

110

Yang-Mills connections with isolated singularities

by

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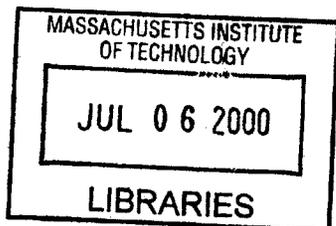
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Abstract

In this thesis, I proved a uniqueness theorem of tangent connections at the singularity for a Yang-Mills connection with an isolated singularity, if every tangent connection at the singularity of the original connection has also isolated singularity. We also obtained a convergence estimate of the connection to the tangent connection as the radius goes to zero. This result shows a local cone-like structure of such Yang-Mills connections at the singularity. If ‘minimal’ Yang-Mills connections can be suitably defined, this result should apply naturally to them. Our method consists of constructing and proving existence of good gauges under which Yang-Mills equations become elliptic and reducing the problem to the asymptotics problem of a nonlinear damped evolution equation which is essentially treated in Leon Simon’s result in 1983. We also gave an application of our methods to the Yang-Mills flow and proved that the Yang-Mills flow exists for all time and has asymptotic limit if the initial value is close to a smooth local minimizer of the Yang-Mills functional.

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Dedicated to my parents, Zhongyao Yang and Dabai Hu

Contents

Introduction	9
1 Background on Yang-Mills connections	13
1.1 Yang-Mills connections and some properties	13
1.2 Stationarity and Monotonicity formulae	18
1.3 Compactness theorem for Yang-Mills connections	22
1.4 Tangent Yang-Mills connections	22
1.5 Statement of the main theorem	25
2 Constructions of Gauges	27
2.1 A change of coordinates	27
2.2 Some properties of gauges	32
2.3 Connections in standard form	34
2.4 Standard form gauges with bounds	39
2.5 Growth and bounds of solutions	43
3 An asymptotic convergence result	47
3.1 A type of nonlinear evolution equations	47
3.2 Growth estimates	51
3.3 Łojasiewicz inequalities	52
3.4 Proof of Theorem 3	54
3.5 Completion of Proof of Theorem 2	55
4 A result for Yang-Mills flows	61
4.1 Yang-Mills flows	61
4.2 Proof of Theorem 4	63
Bibliography	65

Introduction

The work of Leon Simon in [SL1] proved a general asymptotic convergence theorem for the solutions to a class of nonlinear evolution equations of the form

$$\ddot{u} - \dot{u} - \mathcal{M}(u) + \mathcal{R}(u) = f, \quad (0.1)$$

where $u = u(x, t)$, $(x, t) \in M \times [0, T)$, $u(\cdot, t)$ is a smooth section of a vector bundle E on M ; $f = f(x, t)$ is a given function of exponential decay with respect to t , and $\mathcal{M}(u) = -\text{grad } \mathcal{E}$ is the second order elliptic Euler-Lagrange operator of an “energy functional” $\mathcal{E}(u) = \int_M E(x, u, \nabla u)$ on M . The term $\mathcal{R}(u)$ is of nonlinear nature and is ‘negligible’ when C^k - norm of u is small. In [SL1], it is proved that a ‘complete’ solution of (0.1) which satisfies $\mathcal{E}(u(t)) - \mathcal{E}(0) \geq -\varepsilon$ for ε sufficiently small, which is suitably small on an initial time interval and which does not grow too fast, must converge asymptotically to a solution of the stationary equation $\mathcal{M}(w) = 0$. This type of evolution equations arise naturally in certain elliptic geometric variation problems, especially in problems associated with tangent cone or tangent maps. Such tangent objects are defined as (weak) limits of dilations of original object at a fixed point. The existence of such a (weak) limit is usually given by appropriate monotonicity formulae. The general results about asymptotics of (0.1) were applied in [SL1] to show that generalized stationary varifold with a tangent cone which has isolated singularity at one point has unique tangent cone and has a asymptotic cone structure at the point. For energy minimizing harmonic maps between Riemannian manifolds, assume the target manifold has analytic structure and analytic metric. If at a certain point on the domain, the harmonic map has a tangent map which has isolated singularity, then such a harmonic map has a unique tangent map and an asymptotic cone structure at the fixed point, as proved in [SL1]. The main theorem in [SL1] is proved under rather general assumptions; important techniques used in there include blow-up techniques and an infinite dimensional generalization of Łojasiewicz inequality to the infinite dimensional case.

Analogous asymptotics problems in differential geometry were also investigated in [CT] and [MMR]. The work [CT] proved a uniqueness theorem for cone structures at infinity of Ricci flat manifolds under an integrability assumption on the cone manifold. The work [MMR] study the asymptotic limits of anti-self-dual connections with finite energy on a cylindrical 4-manifolds and gave applications to computations of Donaldson invariants.

In Tian [T], stationary admissible Yang-Mills connections on a vector bundle E (with compact structure group) on a Riemannian manifold M , which might admit singularities of codimension 4, are defined and structures of the blow up loci of them are studied. We can have a local picture of these admissible Yang-Mills connections by looking at limits of sequences of Yang-Mills connections which are dilations of the original connection at a fixed point on M , which are called tangent connections of the original connection at that point. Investigation of tangent connections is meaningful only when the dimension of M is greater than 4, for in dimension 4, all tangent connections will be trivial by Uhlenbeck's removable singularity theorem. Tangent connections at a point may not be unique. If we have the uniqueness, then we have an asymptotic cone structure of the connection at the given point, which gives a good local characterization of the connection.

In this paper, we proved the uniqueness of tangent Yang-Mills connections at the singularity of a stationary Yang-Mills connection with isolated singularity for M of dimension greater than 4, under the assumption that every tangent connection at the singularity of the original connection also has isolated singularity. Our condition can also be construed as: for every sequence of dilated connections of the original connection at the singularity, there exists a subsequence which converges in C^∞ on compact subsets of $\mathbb{R}^n \setminus \{0\}$ to a tangent connection which also has isolated singularity at 0. We conceive that such an assumption should hold naturally for suitably defined 'minimal' Yang-Mills connections on a 5-manifold. However, there is still analytical work to be done in order to define minimality properly and prove there is indeed no loss of energy in that case. The difficulty lies in the choice of good spaces of connections and good gauges. We hope to address this issue in a subsequent paper.

Since we are considering a gauge invariant functional, i.e., the Yang-Mills functional, the Yang-Mills equation is degenerate elliptic and the associated evolution equation (after a change of coordinates) is not of the form (0.1) unless we fix the choice of certain gauge. In this thesis we proved the existence of a gauge under which the evolution equation has the desired form and there is a certain growth control of the connection which will allow us to modify [SL1]'s proof to prove the convergence result in our case. Our proof borrowed ideas from [MMR], [CT] and [SL1]. We exploited properties of gauge transformations, monotonicity formulae and various elliptic estimates.

Using our methods, we are also able to prove a result for the Yang-Mills flow. We showed that the Yang-Mills flow exists for all time and has asymptotic limit if the initial value is close to a smooth local minimizer of the Yang-Mills functional.

This thesis is organized into four parts. In this first chapter we describe some known properties of Yang-Mills connections, especially the monotonicity formula and the compactness theorem. Then we state our main theorem, the uniqueness of the tangent connections for Yang-Mills connections with isolated singularities. In the second chapter, we derive the existence of gauge in 'standard form' around a fixed connection, and we treated some properties of gauge transformations, especially those

about how we can bound them. Then we prove a proposition which allows us to choose a good gauge for our purpose and reduce the main theorem to a theorem in the third chapter. The third chapter modifies the proof in [SL1] to prove the theorem which applies to give us existence of asymptotic limits of our evolution equation and completes the proof. In the fourth and final chapter, we proved the result stated above for the Yang-Mills flow.

Chapter 1

Background on Yang-Mills connections

The structure of this chapter is organized as follows. In Section 1.1, we introduced Yang-Mills functional, gauge transformations and Yang-Mills equation on a vector bundle with compact structure group. We defined stationarity of connections and proved the monotonicity formula for them in Section 1.2. In Section 1.3, we stated a priori estimates and compact theorems for Yang-Mills connections. In Section 1.4, we defined tangent connections via the monotonicity formula and showed that it is of cone-like structure up to gauge equivalence. In Section 1.5, we stated our main theorem about uniqueness of tangent connections.

1.1 Yang-Mills connections and some properties

We assume M is an n -dimensional manifold and P is a principle bundle on M with structure group G . We also assume G is compact with Lie algebra \mathfrak{g} . Let E be a vector bundle associated to P with a faithful linear representation $\rho : G \rightarrow \text{GL}(V)$, where V is the fiber type of E . $\text{Aut } P = P \times_{\text{Ad}} G$ is the principal bundle associated to the Ad representation and we denote by \mathfrak{g}_E the associated bundle to $\text{Aut } P$ with the differential of ρ , $d\rho : \mathfrak{g} \rightarrow \text{End}(V)$. Then \mathfrak{g}_E is a subbundle of $\text{End}(E)$. Our main objects for consideration will be (G -)connections A on E . Let A be a connection on E which corresponds to a covariant derivative $\nabla_A : \Gamma(E) \rightarrow \Omega^1(E)$, which is G -equivariant. Under a trivialization $\phi : E|_U \rightarrow U \times V$ on an coordinate open subset $U \in M$, A has local expression $\nabla_A = d + A_\phi$, where $A_\phi \in \Omega^1(\mathfrak{g}_E|_U)$. We often suppress the subscript ϕ and use A for this local expression when the trivialization is clear in the contexts.

We note that there is a unique extension of ∇_A ,

$$\nabla_A : \Gamma(\mathcal{T}(E)) \rightarrow \Gamma(\mathcal{T}(E) \otimes \Omega^1(M))$$

for the total tensor bundle

$$\mathcal{T}(E) = \bigoplus_{i,j=0}^{\infty} (E^*)^{\otimes i} \otimes E^{\otimes j}$$

which is defined via

$$\begin{aligned} \langle \nabla_A(\alpha), \eta \rangle &= -\langle \alpha, \nabla_A \eta \rangle, \\ \nabla_A(\beta \otimes \gamma) &= \nabla_A \beta \otimes \gamma + \beta \otimes \nabla_A \gamma, \end{aligned}$$

where $\alpha \in \Gamma(E^*)$, $\beta \in \Gamma(E)$, $\beta, \gamma \in \Gamma(\mathcal{T}(E))$. In particular, if $\alpha \in \Gamma(\text{End}(E))$, then in local coordinates,

$$\nabla_A(\alpha) = d\alpha + [A, \alpha],$$

where $[\cdot, \cdot] : \Omega^*(\text{End}(E)) \times \Omega^*(\text{End}(E)) \rightarrow \Omega^*(\text{End}(E))$ is the super Lie brackets defined by

$$[\tau \otimes \alpha, \eta \otimes \beta] = \tau \wedge \eta \otimes (\alpha \circ \beta - (-1)^{\deg \tau \deg \eta} \beta \circ \alpha)$$

for $\tau, \eta \in \Omega^*(M)$ and $\alpha, \beta \in \text{End}(E)$. We define $d_A : \Omega^p(\mathcal{T}(E)) \rightarrow \Omega^{p+1}(\mathcal{T}(E))$ by $d_A = \wedge \circ \nabla_A$, i.e.,

$$\begin{aligned} d_A &= \nabla_A : \Gamma(\mathcal{T}(E)) \rightarrow \Omega^1(\mathcal{T}(E)), \\ d_A(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d_A \beta, \end{aligned}$$

for $\alpha \in \Omega^*(M)$, $\beta \in \Gamma(\mathcal{T}(E))$. In particular, in local coordinates,

$$d_A(\alpha) = d\alpha + A \wedge \alpha \tag{1.1}$$

$$d_A(\beta) = d\beta + [A, \beta] \tag{1.2}$$

for $\alpha \in \Omega^*(E)$ and $\beta \in \Omega^*(\text{End}(E))$.

