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Large N Duality, Lagrangian Cycles, and Algebraic Knots

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Abstract: We consider knot invariants in the context of large N transitions of topological strings. In particular we consider aspects of Lagrangian cycles associated to knots in the conifold geometry. We show how these can be explicitly constructed in the case of algebraic knots. We use this explicit construction to explain a recent conjecture relating study of stable pairs on algebraic curves with HOMFLY polynomials. Furthermore, for torus knots, using the explicit construction of the Lagrangian cycle, we also give a direct A-model computation and recover the HOMFLY polynomial for this case.

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1. Introduction

The idea that knot invariants can be captured by physical theories dates back to the work of Witten [43] on the relation between Wilson loop observables of Chern-Simons quantum field theory for $U(N)$ gauge theories and quantum $U(N)$ invariants. It was later noted by Witten [44] that the Chern-Simons theory, in turn, describes the target space physics of A-model topological strings, in the presence of D-branes. In particular, if we have a stack of N D-branes wrapping a three manifold $M^3 \subset T^*M$ viewing T^*M as a Calabi-Yau threefold, the large N perturbative Feynman diagrams, i.e. ‘t Hooft diagrams (known in math literature as ‘ribbon graphs’) can be viewed as degenerate versions of holomorphic maps from Riemann surfaces with boundaries to T^*M , where the boundary of the Riemann surface is restricted to lie on M . It was later conjectured in [14] that in the special case where $M = S^3$ at large N the geometry undergoes a transition, where S^3 shrinks and an S^2 is blown up with size equal to Ng_s , where g_s is the string (or Chern-Simons) coupling constant. This is the small resolution of the conifold. Furthermore in this new geometry there are no more D-branes. In other words the partition function of the Chern-Simons theory is equivalent to the *closed* topological A-model involving Riemann surfaces without boundaries, on the resolved conifold. This large N equivalence was checked by computing the partition function on both sides and observing their equality.

One can also extend this equivalence to the computation of the Wilson loop observables for knots, by adding to both sides suitable ‘spectator D-branes’ [35]. Namely for each knot $K \subset S^3$, consider the canonical Lagrangian $L_K \subset T^*S^3$ which intersects S^3 along the knot K . Note that L_K has the topology $S^1 \times R^2$. A spectator brane is a D-brane supported on L_K in the topological A-model with target space T^*S^3 . The insertion of the spectator brane leads to the insertion of Wilson loop observables on the Chern-Simons side. On the other side the original stack of N D-branes has disappeared but the spectator D-branes L_K , which have the imprint of the knot, survive. We thus end up with the open topological A-model on the resolved conifold, in the presence of D-branes wrapping L_K . This equivalence was checked for the unknot in [35]. Moreover this equivalence leads to integrality predictions for the coefficients of HOMFLY polynomials (and their colored versions) [23,35], which has been proven to be true [27–29]. The integrality structure follows from the fact that on the resolved side the computation of the amplitudes captures the content of BPS particles represented by M2 branes ending on L_K , and one is simply counting them. For example, for the unknot the partition function is captured by the fact that there are two M2 branes ending on L_K .

This leaves open the problem of directly computing the topological A-model for the resolved conifold in the presence of branes wrapping L_K . The difficulty in performing this task is two-fold: first we have to identify the Lagrangian subspace L_K , and second set up a computation for the A-model amplitudes. The difficulty with the first task is that before transition L_K intersects S^3 , and thus as S^3 shrinks L_K becomes singular and its continuation on the resolved side is delicate (though there has been progress along these lines in [22,42]. However, it was further noted in [30] that to make this more well defined, and also in order for the framing dependence to come out accurately we need to lift the original Lagrangian L_K , so that it no longer touches the S^3 , but is separated

from it by a cylinder which ends on the one hand on the knot in S^3 and on the other to the non-trivial circle in L_K . In this way the L_K is non-singular as S^3 shrinks and the process of identifying it on the resolved side is more straightforward. We will clarify this construction later in this paper.

The second task is to compute the A-model amplitudes. When there are enough symmetries this in principle can be done in two ways: Either by direct computation using localization techniques, or by enumerating BPS particles ending on L_K .

The enumeration of BPS particles corresponding to M2 branes ending on L_K is particularly simple for special knots, including the unknot. For example for the unknot there are two BPS particles. One corresponds to a disc which lives on the \mathbf{C}^2 fiber of one point in \mathbf{P}^1 . The disc intersects L_K on a circle where it ends. The other particle is made of the bound state of this disc with an M2 brane wrapping \mathbf{P}^1 . This follows from the fact that the binding process is local, and we already have the disc ending on L_K and the closed M2 brane on \mathbf{P}^1 each as BPS states, and they intersect transversally (relative codimension 4). Thus they form a unique bound state. From this we can recover the HOMFLY polynomial for the unknot. In some sense the unknot is ‘planar’ in that the BPS structures are captured by objects living on the fiber or on the base independently, and simply glued together.

The question remains as to which knots are ‘planar’ in this sense? The natural answer ends up being the class of knots known as algebraic knots, which can be defined by the holomorphic function of two variables. One considers in complex dimension 2 a holomorphic function $f(x, y)$ with a singularity structure at the origin. The intersection of

$$f(x, y) = 0$$

with a large 3-sphere

$$|x|^2 + |y|^2 = r$$

for large r gives a knot K_f on S^3 . It turns out that for these knots the corresponding Lagrangian L_K can be constructed explicitly. Moreover, just as in the case of the unknot they are ‘planar’. In particular, the primitive holomorphic curve ending on it lives on a fiber over a single point of \mathbb{P}^1 . Moreover, identifying the fiber with complex coordinates (x, y) the basic holomorphic curve for the M2 brane is exactly $f(x, y) = 0$ and it intersects L_K on the large three sphere along an S^1 . The novelty, as compared to the case of unknot, is that there could be more than one M2 brane bound state on $f(x, y) = 0$ curve. Enumeration of such bound states turns out to map to a mathematical problem recently studied in [33]. However in the more general case, we have more possibilities for forming bound states, not just the single disc as in the case of the unknot. Furthermore, just as in the case of the unknot, for each such disc we can form bound states of this open M2 brane with a closed M2 brane wrapping \mathbf{P}^1 . The number of bound states depends on the intersection number of the \mathbf{P}^1 with the corresponding transverse bound state. For each intersection point, we get a bound state. Considering all such BPS states wrapping the fiber and base we get the enumeration of BPS states in this geometry which leads to the evaluation of the HOMFLY polynomial for such knots. This turns out to explain the conjecture of [33] relating the HOMFLY polynomial for algebraic knots with computations done for stable pairs associated to the corresponding curve.

We explain in detail how these bound states can be evaluated for the case of the torus knots where

$$f(x, y) = x^r - y^s.$$

Furthermore, for these cases, using the explicit construction of the Lagrangian cycles L_K we are able to also directly compute the A-model amplitudes as well and rederive the HOMFLY polynomials for torus knots.

Note that a general question one can ask in this context is whether the lagrangian A-model amplitudes associated to a knot K are invariant under small deformations of K preserving the isotopy class. In particular, for algebraic knots such deformations are induced by deformations of the plane curve preserving the analytic type of the singularity at the origin. As shown in detail in Sect. 3, each such deformation results in a small deformation of the associated lagrangian cycle. Then the question of invariance under small deformations translates into the invariance of lagrangian Gromov-Witten invariants under deformations of the lagrangian cycle. A definite answer requires a rigorous definition of such invariants, which is beyond the goal of the present work, hence this question will be left open.

The organization of this paper is as follows. Section 2 is a review of large N duality for the unknot, including the construction of toric lagrangian cycles on the resolved conifold. The main goal of this discussion is to motivate the general idea of lifting conormal bundle lagrangian cycles in the deformed conifold. Section 3 presents an explicit construction of such a lift for algebraic knots, as well as the corresponding lagrangian cycles in the resolved conifold. Section 4 provides a physical explanation for the conjecture of Oblomkov and Shende [33] relating HOMFLY polynomials of algebraic knots to certain generating functions associated to Hilbert schemes of plane curve singularities. In particular, the generating functions employed in [33] are identified with counting functions for open M2-brane microstates with boundary on an M5-brane wrapping a lagrangian cycle. Section 5 is a reprise of Sect. 4 in more mathematical dialect. Finally, Sect. 6 consists of detailed computations of open topological A-model amplitudes for lagrangian cycles corresponding to (s, r) -torus knots. The main result is a geometric derivation of the Chern-Simons S -matrix formula found in [5,40] by manipulations of open Gromov-Witten invariants.

2. Large N Duality and Lagrangian Cycles for the Unknot

The conifold transition is a topology changing process relating the smooth hypersurface X_μ ,

$$xz - yw = \mu \tag{2.1}$$

in \mathbb{C}^4 with $\mu \in \mathbb{C} \setminus \{0\}$ to the small resolution Y of the singular threefold X_0 obtained at $\mu = 0$. In fact there exist two such isomorphic resolutions related by a toric flop. For concreteness, let Y be the resolution obtained by blowing-up the subspace $y = z = 0$ in \mathbb{C}^4 . Then Y is determined by the equations

$$x\lambda = w\rho, \quad z\rho = y\lambda \tag{2.2}$$

in $\mathbb{C}^4 \times \mathbb{P}^1$ and there is a natural map $\sigma : Y \rightarrow X_0$ which contracts the rational curve $y = z = 0$ on Y . It can be easily seen that Y is isomorphic to the total space of the rank two bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and the curve $y = z = 0$ is identified with its zero section, which is the only compact holomorphic curve on Y .

The deformed conifold $X_\mu, \mu \neq 0$, equipped with the symplectic form

$$\omega_{X_\mu} = \omega_{\mathbb{C}^4} \Big|_{X_\mu}, \quad \omega_{\mathbb{C}^4} = \frac{i}{2} (dx \wedge d\bar{x} + dy \wedge d\bar{y} + dz \wedge d\bar{z} + dw \wedge d\bar{w})$$

is symplectomorphic to the total space X of the cotangent bundle T^*S^3 . For $\mu \in \mathbb{R}_{>0}$, this can be seen explicitly [38,39] observing that Eq. (2.1) becomes

$$\sum_{i=1}^4 z_i^2 = \mu$$

in the coordinates

$$x = z_1 + iz_2, \quad z = z_1 - iz_2, \quad y = -z_3 - iz_4, \quad w = z_3 - iz_4.$$

Writing $z_j = x_j + iy_j$ $j = 1, \dots, 4$, with (x_j, y_j) real coordinates on \mathbb{C}^4 , Eq. (2.1) is further equivalent to

$$\vec{x} \cdot \vec{y} = 0, \quad |\vec{x}|^2 - |\vec{y}|^2 = \mu. \tag{2.3}$$

Here \cdot denotes the Euclidean inner product on \mathbb{R}^4 and $|\cdot|$ the Euclidean norm.

On the other hand, the total space X of the cotangent bundle T^*S^3 is identified with the subspace $\{(\vec{u}, \vec{v})\} \subset \mathbb{R}^4 \times \mathbb{R}^4$ satisfying

$$|\vec{u}| = 1, \quad \vec{u} \cdot \vec{v} = 0. \tag{2.4}$$

The canonical symplectic form on $X = T^*S^3$ is then obtained by restriction from the ambient space,

$$\omega_X = \left(\sum_{j=1}^4 dv_j \wedge du_j \right) \Big|_X. \tag{2.5}$$

According to Eq. (2.3), $\vec{x} \neq 0$ on X_μ since $\mu \in \mathbb{R}_{>0}$. Therefore there is a well defined map

$$\phi_\mu : X_\mu \rightarrow X \quad \phi_\mu(\vec{x}, \vec{y}) = \left(\frac{x_j}{|\vec{x}|}, -|\vec{x}|y_j \right). \tag{2.6}$$

It is straightforward to check that this map is a diffeomorphism, its inverse being given by

$$\phi_\mu^{-1}(\vec{u}, \vec{v}) = (f_\mu(\vec{v})\vec{u}, -f_\mu(\vec{v})^{-1}\vec{v}), \quad f_\mu(\vec{v}) = \sqrt{\frac{\mu + \sqrt{\mu^2 + 4|\vec{v}|^2}}{2}}. \tag{2.7}$$

It is also straightforward to check that

$$\phi_\mu^*(\omega_X) = \frac{i}{2} \omega_{\mathbb{C}^4} \Big|_{X_\mu}.$$

Therefore ϕ_μ is indeed a symplectomorphism.

A similar construction yields a symplectomorphism $\phi_0 : X_0 \setminus \{0\} \rightarrow X \setminus \{\vec{v} = 0\}$ between the complement of the singular point in X_0 and the complement of the zero section in $X = T^*S^3$. Observing that $\vec{x} \neq 0$ on $X_0 \setminus \{0\}$, ϕ_0 is given exactly by the same formula as ϕ_μ , $\mu > 0$. The same computation shows that ϕ_0 is a symplectomorphism if $X_0 \setminus \{0\}$ is equipped with the symplectic structure obtained by restriction from \mathbb{C}^4 .

Note also that there is an antiholomorphic involution

$$(x, y, z, w) \mapsto (\bar{z}, -\bar{w}, \bar{x}, -\bar{y}) \tag{2.8}$$

on \mathbb{C}^4 which preserves X_μ with $\mu \in \mathbb{R}_{\geq 0}$. Therefore there are induced antiholomorphic involutions $\tau_\mu : X_\mu \rightarrow X_\mu, \mu \in \mathbb{R}_{\geq 0}$. For $\mu > 0$, the fixed locus S_μ of τ_μ is isomorphic to the three-sphere $|x|^2 + |z|^2 = \mu$ in \mathbb{C}^2 . By construction, S_μ is a special lagrangian cycle on X_μ and the image $\phi_\mu(S_\mu)$ is the zero section $S = \{\vec{v} = 0\}$ of the cotangent bundle T^*S^3 .

2.1. Large N duality for the unknot. The primary example of large N duality for topological strings [14] is an equivalence between the large N limit of the topological **A**-model on X_μ with N lagrangian branes on S_μ and the topological **A**-model on Y . The partition function of the latter is given by

$$Z_Y(q, Q) = \prod_{n \geq 1} (1 - Q(-q)^n)^n,$$

where q and Q are related to the string coupling constant g_s and the symplectic area t_0 of C_0 by $q = e^{ig_s}, Q = e^{-t_0}$.

According to [44] the topological **A**-model on X_μ with N lagrangian branes on the sphere S_μ is equivalent to $U(N)$ Chern-Simons theory on S_μ . The level k of the Chern-Simons theory is related to the string coupling constant. The partition function of the Chern-Simons theory on S_μ is naturally expanded in terms of the large N variables,

$$g_s = \frac{2\pi}{k + N}, \quad \lambda = \frac{2\pi N}{k + N}.$$

Then large N duality [14] suggests that the theory on S_μ where there is brane, is equivalent to the one after geometric transition where the branes have disappeared and been replaced by a blown up 2-sphere. This duality thus identifies the analytic part of the Chern-Simons large N expansion with the closed topological string amplitude $Z_Y(q, Q)$ on the resolved side¹

$$Z_{CS}(g_s, \lambda) = Z_Y(q, Q) \Big|_{q=e^{ig_s}, Q=e^{i\lambda}}.$$

Large N duality has been extended to Chern-Simons theory with Wilson loops in [35]. The main idea is that given a smooth knot $K \subset S^3$ the total space of the conormal bundle N_K^* to K in S^3 is a lagrangian cycle in $X = T^*S^3$. Since $\phi_\mu : X_\mu \rightarrow X$ is a symplectomorphism, the inverse image $N_\mu = \phi_\mu^{-1}(L)$ is a lagrangian cycle on X_μ . According to [35], a configuration of N branes on S_μ and M branes on N_μ has a complex bosonic open string mode localized on their intersection which transforms in the bifundamental representation of $U(N) \times U(M)$. Integrating out this mode yields a series of Wilson line corrections to Chern-Simons theory on S_μ of the form

$$\sum_{n \geq 1} \frac{1}{n} \text{Tr}(U^n) \text{Tr}(V^{-n}). \tag{2.9}$$

¹ The non-analytic part of the Chern-Simons function can be identified by the same change of variables with the polynomial part of the $N = 2$ prepotential of a IIA compactification on Y .

Here U is the holonomy of the Chern-Simons gauge field A on K and V is the holonomy on K of an arbitrary background flat gauge field on N_μ . This integrating out can also be explained in terms of the annulus contributions to the amplitudes where one boundary of the annulus ends on S_μ and the other ends on N_μ . These are ‘holomorphic’ annuli which have zero width, corresponding to the fact that in the dual channel there are massless bi-fundamental particles of $U(N) \times U(M)$ going in the loop.

Therefore in the presence of the M noncompact branes on N_μ , the (analytical part of the) topological open string partition function becomes

$$Z_{CS}(g_s, \lambda) \left\langle \exp \left(\sum_{n \geq 1} \frac{1}{n} \text{Tr}(U^n) \text{Tr}(V^{-n}) \right) \right\rangle, \tag{2.10}$$

where $\langle \cdot \rangle$ denote the expectation values of Wilson line operators in $U(N)$ Chern-Simons theory on S^3 . The main question is then to construct a dual topological string model on the resolution Y , extending the results of [14].

This problem was solved in [35] for the case when K is the unknot. For concreteness let $K \subset S^3$ be determined by the equations

$$y = w = 0, \quad |x| = |z| = \sqrt{\mu} \tag{2.11}$$

on X_μ . Omitting the details, a straightforward computation shows that the inverse image $\phi_\mu^{-1}(N_K^*)$ is the lagrangian cycle N_μ in X_μ determined by the equations

$$y = \bar{w}, \quad |x| = |z|. \tag{2.12}$$

Assuming K to be trivially framed, the large N expansion of the partition function (2.10) is in this case

$$Z_{CS}(g_s, \lambda) \exp \left[-i \sum_{n \geq 1} \frac{e^{in\lambda/2} - e^{-in\lambda/2}}{2n \sin(ng_s/2)} \text{Tr}(V^{-n}) \right]. \tag{2.13}$$

In order to find a large N duality interpretation, note that the above partition function is related by analytic continuation to

$$Z_{CS}(g_s, \lambda) \exp \left[-i \sum_{n \geq 1} \frac{\text{Tr}(V^n) + \text{Tr}(V^{-n})}{2n \sin(ng_s/2)} e^{in\lambda/2} \right]. \tag{2.14}$$

This expression is then identified with a series of open Gromov-Witten invariants of a lagrangian cycle in Y determined by the equations

$$|\lambda| = |\rho|, \quad x\lambda = \bar{y}\rho. \tag{2.15}$$

By construction, this cycle intersects the zero section C_0 along the circle $|\lambda| = |\rho|$, dividing it into two discs D_\pm with common boundary. The terms weighted by $\text{Tr}(V^n), \text{Tr}(V^{-n})$ in the exponent of (2.14) represent open Gromov-Witten invariants with positive, respectively negative, winding numbers along the circle $|\lambda| = |\rho|$. This was confirmed by virtual localization computations in [19,24]. In particular, the terms with positive winding numbers are obtained by summing over multicovers of D_+ while those with negative winding numbers are obtained from multicovers of D_- .

The main difficulty in extending the above results to more general knots in S^3 resides in the identification of the lagrangian cycle M in Y associated to a given knot K . Ideally there should be a natural geometric relation between the cycle $M \subset Y$ and the specialization $N_0 \subset X_0$ of $N_\mu \subset X_\mu$ as $\mu \rightarrow 0$, exploiting the fact that the conifold transition is a basic example of symplectic surgery [38,39]. In symplectic geometry the blow-up of X_0 as a symplectic manifold depends on a positive real parameter $\epsilon \in \mathbb{R}_{>0}$ which measures the symplectic area of the exceptional curve $C_0 \subset Y$. More precisely let ω_0 denote the symplectic form $\omega_{\mathbb{C}^4}|_{X_0 \setminus \{0\}}$ on the complement of the conifold singularity in X_0 . Then the blow-up of X_0 is a family of symplectic Kähler manifolds $Y_\epsilon = (Y, \omega_{Y,\epsilon})$ such that the resulting family of symplectic Kähler forms $\omega_{Y,\epsilon}|_{Y \setminus \{C_0\}}$ on the complement of C_0 degenerates to $\sigma^* \omega_0$ at $\epsilon = 0$. This yields a more symmetric picture of the conifold transition, involving two families of symplectic manifolds X_μ, Y_ϵ satisfying a natural compatibility condition at $\mu = 0, \epsilon = 0$ respectively. This process is schematically summarized by the following diagram:

$$\begin{array}{ccc}
 Y_0 & \overset{\sim}{\leftarrow} & Y_\epsilon \\
 \sigma \downarrow & & \\
 X_\mu & \rightsquigarrow & X_0
 \end{array} \tag{2.16}$$

where $\sigma : Y_0 \rightarrow X_0$ is the blow-up map. Note that all Y_ϵ with $\epsilon \geq 0$ are identical as complex manifolds, but not as symplectic manifolds. The symplectic structure is degenerate at $\epsilon = 0$ since C_0 has zero symplectic area with respect to ω_0 .

