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Modified Kadomtsev-Petviashivili Equation for Tsunami over Irregular Seabed

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Abstract We derive an asymptotic equation governing the trans-ocean propagation of tsunami from source to the continental shelf. Focus is on disturbances originated from a slender fault of finite length. The variable sea depth is assumed to consist of a slowly varying mean and random fluctuations. The method of multiple scales is used to derive a Kadomtsev-Petviashivili equation with variable coefficients. Modifications by one and two dimensional random irregularities are shown to affect the wave speed, dissipation and additional dispersion. The result can be used to facilitate physical insight with modest numerical efforts.

Keywords Tsunami, · Kadomtsev-Petviashivili Equation · Nonlinear dispersive waves

1 Introduction

Vast devastations inflicted by recent tsunamis have prompted international efforts to model the hydrodynamics theoretically to help of planning the warning system and coastal defense. Due to the complexity of the physics and bathymetry, various approximations are needed to enable rapid computations. Numerical models often require empirical fittings for comparison with measured data (see [2], [29] for a survey) . For reliable interpretation of observations and confidence in numerical predictions, analytical theories of different aspects of tsunami are very helpful. Outstanding examples include the nonlinear theory of Carrier and Greenspan [5] on fully reflected wave on a sloping beach. Another landmark is Kajiura's theory [21] for long-distance propagation due to localized and instantaneous rupture on the seabed. For the ide-

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alized case of seabed of arbitrary but constant depth he obtained the exact linear solution and explained the key features of the leading wave from both line and point sources. It was later shown for one dimensional case that weak nonlinearity can accumulate with the travel distance so that both dispersion and nonlinearity can become comparably important [18]. Now there are several large-scale computational schemes incorporating Boussinesq approximation accounting for weak dispersion and nonlinearity for the offshore, combined with empirical models of breaking and runup near the coast. These programs have already been applied to simulate DART buoy data of tsunamis in Indian Ocean and Tohoku ([7], [45] [26], [17], [46], etc). However to fully understand the many facets of tsunami physics regarding the source, trans-ocean propagation and amplification near coast and runup on the shore, various analytical efforts emphasizing certain specific environmental factors are flourishing (e.g., [23],[6],[42], etc.).

Since the ultimate interest of tsunami prediction is its impacts on the shoreline, various mathematical studies, stimulated by related advances in other branches of physics have been devoted to idealized and simplified geometries. For example, Madsen and Mei [30] investigated, by numerical integration of Boussinesq equations, the fission of solitons when a long pulse climbs a shelf. To enable deeper understanding, Johnson [20], Kakutani[22] and Tappert and Zabusky [43] have developed asymptotic theories for slowly varying depth and derived a KdV equation with variable coefficients. Depth variation in a large ocean basin can be highly irregular. However, small scale fluctuations are not easy to accounted for in computational models. Statistical theories have been explored only by a few authors. An early advance was made by Elter & Molyneaux [14] who used the diagrammatic technique of Feyman on the linearized long wave equations and treated tsunami from a point source over weak random depth perturbations. As a problem of mathematical physics, Kawahara [25] derived the evolution equation for two-dimensional long waves in a sea of constant mean depth with random perturbations. The result is a modified KdV equation with additional terms representing dissipation by random scattering. To examine the effect of depth disorder on solitons, Mei and Li [34] found that the Kawahara equation can be expressed in a different but equivalent form, and examined soliton fission via numerical examples. There are more theoretical studies of periodic waves over a randomly rough seabed,[40], [31],[33], [16], [37], [38]. Dutykh & Labart [13] reported a numerical study of long waves over random seabed by the Monte Carlo method. The scarcity of statistical theories may be due to the fact that information such as the wavenumber spectra of sea bathymetry is readily available only for some ocean basins ([3],[15]). Such information has been only recently used in the studies of long-lasting tides and internal waves ([4], [28]), but not yet applied to tsunamis.

As the physics of trans-ocean tsunami must involve different types and scales of faults and bathymetry, it is interesting to study the combined effects of random irregularities and gradual depth variation. One of our objectives here is to develop an asymptotic theory which is relatively simple to compute in order to capture the essential physics. In this article we treat an example of a slender fault and variable bathymetry. Three contrasting length scales are involved: the macro-scale for propagation distance, the meso-scale of fault length, and the micro-scale common to waves, the fault width and the correlation length of the random irregularities. The evolution of small-amplitude waves will be considered over slowly varying seabed perturbed by weakly random irregularities.

The vertical displacement of the free surface above the fault is usually much smaller than the sea depth which is in turn much smaller than the wavelength. It is easily reasoned that nonlinearity and dispersion become comparable as tsunami enters the shallow water. Hence the long-wave approximation of Boussinesq accounting for weak nonlinearity and dispersion should be appropriate[18]. Several full-scale numerical models have been developed along this line [7], [26], [17], [45], [46].

Earthquakes from faults of finite size generate waves in two dimensions. From past records, it is known that some fault zones are elongated where the width W is far smaller than the length L. Examples of such earthquakes are : Chile (1960): $W \times L = 200$ km$\times$ 800 km, Alaska (1964): 100 km$\times$ 700 km, and Sumatra, Indian Ocean (2004): 200 km$\times$ 1200 km. It is interesting to analyze the directional bias resulted from this anisotropy. In this work, we describe an asymptotic theory based on a simple model. Starting from the general Boussinesq equations in two dimensions, we shall derive a compact equation of Kadomtsev-Petviashivili (K-P) type which accounts for weak nonlinearity and dispersion.