For a connection A on E , the operator $d_A \circ d_A : \Omega^*(E) \rightarrow \Omega^{*+2}(E)$ happens to be given by an algebraic operator $F_A \wedge : \Omega^*(E) \rightarrow \Omega^{*+2}(E)$, thanks to the fact $d \circ d = 0$. Here $F_A \in \Omega^2(\text{End}(E))$ is the curvature of A and locally has the expression

$$F_A = dA + A \wedge A = dA + \frac{1}{2}[A, A]. \tag{1.3}$$

The curvature F_A satisfies the Bianchi identity

$$d_A F_A = 0. \tag{1.4}$$

For a fixed connection A on E , every connection B on E gives $\nabla_B = \nabla_A + B'$, where $B' \in \Omega^1(\text{End}(E))$. Hence the total space \mathcal{A} of all connections on E has an affine space structure modeled on $\Omega^1(\text{End}(E))$. We denote by $\text{Diff}_G(P)$ the group of all automorphisms of bundle P , which consists of diffeomorphisms of P which preserves

fibers and are G -equivariant. Denote by $\text{Diff}(M)$ the group of diffeomorphisms of M . Then we have the following exact sequence,

$$1 \rightarrow \Gamma(\text{Aut } P) \rightarrow \text{Diff}_G(P) \xrightarrow{\pi} \text{Diff}(M)$$

where π is the projection to the base manifold M .

Suppose $\tilde{\psi} \in \text{Diff}_G(P)$ and $\pi(\tilde{\psi}) = \psi \in \text{Diff}(M)$. There is a natural action of $\tilde{\psi}$ on $E = P \times_{\text{Ad}} V$ which gives a G -equivariant bundle isomorphism. We can define pullbacks and pushforwards by $\tilde{\psi}$ of tensors on M with values in $\mathcal{T}(E)$ as follows,

$$\begin{aligned} \tilde{\psi}^*(\tau \otimes \eta \otimes \alpha \otimes \beta) &= (\tilde{\psi}^{-1})_*(\tau) \otimes \psi^*(\eta) \otimes \tilde{\psi}^{-1}(\alpha) \otimes (\tilde{\psi})^*(\beta) \\ \tilde{\psi}_*(\tau \otimes \eta \otimes \alpha \otimes \beta) &= (\tilde{\psi})_*(\tau) \otimes (\psi^{-1})^*(\eta) \otimes \tilde{\psi}(\alpha) \otimes (\tilde{\psi}^{-1})^*(\beta) \end{aligned}$$

for $\tau \in \Gamma(\wedge^* T^* M)$, $\eta \in \Gamma(\wedge^* T^* M)$, $\alpha \in \Gamma(E^{\otimes \cdot})$ and $\beta \in \Gamma((E^*)^{\otimes \cdot})$. The pull back of connections on E under ψ is defined by

$$\nabla_{\tilde{\psi}^* A}(v) = \tilde{\psi}^*(\nabla_A(\tilde{\psi}_* v))$$

for any connection A and any $v \in \Gamma(\mathcal{T}(E))$.

Lemma 1 *The above formula defines a G -connection $\tilde{\psi}^* A$ and*

$$F_{\tilde{\psi}^* A} = \tilde{\psi}^* F_A$$

Proof. We need to check that $\nabla_{\tilde{\psi}^* A}$ satisfies Leibnitz formula and is G -equivariant.

$$\begin{aligned} \nabla_{\tilde{\psi}^* A}(fv) &= \tilde{\psi}^*(\nabla_A(\tilde{\psi}_*(fv))) \\ &= \tilde{\psi}^*(\nabla_A((f \circ \psi^{-1})\tilde{\psi}_* v)) \\ &= \tilde{\psi}^*(d(f \circ \psi^{-1}) \otimes \tilde{\psi}_* v + f \circ \psi^{-1} \nabla_A(\tilde{\psi}_* v)) \\ &= df \otimes v + f \tilde{\psi}^* \nabla_A(\tilde{\psi}_* v). \end{aligned}$$

Since $\tilde{\psi}$ and ∇_A are G -equivariant, so is $\nabla_{\tilde{\psi}^* A}$. Now we observe $d_{\tilde{\psi}^* A}(v) = \tilde{\psi}^*(d_A(\tilde{\psi}_* v))$ because $d_A = \wedge \circ \nabla_A$ and \wedge commutes with ψ^* . Therefore as an operator on $\Omega^*(E)$,

$$\begin{aligned} F_{\tilde{\psi}^* A} &= d_{\tilde{\psi}^* A} \circ d_{\tilde{\psi}^* A} = \tilde{\psi}^* \circ d_A \circ \tilde{\psi}_* \tilde{\psi}^* \circ d_A \circ \tilde{\psi}_* \\ &= \tilde{\psi}^* \circ d_A \circ d_A \circ \tilde{\psi}_* = (\tilde{\psi}^* F_A). \end{aligned}$$

This implies $F_{\tilde{\psi}^* A} = \tilde{\psi}^* F_A \in \Omega^2(\mathfrak{g}_E)$. \square

As a special case of above, the natural action of the gauge group $\text{Aut } P$ on \mathcal{A} is given by

$$\nabla_{g(A)}(v) = g \circ \nabla_A \circ g^{-1}(v)$$

for $g \in \Gamma(\text{Aut } P)$, $A \in \mathcal{A}$ and $v \in \Gamma(E)$. (Another convention in literature is to let $\nabla_{g(A)} = g^{-1} \circ \nabla_A \circ g$, this is essentially equivalent and will not affect our result

generally.) This action actually corresponds to change of trivializations on E . Under local trivialization, if we write $\nabla_A = d + A$, $\nabla_{g(A)} = d + g(A)$, then

$$g(A) = gAg^{-1} - dg g^{-1}. \quad (1.5)$$

We also have from the lemma that F_A changes as a tensor and for $g \in \Gamma(\text{Aut } P)$,

$$F_{g(A)} = gF_A g^{-1}. \quad (1.6)$$

Now we assume that M is oriented and is given a Riemmanian structure g and assume that E is given a G -invariant metric h . h induces a Ad_G -invariant metric, still denoted h , on $\mathfrak{g}_E \subset \text{End}(E)$. If G is simple, then this metric coincides with the metric on \mathfrak{g}_E induced by the Killing form \mathfrak{g} up to a positive constant.

The Yang-Mills functional $\text{YM} : \mathcal{A} \rightarrow \mathbb{R}$ is defined by

$$\text{YM}(A) = \int_M |F_A|^2 dV_g$$

Here $|\cdot|^2$ is given by the bilinear form

$$\langle \cdot, \cdot \rangle_x : (\Lambda^* T_x M \otimes \text{End}(E)_x) \times (\Lambda^* T_x M \otimes \text{End}(E)_x) \rightarrow \mathbb{R} \quad (1.7)$$

$$\langle \tau \otimes \alpha, \eta \otimes \beta \rangle = \langle \tau, \eta \rangle_g \cdot \langle \alpha, \beta \rangle_h \quad (1.8)$$

for $\tau, \eta \in \Lambda^* T_x M$, $\alpha, \beta \in \text{End}(E)_x$. Similarly, we define a bilinear form

$$\langle \cdot, \cdot \rangle_{\wedge, x} : (\Lambda^* T_x M \otimes \text{End}(E)_x) \times (\Lambda^* T_x M \otimes \text{End}(E)_x) \rightarrow \Lambda^* T_x M \quad (1.9)$$

$$\langle \tau \otimes \alpha, \eta \otimes \beta \rangle_{\wedge, x} = \langle \alpha, \beta \rangle_h \tau \wedge \eta \quad (1.10)$$

for $\tau, \eta \in \Lambda^* T_x M$, $\alpha, \beta \in \text{End}(E)_x$. We call A a *Yang-Mills connection* if and only if A is a critical point of YM on \mathcal{A} . In other words, A is Yang-Mills if and only if for any continuously differentiable family of connections $\{A_t\}_{-\varepsilon < t < \varepsilon}$

$$\left. \frac{d}{dt} \right|_{t=0} \text{YM}(A_t) = 0$$

A Yang-Mills connection A satisfies the following Yang-Mills equation, which is the Euler-Lagrange equation for YM ,

$$d_A^* F_A = 0. \quad (1.11)$$

In (1.11), $d_A^* : \Omega^*(\text{End}(E)) \rightarrow \Omega^{*-1}(\text{End}(E))$ is the formal adjoint of d_A with respect to the metric $(\cdot, \cdot) : \Omega^*(\text{End}(E)) \times \Omega^*(\text{End}(E)) \rightarrow \mathbb{R}$ defined by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle_x dV_g(x).$$

The Bianchi identity (1.4) and Yang-Mills equation (1.11) together give a system of equations which is a nonlinear analogue of the equation for harmonic forms. We define an auxiliary bilinear form $(\cdot, \cdot)_\wedge : \Omega^*(\text{End}(E)) \times \Omega^*(\text{End}(E)) \rightarrow \mathbb{R}$ by

$$(\alpha, \beta)_\wedge = \int_M \langle \alpha, \beta \rangle_{\wedge, x} dV_g(x).$$

We observe that $(\alpha, \beta)_\wedge = 0$ unless $\deg \alpha + \deg \beta = n$. We extend the Hodge star operator $*$ on forms linearly to bundle valued forms and observe that

$$\begin{aligned} (\alpha, \beta) &= (\alpha, * \beta)_\wedge = (-1)^{\deg \alpha (n - \deg \alpha)} (* \alpha, \beta)_\wedge, \\ (* \alpha, \beta) &= (\alpha, \beta)_\wedge. \end{aligned}$$

Lemma 2 *Under local coordinates, if $\alpha \in \Omega^p(\text{End}(E))$, then*

$$d_A^* \alpha = d^* \alpha + (-1)^{(p-1)n+1} * [A, * \alpha] = (-1)^{(p-1)n+1} * d_A * \alpha$$

Proof. We prove the claim by direct computation,

$$\begin{aligned} (d_A^* \alpha, \beta) &= (\alpha, d_A \beta) = (\alpha, d\beta + [A, \beta]) \\ &= (d^* \alpha, \beta) + (-1)^{p(n-p)} (* \alpha, [A, \beta])_\wedge \\ &= (d^* \alpha, \beta) + (-1)^{p(n-p)} ([* \alpha, A], \beta)_\wedge \\ &= (d^* \alpha, \beta) + (-1)^{p(n-p)} (* [* \alpha, A], \beta) \\ &= (d^* \alpha + (-1)^{p(n-p)+1 \cdot (n-p)+1} * [A, * \alpha], \beta) \\ &= (d^* \alpha + (-1)^{(p-1)n+1} * [A, * \alpha], \beta) \end{aligned}$$

□

Examples. 1) Yang-Mills connections on a four-dimensional manifold have been the subject of many important and fruitful works. This dimension is quite special, essentially due to the decomposition $SO(4) = SO(3) \times SO(3)$ or $Spin(4) = Spin(3) \times Spin(3)$. The Hodge operator $*$ satisfies $*^2 = \text{Id} : \Lambda^2 M \rightarrow \Lambda^2 M$. This leads us to define a smooth connection to be *anti-self-dual* (ASD) if

$$*F_A = -F_A.$$

By the above lemma, we have

$$d_A^* F_A = - * d_A * F_A = * d_A F_A = 0,$$

by the second Bianchi identity. If $G = SU(k)$ and $k \geq 2$, we are actually able to show that ASD connections are minimizers of Yang-Mills functional among smooth connections on the same bundle E by the following argument. Decompose $F_A =$

$F_A^+ + F_A^-$ by anti-self-dual and self-dual parts. Then since the Killing form on $SU(k)$ is given by $\langle A, B \rangle = -\text{tr}(AB)$, we have

$$\begin{aligned} \int |F_A|^2 dV_g &= \int -\text{tr}(F_A \wedge *F_A) = \int -\text{tr}(F_A^+ \wedge F_A^+) + \text{tr}(F_A^- \wedge F_A^-) \\ &\geq \int \text{tr}(F_A^+ \wedge F_A^+) + \text{tr}(F_A^- \wedge F_A^-) = \int \text{tr}(F_A \wedge F_A) = -4\pi^2 c_2(E). \end{aligned}$$

The equality holds if and only A is ASD. Here we used the property

$$\langle F_A^+, F_A^- \rangle = \text{tr}(F_A^+ \wedge F_A^-) = \text{tr}(*F_A^+ \wedge F_A^-) = -\langle F_A^+, F_A^- \rangle = 0.$$

Uhlenbeck's removable singularity theorem [U1] asserted that if A is a Yang-Mills connection on E , a vector bundle on a four dimensional manifold M , such that A is smooth outside a point $x \in M$, $\text{YM}(A) \leq \infty$, then the bundle $E|_{M \setminus \{x\}}$ can be extended to a bundle E' on M and A can be extended to a smooth connection A' on E' . This remarkable theorem together with compactness results (for example, [U2]) allow us to give a good compactification of the moduli space of ASD connections, which has important applications to four dimensional differential topology such as defining the Donaldson invariants.