In this framework, a natural formulation of large N duality for knots requires two families of lagrangian cycles $N_\mu \subset X_\mu, M_\epsilon \subset Y_\epsilon$ such that the degenerations N_0, M_0 are related by $\sigma(M_0) = N_0$, at least on the complement of the exceptional curve C_0 . In the case of the unknot reviewed above, the specialization of the cycle N_μ in Eq. (2.12) at $\mu = 0$ is the singular lagrangian cone $N_0 \subset X_0$ determined by

$$y = \bar{w}, \quad |x| = |z|. \tag{2.17}$$

At the same time, the cycle M constructed in Eq. (2.15) is lagrangian with respect to any symplectic Kähler form $\omega_{Y,\epsilon}$ because it is the fixed point set of an antiholomorphic involution. The image of M via the blow-up map σ is precisely the singular lagrangian cycle N_0 determined by the same Eqs. (2.17). Therefore the compatibility condition at $\mu = 0, \epsilon = 0$ is satisfied. For illustration, the resulting geometric picture is represented in Fig. 1.

Since the knot K is contracted in this process it is not clear how such a construction can be extended to more general knots especially such that the resulting open string Gromov-Witten theory on Y is tractable. A related problem is that the analytic continuation required by a proper enumerative interpretation of the partition function does not have a direct geometric interpretation.

Both these problems lead to the idea [2,30] that a better formulation of large N duality would be obtained using lagrangian cycles supported in the complement of the zero section $C_0 \subset Y$, respectively $S^3 \subset X_\mu$. Said differently, this means that the lagrangian cycle $N = N_K^*$ must be lifted to a lagrangian cycle disjoint from the zero section prior to the transition. Accordingly, the corresponding lagrangian cycle in Y will be lifted to a cycle disjoint from the zero section C_0 . Moreover, once properly lifted, these cycles

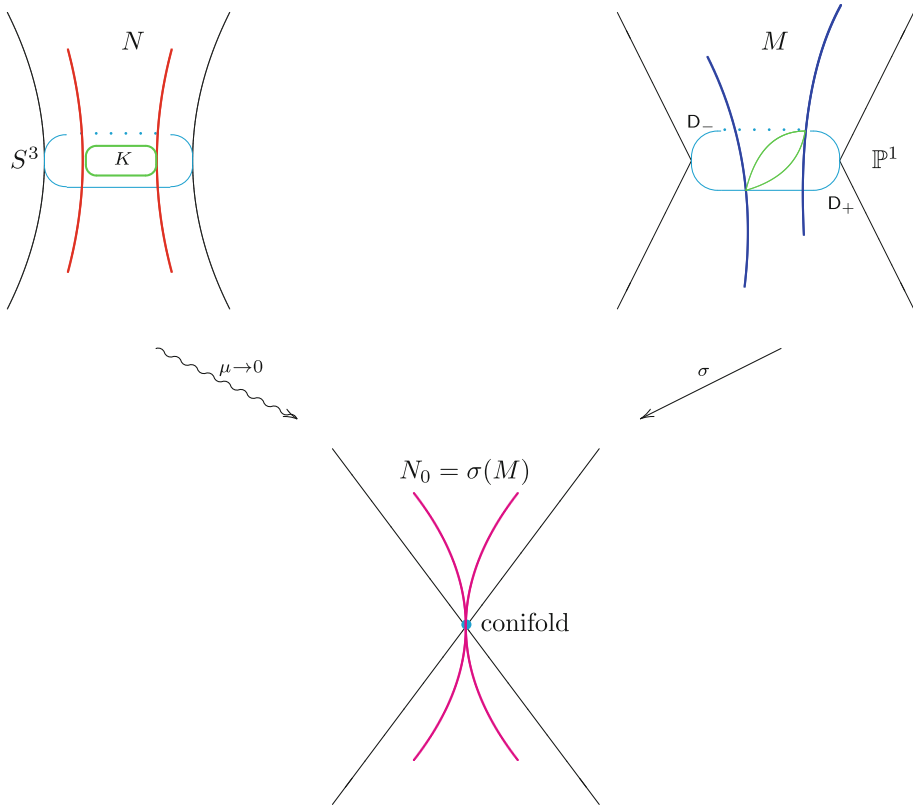


Fig. 1. Conifold transition for unlifted lagrangian cycles

should form families naturally related by symplectic surgery as shown in the following diagram:

$$\begin{array}{ccccc}
 & & Y_0 & \leftarrow & Y_\epsilon & & \\
 & & \downarrow \sigma & \swarrow & \nwarrow & & \\
 X_\mu & \rightsquigarrow & X_0 & & M_0 & \rightsquigarrow & M_\epsilon \\
 \uparrow & & \downarrow \sigma & \swarrow & \nwarrow & & \\
 L_\mu & \rightsquigarrow & L_0 & & & &
 \end{array} \tag{2.18}$$

That there is such a lift can be argued as follows: Assume that with the proper choice of metric, the lagrangian cycle L is actually *special lagrangian* [41]. In this case it is known that the dimension of moduli of L is equal to the dimension of H_1 . This is rigorously the case for compact lagrangians, and we assume it to hold for non-compact ones as well where we have imposed suitable finiteness conditions on the norm of deformations of the lagrangian. Since the topology of L is $R^2 \times S^1$, there is exactly 1 deformation. This corresponds to moving the special lagrangian in the direction determined by the 1-form dual S^1 , by identifying the infinitesimal normal deformation to the lagrangian

with its cotangent space. It is this deformation that lifts L off S_μ . Moreover it suggests that there is a unique such canonical lift for special lagrangian cycles. Even though we will mainly deal with just lagrangian ones, this suggests that the choice of the special lagrangian ones make the constructions more ‘canonical’.

Accepting the idea of lifting lagrangian cycles, a legitimate question is how can one then obtain the Wilson loop corrections (2.9), given that L_μ and S_μ do not intersect. This is also natural. Lifting the lagrangian brane off of S_μ is simply giving the bi-fundamental particles a mass given by the amount of lifting. In other words, the annuli for which the dual channel corresponded to bi-fundamental strings, now have a finite width depending on the amount of lift. These corrections can now be interpreted as ‘honest’ instantons, i.e. holomorphic cylinders which on the one hand end on S_μ and on the other hand on L_μ . Such corrections were predicted in [44] assuming that there are finitely many rigid holomorphic Riemann surfaces $C_\mu^{(\alpha)}$ in X_μ with boundary components on S_μ, L_μ . Each such surface gives rise to a series of Wilson loop corrections by summing over multicovers. In particular, a rigid holomorphic cylinder C_μ in X_μ with boundary components in S_μ, L_μ yields a series of instanton corrections

$$\sum_{n \geq 1} \frac{e^{-t_C}}{n} \text{Tr}(U^n) \text{Tr}(V^n), \tag{2.19}$$

where t_C is the symplectic area of the cylinder C_μ , and can be interpreted as the mass of the bi-fundamental state (where we have changed the variables by $V \mapsto V^{-1}$). Note that the factor e^{-t_C} can be absorbed by a redefinition of the holonomy variable V , hence it will be omitted from now on. Figure 2 is a schematic representation of the surgery process in terms of lifted lagrangian cycles.

To summarize, *for a given knot $K \in S^3$, large N duality requires a family of lagrangian cycles $L_\mu \subset X_\mu$, disjoint from S_μ , such that there is a unique rigid holomorphic cylinder C_μ in X_μ with boundary components in S_μ, L_μ .* Moreover, the boundary component in S_μ must be isotopic to the given knot K . Note that the rigidity assumption on C_μ is not needed if there exists a torus action on X_μ preserving L_μ . In this case it suffices to require C_μ to be the unique torus invariant holomorphic cylinder satisfying the above boundary conditions. The latter is a weaker condition, much easier to check in practice. Note however, that general sufficient uniqueness conditions are not known, even in the presence of a torus action. Uniqueness has to be checked directly on a case by case basis, using the explicit construction of the S^1 -invariant lagrangian cycle. Concrete examples and more details are presented in Sect. 6.2. If C_μ is found to be the only S^1 -invariant holomorphic cylinder, the series (2.19) follows by a virtual localization computation analogous to [19], as shown for example in [11].

A concrete construction of such families of lagrangian cycles for algebraic knots is presented in Sect. 3. By analogy with the unknot, the cycles L_μ will be obtained by taking inverse images $\phi_\mu^{-1}(L)$ of a fixed lift $L \subset X$ of N_K^* in T^*S^3 . Uniqueness and rigidity of the associated holomorphic cylinders will be proven only for torus knots in Sect. 6 and conjectured to hold for all algebraic knots.

The family of lagrangian cycles $M_\epsilon \subset Y$ related to L_μ by geometric transition is expected to have a similar property. Namely there should exist a unique holomorphic disc D_ϵ in Y with boundary $\eta_\epsilon \subset M_\epsilon$. Note that D_ϵ may have isolated singularities away from the boundary. Then large N duality predicts an identification between the Chern-Simons partition function on S^3 , including the instanton corrections (2.19), and the partition function of Gromov-Witten theory on Y with lagrangian boundary

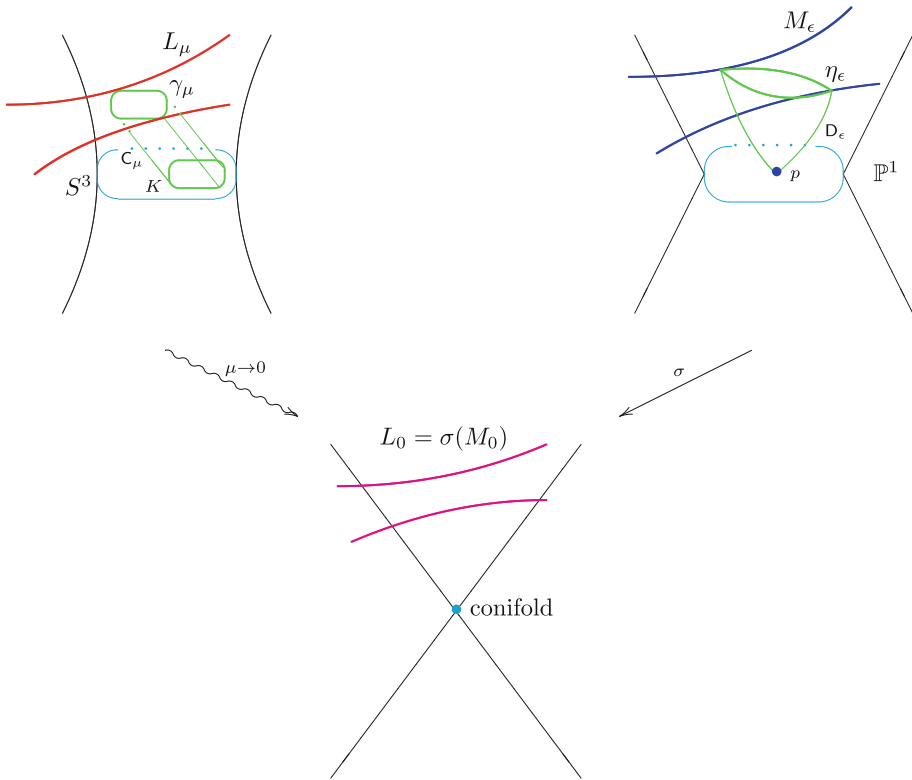


Fig. 2. Conifold transition for lifted lagrangian cycles

conditions on M_ϵ . Again, if there is a torus action on Y preserving M_ϵ , it suffices for D_ϵ to be the unique torus invariant disc with boundary on M_ϵ .

As a first example, the above program will be carried out in detail in the next subsection for an unknot of the form (2.11). In this case the cycles M_ϵ, L_μ will be explicitly constructed employing toric methods [4]. It will be shown that both cycles are preserved by a circle action determined by an action on \mathbb{C}^4 of the form

$$(x, y, z, w) \mapsto (e^{-in_1\varphi} x, e^{-in_2\varphi} y, e^{in_1\varphi} z, e^{in_2\varphi} w). \tag{2.20}$$

Note that the action on Y is uniquely determined by the condition that the blow-up equations (2.2) be left invariant. In particular it yields the circle action

$$[\lambda, \rho] \mapsto [\lambda, e^{i(n_1+n_2)\varphi} \rho]$$

on \mathbb{P}^1 . Assuming the unknot trivially framed, the Chern-Simons expectation value of the instanton corrections (2.19) is

$$\exp \left[i \sum_{n \geq 1} \frac{(1 - e^{in\lambda})}{2n \sin(n g_s / 2)} \text{Tr}(V^{-n}) \right]. \tag{2.21}$$

The open Gromov-Witten invariants with boundary condition on M_ϵ can be computed in close analogy with [19]. As explained in [19], the result depends on the choice

of a torus action, reflecting the fact that the moduli space of stable maps with lagrangian boundary conditions is non-compact. This dependence is related by large N duality to the framing dependence of knot invariants in Chern-Simons theory [30]. Choosing the torus action $(n_1, n_2) = (1, 0)$, which corresponds to the trivial framing, the result takes the simple form

$$\exp \left[i \sum_{n \geq 1} \frac{(1 - Q^n)}{2n \sin(n g_s / 2)} \text{Tr}(V^{-n}) \right]. \tag{2.22}$$

This is in agreement with Eq. (2.21) via the change of variable $Q = e^{i\lambda}$. Note that the term involving a single $\text{Tr}(V^{-1})$ in the exponent has the form (up to an overall factor of $q^{1/2}$)

$$(1 - Q)/(1 - q) = \frac{1}{(1 - q)} - \frac{Q}{(1 - q)},$$

where $q = \exp(i g_s)$. Each of these two terms was interpreted in [35] as the contribution of an M2 brane ending on the Lagrangian brane corresponding to the unknot. The two terms differ by a factor of Q indicating that one of the two M2 branes is in addition wrapped around the \mathbf{P}^1 . The minus sign in front of the second term can be interpreted as the fermion number associated with the M2 wrapped around \mathbf{P}^1 . Moreover the term

$$\frac{1}{(1 - q)} = 1 + q + q^2 + \dots$$

signifies the fact that an M2 brane particle has one mode for each positive integer $n > 0$. Each such n corresponds to the spin of the M2 brane on a plane, in the presence of a magnetic flux. Such configurations admit an alternative description in terms of D-branes in type IIA string theory employing M-theory/IIA duality. This duality relates M-theory compactified on a product threefold $Y \times S^1_M$, with Y a Calabi-Yau threefold, to Type IIA string theory on Y . The extra circle S^1_M present in M-theory is often referred to in the physics literature as the ‘M-theory circle’ or the ‘eleventh circle’. In the type IIA perspective since the rotations around the eleventh circle generate a subgroup of the Lorentz group, n can also be identified with the D0 brane charge [3, 7, 12]. The fact that there are two BPS states for the unknot will be explained in the next subsection.

2.2. Toric lagrangian cycles in the resolved conifold. The construction of the lifted lagrangian cycles M_ϵ, L_μ will be carried out in detail below for the unknot using toric geometry as in [4]. The gauged linear sigma model which flows to Y is a two dimensional $U(1)$ gauge theory containing four chiral superfields Z_1, \dots, Z_4 with charges

$$U(1) \quad \begin{matrix} Z_1 & Z_2 & Z_3 & Z_4 \\ 1 & 1 & -1 & -1. \end{matrix}$$

and trivial superpotential. The D-term equation is

$$|Z_1|^2 + |Z_2|^2 - |Z_3|^2 - |Z_4|^2 = \epsilon, \tag{2.23}$$

where $\epsilon \in \mathbb{R}_{>0}$ is an FI parameter. The symplectic quotient construction yields a family of symplectic Kähler manifolds $Y_\epsilon = (Y, \omega_{Y,\epsilon})$. The exceptional curve C_0 is given

by $Z_3 = Z_4 = 0$, and has symplectic area proportional to ϵ . The contraction map $\sigma : Y \rightarrow X_0$ is determined by the $U(1)$ -invariant monomials

$$x = Z_3 Z_1, \quad y = Z_4 Z_1, \quad z = Z_4 Z_2, \quad w = Z_3 Z_2,$$

which satisfy the relation $xz = yw$.

Lagrangian cycles in Y are constructed by a gauged linear sigma model with boundary, which is expected to flow to a boundary conformal field theory in the infrared limit. In particular consider the cycles M_ϵ defined by the boundary D-term equations,

$$|Z_2|^2 - |Z_4| = 0, \quad |Z_3|^2 - |Z_4|^2 = c, \tag{2.24}$$

where $c \in \mathbb{R}_{>0}$ is a boundary FI parameter, and the phase condition

$$Z_1 Z_2 Z_3 Z_4 = |Z_1 Z_2 Z_3 Z_4|. \tag{2.25}$$

On the open subset $Z_i \neq 0$, where all angular coordinates $\theta_i, i = 1, \dots, 4$, are well defined this condition is equivalent to $\theta_1 + \dots + \theta_4 = 0$. A detailed construction of the boundary gauged linear sigma models has been carried out in [15–17, 31]. The boundary FI parameter $c > 0$ will be kept fixed throughout this discussion.

In order to understand the geometry of M_ϵ , note that Eqs. (2.23), (2.24) imply

$$|Z_1|^2 - |Z_3|^2 = \epsilon.$$

Since $\xi, c > 0$, it follows that Z_1, Z_3 cannot vanish on M_ϵ . Then the phase θ_1 can be set to 0 by $U(1)$ gauge transformations, and the phase relation (2.25) reduces to

$$Z_2 Z_3 Z_4 = |Z_2 Z_3 Z_4|.$$

As emphasized in the previous subsection, it is important to note that M_ϵ is preserved by any circle action $S^1 \times Y \rightarrow Y$ of the form

$$(Z_1, Z_2, Z_3, Z_4) \mapsto (Z_1, e^{i(n_1+n_2)\varphi} Z_2, e^{-in_1\varphi} Z_3, e^{-in_2\varphi} Z_4) \tag{2.26}$$

with $n_1, n_2 \in \mathbb{Z}$. It is straightforward to check that this is in agreement with the action (2.20) on the invariant monomials. It is also important to note that M_ϵ intersects the plane $Z_2 = 0$ along a circle S_c^1 given by

$$|Z_2| = |Z_4| = 0, \quad |Z_3|^2 = c, \quad |Z_1|^2 = \epsilon + c.$$

Since Z_2, Z_4 are set to 0, the intersection is indeed a circle parameterized by the angular variable θ_3 . Moreover, there is a holomorphic disc D_ϵ with boundary on M_ϵ defined by

$$|Z_2| = |Z_4| = 0, \quad |Z_3|^2 \leq c, \quad |Z_1|^2 = \epsilon + c.$$

Reasoning by analogy with [10, 11] it can be checked that D_ϵ is the only Riemann surface in Y with boundary on M_ϵ preserved by a torus action of the form (2.26) with $n_1 \neq 0$.

Next note that setting $\epsilon = 0$ in the above construction yields a lagrangian cycle M_0 on the singular conifold X_0 . In terms of the invariant monomials (x, y, z, w) , the defining equations of M_0 in X_0 are

$$y - \bar{w} = 0, \quad |x| - |z| = c. \tag{2.27}$$

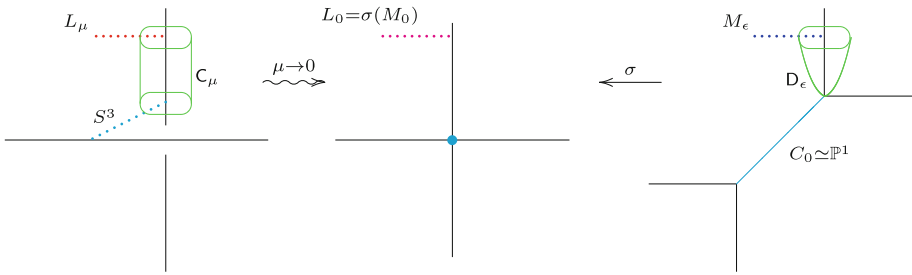


Fig. 3. Conifold transition for toric lagrangian cycles

Since $c > 0$, x cannot vanish, hence M_0 is contained in the complement of the singular point $x = y = z = w = 0$. Moreover, it is easy to check that M_0 is lagrangian with respect to the symplectic form ω_0 obtained by restricting the standard symplectic form $\omega_{\mathbb{C}^4}$ to the complement of the singular point in X_0 . Note also that Eq. (2.27) yields Eq. (2.15) at $c = 0$, confirming that the present construction is a lifted version of the previous one.

The family of lagrangian cycles $L_\mu \subset X_\mu, \mu > 0$ is defined by the same equations, (2.27), now interpreted as equations on the deformation X_μ . It is straightforward to check that L_μ is lagrangian with respect to the symplectic form $\omega_{\mathbb{C}^4}|_{X_\mu}$ and it is preserved by the torus action (2.20). The resulting transition between lagrangian cycles is schematically represented in Fig. 3.