The standard K-P equation has constant coefficients and is the extension of the celebrated Kortweg-deVries equation from one to two dimensions. Their exact solutions have been the highlight of applied mathematics in the past few decades. Many results on soliton evolution and interactions have profound impact in different branches of theoretical physics and are well documented in monographs ([1], [39]). For applications in engineering and/or geophysics, these equations must often be modified to allow forcing terms, variable coefficients and/or complex boundary conditions [11],[10],[24],[27]. Recently the present authors have made extensions of the K-P equation to study two dimensional tsunami emanating from a slender fault and advancing across a large ocean basin onto a shallow continental shelf [35]. The objective is to see how the tsunami front evolves in that environment before it arrives the shore where breaking and runup occurs. In this article we extend the asymptotic basis further to account for scattering by random roughness on the seabed, as a preparation for further numerical studies. In particular we assume the mean sea depth to decrease slowly in one direction only but perturbed by random roughness of small amplitude. Under the following assumptions : wave amplitude $\ll$ sea depth $\ll$ fault width (dominant wave length) $\ll$ fault length $\ll$ ocean size, the method multiple scales is employed. In nature the continental slope is rather steep and causes reflection which is not modeled here. Numerical studies will be reported in the future.

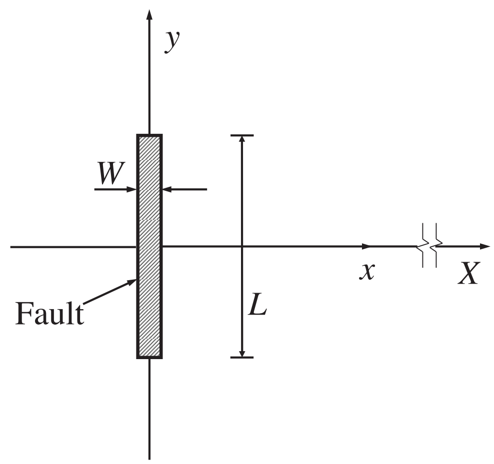

Fig. 1 Definition sketch for a slender fault

2 Boussinesq Equations for Shallow Water Waves over a Two-Dimensional Seabed

We first recall the the Boussinesq equations for two dimensional long waves over a general smooth bathymetry where the total still water depth consists of a slowly varying mean $h(x)$ and random perturbations $b(x, y)$,

$$H(x, y) = h(x) + b(x, y). \tag{2.1}$$

The duration of the earthquake is usually just a few minutes which is much shorter than the time scale of tsunami. After the earthquake is over, the equation of mass conservation is

$$\zeta_t + [(H + \zeta)u]_x + [(H + \zeta)v]_y = 0, \qquad t > 0; \tag{2.2}$$

subject to the initial conditions:

$$\zeta(\mathbf{x}, 0) = Z_0(\mathbf{x}),\ u(\mathbf{x}, 0) = v(\mathbf{x}, 0) = 0, \quad t = 0; \tag{2.3}$$

i.e., the free surface mimics the ground uplift $Z_0(\mathbf{x})$. The momentum equations are

$$\begin{aligned} &u_t + uu_x + vu_y + g\zeta_x \\ &= \frac{H}{2}[(Hu_t)_x + (Hv_t)_y]_x - \frac{H^2}{6}[u_{tx} + v_{ty}]_x, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &v_t + uv_x + vv_y + g\zeta_y \\ &= \frac{H}{2}[(Hu_t)_x + (hv_t)_y]_y - \frac{H^2}{6}[u_{tx} + v_{ty}]_y. \end{aligned} \tag{2.5}$$

Let us define dimensionless variables denoted by primes as follows,

$$x' = \frac{x}{W},\ y' = \frac{y}{W},\ t' = \frac{\sqrt{gh_0}}{W}t,\quad (\zeta', Z_0') = \frac{\zeta, Z_0}{A},$$
$$u' = \frac{u}{\frac{A}{h_0}\sqrt{gh_0}},\ v' = \frac{v}{\frac{A}{h_0}\sqrt{gh_0}},\quad H' = \frac{H}{h_0}. \tag{2.6}$$

where A and h_0 are reference wave amplitude and mean still water depth at the origin. By omitting primes for brevity, the dimensionless equations accurate to $O(\epsilon, \mu^2)$ are

$$\zeta_t + ((H + \epsilon\zeta)u)_x + (H + \epsilon\zeta)v)_y = 0, \tag{2.7}$$

$$u_t + \epsilon\,(uu_x + vu_y) + \zeta_x = \mu^2\frac{H^2}{3}[u_{txx} + v_{txy}], \tag{2.8}$$

and

$$v_t + \epsilon(uv_x + vv_y) + \zeta_y = \mu^2\frac{H^2}{3}[u_{txy} + v_{tyy}]. \tag{2.9}$$

The two small paramaters are

$$\epsilon = \frac{A}{h_0},\quad \mu^2 = \frac{h_0^2}{W^2}; \tag{2.10}$$

which are the measures of nonlinearity and dispersion respectively.

Now let us assume that the mean sea depth is one-dimensional, i.e., $h(\mu^2 x) = h(X)$ which varies slowly in one direction only where $X = \mu^2 x$, from $h(0) = 1$. The random fluctuations are assumed to be small and can be two-dimensional in general $\mu b(\mathbf{x})$ i.e.,

$$H = h(X) + \mu b(x, y),\ \text{with}\quad X = \mu^2 x, \tag{2.11}$$

where $b(\mathbf{x})$ is a stationary random variable of $\mathbf{x}$ with zero mean ($\langle b(\mathbf{x})\rangle = 0$).

By cross differentiation we can get a combined equation accurate to $O(\epsilon, \mu^2)$,

$$h(\zeta_{xx} + \zeta_{yy}) - \zeta_{tt}$$
$$= \mu^2 h_X u_t - \mu[(bu_t)_x + (bv_t)_y] - \epsilon h(uu_x)_x + \epsilon(\zeta u)_{xt} + \mu^2\frac{h^2}{3}u_{txxx} \tag{2.12}$$

The initial condition (2.3) remains.