2) Connections on manifolds of dimension at least 5 will be our main concern in this paper since this is the case where singularities are likely to occur. There is an analogy to ASD as defined in [T]. Assume Ω is a closed $(n-4)$ calibration form (see [HL] for example) on M , A is called Ω -ASD if

$$\Omega \wedge F_A = -*F_A. \quad (1.12)$$

There is topological expression of $\text{YM}(A)$ for an Ω -ASD connection A in terms of Chern classes and the class represented by Ω (See [T]). On a Kähler manifold of dimension $2m$, if we take $\Omega = \frac{\omega^{m-2}}{(m-2)!}$ (which is a calibration form by the Wirtinger inequality, see [HL] for example), then A is Ω -ASD if and only the bundle E has a holomorphic structure, and A is Hermitian-Yang-Mills under this structure. In this case A is an absolute minimizer of YM among smooth connections on E . In [T], (A, S) is defined as an *admissible Yang-Mills connection* if $S \subset M$, $H^{n-4}(S) = 0$, A is a smooth connection on $M \setminus S$ with $d_A^* F_A = 0$ on $M \setminus S$ and $\text{YM}(A) < \infty$.

1.2 Stationarity and Monotonicity formulae

We shall derive the variation formula of the Yang-Mills functional under variations induced by vector fields on the underlying manifold. Fix a smooth covariant derivative ∇_{A_0} on E . Let $\{\psi_t\}_{-\varepsilon < t < \varepsilon}$ be a 1-parameter family of compactly supported diffeomorphisms of M with $\psi_0 = \text{Id}$. We have a lifting to a 1-parameter family $\{\tilde{\psi}_t\}$ in $\text{Diff}_G(P)$ by letting $\tilde{\psi}_t(p) = \tau_t^0(p)$ for $p \in P$. Here $\tau_t^0 : P \rightarrow P$ is the parallel transport

with respect to ∇_{A_0} defined by horizontal lift of the path $\psi_t(\pi(p))$, $-\varepsilon < t < \varepsilon$. Since parallel transport commutes with G -action, we see that $\tilde{\psi}_t \in \text{Diff}_G(P)$. Hence for any connection A on E , by Lemma 1, we obtain a family of connections $A_t = \tilde{\psi}_t^*(A)$. And we have $F_{A_t} = \tilde{\psi}_t^*(F_A)$. If A is Yang-Mills, then

$$\left. \frac{d}{dt} \right|_{t=0} \text{YM}(A_t) = \left. \frac{d}{dt} \right|_{t=0} \int_M |F_{A_t}|^2 dV_g = 0$$

Assume $\{e_i\}_{i=1}^n$ is a local orthonormal basis for TM around $x \in M$, then

$$\begin{aligned} |F_{A_t}(x)|^2 &= |\tilde{\psi}_t^* F_A(x)|^2 = \sum_{1 \leq i < j \leq n} \langle (\tilde{\psi}_t^* F_A)(e_i, e_j)(x), (\tilde{\psi}_t^* F_A)(e_i, e_j)(x) \rangle_{x,h} \\ &= \sum_{1 \leq i < j \leq n} |\tilde{\psi}_t^*(F_A(\psi_{t*} e_i, \psi_{t*} e_j))(\psi_t(x))|_{x,h}^2 \\ &= \sum_{1 \leq i < j \leq n} |F_A(\psi_{t*} e_i, \psi_{t*} e_j)(\psi_t(x))|_{\psi_t(x),h}^2 \end{aligned}$$

We have the last step because as parallel transports, $\tilde{\psi}_t$ preserves the product h on E . By change coordinates $x \rightarrow \psi_t^{-1}(x)$ and assume that

$$\left. \frac{d}{dt} \right|_{t=0} \psi_t = X$$

we have

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} |F_{A_t}(\psi_t^{-1}(x))|^2 &= \left. \frac{d}{dt} \right|_{t=0} \sum_{1 \leq i < j \leq n} |F_A(\psi_{t*} e_i(\psi_t^{-1}(x)), \psi_{t*} e_j(\psi_t^{-1}(x)))(x)|_{x,h}^2 \\ &= -4 \sum_{1 \leq i < j \leq n} \langle F_A([X, e_i], e_j), F_A(e_i, e_j) \rangle_{x,h} \end{aligned}$$

Therefore, if X is supported on a neighborhood on which e_i are defined, we have

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \right|_{t=0} \int_M |F_{A_t}|^2 dV_g \\ &= \left. \frac{d}{dt} \right|_{t=0} \int_M |F_{A_t}(\psi_t^{-1}(x))|^2 J(\psi_t^{-1}) dV_g \\ &= - \int_M |F_A|^2 \text{div } X + 4 \sum_{1 \leq i < j \leq n} \langle F_A([X, e_i], e_j), F_A(e_i, e_j) \rangle dV_g \end{aligned} \quad (1.13)$$

where J denote the Jacobian of maps and we used

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \psi_{t*} e_i(\psi_t^{-1}(x)) &= -[X, e_i](x) \\ \left. \frac{d}{dt} \right|_{t=0} J(\psi_t^{-1}) &= -\text{div } X \end{aligned}$$

Since $[X, e_i] = \nabla_X e_i - \nabla_{e_i} X$, where ∇ is the Levi-Civita connection on M , we have

$$\begin{aligned}
& \sum_{1 \leq i < j \leq n} \langle F_A([X, e_i], e_j), F_A(e_i, e_j) \rangle \tag{1.14} \\
&= - \sum_{1 \leq i < j \leq n} \{ \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) - F_A(\nabla_X e_i, e_j), F_A(e_i, e_j) \rangle \} \\
&= - \sum_{1 \leq i < j \leq n} \{ \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) - \langle \nabla_X e_i, e_k \rangle F_A(e_k, e_j), F_A(e_i, e_j) \rangle \} \\
&= - \sum_{1 \leq i < j \leq n} \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \rangle
\end{aligned}$$

where we used $\langle \nabla_X e_i, e_k \rangle + \langle \nabla_X e_k, e_i \rangle = 0$. In summary, we have the following *first variation formula* (with respect to variations defined by vector fields as above) for smooth Yang-Mills connections

$$\int_M |F_A|^2 \operatorname{div} X - 4 \sum_{1 \leq i < j \leq n} \langle F_A(\nabla_{e_i} X, e_j), F_A(e_i, e_j) \rangle dV_g \tag{1.15}$$

We observe that (1.15) is independent of the choice of the basis e_i and in above we only used the first covariant derivative of X , therefore this formula is true for any compactly supported C^1 vector field X on M . We also note that (1.15) doesn't on the covariant derivative ∇_{A_0} which we used to lift ψ_t to $\tilde{\psi}_t$.

Definition. A connection A is *stationary* if the first variation formula is true for A with any compactly supported C^1 vector field X on M .

Assume A is a stationary connection. We have the following important monotonicity formula by Price [P] (also see Tian [T]). Let $\operatorname{inrad}(x)$ denote the injective radius of $x \in M$.

Proposition 1 *For any $x \in M$, there exists positive constants $\Lambda = \Lambda(x)$ and $r_x < \operatorname{inrad}(x)$ which only depend the supremum bound of curvature of M , such that if $0 < \sigma < \rho \leq r_x$, then*

$$\begin{aligned}
& \rho^{4-n} e^{\Lambda\rho} \int_{B_\rho(x)} |F_A|^2 dV_g - \sigma^{4-n} e^{\Lambda\sigma} \int_{B_\sigma(x)} |F_A|^2 dV_g \tag{1.16} \\
& \geq 4 \int_{B_\rho(x) \setminus B_\sigma(x)} r^{4-n} \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 dV_g
\end{aligned}$$

$$\begin{aligned}
& \rho^{4-n} e^{-\Lambda\rho} \int_{B_\rho(x)} |F_A|^2 dV_g - \sigma^{4-n} e^{-\Lambda\sigma} \int_{B_\sigma(x)} |F_A|^2 dV_g \tag{1.17} \\
& \leq 4 \int_{B_\rho(x) \setminus B_\sigma(x)} r^{4-n} \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 dV_g
\end{aligned}$$

Proof. For a normal coordinate neighborhood $B_{r_0}(0)$ around x , we choose an orthonormal basis $\{e_1, e_2, \dots, e_{n-1}, \frac{\partial}{\partial r}\}$, where r is the distance to x . We shall choose the test vector field of the form

$$X = \xi(r)r \frac{\partial}{\partial r}$$

where $\xi(r)$ is a function with support in $[-1, r_0]$. We have

$$\nabla_{\frac{\partial}{\partial r}} X = (\xi + r\xi') \frac{\partial}{\partial r}$$

and

$$\nabla_{e_i} X = \xi(e_i + O(r^2))$$

by properties of normal coordinates. Substitute the above in (1.16), we obtain

$$\int_{B_\rho(x)} |F_A|^2 (\xi' r + (n-4)\xi + O(r^2)\xi) dV_g = 4 \int_{B_\rho(x)} \xi' r \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 dV_g \quad (1.18)$$

Now we set $\xi = \xi(r, \tau) = \eta(\frac{r}{\tau})$ for $0 < \tau < r_0$, where $\eta \in C^\infty([-1, \infty))$, $\eta(r) = 1$, for $r \leq 1$, $\eta(r) = 0$ for $r \geq 1 + \varepsilon$, $0 < \varepsilon < 1$ and $\eta'(r) \leq 0$. Then $\frac{\partial}{\partial r} \xi = -\frac{\tau}{r} \frac{\partial}{\partial \tau} \xi$. Substitute into (1.18), we have

$$\tau I'(\tau) + ((4-n) + O(\tau^2))I(\tau) = 4\tau J'(\tau) \quad (1.19)$$

where

$$I(\tau) = \int_{B_{r_0}(x)} \xi(r, \tau) |F_A|^2 dV_g$$

$$J(\tau) = \int_{B_{r_0}(x)} \xi(r, \tau) \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 dV_g$$

Let $\varepsilon \rightarrow 0$ and choose η accordingly so that $\eta \rightarrow \chi_{[-1,1]}$ in the space of functions of bounded variations, (1.19) gives for a.e. $r_0 > \tau > 0$,

$$\tau \int_{\partial B_\tau(x)} |F_A|^2 - ((n-4) + O(r_0^2)) \int_{B_\tau(x)} |F_A|^2 \geq 4\tau \int_{\partial B_\tau(x)} \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 \quad (1.20)$$

Assume that if $\tau \leq r_0$, then $|O(\tau^2)| \leq \Lambda' \tau^2$, here Λ' depends on supremum norm of the curvature of M around x . Multiplying both sides of (1.20) by $\tau^{3-n} e^{\Lambda' r_0^2 \tau}$ yields

$$\frac{d}{d\tau} (e^{\Lambda' r_0^2 \tau} \tau^{4-n} \int_{B_\tau(x)} |F_A|^2) \geq 4e^{\Lambda' r_0^2 \tau} \tau^{4-n} \int_{\partial B_\tau(x)} \left| \frac{\partial}{\partial r} \lrcorner F_A \right|^2 \quad (1.21)$$

Integrating in τ from σ to ρ and setting $\Lambda = \Lambda' r_0^2$, $r_x = r_0$ will give us (1.16). (1.17) may be obtained similarly. \square

1.3 Compactness theorem for Yang-Mills connections

We have the following a priori estimate for smooth Yang-Mills connections proven by Uhlenbeck and also by Nakajima [Na]. The proof of it needs a Weitzenböck formula for bundle-valued 2 forms. It can be proven by using a method similar to Schoen's methods ([S1] or [SY]) in proving an a priori estimate for harmonic maps.

