



(a) Time evolution of bounds computed by three algorithms. The ν upper bounds are plotted.

(b) Bounds computed by the three algorithms with gridded fixed values for p within the uncertainty set (cyan) at $k = 4$.

Fig. 2 Outer ellipsoids with an uncertain parameter $p \in [0.9, 1.1]$ and $\alpha = 1/\sqrt{2}$. The ellipsoidal uncertain initial state is centered at $(0, 0)$ with $E_0 = [2\ 0; 0\ 1]$, and the horizon $s = 3$ for Algorithm III. For the initial estimates, extreme uncertainties are used in the LMI optimization



(a) Time evolution of bounds computed by three algorithms. The ν upper bounds are plotted.

(b) Bounds computed by the three algorithms with gridded fixed values for p within the uncertainty set (cyan) at $k = 4$.

Fig. 3 Outer ellipsoids with time-invariant $p \in [-0.3, 0.3]$ and $\alpha = 1$. The ellipsoidal uncertain initial state is centered at $(0, 0)$ with $E_0 = [2\ 0; 0\ 1]$, and the horizon $s = 3$ for Algorithm III. For the initial estimates, extreme uncertainties are used in the LMI optimization

5.2 Consensus in a network

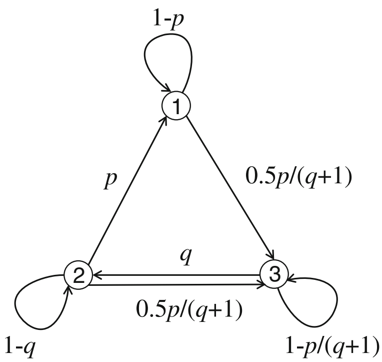
Consider a three-state network with uncertain parameters p and q, as shown in Fig. 4. The state-space equation can be written as

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \end{bmatrix} = \begin{bmatrix} 1-p & p & 0 \\ 0 & 1-q & q \\ 0.5p/(q+1) & 0.5p/(q+1) & 1-p/(q+1) \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix},$$

where $x_{1,k}$, $x_{2,k}$, and $x_{3,k}$ are the states 1, 2, and 3 at time k, respectively. The state-transition matrix is rational with respect to the parameters, and can be written by an LFT similarly as in Example 5.1.

For $0<p<1$ and $0<q<1$, the network is strongly connected and the state can achieve a consensus (Fig. 5). Each of Algorithms I–III gives a sequence of ellipsoids converging to a line in three-dimensional space, as shown Fig. 6, that indicates that consensus is achieved.

Fig. 4 A network with three states. The *numbers* on the links indicate the transition probabilities



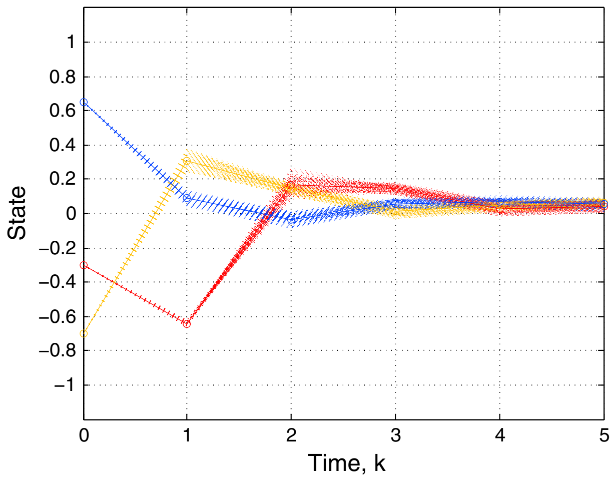


Fig. 5 Trajectories with time-invariant parameters p and q with an arbitrary initial condition. *Solid lines* are with nominal parameter values and *dotted lines* are with arbitrary parameters in the given uncertainty set $0.8 \le p \le 0.9$ and $0.6 \le q \le 0.8$. States $x_{1,k}$, $x_{2,k}$ and $x_{3,k}$ are plotted with *red*, *orange*, and *blue*, respectively. (Color figure online)