3 Variable Depth and Slender Fault

Let us focus on the case of slender fault with length $L = W/\mu \gg W$. Hence we define the slow coordinate $Y = \mu y$ and let $Z = Z(x, Y, t)$. Because Z is independent of fast y, $\frac{\partial \zeta}{\partial y} = \mu \frac{\partial \zeta}{\partial Y} = O(\mu)$ so that $v = O(\mu)$ is small and u, ζ depend on Y and not on y. Thus at the leading order the approximate equation is

$$h\zeta_{xx} - \zeta_{tt} \approx 0 \tag{3.1}$$

with a relative error of order $O(\mu)$. In view of the weak dependence of h on x, we can introduce the new coordinate

$$\bar{x} = \int_0^x \frac{dx'}{\sqrt{h(X')}}, \tag{3.2}$$

In physical dimensions this definition reads

$$\frac{\bar{x}}{\sqrt{gh_0}} = \int_0^x \frac{dx'}{\sqrt{gh(X')}}, \tag{3.3}$$

and represents the time for waves to travel from the origin to x at the local speed $\sqrt{gh}$.

Note that the $x-$derivatives become

$$\frac{\partial F}{\partial x} = \frac{1}{\sqrt{h}} \frac{\partial F}{\partial \bar{x}}, \quad \frac{\partial^2 F}{\partial x^2} = \frac{1}{h} \frac{\partial^2 F}{\partial \bar{x}^2} - \frac{\mu^2}{2} \frac{h_X}{h^{3/2}} \frac{\partial F}{\partial \bar{x}} \tag{3.4}$$

Note also that $b(\mathbf{x}) = b(x, y)$, hence v depends on x, y at order $O(\mu)$. We now perform the multiple scale expansions

$$\zeta = \zeta^{(0)} + \mu\zeta^{(1)} + \cdots, \quad u = u_0 + \mu u_1 + \cdots, \quad v = \mu v_1 + \cdots \tag{3.5}$$

At the leading order, $\zeta^{(0)}$ satisfies

$$\frac{\partial^2 \zeta^{(0)}}{\partial \bar{x}^2} - \frac{\partial^2 \zeta^{(0)}}{\partial t^2} = 0, \tag{3.6}$$

which has the well-known solution

$$\zeta^{(0)} = \zeta_+^{(0)}(\bar{x} - t, X, Y) + \zeta_-^{(0)}(\bar{x} + t, X, Y) \tag{3.7}$$

Waves from the fault now appear as two separate pulses propagating away from each other in opposite directions. From now on we only focus attention to the evolution of the right-going disturbance $\zeta_+^{(0)}$ for $X > 0$. The following condition will be imposed:

$$\zeta_+^{(0)}(\bar{x} - t, 0, Y) \equiv \zeta_+^{(0)}(\bar{x} - t, Y) \equiv \frac{1}{2} Z_0(\bar{x}, Y), \quad X \to 0. \tag{3.8}$$

It is easy to see that

$$(u_+^{(0)}, v_+^{(0)}) = \left(\frac{\zeta_+^{(0)}}{\sqrt{h(X)}}, 0 \right). \tag{3.9}$$

For simplicity we still use $\zeta^{(0)}$ to denote the right-going disturbance $\zeta_+^{(0)}(\bar{x} - t, X, Y)$.

At $O(\mu)$, $\zeta^{(1)}$ is random and governed by the stochastic differential equation

$$\left(\frac{\partial^2 \zeta^{(1)}}{\partial \bar{x}^2} + \frac{\partial^2 \zeta^{(1)}}{\partial \bar{y}^2}\right) - \frac{\partial^2 \zeta^{(1)}}{\partial t^2} = -\frac{1}{h}\frac{\partial}{\partial \bar{x}}\left[b(x,y)\frac{\partial \zeta^{(0)}}{\partial \bar{x}}\right], \tag{3.10}$$

subject to the homogeneous initial conditions

$$\zeta^{(1)}(\bar{x}, \bar{y}, X, Y, t=0) = \frac{\partial \zeta^{(1)}}{\partial t}(\bar{x}, \bar{y}, X, Y, t=0) = 0. \tag{3.11}$$

At $O(\mu^2)$, $\zeta^{(2)}$ is governed by

$$\begin{aligned}\frac{\partial^2 \zeta^{(2)}}{\partial \bar{x}^2} + \frac{\partial^2 \zeta^{(2)}}{\partial \bar{y}^2} - \frac{\partial^2 \zeta^{(2)}}{\partial t^2} &= -2\sqrt{h}\frac{\partial^2 \zeta^{(0)}}{\partial \bar{x}\partial X} - h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} \\ &- \frac{h_X}{2\sqrt{h}}\frac{\partial \zeta^{(0)}}{\partial \bar{x}} - \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial^2 (\zeta^{(0)})^2}{\partial \bar{x}^2} - \frac{h}{3}\frac{\partial^3 \zeta^{(0)}}{\partial \bar{x}^4} \\ &- \frac{1}{h}\left[\frac{\partial}{\partial \bar{x}}\left(b\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right) + \frac{\partial}{\partial \bar{y}}\left(b\frac{\partial \zeta^{(1)}}{\partial \bar{y}}\right)\right] - \frac{\partial b}{\partial y}\frac{\partial \zeta^{(0)}}{\partial Y}\end{aligned} \tag{3.12}$$

The stochastic average is

$$\begin{aligned}\frac{\partial^2 \langle \zeta^{(2)} \rangle}{\partial \bar{x}^2} + \frac{\partial^2 \langle \zeta^{(2)} \rangle}{\partial \bar{y}^2} - \frac{\partial^2 \langle \zeta^{(2)} \rangle}{\partial t^2} &= -2\sqrt{h}\frac{\partial^2 \zeta^{(0)}}{\partial \bar{x}\partial X} - h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} \\ &- \frac{h_X}{2\sqrt{h}}\frac{\partial \zeta^{(0)}}{\partial \bar{x}} - \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial^2 (\zeta^{(0)})^2}{\partial \bar{x}^2} - \frac{h}{3}\frac{\partial^3 \zeta^{(0)}}{\partial \bar{x}^4} \\ &- \frac{1}{h}\left[\frac{\partial}{\partial \bar{x}}\left\langle b\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle + \frac{\partial}{\partial \bar{y}}\left\langle b\frac{\partial \zeta^{(1)}}{\partial \bar{y}}\right\rangle\right]\end{aligned} \tag{3.13}$$

since $\langle b \rangle = 0$.

