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Lozenge Tilings and Hurwitz Numbers

Jonathan Novak¹

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Abstract We give a new proof of the fact that, near a turning point of the frozen boundary, the vertical tiles in a uniformly random lozenge tiling of a large sawtooth domain are distributed like the eigenvalues of a GUE random matrix. Our argument uses none of the standard tools of integrable probability. In their place, it uses a combinatorial interpretation of the Harish-Chandra/Itzykson-Zuber integral as a generating function for desymmetrized Hurwitz numbers.

Keywords Random tilings · Random matrices · Hurwitz numbers

1 Introduction

Let

$$\begin{array}{cccc} & b_1^{(1)} & & \\ & b_1^{(2)} & b_2^{(2)} & \\ & b_1^{(3)} & b_2^{(3)} & b_3^{(3)} \\ & \vdots & \vdots & \vdots & \ddots \end{array} \quad (1.1)$$

be a triangular array of integers, the elements of which are strictly decreasing along rows. The array (1.1) gives rise to a sequence $\Omega^{(N)}$ of planar domains via the following construction. Fix a coordinate system in the plane whose axes meet at a 120° angle. We specify $\Omega^{(N)}$ by specifying its boundary, which consists of two piecewise linear components. One component of $\partial\Omega^{(N)}$ —the lower boundary—is simply the horizontal axis in the plane. The other component—the upper boundary—is built in three steps. First, construct the line parallel to the lower boundary passing through the point $(0, N)$. Second, affix N outward-facing unit triangles to this line such that the midpoints of their bases have horizontal coordinates

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$b_1^{(N)} > \dots > b_N^{(N)}$. Finally, erase the bases of these triangles. We will refer to $\Omega^{(N)}$ as the *sawtooth domain* of rank N with boundary conditions $(b_1^{(N)}, \dots, b_N^{(N)})$.

A *lozenge* is a unit rhombus in the plane whose sides are parallel to one of the coordinate axes, or to the line bisecting the obtuse angle between them. Lozenges are thus divided into three classes: left-leaning, right-leaning, and vertical. Given a lozenge tiling of $\Omega^{(N)}$, as in Fig. 1, the horizontal line through $(0, k)$ “threads” exactly k vertical tiles, or “beads”, and the beads on adjacent threads interlace, as in Fig. 2.

Let $T^{(N)}$ be a uniformly random lozenge tiling of $\Omega^{(N)}$, and let $b_{k1}^{(N)} > \dots > b_{kk}^{(N)}$ be the horizontal coordinates of the centroids of the beads on the k th thread through $T^{(N)}$. The main result of this note is a limit theorem for the k -dimensional random vector $(b_{k1}^{(N)}, \dots, b_{kk}^{(N)})$, in the regime where $N \rightarrow \infty$ with k fixed.

Suppose there exists a positive integer M such that, for each $N \geq 1$,

$$\{b_1^{(N)} > \dots > b_N^{(N)}\} \subseteq \{MN > \dots > -MN\}.$$

Let $\nu^{(N)}$ be the probability measure which places mass $1/N$ at each of the points $b_i^{(N)}/N$. Suppose that $\nu^{(N)}$ converges weakly to ν , the probability measure on $[-M, M]$ with moment sequence $\psi_1, \psi_2, \psi_3, \dots$

Theorem 1 For each $N \geq 1$ and $1 \leq k \leq N$, set

$$\tilde{b}_{kl}^{(N)} = \frac{b_{kl}^{(N)} - (\psi_1 - \frac{1}{2})\sqrt{N}}{\psi_2 - \psi_1^2 - \frac{1}{12}}, \quad 1 \leq l \leq k.$$

For any fixed k , the random vector $(\tilde{b}_{k1}^{(N)}, \dots, \tilde{b}_{kk}^{(N)})$ converges weakly to the ordered list of eigenvalues of a $k \times k$ GUE random matrix as $N \rightarrow \infty$.

Note that ψ_1 and $\psi_2 - \psi_1^2$ are, respectively, the mean and variance of ν , while the numbers $1/2$ and $1/12$ are the mean and variance of the uniform probability measure on $[0, 1]$.