Again, comparison with Eq. (2.12) shows that the cycle L_μ is a lift of the (inverse image of the) conormal bundle $\phi_\mu^{-1}(N_K^*)$. Moreover there is a unique torus invariant holomorphic cylinder C_μ in X_μ with one boundary component in L_μ and the second contained in the vanishing cycle S_μ . This is obtained intersecting the two lagrangian cycles, L_μ, S_μ with the holomorphic curve $C_\mu \subset X_\mu$ given by

$$y = 0, \quad xz = \mu.$$

One then finds two circles determined by the equations

$$C_\mu \cap L_\mu : \quad y = w = 0, \quad xz = \mu, \quad |x| = \frac{c + \sqrt{c^2 + 4\mu^2}}{2},$$

$$C_\mu \cap S_\mu : \quad |x| = |z| = \sqrt{\mu}, \quad y = w = 0.$$

The cylinder C_μ is given by

$$y = w = 0, \quad xz = \mu, \quad \sqrt{\mu} \leq |x| \leq \frac{c + \sqrt{c^2 + 4\mu^2}}{2}.$$

A different construction of lagrangian cycles for more general knots will be presented in the next section.

3. Algebraic Knots, Lagrangian Cycles and Conifold Transitions

The goal of this section is to present a construction of lagrangian cycles in T^*S^3 lifting the conormal bundle N_K^* of any knot $K \subset S^3$. Note that such a construction was previously carried out in [22], where it was also proven that the resulting lagrangian cycles

are related to totally real cycles on the resolved conifold via the conifold transition. Moreover, there is a well defined Gromov-Witten theory with boundary conditions on the totally real cycles, constructed in [22] via symplectic methods. As discussed in more detail below, the construction employed in this paper is a generalization of [22] motivated by the large N duality considerations explained in Sect. 2.1. In particular, in this approach the lagrangian cycles associated to algebraic knots are naturally equipped with holomorphic cylinders with one boundary component in the lifted conormal bundle, the second boundary component being a knot in S^3 in the isotopy class of K . It will also be shown that these cycles are related by the conifold transition to lagrangian cycles in the small resolution of the conifold. For K algebraic, the construction also yields a singular holomorphic disc D_ϵ in the resolved conifold with boundary on the corresponding lagrangian cycles. Furthermore, if K is a torus knot, the resulting Gromov-Witten theory on the resolution with lagrangian boundary conditions turns out to be computable using a virtual localization approach similar to [19] and [24].

The notation and geometric set-up is as in the previous section. The total space of the cotangent bundle T^*S^3 is denoted by X and will be identified with the subspace of $\mathbb{R}^4 \times \mathbb{R}^4$ determined by Eqs. (2.4). In this presentation, the canonical symplectic form ω_X is given by Eq. (2.5). The natural projection map $X \rightarrow S^3$ is denoted by π and the zero section is denoted by S .

*3.1. Knots and lagrangian cycles in T^*S^3 .* Consider a smooth closed curve $\gamma : S^1 \rightarrow X$ such that the projection $\pi \circ \gamma : S^1 \rightarrow S^3$ is a smooth knot K in S^3 . In particular, γ intersects each fiber of $X \rightarrow S^3$ at most once, otherwise its projection to S^3 would have self-intersection points. Suppose the map γ is given by

$$\theta \in S^1 \rightarrow (\vec{u}, \vec{v}) = (\vec{f}(\theta), \vec{g}(\theta)),$$

where $\vec{f}(\theta) = (f_j(\theta))$, $\vec{g}(\theta) = (g_j(\theta))$, $j = 1, \dots, 4$, are smooth periodic functions of θ .

The total space of the conormal bundle N_K^* to K in S^3 is defined by the equations

$$\vec{u} = \vec{f}(\theta), \quad \vec{f}'(\theta) \cdot \vec{v} = 0,$$

where $\vec{f}'(\theta) = d\vec{f}(\theta)/d\theta$. Then a straightforward computation yields

$$\omega_X|_{N_K} = \left(d \sum_{j=1}^4 v_j du_j \right)|_{N_K} = d \left(\sum_{j=1}^4 v_j \dot{f}_j d\theta \right) = 0,$$

confirming that N_K^* is a lagrangian cycle in X .

Now consider the three-cycle $L_\gamma \subset T^*S^3$ determined by the equations

$$\vec{u} = \vec{f}(\theta), \quad \vec{f}'(\theta) \cdot (\vec{v} - \vec{g}(\theta)) = 0. \tag{3.1}$$

By construction L_γ is a cycle in the total space of the restriction $T^*S^3|_K$. The restriction of the canonical projection $\pi : T^*S^3 \rightarrow S^3$ yields a projection $\pi_{L_\gamma} : L_\gamma \rightarrow K$. The fiber of π_{L_γ} over a point $p \in K$ is the two-plane in $T_p^*S^3$ determined by the second

equation in (3.1), which is linear in v_j . Basically, L_γ is obtained by a fiberwise translation of N_K by a translation vector depending on the point $p \in K$. The restriction of the canonical symplectic form to L_γ is given by

$$\omega_X|_{L_\gamma} = \left(d \sum_{j=1}^4 v_j du_j \right) |_{L_\gamma} = d \left(\sum_{j=1}^4 v_j \dot{f}_j(\theta) d\theta \right).$$

Using the second equation in (3.1),

$$\sum_{j=1}^4 v_j \dot{f}_j(\theta) d\theta = \sum_{j=1}^4 g_j(\theta) \dot{f}_j(\theta) d\theta$$

on L_γ . Therefore

$$\omega_X|_{L_\gamma} = d \left(\sum_{j=1}^4 g_j(\theta) \dot{f}_j(\theta) d\theta \right) = 0.$$

In conclusion, L_γ is a lagrangian cycle on T^*S^3 . Note that the intersection of L_γ with the zero section $\vec{v} = 0$ is determined by the equations

$$\vec{u} = \vec{f}(\theta), \quad \vec{f}(\theta) \cdot \vec{g}(\theta) = 0.$$

For sufficiently generic $\vec{f}(\theta), \vec{g}(\theta)$ this intersection will be empty, such that L_γ is a lift of the conormal bundle N_K^* off the zero section.

Note also that the lift constructed in [22] is a special case of the above construction obtained by setting $\vec{g}(\theta) = \vec{f}(\theta)$. The main reason for the above generalization is that at least for algebraic cycles it also yields specific holomorphic open string instantons interpolating between the lifted conormal bundle and the vanishing cycle S^3 in the deformed conifold. This will be explained next.

3.2. Lagrangian cycles for algebraic knots. So far this construction is fairly general and can be applied to any knot in $K \subset S^3$, for any lift $\gamma : S^1 \rightarrow X$ satisfying the above conditions. In the special case when K is an algebraic knot there is a preferred construction of the lift γ motivated by AdS/CFT correspondence. The main idea is to obtain a one-cycle γ as in Sect. 3.1 by intersecting an S^2 -bundle $P_a \subset T^*S^3$ of radius $a > 0$ with the image $\phi_\mu(C_\mu)$ of a certain holomorphic curve $C_\mu \subset X_\mu$ associated to K as explained below. Here $\phi_\mu : X_\mu \rightarrow X$ is the symplectomorphism given in Eq. (2.6).

Suppose K is the link of the plane curve singularity $f(x, y) = 0$ in \mathbb{C}^2 , where $f(x, y)$ is a polynomial with complex coefficients of the form

$$f(x, y) = \sum_{k,l \geq 0} a_{k,l} x^k y^l.$$

For simplicity assume that the curve $f(x, y) = 0$ is irreducible and smooth away from $x = y = 0$, and K is connected. Let

$$g(z, w) = \sum_{k,l \geq 0} (-1)^l \bar{a}_{k,l} z^k w^l,$$

and consider the complete intersection $Z_\mu \subset X_\mu$ determined by

$$f(x, y) = 0, \quad g(z, w) = 0. \tag{3.2}$$

Suppose $f(x, y)$ is sufficiently generic such that Z_μ is smooth for generic $\mu > 0$. Note that Z_μ may have several distinct connected components even though the plane curve $f(x, y) = 0$ is assumed irreducible. For example consider the case of torus knots, $f(x, y) = x^r - y^s$ with (r, s) coprime positive integers. Then Eqs. (3.2) imply

$$(xz)^r - (-yw)^s = 0,$$

and substitution in the deformed conifold equation, $xz - yw = \mu$, yields

$$(xz)^r - (\mu - xz)^s = 0.$$

Therefore $xz = \eta$, where η is a solution of the polynomial equation $t^r - (\mu - t)^s = 0$. Each such solution η determines a connected component of Z_μ of the form

$$(x, y, z, w) = (t^s, t^r, \eta t^{-s}, (\mu - \eta)t^{-r})$$

with $t \in \mathbb{C} \setminus \{0\}$.

By construction Z_μ is preserved by the antiholomorphic involution $\tau_\mu(x, y, z, w) = (\bar{z}, -\bar{w}, \bar{x}, -\bar{y})$ defined in (2.8). Then each connected component of the intersection of Z_μ with the fixed point locus $S_\mu = X_\mu^{\tau_\mu}$ is isomorphic to the one-cycle

$$|x|^2 + |y|^2 = \mu, \quad f(x, y) = 0 \tag{3.3}$$

in \mathbb{C}^2 . For sufficiently small $\mu > 0$, this is the link of the plane curve singularity $f(x, y) = 0$ in \mathbb{C}^2 .

Note that the symplectomorphism ϕ_μ maps S_μ to the zero section $S = \{\vec{v} = 0\}$ in $X = T^*S^3$. Now let $P_a = \{|\vec{v}| = a\}$, $a > 0$, be the sphere bundle of radius a in $X = T^*S^3$, and $B_a \subset X$ be the bounding disc bundle,

$$B_a = \{(\vec{u}, \vec{v}) \mid |\vec{v}| \leq a\}.$$

Suppose there is a connected component C_μ of Z_μ with nontrivial intersection with the vanishing cycle S_μ . As observed above each connected component of the intersection must be isomorphic to the link (3.3) of the plane curve singularity. Since $\phi_\mu(C_\mu)$ has nontrivial intersection with the zero section $S \subset X$, it will also intersect all sphere bundles $P_a \subset X$ for sufficiently small values of $a \in \mathbb{R}_{>0}$. In fact for sufficiently small $a > 0$ the intersection $\phi_\mu(C_\mu) \cap P_a$ will be foliated by disjoint connected one-cycles $\gamma_{\mu,a'} = \phi_\mu(C_\mu) \cap P_{a'}$, $0 \leq a' \leq a$. Then applying the construction in Sect. 3.1 to $\gamma_{\mu,a}$ yields a lagrangian cycle $L_{\gamma_{\mu,a}} \subset X$. The inverse image $L_{\mu,a} \subset X_\mu = \phi_\mu^{-1}(L_{\gamma_{\mu,a}})$ is a lagrangian cycle in X_μ intersecting C_μ along the one-cycle $\phi_\mu^{-1}(\gamma_{\mu,a})$. Moreover, by construction there is a holomorphic cylinder in $C_{\mu,a} \subset X_\mu$ contained in C_μ , with one boundary component in S_μ and the second boundary component in $L_{\mu,a}$. This is precisely the basic set-up of large N duality in terms of lifted lagrangian cycles described in Sect. (2.1), above Eq. (2.19). In order to keep the notation simple $L_{\mu,a}$, $C_{\mu,a}$ will be simply denoted by L_μ , C_μ the a -dependence being implicitly understood.

Several important questions must be addressed at this point:

- (i) The first concerns a rigorous definition of Gromov-Witten invariants with boundary conditions defined by the lagrangian cycles L_μ . This problem will not be solved in this paper, although consistency of large N duality arguments predict that such a rigorous construction should exist. Supporting evidence for this conjecture can be found in [22], where compact moduli spaces of stable holomorphic maps with lagrangian boundary conditions are constructed in a similar context. Additional evidence will be presented in Sect. 6, where explicit virtual localization computations will be carried out for polynomials $f(x, y) = x^r - y^s$ assuming foundational issues without proof. The results will be shown to be in agreement with HOMFLY polynomials of torus knots.
- (ii) The second question concerns deformation invariance under small complex deformations of the curve C preserving the analytic type of the singularity at the origin, hence also the isotopy class of K . Note that a small perturbation of the polynomial $f(x, y)$ results by the above construction in a small perturbation of the lagrangian cycle L_μ . Therefore the question translates into deformation invariance of lagrangian Gromov-Witten invariants. A clear answer would require a rigorous definition of the theory, which is not available. However, such invariants are expected in general to exhibit a chamber structure under deformations of lagrangian cycles. Therefore it is reasonable to expect that invariance will hold at least under sufficiently small perturbations.
- (iii) The third problem is whether one can construct a family of lagrangian cycles M_ϵ on Y completing the geometric transition picture represented in (2.18). This will be shown to be the case for any algebraic knot in the next subsection, with the caveat that the resulting Gromov-Witten theory with lagrangian boundary conditions on M_ϵ is tractable only for torus knots.

A first step towards completing the diagram (2.18) is to understand the specialization of the above construction at $\mu = 0$. The specialization of Z_μ is a reducible curve Z_0 in the singular conifold X_0 with at least two irreducible components C^\pm given by

$$f(x, y) = 0, \quad z = w = 0,$$

respectively

$$g(z, w) = 0, \quad x = y = 0.$$

These components meet at the conifold singularity, which is also a singular point of Z_0 . By construction, the antiholomorphic involution $\tau_0 : X_0 \rightarrow X_0$ exchanges C^\pm .

For concreteness, consider again the example of torus knots, $f(x, y) = x^r - y^s$. In this case the defining equations of Z_0 imply that $t = xz$ must be a solution of the polynomial equation $t^r - (-t)^s = 0$. Therefore $xz = 0$ or $xz = \eta$ with $\eta^{r-s} = (-1)^{s+1}$. This implies that Z_0 has $r - s + 1$ connected components. The connected component corresponding to $xz = 0$ is the union of the two irreducible components C^\pm defined above, which intersect at the singular point $x = y = z = w = 0$. Each connected component corresponding to $xz = \eta$ is determined by the equations

$$xz = yw = \eta, \quad x^r = y^s.$$

Since these equations are invariant under the \mathbb{C}^\times -action

$$(x, y, z, w) \mapsto (\alpha^s x, \alpha^r y, \alpha^{-s} z, \alpha^{-r} w)$$

and x, y, z, w cannot vanish, each such component is isomorphic to \mathbb{C}^\times .

Returning to the general case, let γ^\pm be the one-cycles obtained by intersecting the inverse images $\phi_0(\mathbb{C}^\pm \setminus \{0\})$ with the sphere bundle P_a . It is straightforward to check that τ_0 exchanges the image cycles $\phi_0(\gamma^\pm)$. Applying the construction of Sect. 3.1 to the cycle γ^+ , one obtains a lagrangian cycle L_{γ^+} in X . The inverse image $L_0 = \phi_0^{-1}(L_{\gamma^+})$ is a lagrangian cycle in X_0 . For sufficiently small $\mu \in \mathbb{R}_{>0}$ there exists an irreducible component C_μ of Z_μ such that the intersection $\phi_\mu(C_\mu) \cap P_a$ has a connected component γ_μ which specializes to γ^+ at $\mu = 0$. The resulting family of lifted lagrangian cycles L_μ specializes to L_0 at $\mu = 0$. This completes the bottom part of diagram (2.18). The remaining part will be constructed in the next subsection.

3.3. *Lagrangian cycles in the resolved conifold.* Recall the resolved conifold Y is determined by Eqs. (2.2) in $\mathbb{C}^4 \times \mathbb{P}^1$ and $\sigma : Y \rightarrow X_0$ denotes the natural contraction map to the singular conifold. The family of symplectic manifolds Y_ϵ in diagram (2.18) is determined by the symplectic forms

$$\omega_{Y,\epsilon} = (\omega_{\mathbb{C}^4} + \epsilon^2 \omega_{\mathbb{P}^1})|_Y,$$

where $\omega_{\mathbb{C}^4}$ is the standard symplectic form on \mathbb{C}^4 and $\omega_{\mathbb{P}^1}$ is the Fubini-Study form on \mathbb{P}^1 .

The family of lagrangian cycles $M_\epsilon \subset Y_\epsilon$ will be constructed using [32, Lem. 7.11], which provides a geometric relation between the symplectic structures on Y_ϵ, X_0 . First it will be helpful to recall the statement of [32, Lem. 7.11] for the blow-up $\eta : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$ at the origin. Consider the following one parameter family of symplectic forms on the blow-up

$$\omega_{\tilde{\mathbb{C}}^2,\epsilon} = (\omega_{\mathbb{C}^2} + \epsilon^2 \omega_{\mathbb{P}^1})|_{\tilde{\mathbb{C}}^2}.$$

For any $\epsilon \in \mathbb{R}_{>0}$, let $B(\epsilon) \subset \mathbb{C}^2$ be the ball $|z|^2 + |y|^2 \leq \epsilon^2$ and $\tilde{B}(\epsilon) = \eta^{-1}(B(\epsilon))$ be its inverse image in $\tilde{\mathbb{C}}^2$. Note that there is a radial map $\rho_\epsilon : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}^2 \setminus B(\epsilon)$,

$$\rho_\epsilon(y, z) = \frac{\sqrt{|z|^2 + |y|^2 + \epsilon^2}}{\sqrt{|z|^2 + |y|^2}}(y, z).$$

Then [32, Lem. 7.11] proves that the map $\psi_\epsilon : \tilde{\mathbb{C}}^2 \setminus E \rightarrow \mathbb{C}^2 \setminus B(\epsilon)$,

$$\psi_\epsilon = \rho_\epsilon \circ \eta|_{\tilde{\mathbb{C}}^2 \setminus E}$$

is a symplectomorphism for any $\epsilon \in \mathbb{R}_{>0}$, where $E \subset \tilde{\mathbb{C}}^2$ denotes the exceptional curve.

In order to apply [32, Lem. 7.11] to the present situation, note that Y can be regarded as the quadric hypersurface $x\lambda = w\rho$, in the fourfold Z determined by $z\rho = y\lambda$ in $\mathbb{C}^4 \times \mathbb{P}^1$. Obviously, $Z \simeq \mathbb{C}^2 \times \tilde{\mathbb{C}}^2$, where $\tilde{\mathbb{C}}^2$ is the one point blow-up of \mathbb{C}^2 at the origin. Next note that the map $\varrho_\epsilon = 1_{\mathbb{C}^2} \times \rho_\epsilon : \mathbb{C}^2 \times (\mathbb{C}^2 \setminus \{0\}) \rightarrow \mathbb{C}^2 \times (\mathbb{C}^2 \setminus B(\epsilon))$ preserves the nodal threefold $X_0 \subset \mathbb{C}^2 \times \mathbb{C}^2$, mapping $X_0 \setminus \{0\}$ to the open subset $X_0(\epsilon) = X_0 \setminus X_0 \cap (\mathbb{C}^2 \times B(\epsilon))$. Note also that the exceptional $(-1, -1)$ curve $C_0 \subset Y$ coincides with the curve $\{0\} \times E \subset \mathbb{C}^2 \times \tilde{\mathbb{C}}^2$. This implies that the complement of the zero section $Y \setminus C_0$ coincides with the open subset $Y \cap \mathbb{C}^2 \times (\tilde{\mathbb{C}}^2 \setminus E)$. Then the map

$$\phi_\epsilon : Y \setminus C_0 \rightarrow X_0(\epsilon), \quad \phi_\epsilon = (\varrho_\epsilon \circ \sigma)|_{Y \setminus C_0} \tag{3.4}$$

is a symplectomorphism.

Returning to the construction of the lagrangian cycle $M_\epsilon \subset Y_\epsilon$, recall that the family of complete intersection curves $C_\mu \subset X_\mu$ given by

$$f(x, y) = 0, \quad g(z, w) = 0$$

specializes to a reducible curve at $\mu = 0$ with two components

$$\begin{aligned} C^+ : \quad & f(x, y) = 0, \quad z = w = 0, \\ C^- : \quad & g(z, w) = 0, \quad x = y = 0. \end{aligned}$$

The intersection of $\phi_0(C^+)$ with the sphere bundle P_a yields a one-cycle γ^+ , the limit of the cycles γ_μ^+ as $\mu \rightarrow 0$. The corresponding lagrangian cycle L_{γ^+} is the limit of L_μ^+ as $\mu \rightarrow 0$.

Now consider the one-cycle

$$\gamma_\epsilon^+ = \phi_0 \circ \varrho_\epsilon \circ \phi_0^{-1} \circ \gamma^+ : S^1 \rightarrow X$$

on X obtained by applying the radial map to the inverse image $\phi_0^{-1} \circ \gamma^+$ of the path γ^+ . Then set

$$M_\epsilon = \phi_\epsilon^{-1}(\phi_0^{-1}(L_{\gamma_\epsilon^+})) = \sigma^{-1}(\varrho_\epsilon^{-1}(\phi_0^{-1}(L_\epsilon))), \tag{3.5}$$

where $L_{\gamma_\epsilon^+} \subset X$ is the lagrangian cycle obtained by applying the construction of Sect. 3.1 to γ_ϵ^+ . By construction $L_{\gamma_\epsilon^+}$ intersects the dilation $\varrho_\epsilon(\phi_0(C^+))$ of the curve $\phi_0(C^+)$ along the cycle γ_ϵ^+ . Therefore the inverse image $\varrho_\epsilon^{-1}(\phi_0^{-1}(L_{\gamma_\epsilon^+}))$ intersects the plane curve $C^+ \subset X_0$ along the cycle $\phi_0^{-1} \circ \gamma_\epsilon^+$.