Proposition 2 *Assume A is a smooth Yang-Mills connection. There exists $\varepsilon = \varepsilon(n) > 0$ and $C = C(M, n) > 0$ such that for any $x \in M$, $\rho < r_x$, if $\rho^{4-n} \int_{B_\rho(x)} |F_A|^2 dV_g \leq \varepsilon$, then*

$$\rho^2 |F_A|(x) \leq C \left(\int_{B_\rho(x)} |F_A|^2 dV_g \right)^{\frac{1}{2}} \quad (1.22)$$

Compactness theorems about Yang-Mills connections are proven in Uhlenbeck [U2], Nakajima [Na], Tian [T]. The proof of the compactness theorem stated below employs the above a priori estimate and the monotonicity formula.

Proposition 3 *Assume $\{A_i\}$ is a sequence of smooth Yang-Mills connections on E , with $YM(A_i) \leq \Lambda$, where Λ is a constant. Then there exist a closed subset S of M with $H^{n-4}(S) < \infty$, constant $\varepsilon > 0$, a nonnegative H^{n-4} -integrable function Θ on S , a subsequence $\{A_{i_j}\}$, gauge transformations $\sigma_j \in \Gamma(\text{Aut } P)$ and a smooth Yang-Mills connection on $M \setminus S$, such that the following holds:*

- (1) *On any compact set $K \subset M \setminus S$, $\sigma_j(A_{i_j})$ converges to A in C^∞ topology.*
- (2) *$|F_{A_{i_j}}|^2 dV_g \rightarrow \Theta H^{n-4} \lfloor S + |F_A|^2 dV_g$ as measures on U .*

The closed set S of Hausdorff codimension ≥ 4 may be given by

$$S = \{x \in M : \liminf_{\rho \rightarrow 0, j \rightarrow \infty} \rho^{4-n} \int_{B_\rho(x)} |F_{A_{i_j}}|^2 \geq \varepsilon\} \quad (1.23)$$

where ε is as in Prop. 2. S is called the *blow-up locus* of the sequence $\{A_{i_j}\}$. This theorem remains true if we relax the requirement that A_i are smooth to that A_i are stationary admissible Yang-Mills connections with singularity sets S_i and $H^{n-4}(S_i) = 0$. In [T], it is proved the the blow-up locus of a sequence of stationary Yang-Mills connections is an $(n - 4)$ -rectifiable set.

1.4 Tangent Yang-Mills connections

Let A be a stationary admissible Yang-Mills connection on (M, g) with $H^{n-4}(S) = 0$ for the singular set S . Assume $x \in M$, for any $\lambda \in (0, 1)$, let $\tau_\lambda : T_x M \rightarrow T_x M$ be

the scaling map given by $\tau_\lambda : v \rightarrow \lambda v$ and $\tilde{\tau}_\lambda : B_{r_x}(x) \rightarrow B_{r_x}(x)$ be the map induced on neighborhood of x in M via $\exp_x : T_x M \rightarrow M$,

$$\tilde{\tau}_\lambda(y) = (\exp_x)\tau_\lambda(\exp_x)^{-1}$$

where $r_x < \text{inrad}(x)$ is the constant as in Prop. 1. We also view $\tilde{\tau}_\lambda$ as a map on bundle E via a local trivialization. Define scaling of the metric and A by

$$g_\lambda = \lambda^{-2}g, \quad A_\lambda = \tilde{\tau}_\lambda^* A \quad (1.24)$$

Then A_λ is a stationary admissible Yang-Mills connection with respect to g_λ and the singularity of A_λ has zero H^{n-4} -measure as well. By monotonicity formula (1.16),

$$\int_{B_r(0, g_\alpha)} |F_{A_\lambda}|^2 dV_g = \lambda^{4-n} \int_{B_{\lambda r}(0)} |F_A|^2 dV_g \leq C \int_{B_r(0)} |F_A|^2 dV_g = \lambda \quad (1.25)$$

for $0 < r < r_x$. Hence for any sequence $\lambda_i \rightarrow 0$, A_{λ_i} satisfies the assumption of Prop. 3 (although the metric is changing with λ_i , we observe that g_λ converges to the standard flat metric on \mathbb{R}^n as $\lambda \rightarrow 0$ and the compactness theorem Prop. 3 remains true). There exists a subsequence, still denoted $\{\lambda_i\}$, blow-up locus $S_0 \subset \mathbb{R}^n = T_x M$ with a positive H^{n-4} -integrable density function Θ , gauge transformations σ_i , such that $\sigma_i(A_{\lambda_i})$ converges to a Yang-Mills connection A_0 in C^∞ topology on any subset of $\mathbb{R}^n \setminus S_0$. We observe that A_{λ_i} and σ_i are defined on $B_{\lambda_i^{-1}}(0)$ and the limiting connection A_0 is an admissible Yang-Mills connection defined on $\mathbb{R}^n \setminus S_0$ with respect to the standard metric on \mathbb{R}^n . The triple (A_0, S_0, Θ) is called a *tangent connection* of A at 0. It was shown in [T] 5.3.1 that

Lemma 3 *With notations as above, we have*

$$\frac{\partial}{\partial r} \rfloor F_{A_0} = 0, \quad (1.26)$$

$$\frac{\partial}{\partial r} \Theta = 0, \quad a \cdot S_0 = S_0, \quad \forall a > 0. \quad (1.27)$$

In other words, the radial part of the curvature of A_0 vanishes and S_0, Θ are radially symmetric.

Proof. For simplicity, we assume $H^{n-4}(S_0) = 0$ and only show (1.26) here. In the general case, (1.26) together with (1.27) may be shown with a generalized monotonicity formula. Now (1.16) implies that (notice $\Lambda = 0$ since the metric on \mathbb{R}^n is flat)

$$\begin{aligned} & \rho^{4-n} \int_{B_\rho(0)} |F_{A_0}|^2 dx - \sigma^{4-n} \int_{B_\sigma(0)} |F_{A_0}|^2 dx \\ & \geq 4 \int_{B_\rho(0) \setminus B_\sigma(0)} r^{4-n} \left| \frac{\partial}{\partial r} \rfloor F_{A_0} \right|^2 dx \end{aligned} \quad (1.28)$$

However, for any $\rho > 0$, we have

$$\begin{aligned} \rho^{4-n} \int_{B_\rho(0)} |F_{A_0}|^2 dx &= \lim_{i \rightarrow \infty} (\lambda_i \rho)^{4-n} \int_{B_{\lambda_i \rho}(x)} |F_A|^2 dV_g \\ &= \lim_{r \rightarrow \infty} \int_{B_r(x)} |F_A|^2 dV_g > 0 \end{aligned}$$

Therefore, both sides of (1.28) are zero and

$$\int_{B_\rho(0) \setminus B_\sigma(0)} r^{4-n} \left| \frac{\partial}{\partial r} \lrcorner F_{A_0} \right|^2 dx = 0 \quad (1.29)$$

This implies (1.26). \square

The following lemma implies that tangent connections of Yang-Mills connections are radially symmetric up to gauge equivalence, at least if they are smooth on $\mathbb{R}^n \setminus \{0\}$. Using this we shall usually assume a tangent connection is the pullback of a connection on S^{n-1} .

Lemma 4 *If A is a smooth connection on the trivial bundle $(\mathbb{R}^n \setminus \{0\}) \times V$ such that*

$$\frac{\partial}{\partial r} \lrcorner F_A = 0, \quad (1.30)$$

then there exists a smooth gauge transformation σ on $(\mathbb{R}^n \setminus \{0\}) \times V$ such that $\sigma(A) = p^(A_0)$ for a smooth connection A_0 on the trivial bundle $S^{n-1} \times V$, where $p : (\mathbb{R}^n \setminus \{0\}) \times V \rightarrow S^{n-1} \times V$ is the radial projection.*

Proof. By changing coordinates via $\phi : x \mapsto (\omega, t)$ on $\mathbb{R}^n \setminus \{0\}$, where $\omega = \frac{x}{|x|}$ and $t = -\log|x|$, we may assume A is a smooth connection on $S^{n-1} \times \mathbb{R}$ and

$$\frac{\partial}{\partial t} \lrcorner F_A = 0. \quad (1.31)$$

Decompose $A = a(t) + \beta(t)dt$ (under the trivialization of the bundle), where $a(t) \in \Omega^1(\mathfrak{g}_E|_{S^{n-1}})$, $\beta(t) \in \Gamma(\mathfrak{g}_E|_{S^{n-1}})$. Let $\sigma \in \Gamma(\text{Aut } P)$ with $\sigma(\cdot, t) \in \Gamma(\text{Aut } P|_{S^{n-1}})$ for fixed t , be a solution to the following ODE

$$\frac{\partial \sigma(t)}{\partial t} = \sigma(t)\beta(t), \quad t \in \mathbb{R} \quad (1.32)$$

$$\sigma(1) = \text{Id} \in \Gamma(\text{Aut } P|_{S^{n-1}}) \quad (1.33)$$

Then,

$$\sigma(A) = \sigma(t)(a(t)) + (\sigma(t)\beta(t)\sigma(t)^{-1} - \frac{\partial \sigma(t)}{\partial t} \sigma(t)^{-1})dt = \sigma(t)(a(t))$$

has no radial part. And (1.30) gives that

$$0 = \frac{\partial}{\partial t} \lrcorner F_A = \frac{\partial a}{\partial t} - d_a \beta = \sigma(t)^{-1} \left(\frac{\partial(\sigma(t)(a))}{\partial t} \right) \sigma(t)$$

Hence $\frac{\partial(\sigma(t)(a))}{\partial t} = 0$ and $\sigma(A) = \sigma(t)(a(t)) = p^*(A_0)$ for some connection A_0 on S^{n-1} .

\square

1.5 Statement of the main theorem

Assume M is of dimension $n \geq 5$. Now we can state our main result as follows.

Theorem 1 *Let \mathbb{A} be a smooth stationary Yang-Mills connection on bundle E on $M \setminus \{x\}$. Assume that every tangent connection $(\mathbb{A}_0, S_0, \Theta)$ of \mathbb{A} at x has isolated singularity at 0, i.e. $S_0 \subset \{0\}$, \mathbb{A}_0 is smooth on $\mathbb{R}^n \setminus \{0\}$ and $\Theta \equiv 0$. Then the tangent connection of \mathbb{A} is unique up to gauge transformations. Let $\mathbb{A}_0 = p^*(A_0)$ be the tangent connection in a suitable gauge, i.e., the pull back a Yang-Mills connection A_0 on S^{n-1} under the radial projection map $p : \mathbb{R}^n \setminus \{x\} \rightarrow S^{n-1}$. Then there exists gauge transformation τ on $M \setminus \{x\}$ such that*

$$\tau(\mathbb{A})_r|_{S^{n-1}} \xrightarrow{C^\infty} A_0 \quad \text{as } r \rightarrow 0$$

where $\tau(\mathbb{A})_r$ is the rescaled connection under the map $(t, x) \mapsto (rt, x)$. Furthermore, there exists constant C_k and $\alpha > 0$, such that

$$|\tau(\mathbb{A})_r|_{S^{n-1}} - A_0|_{C^k(S^{n-1})} \leq C_k |\log r|^{-\alpha}$$

Remark. 1) The assumption in the theorem is equivalent to that for every sequence of dilated connections of the original connection at the singularity, there exists a subsequence which converges in C^∞ on compact subsets of $\mathbb{R}^n \setminus \{0\}$ to a tangent connection which has isolated singularity at 0.