Given that the law of large numbers for $T^{(N)}$ manifests as the convergence of the height function of the normalized tiling $N^{-1}T^{(N)}$ to a deterministic limit, the so-called *limit shape* [4, 15, 21], the $N^{-1/2}$ scaling in Theorem 1 is natural. Indeed, as discussed in [20], the arctic curve separating the frozen and liquid regions of $T^{(N)}$ which emerge as $N \rightarrow \infty$ resembles a parabola near the point where it is tangent to the lower boundary of $\Omega^{(N)}$. For boundary conditions producing an arctic curve which actually is a parabola, see [16, 17].

The connection between the joint distribution of vertical tiles near the frozen boundary and GUE eigenvalues was first studied by by Okounkov and Reshetikhin [20]. For a special

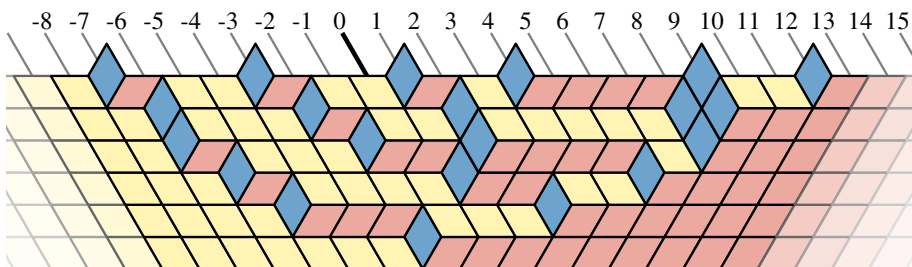


Fig. 1 A lozenge tiling of a sawtooth domain of rank 6

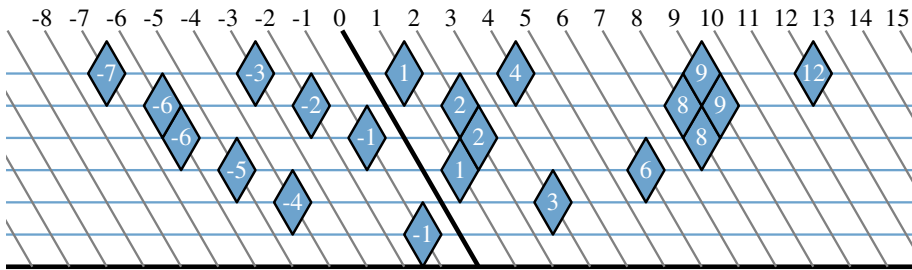


Fig. 2 Interlacing beads and their coordinates

class of boundary conditions, Theorem 1 was proved by Johansson and Nordenstam [13]. In a slightly different (but equivalent) form, Theorem 1 was obtained in full generality by Gorin and Panova [6] as a consequence of their general approach to Schur function asymptotics. In this note, we present a different approach to Theorem 1 in which the usual tools of integrable probability (e.g. determinantal processes, steepest descent analysis) play no role. Instead, our argument is based on the combinatorial interpretation of the Harish-Chandra/Itzykson-Zuber integral discovered in [7].

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2 Proof of Theorem 1

Let us replace the k -dimensional random vector $(b_{k1}^{(N)}, \dots, b_{kk}^{(N)})$ with the random Hermitian matrix

$$B_k^{(N)} = U_k \begin{bmatrix} b_{k1}^{(N)} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & b_{kk}^{(N)} \end{bmatrix} U_k^{-1},$$

where U_k is a random matrix drawn from normalized Haar measure on the unitary group $U(k)$. By the Laplace transform of $B_k^{(N)}$, we mean the function on $k \times k$ complex semisimple matrices A defined by

$$A \mapsto \mathbf{E} \left[e^{\text{Tr} AB_k^{(N)}} \right],$$

where \mathbf{E} denotes expectation. In the case $k = 1$, this function coincides with the classical two-sided Laplace transform encoding the distribution of the horizontal coordinate of the bottom bead.

