COMPARISON OF KANE'S DYNAMICAL EQUATIONS TO TRADITIONAL DYNAMICAL TECHNIQUES

by

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ABSTRACT

A method of formulating equations of motion known as Kane's Dynamical Equations is compared to traditional methods of dynamics, specifically Lagrange's equations. A derivation of both Lagrange's equations and Kane's Dynamical Equations is presented. In addition, when a system is holonomic and when using Kane's Dynamical Equations where the generalized speeds are chosen to be the total time derivatives of the generalized coordinates, it is shown that Lagrange's equations are identical to Kane's Dynamical Equations. Example problems are presented using both methods. Conclusions are drawn to show that in deriving the equations of motion for simple dynamic problems, the most appropriate choices for the generalized speeds are the total time derivatives of the generalized coordinates. When this is the case for a holonomic system, Kane's Dynamical Equations are equivalent to Lagrange's equations. Finally, the advantage that Kane's Dynamical Equations afford is their ability to eliminate constraint forces while still being able to formulate equations of motion for nonholonomic systems.

Thesis Supervisor: Dr. James H. Williams, Jr.
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I. Introduction

Understanding the world about us is the goal of a scientist. Describing the physical motion of objects in mathematical terms is the goal of a dynamicist.

One traditional method used to accomplish this understanding is Newton's laws, another is Lagrange's equations. Lagrange's equations, developed in 1788 by Joseph Louis Lagrange utilize variational principles in order to derive equations of motion for a system. Thus, Lagrange's equations have been a tool in dynamics for over 200 years.

Recently, within the last 40 years, a new method of analyzing problems of motion has arisen. This method, known as Kane's Dynamical Equations, has been developed by Professor Thomas R. Kane of Stanford University. Kane states that this method replaces the virtual quantities of variational mechanics with specific, known quantities.[5] By doing this, the equations of motion for a system can not only be developed more easily, but they can also be solved more efficiently, than when traditional methods are used.

This report endeavors to compare Kane's method of dynamics to traditional methods on an elementary level. First, The focus will be on the derivation of the two methods, and then on their application. Initially, an historical background of the origin of Kane's equations will be recounted. Then the derivation of Lagrange's equations from D'Alembert's principle will be reviewed, followed by the derivation of Kane's Dynamical Equations. Then a group of example problems that are solved using both Lagrange's equations and Kane's Dynamical Equations will be presented. The final section will give a comparative analysis and discussion of the two techniques, based both on their derivations and applications.

II. Background

After more than 200 years of effectively using Lagrange's equations to formulate problems of dynamics, why is there a necessity for a new method of performing dynamics? According to Kane[5] traditional dynamics involves a great amount of "experience, intuition, and aptitude" in order to solve problems. Kane believes that the formulation of equations of motion using variational principles is difficult to do with large complicated systems. In fact, Kane's Dynamical Equations were developed from dynamical applications for the aerospace industry.

Kane's idea is to replace the "*mystical*" quantities in lagrangian analysis known as virtual displacements with a known quantity. A virtual displacement is defined to be an infinitesimal displacement that a system can perform without violating the kinematic constraints of the system. Usually these displacements are defined by using the variational operator on the generalized coordinates. For example, if θ were a generalized coordinate for a system, then the virtual variation of θ would be $\delta\theta$. Consideration of these quantities is necessary in order to form equations of motion using Lagrange's equations.

By replacing these virtual variations with a more tangible quantity, Kane's goal is to create a method of dynamics that is both more systematic and more physically intuitive. Specifically, Kane replaces virtual variations with partial velocities.

Kane also states that Lagrange's equations are inadequate in formulating equations of motion for complex multi-body dynamics problems that can be solved easily.[3] Kane feels that his method can produce the equations of motion for a system in such a way that they can be solved more easily than equations of motion produced by Lagrange's equation.

III. Review of Lagrange's Equations

Lagrange's equations are derived in the following manner. Let $\mathbf{r}_i$ be the position vector of the ith particle at any instant in time. This position vector is expressed in terms of generalized coordinates

$$\mathbf{r}_i = f(q_1, q_2, \ldots, q_{3N-k}, t) \tag{3.1}$$

where q_j are the generalized coordinates, N is the number of particles in the system, and k is the number of kinematical constraints. For each particle, three scalars are needed to define the particle's position at any instant in time. However, when the position of a particle is restricted due to geometric constraints within the system, the number of scalars needed to define the particle's position can be reduced. Furthermore, the number of scalars is reduced when a holonomic constraint can be determined. A holonomic constraint is an equation that expresses a geometric constraint within a system in terms of the generalized coordinates and time. Thus, the number of degrees of freedom that a system has is defined to be

$$p = 3N - k \tag{3.2}$$

The velocity of the ith particle can be expressed as a sum of partial derivatives of the ith position vector $\mathbf{r}_i$

$$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_{k=1}^{p} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \tag{3.3}$$

D'Alembert's principle states that equations of motion can be obtained in an inertial reference frame if the actual external forces are in equilibrium with the inertia forces acting on each particle of the system

$$\sum_{i=1}^{N} \mathbf{F}_i - \dot{\mathbf{p}}_i = 0 \tag{3.4}$$

where ($\dot{}$) denotes the total time derivative, $\mathbf{F}_i$ is a force acting on a particle, $\mathbf{p}_i$ is the linear momentum of the particle, and N is the number of particles in the system. Taking the dot product of both sides of equation 3.4 with the virtual displacement $\delta\mathbf{r}_i$ gives

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{3.5}$$

where $\delta\mathbf{r}_i$ is an admissible variation of the particle. The first dot product of equation 3.5 will be examined first

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i \tag{3.6}$$

The arbitrary virtual displacement $\delta\mathbf{r}_i$ can be related to the generalized virtual displacements δq_i by

$$\delta\mathbf{r}_i = \sum_{j=1}^{p} \frac{\partial\mathbf{r}_i}{\partial q_j} \delta q_j \tag{3.7}$$

The generalized coordinates can then be used to express the virtual work of the force $\mathbf{F}_i$

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{p} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{p} Q_j \, \delta q_j \qquad (3.8)$$

where Q_j are the components of the generalized force.

Now, the second dot product of equation 3.2 will be examined. Assuming the particles of the system to have a constant mass m_i, this dot product can be expressed as

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \qquad (3.9)$$

Substitution of equation 3.7 for the virtual displacements into equation 3.9 yields

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \qquad (3.10)$$

Examining the dot product

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \qquad (3.11)$$

and using the identity for integration by parts, this term becomes

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^{N} \left\{ \frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right\} \qquad (3.12)$$

First, in the last term of equation 3.12, the differentiation with respect to t and q_j can be interchanged to obtain

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \frac{\partial \mathbf{v}_i}{\partial q_j} \tag{3.13}$$

Second, examining equation 3.3 for the velocity of a particle, the following relation can be developed

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}\right)}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{3.14}$$

Substituting equations 3.13 and 3.14 into equation 3.12 gives

$$\sum_{i=1}^{N} \mathrm{m}_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_{i=1}^{N}\left\{\frac{d}{dt}\left(\mathrm{m}_i\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) - \mathrm{m}_i\mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right\} \tag{3.15}$$

and using this result, equation 3.10 is equivalent to

$$\sum_{j=1}^{p}\left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_{i=1}^{N}\frac{1}{2}\mathrm{m}_i(\mathbf{v}_i \cdot \mathbf{v}_i)\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{N}\frac{1}{2}\mathrm{m}_i(\mathbf{v}_i \cdot \mathbf{v}_i)\right)\right\}\delta q_j \tag{3.16}$$

Using the equation for the system's kinetic coenergy, T^*

$$T^* = \sum_{i=1}^{N}\frac{1}{2}\mathrm{m}_i(\mathbf{v}_i \cdot \mathbf{v}_i) \tag{3.17}$$

and equation 3.5, D'Alembert's principle becomes

$$\sum_{j=1}^{p}\left[\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_j}\right) - \frac{\partial T^*}{\partial q_j} - Q_j\right]\delta q_j = 0 \tag{3.18}$$

If, the system is holonomic, then equations 3.1 implicitly contain the constraint conditions through the independence of the variables q_j. Thus, each virtual displacement is independent from one another, and equation 3.18 can be satisfied only when

$$\sum_{j=1}^{p} \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_j} \right) - \frac{\partial T^*}{\partial q_j} = Q_j \tag{3.19}$$

is satisfied.

Now assume that the forces $\mathbf{F}_i$ are derivable from a scalar potential function, V, which is a function $f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$

$$\mathbf{F}_i = -\nabla_i V \tag{3.20}$$

The generalized forces Q_j can now be written as

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_{i=1}^{N} \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{3.21}$$

This expression is exactly the same as a partial derivative of a function $f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ with respect to q_j

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{3.22}$$

Substitution of equation 3.22 into equation 3.19 yields

$$\sum_{j=1}^{p} \frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_j} \right) - \frac{\partial (T^* - V)}{\partial q_j} = 0 \tag{3.22}$$

Also, since V, as defined, does not depend on $\dot{q}_j$, a V can also be included in the partial derivative with respect to $\dot{q}_j$

$$\sum_{j=1}^{p} \frac{d}{dt}\left(\frac{\partial (T^* - V)}{\partial \dot{q}_j}\right) - \frac{\partial (T^* - V)}{\partial q_j} = 0 \qquad (3.23)$$

Finally, defining a term L known as the lagrangian to be

$$L = T^* - V \qquad (3.24)$$

Lagrange's equations are known as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, \ldots, p) \qquad (3.25)$$

However, if not all the forces acting on the system are conservative, then Lagrange's equations are written in the following manner

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad (j = 1, \ldots, p) \qquad (3.26)$$

where the lagrangian accounts for the potential of the conservative forces, and Q_j accounts for the forces not derivable from a potential.

IV. Derivation of Kane's Dynamical Equations

In this section, Kane's Dynamical Equations will be derived following Kane.[3] After this derivation, parallels will be drawn between Kane's Dynamical Equations and Lagrange's Equations.

If $\mathbf{R}_i$ is the resultant of all contact forces and distance forces acting on the ith particle P_i of a system S and $\mathbf{a}_i$ is the acceleration of P_i in an inertial reference frame **N**, then, in accordance with Newton's second law,

$$\mathbf{R}_i - m_i \mathbf{a}_i = 0 \qquad (i = 1, \ldots, \upsilon) \qquad (4.1)$$

where m_i is the mass of P_i and υ is the number of particles in S.

The partial velocities of a particle are defined in the following manner. If $\mathbf{b}_i$ is the position vector of the ith particle in a system S with p degrees of freedom, then time differentiation of $\mathbf{b}_i$

$$\frac{d\mathbf{b}_i}{dt} = \dot{\mathbf{b}}_i = \sum_{r=1}^{p} \frac{\partial \mathbf{b}_i}{\partial q_r} \dot{q}_r + \frac{\partial \mathbf{b}_i}{\partial t} \qquad (r = 1, \ldots, p) \qquad (4.2)$$

Equation 4.2 is an expression for the velocity of the ith particle, and is identical to equation 3.3. The generalized coordinates are chosen in the same manner as if Lagrange's equations were being used, and the number of degrees of freedom for the system, p, can be determined using equation 3.5. Furthermore, the velocity of the ith particle P_i can be expressed in an alternative manner as

$$\dot{\mathbf{b}}_i = \tilde{\mathbf{v}}_r^{P_i} u_r + \tilde{\mathbf{v}}_t^{P_i} \qquad (r = 1, \ldots, p) \qquad (4.3)$$

where $\tilde{\mathbf{v}}_r^{P_i}$ is the r[th] nonholonomic partial velocity of the particle P_i, $\tilde{\mathbf{v}}_t^{P_i}$ is known as the remainder* , and u_r are the generalized speeds . Equation 4.3 is central to Kane's Dynamical Equations, because most often, the partial velocities are derived from the observation of this equation.

The dot product of equations 4.1 with the partial velocities $\tilde{\mathbf{v}}_r^{P_i}$ of P_i in **N** gives

$$\sum_{i=1}^{\upsilon} \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i + \sum_{i=1}^{\upsilon} \tilde{\mathbf{v}}_r^{P_i} \cdot (-m_i \mathbf{a}_i) = 0 \qquad (i = 1, \ldots, p) \qquad (4.4)$$

where p is the number of degrees of freedom of the system.