Now recall that the threefold Y is determined by the equations

$$x\lambda = w\rho, \quad z\rho = y\lambda$$

in $\mathbb{C}^4 \times \mathbb{P}^1$, where $[\lambda, \rho]$ are homogeneous coordinates on \mathbb{P}^1 . The blow-up map $\sigma : Y \rightarrow X_0$ is the restriction of the natural projection $\mathbb{C}^4 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ to Y . Then the scheme theoretic inverse image $\sigma^{-1}(C^+)$ is determined by equations

$$z = 0, \quad w = 0, \quad f(x, y) = 0$$

in Y . Therefore it has two irreducible components, a component C determined by

$$f(x, y) = 0, \quad \lambda = 0$$

and the exceptional curve $C_0 \subset Y$ given by

$$x = 0, \quad y = 0.$$

The first component C will be called the strict transform of C^+ in Y . Note that the curve C^+ does not undergo an embedded blow-up in this process, in contrast to the strict transform of the plane curve $\{f(x, y) = 0\}$ under the blow-up of plane at the origin. In the latter case, the inverse image of the plane curve under the blow-up map is determined by

$$f(x, y) = 0, \quad x\lambda = y\rho$$

in $\mathbb{C}^2 \times \mathbb{P}^1$. Writing these equations in the affine coordinate charts $\lambda \neq 0$, respectively $\rho \neq 0$, it can be easily shown that the inverse image has again two components, an

irreducible plane curve C' and a divisor of the form mE , where m is the multiplicity of the plane curve at the origin, and E the exceptional curve. The strict transform C' is not isomorphic to the curve $\{f(x, y) = 0\}$ in this case.

Since $\sigma : Y \setminus C_0 \rightarrow X_0 \setminus \{0\}$ is an isomorphism of complex manifolds, it follows that M_ϵ intersects the strict transform $C \subset Y$ of C^+ along the cycle $\eta_\epsilon = \sigma^{-1} \circ \phi_0^{-1} \circ \gamma_\epsilon^+$. The strict transform C is the plane singular curve cut by the equations

$$f(x, y) = 0, \quad \lambda = 0$$

on Y . Therefore it is a singular plane curve isomorphic to C^+ , contained in the fiber $\lambda = 0$ of Y over \mathbb{P}^1 . The singular point $p \in C$ is the unique point of intersection with the zero section, $x = y = 0, \lambda = 0$. The cycle η_ϵ divides C into two connected components, the component containing p being a singular holomorphic disc D_ϵ in Y_ϵ with boundary $\eta_\epsilon \subset M_\epsilon$. This is precisely the geometric set-up outlined in diagram (2.18). In order to obtain a complete large N duality picture, one should prove that the holomorphic disc D_ϵ is rigid, which is a difficult technical question for general algebraic knots. Section 6 will provide an affirmative answer for torus knots, leaving the general case for future work.

Assuming that D_ϵ is rigid, the next problem is the computation of its multicover contributions to the Gromov-Witten theory with lagrangian boundary conditions on M_ϵ . One angle on this problem is to try to generalize the computations of [19] based on stable maps with lagrangian boundary conditions to the present case. This approach requires a torus action preserving M_ϵ, D_ϵ , which is the case only for torus knots. In this case, the details of the virtual localization computation are presented in Sect. 6, the resulting formulas being in agreement with large N duality predictions.

A second approach follows from string duality considerations as in [13,35], converting the calculation of topological open **A**-model amplitudes to D-brane bound state counting. In this framework, the topological amplitudes are expressed in terms of BPS states as in Donaldson-Thomas type invariants, making a direct connection with the [33]. This will be discussed next.

4. D-brane Bound States and the Hilbert Scheme

The goal of this section is to provide a physical explanation for the work of Oblomkov and Shende [33] on plane curve singularities in the framework of large N duality. The geometric set-up will be the same as in Sect. 3.3, namely a lagrangian cycle $M_\epsilon \subset Y_\epsilon$ intersecting a singular plane curve $C \subset Y$ along a smooth connected one-cycle $\eta : S^1 \rightarrow M_\epsilon$. The curve C is given by

$$f(x, y) = 0, \quad \lambda = 0,$$

on Y and it will be assumed that it has only one singular point p , given by $x = y = 0, \lambda = 0$. The cycle η_ϵ divides C into two connected components, the component containing p being a holomorphic disc D_ϵ with boundary on M_ϵ . Note that $M_\epsilon \simeq \mathbb{R}^2 \times S^1$ and the cycle η_ϵ is a generator of $H_1(M_\epsilon) \simeq \mathbb{Z}$. It will be assumed that D_ϵ is rigid, which is in fact proven in Sect. 6 for curves of the form $f(x, y) = x^r - y^s$. The subscript ϵ will be dropped in this section because all considerations below hold for any fixed arbitrary value of $\epsilon > 0$.

According to [23,30,35], string duality transformations show that open topological string amplitudes with lagrangian boundary conditions on M are determined by

counting supersymmetric M2-brane or D2-D0 bound states in different duality frames. This is achieved by studying the low-energy effective action for type IIA D4-brane wrapped on the lagrangian cycle M , resulting in a string-like object in the four transverse dimensions. Open topological string amplitudes with boundary conditions on M determine certain holomorphic couplings in the low energy effective action of this string. Since $H_1(M) \simeq \mathbb{Z}$ is generated by η , open string instantons with fixed genus $g \in \mathbb{Z}_{\geq 0}$ and $h = 1$ boundary components are topologically classified by the wrapping number $d \in \mathbb{Z}_{\geq 0}$ on the holomorphic curve C_0 and the winding number $k \in \mathbb{Z}_{\geq 1}$ about the cycle η . The corresponding Gromov-Witten invariants with lagrangian boundary conditions will be denoted by $GW_{g,1}(d, k)$. Only topological open string amplitudes with winding number $k = 1$ will be considered in the following, because we are interested in the Wilson loop observables in the fundamental representation. Moreover we can assume, with no loss of generality that we have only 1 spectator lagrangian A-brane and we replace $\text{Tr}V$ with V . According to [35], these amplitudes determine terms of the form

$$\int d^4x d^4\theta \delta^{(2)}(x) \delta^{(2)}(\theta) F_{g,1}(t, V) (W^2)^g$$

in the effective action of the string, where

$$F_{g,1}(t, V) = \sum_{d \geq 0} g_s^{2g-1} e^{-dt} GW_{g,1}(d, 1) V.$$

Here $W_{\alpha\beta}$, where α, β are symmetric spinor indices, denotes the four dimensional graviphoton multiplet, and t denotes the vector multiplet whose top component is the Kähler modulus of the zero section $C_0 \subset Y$. As in Sect. 2.1, V is the holonomy of a background flat $U(1)$ gauge field on the D4-brane. The four dimensional superspace integral is restricted to the string world-sheet by the δ -functions $\delta^{(2)}(x), \delta^{(2)}(\theta)$.

The M-theory lift of this configuration is an M5-brane wrapping the same lagrangian cycle M . Holomorphic IIA world-sheet instantons lift to open M2-branes with boundary on M wrapping the disc \mathbb{D} and the zero section C_0 . The low energy effective theory of the M5-brane is now a three dimensional theory containing a spectrum of supersymmetric particles corresponding to bound states of open M2-branes. The low energy degrees of freedom include a three-dimensional $N = 2 U(1)$ vector multiplet, the reduction of the M5-brane self-dual tensor multiplet on a harmonic generator of $H^1(M_\epsilon) \simeq \mathbb{Z}$. In addition the space-time effective action includes a $U(1)$ gauge field in the supergravity multiplet. The three-dimensional BPS particles carry integer charges $(k, d) \in \mathbb{Z}$ with respect to these gauge fields. Geometrically, k, d are the M2-brane multiplicity on the disc \mathbb{D} , respectively the zero section C_0 . The five dimensional $SO(4)$ little group is broken to $U(1) \times U(1)$ by the M5-brane, the first factor being the little group in the M5 three dimensional effective theory. The second factor is generated by rotations in the two transverse dimensions. Therefore the BPS spectrum is graded by two spin quantum numbers $\sigma, j \in \mathbb{Z} + 1/2$. The degeneracies of BPS states will be denoted accordingly by $N_{k,d,\sigma,j}$. By analogy with [13] a Schwinger computation shows that the couplings $F_{g,1}(t, V)$ are given by

$$F_{g,1}(t, V) = \sum_{d \geq 0} \sum_{\sigma \in \mathbb{Z}} \frac{N_{d,\sigma}}{2 \sin(g_s/2)} e^{-dt+i\sigma g_s V}, \tag{4.1}$$

where $N_{d,\sigma}$ is a BPS index given by

$$N_{d,\sigma} = \sum_{j \in \mathbb{Z} + 1/2} (-1)^{2j+1} N_{1,d,\sigma,j}.$$

In particular note that the coefficients $N_{d,\sigma}$ have to be integral but not necessarily positive.

The above expression is the restriction of [35, Eq. 4.4] to open string amplitudes of winding number 1, therefore the sum over the integer $n \geq 1$ corresponding to degree n multicover contributions collapses to a single term, $n = 1$. A more convenient form of Eq. (4.1) can be obtained by a change of variables $q = e^{i g s}$, $Q = e^{-t}$, and a redefinition of the spin quantum number, $\sigma = s + 1/2$, $s \in \mathbb{Z}$. Then Eq. (4.1) becomes

$$F_{g,1}(t, V) = \sum_{d \geq 0} \sum_{\sigma \in \mathbb{Z}} \frac{N_{d,s} q^s Q^d}{1 - q} V. \tag{4.2}$$

In order to compute the BPS numbers $N_{d,s}$, one has to count spin $s + 1/2$ bound states of an open M2-brane wrapping the singular holomorphic disc D and d M2-branes wrapping the zero section C_0 . Note that using the Large N duality this implies that the expectation value of the Wilson loop in the fundamental representation of the knot, which is known as the HOMFLY polynomial of the knot, is given by this expression (recalling that $\text{Tr}U$ is paired up with $\text{Tr}V$):

$$\langle \text{Tr}U \rangle = \sum_{d \geq 0} \sum_{\sigma \in \mathbb{Z}} \frac{N_{d,s} q^s Q^d}{1 - q}. \tag{4.3}$$

It is known [13, 14] that d M2-branes wrapping the compact curve C_0 form supersymmetric bound states only if $d = 1$, in which case there is a single spin 0 state. Therefore the main problem is to understand bound state counting for open M2-branes wrapping the singular disc D . This is most efficiently done reducing the problem to counting D2-D0 bound states in a suitable weakly coupled Type IIA limit. More precisely, one can choose the M-theory circle such that the M5-brane is mapped again to a D4-brane, but the M2-branes yield open D2-branes with boundary on the D4-brane. Furthermore the $d = 0$ truncation of the right-hand side of Eq. (4.2) is interpreted as a partition function of the form

$$\sum_{n \geq 0} C_n q^n \tag{4.4}$$

counting supersymmetric states of open D2-branes wrapping D bound to an arbitrary number n of D0-branes [13, 20]. The coefficients C_n in Eq. (4.4) are BPS indices counting states weighted by a sign determined by their spin. This index can be exactly computed in the semiclassical limit, in which case C_n equals the Euler characteristic of the moduli space of supersymmetric D-brane configurations. In order to understand the structure of such moduli spaces, it is helpful to consider first configurations of n D0-brane bound to a D2-brane wrapping a compact holomorphic curve Z in a smooth projective Calabi-Yau threefold, which may be assumed arbitrary for the purposes of a general presentation. Once the main ideas are clearly illustrated in a general context, the discussion will be specialized to D2-branes wrapping the disc D in Y . According to [20, 21, 36], such configurations are mathematically modeled by an abelian vortex configuration of degree n on Z . The basics of this formalism will be reviewed in some detail below.

4.1. *D2-D0 bound states, vortices, and stable pairs.* A degree n abelian vortex is a triple pair (\mathcal{L}, A, s) , where \mathcal{L} is a complex line bundle on Z with first Chern class $c_1(\mathcal{L}) = n$, A is a $U(1)$ connection on \mathcal{L} and s is a section of \mathcal{L} satisfying $D_A s = 0$. This naturally captures the data of the choice of the gauge field on the D2 brane, as well as the geometry of the D0 brane, which can be identified with s . It corresponds to the bifundamental field charged under the D2-brane $U(1)$ stretched between D0-brane and D2-brane. The moduli space of triples (\mathcal{L}, A, s) modulo unitary gauge transformations is isomorphic to the moduli space of pairs (\mathcal{L}, s) modulo complexified gauge transformations, where \mathcal{L} is a holomorphic line bundle on Z and $s \in H^0(Z, \mathcal{L})$ is a nontrivial holomorphic section. The relation between differential geometric and algebraic geometric data follows as usual observing that any connection A on a C^∞ complex line bundle \mathcal{L} determines a Dolbeault operator $\bar{\partial}_A$.

In the algebraic formulation, note that the zero locus of s is a degree n effective divisor (s) on Z , that is a formal linear combination of points $\sum_{i=1}^k n_i p_i$, where $n_i \geq 1$ and $\sum_{i=1}^k n_i = n$. The points p_1, \dots, p_k represent the locations of the D0-branes and the integers $n_i \geq 1, i = 1, \dots, k$, the D0-brane multiplicity at each point. Assigning to each pair (\mathcal{L}, s) the divisor (s) yields an isomorphism between the moduli space of isomorphism classes of pairs (\mathcal{L}, s) and the symmetric product $S^n(Z) = Z^n/S_n$, where S_n is the permutation group on n letters.

From the sheaf theoretic point of view, a pair (\mathcal{L}, s) can be uniquely characterized up to gauge transformations by specifying the germs of local holomorphic sections of \mathcal{L} near each point p of Z . The simplest case is the trivial vortex configuration, when \mathcal{L} is isomorphic to the trivial line bundle \mathcal{O}_Z on Z and s is constant. The germs of local sections of \mathcal{O}_Z near each point $p \in Z$ are simply germs of local holomorphic functions with no restrictions on the vanishing order at p . For a configuration (\mathcal{L}, s) with $n > 0$ the same holds locally near any point $p \in Z, p \neq p_i, i = 1, \dots, k$. Near one of the points p_i , the germs of holomorphic sections of \mathcal{L} are identified with germs of meromorphic functions with at most a pole of order n_i at p_i . In this local picture the section s corresponds to the natural inclusion of the local sections of \mathcal{O}_Z in the set of local sections of \mathcal{L} . Note that the complement is a finite dimensional vector space of dimension n_i . In terms of a local coordinate z_i centered at p_i , this vector space is generated by sections of the form $\{z_i^{-l}\}, l = 1, \dots, n_i$. More abstractly, the local sections of \mathcal{L} near p_i form a rank 1 module over the local ring of functions \mathcal{O}_Z generated by $\{z_i^{-n_i}\}$.

It may be also helpful to note that there is a dual mathematical model for D2-D0 configurations. In the dual model, n_i D0-branes located at p_i are described by the set of local holomorphic functions which vanish at least to order n_i at p_i . This set is the ideal generated by $z_i^{n_i}$ in the ring of local functions near p_i . The geometric object characterized by this local behavior is the dual line bundle \mathcal{L}^{-1} , which is a sub-sheaf of the trivial line bundle \mathcal{O}_Z . In more abstract language, $\mathcal{L}^{-1} \subset \mathcal{O}_Z$ is the defining ideal sheaf of the effective divisor $(s) = \sum_{i=1}^k n_i p_i$.

Similar considerations apply [36,37] to a singular curve Z , abelian vortices being generalized to stable pairs. This essentially means that one has to allow the gauge field A to develop singularities at the singular points of the curve Z . While a complete analytic treatment of such singularities would be quite difficult, the sheaf theoretic point of view discussed above leads to an efficient construction of the moduli space. A single D0-brane supported at a smooth point p was previously identified with the module of local meromorphic functions with at most a simple pole at p . If p is a singular point of Z , a single D0-brane at p is still defined by a module of local meromorphic functions,

but this module may have more than one generator. Conceptually, this may be easily understood employing the dual model. Consider for example the plane curve singularity $x^3 = y^2$. A D0-brane with multiplicity 1 located at the singular point $x = y = 0$ cannot be described as the zero locus of a single local holomorphic function. If one simply sets $x = 0$ or $y = 0$ the defining equation of the curve reduces to $y^2 = 0$, respectively $x^3 = 0$. According to the previous paragraph this is in fact a D0-brane configuration with multiplicity 2, respectively 3. A single D0-brane is the zero locus of two local functions, (x, y) which generate an ideal in the ring of local holomorphic functions. The dual stable pair is given by a local module over the ring of local functions generated by two elements.

Informally, the main idea of this construction is that at a singular point p of Z the rank of the Chan-Paton line bundle \mathcal{L} on Z is allowed to jump in a controlled way, depending on the analytic type of the singularity at p . Effectively, the single D2-brane on Z behaves locally at p as a stack of D2-branes with higher multiplicity $m \geq 1$. For a fixed number of D0-branes m may take finitely many values determined by n and the singularity type. This point of view will be very useful in understanding bound state formation for D2-branes wrapping different holomorphic curves with transverse intersection.

A consequence of the above discussion is that the moduli space of D2-D0-brane configurations supported on Z is no longer isomorphic to the symmetric product $S^n(Z)$. It has been shown in [37] that the moduli space of D2-D0-brane configurations supported on Z is in this case isomorphic to the Hilbert scheme $\mathcal{H}^n(Z)$ of n points on Z . The rigorous definition of the Hilbert scheme is not needed for the purpose of the present discussion, but it may be helpful to note that there is a natural map $\pi : \mathcal{H}^n(Z) \rightarrow S^n(Z)$ forgetting the extra algebraic structure associated to each singular point. From a physical point of view this means that the D0-branes are treated simply as non-interacting particles ignoring interactions due to open string effects.

Analogous considerations hold for D2-branes wrapping a smooth holomorphic disc D with boundary on a lagrangian cycle. The holomorphic line bundle \mathcal{L} must now be equipped with a trivialization on the boundary of the disc $\partial D \simeq S^1$, which is part of the boundary conditions on the D2-brane fields. Complex line bundles on the disc with boundary trivialization are topologically classified by the first Chern class, which takes values in the relative homology group $H_2(D, \partial D) \simeq \mathbb{Z}$. Moreover, since the section $s \in H^0(\mathcal{L})$ must be compatible with the trivialization, the number of zeroes of s , counted with multiplicity must equal the first Chern class n . Summing over all Chern classes yields the partition function

$$\sum_{n \geq 0} C_n q^n = \sum_{n \geq 0} \chi(S^n(D)) q^n = \frac{1}{1 - q}, \tag{4.5}$$

since the symmetric power $S^n(D)$ is contractible for any $n \geq 0$. Note that this result is the same as the winding number one partition function of a single lagrangian brane in \mathbb{C}^3 given [1] by the topological vertex $C_{\emptyset, \emptyset, \square}(q)$. As observed in [18, Sect. 4], the above formula can be alternatively interpreted as the Hilbert series of the ring $\mathbb{C}[t]$ of polynomial functions on the complex line \mathbb{C} . By definition, the Hilbert series of a polynomial ring R is

$$H_R(q) = \sum_{n \geq 0} c_n(R) q^n,$$

where $c_n(R)$ is the number of degree n monomials in R . Obviously, $H_{\mathbb{C}[t]}(q)$ is equal to the above partition function. It was explained in [18, Sect. 4], that $H_{\mathbb{C}[t]}(q)$ can be also

interpreted as a counting function of states in the Hilbert space \mathcal{H} of a single quantum harmonic oscillator.

Next suppose \mathbb{D} has singular points away from the boundary. Without any loss of essential information, one may assume that \mathbb{D} has only one singular point p . Several singular points may be treated analogously with no new conceptual issues.

In this case the Chern class of a singular vortex configuration admits a splitting, $n+l$, where $n \in \mathbb{Z}$ is determined as above by the trivialization on the boundary, and $l \in \mathbb{Z}$ is a contribution supported at the singular point. A rigorous account of this splitting is provided at the end of Sect. 5, where it is also shown that l takes finitely many values. In addition, one has to specify the multiplicity $m \geq 1$ of the singular vortex at p , as discussed above. Therefore the partition function will be in general of the form

$$\sum_{l \geq 0} \sum_{m \geq 1} f_{l,m}(q), \tag{4.6}$$

where only finitely many terms are nontrivial. Note that each term $f_{l,m}(q)$ is a power series in q because for fixed values of (l, m) one has to sum over all possible boundary trivializations, as in the smooth case. More detailed information on the terms $f_{l,m}(q)$ requires a more involved technical analysis, as shown for specific examples in Sect. 4.3. A more immediate task at this stage is however to explain how the above general reasoning can be applied to more general M2-brane configurations supported on intersecting curves.