The Laplace transform of $B_k^{(N)}$ depends only on the eigenvalues of A , and thus may be considered as a function of k complex variables. This function is analytic, because the distribution of $B_k^{(N)}$ in $H(k)$, the space of $k \times k$ Hermitian matrices, is compactly supported. Explicitly,

$$L_k^{(N)}(a_1, \dots, a_k) = \sum_{\{b_1 > \dots > b_k\} \subset \mathbb{Z}} \mathbf{P} \left(b_{k1}^{(N)} = b_1, \dots, b_{kk}^{(N)} = b_k \right) \times \int_{U(k)} e^{\text{Tr} \text{diag}(a_1, \dots, a_k) U \text{diag}(b_1, \dots, b_k) U^{-1}} dU,$$

where the sum is over all k -point particle configurations on the integer lattice and \mathbf{P} is the uniform probability measure on lozenge tilings of $\Omega^{(N)}$. The integral over $U(k)$ is just the Laplace transform of the uniform probability measure on the set of $k \times k$ Hermitian matrices with eigenvalues $b_1 > \dots > b_k$. That is, $L_k^{(N)}$ is the Laplace transform of a mixture of *orbital measures*. If $k = N$, the bead locations are deterministic, and we are dealing with the Laplace transform of a pure orbital measure. The following proposition reduces our workload to the analysis of the Laplace transforms of pure orbital measures.

Proposition 2 For any integers $1 \leq k \leq N$,

$$L_k^{(N)}(a_1, \dots, a_k) = \left(\prod_{i=1}^k \frac{a_i}{e^{a_i} - 1} \right)^{N-k} L_N^{(N)}(a_1, \dots, a_k, 0, \dots, 0).$$

Proof The proof is a combination of three standard facts from the representation theory of the complex general linear group $GL(N)$.

First, the isomorphism classes of irreducible rational representations of $GL(N)$ are indexed by N -point particle configurations on \mathbb{Z} . This is a classical result, see e.g. [23].

Second, given a particle configuration $\{b_1 > \dots > b_N\} \subset \mathbb{Z}$, the corresponding normalized irreducible character

$$\frac{\chi^{(b_1, \dots, b_N)}(e^{a_1}, \dots, e^{a_N})}{\chi^{(b_1, \dots, b_N)}(1, \dots, 1)}$$

equals the twisted Laplace transform

$$\prod_{1 \leq i < j \leq N} \frac{a_i - a_j}{e^{a_i} - e^{a_j}} \int_{U(N)} e^{\text{Tr} \text{diag}(a_1, \dots, a_N) U \text{diag}(b_1, \dots, b_N) U^{-1}} dU$$

of the uniform measure on Hermitian matrices with spectrum $\{b_1 > \dots > b_N\}$. This identity is independently due to Harish-Chandra [10], and Itzykson and Zuber [12]—it is the $U(N)$ case of the Kirillov character formula [14].

The third and final ingredient is the branching rule for irreducible characters of $GL(N)$ under restriction to $GL(N - 1)$:

$$\chi^{(b_1, \dots, b_N)}(e^{a_1}, \dots, e^{a_{N-1}}, 1) = \sum_{\{c_1 > \dots > c_{N-1}\} \subset \mathbb{Z}} \chi^{(c_1, \dots, c_{N-1})}(e^{a_1}, \dots, e^{a_{N-1}}),$$

where the sum is over all configurations of $N - 1$ particles on \mathbb{Z} which interlace with the configuration $\{b_1 > \dots > b_N\}$. A proof of the branching rule may be found in [5, Chapter 8]. Iterating the branching rule $N - k$ times and applying the Harish-Chandra formula yields the stated formula for $L_k^{(N)}$ in terms of $L_N^{(N)}$. □

Consider the analytic function $\mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^N \rightarrow \mathbb{C}$ defined by

$$(\mathbb{z}; a_1, \dots, a_N; b_1, \dots, b_N) \mapsto \int_{U(N)} e^{\mathbb{z} \text{Tr} \text{diag}(a_1, \dots, a_N) U \text{diag}(b_1, \dots, b_N) U^{-1}} dU.$$