In accordance with the following definitions:

$$\tilde{F}_r = \sum_{i=1}^{\upsilon} \tilde{\mathbf{v}}_r^{P_i} \cdot \mathbf{R}_i \qquad (i = 1, \ldots, p) \qquad (4.5)$$

$$\tilde{F}_r^* = \sum_{i=1}^{\upsilon} \tilde{\mathbf{v}}_r^{P_i} \cdot (-m_i \mathbf{a}_i) \qquad (i = 1, \ldots, p) \qquad (4.6)$$

the first sum of equations 4.4 $\tilde{F}_r$; is the generalized active the second sum $\tilde{F}_r^*$ is the generalized inertial forces. Thus, equation 4.4 leads directly to

$$\tilde{F}_r + \tilde{F}_r^* = 0 \qquad (4.7)$$

Equations 4.7 are known as Kane's Dynamical Equations.

* The naming of this term is a recent addition to Kane's work, and was expressed during a phone conversation with Professor Kane on April 24, 1991.

Some relations can be drawn between equations 4.2 and 4.3, which are both expressions for the velocity of a particle. First, examining the terms of equations 4.2 and 4.3, the last terms are equivalent

$$\frac{\partial \mathbf{b}_i}{\partial t} = \tilde{\mathbf{v}}_t^{P_i} \tag{4.8}$$

Equation 4.8 shows that for a particle's velocity according to equation 4.3, the remainder, $\tilde{\mathbf{v}}_t^{P_i}$ is equivalent to the partial derivative of the position vector with respect to time. Second, since these two quantities are equal, then the following relation must also be true

$$\frac{\partial \mathbf{b}_i}{\partial q_r} \dot{q}_r = \tilde{\mathbf{v}}_r^{P_i} u_r \tag{4.9}$$

Many times the generalized speeds u_r are chosen to be the total time derivatives of the generalized coordinates. When this is the case, then

$$\frac{\partial \mathbf{b}_i}{\partial q_r} = \tilde{\mathbf{v}}_r^{P_i} \tag{4.10}$$

Recalling equation 3.7 from the derivation of Lagrange's equations

$$\delta \mathbf{r}_i = \sum_{j=1}^{p} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \tag{4.11}$$

which relates the arbitrary virtual variations to the generalized virtual displacements, a correlation between Kane's Dynamical Equations and Lagrange's equations can be shown.

In the derivation of Lagrange's equations, D'Alembert's principle is dotted on both sides by the virtual variations, $\delta \mathbf{r}_i$ and then equation 4.14 is used to substitute for these variations. This manipulation of D'Alembert's principle results in the following equation

$$\left\{ \sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - \sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right\} \delta q_j = 0 \qquad (4.12)$$

Furthermore, if the system is holonomic, the virtual variations δq_j are arbitrary, and equation 4.12 can only be satisfied when

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} - \sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = 0 \qquad (4.13)$$

is satisfied.

Returning to Kane's Dynamical Equations, equation 4.4 can be altered by substituting equation 4.10 for the partial velocities

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \frac{\partial \mathbf{b}_i}{\partial q_r} + \sum_{i=1}^{N} (- \mathrm{m}_i \, \mathbf{a}_i) \cdot \frac{\partial \mathbf{b}_i}{\partial q_r} = 0 \qquad (4.14)$$

Equation 4.14 proves that when using Kane's Dynamical Equations, if the mass of the system is constant and the generalized speeds are chosen to be the time derivatives of the generalized coordinates, then Kane's Dynamical Equations are equivalent to Lagrange's Equations.

V. Example Problems

A. Simple Pendulum Problem Using Kane's Dynamical Equations

Problem:

A pendulum is free to move in a plane. A mass m is fixed to the end of a thin rod of length L. The mass of the rod can be assumed to be negligible. Using θ, the angle of the pendulum with respect to its downward equilibrium position, derive the equation(s) of motion of the mass m with respect to an earth-fixed reference frame.

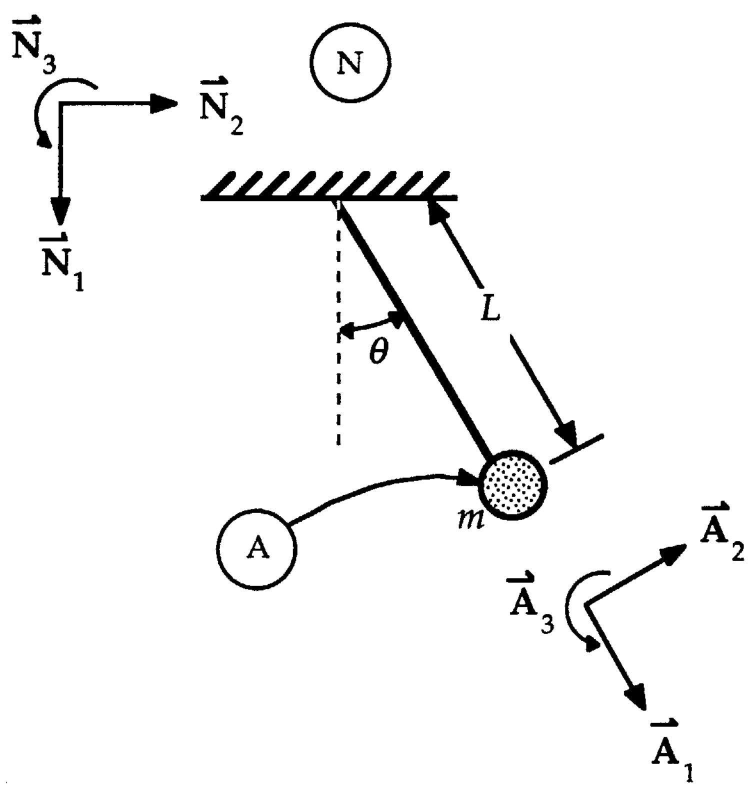

Figure 5.a.1: A planar pendulum of mass m and length L.

Terms Used in Solution of Simple Pendulum Problem Using Kane's Method

N_1, N_2, N_3 Set of mutually perpendicular unit vectors attached to an earth-fixed reference frame, where N_1 and N_2 are in the plane of the motion of the pendulum, and N_1 is oriented downward.

$\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ — Set of mutually perpendicular unit vectors attached to the center of the mass m at the end of the pendulum, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are in the plane of the motion of the pendulum, and $\mathbf{A}_1$ is oriented along the length of the pendulum.

θ — The angle between the pendulum and vertical, as defined in the earth-fixed reference frame.

q_1 — Kane's notation for the generalized coordinate of this problem.

u_1 — Kane's notation for the generalized speed. For this problem, it is simply the time derivative of the generalized coordinate.

F_r — The generalized active force. This force is the first term of Kane's Dynamical Equations, and represents the external forces acting on the particles and/or rigid bodies of the system.

F_r^* — The generalized inertial force. This force is the second term in Kane's dynamical Equations, and represents the effects of the forces on the particles and/or rigid bodies of the system ($F_r = -F_r^*$).

$\mathbf{v}^{P_i}$ — The velocity of the i[th] particle.

$\mathbf{v}_r^{P_i}$ — This is the partial velocity of the i[th] particle with respect to the r[th] generalized speed. This term is used to calculate the generalized active and inertial forces.

${}^{N}\mathbf{v}^{m}$ — This is Kane's notation for the velocity of the mass m in the **N** reference frame, or earth-fixed reference frame.

$\mathbf{R}_i$ — The resultant of all contact forces (for example friction) and distance forces (for example gravitational and magnetic) on the i[th] particle.

$\mathbf{R}_i^*$ — The inertia force, which is equal to the mass of the i[th] particle times its acceleration.

${}^{N}\mathbf{a}^{m}$ — This is Kane's notation for the acceleration of mass m in the **N** reference frame, or earth-fixed reference frame.

Solution:

Initially, global and local coordinate systems must be assigned to the elements in the problem. As shown in Figure 1, **N** will serve as the earth-fixed reference frame or inertial reference frame, and **A** will be attached to the center of mass m. Both coordinate systems consist of a triad of mutually perpendicular vectors, and both $\mathbf{N}_3$ and $\mathbf{A}_3$ project directly out of the page.

According to Kane, only one generalized coordinate would exist. This is due to the fact that the system consists of only one particle, the mass m, and one degree of freedom, angular rotation in one plane. Thus, the angle θ is the only necessary information required to determine the status of the system at any given instant in time. The generalized coordinate would be:

$$q_1 = \theta \qquad (5.a.1)$$

The next step is to define the generalized speeds. In this case, the generalized speed is simply the time-derivative of the generalized coordinate:

$$u_1 = \dot{\theta} \qquad (5.a.2)$$

After defining the reference frames, the generalized coordinates, and the generalized speeds, Kane's Dynamical Equations can be used to derive the equation(s) of motion.

$$\tilde{F}_r + \tilde{F}_r^* = 0 \qquad (5.a.3)$$

For this case, r is equal to 1, because there is only one generalized coordinate, resulting in a single equation of motion for the system. Also, the tilde notation can be removed, signifying that the system is holonomic. Equation 5.a.3 simplifies to

$$F_1 + F_1^* = 0 \qquad (5.a.4)$$

First, we will determine the generalized active forces, F_r. These forces are all the forces acting on the mass *m*.

$$F_r = \sum_{i=1}^{\upsilon} v_r^{P_i} \cdot \mathbf{R}_i \qquad (5.a.5)$$

For a system with multiple particles or rigid bodies, a separate generalized active force would be derived for each of the particles or bodies.

In order to isolate the external forces of each system element with respect to each of the generalized coordinates, two terms must be calculated. First the partial velocities of the particle with respect to each of the generalized speeds must be determined. Obviously in order to do this the velocities of each particle of the system must be determined:

$$\mathbf{v}^{P_i} \equiv \text{velocity of the i}^{\text{th}} \text{ particle} \qquad (5.a.6)$$

For our system of the pendulum, we have only one particle to analyze, the mass m. By inspection, the velocity of the mass is determined to be:

$$\mathbf{v}^{P_i} \equiv {}^{N}\mathbf{v}^{m} \equiv \text{velocity of the mass } m \text{ in the reference frame } \mathbf{N} \qquad (5.a.7)$$

$${}^{N}\mathbf{v}^{m} = u_1 \, L \, \mathbf{A}_2 \qquad (5.a.8)$$

Once the velocity of the particle is determined the next step is to take the partial of the velocity with respect to each of the generalized speeds. According to Kane this is a necessary step , because "the use of partial angular velocities and partial velocities greatly facilitates the formulation of equations of motion". The partial velocities of a system are defined to be:

$\mathbf{v}_r^{P_i} \equiv$ the partial velocity of the ith particle with respect to the rth generalized speed

For our system the one partial velocity is:

$$\mathbf{v}_1^m = L\,\mathbf{A}_2 \tag{5.a.9}$$

Now that the first term of the generalized active force is defined, the second term $\mathbf{R}_i$ must be defined. $\mathbf{R}_i$ is the resultant of all contact forces (for example, friction forces) and distance forces (for example, gravitational and magnetic forces). Once again, with our system we have only one resultant force to calculate, because there is only one particle in the system. The best way to determine this force is through a free body diagram:

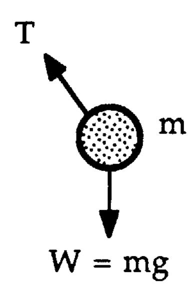

Figure 5.a2: Free body diagram of the forces acting on the mass *m*.