4.2. Intersecting M2-brane bound states. The relevant intersecting curve configurations for large N duality consist of a singular holomorphic disc \mathbb{D} as above meeting a smooth $(-1, -1)$ rational curve C_0 at the singular point p . One then has to count bound states of $k = 1$ open M2-branes wrapping \mathbb{D} and d closed M2-branes wrapping the zero section C_0 . As shown in [13, 14], M2-branes wrapping a $(-1, -1)$ curve C_0 with multiplicity $d \geq 1$ form bound states only for $d = 1$, in which case the spectrum consists of one BPS state of spin 0.

In addition, when an M2-brane wrapping \mathbb{D} is added to the system one can form new bound states binding a membrane wrapping C_0 to the membrane wrapping \mathbb{D} . If \mathbb{D} were smooth, the intersection between the two M2-branes would be modeled by a curve with a simple nodal singularity $xy = 0$. This configuration can be viewed as a limit of a single M2-brane wrapping the smooth curve $xy = \epsilon$ as $\epsilon \rightarrow 0$. Therefore two intersecting M2-branes form in this case a single bound state.

However in the case of interest here \mathbb{D} is a singular disk, which has local multiplicity $m \geq 1$ at the singular point p , even though its generic multiplicity 1. In other words m counts the number of ‘points’ at p . What this means is that if we were to consider an annulus which ends on one end on the D2-D0 brane bound state on one side, and on a transverse D-brane intersecting the curve at p on the other, m counts the Witten index for it. Therefore a membrane wrapping C_0 may bind to the singular membrane in m distinct ways, depending on which local branch it is attached to. This results in a spectrum of m BPS states in the low energy effective action. More generally, d membranes wrapping C_0 can bind in $\binom{m}{d}$ distinct ways to the singular open membrane, resulting in as many BPS particles. In particular, if $d > m$ no irreducible bound state may be constructed. Therefore the partition function for such configurations must take the general form

$$\sum_{l \geq 0} \sum_{m \geq 1} \sum_{d=0}^m \binom{m}{d} (-Q)^d f_{l,m}(q) = \sum_{m \geq 1} f_m(q) (1 - Q)^m, \tag{4.7}$$

where

$$f_m(q) = \sum_{l \geq 0} f_{l,m}(q).$$

Here we used the fact, already seen for the unknot, that the fermion parity of the M2 brane wrapping \mathbf{P}^1 is -1 leading to $-Q$ for each such state in the above formula. Moreover, Eq. (4.1) predicts that

$$f_m(q) = \frac{g_m(q)}{1 - q}$$

with $g_m(q)$ a polynomial with integral coefficients. These predictions will be confirmed by explicit computations for plane curves of the form

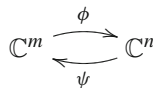
$$x^r - y^s = 0$$

in the next section.

For completeness, it is worth noting that the combinatorial factors $\binom{m}{d}$ admit a geometric interpretation in the weakly coupled IIA limit mapping M2-branes to D2-branes. Then the massless spectrum of open string stretching between a D2-brane on D and d D2-branes on C_0 consists of an $N = 2$, $d = 4$ hypermultiplet reduced to one dimension. The bosonic components are two complex scalar fields ϕ, ψ transforming in the bifundamental representation of the D-brane gauge fields and its dual. Again, the singular D2-brane effectively has multiplicity m at the singular point even if it is generically of rank one. Therefore ϕ, ψ may be identified with linear maps $\phi : \mathbb{C}^m \rightarrow \mathbb{C}^n, \psi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ respectively. Then the F-term equations are simply

$$\psi \circ \phi = 0, \quad \phi \circ \psi = 0.$$

This implies that the moduli space of flat directions modulo gauge transformations is isomorphic to a moduli space of representations of a quiver of the form



subject to the F-term equations. The stability conditions are determined as usual by the D-term equations,

$$|\phi|^2 - |\psi|^2 = \xi.$$

The subtle aspect here is that even though the singular D2-brane has multiplicity m at p , one should only mod out by $U(1) \times U(d)$ gauge transformations since the brane has generic of rank 1. Moreover, since the diagonal $U(1)$ subgroup acts trivially on ϕ, ψ , it suffices to mod out by $U(d)$ gauge transformations.

A straightforward analysis of the resulting stability condition shows that $\phi = 0$ and ψ must be surjective for $\xi < 0$ and $\psi = 0$ and ϕ must be surjective for $\xi > 0$. Therefore if $\xi > 0$, the moduli space of stable representations modulo $U(d)$ gauge transformations is isomorphic to the grassmannian $G(m, d)$ of d -dimensional quotients of \mathbb{C}^m if $d \leq m$ and empty if $d > m$. If $\xi < 0$ the moduli space is just a point if $d \geq m$ and empty if $d < m$.

In string theory the FI term ξ is determined by the expectation value of the background fields on Y , such as the metric and B-field. The previous paragraph implies that for any choice of background fields such that $\xi > 0$ the weakly coupled IIA analysis agrees with M-theory considerations. Namely, the moduli space of expectation values of open string modes is isomorphic to the grassmannian $G(m, d)$ which has Euler character $\binom{m}{d}$. This is precisely the number of bound states predicted by M-theory arguments.

4.3. *Curves of type (r, s).* Returning to the setup described at the beginning of this section, consider a singular curve C of the form

$$x^r - y^s = 0, \quad \lambda = 0$$

in a resolved conifold Y . Here (r, s) are coprime positive integers and it will be assumed that $r > s \geq 1$. Note that C has only one singular point p given by $x = y = 0, \lambda = 0$. The construction of Sect. 3.3 produces a lagrangian cycle $M \subset Y$ which intersects C along a smooth connected one cycle η . Therefore C is divided into two connected components, the holomorphic disc D being the component containing the singular point p . Note that D is preserved by the circle action

$$(x, y, \zeta) \mapsto (e^{-is\varphi}x, e^{-ir\varphi}y, e^{i(r+s)\varphi}\zeta)$$

which fixes only the singular point p . This action yields a natural action on the moduli space of vortices, and Euler character computations localize to the fixed point set. As shown at the end of Sect. 5, the fixed point set in the moduli space of vortices is discrete and consists of vortex configurations centered at the singular point p . Since p is away from the boundary of D , the localization computation of the partition function (4.6) yields the same answer as the localization computation for vortices on the open curve C . Therefore for computational purposes one may work with stable pairs on C . This yields an explicit computational algorithm for the terms $f_{l,m}(q)$ in (4.6) which is summarized below.

The first term $f_{0,1}(q)$ in (4.6) represents the contribution of topologically trivial gauge field configurations. All terms $f_{0,m}(q), m \geq 2$ are obviously zero since the trivial line bundle has multiplicity 1 at p . Just as in the smooth case, $f_{0,1}(q)$ is given by the Hilbert series of the ring R_C of regular functions on C . Since (r, s) are coprime, the curve C may be given in parametric form as $(x, y) = (t^s, t^r)$. Therefore R_C is isomorphic to the subring $\mathbb{C}[t^r, t^s] \subset \mathbb{C}[t]$ spanned by polynomials of the form $p(t^r, t^s)$ with $p(x, y)$ an arbitrary polynomial of two variables. It will be convenient to identify the set of monomials $t^n \in \mathbb{C}[t^r, t^s]$ with the set of exponents $n \in \mathbb{Z}_{\geq 0}$, which will be denoted by $\Lambda(r, s)$. Note that the complement $\Xi(r, s) = \mathbb{Z}_{\geq 0} \setminus \Lambda(r, s)$ is a finite set. Therefore $f_{0,1}(q)$ can be identified with the germ turn generated by x, y with weights s, r respectively, modulo a relation of degree sr :

$$f_{0,1}(q) = \sum_{n \in \Lambda(r,s)} q^n = \frac{(1 - q^{rs})}{(1 - q^r)(1 - q^s)} = \frac{1}{1 - q} - \sum_{n \in \Xi(r,s)} q^n. \quad (4.8)$$

By comparison with the formula (4.5) it follows that the effect of the singularity in the topologically trivial sector is to remove the states in the Hilbert space of the harmonic oscillator with quantum numbers $n \in \Xi(p, q)$. Note that Eq. (4.8) is a specialization of a general result for the Alexander polynomial of torus knots proven in [6].

For concreteness, suppose $(r, s) = (4, 3)$. Then $\Lambda(4, 3)$ is the set

$$0, \quad 3, 4, \quad 6, 7, \dots$$

and the complement $\Xi(3, 4)$ is the finite set

$$1, 2, \quad 5.$$

Therefore in this case

$$f_{0,1}(q) = \frac{1}{1-q} - (q + q^2 + q^5) = \frac{1 - q + q^3 - q^5 + q^6}{1 - q}.$$

The terms $f_{l,m}(q)$ corresponding to topologically nontrivial sectors are constructed in a similar manner in terms of partial fillings of $\Lambda(r, s)$. A partial filling of $\Lambda(r, s)$ is a subset

$$\Lambda(r, s) \subseteq \Lambda'(r, s) \subseteq \mathbb{Z}_{\geq 0}$$

with the property that if $\Lambda'(r, s)$ contains some $n' \in \Xi(r, s)$, then it must contain all its translates $n' + n$ by arbitrary elements $n \in \Lambda(r, s)$. Each partial filling is obtained by adding finitely many elements in $\Xi(r, s)$ to $\Lambda(r, s)$ subject to this selection rule. For example all possible partial fillings in the case $(r, s) = (4, 3)$ are

$$\begin{aligned} \Lambda'(4, 3)_{(1)} &: \quad \mathbf{0}, \mathbf{1}, \quad 3, 4, \underline{5}, 6, 7, \dots, \\ \Lambda'(4, 3)_{(2)} &: \quad \mathbf{0}, \quad \underline{2}, 3, 4, \underline{5}, 6, 7, \dots, \\ \Lambda'(4, 3)_{(3)} &: \quad \mathbf{0}, \quad 3, 4, \underline{5}, 6, 7, \dots, \\ \Lambda'(4, 3)_{(4)} &: \quad \mathbf{0}, \underline{\mathbf{1}}, \underline{\mathbf{2}}, 3, 4, \underline{5}, 6, 7, \dots, \end{aligned}$$

the extra elements being underlined in each case. What this means is for example for $\Lambda'(4, 3)_{(1)}$ the line bundle has one additional section s' represented by $\mathbf{1}$, which does not vanish at the origin as we put $x = y = 0$. The additional element in the ring given by $s'y$ given by $\underline{5}$, does vanish at the origin, as it vanishes as we set $y = 0$. Similar considerations apply to the rest. A disallowed filling is for example

$$0, 1, \quad 3, 4, \quad , 6, 7, \dots,$$

since the translation of 1 by 4 is $1 + 4 = 5$, which is missing in the above sequence. This is consistent with the fact that we can multiply a section by the holomorphic functions of x, y and still get a section of the same bundle, and so 5 should also have been in the sequence of the sections of the line bundle.

Note that any partial filling $\Lambda'(r, s)$ contains a unique finite subset $\Gamma(r, s)$ consisting of all elements n' which cannot be decomposed as

$$n' = n'' + n$$

with $n'' \in \Lambda'(r, s)$ and $n \in \Lambda(r, s), n \neq 0$. Moreover it is easy to show that any element $n' \in \Lambda'(r, s)$ can be written as $n' = n'' + n$ with $n'' \in \Gamma(r, s)$ and $n \in \Lambda(r, s)$. The elements of $\Gamma(r, s)$ will be called the generators of $\Lambda'(r, s)$. In the above example the generators are marked in each case with boldface characters.

The first Chern class l of the vortex corresponding to $\Lambda'(r, s)$ is the number of elements in the complement $\Lambda'(r, s) \setminus \Lambda(r, s)$, which is the same as the number of additional sections we have introduced while the multiplicity m is the number of generators, which

is also the number of sections which do not vanish at p as we set $x = y = 0$. In the above example,

$$l_{(1)} = l_{(2)} = 2, \quad l_{(3)} = 1, \quad l_{(4)} = 3,$$

and

$$m_{(1)} = m_{(2)} = m_{(3)} = 2, \quad m_{(4)} = 3.$$

Note that 0 is always a generator, and never an extra element. The pair (l, m) assigned to a partial filling $\Lambda'(r, s)$ will be called below the type of the partial filling.

The terms $f_{(l,m)}(q)$ are then obtained by summing the Hilbert series of all modules associated to partial fillings $\Lambda'(r, s)$ of fixed type (l, m) . That is

$$f_{(l,m)}(q) = q^l \sum_{\Lambda'(r,s) \text{ of type } (l,m)} \sum_{n \in \Lambda'(r,s)} q^n. \tag{4.9}$$

The factor q^l reflects the fact that all such configurations have first Chern class l . For $(r, s) = (4, 3)$ the resulting contributions are

$$\begin{aligned} f_{1,2}(q) &= q \sum_{n \in \Lambda'(4,3)_{(3)}} q^n = q \left(\frac{1}{1-q} - q - q^2 \right), \\ f_{2,2}(q) &= q^2 \sum_{n \in \Lambda'(4,3)_{(1)}} q^n + q^2 \sum_{n \in \Lambda'(4,3)_{(2)}} q^n = \left(\frac{1}{1-q} - q^2 \right) + q^2 \left(\frac{1}{1-q} - q \right), \\ f_{3,3}(q) &= q^3 \sum_{n \in \Lambda'(4,3)_{(4)}} q^n = \frac{q^3}{1-q}, \end{aligned}$$

all other terms being trivial. Then the coefficients $f_m(q)$ in Eq. (4.7) are

$$\begin{aligned} f_1(q) &= \frac{1 - q + q^3 - q^5 + q^6}{1 - q}, \\ f_2(q) &= \frac{q + q^2 - q^3 + q^4 + q^5}{1 - q}, \\ f_3(q) &= \frac{q^3}{1 - q}. \end{aligned}$$

The HOMFLY polynomial of the (r, s) torus knot is given by

$$H_{(r,s)}(q, Q) = \left(\frac{Q}{q} \right)^{(r-1)(s-1)/2} \frac{1}{1 - q^r} \sum_{j=0}^{r-1} \frac{q^{sj+(r-1-j)(r-j)/2}}{[j]![r-1-j]!} \prod_{i=j+1-r}^j (q^i - Q), \tag{4.10}$$

where $[0]! = 1$ and $[j]! = (1 - q^j)[j - 1]!$ for all $j \geq 1$. Then a straightforward computation yields

$$\sum_{m=1}^3 (1 - Q)^m f_m(q) = \left(\frac{q}{Q} \right)^6 H_{(4,3)}(q, Q),$$

confirming large N duality for the $(4, 3)$ torus knot. Note that [33, Thm. 19] proves the agreement between formula (4.10) and the stable pair localization computation for all (r, s) . The examples considered in this section are meant to explain the localization computation in a physical context.

The next case treated explicitly here is $(r, s) = (2, 2k + 1)$, $k \geq 1$. Then the HOMFLY polynomial (4.10) reduces to

$$\begin{aligned} H_{(2, 2k+1)}(q, Q) &= \left(\frac{Q}{q}\right)^k \frac{1-Q}{1-q} \frac{1-q^{2k+2}-qQ(1-q^{2k})}{1-q^2} \\ &= \left(\frac{Q}{q}\right)^k \frac{1-Q}{1-q} \left[1 + (q-Q) \sum_{j=0}^{k-1} q^{2j}\right]. \end{aligned} \tag{4.11}$$

The subset $\Lambda(2k + 1, 2) \subset \mathbb{Z}_{\geq 0}$ consists of the following elements:

$$0, \quad 2, \quad \dots \quad 2k, \quad 2k + 1, \quad 2k + 2, \dots,$$

its complement $\Xi(2k + 1, 2)$ being

$$1, \quad 3, \quad \dots \quad 2k - 1.$$

There are $k + 1$ partial fillings $\Lambda'(2k + 1, 2)_{(j)}$, $0 \leq j \leq k + 1$ as follows:

$$\begin{aligned} \Lambda'(2k + 1, 2)_{(0)} &= \Lambda(2k + 1, 2), \\ \Lambda'(2k + 1, 2)_{(1)} &= \Lambda(2k + 1, 2) \cup \{1, \dots, 2k - 1\}, \\ &\vdots \\ \Lambda'(2k + 1, 2)_{(j)} &= \Lambda(2k + 1, 2) \cup \{2j - 1, \dots, 2k - 1\}, \\ &\vdots \\ \Lambda'(2k + 1, 2)_{(k)} &= \Lambda(2k + 1, 2) \cup \{2k - 1\}. \end{aligned}$$

Each $\Lambda'(2k + 1, 2)_{(j)}$, $1 \leq j \leq k$ has two generators, $0, 2j - 1$ and the complement of $\Lambda(2k + 1, 2)$ contains $k - j + 1$ elements. Therefore

$$l_{(j)} = k - j + 1, \quad m_{(j)} = 2$$

for all $1 \leq j \leq k$. Obviously, $l_{(0)} = 0, m_{(0)} = 1$. Therefore

$$\begin{aligned} f_1(q) &= \sum_{n \in \Lambda(2k+1, 2)} q^n = \frac{1}{1-q} - \sum_{j=1}^k q^{2j-1} = \frac{1+q^{2k+1}}{1-q^2}, \\ f_2(q) &= \sum_{j=1}^k q^{k-j+1} \left(\frac{1}{1-q} - \sum_{i=1}^{j-1} q^{2i-1} \right), \\ &= \sum_{j=1}^k q^{k-j+1} \frac{1+q^{2j-1}}{1-q^2} = \frac{q(1-q^{2k})}{(1-q)(1-q^2)}. \end{aligned}$$

Note that

$$f_1(q) + f_2(q) = \frac{1 - q^{2k+2}}{(1 - q)(1 - q^2)}.$$

Then a straightforward computation yields

$$\sum_{m=1}^2 (1 - Q)^m f_m(q) = \left(\frac{q}{Q}\right)^k H_{(2,2k+1)}(q, Q).$$

5. A Summary for Mathematicians

This section recapitulates the previous, in language perhaps more amenable to mathematicians. As before, the goal is to explain how a conjecture of Oblomkov and the second author [33] is related to a certain series of string dualities. On the one hand, this provides a physics proof of the conjecture. On the other, the conjecture was proven (in the mathematical sense) for torus knots in [33]. This may then be viewed as confirming evidence for the string dualities which occur in the discussion below.

We recall the conjecture in question. Let C be a curve in \mathbb{C}^2 , say given by $f(x, y) = 0$. Assume C passes through the origin. Then the intersection of C with the boundary of a small ball around the origin gives a link in the 3-sphere. Note this link has a natural orientation since it bounds a complex variety, and in fact a natural framing (though we will not use this). Recall that the HOMFLY polynomial is an invariant of links which assigns to a link L a certain rational function $H(L)$ in the variables $q^{\pm 1/2}, Q^{\pm 1/2}$, characterized by the following skein relation:

$$Q^{1/2} H(\text{crossing}) - Q^{-1/2} H(\text{crossing}) = (q^{1/2} - q^{-1/2}) H(\text{cup}), \tag{5.1}$$

$$Q^{1/2} - Q^{-1/2} = (q^{1/2} - q^{-1/2}) H(\text{circle}). \tag{5.2}$$

On C , we consider the moduli space $C^{[n]}$ parameterizing pairs (F, s) , where F is a torsion free sheaf, s is a section $s : \mathcal{O}_C \rightarrow F$, and $\dim F/s\mathcal{O}_C = n$. Note that in [33] the same notation was used for the Hilbert scheme of n points on C ; as shown in [37] these spaces are isomorphic for Gorenstein (and in particular planar) curves C . By $C_0^{[n]}$ we denote the space of such pairs in which the section vanishes only at the origin. By $C_0^{[n];m}$ we denote the locus, where $m = \dim_{\mathbb{C}} F/(x, y)F$. Let $\mu = \dim \mathbb{C}[[x, y]]/(\partial_x f, \partial_y f)$ be the Milnor number of the singular point. We can now state:

Conjecture 5.1 [33].

$$H(\text{the link of } C) = \left(\frac{Q}{q}\right)^{\mu-1} \sum_{n,m} q^n (1 - Q)^m \chi(C_0^{[n];m}).$$

One contribution of the present article is to explain how a certain sequence of string dualities connects the left of the conjecture to the right. The HOMFLY polynomial enters physics through Witten’s observation [43] that it computes the expectation value of the knot viewed as a Wilson line in the Chern-Simons gauge theory on the three sphere. Here the gauge group is $U(N)$, and its holonomy around the knot is traced in the fundamental representation. Witten later [44] explained that this theory was equivalent to the type

IIA topological string theory on T^*S^3 , with N lagrangian D-branes on S^3 . The Wilson loop expectations (and hence the HOMFLY polynomial) are reproduced by introducing [35] the conormal bundle of the knot as a lagrangian brane $N^*K \subset T^*S^3$ and counting open strings with one end on N^*K and the other on S^3 .