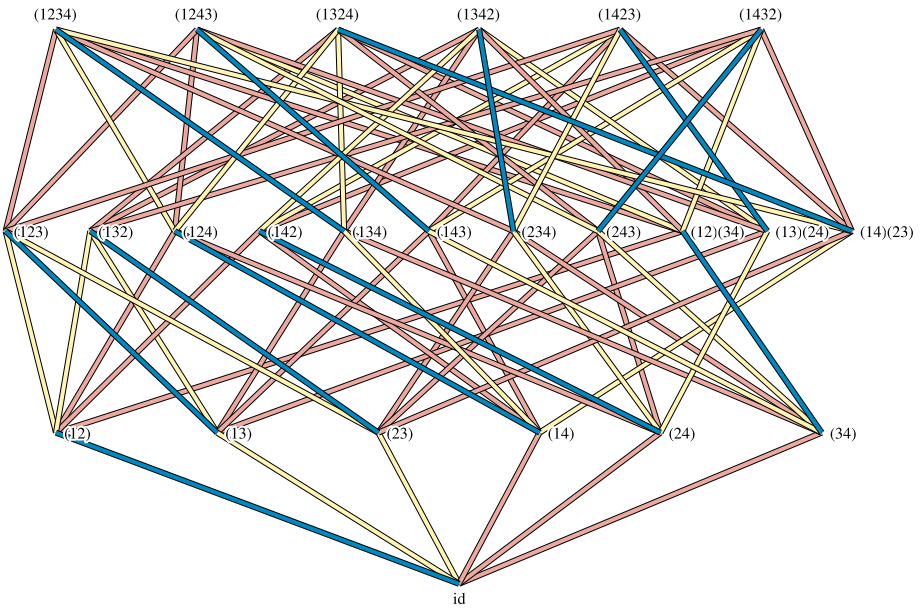


Fig. 3 $S(4)$ with the Biane–Stanley edge-labelling

This is the famous Harish-Chandra/Itzykson-Zuber integral. The parameter z may be called the *coupling constant*, as a reference to its origin in the spectral analysis of coupled random semisimple matrices with AB -interaction [1, 12].

The HCIZ integral enjoys a natural $S(N) \times S(N)$ symmetry: it is invariant under permutation of the a 's amongst themselves, and the b 's amongst themselves. Combining this symmetry with the fact that the Newton power-sums form a linear basis of the algebra of symmetric polynomials, we may present the Maclaurin series of the logarithm of the HCIZ integral in the form

$$\begin{aligned} & \log \int_{U(N)} e^z \text{Tr} \text{diag}(a_1, \dots, a_N) U \text{diag}(b_1, \dots, b_N) U^{-1} dU \\ &= \sum_{d=1}^{\infty} \frac{z^d}{d!} \sum_{\alpha, \beta \vdash d} C_N(\alpha, \beta) p_{\alpha}(a_1, \dots, a_N) p_{\beta}(b_1, \dots, b_N), \end{aligned}$$

where the internal sum is over all pairs of Young diagrams with d cells.

The coefficients $C_N(\alpha, \beta)$ have the following combinatorial interpretation. Consider the Cayley graph of the symmetric group $S(d)$ as generated by the conjugacy class of transpositions. Equip this graph with the Biane–Stanley edge labelling [2, 22], wherein each edge corresponding to the transposition $(s\ t)$ is tagged with t , the larger of the two numbers interchanged. The $d = 4$ case is shown in Fig. 3, where 2-edges are drawn in blue, 3-edges in yellow, and 4-edges in red. A walk on the Cayley graph is said to be *monotone* if the labels of the edges it traverses form a weakly increasing sequence. A walk is *transitive* if its steps and endpoints together generate a transitive subgroup of $S(d)$. Given two partitions $\alpha, \beta \vdash d$, and a nonnegative integer r , let $\bar{H}^r(\alpha, \beta)$ be the number of r -step monotone, transitive walks on $S(d)$ which begin at a permutation of cycle type α and end at a permutation of cycle type β .

Theorem 3 ([7]) *For any $1 \leq d \leq N$, and any $\alpha, \beta \vdash d$, we have*

$$C_N(\alpha, \beta) = \frac{1}{N^d} \sum_{r=0}^{\infty} (-1)^r \frac{\vec{H}^r(\alpha, \beta)}{N^r}.$$