By examining the diagram the resultant force, $\mathbf{R}_1$ on the mass m is:

$$\mathbf{R}_1 = m\,g\,\mathbf{N}_1 - \mathrm{T}\,\mathbf{A}_1 = m\,g\,(\cos q_1\,\mathbf{A}_1 - \sin q_1\,\mathbf{A}_2) - \mathrm{T}\,\mathbf{A}_1 \tag{5.a.10}$$

Thus, referring to equation 5.a.5 the only active generalized force for the pendulum system is:

$$F_1 = L\,\mathbf{A}_2 \cdot [\,m\,g\,(\cos q_1\,\mathbf{A}_1 - \sin q_1\,\mathbf{A}_2) - T\mathbf{A}_1\,] \qquad (5.a.11)$$

$$F_1 = -L\,m\,g\sin q_1 \qquad (5.a.12)$$

The next task is to derive the second term of Kane's Dynamical Equations, the generalized inertia force:

$$F_r^* = \sum_{i=1}^{\upsilon} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i^* \qquad (5.a.13)$$

where $\mathbf{R}_i^*$ is the inertia force acting on particle P_i.

We have already determined the partial velocities when the generalized active force was calculated as shown in equation 17. The only term that needs to be defined is the inertia force, $\mathbf{R}_i^*$. Kane defines the inertia force to be:

$$\mathbf{R}_i^* = -M\,\mathbf{a}_i \qquad (5.a.14)$$

where M is the mass of the i^{th} particle, $\mathbf{a}_i$ is the particle's acceleration. The acceleration of the mass m is simply the time derivative of the velocity:

${}^{N}\mathbf{a}^{m}$ ≡ the acceleration of the mass m in the **N** reference frame

$${}^{N}\mathbf{a}^{m} = \frac{{}^{N}d\,{}^{N}\mathbf{v}^{m}}{dt} = \frac{{}^{N}d\,(u_1\,L\,\mathbf{A}_2)}{dt} \qquad (5.a.15)$$

$${}^{N}\mathbf{a}^{m} = \frac{{}^{N}d\,[\,u_1\,L\,(\cos q_1\,\mathbf{N}_1 + \sin q_1\,\mathbf{N}_2)]}{dt} \qquad (5.a.16)$$

$${}^{N}\mathbf{a}^{m} = \dot{u}_1\,L\,(\cos q_1\,\mathbf{N}_1 + \sin q_1\,\mathbf{N}_2) + u_1^2\,L\,(-\sin q_1\,\mathbf{N}_1 + \cos q_1\,\mathbf{N}_2) \qquad (5.a.17)$$

$$ {}^{N}\mathbf{a}^{m} = \dot{u}_1 L \mathbf{A}_2 + u_1^2 L \mathbf{A}_1 \tag{5.a.18} $$

Now that the acceleration of the mass m has been determined, the inertia force of m can be calculated. Recalling equation 5.a.14

$$ \mathbf{R}_1^* = -m (\dot{u}_1 L \mathbf{A}_2 - u_1^2 L \mathbf{A}_1)] \tag{5.a.19} $$

and substituting equation 5.a.19 into 5.a.13 the generalized inertial force can be resolved

$$ \mathrm{F}_1^* = L \mathbf{A}_2 \cdot [-m (\dot{u}_1 L \mathbf{A}_2 - u_1^2 L \mathbf{A}_1)] \tag{5.a.20} $$

$$ \mathrm{F}_1^* = -m L^2 \dot{u}_1 \tag{5.a.21} $$

Finally, after having computed the generalized active forces and the generalized inertia forces, Kane's Dynamical Equations can be determined. Recalling equation 5.a.4

$$ \mathrm{F}_1 + \mathrm{F}_1^* = 0 \tag{5.a.22} $$

$$ -m L \mathrm{g} \sin q_1 - m L^2 \dot{u}_1 = 0 \tag{5.a.23} $$

$$ \dot{u}_1 + \frac{\mathrm{g}}{L} \sin q_1 = 0 \tag{5.a.24} $$

Replacing the generalized speed and generalized coordinate with equations 5.a.1 and 5.a.2

$$ \ddot{\theta} + \frac{\mathrm{g}}{L} \sin \theta = 0 \tag{5.a.25} $$

which is the only equation of motion for the system.

B. Simple Pendulum Problem Using Lagrange's Equations

Problem:

A pendulum is free to move in a plane. A mass m is fixed to the end of a thin rod of length L. The mass of the rod can be assumed to be negligible as compared to mass fixed to the rod. Using θ, the angle of the pendulum with respect to vertical, derive the equation(s) of motion of the mass m with respect to an earth-fixed reference frame.

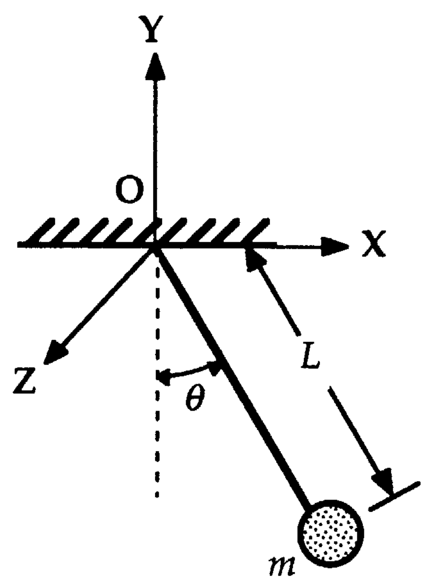

Figure 5.b.1: A planar pendulum of mass m and length L.

Solution:

An inertial reference frame is attached to the point at which the pendulum is fixed. The Y axis extends upwards from that point and the **Z** axis projects directly out of the page. The generalized coordinate of this problem will be θ, just as in the solution using Kane's method.

$$q_1 = \theta \tag{5.b.1}$$

The next step is to construct the generalized force Ξ_1. This is accomplished by observing the system and looking for any non-conservative forces associated with the generalized coordinate θ. Realizing that the kinetic coenergy and potential energy are conserved within the system and using the variational operator, the generalized force is determined to be

$$\Xi_1 \, \delta q_1 = 0 \tag{5.b.2}$$

$$\Xi_1 = 0 \tag{5.b.3}$$

Now, the lagrangian L for the system must be determined, using the equation

$$L = T^* - V \tag{5.b.4}$$

where T^* is the kinetic coenergy of the system and V is the potential energy. The kinetic coenergy of the system is defined to be

$$T^* = \sum_{i=1}^{N} \frac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i) \tag{5.b.5}$$

For the system of the pendulum, the velocity of the mass m is found by taking the time derivative of the position vector $\mathbf{R}_1$

$$\mathbf{R}_1 = L \sin\theta \, \mathbf{X} - L \cos\theta \, \mathbf{Y} \tag{5.b.6}$$

$$\mathbf{v}_1 = \dot{\mathbf{R}}_1 = \dot{\theta} L (\cos\theta \, \mathbf{X} - \sin\theta \, \mathbf{Y}) \tag{5.b.7}$$

Using this velocity, the kinetic coenergy of the system, T^* is

$$T^* = \frac{1}{2} m_1 [\dot{\theta} L (\cos\theta \, \mathbf{X} - \sin\theta \, \mathbf{Y}) \cdot \dot{\theta} L (\cos\theta \, \mathbf{X} - \sin\theta \, \mathbf{Y})] \tag{5.b.8}$$

$$T^* = \frac{1}{2} m_1 \; [(\dot{\theta} L)^2 (\cos^2\theta + \sin^2\theta)] \tag{5.b.9}$$

$$T^* = \frac{1}{2} m_1 (\dot{\theta} L)^2 \tag{5.b.10}$$

The potential energy, in this instance, is invested in the height of the pendulum's mass

$$V = m\, g\, h \tag{5.b.11}$$

where h is the height of the mass above its stable position.

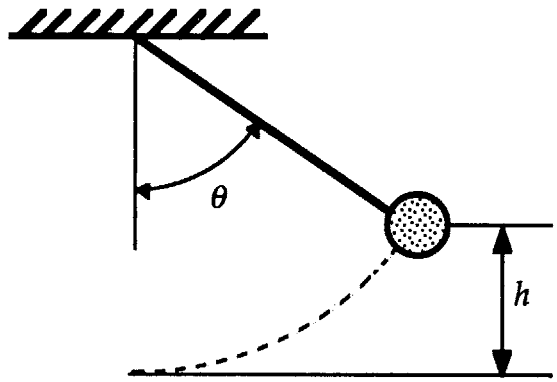

Figure 5.b.2: Description of the height of the pendulum

The distance h is

$$h = L\,(\,1 - \cos\theta\,) \tag{5.b.12}$$

$$V = m\, g\, L\,(\,1 - \cos\theta\,) \tag{5.b.13}$$

Substituting for the potential energy and kinetic coenergy, and the generalized forces, the lagrangian is

$$L = \frac{1}{2}\, m\,(\,\dot{\theta}\, L\,)^2 - m\, g\, L\,(\,1 - \cos\theta\,) \tag{5.b.14}$$

Substitution of the lagrangian into Lagrange's equation results in the following equation of motion

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0 \tag{5.b.15}$$

C. Mass-Spring-Damper Problem Using Kane's Dynamical Equations

Problem:

A mass M is constrained to move in a plane. The mass is connected to a spring and a damper which are both connected to ground. When this system is at equilibrium, the distance x is equal to 0. The distance x is defined to be the distance of the center of mass M from its position when the system is at equilibrium. Using x, the distance of the mass from its equilibrium position, derive the equation(s) of motion of the mass M with respect to an earth-fixed reference frame when the mass is subjected to a force $f(t)$.

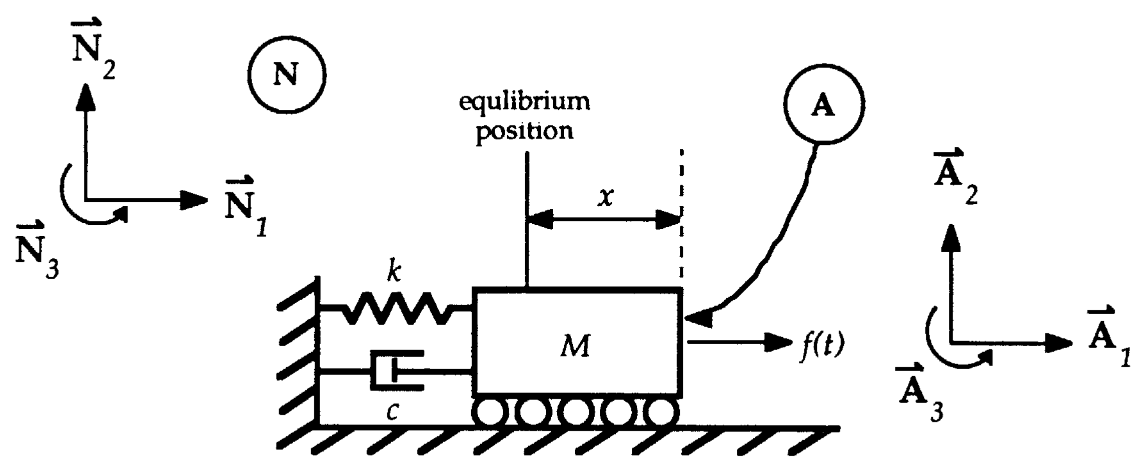

Figure 5.c.1: A mass-spring-damper system.

As before, the first item of business is to define the necessary reference frames. The two reference frames, **A** and **N**, are an intermediate reference frame and an inertial reference frame, respectively. **A** is attached to the center of the mass M, and moves laterally with the mass.

As in the simple pendulum problem, there will be only one generalized coordinate, which is the distance x.

$$q_1 = x \tag{5.c.1}$$

The generalized speed will simply be the total time derivative of the generalized coordinate

$$u_1 = \dot{x} \tag{5.c.2}$$

Since there is only one generalized coordinate, Kane's Dynamical Equations will take the form of

$$F_1 + F_1^* = 0 \tag{5.c.3}$$

The generalized active force, F_1 of equation 5.c.3 will be determined first.

$$F_r = \sum_{i=1}^{\upsilon} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i \tag{5.c.4}$$

The velocity of the mass M is defined to be

$$^N\mathbf{v}^M = u_1 \, \mathbf{N}_1 \tag{5.c.5}$$

By inspection of equation 5.c.5, the partial velocity of the of the mass M with respect to the only generalized speed u_1 is

$$^N\mathbf{v}_1^M = \mathbf{N}_1 \tag{5.c.6}$$

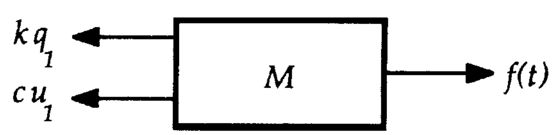

Figure 5.c.2: Free body diagram of the forces acting on the mass M.