2) The theorem implies a connection satisfies the assumptions of Theorem 1 locally has a asymptotic cone-like structure at the singularity. If \mathbb{A} has a discrete set of isolated singularities, then the theorem applies to every singularity and gives local asymptotic cone-like structure of the connection at each of them. It would be interesting to construct examples of such connections with isolated singularities on compact manifolds.

3) We have a remark at the end of Chapter 3 about the slow convergence rate to the tangent connection.

Chapter 2

Constructions of Gauges

This chapter is mainly about construction of gauges in our proof. In Section 2.1, we derived the Yang-Mills equations under a change of coordinates (geometrically we change a punctured disk into a cylinder) and restate Theorem 1 under this coordinates. In Section 2.2, we described some ways to bound gauges and tangent connections. In Section 2.3, we showed the existence of a ‘standard form’ gauge for connections on a cylinder and gave some a priori elliptic estimates for connections in standard form. In Section 2.4, we showed the existence of a standard form gauge on a time interval with a certain condition at the end of the maximal existence interval of the above gauge. In Section 2.5, we derived estimates of the growth and bound of the connection under the above gauge at the end of the maximal time interval. We leave the completion of the proof of Theorem 1 until Chapter 3.

2.1 A change of coordinates

Since our problem in Theorem 1 is of local nature, we may assume E is a trivial bundle with product metric on $B_2(0)$ which has standard flat metric. Nonstandard metrics only give rise to a perturbation term which do not affect our proof following essentially (also see the remark following Lemma 5) . To facilitate our proof later, we make a change of coordinates. Consider the diffeomorphism map $\phi : B_2(0) \rightarrow S^{n-1} \times [0, \infty)$ defined by $\phi(x) = (\omega(x), t(x))$, where $\omega(x) = \frac{x}{|x|} \in S^{n-1}$, $t(x) = -\log(|x|) \in \mathbb{R}$. With a little ambiguity, we use E also to represent $(\phi^*)^{-1}(E)$. We have $(\phi^*)^{-1}(\mathbb{A}) = A(t) + \beta(t)dt$, where $A \in \Omega^1(\text{End } E)$, $\beta \in \Omega^0(\text{End } E)$.

Lemma 5 *The Yang-Mills equation $d_{\mathbb{A}}^* F_{\mathbb{A}} = 0$ is equivalent to the following system of equations,*

$$\ddot{A} - (n-4)\dot{A} - d_A^* F_A - d_A \dot{\beta} + (n-4)d_A \beta + *[\beta, *d_A \beta] = 0 \quad (2.1)$$

$$d_A^*(\dot{A} - d_A \beta) = 0 \quad (2.2)$$

Proof. Recall that from Lemma 2, $d_B^*(\xi) = d^*\xi \pm *[B, *\xi] = \pm(*d*\xi + *[B, *\xi])$, for $\xi \in \Omega^*(\text{End } E)$. The signs depend on n and $\deg \xi$. We know that d commutes with $(\phi^*)^{-1}$, therefore we only need to investigate the behavior of $*$ under $(\phi^*)^{-1}$. Let g_0 be the standard metric in \mathbb{R}^n , g the standard product metric on $S^{n-1} \times \mathbb{R}$, and $\tilde{g} = (\phi^*)^{-1}(g_0)$ be the pushforward metric of g_0 . Let $*_{n0}$, $*_n$, $\tilde{*}_n$ and $*$ be the Hodge operator associated to respectively g_0 , g , \tilde{g} and the standard metric S^{n-1} . Let d_{n0} , d_n and d be the exterior differential on respectively \mathbb{R}^n , $S^{n-1} \times \mathbb{R}$ and S^{n-1} . If $(d\omega^i, dt)$ is local orthonormal basis of $T^1(S^{n-1} \times \mathbb{R})$ with respect to g , it is easy to see that $(e^{-t}d\omega^i, -e^{-t}dt)$ constitute an orthonormal basis for \tilde{g} . Hence

$$\tilde{g} = e^{2t}g, \quad \langle \alpha, \beta \rangle_{\tilde{g}} = e^{2 \deg \beta t} \langle \alpha, \beta \rangle_g, \quad dV_{\tilde{g}} = e^{-nt}dV_g \quad (2.3)$$

Therefore

$$\alpha \wedge \tilde{*}_n \beta = \langle \alpha, \beta \rangle_{\tilde{g}} dV_{\tilde{g}} = e^{2 \deg \beta t} \langle \alpha, \beta \rangle_g e^{-nt} dV_g = e^{-(n-2 \deg \beta)t} \alpha \wedge *_n \beta$$

That is

$$\tilde{*}_n \beta = e^{-(n-2 \deg \beta)t} *_n \beta$$

We can now begin the calculation, first

$$d_{\mathbb{A}}^* F_{\mathbb{A}} = \pm(*_{n0}d_{n0} *_n F_{\mathbb{A}} + *_n0[\mathbb{A}, *_n0 F_{\mathbb{A}}])$$

and we have

$$(\phi^*)^{-1}(F_{\mathbb{A}}) = F_A - (\dot{A} - d_A \beta) dt$$

For simplicity, let $\eta = (\dot{A} - d_A \beta) dt$.

$$\begin{aligned} (\phi^*)^{-1}(*_{n0}d_{n0} *_n F_{\mathbb{A}}) &= \tilde{*}_n d_n \tilde{*}_n (F_A - \eta dt) = e^{(n-2)t} *_n d_n e^{-(n-4)t} *_n (F_A - \eta dt) \\ &= -e^{2t} \{ *d * F_A + \dot{\eta} - (n-4)\eta + (*d * \eta) \wedge dt \} \\ (\phi^*)^{-1}(*_{n0}[\mathbb{A}, *_n0 F_{\mathbb{A}}]) &= \tilde{*}_n [A + \beta dt, \tilde{*}_n (F_A - \eta dt)] \\ &= e^{2t} *_n [A + \beta dt, *_n (F_A - \eta dt)] \\ &= e^{2t} \{ *[A, *\eta] dt - (*[A, *F_A] - *[\beta, *\eta]) \} \end{aligned}$$

Combining the above two equalities, we obtain

$$(\phi^*)^{-1}(d_{\mathbb{A}}^* F_{\mathbb{A}}) = \pm e^{2t} \{ (*d * \eta + *[A, *\eta]) \wedge dt - (\dot{\eta} - (n-4)\eta - d_A^* F_A - *[\beta, *\eta]) \}$$

Hence \mathbb{A} is Yang-Mills if and only if the dt part and the vertical part without dt above are zero. This gives the two equations in the lemma. \square

Since this cylindrical coordinate representation is quite convenient for us, we shall restate Theorem 1 in terms of this coordinate system. For this we assume from now on in this chapter $M = S^{n-1}$, and E is a vector bundle on M with compact structure group G . We shall consider connections on the bundle $E \times [0, \infty)$ on $M \times [0, \infty)$. We let the bundle $E \times [0, \infty)$ and the manifold $M \times [0, \infty)$ have the product metric.

For notations, we use caligraphic letters \mathbb{A} , \mathbb{A}_0 , etc to represent connections on $E \times I$, where I is an interval of possibly infinite length (occasionally they also represent connections on a bundle on $U \subset \mathbb{R}^n$); we use A , A_0 , B , etc to represent connections on E .

Remark. 1) E shall be trivial in our proof of Theorem 1. However, almost all arguments (except those about stationarity as in Section 3.5) we used work for E nontrivial, in which case we can just fix a smooth connection \underline{A} on E and replace in our arguments A by $A - \underline{A}$, d and d^* by $d_{\underline{A}}$ and $d_{\underline{A}}^*$ etc.

2) In general, if the metric on $B_1(0)$ is not standard, in normal coordinates it is of the form $g_0 + O(r^2)$. Under our change of coordinates, it only makes a difference with exponential decay since $r = e^{-t}$. Such a fast decaying perturbation will not affect the validity of our arguments later.

In our conformal change of coordinates $B_2(0) \rightarrow S^{n-1} \times [0, \infty)$, $g = e^{-2t}\tilde{g}$ from (2.3), therefore we define the Yang-Mills functional on $M \times [0, \infty)$ by

$$\text{YM}(\mathbb{A}) = \int_{M \times [0, \infty)} |F_{\mathbb{A}}(x)|^2 e^{-(n-4)t} dx dt$$

where $|\cdot|$ is taken under the standard product metric. The Yang-Mills functional on E is the usual one

$$\text{YM}(A) = \int_M |F_A|^2 dx$$

Notice that if $\mathbb{A} = A(t) + \beta(t)dt$, then the Euler-Lagrange equation of YM on $E \times [0, \infty)$ is given by (2.1) and (2.2).

Given a connection \mathbb{A} on $E \times [0, \infty)$ and $\lambda > 0$, we define the rescaling of \mathbb{A} by $\mathbb{A}_\lambda = \lambda^*(\mathbb{A})$, where $\lambda : E \times [0, \infty) \rightarrow E \times [0, \infty)$ is the translation $\lambda : (v, t) \mapsto (v, t + \lambda), \forall v \in E, t \geq 0$.

Let \mathbb{A} be a smooth Yang-Mills connection on the bundle $E \times [0, \infty)$. We call the following requirements property (C):

For any sequence of positive numbers $\{\lambda_i\}$ going to infinity, there exists a subsequence $\{\lambda_{i_j}\}$ such that after smooth gauge transformations, the sequence $\{\mathbb{A}_{\lambda_{i_j}}\}$ converges in C^∞ on compact subsets of $M \times [0, \infty)$ to some connection \mathbb{A}_0 on $E \times [0, \infty)$ which is the pullback of a Yang-Mills connection A_0 on E under $p : E \times [0, \infty) \rightarrow E$.

By abuse of terminology, we shall call such an A_0 a *tangent connection* (at infinity) of \mathbb{A} .

Assume $\mathbb{A} = \phi^*(\mathbb{A}_1)$ for some Yang-Mills \mathbb{A}_1 on $B_r(0) \subset \mathbb{R}^n$ satisfies the hypotheses of Theorem 1, then by compactness theorem and Lemma 4, we see that \mathbb{A} satisfies property (C) above. Now we can state our theorem as follows.

Theorem 2 *Let \mathbb{A} be a smooth Yang-Mills connection with property (C) as above and the pull back connection \mathbb{A}_1 is stationary on $B_r(0)$, then \mathbb{A} has a unique tangent connection A_0 up to smooth gauge transformations on E . And there exists gauge transformation σ on $E \times [0, \infty)$ and constants $C_k > 0$ for any positive integer k and $\alpha > 0$, depending on \mathbb{A} , such that*

$$|\sigma(\mathbb{A})|_{E \times \{t\}} - A_0|_{C^k(M)} \leq C_k |t|^{-\alpha} \quad (2.4)$$

where we view $\sigma(\mathbb{A})|_{E \times \{t\}}$ as a connection on E via the identification $E \times \{t\} \cong E$ and C^k norm is taken with respect to a fixed smooth connection on E .