The number $H^r(\alpha, \beta)$, which counts walks as above, but without the monotonicity constraint, is a *double Hurwitz number*. The double Hurwitz numbers are important quantities in classical and modern enumerative geometry, see [8, 19]. Reversing a classical construction due to Hurwitz [11], we have that

$$\frac{1}{d!} H^r(\alpha, \beta) = \sum_{(X, f)} \frac{1}{|\text{Aut}(X, f)|},$$

where the sum runs over all isomorphism classes of pairs (X, f) in which X is a compact, connected Riemann surface and $f : X \rightarrow \mathbf{P}^1$ is a degree d mapping to the Riemann sphere with profile α over ∞ , profile β over 0 , and simple ramification over the r th roots of unity. By the Riemann–Hurwitz formula, such a branched covering exists if and only if

$$g = \frac{r + 2 - \ell(\alpha) - \ell(\beta)}{2}$$

is a non-negative integer, in which case g is the genus of X . Here $\ell(\alpha)$ is the number of parts in the partition $\alpha \vdash d$, and likewise for $\ell(\beta)$. We write $H^r(\alpha, \beta) = H_g(\alpha, \beta)$, with the understanding that r and g determine one another via Riemann–Hurwitz.

Following the terminology of [7], we refer to the numbers $\vec{H}^r(\alpha, \beta) = \vec{H}_g(\alpha, \beta)$ as the *monotone double Hurwitz numbers*. The expansion in Theorem 3 may equivalently be written

$$C_N(\alpha, \beta) = (-1)^{\ell(\alpha)+\ell(\beta)} N^{2-d-\ell(\alpha)-\ell(\beta)} \sum_{g \geq 0} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}}. \tag{2.1}$$

This expansion renders the asymptotics of the HCIZ integral transparent in virtually any scaling regime. In particular, one obtains the following limits.

Proposition 4 *Under the hypotheses of Theorem 1, for any fixed $d \in \mathbb{N}$ and $a_1, \dots, a_k \in \mathbb{C}$, we have*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\alpha, \beta \vdash d} C_N(\alpha, \beta) p_\alpha(a_1, \dots, a_k) p_\beta(b_1^{(N)}, \dots, b_N^{(N)}) \\ &= p_d(a_1, \dots, a_k) \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \vec{H}_0(d, \beta) \psi_\beta, \end{aligned}$$

where $\psi_\beta = \prod_i \psi_{\beta_i}$.

Proof According to (2.1), we have

$$\begin{aligned} & \frac{1}{N} \sum_{\alpha, \beta \vdash d} C_N(\alpha, \beta) p_\alpha(a_1, \dots, a_k) p_\beta(b_1^{(N)}, \dots, b_N^{(N)}) \\ &= \sum_{\alpha \vdash d} (-1)^{\ell(\alpha)} \frac{p_\alpha(a_1, \dots, a_k)}{N^{\ell(\alpha)-1}} \sum_{\beta \vdash d} (-1)^{\ell(\beta)} \frac{p_\beta\left(\frac{b_1^{(N)}}{N}, \dots, \frac{b_N^{(N)}}{N}\right)}{N^{\ell(\beta)}} \sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} \end{aligned}$$

for any $N \geq d$. From the definition of $\vec{H}_g(\alpha, \beta)$, we have the upper bound

$$\vec{H}_g(\alpha, \beta) \leq (d!)^{2g+\ell(\alpha)+\ell(\beta)} \leq (d!)^{2g+2d}.$$

Thus

$$\sum_{g=0}^{\infty} \frac{\vec{H}_g(\alpha, \beta)}{N^{2g}} = \vec{H}_0(\alpha, \beta) + O\left(\frac{1}{N^2}\right)$$

as $N \rightarrow \infty$, uniformly in α, β .

The weak convergence of $\nu^{(N)}$ to ν , the measure on $[-M, M]$ with moments $\{\psi_m : m \in \mathbb{N}\}$, is equivalent to the limits

$$\lim_{N \rightarrow \infty} \frac{p_m\left(\frac{b_1^{(N)}}{N}, \dots, \frac{b_N^{(N)}}{N}\right)}{N} = \psi_m, \quad m \in \mathbb{N}.$$