By examining the free body diagram of the mass M, $\mathbf{R}_1$ the resultant of all contact and distance forces acting on M is

$$\mathbf{R}_1 = f(t)\, \mathbf{N}_1 - kq_1 \, \mathbf{N}_1 - cu_1 \, \mathbf{N}_1 \tag{5.c.7}$$

$$\mathbf{R}_1 = (f(t) - kq_1 - cu_1)\, \mathbf{N}_1 \tag{5.c.8}$$

Using equations 5.c.6 and 5.c.8, the generalized active force is

$$F_1 = \mathbf{N}_1 \cdot (f(t) - kq_1 - cu_1)\, \mathbf{N}_1 \tag{5.c.9}$$

$$F_1 = f(t) - kq_1 - cu_1 \qquad (5.c.10)$$

The next step is to calculate the generalized inertia forces, in this case, $\mathbf{F}_1^*$. Recalling that

$$F_r^* = \sum_{i=1}^{\nu} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i^* \qquad (5.c.11)$$

The only quantity that needs to be determined is $\mathbf{R}_1^*$, the inertia forces acting on M

$$\mathbf{R}_1^* = -M\,\mathbf{a}_1 \qquad (5.c.12)$$

The acceleration $\mathbf{a}_1$ of the mass M is the time derivative of its velocity

$$\mathbf{a}_1 = \frac{{}^N d\,{}^N\mathbf{v}^M}{dt} = \frac{{}^N d\,[\,u_1\,\mathbf{N}_1\,]}{dt} = \dot{u}_1\,\mathbf{N}_1 \qquad (5.c.13)$$

Thus, the inertia force acting on M is

$$\mathbf{R}_1^* = -M\,\dot{u}_1\,\mathbf{N}_1 \qquad (5.c.14)$$

Using equations 5.c.6 and 5.c.14, the generalized inertia force for the mass M is

$$F_1^* = \mathbf{N}_1 \cdot -M\,\dot{u}_1\,\mathbf{N}_1 = -M\,\dot{u}_1 \qquad (5.c.15)$$

Finally, substitution of the generalized active force (5.c.10) and of the generalized inertia force (5.c.15) into Kane's Dynamical Equations for this system (5.c.3) result in the system's only equation of motion

$$M\,\dot{u}_1 + c\,u_1 + k\,q_1 = f(t) \qquad (5.c.16)$$

Back-substituting of the original values for the generalized speed and generalized coordinate gives

$$M\,\ddot{x} + c\,\dot{x} + k\,x = f(t) \qquad (5.c.17)$$

D. Mass-Spring-Damper Problem Using Lagrange's Equations

Problem:

A mass M is constrained to move in a plane. The mass is connected to a spring and a damper which are both connected to ground. When this system is at equilibrium, the distance x is equal to 0. Using x, the distance of the mass from its equilibrium position, derive the equation(s) of motion of the mass M with respect to an earth-fixed reference frame when the mass is subjected to a force $f(t)$.

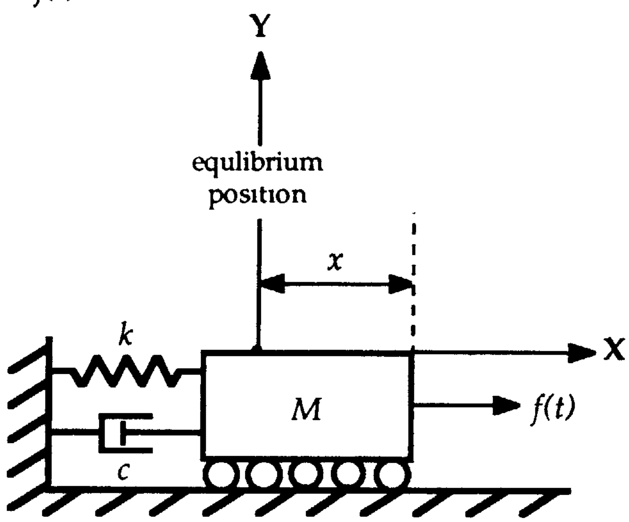

Figure 5.d.1: A mass-spring-damper system.

Only one reference frame is needed for this problem. An inertial reference frame is fixed to the point at which mass M is in equilibrium when no external forces are acting upon it. The generalized coordinate of this problem will be x, just as in the solution using Kane's method.

$$q_1 = x \tag{5.d.1}$$

The next step is to construct the generalized force Ξ_1. This is accomplished by observing the system and looking for any non-conservative forces associated with the generalized coordinate x. In this case, there are two non-conservative forces in the system: the system input $f(t)$ and the force associated with the damper.

$$\Xi_1 \, \delta x = f(t)\, \delta x - c\, \dot{x}\, \delta x \qquad (5.d.2)$$

$$\Xi_1 = f(t) - c\, \dot{x} \qquad (5.d.3)$$

Now, the lagrangian L for the system must be determined using the equation

$$L = T^{*} - V \qquad (5.d.4)$$

The kinetic coenergy, T^{*} is

$$T^{*} = \frac{1}{2} M \, (\mathbf{v} \cdot \mathbf{v}) \qquad (5.d.5)$$

where $\mathbf{v}$ is the velocity of the mass. The velocity of the mass is simply the time derivative of the generalized coordinate x, which would make the kinetic coenergy

$$T^{*} = \frac{1}{2} M \, (\dot{x}\, \mathbf{X} \cdot \dot{x}\, \mathbf{X}) = \frac{1}{2} M \dot{x}^2 \qquad (5.d.6)$$

The potential energy of the system is simply the energy invested into the spring. The constitutive relationship for a spring is

$$\mathbf{F} = k\, \mathbf{x} \qquad (5.d.7)$$

The energy invested to change the position of the spring is

$$V = \frac{1}{2} k\, x^2 \qquad (5.d.8)$$

Using equations 5.d.6 and 5.d.8, the lagrangian is

$$L = \frac{1}{2} M x^2 - \frac{1}{2} k\, x^2 \qquad (5.d.9)$$

Substitution of the lagrangian into Lagrange's equations gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \Xi_x \qquad (5.d.10)$$

$$\frac{d}{dt}(M\dot{x}) - (-kx) = f(t) - c\dot{x} \qquad (5.d.11)$$

$$M\ddot{x} + c\dot{x} + kx = f(t) \qquad (5.d.12)$$

which is the only equation of motion for the system.

E. Sphere on Rotating Platform Using Kane's Dynamical Equations

Problem:

A uniform sphere of mass m and radius a is on top of a rotating platform. The platform is rotating about a fixed point **O** at a rotational velocity of Ω. The sphere rolls on the platform without slipping. Find the equation(s) of motion for the center of the sphere, C.

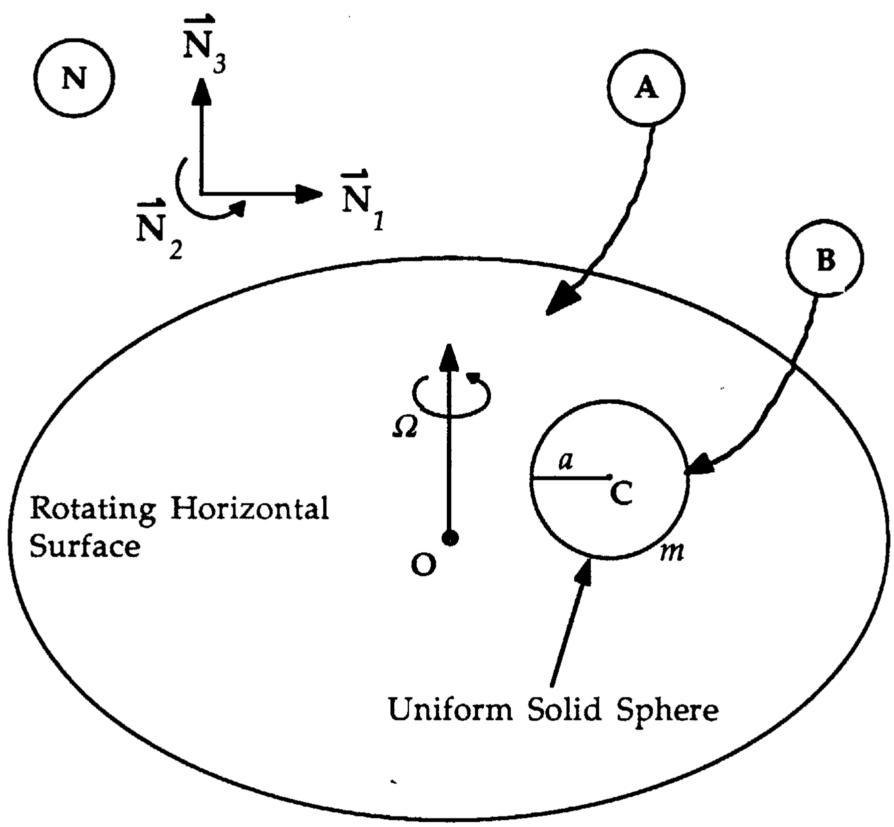

Figure 5.e.1: A sphere on rotating platform.

Solution:

Initially, a reference frame is assigned to three bodies. **N** is an inertial reference frame, **A** is a reference frame fixed to the rotating platform, and **B** is a reference frame fixed to the sphere. The next step is to make intelligent choices for the generalized coordinates and the generalized speeds.

For this problem, there are five degrees of freedom for the sphere. It can rotate about all three inertial axes, and can have translational motion in the N_1 and N_2 directions. However, two of the rotational motions are governed by the no-slip condition for the sphere, or in an alternative manner, the two translational motions can be governed by two of the rotational motions. Thus, leaving three degrees of freedom for the system.

The three generalized coordinates that will be chosen are two translational motions in the N_1 and N_2 directions and one rotational motion about the N_3 axis. The generalized coordinates are defined to be

$$q_1 = x \tag{5.e.1}$$

$$q_2 = y \tag{5.e.2}$$

$$q_3 = \theta \tag{5.e.3}$$

and can be seen in figure 5.e.2.

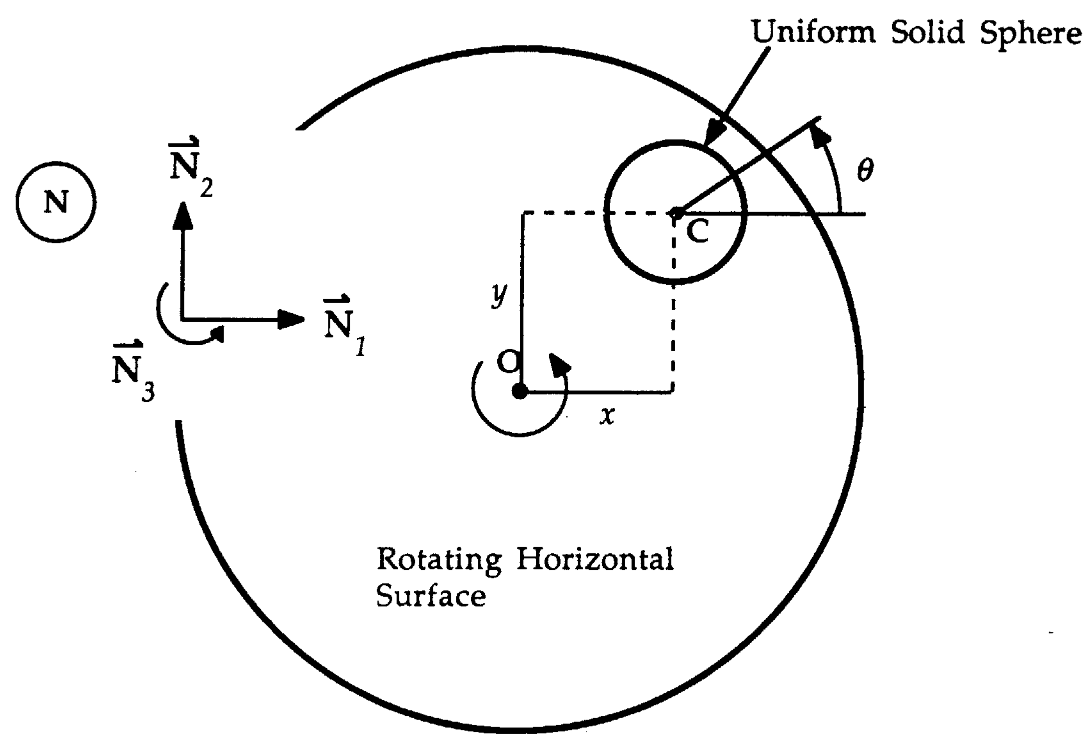

Figure 5.e.2: Diagram of generalized coordinates.