It will be shown that the right-hand side is a function of $\sigma = \bar{x} - t$. To ensure that $\langle \zeta^{(2)} \rangle$ is solvable it must be set to zero, hence

$$\begin{aligned}-h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} = \frac{\partial}{\partial \sigma}&\left\{2\sqrt{h}\frac{\partial \zeta^{(0)}}{\partial X} + \frac{h_X}{2\sqrt{h}}\zeta^{(0)} + \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial (\zeta^{(0)})^2}{\partial \sigma} + \frac{h}{3}\frac{\partial^3 \zeta^{(0)}}{\partial \sigma^3}\right\} \\ &+ \frac{1}{h}\left[\frac{\partial}{\partial \bar{x}}\left\langle b\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle + \frac{\partial}{\partial \bar{y}}\left\langle b\frac{\partial \zeta^{(1)}}{\partial \bar{y}}\right\rangle\right], \qquad t > 0.\end{aligned} \tag{3.14}$$

This is the new K-P equation which must be subjected to the initial condition (3.8) and the boundary conditions,

$$\zeta^{(0)} \to 0, \quad \sigma, Y \to \pm\infty. \tag{3.15}$$

If the fault is infinitely long and its displacement is uniform in Y, the problem becomes one-dimensional. If random fluctuations are also absent, (3.14) reduces to

$$2\sqrt{h}\frac{\partial \zeta^{(0)}}{\partial X} + \frac{h_X}{2\sqrt{h}}\zeta^{(0)} + \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial (\zeta^{(0)})^2}{\partial \sigma} + \frac{h}{3}\frac{\partial^3 \zeta^{(0)}}{\partial \sigma^3} = 0. \tag{3.16}$$

after integration in σ and applying the condition that the result vanishes at $\sigma \to \infty$. This is the one dimensional KdV equation derived by Johnson [19].

We now evaluate the random forcing terms for one and two dimensional disorder separately in the following sections.

4 One Dimensional Roughness

4.1 Solution for $\zeta^{(1)}$

In the special case where $b = b(x)$ is one dimensional. For brevity we denote $b(x(\bar{x})) = \bar{b}(\bar{x})$. Then $\zeta^{(1)}$ is governed by

$$\frac{\partial^2 \zeta^{(1)}}{\partial \bar{x}^2} - \frac{\partial^2 \zeta^{(1)}}{\partial t^2} = \frac{1}{h}\frac{\partial}{\partial \bar{x}}\left[\bar{b}(\bar{x})\frac{\partial \zeta^{(0)}}{\partial \bar{x}}\right], \tag{4.1}$$

subject to the homogeneous initial conditions

$$\zeta^{(1)}(\bar{x}, X, Y, t=0) = \frac{\partial \zeta^{(1)}}{\partial t}(\bar{x}, X, Y, t=0) = 0. \tag{4.2}$$

From the well known theory of d'Alembert, the solution is given by

$$\begin{aligned}
\zeta^{(1)}(\bar{x}, t) =& -\frac{1}{2h}\int_0^t dt' \int_{\bar{x}-t+t'}^{\bar{x}+t-t'} \frac{\partial}{\partial \bar{x}'}\left(\bar{b}(\bar{x}')\frac{\partial \zeta^{(0)}}{\partial \bar{x}'}\right) d\bar{x}' \\
=& -\frac{1}{2h}\int_0^t \left[\bar{b}(\bar{x}+t-t')\frac{\partial \zeta^{(0)}(\bar{x}+t-t', t')}{\partial \bar{x}} - \bar{b}(\bar{x}-t+t')\frac{\partial \zeta^{(0)}(\bar{x}-t+t', t')}{\partial \bar{x}}\right] dt' \\
=& -\frac{1}{2h}\int_0^t \left[\bar{b}(\bar{x}+t-t')\frac{\partial \zeta^{(0)}(\bar{x}+t-2t')}{\partial \bar{x}} - \bar{b}(\bar{x}-t+t')\frac{\partial \zeta^{(0)}(\bar{x}-t)}{\partial \bar{x}}\right] dt' \\
=& -\frac{1}{2h}\int_0^t \left[\bar{b}(\bar{x}+\xi)\frac{\partial \zeta^{(0)}(\bar{x}-t+2\xi)}{\partial \bar{x}} - \bar{b}(\bar{x}-\xi)\frac{\partial \zeta^{(0)}(\bar{x}-t)}{\partial \bar{x}}\right] d\xi \\
=& \zeta^{(1)}_{bw} + \zeta^{(1)}_{fw}
\end{aligned} \tag{4.3}$$

after replacing $t - t' = \xi$, where

$$\zeta^{(1)}_{bw} = -\frac{1}{2h}\int_0^t \left[\bar{b}(\bar{x}+\xi)\frac{\partial \zeta^{(0)}(\bar{x}-t+2\xi)}{\partial \bar{x}} d\xi, \quad \zeta^{(1)}_{fw} = \frac{1}{2h}\int_0^t \bar{b}(\bar{x}-\xi)\frac{\partial \zeta^{(0)}(\bar{x}-t)}{\partial \bar{x}}\right] d\xi. \tag{4.4}$$

Note that $\zeta^{(0)}(\bar{x}, t) = \zeta^{(0)}(\bar{x}-t)$ has been used.

At $O(\mu^2)$, Eq. (3.14) becomes

$$-h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} = \frac{\partial}{\partial \sigma}\left[2h^{\frac{1}{2}}\frac{\partial \zeta^{(0)}}{\partial X} + \frac{h_X}{2h^{\frac{1}{2}}}\zeta^{(0)} + \frac{1}{3}h\frac{\partial^3 \zeta^{(0)}}{\partial \sigma^3} + \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial {\zeta^{(0)}}^2}{\partial \sigma}\right] + \frac{1}{h}\frac{\partial}{\partial \bar{x}}\left\langle \bar{b}\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle, \tag{4.5}$$

where $\partial \zeta^{(1)}/\partial \bar{y} = 0$ has been used.