□

The numbers $\vec{H}_g(d, \beta)$ are *one-part* monotone double Hurwitz numbers; their classical counterparts $H_g(d, \beta)$ are analyzed in [8]. The sum

$$K_d = \sum_{\beta \vdash d} (-1)^{1+\ell(\beta)} \vec{H}_0(d, \beta) \psi_\beta$$

which emerges in Proposition 4 is an element of $\mathbb{Z}[\psi_1, \dots, \psi_d]$, homogeneous of degree d with respect to the grading $\deg(\psi_m) = m$. In fact, K_d is, up to a simple factor, the d th *free cumulant* κ_d of the measure ν :

$$K_d = (d - 1)! \kappa_d. \tag{2.2}$$

We recall that the free cumulants of a probability measure are obtained by replacing the lattice of all partitions with the lattice of noncrossing partitions in the moment-cumulant formula, see e.g. [18]. The identity (2.2) may be established in a purely combinatorial way, by viewing the noncrossing partition lattice $NC(d)$ as the set of geodesic paths $(1) \dots (d) \rightarrow (1 \dots d)$ on the Cayley graph of $S(d)$ and using the Kreweras antiautomorphism. For our purposes, we only require explicit knowledge of K_1 and K_2 , which can be computed directly from the definition of the monotone double Hurwitz numbers:

$$\begin{aligned} \vec{H}_0(1, 1) = 1 &\implies K_1 = \psi_1 \\ \vec{H}_0(2, 2) = \vec{H}_0(2, 11) = 1 &\implies K_2 = \psi_2 - \psi_1^2. \end{aligned}$$

We thus leave the proof of (2.2) to the interested reader.

The absolute summability of the series

$$\sum_{d=1}^{\infty} \frac{z^d}{d!} K_d$$

follows from [7, Theorem3.4]. Arguing as in [7, Theorem4.1], Proposition 4 may be promoted to the following scaling limit of the HCIZ integral, which is closely related to the results of [3, 9].

Proposition 5 *Let $k \in \mathbb{N}$ be fixed. Under the assumptions of Theorem 1, there exists $\varepsilon > 0$ such that*

$$\frac{1}{N} \log \int_{U(N)} e^{z \operatorname{Tr} \operatorname{diag}(a_1, \dots, a_k, 0, \dots, 0) U \operatorname{diag}(b_1^{(N)}, \dots, b_N^{(N)}) U^{-1}} dU \rightarrow \sum_{d=1}^{\infty} \frac{z^d}{d!} p_d(a_1, \dots, a_k) K_d,$$

uniformly on compact subsets of $\{(z; a_1, \dots, a_k) \in \mathbb{C} \times \mathbb{C}^k : |z a_i| < \varepsilon\}$.

Tuning the coupling constant to $z = N^{-1/2}$, Proposition 5 yields the following corollary.

Corollary 6 *Let $k \in \mathbb{N}$ be fixed. Under the assumptions of Theorem 1, we have the $N \rightarrow \infty$ asymptotic expansion*

$$\log \int_{U(N)} e^{\frac{1}{\sqrt{N}} \operatorname{Tr} \operatorname{diag}(a_1, \dots, a_k, 0, \dots, 0) U \operatorname{diag}(b_1^{(N)}, \dots, b_N^{(N)}) U^{-1}} dU \sim \sum_{d=1}^{\infty} \frac{K_d}{d!} p_d(a_1, \dots, a_k) N^{1-\frac{d}{2}},$$

uniformly on compact subsets of \mathbb{C}^k .

Combining Corollary 6 with the fact that

$$\log \frac{a}{e^a - 1} = -\log \frac{e^a - 1}{a} = -\frac{1}{2} \frac{a^1}{1!} - \frac{1}{12} \frac{a^2}{2!} + \dots$$

is negative one times the generating function for the classical cumulants c_1, c_2, \dots of uniform measure on $[0, 1]$, Proposition 2 yields the asymptotic expansion

$$\log L_k^{(N)} \left(\frac{a_1}{\sqrt{N}}, \dots, \frac{a_k}{\sqrt{N}} \right) \sim \sum_{d=1}^{\infty} \frac{K_d - c_d}{d!} p_d(a_1, \dots, a_k) N^{1-\frac{d}{2}},$$

uniformly on compact subsets of \mathbb{C}^k . In particular,

$$\begin{aligned} \log L_k^{(N)} \left(\frac{a_1}{\sqrt{N}}, \dots, \frac{a_k}{\sqrt{N}} \right) &= \sqrt{N} \left(\psi_1 - \frac{1}{2} \right) p_1(a_1, \dots, a_k) \\ &\quad + \frac{1}{2} \left(\psi_2 - \psi_1^2 - \frac{1}{12} \right) p_2(a_1, \dots, a_k) + O \left(\frac{1}{\sqrt{N}} \right) \end{aligned}$$