Now that the generalized coordinates are defined, the generalized speeds must be defined. In this case the generalized speeds will be the total time derivatives of the generalized coordinates.

$$u_1 = \dot{q}_1 \qquad (5.e.4)$$

$$u_1 = \dot{q}_2 \qquad (5.e.5)$$

$$u_1 = \dot{q}_3 \qquad (5.e.6)$$

The next step is to find the appropriate quantities for Kane's Dynamical Equations

$$F_r + F_r^* = 0 \qquad (r = 1, \ldots, 3) \qquad (5.e.7)$$

However, in this example, the method of formulating these equations will be slightly different. Whereas before, first the generalized active forces were determined and then the generalized inertial forces were determined separately, now the individual quantities needed for each of these forces will be calculated. This translates to a method by which the velocities, partial velocities, accelerations, reaction forces, and inertial forces are determined first, and then the generalized active forces and generalized inertial forces are calculated, which results directly to Kane's Dynamical Equations for the motion of the system.

The first quantities which will be calculated, are the translational and rotational velocities of the sphere. The translational velocity of the sphere will be the velocity of the center of the sphere, B^*

$$^N\mathbf{v}^{B^*} = u_1 \mathbf{N}_1 + u_2 \mathbf{N}_2 \qquad (5.e.8)$$

and the rotational velocity of the sphere is a function of the three rotations about each inertial axis

$$^N\omega^B = \omega_1 \mathbf{N}_1 + \omega_2 \mathbf{N}_2 + \omega_3 \mathbf{N}_3 \qquad (5.e.9)$$

However, two of the rotations, ω_1 and ω_2 are functions of the translational generalized coordinates q_1 and q_2, due to the no-slip rolling constraint. This constraint can be determined by the velocity of the contact point of the sphere and the rotating platform. At this point, the velocity of the sphere and the velocity of the rotating platform are equal. The velocity of the sphere at the contact point is

$$^{N}\mathbf{v}^{c} = {}^{N}\mathbf{v}^{B^*} + [\,{}^{N}\omega^{B} \times (-a\,\mathbf{N}_3)] \tag{5.e.10}$$

and the velocity of the rotating platform at the contact point is

$$^{N}\mathbf{v}^{c} = \Omega\,\mathbf{N}_3 \times (q_1\,\mathbf{N}_1 + q_2\,\mathbf{N}_2) \tag{5.e.11}$$

Equating these values for the velocity of the contact point and then solving to find ω_1 and ω_2 gives

$$^{N}\omega^{B} \times (-a\,\mathbf{N}_3) = \Omega\,\mathbf{N}_3 \times (q_1\,\mathbf{N}_1 + q_2\,\mathbf{N}_2) - {}^{N}\mathbf{v}^{B^*} \tag{5.e.12}$$

$$-a\,({}^{N}\omega^{B} \times \mathbf{N}_3) = -\Omega\,q_2\,\mathbf{N}_1 + \Omega\,q_1\,\mathbf{N}_2 - u_1\,\mathbf{N}_1 - u_2\,\mathbf{N}_2 \tag{5.e.13}$$

$$^{N}\omega^{B} \times \mathbf{N}_3 = \frac{\Omega\,q_2 - u_1}{a}\,\mathbf{N}_1 + \frac{-\Omega\,q_1 + u_2}{a}\,\mathbf{N}_2 \tag{5.e.14}$$

where ω_1 and ω_2 are

$$\omega_1 = \frac{\Omega\,q_1 - u_2}{a} \tag{5.e.15}$$

$$\omega_2 = \frac{\Omega\,q_2 + u_1}{a} \tag{5.e.16}$$

Using ω_1 and ω_2, the rotational velocity of the sphere is

$$^{N}\omega^{B} = \frac{\Omega\,q_1 - u_2}{a}\,\mathbf{N}_1 + \frac{\Omega\,q_2 + u_1}{a}\,\mathbf{N}_2 + u_3\,\mathbf{N}_3 \tag{5.e.17}$$

Now a table can be constructed of the partial velocities for the sphere. These partial velocities are of the rotational and translational velocities, equations 5.e.17 and 5.e.8, with respect to each of the generalized speeds.

	u_1	u_2	u_3
${}^N\mathbf{v}^{B^*}$	$\mathbf{N}_1$	$\mathbf{N}_2$	0
${}^N\boldsymbol{\omega}^B$	$\frac{1}{a}\mathbf{N}_2$	$-\frac{1}{a}\mathbf{N}_1$	$\mathbf{N}_3$

Table 5.e.1: The partial velocities of ${}^N\mathbf{v}^{B^*}$ and ${}^N\boldsymbol{\omega}^B$ with respect to the generalized speeds.

Now that the partial angular and translational velocities have been calculated, the next step is to determine the sphere's accelerations, both angular and translational. The translational acceleration of the sphere is just the time-derivative of the velocity.

$$ {}^N\mathbf{a}^{B^*} = \frac{d}{dt}(u_1\,\mathbf{N}_1 + u_2\,\mathbf{N}_2) = \dot{u}_1\,\mathbf{N}_1 + \dot{u}_2\,\mathbf{N}_2 \qquad (5.e.18) $$

and the angular acceleration is the time derivative of the angular velocity

$$ {}^N\boldsymbol{\alpha}^B = \frac{d}{dt}\left(\frac{\Omega q_1 - u_2}{a}\,\mathbf{N}_1 + \frac{\Omega q_2 + u_1}{a}\,\mathbf{N}_2 + u_3\,\mathbf{N}_3\right) \qquad (5.e.19) $$

$$ {}^N\boldsymbol{\alpha}^B = \frac{\Omega u_1 - \dot{u}_2}{a}\,\mathbf{N}_1 + \frac{\Omega u_2 + \dot{u}_1}{a}\,\mathbf{N}_2 + \dot{u}_3\,\mathbf{N}_3 \qquad (5.e.20) $$

The next step is to calculate the reaction forces acting on the center of the sphere. The only reaction force acting on the center of the sphere is the effect of gravity.

$$ \mathbf{R} = m\,\mathrm{g}\,\mathbf{N}_3 \qquad (5.e.21) $$

Now, the inertial force will be determined. Since the sphere is being examined as a rigid body, there are rotational and translational components to the inertial forces. The translational component of the inertial force is

$$ \mathbf{R}^* = m\,{}^N\mathbf{a}^{B^*} = m\,(\dot{u}_1\,\mathbf{N}_1 + \dot{u}_2\,\mathbf{N}_2) \qquad (5.e.22) $$

The inertial torque, denoted as $\mathbf{T}^*$,is found using

$$\mathbf{T}^* = -\sum_{i=1}^{\upsilon} m_i \mathbf{r}_i \times \mathbf{a}_i \qquad (5.e.23)$$

$\mathbf{T}^*$ can also be written in another form using the angular accelerations and angular velocities of the rigid bodies

$$\mathbf{T}^* = -\boldsymbol{\alpha} \cdot \mathbf{I} - \boldsymbol{\omega} \times \mathbf{I} \cdot \boldsymbol{\omega} \qquad (5.e.24)$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\omega}$ are the angular acceleration of a rigid body in **N** and the angular velocity of a rigid body in **N**, and **I** is central inertial tensor of a rigid body. In order to express $\mathbf{T}^*$ in a useful manner, the following definitions will be needed. Assuming that $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ are mutually perpendicular unit vectors that are aligned along the principle axes of a rigid body, then α_j, ω_j, and I_j are defined to be

$$\alpha_j = \boldsymbol{\alpha} \cdot \mathbf{c}_j \qquad (5.e.25)$$

$$\omega_j = \boldsymbol{\omega} \cdot \mathbf{c}_j \qquad (5.e.26)$$

$$I_j = \mathbf{c}_j \cdot \mathbf{I} \cdot \mathbf{c}_j \qquad (5.e.27)$$

Using these components of the angular acceleration, angular velocity and the inertia tensor, equation 5.e.24 can be written as

$$\begin{aligned} \mathbf{T}^* = & -[\, \alpha_1 I_1 - \omega_2 \omega_3 (I_2 - I_3)\,]\, \mathbf{c}_1 \\ & -[\, \alpha_2 I_2 - \omega_3 \omega_1 (I_3 - I_1)\,]\, \mathbf{c}_2 \\ & -[\, \alpha_3 I_3 - \omega_1 \omega_2 (I_1 - I_2)\,]\, \mathbf{c}_3 \end{aligned} \qquad (5.e.28)$$

For this problem, by taking advantage of the symmetry of the sphere, equation 5.e.28 reduces to

$$\mathbf{T}^* = -\alpha_1 I_1 - \alpha_2 I_2 - \alpha_3 I_3 \qquad (5.e.29)$$

Using this equation, and the definitions of equations 5.e.25, 5.e.26, and 5.e.27, the inertial torque is

$$\mathbf{T}^* = \frac{I_1}{a}(\Omega u_1 - \dot{u}_2)\mathbf{N}_1 + \frac{I_2}{a}(\Omega u_2 + \dot{u}_1)\mathbf{N}_2 + I_3 \dot{u}_3 \mathbf{N}_3 \quad (5.e.30)$$

The next step is to refer to the table of partial velocities, the resultant force **R**, the inertial force $\mathbf{R}^*$, and the inertial torque $\mathbf{T}^*$ in order to determine the generalized active forces and the generalized inertial forces. Recalling that the generalized active forces are defined as

$$F_r = \sum_{i=1}^{\upsilon} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i \qquad (5.e.31)$$

and using the partial velocities in table 5.e.1, the generalized active forces in this case are

$$F_1 = \mathbf{N}_1 \cdot m\,g\,\mathbf{N}_3 = 0 \qquad (5.e.32)$$

$$F_2 = \mathbf{N}_2 \cdot m\,g\,\mathbf{N}_3 = 0 \qquad (5.e.33)$$

$$F_3 = 0 \cdot m\,g\,\mathbf{N}_3 = 0 \qquad (5.e.34)$$

Taking into consideration the inertial effects of rigid body rotation, the equation for the generalized active force becomes

$$F_r^* = \sum_{i=1}^{\nu} \omega_r^{P_i} \cdot \mathbf{T}^* + \sum_{i=1}^{\nu} \mathbf{v}_r^{P_i} \cdot \mathbf{R}^* \qquad (5.e.35)$$

Performing the dot products, the generalized inertia forces become

$$F_1^* = m\,\dot{u}_1 + \frac{I_2}{a^2}(\dot{u}_1 - \Omega u_2) \qquad (5.e.36)$$

$$F_2^* = m\,\dot{u}_2 + \frac{I_1}{a^2}(\dot{u}_2 - \Omega u_1) \qquad (5.e.37)$$

$$F_3^* = I_3\,\dot{u}_3 \qquad (5.e.38)$$

Realizing that the generalized active forces are zero, the generalized inertia forces are actually the equations of motion for the system. Substituting for the generalized speeds and for the moment of inertia of the sphere, which is

$$I_1 = I_2 = I_3 = \frac{2}{5} m\, a^2 \tag{5.e.39}$$

the equations of motion for the center of the sphere become

$$\ddot{x} + \frac{2}{7}\,\Omega\,\dot{y} = 0 \tag{5.e.40}$$

$$\ddot{y} - \frac{2}{7}\,\Omega\,\dot{x} = 0 \tag{5.e.41}$$

$$\ddot{\theta} = 0 \tag{5.e.42}$$

VI. Discussion of Methods and Conclusions

This section will be devoted to discussing the differences and similarities between Kane's Dynamical Equations and Lagrange's Equations.