As N grows large one may take the view [14] that the S^3 shrinks and the space $X = T^*S^3$ is ultimately replaced by the small resolution of the conifold, i.e., by the total space Y of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over \mathbb{P}^1 . The D-branes on S^3 vanish along with the S^3 . Attempting to follow N^*K through the conifold transition is problematic, since it meets the collapsing S^3 . Instead, it is better to first deform it off the S^3 . In the case that the knot K arose as an algebraic knot, we have explicitly constructed such a deformation in Sect. 3.2, and followed it through the conifold transition. The essential feature of the resulting $L_K \subset Y$ is that it intersects the fiber over infinity in a single circle; the unique holomorphic curve passing through this circle is the singular curve itself. Conjecturally this curve is in fact the *only* irreducible curve with boundary on Y ; this is proven in the next section in the case where K arose from the curve $x^r = y^s$.

At this stage we see that the HOMFLY polynomial of K should be computed by counting curves in Y with boundary on L_K . The mathematical foundations of open Gromov-Witten theory are not presently available, but nonetheless for torus knots it is possible to describe the inevitable result of torus localization of the virtual class as in [19]. This is done in the subsequent section, yielding agreement with known formulas for the HOMFLY polynomial in these cases.

According to [13, 23, 30, 35], we may lift to M-theory. Indeed, the topological string computes certain supersymmetric quantities in the full type IIA string theory on $Y \times \mathbb{R}^{3,1}$, which in turn is viewed as a limit of M-theory on $Y \times \mathbb{R}^{4,1}$. The variables work out so that the coefficient of $q^r Q^s$ in $(1 - q)H$ counts certain M2-branes. More precisely, one considers an M5-brane $L \times \mathbb{R}^{2,1}$ for some $\mathbb{R}^{2,1} \subset \mathbb{R}^{4,1}$. Note that this brane breaks the symmetry group of the $\mathbb{R}^{4,1}$ to $Spin(2, 1) \times Spin(2)$; we will only be interested in the $Spin(2) \times Spin(2)$ action. This group acts on all spaces of BPS states of M2 branes with boundary on this M5 brane, so these acquire a bigrading by the characters q, t of the group. (Here t is the character of the rotation transverse to the Lagrangian.) The M2 brane states also carry two additional gradings, corresponding to the class of the brane in $H_2(Y, L) = \mathbb{Z}\mathbb{P}^1 \oplus \mathbb{Z}\mathbb{D}$, where \mathbb{D} is the class of the singular disc bounding the lagrangian. Writing $N_{d', d, \sigma, j}$ for the space of states of character $q^\sigma t^j$ and homology class $d\mathbb{P}^1 + d'\mathbb{D}$, the prediction of the above dualities is that the HOMFLY polynomial is (up to an appropriate $q \cdot Q$) given by

$$H(q, Q) = \frac{1}{1 - q} \sum N_{1, d, \sigma, j} Q^d q^\sigma (-1)^{2j+1}. \tag{5.3}$$

The geometry of M2-branes is not well enough understood that the $N_{1, d, \sigma, j}$ may be computed directly. However, according to [13, 23, 30, 35] the above index may be computed in a *different* type IIA limit of the M-theory, in which one of the dimensions of the \mathbb{R}^4 is compactified on a circle, and the different momenta modes around this circle are converted into bound states of D0-branes to D2-branes. The D2-branes must of course still have boundary along the Lagrangian. (The reason one is free to compute in any limit one likes is that all the states in question are BPS.) In the large volume limit, the space of D2-D0 branes is understood to be mathematically modelled by the space of stable pairs [20, 21, 36], and the index above is just its (appropriately weighted) Euler characteristic. These spaces *are not* identical to the spaces $C_0^{[n]; m}$. In an upcoming paper

of the first author, they will be shown to be related by wall crossing, and as a consequence Conjecture 5.1 will be deduced from (5.3).

Here we argue instead at the level of M-theory. Since the coefficient of Q^d is counting bound states formed by one M2-brane wrapping D and d M2 branes wrapping \mathbb{P}^1 , we must analyse how this configuration may occur. It is already known that M2 branes wrapping the \mathbb{P}^1 may not bind to each other; this for instance follows by running through the above series of dualities for the partition function of Chern-Simons theory itself; the consequence being that in the absence of Lagrangians there is a unique BPS state consisting of a single M2 brane wrapping the \mathbb{P}^1 . Thus each of the M2 branes wrapping the \mathbb{P}^1 must bind to D .

Let $g_m(q)$ be the generating polynomial of which the coefficient of q^n is the number of spin n M2 branes wrapping D which can bind up to m M2 branes on \mathbb{P}^1 . Then, since the M2 branes on \mathbb{P}^1 are indistinguishable and have fermion number -1 , the generating polynomial of bound states of such a brane to some number of branes wrapping \mathbb{P}^1 is just $(1 - Q)^m g_m(q)$. Thus (5.3) can be rewritten as

$$H(q, Q) = \frac{1}{1 - q} \sum_m g_m(q)(1 - Q)^m. \tag{5.4}$$

To relate this to Conjecture 5.1, it remains only to explain why

$$\frac{g_m(q)}{1 - q} = \sum_n q^n \chi(C_0^{[n];m}), \tag{5.5}$$

i.e., why the BPS M2-branes which can bind exactly m M2 branes on \mathbb{P}^1 may be computed by stable pairs $s : \mathcal{O}_C \rightarrow F$, where $\dim F/(x, y)F = m$. As mentioned above, the relation to stable pairs and the appearance of the $1/(1 - q)$ is standard: the M2 brane has momentum modes around the circle which become D2-D0 bound states, and the particular index being computed becomes the Euler number of the stable pairs space. One might worry about the appropriate boundary conditions for the sheaf and the section. But whatever boundary conditions are chosen we will surely want all zeroes of the section s to lie in the connected component D of $C \setminus L$ containing the origin. Assuming we choose L sufficiently near the origin that D is contractible, then the space of such pairs contracts to $C_0^{[m]}$.

The essential thing to explain is what binding to m branes on \mathbb{P}^1 has to do with the number of generators $\dim F/(x, y)F = m$. To count the number of ways an M2 brane on \mathbb{P}^1 may bind to a given M2 brane on D , we first pass to the IIA theory and compute instead the number of ways the D2 brane on \mathbb{P}^1 may bind to a D2 brane F . The heuristic given in the previous section is that the virtual number of points on F at the origin which are available for the branes on \mathbb{P}^1 is just $\dim F/(x, y)F$. To elaborate on this slightly, the ‘‘open strings are Ext’’ philosophy here specializes to the statement that the space of open strings from F to a brane $\mathcal{O}_{\mathbb{P}^1}$ is $\text{Ext}_Y^1(F, \mathcal{O}_{\mathbb{P}^1})$. This immediately localizes to the intersection of \mathbb{P}^1 and D ; since this is a point the local to global spectral sequence collapses and we are reduced to computing Ext of modules in the complete local ring. Let us give coordinate z to the \mathbb{P}^1 direction; then we are computing

$$\text{Ext}_{\mathbb{C}[[x, y, z]]}^1(F, \mathbb{C}[[z]]) = \text{Hom}_{\mathbb{C}[[x, y]]}(F, \mathbb{C}) = (F/(x, y)F)^\vee.$$

One can also compute open strings in the other direction, $\text{Ext}_Y^1(\mathcal{O}_{\mathbb{P}^1}, F)$; the result is that this canonically parameterizes the nontrivial syzygies of the completion of F as a

module over $\mathbb{C}[x, y]$. Because the curve is planar, this space has the same dimension as that parameterizing the generators.

We make a quick note about the sample computations in the previous section of the right-hand side of Conjecture 5.1. The main point is a certain invariant introduced there did not in fact require the existence of a torus action. Let R be the complete local ring of the singularity; then the Jacobian factor parameterizes R -modules M such that $\mathbb{C}[[t]] \supset M \supset R$. Note that given a stable pair (M, s) one has an abstract isomorphism $M \otimes_R \mathbb{C}((t)) = \mathbb{C}((t))$; requiring $s \mapsto 1$ fixes the isomorphism. In other words a stable pair with quotient supported at 0 is equivalent data to a rank one R -submodule of $\mathbb{C}((t))$. Let M be such a module, then $M\mathbb{C}[[t]] = t^{-k}\mathbb{C}[[t]]$ for some k , and $\mathbb{C}[[t]] \subset t^k M \subset R$. Thus there is a map from the space of stable pairs to the Jacobian factor. The fiber over some module M is just the set of elements in M , up to constant multiple. It is straightforward to see that the space of elements with leading term t^a is a vector space and hence has Euler characteristic one. On the other hand $\dim_{\mathbb{C}} M/t^a R = a + \dim_{\mathbb{C}} M/R$. Thus the contribution of M to the Euler numbers of pairs spaces is $q^{\dim_{\mathbb{C}} M/R} \mathcal{H}_M(q)$, where $\mathcal{H}_M(q)$ is the Hilbert function of M . Note that $\dim_{\mathbb{C}} M/R$ is the “ $l(M)$ ” of the previous section. Let us also write $m(M)$ for the number of generators, and $\mathcal{J}^{l,m}$ for the locus in the Jacobian factor of modules with m generators. Then the f_m of the previous section are

$$f_m(q) = \sum_n q^n \chi(C_0^{[n];m}) = \int_{\mathcal{J}^{l,m}} q^{\ell(M)} \mathcal{H}_M(q) d\chi(M).$$

The integral is with respect to Euler characteristic, and has the meaning that we sum possible $\mathcal{H}_M(q)$ weighted by the Euler characteristic of the locus of modules with this Hilbert series. One may as desired further stratify by l , and introduce

$$f_{l;m} = \int_{\mathcal{J}^{l,m}} \mathcal{H}_M(q) d\chi(M)$$

in order to write

$$\sum_{m,n} q^n (1 - Q)^m \chi(C_0^{[n];m}) = \sum_{l,m} q^l (1 - Q)^m f_{l;m}(q).$$

6. Large N Duality and Topological Amplitudes for Torus Knots

The main goal of this section is to generalize the large N duality results for the unknot reviewed in Sect. 2 to arbitrary (s, r) torus knots. The open topological **A**-model amplitudes for lagrangian cycles associated to torus knots will be explicitly computed on both sides of the transition employing an equivariant virtual localization approach analogous to [19]. Note that the mirror topological **B**-model has been studied in [5], reproducing the HOMFLY polynomials via a matrix model approach.

The first task, carried out in Sect. 6.1, is to write down an explicit analytic presentation the lagrangian cycles $L_\mu \subset X_\mu$, constructed in Sect. 3 and show that they are preserved by the circle action

$$(x, y, z, w) \mapsto (e^{is\varphi} x, e^{ir\varphi} y, e^{-is\varphi} z, e^{-ir\varphi} w). \tag{6.1}$$

Then it shown in Sect. 6.2 that on the deformation side the open string instanton corrections to Chern-Simons theory are encoded in a formula of the form (2.19). The next

Sect. 6.3 contains the virtual localization computation of Gromov-Witten invariants on the resolution Y with lagrangian boundary conditions on M_ϵ . In particular it is shown that M_ϵ is preserved by the circle induced by (6.1) and the disc D_ϵ obtained in Sect. 3.3 is the only circle invariant Riemann surface in Y with boundary in M_ϵ . The tangent-obstruction complex for circle invariant stable maps with lagrangian boundary conditions on M_ϵ is derived by linearizing the defining equations of M_ϵ near the one-cycle $\eta_\epsilon = \partial D_\epsilon$. The final details of the localization computation are given in Sect. 6.4. The main result is that the winding number one \mathbf{A} -model partition function of the lagrangian cycle M_ϵ is in agreement with the HOMFLY polynomial of the (s, r) torus knot up to an overall sign depending on orientations. The proof is essentially an open \mathbf{A} -model reflection of the Chern-Simons S -matrix formula [5, 40] relating the HOMFLY polynomial of a torus knot to the colored invariants of the unknot [5, 40].

6.1. *Lagrangian cycles for torus knots.* Lagrangian cycles for torus knots are obtained as a special case of the construction explained in Sect. 3.2 for general algebraic knots. Consider the family of the curves $Z_\mu \subset X_\mu$,

$$Z_\mu : \quad x^r - \alpha^r y^s = 0, \quad z^r - \alpha^r (-w)^s = 0, \tag{6.2}$$

where (r, s) are coprime integers with $r > s \geq 1$ or $r = s = 1$, and $\alpha \in \mathbb{R} \setminus \{0\}$ is a fixed nonzero real number. As explained in Sect. 3.2, the specialization of Z_μ at $\mu = 0$ has $r - s + 1$ connected components classified by the distinct roots of the equation $\eta^r - \alpha^{2r} (-\eta)^s = 0$. The component corresponding to $\eta = 0$ is a union of two irreducible components C^\pm given by

$$x^r - \alpha^r y^s = 0, \quad z = w = 0,$$

respectively

$$z^r - \alpha^r (-y)^s = 0, \quad x = y = 0.$$

The remaining $r - s$ are disjoint smooth components isomorphic to \mathbb{C}^\times . Let $\gamma^+ : S^1 \rightarrow X$ be a parametric presentation of the intersection $\phi_0(C^+ \setminus \{0\}) \cap P_a$, where $P_a \subset X$ is the sphere bundle $|\bar{v}| = a$. Its inverse image $\phi_0^{-1} \circ \gamma^+ : S^1 \rightarrow X_0$ has a parametric presentation of the form

$$(x, y, z, w) = (\alpha b_1^s e^{is\theta}, b_1^r e^{ir\theta}, 0, 0), \tag{6.3}$$

where b_1 must be a solution of the equation

$$b_1^{2r} + \alpha^2 b_1^{2s} = 4a. \tag{6.4}$$

Some elementary real analysis shows that this equation has a unique positive real solution for any fixed $\alpha \neq 0, a > 0$. Then the construction of Sect. 3.1 then yields a lagrangian cycle $L_{\gamma^+} \subset X$. It will be checked below that L_{γ^+} does not intersect the zero section, hence its inverse image $L_0 = \phi_0^{-1}(L_{\gamma^+})$ is a lagrangian cycle on X_0 supported away from the conifold singularity.

For $\mu > 0$ the connected components of Z_μ are in one-to-one correspondence with distinct roots of the equation $\eta^r - \alpha^{2r} (\mu - \eta)^s = 0$. Each such component is given by

$$(x, y, z, w) = (t^s, t^r, \eta t^{-s}, (\mu - \eta)t^{-r}).$$

In particular for sufficiently small $\mu > 0$ there exists a continuous family $\eta(\mu)$ of roots specializing to $\eta = 0$ at $\mu = 0$. Let $C_\mu \subset X_\mu$ be the corresponding components of

Z_μ . Each connected component of the intersection $C_\mu \cap \phi_\mu^{-1}(P_a)$ must be an orbit of the circle action (6.1) since both $C_\mu, \phi_\mu^{-1}(P_a) \subset X_\mu$ are invariant cycles. Taking into account the parametric presentation of C_μ , each intersection component must be of the form

$$(x, y, z, w) = (\alpha b_1^s e^{is\theta}, b_1^r e^{ir\theta}, \alpha b_2^s e^{-is\theta}, -b_2^r e^{-ir\theta}) \tag{6.5}$$

with θ an angular coordinate on S^1 and $b_1, b_2 \in \mathbb{R}_{>0}$. The parameters b_1, b_2 must satisfy the condition

$$\alpha^2(b_1^s + b_2^s)^2 + (b_1^r + b_2^r)^2 = 2(\mu + \sqrt{\mu^2 + 4a^2}), \tag{6.6}$$

which follows from the defining equation of $\phi_\mu^{-1}(P_a) \subset X_\mu$, and

$$\alpha^2(b_1 b_2)^s + (b_1 b_2)^r = \mu, \tag{6.7}$$

which follows from $xz - yw = \mu$. By continuity, for sufficiently small $\mu > 0$ the intersection of $\phi_\mu(C_\mu)$ with P_a consists of two connected one-cycles γ_μ^\pm conjugate under the antiholomorphic involution τ_μ defined in (2.8). Moreover γ_μ^+ specializes to the cycle γ^+ constructed above, while γ_μ^- specializes to its conjugate, which is the intersection of $\phi_0(C^- \setminus \{0\})$ with P_a . In fact this picture can be confirmed by detailed analytic computations which will be omitted in the interest of brevity.

Applying the construction of Sect. 3.2 to the one-cycles $\gamma_\mu^+ : S^1 \rightarrow X$ yields a family of lagrangian cycles $L_{\gamma_\mu^+} \subset X$. The lagrangian cycles $L_\mu \subset X_\mu$ are the inverse images, $L_\mu = \phi_\mu^{-1}(L_{\gamma_\mu^+})$, via the symplectomorphisms $\phi_\mu : X_\mu \rightarrow X$.

The next task is to check that the lagrangian cycles L_μ, L_0 are invariant under the circle action (6.1) and do not intersect the zero section. Since the arguments are very similar, it suffices to present the details in one case only, say L_μ . The explicit form of the circle action on X is

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &\mapsto R(s\varphi) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, & \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} &\mapsto R(r\varphi) \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}, \\ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &\mapsto R(s\varphi) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, & \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} &\mapsto R(r\varphi) \begin{bmatrix} v_3 \\ v_4 \end{bmatrix}, \end{aligned} \tag{6.8}$$

where

$$R(\varphi) = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix}.$$

According to Eqs. (3.1), the defining equation of $L_{\gamma_\mu^+}$ is

$$\vec{u} = \vec{f}(\theta), \quad \dot{\vec{f}}(\theta) \cdot (\vec{v} - \vec{g}(\theta)) = 0,$$

where the functions $\vec{f}(\theta), \vec{g}(\theta)$ are determined by Eq. (6.5). One then finds

$$\begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} = \alpha \frac{b_1^s + b_2^s}{c} R(s\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} f_3(\theta) \\ f_4(\theta) \end{bmatrix} = -\frac{b_1^r + b_2^r}{c} R(r\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{6.9}$$

and

$$\begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} = -\alpha \frac{b_1^s - b_2^s}{4} c R(s\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} g_3(\theta) \\ g_4(\theta) \end{bmatrix} = \frac{b_1^r - b_2^r}{4} c R(r\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{6.10}$$

with $c = \sqrt{2(\mu + \sqrt{\mu^2 + 4a^2})}$. Then it is straightforward to check that $L_{\gamma_\mu^+}$ is preserved by the torus action using the elementary identity $R(\varphi)R(\varphi') = R(\varphi + \varphi')$. In fact $L_{\gamma_\mu^+}$ admits a parameterization of the form

$$\vec{u} = \vec{f}(\theta), \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = R(s\theta) \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix} \quad \begin{bmatrix} v_3 \\ v_4 \end{bmatrix} = R(r\theta) \begin{bmatrix} v_{30} \\ v_{40} \end{bmatrix},$$

where $(v_{10}, v_{20}, v_{30}, v_{40})$ are real parameters satisfying

$$\begin{aligned} \alpha(b_1^s + b_2^s)v_{10} - (b_1^r + b_2^r)v_{30} &= 0, \\ s\alpha(b_1^s + b_2^s)v_{20} - r(b_1^r + b_2^r)v_{40} &= -\frac{c}{4}[\alpha s(b_1^{2s} - b_2^{2s}) + r(b_1^{2r} - b_2^{2r})]. \end{aligned} \tag{6.11}$$

The first equation in (6.11) follows from the defining equation $\vec{u} \cdot \vec{v} = 0$ of X , and the second from the equation $\vec{f}(\theta) \cdot (\vec{v} - \vec{g}(\theta)) = 0$. Note that Eqs. (6.11) define a real 2-plane in the fiber of T^*S^3 over the point $\vec{u}_0 = \vec{f}(0)$. The points in this plane are in one-to-one correspondence with orbits of the circle action on the lagrangian cycle.

Note also that the intersection of $L_{\gamma_\mu^+}$ with the zero section $\vec{v} = 0$ is determined by the equation

$$\vec{f}(\theta) \cdot \vec{g}(\theta) = 0$$

which yields

$$s\alpha^2(b_1^{2s} - b_2^{2s}) + r(b_1^{2r} - b_2^{2r}) = 0. \tag{6.12}$$

Since b_1, b_2 satisfy simultaneously Eqs. (6.6)–(6.7), Eq. (6.12) will have no solutions for generic values of $\mu, a > 0$. Therefore in the generic case, this intersection with the zero section is empty.