Next we shall describe the various norms of connections we are going to work with. Fix $\mu \in (0, 1)$ and k a nonnegative integer. Choose a finite covering of coordinate opens $\{U_i\}_{i=1}^m$, with coordinate maps $\phi_i : U_i \rightarrow V_i \subset \mathbb{R}^n$, such that there are trivialisations $\psi_i : E|_{U_i} \rightarrow U_i \times V$. We then choose a partition of unity $\{\chi_i\}$ subordinate to $\{U_i\}$, i.e., $\chi_i \in C^\infty(M)$, $\text{supp } \chi_i \subset U_i$ and $\sum \chi_i = 1$. We define the $C^{k,\mu}$ norm on mixed tensors in $\mathcal{T}(E) \otimes \Lambda^*M$ by

$$|\tau \otimes \alpha|_{C^{k,\mu}(M)} = \sum_{i=1}^m |\chi_i(\tau \otimes \alpha)|_{C^{k,\mu}(U_i)}$$

where $\chi_i(\tau \otimes \alpha)$ is viewed as a map from U_i to $\mathcal{T}(V) \otimes \Lambda^*\mathbb{R}^n$ via the trivialisations ϕ_i and ψ_i and $C^{k,\mu}$ norm is then taken for the standard vector-valued functions on \mathbb{R}^n . Similarly we may define $C^{k,\mu}$ norms for tensors of a bundle on a manifold with boundary, say $E \times I$ on $M \times I$, if we choose trivialisations at the boundary suitably. It can be checked that this definition of norm depends on different choices of the trivialisations only up to equivalence of norms. We also observe that the usual Schauder theories for elliptic operators on Hölder spaces hold through for these bundle valued Hölder spaces (by a partition argument, for example). Later on, we shall assume $k \geq 3$ is some constant.

Remark. The reason we use Hölder norms rather than Sobolev norms is that the restriction to submanifolds of $C^{k,\mu}$ functions are $C^{k,\mu}$ functions, which makes many statements simpler. However, it should be observed that the Sobolev norms also work for the proof.

For simplicity of notations, we let

$$\begin{aligned} |A|_{C^{k,\mu}} &:= |A|_{C^{k,\mu}(M)}, \\ |A(t)|_{C^{k,\mu}(I)} &:= \sup_{t \in I} |A(t)|_{C^{k,\mu}(M)}, \\ |\mathbb{A}|_{C^{k,\mu}(I)}^* &:= |\mathbb{A}|_{C^{k,\mu}(M \times I)}. \end{aligned}$$

Here the presence of $*$ emphasizes the fact that we take the derivative with respect to time t into account, while as the absence of $*$ expresses the opposite. We define the following anisotropic Hölder norms which shall be useful for us in later statement of propositions.

$$|a(t)|_{C^{(k,l),\mu}(I)} := \sum_{0 \leq j \leq k, 0 \leq i+j \leq k+l} \sup_I |\nabla_M^i \frac{\partial^j}{\partial t^j} a(t)|_{C^{k+l-i-j,\mu}(M)} \quad (2.5)$$

The space $C^{(k,l),\mu}(I)$, of course, is defined by those sections such that the norm as in (2.5) is finite. The difference between $C^{(k,l),\mu}(I)$ and $C^{k+l,\mu}(I)$ norms are that

the former only take up to k -th derivative to t , thus distinguishing the ‘time’ direction t and the ‘spatial’ direction on M . Note that $|a(t)|_{C^{(k,0),\mu}(I)} = |a(t)|_{C^{k,\mu}(I)}^*$ and $|a(t)|_{C^{(0,k),\mu}(I)} = |a(t)|_{C^{k,\mu}(I)}$. We also define the following space of connections:

$$\mathcal{S}^{k,\mu}(I) = \{A + \beta dt : A(t) \in C^{(k,0),\mu}(I), \beta(t) \in C^{(k-1,1),\mu}(I)\} \quad (2.6)$$

And we let the norm be

$$|A + \beta dt|_{\mathcal{S}^{k,\mu}(I)} = |A|_{C^{(k,0),\mu}(I)} + |\beta|_{C^{(k-1,1),\mu}(I)} \quad (2.7)$$

The natural space of gauges for connections in $\mathcal{S}^{k,\mu}(I)$ is given by gauges in $C^{(k,1),\mu}(I)$. We see from the formula

$$g(A + \beta dt) = gAg^{-1} - dg g^{-1} + (g\beta g^{-1} - \frac{\partial g}{\partial t} g^{-1}) dt \quad (2.8)$$

that $C^{(k,1),\mu}(I)$ gauges act on $\mathcal{S}^{k,\mu}(I)$ continuously.

For notations of L^2 and Sobolev norms, by fixing smooth covariant derivatives on E and on $E \times I$, we define

$$\begin{aligned} \|A\| &= \left(\int_M |A|^2 d\sigma \right)^{\frac{1}{2}}, \\ \|\mathbb{A}\|_I &= \left(\int_{M \times I} |\mathbb{A}|^2 dt d\sigma \right)^{\frac{1}{2}}, \\ \|A\|_{H^l} &= \left(\sum_{0 \leq |i| \leq l} \int_M |\nabla^i A|^2 d\sigma \right)^{\frac{1}{2}}, \\ \|\mathbb{A}\|_{H^l(I)} &= \left(\sum_{0 \leq |i| \leq l} \int_{M \times I} |\nabla^i \mathbb{A}|^2 d\sigma \right)^{\frac{1}{2}}. \end{aligned}$$

We often abuse notation and use A to represent the connection $p^*(A)$ on $M \times I$; the meaning should be clear from the context. The gauge transformations are usually assumed to have one higher order of smoothness than the connections they act on; hence by $\Gamma(\text{Aut } P)$ we often mean $C^{k+1,\mu}(M, \text{Aut } P)$. Let $\text{Stab}(A) = \{\sigma \in \Gamma(\text{Aut } P) : \sigma(A) = A\}$.

Assume \mathbb{A} satisfies property (C), then we obviously have

Lemma 6 *Given $L > 0$. For any sequence of positive numbers $\{R_i\}$ going to infinity, there exists a subsequence i' , gauge transformations $\sigma_{i'} \in \Gamma(\text{Aut } P \times [R_{i'}, R_{i'} + L])$ and a tangent Yang-Mills A on M of \mathbb{A} , such that*

$$\lim_{i' \rightarrow \infty} |\sigma_{i'}(\mathbb{A}) - A|_{C^{k,\mu}([R_{i'}, R_{i'} + L])}^* = 0$$

2.2 Some properties of gauges

We give some well-known yet very useful facts about norms of gauges in the next lemma.

Lemma 7 *Let \mathbb{A}_0 and \mathbb{A}_1 be connections in $C^{k,\mu}(I)$. Let σ be a gauge on I ,
a) If $\max\{|\mathbb{A}_0|_{C^{k,\mu}(I)}^*, |\sigma(\mathbb{A}_0)|_{C^{k,\mu}(I)}^*\} \leq C$. Then there exists $C_1 = C_1(k, C)$ such that*

$$|\sigma|_{C^{k+1,\mu}(I)}^* \leq C_1$$

b) If $|\sigma|_{C^{k,\mu}(I)}^ \leq C$, then there exists $C_1 = C(k)C^2$ such that*

$$|\sigma(\mathbb{A}_1) - \sigma(\mathbb{A}_0)|_{C^{k,\mu}(I)}^* \leq C_1 |\mathbb{A}_0 - \mathbb{A}_1|_{C^{k,\mu}(I)}^*$$

Proof. a) We have

$$d_{\mathbb{A}_0} \sigma = (\sigma(\mathbb{A}_0) - \mathbb{A}_0) \cdot \sigma \tag{2.9}$$

We also observe the fact that $|\sigma|_{C^0} \leq C$ since the fiber of $\text{Aut } P$, G , is compact. Therefore we can apply a bootstrapping procedure to (2.9) and prove the required estimates. This estimate is standard, see for example [DK].

b) We have

$$\sigma(\mathbb{A}_1) - \sigma(\mathbb{A}_0) = \sigma \cdot (\mathbb{A}_1 - \mathbb{A}_0) \cdot \sigma^{-1} \tag{2.10}$$

Notice that $d(\sigma^{-1}) = -\sigma^{-1} d\sigma \sigma^{-1}$ and therefore,

$$|\sigma^{-1}|_{C^{k,\mu}(I)}^* \leq C(k) |\sigma|_{C^{k,\mu}(I)}^* \tag{2.11}$$

Combining (2.10) and (2.11), we have the required bounds. \square

Remark. The above lemma also holds for other suitable spaces of connections and natural spaces of gauges acting on them. For examples, the lemma is true for connections in $\mathcal{S}^{k,\mu}(I)$ and gauges in $C^{(k,1),\mu}(I)$, while as in (b) we only need $C^{(k,0),\mu}$ bound of gauges; for connections in $C^{k,\mu}(M)$ and gauges in $C^{k+1,\mu}(M)$, while as in (b) we only need $C^{k,\mu}(M)$ bound of gauges. Such bounds for gauges and gauge transformed differences between connections are crucial to us and frequently used in below and we shall use them often implicitly.

The following lemma asserts the existence of gauge under which the Hölder norms of the connection is minimal.

Lemma 8 *Assume A is a connection on E with $|A|_{C^{k,\mu}} < \infty$. Then there exists $g \in C^{k+1,\mu}(\text{Aut } P)$ with*

$$|g(A)|_{C^{k,\mu}} = \inf_{h \in C^{k+1,\mu}(\text{Aut } P)} |h(A)|_{C^{k,\mu}}$$

Proof. Choose a sequence $g_n \in C^{k+1,\mu}(\text{Aut } P)$ such that

$$\lim_{n \rightarrow \infty} |g_n(A)|_{C^{k,\mu}} = \inf_{h \in C^{k+1,\mu}(\text{Aut } P)} |h(A)|_{C^{k,\mu}} := \delta$$

By Lemma 7, we have $|g_n|_{C^{k+1,\mu}} \leq C$. Hence there exists a subsequence $g_{n'}$ such that $g_{n'} \rightarrow g$ in C^{k+1} and we have $|g|_{C^{k+1,\mu}} \leq C$. Now $g_{n'} A g_{n'}^{-1} \rightarrow g A g^{-1}$ in $C^{k,\mu}$ and $d g_{n'} g_{n'}^{-1} \rightarrow d g g^{-1}$ in C^k . These imply that $g_{n'}(A) \rightarrow g(A)$ in C^k , therefore,

$$|g(A)|_{C^{k,\mu}} \leq \liminf_{n \rightarrow \infty} |g_{n'}(A)|_{C^{k,\mu}} = \delta$$

□

Assume \mathbb{A} is a Yang-Mills connection with property (C). In the following lemma we give an interesting fact about the set of tangent connections of \mathbb{A} . It may be viewed as compactness of the tangent connections up to gauge equivalence. Although the lemma itself is trivial in view of Theorem 1, the idea and result of its proof is useful in later parts.