Substituting (4.3) into the right hand side of (4.5), we get

$$\left\langle \bar{b}(\bar{x}) \frac{\partial \zeta^{(1)}}{\partial \bar{x}} \right\rangle = \left\langle \bar{b}(\bar{x}) \frac{\partial \zeta^{(1)}_{bk}}{\partial \bar{x}} \right\rangle + \left\langle \bar{b}(\bar{x}) \frac{\partial \zeta^{(1)}_{fw}}{\partial \bar{x}} \right\rangle \tag{4.6}$$

where

$$\left\langle \bar{b}(\bar{x}) \frac{\partial \zeta^{(1)}_{bk}}{\partial \bar{x}} \right\rangle = -\frac{1}{2h} \int_0^\infty \left\langle \bar{b}(\bar{x}) \frac{d\bar{b}(\bar{x}+\xi)}{d\bar{x}} \right\rangle \frac{\partial \zeta^{(0)}(\bar{x}-t+2\xi)}{\partial \bar{x}}\, d\xi$$
$$-\frac{1}{2h} \int_0^\infty \left\langle \bar{b}(\bar{x}) \bar{b}(\bar{x}+\xi) \right\rangle \frac{\partial^2 \zeta^{(0)}(\bar{x}-t+2\xi)}{\partial \bar{x}^2}\, d\xi. \tag{4.7}$$

$$\left\langle \bar{b}(\bar{x}) \frac{\partial \zeta^{(1)}_{fw}}{\partial \bar{x}} \right\rangle = +\frac{1}{2h} \int_0^\infty \left\langle \bar{b}(\bar{x}) \frac{db(\bar{x}-\xi)}{d\bar{x}} \right\rangle \frac{\partial \zeta^{(0)}(\bar{x}-t)}{\partial \bar{x}}\, d\xi$$
$$+\frac{1}{2h} \int_0^\infty \left\langle \bar{b}(\bar{x}) \bar{b}(\bar{x}-\xi) \right\rangle \frac{\partial^2 \zeta^{(0)}(\bar{x}-t)}{\partial \bar{x}^2}\, d\xi, \tag{4.8}$$

(4.9)

To follow the wave for a long time we are interested in finite $\bar{x}-t = O(1)$ but large t. Hence we have replaced the upper limit of the time integral by ∞.

Let Γ be the correlation function[1]

$$\Gamma(\bar{x}-\bar{x}') \equiv \Gamma(p) = \left\langle \bar{b}(\bar{x}) \bar{b}(\bar{x}') \right\rangle, \tag{4.10}$$

whose first derivative has the property (Papoulis, 1965),

$$\frac{d\Gamma(p)}{dp} = \Gamma'(\bar{x}'-\bar{x}) = \left\langle \bar{b}(\bar{x}) \frac{d\bar{b}(\bar{x}')}{d\bar{x}'} \right\rangle = -\left\langle \frac{d\bar{b}(\bar{x})}{d\bar{x}} \bar{b}(\bar{x}') \right\rangle. \tag{4.11}$$

By using (4.10), (4.11) and letting $\sigma = \bar{x}-t$, (4.7) and (4.8) become

$$\left\langle \bar{b}(x) \frac{\partial \zeta_{bk}}{\partial \bar{x}} \right\rangle = -\frac{1}{2h} \int_0^\infty \frac{d\Gamma(\xi)}{d\xi} \frac{\partial \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma}\, d\xi - \frac{1}{2h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma^2}\, d\xi, \tag{4.12}$$

$$\left\langle \bar{b}(\bar{x}) \frac{\partial \zeta_{fw}}{\partial \bar{x}} \right\rangle = -\frac{1}{2h} \int_0^\infty \frac{d\Gamma(\xi)}{d\xi} \frac{\partial \zeta^{(0)}(\sigma)}{\partial \sigma}\, d\xi + \frac{1}{2h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma)}{\partial \sigma^2}\, d\xi. \tag{4.13}$$

Integrating the first term in (4.12) by parts, we get

$$\left\langle \bar{b}(x) \frac{\partial \zeta_{bk}}{\partial \bar{x}} \right\rangle = -\frac{1}{2h} \int_0^\infty \frac{\partial \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma}\, d\Gamma(\xi) - \frac{1}{2h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma^2}\, d\xi$$
$$= \frac{\Gamma(0)}{2h} \frac{\partial \zeta^{(0)}}{\partial \sigma} + \frac{1}{h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma^2}\, d\xi - \frac{1}{2h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma^2}\, d\xi$$
$$= \frac{\Gamma(0)}{2h} \frac{\partial \zeta^{(0)}}{\partial \sigma} + \frac{1}{2h} \int_0^\infty \Gamma(\xi) \frac{\partial^2 \zeta^{(0)}(\sigma+2\xi)}{\partial \sigma^2}\, d\xi. \tag{4.14}$$

[1] Alternatively we can define the correlation as $\widehat{\Gamma}(x'-x) = \langle b(x) b(x') \rangle$ which is related to $\Gamma(\bar{x}-\bar{x}')$ by $\widehat{\Gamma}(x'-x) = \Gamma(h(X)(\bar{x}'-\bar{x}))$.

Integrating (4.13) by parts, we find

$$\left\langle \bar{b}(\bar{x})\frac{\partial \zeta_{fw}}{\partial \bar{x}} \right\rangle = \frac{\Gamma(0)}{2h}\frac{\partial \zeta^{(0)}}{\partial \sigma} + \frac{1}{2h}\frac{\partial^2 \zeta^{(0)}}{\partial \sigma^2}\int_0^\infty \Gamma(\xi)d\xi \tag{4.15}$$

We comment that

$$\int_0^\infty \Gamma(\xi)d\xi = S(0) \tag{4.16}$$

is the value of the wave number spectrum $S(k)$ at $k = 0$. For a random seabed of finite length, the power spectrum computed by Fourier series must have the property $S(0) = 0$. Hence the second term in (4.15) is of no real consequence.