6.2. Open string A-model on the deformation. Now consider an open **A**-model with target space X_μ and lagrangian branes on the lagrangian cycles L_μ, S_μ , where L_μ is defined in Eqs. (6.9), (6.10) and $S_\mu \simeq S^3$ is the fixed point set of the antiholomorphic involution (2.8) on X_μ . Note that both cycles are preserved by the circle action (6.1). Moreover, L_μ intersects an irreducible component of the curve (6.2) along an orbit (6.5) of the S^1 -action. Then it follows that the holomorphic cylinder C_μ given by

$$(x, y, z, w) = (\alpha b_1^s t^s, b_1^r t^r, \alpha b_2^s t^{-s}, -b_2^r t^{-r}), \tag{6.13}$$

with

$$\sqrt{\frac{b_2}{b_1}} \leq |t| \leq 1,$$

has boundary components on S_μ, L_μ respectively. Obviously, \mathbf{C}_μ is preserved by the circle action (6.1). Set $\alpha = 1$ in the following.

Equation (6.13) describes a circle invariant genus zero stable map to X_μ with two boundary components mapped to S_μ, L_μ . According to [44] such instantons are expected to generate Wilson loop corrections to the Chern-Simons action. If \mathbf{C}_μ is the only torus invariant holomorphic cylinder in X_μ with boundary components on L_μ, S_μ , these corrections can be easily evaluated by a virtual localization computation analogous to [19]. Such a computation has been carried out for example in [11] in a similar context. As required in Sect. 2.1, the final formula for the instanton series is of the form

$$Z_{op-inst}(g_s, \mathbf{q}) = \sum_{n \geq 1} \frac{e^{-t_C}}{n} \text{Tr}(U^n) \text{Tr}(V^n), \tag{6.14}$$

where U, V are the holonomy of the gauge fields on S_μ, L_μ about the boundary components of \mathbf{C}_μ , and t_C is the symplectic area of \mathbf{C}_μ .

It remains to show that \mathbf{C}_μ is indeed the unique torus invariant cylinder in X_μ with boundary components in S_μ, L_μ respectively. The argument is analogous with [10], although more technically involved since the present torus action allows continuous families of invariant curves on X_μ . The main steps will be summarized below omitting many computational details.

First note that any invariant map $\mathbb{C}^\times \rightarrow X_\mu$ must be of the form

$$t \mapsto (x, y, z, w) = (\alpha_1 t^s, \alpha_2 t^r, \alpha_3 t^{-s}, -\alpha_4 t^{-r}), \tag{6.15}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constant parameters satisfying

$$\alpha_1 \alpha_3 + \alpha_2 \alpha_4 = \mu. \tag{6.16}$$

Let $C_{\vec{\alpha}} \subset X_\mu, \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ denote the image of this map. Since L_μ, S_μ are preserved by the circle action, any connected Component of the intersection of $C_{\vec{\alpha}}$ with the lagrangian cycles must be an orbit of the form $t = \rho e^{i\theta}$, with $\rho \in \mathbb{R}_{>0}$. Then Eqs. (6.9)–(6.11) for L_μ imply the following conditions:

$$\rho_L^s \alpha_1 + \rho_L^{-s} \bar{\alpha}_3 = \frac{A}{c} (b_1^s + b_2^s), \quad \rho_L^r \alpha_2 + \rho_L^{-r} \bar{\alpha}_4 = \frac{A}{c} (b_1^r + b_2^r), \tag{6.17}$$

where ρ_L is the radius of a component of the intersection $C_{\vec{\alpha}} \cap L_\mu$ and $A = \sqrt{(|\alpha_1 \rho_L^s + \bar{\alpha}_3 \rho_L^{-s}|^2 + |\alpha_2 \rho_L^r + \bar{\alpha}_4 \rho_L^{-r}|^2)}/2$. At the same time any connected component of the intersection $C_{\vec{\alpha}} \cap L_\mu$ must satisfy

$$\alpha_1 \rho_S^{2s} = \bar{\alpha}_3, \quad \alpha_2 \rho_S^{2r} = \bar{\alpha}_4, \tag{6.18}$$

where ρ_S denotes again the radius of the orbit. Equations (6.17), (6.18) imply that $\alpha_i, i = 1, \dots, 4$ must be non-zero real numbers, if $C_{\vec{\alpha}}$ intersects both S_μ, L_μ nontrivially. For example, if $\alpha_1 = 0$, it follows easily that all the remaining coefficients $\alpha_2, \alpha_3, \alpha_4$ must be also trivial, which contradicts relation (6.16). Then by a reparametrization of the domain, the map (6.15) can be set in the form

$$t \rightarrow (\alpha \beta_1^s t^s, \beta_2^r t^r, \alpha \beta_2^s t^{-s}, -\beta_1^r t^{-r}) \tag{6.19}$$

with $\beta_1, \beta_2 \in \mathbb{R}_{>0}, \alpha \in \mathbb{R} \setminus \{0\}$, the intersections with S_μ, L_μ being given by

$$|t| = \sqrt{\frac{\beta_2}{\beta_1}}, \quad |t| = 1$$

respectively. Using again Eqs. (6.9)-(6.11) for L_μ , one finds the following nonempty intersection conditions:

$$\frac{\alpha}{c'}(\beta_1^s + \beta_2^s) = \frac{1}{c}(b_1^s + b_2^s), \quad \frac{1}{c'}(\beta_1^r + \beta_2^r) = \frac{1}{c}(b_1^r + b_2^r), \quad (6.20)$$

$$-\frac{\alpha c'}{4}(\beta_1^s - \beta_2^s) = v_{20}, \quad \frac{c'}{4}(\beta_1^r - \beta_2^r) = v_{40}, \quad (6.21)$$

$$s(b_1^s + b_2^s)v_{20} - r(b_1^r + b_2^r)v_{40} = -\frac{c}{4}[s(b_1^{2s} - b_2^{2s}) + r(b_1^{2r} - b_2^{2r})], \quad (6.22)$$

$$v_{10} = v_{30} = 0, \quad v_{20}^2 + v_{40}^2 = a'^2,$$

where \vec{v}_0 is a vector in the plane (6.11) parametrizing a common circle orbit of L_μ and $C_{\vec{a}}$, and

$$a' = \sqrt{a^2 + |\vec{v}_0|^2}, \quad c' = \sqrt{2(\mu + \sqrt{\mu^2 + 4a'^2})}.$$

Recall that the coefficients b_1, b_2 are given functions of (μ, a) determined by Eqs. (6.6)–(6.7), as explained in Sect. (6.1).

Next note that it suffices to show that the orbit parametrized by the vector \vec{v}_0 coincides with the boundary of the built in cylinder C_μ , since then the two cylinders must coincide by holomorphy. This follows by an elementary but fairly tedious computation in real analysis. The strategy is to solve for (v_{20}, v_{40}) in Eqs. (6.22) and substitute the solutions in Eqs. (6.20)–(6.21). Then one solves for $(\alpha\beta_1^s, \alpha\beta_2^s)$ respectively (β_1^r, β_2^r) in the resulting equations imposing at the same time the positivity conditions $\beta_1, \beta_2 > 0$. Note that this will yield a priori independent expressions of the form

$$\beta_i^r = F_i(c', b_1, b_2), \quad \alpha\beta_i^s = G_i(c', b_1, b_2), \quad (6.23)$$

$i = 1, 2$, where $F_i(c', a, \mu), G_i(c', a, \mu), i = 1, 2$ are explicit functions of (c', a, μ) . Moreover the expressions (6.23) must satisfy the obvious compatibility condition

$$F_1(c', b_1, b_2)^s G_2(c', b_1, b_2)^r = F_2(c', b_1, b_2)^s G_1(c', b_1, b_2)^r. \quad (6.24)$$

A straightforward but fairly long computation shows that the matching condition (6.24) is equivalent to

$$F_+(\eta)^s G_+(\eta)^r = F_-(\eta)^s G_-(\eta)^r,$$

where $\eta = c'^2$,

$$F_\pm(\eta) = \left(1 + \frac{r^2 D^2}{s^2}\right) \frac{D\eta}{2c} \mp \frac{r}{s} BD \pm \left[\left(1 + \frac{r^2 D^2}{s^2}\right) (1 + D^2) \left(\frac{\eta^2}{4c^2} - \tilde{\mu}\eta\right) - B^2 \right]^{1/2},$$

$$G_\pm(\eta) = \left(1 + \frac{r^2 D^2}{s^2}\right) \frac{\eta}{2c} \pm B \pm \frac{rD}{s} \left[\left(1 + \frac{r^2 D^2}{s^2}\right) (1 + D^2) \left(\frac{\eta^2}{4c^2} - \tilde{\mu}\eta\right) - B^2 \right]^{1/2},$$

and

$$D = \frac{b_1^r + b_2^r}{b_1^s + b_2^s} \quad B = \frac{c}{2} \left(\frac{b_1^s - b_2^s}{b_1^s + b_2^s} + \frac{r}{s} D \frac{b_1^r - b_2^r}{b_1^s + b_2^s} \right).$$

One then has to analyze the monotonicity properties of the functions $F_{\pm}(\eta)$, $G_{\pm}(\eta)$ on the intervals where $\beta_1, \beta_2 > 0$. Suppressing the details, which are quite elementary, it follows that for sufficiently small $\mu > 0$, Eq. (6.24) admits only the solution $c' = c$, if $a > 0$ is in addition bounded above by a constant $a_0(r, s)$ depending only on r, s . Returning to the expressions (6.23), this implies in turn that $\beta_i = b_i$ for $i = 1, 2$. Therefore the two orbits indeed coincide.

6.3. Open Gromov-Witten invariants on the resolution. The goal of this section is to compute the Gromov-Witten invariants for stable maps $f : \Sigma \rightarrow Y$ with lagrangian boundary conditions on the cycle M_{ϵ} constructed in Sect. (3.3) for a polynomial $f(x, y)$ of the form

$$f(x, y) = x^r - y^s,$$

with $r > s \geq 1$ coprime. These invariants will be computed assuming the existence of a virtual fundamental cycle and a virtual localization result for the moduli space of such maps, by analogy with [19].

Recall that the main steps in the construction of $M_{\epsilon} \subset Y$ are as follows. Let $C^+ \subset X_0$ be the plane curve determined by

$$f(x, y) = 0, \quad z = w = 0$$

in the singular conifold X_0 . Let $\gamma^+ : S^1 \rightarrow X = T^*S^3$ be the one-cycle obtained by intersecting the sphere bundle P_a , $a > 0$ with the image $\phi_0(C^+)$, where $\phi_0 : X_0 \rightarrow X$ is the symplectomorphism constructed below Eq. (2.6). Let $\gamma_{\epsilon}^+ = \phi_0 \circ \varrho_{\epsilon} \circ \phi_0^{-1} \circ \gamma^+$ be the dilation of γ^+ via the radial map $\varrho_{\epsilon} : X_0 \setminus \{0\} \rightarrow X_0(\epsilon)$,

$$\varrho_{\epsilon}(x, y, z, w) = \left(x, \frac{\sqrt{|z|^2 + |y|^2 + \epsilon^2}}{\sqrt{|z|^2 + |y|^2}} y, \frac{\sqrt{|z|^2 + |y|^2 + \epsilon^2}}{\sqrt{|z|^2 + |y|^2}} z, w \right). \quad (6.25)$$

Applying the construction in Sect. 3.1 to γ_{ϵ}^+ yields a lagrangian cycle $L_{\epsilon} \subset X$. As shown in Eq. (3.5), M_{ϵ} is the inverse image $\sigma^{-1} \circ \varrho_{\epsilon}^{-1} \circ \phi_0^{-1}(L_{\epsilon})$.

The cycle $L_{\epsilon} \subset X$ admits an explicit parametric presentation analogous to the presentation of the cycles $L_{\mu} \subset X_{\mu}$ in Sect. 6.1. Note that the one-cycle $\phi_0^{-1}(\gamma_{\epsilon}^+) = \phi_0^{-1}(\varrho_{\epsilon} \circ \gamma^+)$ is parametrically given by

$$(x, y, z, w) = (b_1^s e^{is\theta}, \sqrt{b_1^{2r} + \epsilon^2} e^{ir\theta}, 0, 0), \quad (6.26)$$

where $b_1 = b_1^+(a)$ is the unique positive real solution of the equation

$$b_1^{2s} + b_1^{2r} = 4a.$$

Then $L_{\epsilon} \subset X$ is given by equations of the form

$$\vec{u} = \vec{f}(\theta), \quad \vec{f}(\theta) \cdot (\vec{v} - \vec{g}(\theta)) = 0, \quad (6.27)$$

where

$$\begin{aligned} \begin{bmatrix} f_1(\theta) \\ f_2(\theta) \end{bmatrix} &= \frac{b_1^q}{c} R(s\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} f_3(\theta) \\ f_4(\theta) \end{bmatrix} &= -\frac{\sqrt{b_1^{2r} + \epsilon^2}}{c} R(r\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix} &= \frac{b_1^s}{4} c R(s\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} g_3(\theta) \\ g_4(\theta) \end{bmatrix} &= -\frac{\sqrt{b_1^{2r} + \epsilon^2}}{4} c R(p\theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

and $c = \sqrt{4a + \epsilon^2}$. Using the above formulas, it is straightforward to show that L_ϵ is invariant under the circle action (6.8).

Now recall that the defining equations of Y in $\mathbb{C}^4 \times \mathbb{P}^1$ are

$$x\lambda = w\rho, \quad y\lambda = z\rho,$$

where $[\lambda, \rho]$ are homogeneous coordinates on \mathbb{P}^1 . There are two affine coordinate patches on Y, U given by $\rho \neq 0$ with coordinates

$$x, y, \zeta = \frac{\lambda}{\rho},$$

and $U' \subset Y$ given by $\lambda \neq 0$, with coordinates

$$z, w, \zeta' = \frac{\rho}{\lambda}.$$

Obviously, the transition functions are

$$w = x\zeta, \quad z = y\zeta, \quad \zeta' = \frac{1}{\zeta}.$$

The strict transform $C \subset Y$ of C^+ is contained in the first patch and has defining equations

$$f(x, y) = 0, \quad \zeta = 0.$$

Moreover, note that Eq. (6.1) also defines a circle action on singular threefold X_0 which preserves C^+ . This lifts to a circle action $S^1 \times Y \rightarrow Y$,

$$(x, y, z, w) \times [\lambda, \rho] \mapsto (e^{is\varphi} x, e^{ir\varphi} y, e^{-is\alpha} z, e^{-ir\alpha} w) \times [e^{-i(r+s)\alpha} \lambda, \rho], \tag{6.28}$$

which preserves C . Since the blow-up map $\sigma : Y \rightarrow X_0$ and the dilation map (6.25) are equivariant, it follows that the action (6.28) preserves M_ϵ . Therefore it also preserves the singular holomorphic disk D_ϵ with boundary on M_ϵ obtained by intersecting M_ϵ and C . Note that D_ϵ is given in parametric form by

$$(x, y, \zeta) = (t^s, t^r, 0), \quad |t| \leq b_1. \tag{6.29}$$

Next one has to show that (6.29) is the unique torus invariant disk instanton $f : \Delta \rightarrow Y$ with lagrangian boundary conditions on M_ϵ . Using Eqs. (6.27) it is straightforward to check that the only coordinate hyperplane in Y intersecting M_ϵ nontrivially is $\lambda = 0$, in which case the intersection is the one-cycle $\eta_\epsilon = D_\epsilon$. All other coordinate hyperplanes, $x = 0, y = 0, \rho = 0$ do not intersect M_ϵ . In particular this implies the image $f(\Delta)$ of such a map cannot be contained in the surface $\rho = 0$. Then torus invariance implies that $f(\Delta)$ is either disjoint from the surface $\rho = 0$, or intersects it transversely at the torus

fixed point $z = w = 0, \rho = 0$. In the first case the fixed point $t = 0$ in the domain must be mapped to the fixed point $x = y = 0, \lambda = 0$ in the target. Moreover, in both cases, the restriction of the map f to the punctured disk $\Delta \setminus \{0\}$ must be of the form

$$(x, y, \zeta) = (\alpha_1 t^{\pm s}, \alpha_2 t^{\pm r}, \alpha_3 t^{\mp(r+s)}) \tag{6.30}$$

for some complex parameters $(\alpha_1, \alpha_2, \alpha_3)$.

If the first case holds, the map f must be of the form

$$(x, y, \zeta) = (\alpha_1 t^s, \alpha_2 t^r, 0)$$

or

$$(x, y, \zeta) = (0, 0, \alpha_3 t^{(r+s)})$$

since $f(0) = (0, 0, 0)$. The second subcase is ruled out because M_ϵ does not intersect the zero section $x = y = 0$. In the first subcase the image $f(\Delta)$ is contained in the surface $\lambda = 0$ which intersects M_ϵ along the boundary of D_ϵ . Therefore $f(\Delta)$ and D_ϵ must have common boundary, which implies they must coincide.

The second case can hold only if $\alpha_3 \neq 0$, which implies that the image $f(\Delta)$ cannot be contained in the surface $\lambda = 0$. Then torus invariance implies that $f(\Delta)$ must be disjoint from the surface $\lambda = 0$ since any common point would have to be a fixed point of the torus action. At the same time the only fixed point in the domain is mapped to the fixed point $z = w = 0, \rho = 0$. Therefore $f(\Delta)$ is contained in the coordinate chart U' . In terms of the coordinates (z, w, ζ') , Eq. (6.30) reads

$$(z, w, \zeta') = (\alpha_1 \alpha_3 t^{\mp r}, \alpha_2 \alpha_3 t^{\mp s}, \alpha_3^{-1} t^{\pm(r+s)}). \tag{6.31}$$

Since $\alpha_3 \neq 0$, the condition $f(0) = (0, 0, 0)$ implies that $\alpha_1 = \alpha_2 = 0$. This is again ruled out since M_ϵ does not intersect the zero section.

In conclusion, D_ϵ is indeed the unique torus invariant holomorphic disc on Y with boundary in M_ϵ . Then the computation of Gromov-Witten invariants reduces to the computation of multicover contributions of D_ϵ via a virtual localization theorem. One then requires an explicit form of lagrangian boundary conditions for an S^1 -invariant stable map $f : \Sigma \rightarrow Y$ which factors through the disc $D_\epsilon \subset Y$. Let $Ann(M_\epsilon) \subset T^*Y|_{M_\epsilon}$ be the subbundle of the cotangent bundle of Y which annihilates the tangent bundle $TM_\epsilon \subset TY|_{M_\epsilon}$. The boundary conditions are determined by a framing of $Ann(M_\epsilon)|_{\eta_\epsilon}$, that is three sections of $T^*Y|_{\eta_\epsilon}$ which form a basis of $Ann(M_\epsilon)$ at any point on $\eta_\epsilon = \partial D_\epsilon$. This computation reduces basically to the linearization of the defining equations of M_ϵ in Y , which is standard differential geometry. Omitting the intermediate steps, the resulting generators are, in local coordinates (x, y, ζ) ,

$$\begin{aligned} \alpha &= b_1^s [2AC + (s-r)(b_1^{2r} + \epsilon^2)B + (s-r)^2 b_1^{2s} (b_1^{2r} + \epsilon^2)] (e^{-is\theta} dx + e^{is\theta} d\bar{x}) \\ &\quad + b_1^{2r} [BC - (s-r)b_1^{2s} A + (s-r)^2 b_1^{2s} (b_1^{2r} + \epsilon^2)] (e^{-ir\theta} dy + e^{ir\theta} d\bar{y}), \end{aligned} \tag{6.32}$$

$$\begin{aligned} \beta &= e^{-i(r+s)\theta} d\bar{\zeta} + \frac{b_1^s \sqrt{b_1^{2r} + \epsilon^2}}{C} \left[\frac{B}{2c^2 b_1^s} e^{-is\theta} dx + \frac{(s-r)b_1^s}{2c^2} e^{is\theta} d\bar{x} \right. \\ &\quad \left. - \frac{1}{4c^2 b_1^r} \left(\frac{2b_1^{2r} + \epsilon^2}{b_1^{2r} + \epsilon^2} A + (s-r)\epsilon^2 \right) e^{-ir\theta} dy \right. \\ &\quad \left. + \frac{1}{4c^2 b_1^r} \left(\frac{\epsilon^2 A}{b_1^{2r} + \epsilon^2} + (s-r)(2b_1^{2r} + \epsilon^2) \right) e^{ir\theta} d\bar{y} \right], \end{aligned} \tag{6.33}$$

where

$$A = 2sb_1^{2s} + (r + s)(b_1^{2r} + \epsilon^2), \quad B = (r + s)b_1^{2s} + 2r(b_1^{2r} + \epsilon^2),$$

$$C = sb_1^{2s} + r(b_1^{2r} + \epsilon^2).$$

In particular, α is real and β is complex.