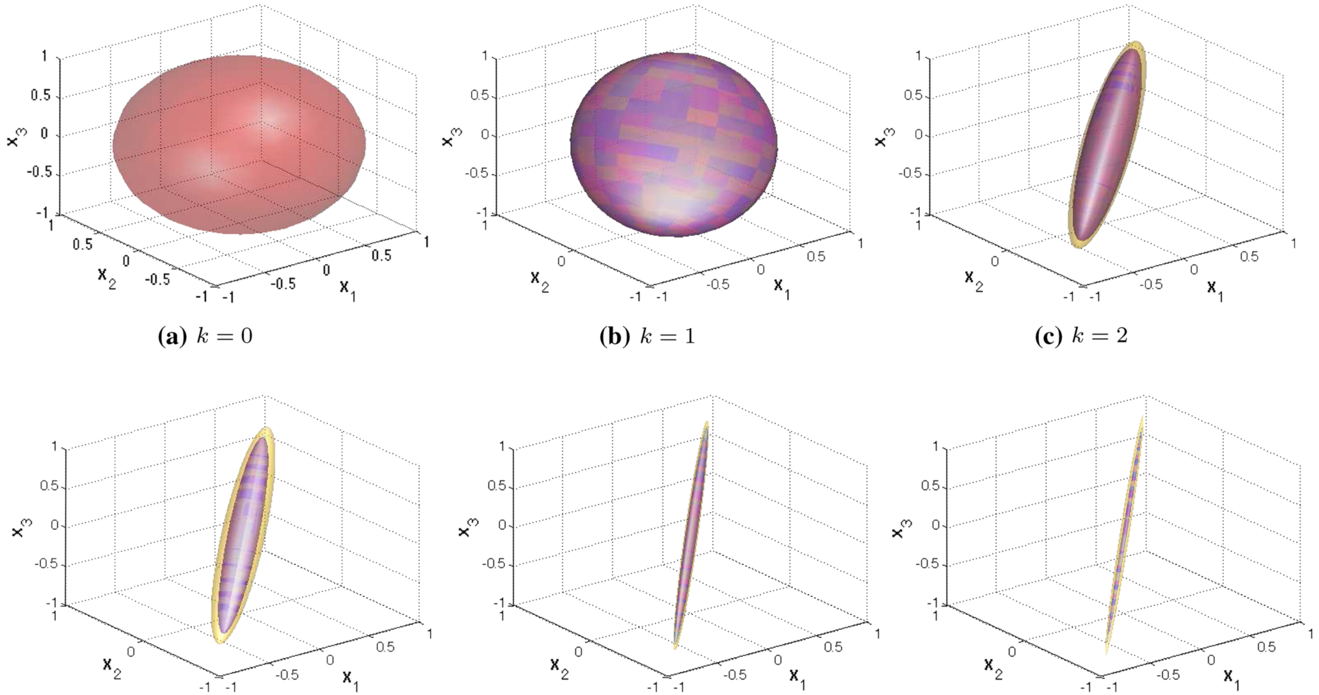


Fig. 6 Time evolution of the bounds computed by the three algorithms. The v upper bounds are plotted. The ellipsoidal uncertain initial state is centered at $(0, 0, 0)$ with $E_0 = [1\ 0\ 0; 0\ 1\ 0;\ 0\ 0\ 1]$, the uncertain parameters $p \in [0.8,\ 0.9]$ and $q \in [0.6,\ 0.8]$, and the horizon $s = 3$ for Algorithm III. For the initial estimate, extreme uncertainties are used in the LMI optimization

6 Conclusions

Algorithms are presented for computing ellipsoidal bounds on the state trajectories of discrete-time linear systems with ellipsoidal uncertainty on the initial state and time-varying or time-invariant real parametric uncertainties. Upper and lower bounds on the minimum size of the ellipsoid were determined by using the skewed structured singular value ν, with rotation and ratios of the axis lengths determined by solving quasi-convex LMI-based optimizations. The algorithms apply to systems with linear fractional dependence on the model parameters, which includes the polynomial and rational dependencies that commonly occur in applications.

Algorithm I has the lowest computational cost, but can be conservative if applied to a system with time-invariant uncertainties, or if the actual reachable sets of states are not ellipsoids. Algorithm II produced tight bounds for polynomial systems with either time-varying or time-invariant parameter uncertainties but is computationally expensive. Algorithm III employs a moving horizon to reduce the computational cost of Algorithm II, while increasing conservatism. The moving horizon can be specified in Algorithm III to trade off computational cost with tightness of the bounds, and this algorithm is the most practical for computing outer ellipsoids for large k for systems with time-invariant uncertainties.

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