Putting (4.14) and (4.15) in (4.6), we get

$$\left\langle \bar{b}(\bar{x})\frac{\partial \zeta^{(1)}}{\partial \bar{x}} \right\rangle = \frac{\Gamma(0)}{h}\frac{\partial \zeta^{(0)}}{\partial \sigma} + \frac{1}{2h}\frac{\partial^2 \zeta^{(0)}(\sigma)}{\partial \sigma^2}\int_0^\infty \Gamma(\xi)d\xi + \frac{1}{2h}\int_0^\infty \Gamma(\xi)\frac{\partial^2 \zeta^{(0)}(\sigma + 2\xi)}{\partial \sigma^2}d\xi. \tag{4.17}$$

Now the right-hand side of (4.5) is a function of $\sigma = \bar{x} - t$, Using $\partial/\partial \bar{x} = \partial/\partial \sigma$, the evolution equation for $\zeta^{(0)}$ by (4.5) becomes

$$-h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} = \frac{\partial}{\partial \sigma}\left[2h^{\frac{1}{2}}\frac{\partial \zeta^{(0)}}{\partial X} + \frac{h_X}{2h^{\frac{1}{2}}}\zeta^{(0)} + \frac{1}{3}h\frac{\partial^3 \zeta^{(0)}}{\partial \sigma^3} + \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial {\zeta^{(0)}}^2}{\partial \sigma} + \frac{\Gamma(0)}{h}\frac{\partial \zeta^{(0)}}{\partial \sigma} + \frac{1}{2h}\int_0^\infty \Gamma(\xi)\frac{\partial^2 \zeta^{(0)}(\sigma + 2\xi)}{\partial \sigma^2}d\xi\right]. \tag{4.18}$$

In the one dimensional limit ($\partial/\partial Y = 0$) and constant mean depth ($h = 1$), the above equation reduces to the modified KdV equation of Kawahara [25], derived differently by Fourier transform. Without randomness, the two -dimensional K-P equation has been used recently to study tsunami climbing on a continental shelf [35].

5 Two Dimensional Roughness

For two dimensional disorder $\zeta^{(1)}$ is governed by (3.10).For brevity again we write $b(x, y)$ as $\bar{b}(\bar{x}, \bar{y})$ and use the Green function [12]

$$\mathcal{G} = H(t - t' - \rho)G, \quad G(\rho, t - t') = \frac{1}{\sqrt{(t - t')^2 - \rho^2}}, \tag{5.1}$$

where

$$\bar{\mathbf{x}} = (\bar{x}, \bar{y}), \quad \bar{\mathbf{x}}' = (\bar{x}', \bar{y}'); \quad \rho^2 = (\bar{x} - \bar{x}')^2 + (y - y')^2 \tag{5.2}$$

For later use we also write $p = \bar{x}' - \bar{x}; \quad q = y' - y, \quad \tau = t - t'$ so that

$$G = G(\rho, \tau) = \frac{1}{2\pi}\frac{1}{\sqrt{\tau^2 - (p^2 + q^2)}} \tag{5.3}$$

Using Green's formula the solution for $\zeta^{(1)}$ is

$$\zeta^{(1)} = \frac{1}{2\pi h}\int_0^t dt' \iint_{\rho<(t-t')} \frac{\partial}{\partial \bar{x}'}\left(\bar{b}(\bar{\mathbf{x}}')\frac{\partial \zeta^{(0)}}{\partial \bar{x}'}\right) G(\bar{\mathbf{x}},\bar{\mathbf{x}}',t-t')dS' \tag{5.4}$$

Changing the integration variable from t' to τ, the first random forcing term in (3.14) is

$$\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle = \frac{1}{2\pi h}\int_0^t d\tau \left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial}{\partial \bar{x}}\iint_{\rho<\tau}\frac{\partial}{\partial \bar{x}'}\left(\bar{b}(\bar{\mathbf{x}}')\frac{\partial \zeta^{(0)}}{\partial \bar{x}'}\right) G(\bar{\mathbf{x}},\bar{\mathbf{x}}',\tau)dS'\right\rangle \tag{5.5}$$

Since

$$\begin{aligned}
&\bar{x}' = \bar{x} + (\bar{x}'-\bar{x}) = \bar{x}+p; \quad \bar{x}'-t = \bar{x}-t-(\bar{x}'-\bar{x}) = \sigma + p;\\
&\qquad \bar{y}' = \bar{y} + (\bar{y}'-\bar{y}) = \bar{y}+q\\
&\zeta^{(0)} = \zeta^{(0)}(\bar{x}'-t) = \zeta^{(0)}(\sigma+(\bar{x}'-\bar{x})) = \zeta^{(0)}(\sigma+p),\\
&\frac{\partial}{\partial \bar{x}'} = \frac{\partial}{\partial p}, \quad \frac{\partial \zeta^{(0)}(\bar{x}'-t)}{\partial \bar{x}'} = \frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}
\end{aligned} \tag{5.6}$$

Note $dS' = d\bar{x}'\,d\bar{y}' = dp\,dq$. The ensemble average in the integrand of (5.5) is

$$\begin{aligned}
\langle(5.5)\rangle &= \left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial}{\partial \bar{x}}\iint_{\rho<\tau}\frac{1}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial \bar{x}'}\left(\bar{b}(\bar{\mathbf{x}}')\frac{\partial \zeta^{(0)}}{\partial \bar{x}'}\right) dS'\right\rangle\\
&= \left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial}{\partial \bar{x}}\iint_{\rho<\tau}\frac{1}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial p}\left(\bar{b}(\bar{\mathbf{x}}+p)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right) dS'\right\rangle\\
&= \iint_{\rho<\tau}\frac{dpdq}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial p}\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial}{\partial \bar{x}}\left(\bar{b}(\bar{\mathbf{x}}+p)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right)\right\rangle.
\end{aligned} \tag{5.7}$$