Referring to the example section, the most immediate difference between the two methods is the relative length of using each method. For the pendulum problem and the mass-spring-damper problem, the derivation of equations of motion using Kane's method was longer than using lagrangian dynamics. There is a primary difference when formulating equations of motion with Kane's Dynamical Equations as opposed to using Lagrange's Equations. When calculating the generalized inertia forces using Kane's Dynamical Equations

$$\tilde{F}_r^* = \sum_{i=1}^{v} \tilde{v}_r^{P_i} \cdot (-m_i a_i) \qquad (i = 1, \ldots, p) \qquad (6.1)$$

the acceleration $\mathbf{a}_i$ must be computed for each particle of the system. In addition, when there are rigid bodies in a system, as was the case with the example of the sphere on the rotating platform, the rotational accelerations must also be computed. The bottom line is that it can be time-consuming to calculate an acceleration. Since accelerations must be computed, it would seem that in certain instances, such as in the example of the pendulum problem, the formulation of the equations of motion can be algebraically time-consuming when Kane's Dynamical Equations are used. Whereas, when Lagrange's equations are used, only the velocities need to be calculated.

Furthermore, Kane strives to remove the mystery from dynamics.[5] In order to accomplish this, Kane replaces the intangible virtual variations of lagrangian dynamics with the more tangible quantities of partial velocities. By doing this, Kane believes that dynamics becomes a systematic science, removing the intuition and experience that is necessary for complex dynamics problems. However, it still seems rather arbitrary as to how the generalized speeds are chosen for a problem. For a simple problem, it seems that the most intelligent choice for the generalized speeds is the total time derivatives of the generalized coordinates. In the appendix a problem where the generalized speeds are not simply the total time derivatives of the generalized coordinates is presented. This problem came directly from a problem set that was used in Professor Kane's class at Stanford. The benefit of assigning the generalized speeds to be something other than the total time derivatives of the generalized speeds becomes apparent in this problem. The expressions for the velocity of the particle and its acceleration become much simpler, because the generalized speeds have replaced lengthy terms in each of the expressions. However, there is no explanation as to when the generalized speeds should be assigned in this manner. It would seem that it is a matter of experience in formulating equations of motion with Kane's Dynamical Equations to be able to decide when the generalized speeds should not be assigned to be simply the total time derivatives of the generalized coordinates.

This paper did little to prove or disprove the notion that Kane's Dynamical Equations produce equations of motion that can be more easily solved than equations of motion produced by Lagrange's Equations. All the problems that were addressed resulted in equations of motion from both methods that were identical. However, the dynamics class at Stanford University that Professor Kane teaches is intensely computer oriented. Many of the problems that are taught in the class are examples of the power that a program called AUTOLEV has in solving equations of motion.

The uniqueness of Kane's equations is also debatable: How unique are Kane's Dynamical Equations? As shown in section IV, the derivation of Kane's Dynamical Equations, these equations can be made to look exactly like Lagrange's Equations. However, Kane's equations can be used to formulate equations of motion for nonholonomic systems, which is an advantage over Lagrange's equations.

Finally, Kane's equations are useful in dynamics. They present dynamics in a new light. They are effective in removing the constraint forces from the system and still preserving an ability to formulate equations of motion for nonholonomic systems. The focus when using Kane's equations is on the rate of change in a system, rather than the equilibrium status of the system. The rate of change of a system is presented by Kane's use of the partial velocities, instead of the virtual variations used by Lagrange's equations. More dynamical quantities are determined when a problem is formulated using Kane's Dynamical Equations. When formulating a problem with Kane's equations, the translational and rotational accelerations must be computed of all the particles and rigid bodies of the system in addition to the translational and rotational velocities.

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Appendix

Additional Example Problems

The following section contains three additional example problems. The first problem is similar to example 5.a of the pendulum, except that the pendulum is now modelled as a rigid body. The second problem is another simple dynamics problem of a bead that is on a tight, thin wire subjected to a force $f(t)$ at a prescribed angle θ. The last problem is a pendulum that is modelled to have a spring modulus of k along its length. The unique concept that is shown by this problem is that the generalized speeds are not simply the total time derivatives of the generalized coordinates.

Motion of a Pendulum Modelled as a Rigid Body

Problem:

A pendulum is free to move in a plane, and is fixed to an inertial reference frame at the point O. The pendulum is a rigid body of mass m and length L, and the center of mass is located at point P, a distance of $L/2$ from either end of the pendulum. Using θ, the angle of the pendulum with respect to its downward equilibrium position, derive the equation(s) of motion of the pendulum with respect to an earth-fixed reference frame.

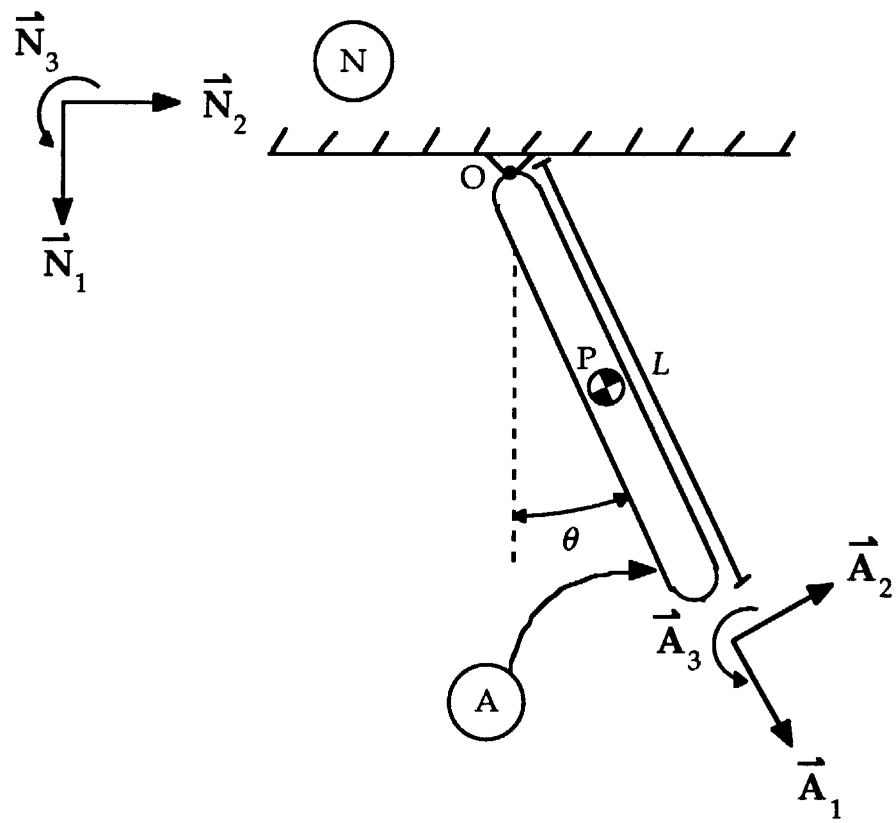

Figure A1.1: Diagram of a rigid body pendulum.

Solution:

This problem is identical to example 5.a. The only difference is that the pendulum is now viewed as a rigid body.

Two reference frames are assigned: an inertial reference frame **N**, and a reference frame **A** fixed to the rigid body of the pendulum, where $\mathbf{A}_1$ is aligned along the length of the pendulum.

The only generalized coordinate is θ

$$q_1 = \theta \tag{1}$$

and the generalized speed is the total time derivative of the generalized coordinate

$$u_1 = \dot{q}_1 \tag{2}$$

The velocity of the center of mass of the pendulum is

$$^{N}\mathbf{v}^{P} = u_1 \frac{L}{2} \mathbf{A}_2 \tag{3}$$

where $^{N}\mathbf{v}^{P}$ is the velocity of the point P in the **N** reference frame. Examining equation 3, the partial velocity of the pendulum with respect to the generalized speed u_1 is

$$^{N}\mathbf{v}_1^{P} = \frac{L}{2} \mathbf{A}_2 \tag{4}$$

The acceleration of the pendulum's center of mass is found in the exact same manner as the acceleration for the pendulum in example 5.a, which is given by

$$^{N}\mathbf{a}^{P} = \dot{u}_1 \frac{L}{2} \mathbf{A}_2 + u_1^2 \frac{L}{2} \mathbf{A}_1 \tag{5}$$

The angular velocity of the rigid body of the pendulum is

$$^{N}\omega^{P} = u_1 \mathbf{N}_3 = u_1 \mathbf{A}_3 \tag{6}$$

and by examining equation 6 the partial angular velocity with respect to the generalized speed u_1 is

$$^{N}\omega_1^{P} = \mathbf{A}_3 \tag{7}$$

The angular acceleration of the pendulum is

$$ {}^{N}\alpha^{P} = \dot{u}_1 \, \mathbf{A}_3 \tag{8} $$

By examining a free-body diagram of the center of mass of the pendulum, the active force $\mathbf{R}_1$ can be determined.

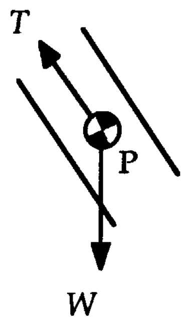

Figure A1.2: Free body diagram of the forces acting on the center of mass of the pendulum.

The active force $\mathbf{R}_1$ is given by

$$ \mathbf{R}_1 = W\,\mathbf{N}_1 - T\,\mathbf{A}_1 = m\,g\,(\cos q_1\,\mathbf{A}_1 - \sin q_1\,\mathbf{A}_2) - T\,\mathbf{A}_1 \tag{9} $$

The last quantities that are needed to formulate the equation of motion for this system are the inertia force $\mathbf{R}^*$ and the inertia torque $\mathbf{T}^*$. The inertial torque must be computed because the pendulum is being modelled as a rigid body, which means that the equation for the generalized inertial force is

$$ F_r^* = \sum_{i=1}^{\nu} \omega_r^{P_i} \cdot \mathbf{T}^* + \sum_{i=1}^{\nu} \mathbf{v}_r^{P_i} \cdot \mathbf{R}^* \tag{10} $$

where in example 5.a the equation was

$$ F_r^* = \sum_{i=1}^{\upsilon} \mathbf{v}_r^{P_i} \cdot \mathbf{R}_i^* \tag{11} $$

The inertial force $\mathbf{R}_1^*$ is

$$ \mathbf{R}_1^* = -m\,{}^{N}\mathbf{a}^{P} = -m\left(\dot{u}_1 \frac{L}{2}\,\mathbf{A}_2 + u_1^2 \frac{L}{2}\,\mathbf{A}_1 \right) \tag{12} $$

and the inertial torque $\mathbf{T}_1^*$ is

$$\mathbf{T}_1^* = -I_3 \, \dot{u}_1 \, \mathbf{A}_3 \tag{13}$$

where I_3 is the moment of inertia of the pendulum about the $\mathbf{A}_3$ axis. The moment of inertia for a rod is

$$I_3 = \frac{1}{12} m L^2 \tag{14}$$

and equation 13 becomes

$$\mathbf{T}_1^* = -\frac{1}{12} m L^2 \dot{u}_1 \mathbf{A}_3 \tag{15}$$

The generalized active force is determined by the dot product of the active force **R** with the partial velocity of the center of mass of the pendulum ${}^N\mathbf{v}_1^P$. This dot product gives

$$F_1 = [m g (\cos q_1 \mathbf{A}_1 - \sin q_1 \mathbf{A}_2) - T \mathbf{A}_1] \cdot \frac{L}{2} \mathbf{A}_2 \tag{16}$$

$$F_1 = - m g \frac{L}{2} \sin q_1 \tag{17}$$

The generalized inertia force for a rigid body is a sum of two dot products between the inertia force $\mathbf{R}^*$ and the partial velocity of the center of mass of the pendulum , and the inertia torque $\mathbf{T}^*$ and the partial angular velocity of the pendulum. This sum of dot products gives

$$F_1^* = \left[- m \left(\dot{u}_1 \frac{L}{2} \mathbf{A}_2 + u_1^2 \frac{L}{2} \mathbf{A}_1 \right) \cdot \frac{L}{2} \mathbf{A}_2 \right] + \left[-\frac{1}{12} m L^2 \dot{u}_1 \mathbf{A}_3 \cdot \mathbf{A}_3 \right] \tag{18}$$

$$F_1^* = - m \dot{u}_1 \frac{L^2}{4} - \frac{1}{12} m L^2 \dot{u}_1 \tag{19}$$

Finally, by adding equations 17 and 19, the equation of motion for the system is

$$- m g \frac{L}{2} \sin q_1 - m \dot{u}_1 \frac{L^2}{4} - \frac{1}{12} m L^2 \dot{u}_1 = 0 \tag{20}$$

$$m\,g\,\frac{L}{2}\sin q_1 + m\,\dot{u}_1\frac{L^2}{3} = 0 \tag{21}$$

$$\dot{u}_1 + \frac{3}{2}\frac{g}{L}\sin q_1 = 0 \tag{22}$$

and substituting for the generalized coordinate and the generalized speed gives

$$\ddot{\theta} + \frac{3}{2}\frac{g}{L}\sin\theta = 0 \tag{23}$$

Bead on a Wire using Kane's Dynamical Equations

Problem:

A bead is free to move along a thin, tight wire. The bead is of mass m, and can be regarded as a particle. The distance x is defined to be the distance of the mass m from the fixed end of the wire as shown in the figure. A force $f(t)$ is acting on the bead at a prescribed angle θ. The effects of gravity can be neglected, and assume that there is no vertical deflection in the wire. Using x, derive the equation(s) of motion for the mass m.