6.3.1. *Deformation theory.* Let $\Delta \subset \mathbb{C}$ be the disk $|t| \leq b_1$. Let $f : \Delta \rightarrow Y$ be the map

$$t \mapsto (x, y, \zeta) = (t^s, t^r, 0). \tag{6.34}$$

Obviously f factors through the disk $\mathbf{C} \subset Y$ mapping the boundary of the disk, $|t| = b_1$ to the boundary $\eta_\epsilon = \partial\mathbf{C} \subset M_\epsilon$. Let f_∂ denote the restriction of f to the boundary. Let $\mathcal{T}_{(\Delta, f)}$ denote the sheaf of germs of holomorphic sections of the bundle f^*T_Y satisfying the boundary conditions

$$f_\partial^*(\alpha)(s|_{\partial\Delta}) = 0, \quad f_\partial^*(\beta)(s|_{\partial\Delta}) = 0, \tag{6.35}$$

where α, β are the generators of the annihilator sub-bundle $Ann(M_\epsilon)|_{\eta_\epsilon}$ given in Eqs. (6.32)–(6.33). Let \mathcal{T}_Δ be the sheaf of germs of holomorphic sections of the tangent bundle T_Δ satisfying the boundary condition

$$\gamma|_{\partial\Delta}(s|_{\partial\Delta}) = 0, \tag{6.36}$$

where $\gamma = td\bar{t} + \bar{t}dt$.

The deformation complex of the stable map (Δ, f) with lagrangian boundary conditions along M_ϵ is

$$0 \rightarrow H^0(\Delta, \mathcal{T}_\Delta) \rightarrow H^0(\Delta, \mathcal{T}_{(\Delta, f)}) \rightarrow Def(\Delta, f)$$

$$\rightarrow H^1(\Delta, \mathcal{T}_\Delta) \rightarrow H^1(\Delta, \mathcal{T}_{(\Delta, f)}) \rightarrow Obs(\Delta, f) \rightarrow 0. \tag{6.37}$$

The computation of the Čech cohomology groups $H^k(\Delta, \mathcal{T}_{(\Delta, f)})$, $k = 0, 1$ is straightforward but fairly tedious, hence details will be omitted. For sufficiently generic $\epsilon > 0$, $H^0(\Delta, \mathcal{T}_{(\Delta, f)})$ is isomorphic to the space of sections of the form

$$s = \sum_{n=0}^{2s} a_n t^n \partial_x + \sum_{n=r-s}^{r+s} b_n t^n \partial_y, \tag{6.38}$$

where

$$b_{n+r} + \bar{b}_{r-n} = 0 \tag{6.39}$$

for all $r - s \leq n \leq r + s$, and the coefficients a_n are determined in terms of b_n by certain linear relations with ϵ -dependent coefficients. The group $H^1(\Delta, \mathcal{T}_{(\Delta, f)})$ is identified with the space of sections of the form

$$s = \sum_{n=1-s}^{-1} \gamma_n t^n \partial_\zeta \tag{6.40}$$

on the punctured disc $\Delta \setminus \{0\}$. In particular, if $s = 1$, this space is trivial.

For $H^i(\Delta, \mathcal{T}_\Delta)$, $i = 0, 1$ one finds that $H^0(\Delta, \mathcal{T}_\Delta)$ is generated by sections of the form

$$a_{-1}\partial_t + a_0t\partial_t + a_1t^2\partial_t$$

with

$$a_{-1} + \bar{a}_1 = 0, \quad a_0 + \bar{a}_0 = 0,$$

while $H^1(\Delta, \mathcal{T}_\Delta)$ is trivial.

6.3.2. *Virtual localization.* Now let $\overline{\mathcal{M}}_{g,1}(Y, M_\epsilon; d, 1)$ be the moduli space of genus $g \geq 0$ stable maps with $h = 1$ boundary components mapped to M_ϵ , in the relative homology class $d[C_0] + [\mathbf{C}] \in H_2(Y, M_\epsilon)$, $d \in \mathbb{Z}_{\geq 0}$. The circle action

$$(x, y, \zeta) \mapsto (e^{-is\varphi}x, e^{-ir\varphi}y, e^{i(r+s)\varphi}\zeta)$$

on Y preserves M_ϵ , hence it induces an action on the moduli space of stable maps. Let also $\overline{M}_{g,1}(Y, d)$ denote the moduli space of genus g stable maps to Y with one marked point in the homology class $d[C_0] \in H_2(Y)$. This moduli space is equipped with a natural evaluation map at the marked point, $ev : \overline{M}_{g,1}(Y, d) \rightarrow Y$.

A map $f : \Sigma \rightarrow Y$ determines a circle fixed point in the moduli space $\overline{\mathcal{M}}_{g,1}(Y, M_\epsilon; d, 1)$ if and only if there exists a circle action on the domain Σ such that f is equivariant. This implies that the domain must be a union $\Sigma = \Sigma_0 \cup_\nu \Delta$, where Σ_0 is a closed nodal Riemann surface without boundary which intersects the disk Δ at a single point ν , which is a simple node of Σ . Moreover the image of the restriction $f|_\Delta$ must coincide with the holomorphic disc D , which has been shown below (6.29) to be the unique torus invariant disc in Y with boundary in M_ϵ . In more detail, the following conditions must hold:

- Δ admits a parameterization $\Delta = \{|t| \leq r_1\}$ such that ν is identified with the point $t = 0$ and

$$f|_\Delta(t) = (t^s, t^r, 0).$$

The circle action on Δ is given by $t \mapsto e^{-i\varphi}t$.

- Note that there is an algebraic torus action $\mathbb{C}^\times \times Y \rightarrow Y$ which agrees with the above real torus action by restriction to the unit circle. Then the data (Σ_0, f_0, ν) , with $f_0 = f|_{\Sigma_0}$ must be a \mathbb{C}^\times -invariant stable map to Y such that $f_0(\nu) = p$, where $p \in Y$ is the point $x = y = \zeta = 0$.

These conditions imply that the fixed locus $\overline{\mathcal{M}}_{g,1}(Y, M_\epsilon; d, 1)^{S^1}$ is isomorphic to the fixed subspace

$$ev_\nu^{-1}(p)^{\mathbb{C}^\times} \subset \overline{M}_{g,1}(Y, d)^{\mathbb{C}^\times}.$$

The deformation complex of a fixed stable map (Σ, f) is

$$\begin{aligned} 0 \rightarrow \text{Aut}(\Sigma) \rightarrow \text{Def}(f) \rightarrow \text{Def}(\Sigma, f) \\ \rightarrow \text{Def}(\Sigma) \rightarrow \text{Obs}(f) \rightarrow \text{Obs}(\Sigma, f) \rightarrow 0, \end{aligned} \tag{6.41}$$

where the notation is self-explanatory. All terms carry natural circle actions since (Σ, f) is a circle invariant map. The fixed part of the deformation complex determines the virtual fundamental cycle on the fixed locus, while the moving part determines the virtual

normal bundle to the fixed locus. Each term will be analyzed below assuming that Σ_0 is nonempty. In the special case $\Sigma_0 = \emptyset$ the deformation complex (6.41) reduces to (6.37) analyzed in the previous subsection.

Given the structure of fixed maps explained above, there is an exact sequence

$$\begin{aligned} 0 \rightarrow Def(f) &\rightarrow H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)}) \oplus Def(f_0) \rightarrow T_p Y \\ &\rightarrow Obs(f) \rightarrow H^1(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)}) \oplus Obs(f_0) \rightarrow 0. \end{aligned} \tag{6.42}$$

This yields the following relations in the representation ring of the circle:

$$\begin{aligned} Obs(f)^f - Def(f)^f &= H^1(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^f - H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^f \\ &\quad + Obs(f_0)^f - Def(f_0)^f, \\ Obs(f)^m - Def(f)^m &= H^1(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^m - H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^m \\ &\quad + Obs(f_0)^m - Def(f_0)^m + T_p Y. \end{aligned} \tag{6.43}$$

Moreover standard arguments imply

$$\begin{aligned} Aut(\Sigma)^{f,m} &= Aut(\Sigma_0, \nu)^{f,m} + Aut(\Delta, 0)^{f,m}, \\ Def(\Sigma)^f &= Def(\Sigma_0, \nu)^f, \\ Def(\Sigma)^m &= Def(\Sigma_0, \nu)^m + T_\nu \Sigma_0 \otimes T_0 \Delta, \end{aligned} \tag{6.44}$$

while the cohomology groups $H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})$ have been determined in Eqs. (6.38), (6.40). There is however a discrete ambiguity in reading off their equivariant content, reflecting a choice of orientation on the moduli space of stable maps with lagrangian boundary conditions [19]. As explained in [19], the difference between these choices is encoded in an overall sign which cannot be fixed in the absence of a rigorous construction of the moduli space equipped with a virtual cycle. Therefore the present computation will be a test of large N duality up to sign. Given Eqs. (6.39), the deformation space (6.38) is isomorphic to a vector space of the form

$$\mathbb{R}\langle \partial_y \rangle \oplus \bigoplus_{n=r-s}^{r-1} \mathbb{C}\langle t^n \partial_y \rangle.$$

At the same time, the obstruction space (6.40) is naturally identified with the complex vector space

$$\bigoplus_{n=1-s}^{-1} \mathbb{C}\langle t^n \partial_\zeta \rangle.$$

This yields the following relations in the representation ring of S^1 :

$$\begin{aligned} H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^m &= \sum_{n=1}^s R^n, & H^1(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^m &= \sum_{n=1}^{s-1} R^{-(r+n)}, \\ H^0(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^f &= \mathbb{R}, & H^1(\Delta, \mathcal{T}_{(\Delta, f|_\Delta)})^f &= 0, \end{aligned} \tag{6.45}$$

where R is the canonical representation of S^1 on \mathbb{C} , and \mathbb{R} denotes the trivial real representation. Note also that $Aut(\Delta)$ is isomorphic to the space of sections of T_Δ of the form $a\partial_t + bt\partial_{\bar{t}}$ with $a \in \mathbb{C}, b \in i\mathbb{R}$. Therefore

$$Aut(\Delta)^f = \mathbb{R}, \quad Aut(\Delta)^m = R.$$

The subgroup of automorphisms preserving the origin, $Aut(\Delta, 0)$ is generated by $t\partial_t$ over \mathbb{R} , therefore it has only a fixed part $Aut(\Delta, 0)^f = \mathbb{R}$.

Collecting all the above results one obtains

$$\begin{aligned} Obs(\Sigma, f)^f - Def(\Sigma, f)^f &= Obs(f_0)^f - Def(f_0)^f + Aut(\Sigma_0, v)^f - Def(\Sigma_0, v)^f \\ &= Obs(\Sigma_0, f_0)^f - Def(\Sigma_0, f_0)^f, \\ Obs(\Sigma, f)^m - Def(\Sigma, f)^m &= Obs(f_0)^m - Def(f_0)^m + Aut(\Sigma_0, v)^m \\ &\quad - Def(\Sigma_0, v)^m + \sum_{n=1}^{s-1} R^{-(r+n)} - \sum_{n=1}^s R^n + T_p Y \\ &\quad - T_v \Sigma_0 \otimes T_0 \Delta. \end{aligned} \tag{6.46}$$

This implies that the virtual fundamental cycle of the fixed locus is the restriction of the natural virtual cycle of the fixed locus $[\overline{M}_{g,1}(Y, d)^{\mathbb{C}^\times}]^{vir}$ with the subspace $ev^{-1}(p)^{\mathbb{C}^\times}$. The equivariant K-theory class of the virtual normal bundle is given by

$$\begin{aligned} N^{vir} &= N_{\overline{M}_{g,1}(Y,d)^{\mathbb{C}^\times} / \overline{M}_{g,1}(Y,d)}^{vir} - T_p Y + R\mathbb{L}^{-1} \\ &\quad + \sum_{n=1}^{s-1} R^{-(r+n)} - \sum_{n=1}^s R^n, \end{aligned}$$

where \mathbb{L} is the tautological line bundle on $\overline{M}_{g,1}(Y, d)$ associated to the marked point. Then the residual formula for open Gromov-Witten invariants is

$$\begin{aligned} GW_{g,1}(d, 1) &= (-1)^{s-1} \frac{\prod_{n=1}^{s-1} (r+n)}{s!} \frac{e_{\mathbb{C}^\times}(T_p Y)}{\alpha} \\ &\quad \int_{[\overline{M}_{g,1}(Y,d)_p^{\mathbb{C}^\times}]^{vir}} \frac{1}{e_{\mathbb{C}^\times}(N_{\overline{M}_{g,1}(Y,d)^{\mathbb{C}^\times} / \overline{M}_{g,1}(Y,d)}^{vir})^{-1} (\alpha - \psi)}, \end{aligned} \tag{6.47}$$

where $\overline{M}_{g,1}(Y, d)_p^{\mathbb{C}^\times}$ denotes the union of connected components of the fixed locus contained in $ev^{-1}(p)$. Standard formal manipulations show that this formula is equivalent to

$$GW_{g,1}(d, 1) = (-1)^{s-1} \frac{\prod_{n=1}^{s-1} (r+n)}{s!} \int_{[\overline{M}_{g,1}(Y,d)]_{\mathbb{C}^\times}^{vir}} \frac{ev^* \phi_{\mathbb{C}^\times}(p)}{\alpha(\alpha - \psi)}, \tag{6.48}$$

where $[\overline{M}_{g,1}(Y, d)]_{\mathbb{C}^\times}^{vir}$ denotes the equivariant virtual cycle of the moduli space, $\phi_{\mathbb{C}^\times}(p) \in H_{\mathbb{C}^\times}^*(Y)$ is the equivariant Thom class of $p \in Y$, and $\alpha = \text{ch}(R)$.

6.4. *Comparison with HOMFLY polynomial.* The goal of this section is to compare the generating function for the open Gromov-Witten invariants $GW_{g,1}(Y, M_\epsilon; d)$ with the HOMFLY polynomial of (s, r) -torus knots. It will be shown that large N duality for (s, r) torus knots follows from known results on large N duality for the unknot. The manipulations of enumerative invariants justifying this statement parallel similar manipulations in Chern-Simons theory relating invariants of (s, r) torus knots to colored invariants of the unknot [5, 40].

The main observation is that the Gromov-Witten invariants given in (6.48) for some coprime (r, s) can be expressed in terms of analogous invariants determined by the curve

$$y = z = w = 0 \tag{6.49}$$

in X_0 and the associated lagrangian cycles. In order to emphasize the dependence on (r, s) , the lagrangian cycles used in the above construction will be denoted by $M_\epsilon^{(s,r)}$, and the corresponding invariants by $GW_{g,1}^{(s,r)}(d, 1)$.

Consider the construction of lagrangian cycles carried out in Sects. 3.1–3.3 for a curve C of the form (6.49). By analogy with Sect. 6.1 one can easily check that the lagrangian cycle $M_\epsilon^{(1,0)}$ obtained in this case is preserved by any circle action on Y of the form

$$(x, y, \zeta) \mapsto (e^{-is\varphi}x, e^{-ir\varphi}y, e^{i(r+s)\varphi}\zeta) \tag{6.50}$$

with $r, s \in \mathbb{Z}$. Moreover, $M_\epsilon^{(1,0)}$ intersects the strict transform of C along an orbit of the torus action, obtaining a unique holomorphic circle invariant disk D_0 on Y with boundary on $M_\epsilon^{(1,0)}$. In this case D_0 is smooth and Gromov-Witten invariants with boundary conditions on $M_\epsilon^{(1,0)}$ can be constructed using virtual localization in close analogy with [19]. Let $\overline{\mathcal{M}}_{g,1}(Y, M_\epsilon; d, k)$ be the moduli space of genus $g \geq 0$ stable maps with $h = 1$ boundary components mapped to $M_\epsilon^{(1,0)}$, in the relative homology class $d[C_0] + k[D_0] \in H_2(Y, M_\epsilon)$, $d \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{>0}$. In contrast with the previous section, the winding number k will be allowed to take arbitrary values in the present context. Then there is a residual formula of the form

$$GW_{g,1}^{(1,0)}(d, k; r, s) = (-1)^{k-1} \frac{\prod_{n=1}^{k-1} (rk + ns)}{(k-1)!} \int_{[\overline{\mathcal{M}}_{g,1}(Y,d)]_{\mathbb{C}^\times}^{vir}} \frac{ev^* \phi_{\mathbb{C}^\times}(p)}{k\alpha(k\alpha - s\psi)}. \tag{6.51}$$

Note that the resulting invariants depend on the choice of torus action on Y preserving $M_\epsilon^{(1,0)}$, which in this case is taken of the form (6.50). As a result the $(1, 0)$ -invariants are rational functions of (r, s) .

Setting $k = s$ in Eq. (6.51), it follows that

$$GW_{g,1}^{(s,r)}(d, 1) = sGW_{g,1}^{(1,0)}(d, s; r, s). \tag{6.52}$$

Now define the generating functions with fixed winding numbers 1, respectively s ,

$$F_1^{(s,r)}(g_s, Q, V) = \sum_{g \geq 0} \sum_{d \geq 0} g_s^{2g-1} Q^d GW_{g,1}^{(s,r)}(d, 1) \text{Tr}(V),$$

$$F_s^{(1,0)}(g_s, Q, V) = \sum_{g \geq 0} \sum_{d \geq 0} g_s^{2g-1} Q^d GW_{g,1}^{(1,0)}(d, s; r, s) \text{Tr}(V^s),$$

where the open string Gromov-Witten are defined by residual formulas (6.48), (6.51) with respect to a circle action of the form (6.50).

Large N duality for the unknot yields the following identity [30, Eq. (5.6)]:

$$F_q^{(1,0)}(g_s, Q, V) = \frac{(-1)^{s-1}}{s} \sum_R \chi_R(C_{(s)}) e^{i(r/s)\kappa_R g_s/2} W_R^{(1,0)}(q, Q) \text{Tr}(V^s), \quad (6.53)$$

the terms in the right-hand side being explained below.

- The sum in the right-hand side of (6.53) is over all Young diagrams R and $\chi_R(C_{(s)})$ denotes the character of the conjugacy class determined by the vector $\vec{k} = (k_j)_{j \geq 1}$, with $k_j = 1$ if $j = s$ and $k_j = 0$ otherwise in the representation determined by R . See [30, Sect. 4.1] for more details.
- $W_R^{(1,0)}(q, Q)$ is the HOMFLY polynomial colored by the representation R of $U(N)$, expressed as a function of the large N Chern-Simons theory on S^3 ,

$$g_s = \left(\frac{2\pi}{k + N} \right), \quad \lambda = \left(\frac{2\pi N}{k + N} \right), \quad q = e^{i g_s} \quad Q = e^{i \lambda}.$$

Up to a normalization factor, $W_R^{(1,0)}(q, Q)$ is given by the quantum dimension of R ,

$$W_R^{(1,0)}(q, Q) = Q^{-|R|/2} \text{dim}_q(R),$$

where $|R|$ is the total number of boxes in the Young diagram R .

- For any Young diagram R , the number κ_R is defined by

$$\kappa_R = |R| + \sum_{i=1}^{l_R} (l_i^2 - 2il_i),$$

where l_R is the number of rows of R and l_i is the length of the i^{th} row, $i = 1, \dots, l_R$.

As explained in [30, Sect. 3.2] the factor $e^{im\kappa_R g_s/2}$ encodes the framing dependence of colored HOMFLY polynomials, m being the framing of the knot with respect to the canonical framing. The expression $e^{i(r/s)\kappa_R g_s/2} W_R^{(1,0)}(q, Q)$ in the right-hand side of Eq. (6.53) must therefore be interpreted as a colored HOMFLY polynomial with fractional framing. The relation between quantum knot invariants with fractional framing and residual open string Gromov-Witten invariants has been observed in a similar context in [8, 9].

Formula (6.53) was initially tested in specific examples for the free term in the λ -expansion of $W_R^{(1,0)}(q, Q)$. The higher order terms were implicitly tested in [9] in the process of finding an enumerative interpretation of the topological vertex [1]. In fact formula (6.53) follows rigorously using more recent results in the mathematical literature [25, 26, 34] on one and two-partition Hodge integrals. Details will be omitted because this is a standard virtual localization computation.

The important fact for the present goal is to note that Eqs. (6.52), (6.53) yield an identity of the form

$$F_1^{(s,r)}(g_s, q, Q) = (-1)^{s-1} \sum_R \chi_R(C_{(s)}) e^{i(r/s)\kappa_R g_s/2} W_R^{(1,0)}(q, Q). \quad (6.54)$$

Now recall that according to [40, Sect. 3.3], [5, Eq. (2.43)], the HOMFLY polynomials of (s, r) torus knots is expressed in terms of colored HOMFLY polynomials of the unknot as follows:

$$W_{\square}^{(s,r)}(q, Q) = \sum_R \chi_R(C_{(s)}) e^{2\pi i(r/s)h_R} \dim_q(R). \tag{6.55}$$

Next note that

$$h_R = \frac{N|R|}{2(k+N)} + \frac{\kappa_R}{2(k+N)},$$

which implies

$$e^{2\pi i(r/s)h_R} = e^{i(r/s)\lambda|R|/2} e^{i(r/s)\kappa_R g_s}.$$

Since only diagrams R with q boxes contribute to the right-hand side of (6.54), (6.55), it follows that

$$F_1^{(s,r)}(g_s, q, Q) = q^{-r/2} (-1)^{s-1} W_{\square}^{(s,r)}(q, Q) \text{Tr}(V).$$

This is the expected large N duality prediction for torus knots. The factor $(-1)^{s-1}$ reflects a specific choice of orientation of the moduli space of stable maps with lagrangian boundary conditions, as explained above.

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