The ensemble average in the integrand above is

$$\langle(5.7)\rangle = \langle \bar{b}(\bar{\mathbf{x}})\bar{b}(\bar{\mathbf{x}}+p)\rangle \frac{\partial^2 \zeta^{(0)}(\sigma+p)}{\partial p^2} + \left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \bar{b}(\bar{\mathbf{x}}')}{\partial \bar{x}}\right\rangle \frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p} \tag{5.8}$$

Let Γ be the correlation function

$$\Gamma(p,q) = \langle \bar{b}(\bar{\mathbf{x}})\bar{b}(\bar{\mathbf{x}}')\rangle \tag{5.9}$$

Using the property of a stationary random process,

$$\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \bar{b}(\bar{\mathbf{x}}')}{\partial \bar{x}}\right\rangle = \left\langle \bar{b}((\bar{x},\bar{y}))\frac{\partial \bar{b}(\bar{x}+p,\bar{y}+q)}{\partial \bar{x}}\right\rangle = \frac{\partial \Gamma(p,q)}{\partial p}. \tag{5.10}$$

we get

$$\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle = \frac{1}{2\pi h}\int_0^t d\tau \iint_{\rho<\tau}\frac{dpdq}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial p}\left(\Gamma(p,q)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right). \tag{5.11}$$

Finally

$$\frac{\partial}{\partial \bar{x}}\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle = \frac{1}{2\pi h}\frac{\partial}{\partial \bar{x}}\int_0^t d\tau \iint_{\rho<\tau}\frac{dpdq}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial p}\left(\Gamma(p,q)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right). \tag{5.12}$$

Similarly

$$\begin{aligned}\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \zeta^{(1)}}{\partial \bar{y}}\right\rangle &= \frac{1}{2\pi h}\int_0^t d\tau \left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial}{\partial \bar{y}}\iint_{\rho<\tau}\frac{\partial}{\partial \bar{x}'}\left(\bar{b}(\bar{\mathbf{x}}')\frac{\partial \zeta^{(0)}}{\partial \bar{x}'}\right)G(\bar{\mathbf{x}},t;\bar{\mathbf{x}}',t')dS'\right\rangle \\ &= \frac{1}{2\pi h}\int_0^t d\tau \iint_{\rho<\tau}\frac{dpdq}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial q}\left(\Gamma(p,q)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right)\end{aligned} \tag{5.13}$$

is a function of $\sigma = \bar{x} - t$ but not of $\bar{y}$ since the primary direction of wave propagation is x. Therefore

$$\frac{\partial}{\partial \bar{y}}\left\langle \bar{b}(\bar{\mathbf{x}})\frac{\partial \zeta^{(1)}}{\partial \bar{y}}\right\rangle = 0. \tag{5.14}$$

The evolution equation (3.14) becomes finally

$$\begin{aligned}-h\frac{\partial^2 \zeta^{(0)}}{\partial Y^2} &= \frac{\partial}{\partial \sigma}\left\{2\sqrt{h}\frac{\partial \zeta^{(0)}}{\partial X} + \frac{h_X}{2\sqrt{h}}\zeta^{(0)} + \frac{\epsilon}{\mu^2}\frac{3}{2h}\frac{\partial (\zeta^{(0)})^2}{\partial \sigma} + \frac{h}{3}\frac{\partial^3 \zeta^{(0)}}{\partial \sigma^3}\right\} \\ &\quad + \frac{1}{2\pi h^2}\frac{\partial}{\partial \bar{x}}\int_0^t d\tau \iint_{\rho<\tau}\frac{dpdq}{\sqrt{\tau^2-\rho^2}}\frac{\partial}{\partial p}\left(\Gamma(p,q)\frac{\partial \zeta^{(0)}(\sigma+p)}{\partial p}\right)\end{aligned} \tag{5.15}$$

which governs the trans ocean propagation of tsunami from a slender fault toward the shallow coast, as long as the total distance is not long enough to warrant the inclusion of earth curvature.

It is useful to rederive the one-dimensional resulat as a special case where $\Gamma = \Gamma(p)$. Eq.(5.11) becomes

$$\begin{aligned}\left\langle \bar{b}(\bar{x})\frac{\partial \zeta^{(1)}}{\partial \bar{x}}\right\rangle =& \frac{1}{2\pi h}\int_0^\infty d\tau \int_{-\tau}^{\tau} dp \int_{-\sqrt{\tau^2-p^2}}^{\sqrt{\tau^2-p^2}}\frac{dq}{\sqrt{\tau^2-p^2-q^2}} \\ &\times \frac{\partial}{\partial p}\left[\Gamma(p)\frac{\partial^2 \zeta^{(0)}(\xi+p)}{\partial p^2}\right].\end{aligned} \tag{5.16}$$

Since

$$\int_{-\sqrt{\tau^2-p^2}}^{\sqrt{\tau^2-p^2}}\frac{dq}{\sqrt{\tau^2-p^2-q^2}} = \pi, \tag{5.17}$$