Lemma 9 *Let $\mathcal{C} = \{ \text{the set of tangent connections of } \mathbb{A} \}$, then there exists $c_1 = c_1(k, \mathbb{A}) > 0$ such that for any $A \in \mathcal{C}$, there exists $g \in C^{k+1,\mu}(\text{Aut } P)$ with $|g(A)|_{C^{k,\mu}} \leq c_1$.*

Proof. Assume the contrary, then by Lemma 8, there exists a sequence of tangent Yang-Mills connections $A_i \in C^{k,\mu}$ of \mathbb{A} , with

$$|A_i|_{C^{k,\mu}} = \inf_{h \in C^{k+1,\mu}(\text{Aut } P)} |h(A_i)|_{C^{k,\mu}},$$

$$\lim_{i \rightarrow \infty} |A_i|_{C^{k,\mu}} \rightarrow \infty.$$

Since A_i are tangent connections of \mathbb{A} , for any sequence $\varepsilon_i \rightarrow 0$, there exists $R_i > 0$, $R_i \rightarrow \infty$ and $g_i \in \Gamma(\text{Aut } P \times [R_i, R_i + 1])$ with

$$|g_i(\mathbb{A}) - A_i|_{C^{k,\mu}}^* \leq \varepsilon_i \tag{2.12}$$

We shall determine ε_i later to produce a contradiction. Applying Lemma 6 to the sequence $\{R_i\}$ with $L = 1$, there exists a subsequence, which we still denote by $\{R_i\}$ for simplicity, a tangent Yang-Mills connection $A \in C^{k,\mu}$ and gauge transformations $g'_i \in \Gamma(\text{Aut } P \times [R_i, R_i + 1])$ such that

$$\lim_{i \rightarrow \infty} |g'_i(\mathbb{A}) - A|_{C^{k,\mu}([R_i, R_i + 1])}^* = 0$$

Denote $c_0 := |A|_{C^{k,\mu}} < \infty$. For $0 < \varepsilon < \min\{\frac{c_0}{2}, 1\}$, choose i sufficiently large such that

$$|A_i|_{C^{k,\mu}} > 2c_0 + 1, \tag{2.13}$$

$$|g'_i(\mathbb{A}) - A|_{C^{k,\mu}([R_i, R_i + 1])}^* \leq \varepsilon. \tag{2.14}$$

Let $h_i = g'_i g_i^{-1}$ be a gauge on I_i . By $dh_i = h_i g_i(\mathbb{A}) - g'_i(\mathbb{A}) h_i$, (2.12), (2.14) and a bootstrapping procedure similar to Lemma 7, we have (denote $I_i = [R_i, R_i + 1]$)

$$\begin{aligned} |h_i|_{C^{k+1,\mu}(I_i)}^* &\leq c_k(|g_i(\mathbb{A})|_{C^{k,\mu}(I_i)} + |g'_i(\mathbb{A})|_{C^{k,\mu}(I_i)}) \leq c_k(2|A|_{C^{k,\mu}} + 2|A_i|_{C^{k,\mu}}) \\ &\text{by (2.13)} \leq 4c_k|A_i|_{C^{k,\mu}} \end{aligned} \quad (2.15)$$

By Lemma 7,

$$\begin{aligned} |h_i(g_i(\mathbb{A})) - h_i(A_i)|_{C^{k,\mu}(I_i)}^* &\leq c_k(|h_i|_{C^{k+1,\mu}(I_i)}^*)^2 |g_i(\mathbb{A}) - A_i|_{C^{k,\mu}(I_i)}^* \\ &\text{by (2.12) and (2.15)} \leq c_k \varepsilon_i |A_i|_{C^{k,\mu}}^2 \leq c_k i^{-1}, \end{aligned}$$

if we choose $\varepsilon_i \leq i^{-1} |A_i|_{C^{k,\mu}}^{-2}$. We next have

$$\begin{aligned} |h_i(A_i)|_{C^{k,\mu}(I_i)}^* &\leq |h_i(g_i(\mathbb{A})) - h_i(A_i)|_{C^{k,\mu}(I_i)}^* + |h_i(g_i(\mathbb{A})) - A_i|_{C^{k,\mu}(I_i)}^* + |A_i|_{C^{k,\mu}} \\ &\leq c_k i^{-1} + \varepsilon + c_0 \leq 2c_0 \end{aligned}$$

if i is sufficiently large and ε is sufficiently small such that $c_k i^{-1} + \varepsilon \leq c_0$. Take $g = h_i(R_i)$, then $g \in C^{k+1,\mu}$. Noticing that $(h_i(A_i))(t) = h_i(t)(A_i) + (-\frac{\partial h_i}{\partial t})dt$, we have

$$|g(A_i)|_{C^{k,\mu}} \leq |h_i(A_i)|_{C^{k,\mu}(I_i)}^* \leq 2c_0 \quad (2.16)$$

This is a contradiction to the fact that A_i is in the minimal gauge and $|A_i|_{C^{k,\mu}} > 2c_0 + 1$. \square

2.3 Connections in standard form

For Yang-Mills connections, there is a standard way of fixing gauge, i.e. the *Coulomb gauge* (also called the *Hodge gauge*). We say B is in Coulomb gauge relative to A if

$$d_A^*(B - A) = 0$$

(see [DK] or [FU] for example). The above Coulomb gauge equation and the Yang-Mills equation (1.11) form an elliptic system. In the following, we find a suitable gauge for a connection \mathbb{A} on $E \times I$, I being an interval of possibly infinite length. Our choices of gauge are based on the ideas in Morgan, Mrowka and Ruberman [MMR, 2.4.3].

Lemma 10 *Assume \mathbb{A} is a smooth connection on $E \times I$ and A_0 is a smooth connection on E . There exists positive ε_1 depending only on A_0 and $C = C(A_0) > 0$ such that if*

$$|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \leq \varepsilon_1, \quad (2.17)$$

then there exists a gauge transformation $\sigma \in C^{(k,1),\mu}(I)$, such that $\sigma(\mathbb{A}) = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$ satisfies

$$d_{A_0}^*(A(t) - A_0) = d_{A_0}^*a(t) = 0 \quad (2.18)$$

$$\beta(t) \in \text{Ker}(d_{A_0})^\perp \quad (2.19)$$

$$|\sigma(\mathbb{A}) - A_0|_{\mathcal{S}^{k,\mu}(I)} \leq C(1 + |I|)|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I)} \quad (2.20)$$

where $a(t) \in \Omega^1(\mathfrak{g}_E)$ and $\text{Ker}(d_{A_0})^\perp \subset \Gamma(\mathfrak{g}_E)$. Furthermore, σ is unique up to the pullback of an element of $\text{Stab}(A_0)$, i.e., up to the action by $p^*(\tau) \in \Gamma(\text{Aut } P \times I)$, $p : \text{Aut } P \times I \rightarrow \text{Aut } P$, and $\tau \in \text{Stab}(A_0) \subset \Gamma(\text{Aut } P)$.

Following the terminology in [MMR], we call the choice of gauge in the above lemma a *standard form* of \mathbb{A} around A_0 . Notice the condition $d_{A_0}^*(A(t) - A_0) = 0$ means $A(t)$ is in Coulomb gauge relative to A_0 . In the lemma, if $|I| = \infty$, then (2.20) has no content.

Proof. We divide the choice of gauge into two steps. First we find a gauge under which the Coulomb gauge condition (2.18) is satisfied. Consider the gauge action map

$$\begin{aligned} \Phi : C^{k+1,\mu}(\text{Aut } P) \times (C^{k,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \cap \text{Ker}(d_{A_0}^*)) &\rightarrow C^{k,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \\ (g, a) &\mapsto d_{A_0}g g^{-1} + g a g^{-1} \end{aligned}$$

The differential of this map at $(0, 0)$ is

$$\begin{aligned} D\Phi : C^{k+1,\mu}(\mathfrak{g}_E) \times C^{k,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) &\rightarrow C^{k,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \\ (h, b) &\mapsto d_{A_0}h + b \end{aligned}$$

It is well-known that the restriction of $D\Phi$ to

$$(C^{k+1,\mu}(\mathfrak{g}_E) \cap \text{Ker}(d_{A_0})^\perp) \times (C^{k,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \cap \text{Im}(d_{A_0}^*))$$

is an invertible map with continuous inverse map for the appropriate norms. The second component above corresponds to the tangent space of gauge equivalent orbits of connection A_0 , and the first component above corresponds to the tangent space of the moduli space of connections (at least when A_0 is irreducible), which is transverse to the gauge orbits. We can write down the inverse of the map as $(D\Phi)^{-1}(a) = (h, b)$, where $h = (d_{A_0}^*d_{A_0})^{-1}(d_{A_0}^*a)$ and $b = a - d_{A_0}h$. Here we used the fact that

$$d_{A_0}^*d_{A_0} : C^{k+1,\mu}(\mathfrak{g}_E) \cap \text{Ker}(d_{A_0})^\perp \rightarrow C^{k-1,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \cap \text{Ker}(d_{A_0}^*) \quad (2.21)$$

is invertible and has a continuous inverse. Using implicit function theorem, we see that a left inverse Ψ of Φ can be defined in a neighborhood of $(0, 0)$ and Ψ is continuous in the Hölder norms.

Now assume $\mathbb{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$ satisfies (2.17). It follows that $|a(t)|_{C^{(k,0),\mu}(I)} \leq \varepsilon_1$, $\forall t \in I$. If ε_1 is sufficiently small, then we can apply Ψ to $a(t)$ to obtain $\Psi(a(t)) = (g(t), a_1(t))$. Viewing g as a gauge transformation of $E \times I$, we have

$$g(\mathbb{A}) = A_0 + a_1(t) + (g(t)\beta(t)g(t)^{-1} - \frac{\partial}{\partial t}g(t)g^{-1})dt$$

Since $a_1(t) \in \text{Ker}(d_{A_0}^*)$, the spatial part $A(t)$ of $g(\mathbb{A})$ is in Coulomb gauge relative to A_0 . It is only left to check that $g \in C^{(k,1),\mu}(I)$ and

$$|g(\mathbb{A}) - A_0|_{S^{k,\mu}(I)} \leq C|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \quad (2.22)$$

For this, we observe first

$$|g(t) - \text{Id}|_{C^{(0,k+1),\mu}} \leq C|a(t)|_{C^{(0,k),\mu}(I)} \leq C|\mathbb{A} - A_0|_{S^{k,\mu}(I)}, \quad (2.23)$$

and we have

$$\frac{\partial}{\partial t}g(t) = D_1\Psi_{a(t)}\left(\frac{\partial}{\partial t}a(t)\right) \quad (2.24)$$

where $\frac{\partial}{\partial t}a(t) \in C^{k-1,\mu}(\Lambda^1 \otimes \mathfrak{g}_E)$ and

$$D_1\Psi : C^{k-1,\mu}(\Lambda^1 \otimes \mathfrak{g}_E) \rightarrow C^{k,\mu}(\text{Aut } P)$$

is a continuous map. Hence $\frac{\partial}{\partial t}g \in C^{(0,k),\mu}(I)$ and

$$\left|\frac{\partial}{\partial t}g\right|_{C^{(0,k),\mu}(I)} \leq C\left|\frac{\partial}{\partial t}a(t)\right|_{C^{(0,k-1),\mu}(I)} \leq C|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \quad (2.25)$$

By taking further derivatives of (2.24), we can derive that $\frac{\partial^j}{\partial t^j}g \in C^{(0,k+1-j),\mu}(I)$ and

$$\left|\frac{\partial^j}{\partial t^j}g\right|_{C^{(0,k+1-j),\mu}(I)} \leq C|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \quad (2.26)$$

for $1 \leq j \leq k$. (2.23) and (2.26) imply that $g \in C^{(k,1),\mu}(I)$ and (2.22) by continuity of gauge actions.

We can now assume $\mathbb{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$, $a(t) \in \text{Ker}(d_{A_0}^*)$ and (2.17) holds. Assume

$$\beta(t) = \beta_0(t) + \beta_1(t), \quad \beta_0(t) \in \text{Ker}(d_{A_0}), \quad \beta_1(t) \in \text{Ker}(d_{A_0})^\perp$$

By (2.21) and $\beta(t) \in C^{(k-1,1),\mu}$, we have $\beta_1 = (d_{A_0}^*d_{A_0})^{-1}(d_{A_0}\beta) \in C^{(k-1,1),\mu}(I)$ and $\beta_0 = \beta - \beta_1 \in C^{(k-1,1),\mu}(I)$. The following ordinary differential equation

$$\begin{cases} \frac{\partial g}{\partial t} = g\beta_0(t), & \forall t \in I \\ g(0) = \text{Id}. \end{cases} \quad (2.27)$$

has a solution $g \in C^{(k,0),\mu}(\text{Aut } P \times I)$. We have

$$\begin{aligned} |g - \text{Id}|_{C^{(k,0),\mu}(I)} &\leq C(|\beta_0|_{C^{(k-1,1),\mu}(I)} + \int_I |\beta_0|_{C^{(k-1,1),\mu}(I)} dt) \\ &\leq C(1 + |I|)|\mathbb{A} - A_0|_{S^{k,\mu}(I)}. \end{aligned} \quad (2.28)$$

Since $\beta_0(t) \in \text{Ker}(d_{A_0}) =$ the Lie algebra of $\text{Stab}(A_0)$, it follows that $g(t) \in \text{Stab}(A_0), \forall t \in I$. i.e.

$$d_{A_0}g(t) = 0. \quad (2.29)$$

By differentiation of (2.29) with respect to t , we have

$$d_{A_0} \frac{\partial^l g(t)}{\partial t^l} = 0, \quad \text{for } 1 \leq l \leq k. \quad (2.30)$$

From (2.29) and (2.30), we have $g \in C^{(k,1),\mu}(I)$ and the following estimates.

$$|g - \text{Id}|_{C^{(k,1),\mu}(I)} \leq C|g - \text{Id}|_{C^{(k,0),\mu}(I)} \leq C(1 + |I|)|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \quad (2.31)$$

(In fact, we have $g \in C^{(k,l),\mu}(I)$ for any $l \geq 1$ and corresponding estimates, but (2.31) suffices for our purpose.) We have the following lemma.