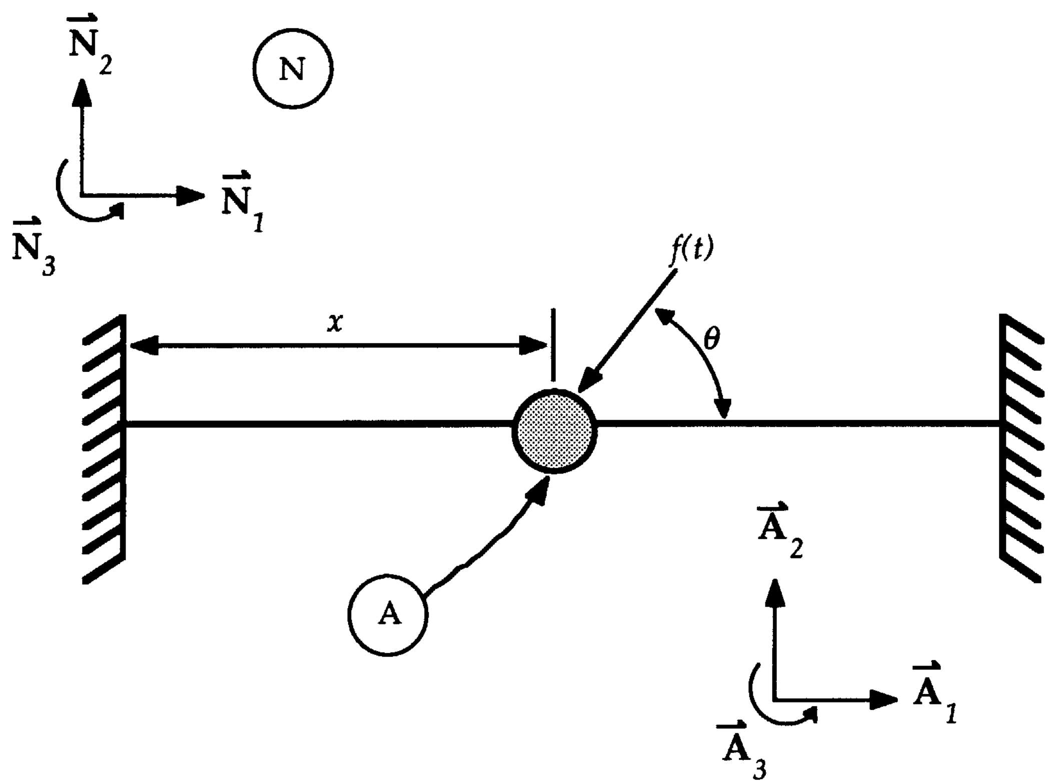

Figure A2.1: Diagram of bead on a wire.

Solution:

A reference frame is first assigned to be an inertial reference frame, N, and then a reference frame is assigned to the mass m, **A**. The reference frame

A is attached to the center of mass of m, and translates along the wire in the $\mathbf{N}_1$ direction.

The next step is to assign values for the generalized coordinates and the generalized speeds. The only generalized coordinate needed to specify the status of the system at any instant in time is x

$$q_1 = x \tag{1}$$

The generalized speed will be assigned to be the total time derivative of the generalized coordinate

$$u_1 = \dot{q}_1 \tag{2}$$

The velocity of the mass is simply

$$^{N}\mathbf{v}^{m} = u_1 \mathbf{N}_1 \tag{3}$$

and by inspection of equation 3 the partial velocity of m with respect to u_1 is

$$^{N}\mathbf{v}^{m} = \mathbf{N}_1 \tag{4}$$

The acceleration of the mass m is the total time derivative of the velocity

$$^{N}\mathbf{a}^{m} = \frac{^{N}d\ ^{N}\mathbf{v}^{m}}{dt} = \frac{^{N}d\,(u_1 \mathbf{N}_1)}{dt} = \dot{u}_1 \mathbf{N}_1 \tag{5}$$

In order to determine the contact and distance forces that act on m a free body diagram is needed.

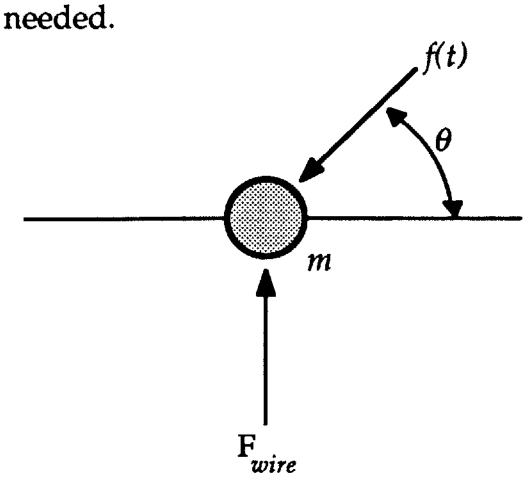

Figure A2.2: Free body diagram of the forces acting on m.

From the free body diagram, it can be seen that the resultant force on the mass m is

$$\mathbf{R}_1 = f(t)(\cos\theta\,\mathbf{N}_1 - \sin\theta\,\mathbf{N}_2) + F_{wire}\,\mathbf{N}_2 \tag{6}$$

The inertia force is found using the acceleration of equation 5

$$\mathbf{R}_1^* = -m_1\,\mathbf{a}_1 = -m\,\dot{u}_1\,\mathbf{N}_1 \tag{7}$$

In order to find the generalized active forces and the generalized inertial forces of the mass m, the dot products of the active force and the inertial force with the partial velocity of the mass m must be computed, respectively. The generalized active force is

$$F_1 = [f(t)(\cos\theta\,\mathbf{N}_1 - \sin\theta\,\mathbf{N}_2) + F_{wire}\,\mathbf{N}_2] \cdot \mathbf{N}_1 \tag{8}$$

$$F_1 = f(t)\cos\theta \tag{9}$$

and the generalized inertial force is

$$F_1^* = -m\,\dot{u}_1\,\mathbf{N}_1 \cdot \mathbf{N}_1 \tag{10}$$

$$F_1^* = -m\,\dot{u}_1 \tag{11}$$

By adding the generalized active force and the generalized inertial force, the equation of motion for the mass m is determined to be

$$f(t)\cos\theta - m\,\dot{u}_1 = 0 \tag{12}$$

and then substituting the original values for the generalized coordinate and the generalized speeds, equation 12 becomes

$$m\,\ddot{x} = f(t)\cos\theta \tag{13}$$

Spring Pendulum Problem using Kane's Dynamical Equations*

Problem:

A mass m is attached to one end of a light, linear spring of modulus k and natural length L. The other end of the spring is supported at an earth-fixed point O in such a way that the spring can rotate about a horizontal axis passing through O.

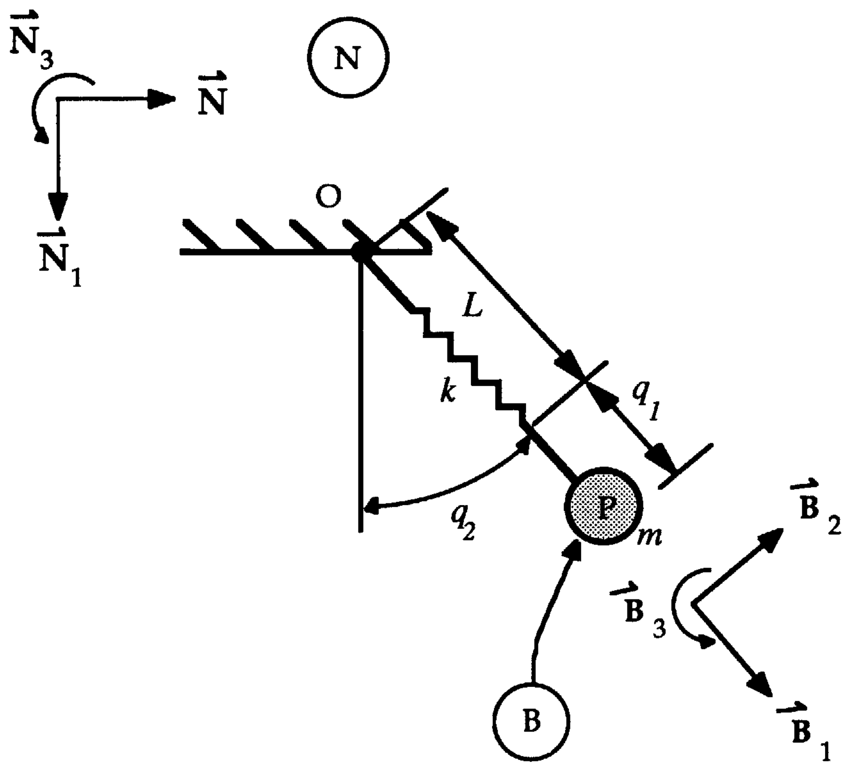

Figure A3.1: Diagram of a spring pendulum.

Letting q_1 measure the stretch of the spring, and q_2 the angle between the axis of the spring and the downward vertical, as indicated in Figure A3.1, and using as generalized speeds quantities u_1 and u_2 defined as

$$u_i = {}^{N}\mathbf{v}^{m} \cdot \mathbf{B}_i$$

* This problem came directly from Professor Kane's problem set that was distributed in the Winter Quarter of 1990, and is problem 9.1.

where ${}^{N}\mathbf{v}^{m}$ is the velocity of the mass m in a reference frame rigidly attached to the earth, and $\mathbf{B}_i$ (i = 1, 2) are unit vectors directed as shown, determine the equation(s) of motion for the mass m.

Solution:

During the formulation of this problem, notes will be made when certain equations are used. These notes will serve as markers that will be discussed after the problem.

The two reference frames that are shown in Figure A3.1 are an inertial reference frame **N** and a reference frame **B** that is rigidly attached to the center of mass of the mass m. The reference frame **B** is oriented so that $\mathbf{B}_1$ points along the axis of the pendulum and $\mathbf{B}_2$ is aligned perpendicular to the length of the pendulum.