we have

$$\begin{aligned}
\left\langle \bar{b}(\bar{x})\frac{\partial \zeta^{(1)}}{\partial \bar{x}} \right\rangle &= \frac{1}{2h}\int_0^\infty d\tau \int_{-\tau}^{\tau} dp \frac{\partial}{\partial p}\left[\Gamma(p)\frac{\partial^2\zeta^{(0)}(\bar{x}+p)}{\partial p^2}\right] \\
&= \frac{1}{2h}\int_0^\infty d\tau \left(\int_{-\tau}^{0} + \int_0^\tau\right) dp \frac{\partial}{\partial p}\left[\Gamma(p)\frac{\partial^2\zeta^{(0)}(\bar{x}+p)}{\partial \bar{x}^2}\right] \\
&= \frac{1}{2h}\int_0^\infty d\tau \int_0^\tau dp \frac{\partial}{\partial p}\left[\Gamma(p)\frac{\partial^2\zeta^{(0)}(\bar{x}+p)}{\partial \bar{x}^2} - \Gamma(p)\frac{\partial^2\zeta^{(0)}(\bar{x}-p)}{\partial \bar{x}^2}\right] \\
&= \frac{1}{2h}\int_0^\infty d\tau \left\{ \Gamma(\tau)\left[\frac{\partial^2\zeta^{(0)}(\bar{x}+\tau)}{\partial \bar{x}^2} + \frac{\partial^2\zeta^{(0)}(\bar{x}-\tau)}{\partial \bar{x}^2}\right] - 2\Gamma(0)\frac{\partial^2\zeta^{(0)}(\bar{x})}{\partial \bar{x}^2}\right\} \\
&= \frac{1}{2h}\left\{ \int_0^\infty \Gamma(\tau)\left[\frac{\partial^2\zeta^{(0)}(\sigma+2\tau)}{\partial \sigma^2} + \frac{\partial^2\zeta^{(0)}(\sigma)}{\partial \sigma^2}\right] d\tau - 2\Gamma(0)\int_0^\infty \frac{\partial^2\zeta^{(0)}(\sigma+\tau)}{\partial \tau^2} d\tau\right\} \\
&= \frac{1}{2h}\left\{ \int_0^\infty \Gamma(\tau)\left[\frac{\partial^2\zeta^{(0)}(\sigma+2\tau)}{\partial \sigma^2} + \frac{\partial^2\zeta^{(0)}(\sigma)}{\partial \sigma^2}\right] d\tau + 2\Gamma(0)\frac{\partial\zeta^{(0)}}{\partial \sigma}\right\}, \qquad (5.18)
\end{aligned}$$

which agrees with (4.17) obtained by using the d'Alembert solution.

6 Concluding Remarks

To provide a theoretical basis for investigating the trans-ocean propagation of tsunami originated from a two -dimensional fault, we use asymptotic analysis to derive the modified Kadomtsev-Petviashivili equation with variable coefficients. The fault is assumed to be elongated so that the propagation is strongly focussed in two opposite directions normal to the fault. Along the principal direction the sea depth is assumed to consist of a slowly varying mean and a random perturbation of known statistical properties. This equation involves only two independent variables X and Y, and is of parabolic type. Its dependence on σ is known from the initial fault displacement. Numerical solution is considerably simpler than the quasi-linear Boussinesq equations in x, y and t. The simplicity also comes from the fact that the modified K-P equation only involves the correlation function of the random seabed instead of the random roughness $b(x, y)$ itself. The equation can be solved in a finite domain moving at the local wave speed instead of the whole domain from the fault to beach. Results will be reported in the near future.

Appendix: Equivalence of Random Forcing Term in 1D

For constant mean depth ($h = 1$), Mei & Li [34] found the random forcing term to be,

$$\begin{aligned}
\left\langle b(x)\frac{\partial \zeta^{(1)}}{\partial x}\right\rangle =& \Gamma(0)\frac{\partial\zeta^{(0)}}{\partial\sigma} + \frac{1}{4}\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}\int_{-\infty}^{\infty}\Gamma(\xi)d\xi \\
&+\frac{1}{8}\int_{-\infty}^{\infty}\Gamma\left(\frac{\sigma-\sigma'}{2}\right)\frac{\partial^2\eta_0}{\partial\sigma'^2}d\sigma' + \frac{1}{4}\int_{-\infty}^{\infty}P\left(\frac{\sigma-\sigma'}{2}\right)\frac{\partial^3\eta_0}{\partial\sigma'^3}d\sigma' \qquad (A.1)
\end{aligned}$$

where $P(\xi)$ is

$$P(\xi) = \int_{|\xi|}^{\infty} \Gamma(u)\, du, \tag{A.2}$$

In this form the physical influence of random scattering is clear. The first term represents reduction of phase speed. The second and the third terms give rise to dissipation while the last term augments dispersion. Let us prove that it is equivalent to (4.17).

Note first that

$$\frac{dP(\xi)}{d\xi} = -\frac{d|\xi|}{d\xi}\Gamma(\xi) = -\mathrm{sgn}(\xi)\Gamma(\xi). \tag{A.3}$$

Let us start from (4.17)

$$\begin{aligned}
&\int_0^{\infty} \Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi\\
=&\frac{1}{2}\int_0^{\infty}\big[1+\mathrm{sgn}(\xi)\big]\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi\\
=&\frac{1}{2}\int_{-\infty}^{\infty}\big[1+\mathrm{sgn}(\xi)\big]\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi\\
=&\frac{1}{2}\int_{-\infty}^{\infty}\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi - \frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, dP(\xi)\\
=&\frac{1}{2}\int_{-\infty}^{\infty}\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi + \frac{1}{2}\int_{-\infty}^{\infty}P(\xi)\frac{\partial^3\zeta^{(0)}}{\partial\sigma^2\partial\xi}(\sigma+2\xi)\, d\xi\\
=&\frac{1}{2}\int_{-\infty}^{\infty}\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi + \int_{-\infty}^{\infty}P(\xi)\frac{\partial^3\zeta^{(0)}}{\partial\sigma^3}(\sigma+2\xi)\, d\xi.
\end{aligned} \tag{A.4}$$

Eq. (4.17) becomes

$$\begin{aligned}
\left\langle b(x)\frac{\partial\zeta^{(1)}}{\partial x}\right\rangle =&\Gamma(0)\frac{\partial\zeta^{(0)}}{\partial\sigma} + \frac{1}{2}\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}\int_0^{\infty}\Gamma(\xi)d\xi + \frac{1}{4}\int_{-\infty}^{\infty}\Gamma(\xi)\frac{\partial^2\zeta^{(0)}}{\partial\sigma^2}(\sigma+2\xi)\, d\xi\\
&+\frac{1}{2}\int_{-\infty}^{\infty}P(\xi)\frac{\partial^3\zeta^{(0)}}{\partial\sigma^3}(\sigma+2\xi)\, d\xi.
\end{aligned} \tag{A.5}$$

By letting $\sigma' = \sigma + 2\xi$, (A.1) is recovered.

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