as $N \rightarrow \infty$. Since a $k \times k$ standard GUE random matrix X_k is characterized by the log-Laplace transform

$$\log \mathbf{E} \left[e^{\operatorname{Tr} A X_k} \right] = \frac{1}{2} \operatorname{Tr} A^2,$$

and since $H(k)$ is a finite-dimensional Euclidean space with the inner product $\langle A, B \rangle = \operatorname{Tr} AB$, Theorem 1 follows from the above quadratic approximation and the Lévy continuity theorem.

References

1. Adler, M., van Moerbeke, P.: The spectrum of coupled random matrices. *Ann. Math.* **149**, 921–976 (1999)
2. Biane, P.: Parking functions of types A and B . *Electron. J. Comb.* **9**, N7 (2002)
3. Collins, B., Śniady, P.: New scaling of Itzykson–Zuber integrals. *Ann. IH Poincaré* **43**, 139–146 (2007)
4. Duse, E., Metcalfe, A.: Asymptotic geometry of discrete interlaced patterns: Part I, [arXiv:1412.6653v1](https://arxiv.org/abs/1412.6653v1)

5. Goodman, R., Wallach, N.: Symmetry. Representations, and Invariants. Graduate Texts in Mathematics, vol. 255. Springer, Berlin (2009)
6. Gorin, V., Panova, G.: Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory, [arXiv:1301.0634v4](https://arxiv.org/abs/1301.0634v4)
7. Goulden, I.P., Guay-Paquet, M., Novak, J.: Monotone Hurwitz numbers and the HCIZ integral. *Ann. Math. Blaise Pascal* **21**, 71–89 (2014)
8. Goulden, I.P., Jackson, D.M., Vakil, R.: Towards the geometry of double Hurwitz numbers. *Adv. Math.* **198**, 43–92 (2005)
9. Guionnet, A., Maida, M.: A Fourier view on the R -transform and related asymptotics of spherical integrals. *J. Funct. Anal.* **222**, 435–490 (2005)
10. Harish-Chandra, : Differential operators on a semisimple Lie algebra. *Am. J. Math.* **79**, 87–120 (1957)
11. Hurwitz, A.: Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten. *Math. Ann.* **55**, 53–66 (1902)
12. Itzykson, C., Zuber, J.-B.: The planar approximation II. *J. Math. Phys.* **21**, 411–421 (1980)
13. Johansson, K., Nordenstam, E.: Eigenvalues of GUE minors. *Electron. J. Probab.* **50**, 1342–1371 (2006)
14. Kirillov, A.A.: Lectures on the Orbit Method. Graduate Studies in Mathematics. AMS, Providence (2004)
15. Kenyon, R., Okounkov, A.: Limit shapes and the complex Burgers equation. *Acta. Math.* **199**, 263–302 (2007)
16. Nordenstam, E., Young, B.: Domino shuffling on Novak half-hexagons and Aztec half-diamonds. *Electron. J. Comb.* **18**, 181 (2011)
17. Nordenstam, E., Young, B.: Correlations for the Novak process. *Discret. Math. Theor. Comput. Sci.* 643–654 (2012)
18. Novak, J., Śniady, P.: What is... a free cumulant? *Not. Am. Math. Soc.* **58**, 300–302 (2011)
19. Okounkov, A.: Toda equations for Hurwitz numbers. *Math. Res. Lett.* **7**, 447–453 (2000)
20. Okounkov, A., Reshetikhin, N.: The birth of a random matrix. *Mosc. Math. J.* **6**, 553–556 (2006)
21. Petrov, L.: Asymptotics of random lozenge tilings via Gelfand-Tsetlin schemes. *Probab. Theory. Relat. Fields* **160**, 429–487 (2014)
22. Stanley, R.P.: Parking functions and noncrossing partitions. *Electron. J. Comb.* **4**, 20 (1997)
23. Weyl, H.: The Classical Groups: Their Invariants and Representations. Princeton University Press, Princeton (1939)