Since the generalized speeds, as defined by

$$u_i = {}^{N}\mathbf{v}^{m} \cdot \mathbf{B}_i \qquad (1)$$

are dependent on the velocity of mass m, the velocity of the mass will be calculated first. The velocity of mass m can be found using the following equation[A]

$${}^{N}\mathbf{v}^{m} = {}^{N}\mathbf{v}^{O} + {}^{N}\omega^{B} \times \mathbf{r}^{OP} + {}^{B}\mathbf{v}^{m} \qquad (2)$$

where ${}^{N}\mathbf{v}^{O}$ is the velocity of the point O in **N**, ${}^{N}\omega^{B}$ is the angular velocity of the reference frame **B** with respect to the inertial reference frame **N**, $\mathbf{r}^{OP}$ is the position vector from point O to point P, and ${}^{B}\mathbf{v}^{m}$ is the velocity of the particle P in the **B** reference frame. The velocity of the point O is zero, since it is fixed in the inertial reference frame **N**

$${}^{N}\mathbf{v}^{O} = 0 \qquad (3)$$

The angular velocity of the reference frame **B** with respect to the inertial frame is

$$ {}^{N}\omega^{B} = \dot{q}_2 \, \mathbf{N}_3 = \dot{q}_2 \, \mathbf{B}_3 \tag{4} $$

The position vector from point O to the particle P is

$$ \mathbf{r}^{OP} = (L + q_1)\, \mathbf{B}_1 \tag{5} $$

The velocity of the particle in the reference frame **B** is

$$ {}^{B}\mathbf{v}^{m} = \dot{q}_1 \, \mathbf{B}_1 \tag{6} $$

Using equations 3, 4, 5 and 6, the velocity of the particle P is

$$ {}^{N}\mathbf{v}^{m} = \dot{q}_2 \, \mathbf{N}_1 \times (L + q_1)\, \mathbf{B}_1 + \dot{q}_1 \, \mathbf{B}_1 \tag{7} $$

$$ {}^{N}\mathbf{v}^{m} = \dot{q}_1 \, \mathbf{B}_1 + \dot{q}_2 (L + q_1)\, \mathbf{B}_2 \tag{8} $$

Using equation 1 and examining equation 8, the generalized speeds can be determined to be[B]

$$ u_1 = \dot{q}_1 \tag{9} $$

$$ u_2 = \dot{q}_2 (L + q_1) \tag{10} $$

Using equations 9 and 10, the velocity of the mass m is

$$ {}^{N}\mathbf{v}^{m} = u_1 \, \mathbf{B}_1 + u_2 \, \mathbf{B}_2 \tag{11} $$

The partial derivatives of the velocity of the mass m with respect to both of the generalized speeds are

$$ {}^{N}\mathbf{v}_1^{m} = \mathbf{B}_1 \tag{12} $$

$$ {}^{N}\mathbf{v}_2^{m} = \mathbf{B}_2 \tag{13} $$

where ${}^{N}\mathbf{v}_1^{m}$ and ${}^{N}\mathbf{v}_2^{m}$ are the partial velocity of the mass m with respect to the generalized speeds u_1 and u_2, respectively.

The next step is to determine the acceleration of the mass m. The acceleration is found by using the following equation[C]

$$ {}^{N}\mathbf{a}^{m} = \frac{{}^{N}d \, {}^{N}\mathbf{v}^{m}}{dt} = \frac{{}^{B}d \, {}^{N}\mathbf{v}^{m}}{dt} + {}^{N}\omega^{B} \times {}^{N}\mathbf{v}^{m} \tag{14} $$

where $\frac{{}^{B}d\,{}^{N}\mathbf{v}^{m}}{dt}$ is the total time derivative of the velocity in the **B** reference frame, ${}^{N}\omega^{B}$ is the angular velocity of the **B** reference frame with respect to the inertial reference frame **N**, and ${}^{N}\mathbf{v}^{m}$ is the velocity of the mass m with respect to the inertial frame **N**. The total time derivative of the velocity is

$$\frac{{}^{B}d\,{}^{N}\mathbf{v}^{m}}{dt} = \dot{u}_1\,\mathbf{B}_1 + \dot{u}_2\,\mathbf{B}_2 \qquad (15)$$

The angular velocity was already found in equation 4, and the velocity of the mass m was found in equation 11. Thus, the acceleration of the mass m is

$${}^{N}\mathbf{a}^{m} = \dot{u}_1\,\mathbf{B}_1 + \dot{u}_2\,\mathbf{B}_2 + \dot{q}_2\,\mathbf{B}_3 \times (u_1\,\mathbf{B}_1 + u_2\,\mathbf{B}_2) \qquad (16)$$

$${}^{N}\mathbf{a}^{m} = (\dot{u}_1 - \dot{q}_2\,u_2)\,\mathbf{B}_1 + (\dot{u}_2 + \dot{q}_2\,u_1)\,\mathbf{B}_2 \qquad (17)$$

From equation 10, the total time derivative of q_2 is

$$\dot{q}_2 = \frac{u_2}{L + q_1} \qquad (18)$$

Using equation 18, the acceleration of the mass m becomes

$${}^{N}\mathbf{a}^{m} = \left(\dot{u}_1 - \frac{u_2^2}{L + q_1}\right)\mathbf{B}_1 + \left(\dot{u}_2 + \frac{u_2\,u_1}{L + q_1}\right)\mathbf{B}_2 \qquad (19)$$

Examining a free body diagram of the mass m,

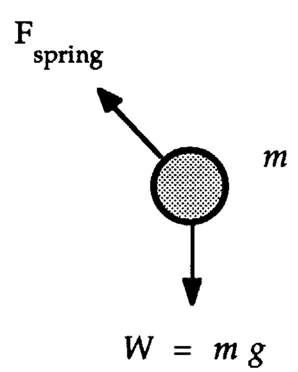

Figure 2: Free body diagram of the forces acting on m.

the active force acting on the mass m is determined to be

$$\mathbf{R}_m = -k\,q_1\,\mathbf{B}_1 - m\,g\,\mathbf{N}_1 \tag{20}$$

$$\mathbf{R}_m = -k\,q_1\,\mathbf{B}_1 - m\,g\,(-\cos q_2\,\mathbf{B}_1 + \sin q_2\,\mathbf{B}_2) \tag{21}$$

The generalized inertial forces are determined in the following manner[D]

$$F_1^* = (-m\,{}^N\mathbf{a}^m) \cdot {}^N\mathbf{v}_1^m \tag{22}$$

$$F_2^* = (-m\,{}^N\mathbf{a}^m) \cdot {}^N\mathbf{v}_2^m \tag{23}$$

$$F_1^* = \left\{ -m\left[\left(\dot{u}_1 - \frac{u_2^2}{L+q_1}\right)\mathbf{B}_1 + \left(\dot{u}_2 + \frac{u_1 u_2}{L+q_1}\right)\mathbf{B}_2\right]\right\} \cdot \mathbf{B}_1 \tag{24}$$

$$F_2^* = \left\{ -m\left[\left(\dot{u}_1 - \frac{u_2^2}{L+q_1}\right)\mathbf{B}_1 + \left(\dot{u}_2 + \frac{u_1 u_2}{L+q_1}\right)\mathbf{B}_2\right]\right\} \cdot \mathbf{B}_2 \tag{25}$$

$$F_1^* = -m\left(\dot{u}_1 - \frac{u_2^2}{L+q_1}\right) \tag{26}$$

$$F_2^* = -m\left(\dot{u}_2 + \frac{u_1 u_2}{L+q_1}\right) \tag{27}$$

The generalized active forces are determined to be[E]

$$F_1 = \mathbf{R}_m \cdot {}^N\mathbf{v}_1^m \tag{28}$$

$$F_2 = \mathbf{R}_m \cdot {}^N\mathbf{v}_2^m \tag{29}$$

$$F_1 = [-k\,q_1\,\mathbf{B}_1 - m\,g\,(-\cos q_2\,\mathbf{B}_1 + \sin q_2\,\mathbf{B}_2)] \cdot \mathbf{B}_1 \tag{30}$$

$$F_2 = [-k\,q_1\,\mathbf{B}_1 - m\,g\,(-\cos q_2\,\mathbf{B}_1 + \sin q_2\,\mathbf{B}_2)] \cdot \mathbf{B}_2 \tag{31}$$

$$F_1 = -k\,q_1 + m\,g\cos q_2 \tag{32}$$

$$F_2 = -m\,g\sin q_2 \tag{33}$$

Finally, the two equations of motion for the system are[F]

$$F_1 + F_1^* = 0 \tag{34}$$

$$F_2 + F_2^* = 0 \tag{35}$$

$$-k\,q_1 + m\,g\cos q_2 - m\left(\dot{u}_1 - \frac{u_2^2}{L+q_1}\right) = 0 \tag{36}$$

$$-m\,g\sin q_2 - m\left(\dot{u}_2 + \frac{u_1 u_2}{L+q_1}\right) = 0 \tag{37}$$

Discussion of the Spring Pendulum Problem

Examining the spring pendulum problem, some parallels can be drawn to traditional dynamical methods. In addition, this problem exhibits some quantities that Kane uses in his derivation of the generalized speeds. Each of the superscripted letters ([A] through [F]) that appeared during the formulation of this problem denoted a specific area that will be discussed in this section.

[A]The equation that is used to calculate the velocity of the mass m is

$${}^{N}\mathbf{v}^{m} = {}^{N}\mathbf{v}^{O} + {}^{N}\boldsymbol{\omega}^{B} \times \mathbf{r}^{OP} + {}^{B}\mathbf{v}^{m} \tag{A1}$$

This equation is identical to equation 2-55 in reference [1]

$$\mathbf{v} = \frac{d\mathbf{R}_o}{dt} + \mathbf{v}_{rel} + \boldsymbol{\omega} \times \mathbf{r} \tag{A2}$$

where the terms correspond to one another in the following manner

$$\frac{d\mathbf{R}_o}{dt} = {}^{N}\mathbf{v}^{O} \tag{A3}$$

$$\mathbf{v}_{rel} = {}^{N}\mathbf{v}^{B} \tag{A4}$$

$$\boldsymbol{\omega} \times \mathbf{r} = {}^{N}\boldsymbol{\omega}^{B} \times \mathbf{r}^{OP} \tag{A5}$$

[B]After the velocity of the mass is determined, the velocity is then used to determine the generalized speeds. Furthermore, Kane defines the generalized speeds in the following manner

$$u_r = \sum_{s=1}^{n} Y_{rs} \dot{q}_s + Z_r \tag{B1}$$

where the quantities Y_{rs} and Z_r are functions of the generalized coordinates and time. In the case of the spring pendulum, the quantities Y_{rs} and Z_r are defined to be

$$Y_{11} = 1 \tag{B2}$$

$$Y^{12} = 0 \tag{B3}$$

$$Y_{21} = 0 \tag{B4}$$

$$Y_{22} = \frac{1}{L + q_1} \tag{B5}$$

$$Z_1 = 0 \tag{B6}$$

$$Z_2 = 0 \tag{B7}$$

[C]Here, the acceleration is defined by Kane to be

$$^{N}\mathbf{a}^{m} = \frac{^{N}d\ ^{N}\mathbf{v}^{m}}{dt} = \frac{^{B}d\ ^{N}\mathbf{v}^{m}}{dt} + {}^{N}\omega^{B} \times {}^{N}\mathbf{v}^{m} \tag{C1}$$

In reference [1], the acceleration is defined to be the total time derivative of the velocity. In order to take the total time derivative, the following operator can be used

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} \right)_{\text{rel}} + \omega \times \tag{C2}$$

When this operator is used on the velocity $^{N}\mathbf{v}^{m}$, the result is equation C1.

[D]In this portion of the formulation, the generalized inertial forces are calculated. The generalized inertial forces are found by calculating the inertial force $\mathbf{R}^{*}$, and then taking the dot product of the inertial force with the partial velocities. In this problem, since the partial velocities are two unit vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ that are perpendicular to one another, the two generalized active forces are two orthogonal components of the inertial force in the $\mathbf{B}_1$ and $\mathbf{B}_2$ directions.

[E]Similarly, the generalized active forces are found by first calculating the active force **R**, and then taking the dot product of the active force and the partial velocities. Again, the two generalized active forces are simply components of the active force in the $\mathbf{B}_1$ and $\mathbf{B}_2$ directions.

[F]Finally, by adding the generalized active forces and the generalized inertial forces the equations of motion for the system are derived. These equations of motion represent the equation $\mathbf{F}=m\mathbf{a}$ in the $\mathbf{B}_1$ and $\mathbf{B}_2$ directions. The total active force **F** and the total inertial force ($m\mathbf{a}$) are found for the mass m, and then the equation $\mathbf{F}=m\mathbf{a}$ is broken into two components corresponding to the $\mathbf{B}_1$ and $\mathbf{B}_2$ directions.