Lemma 11 *Assume A_0 is a connection on E , $g \in \text{Stab}(A_0)$ and $A \in A_0 + \text{Ker}(d_{A_0}^*)$, then*

$$g(A) \in A_0 + \text{Ker}(d_{A_0}^*).$$

If $\beta \in \text{Ker}(d_{A_0})^\perp \subset \Gamma(\mathfrak{g}_E)$, then

$$g(\beta) = g\beta g^{-1} \in \text{Ker}(d_{A_0})^\perp.$$

Proof.

$$\begin{aligned} d_{A_0}^*(g(A) - A_0) &= d_{g(A_0)}^*(g(A) - g(A_0)) = g \circ d_{A_0}^* \circ g^{-1}(g(A - A_0)g^{-1}) \\ &= g(d_{A_0}^*(A - A_0))g^{-1} = 0 \end{aligned}$$

Hence $g(A) \in A_0 + \text{Ker}(d_{A_0}^*)$. Similar to above, we can prove $g \in \text{Stab}(A_0)$ preserves $\text{Ker}(d_{A_0}) \subset \Gamma(\mathfrak{g}_E)$. Since gauge action is orthogonal, g also preserves $\text{Ker}(d_{A_0})^\perp$. \square

We have by (2.27),

$$g(\mathbb{A}) = g(t)(A(t)) + (g\beta(t)g^{-1} - \frac{\partial g}{\partial t}g^{-1})dt = A_0 + a_1(t) + g(\beta_1(t))dt,$$

where $a_1(t) \in \text{Ker}(d_{A_0}^*)$ and $g(\beta_1(t)) \in \text{Ker}(d_{A_0})^\perp$ by the above lemma; hence $g(\mathbb{A})$ is of standard form around A_0 . Now

$$\begin{aligned} |g(\mathbb{A}) - A_0|_{S^{k,\mu}(I)} &\leq |g(t)(A(t) - A_0)|_{C^{(k,0),\mu}(I)} + |g\beta_1 g^{-1}|_{C^{(k-1,1),\mu}(I)} \\ &\text{by Lemma 7} \leq C(1 + |g - \text{Id}|_{C^{(k,0),\mu}(I)})|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \\ &\leq C(1 + |I|)|\mathbb{A} - A_0|_{S^{k,\mu}(I)} \end{aligned}$$

if $|\mathbb{A} - A_0|_{S^{k,\mu}(I)} < 1$, where the last inequality follows from (2.28). \square

For Yang-Mills connections in standard form around a smooth Yang-Mills connection A_0 , we have a priori elliptic estimates. Assume $\mathbb{A} = A(t) + \beta(t)dt \in \mathcal{S}^{k,\mu}(I)$ is in standard form around A_0 . Let $\dot{a} = \frac{\partial}{\partial t}a$. From (2.1), we have $d_A^*(\dot{a} - d_A\beta) = 0$, hence

$$\begin{aligned} \Delta_A\beta &= d_A^*d_A\beta = d_A^*\dot{a} = d_{A_0}^*\dot{a} - *[a, *\dot{a}] \\ \text{by (2.18)} &= -*[a, *\dot{a}] \end{aligned} \quad (2.32)$$

If $|A - A_0|_{C^{k,\mu}}$ is sufficiently small, then $\Delta_A : \text{Ker}(d_{A_0})^\perp \rightarrow \text{Im}(d_A^*)$ is invertible and the inverse G_A has similar smoothing properties as the parametrix $G_{A_0} = (\Delta_{A_0})^{-1}$. Hence

$$\beta = -G_A(*[a, *\dot{a}]), \quad (2.33)$$

From (2.33), we have elliptic estimates for β in terms of norms of a . In particular, if $|a|_{C^{(k,0),\mu}(I)} < 1$, we have

$$|\beta|_{C^{(k-1,2),\mu}} \leq C|a|_{C^{(k,0),\mu}(I)}^2 \quad (2.34)$$

It is important for us to note that β is of quadratic nature in terms of a . Substitute (2.33) into (2.1) and use (2.18), we have

$$\begin{aligned} \ddot{a} - (n-4)\dot{a} - d_A^*F_{A_0+a} - d_{A_0}d_{A_0}^*a + d_A((G_A(*[a, *\dot{a}])))' - (n-4)d_A G_A(*[a, *\dot{a}]) \\ + *[G_A(*[a, *\dot{a}]), *d_A G_A(*[a, *\dot{a}])] = 0. \end{aligned} \quad (2.35)$$

We observe that (2.35) is an elliptic equation of a on $M \times I$ if $|a|_{\mathcal{S}^{k,\mu}(I)}$ is sufficiently small (by using the smoothing properties of G_A and quadratic nature of terms involving β). And we have by (2.34) and (2.35) the following a priori interior estimates for connections in standard form,

$$|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I_s)} \leq C(s)\|\mathbb{A} - A_0\|_I, \quad (2.36)$$

if $|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I)}$ is sufficiently small, where $I_s = [a+s, b-s]$ if $I = [a, b]$ and $s < \frac{b-a}{2}$. By the expression (2.33), we have

$$\|\beta(t)\| \leq c\tau\|a(t)\|, \quad (2.37)$$

if $|a(t)|_{C^{k,\mu}}^* \leq \tau \ll 1$. Combining (2.36) and (2.37) gives

$$|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I_s)} \leq C(s)\|a(t)\|_I \quad (2.38)$$

if $|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I)} \leq \tau \ll 1$. We can do bootstrapping in (2.35) and derive estimates for higher derivatives with the assumption that $|\mathbb{A} - A_0|_{\mathcal{S}^{k,\mu}(I)}$ is sufficiently small.

We also want to mention that if A_1 is a Yang-Mills connection on M in Coulomb gauge around A_0 and $|A_1 - A_0|_{C^{k,\mu}}$ is sufficiently small, then by the Yang-Mills equation and Coulomb gauge condition, we have

$$d_{A_0+a}^*F_{A_0+a} + d_{A_0}d_{A_0}^*a = 0 \quad (2.39)$$

From which and bootstrapping, we have elliptic estimates

$$|A_1 - A_0|_{C^{k+l,\mu}} \leq C(l)\|A_1 - A_0\| \quad (2.40)$$

for $l \geq 0$, if $|A_1 - A_0|_{C^{k,\mu}} \leq \varepsilon$, $\varepsilon = \varepsilon(A_0) > 0$.

2.4 Standard form gauges with bounds for Yang-Mills connections

We shall prove the following key proposition, which provides us with a good gauge (actually, the standard form around some tangent connection) on an interval and we provide here a lower bound of the connection at the end of the maximal existence interval of such a gauge. This design of gauge and the lower bound are motivated by Cheeger and Tian [CT].

Proposition 4 *Assume \mathbb{A} is a connection with property (C). Fix a tangent Yang-Mills connection A_0 of \mathbb{A} . Given $L > 0$, $R_1 > 0$, there exists $\tau_1 = \tau_1(A_0, L) > 0$ and a function $\varepsilon_1 = \varepsilon_1(\tau)$ satisfying $0 < \varepsilon_1(\tau) < \tau$ such that for $0 < \tau \leq \tau_1$ and $0 < \varepsilon \leq \varepsilon_1(\tau)$, there exists $R \geq R_1$, integer $2 < N \leq \infty$, and gauge transformation $g \in \Gamma(\text{Aut } P \times [R, R'])$, where $R' = \frac{2}{3}NL$, such that the following (a) and (b) hold:*

(a) $g(\mathbb{A}) = A(t) + \beta(t)dt$ is of standard form around A_0 .

(b) The following estimates hold,

$$|g(\mathbb{A}) - A_0|_{\mathcal{S}^{k,\mu}([R, R+2L])} \leq \varepsilon \quad (2.41)$$

$$|g(\mathbb{A}) - A_0|_{\mathcal{S}^{k,\mu}([R, R'])} \leq \tau \quad (2.42)$$

$$\left| \frac{\partial}{\partial t} g(\mathbb{A}) \right|_{\mathcal{S}^{k-1,\mu}([R, R'])} \leq \varepsilon \quad (2.43)$$

In particular, $|A(t) - A_0|_{\mathcal{C}^{k,\mu}} \leq \tau$ and $|\frac{\partial^l}{\partial t^l} A(t)|_{\mathcal{C}^{k-l,\mu}} \leq \varepsilon$ for $1 \leq l \leq k$ and $t \in [R, R']$. Furthermore, if N is the maximal integer such that (a) and (b) are satisfied, then (c) if $N < \infty$, we have

$$\sup_{[R'-L, R']} |A(t) - A_0|_{\mathcal{C}^{k,\mu}} \geq c_2 \tau, \quad (2.44)$$

where $0 < c_2 = c_2(A_0, L) < 1$

Remark. The condition on ε and τ means that ε is sufficiently small relative to τ and τ is sufficiently small relative to 1. As a simplification, it can be just said as “If $0 < \varepsilon \ll \tau \ll 1$ depending on A_0, L , then ...”. We shall use this expression later to simplify our statements.

Proof. We first have the following claim:

Assume $L > 0$, $\varepsilon > 0$, there exists $R_2 = R_2(\varepsilon, L) > 0$ such that if $R \geq R_2$, then there exists gauge transformation $g \in \Gamma(\text{Aut } P \times [R, R+L])$ and tangent connection A of \mathbb{A} , such that

$$|g(\mathbb{A}) - A|_{\mathcal{S}^{k,\mu}([R, R+L])} \leq \varepsilon, \quad (2.45)$$

$$|A|_{\mathcal{C}^{k,\mu}} \leq c_1, \quad (2.46)$$

where $c_1 = c_1(k, \mathbb{A})$ is the constant given in Lemma 9.

Assume the claim is not true, then there exist $\varepsilon_j \rightarrow 0$ and $R_j \rightarrow \infty$ such that there does not exist tangent Yang-Mills connection A such that (2.45) and (2.46) are true for R replaced by R_j . By Lemma 6, there exist a subsequence $\{j'\}$ and $g_{j'} \in \Gamma(\text{Aut } P \times [R_{j'}, R_{j'} + L])$ such that $g_{j'}(\mathbb{A})|_{[R_{j'}, R_{j'} + L]} \xrightarrow{S^{k,\mu}} A$, a tangent connection of \mathbb{A} . By virtue of Lemma 9, up to a further $C^{k+1,\mu}$ gauge transformation on M , we may assume $|A|_{C^{k,\mu}} \leq c_1$. Now taking j' large will give us a contradiction.

Assume now $L > 0$, $R_1 > 0$, $0 < \varepsilon \leq \tau$. We consider a number ε_2 with $0 < \varepsilon_2 < \varepsilon$ for convenience and fix $R > \max\{R_1, R_2(3L, \varepsilon_2) + L\}$ (R_2 as in the above claim). More restrictions on the magnitudes of $\varepsilon_2, \varepsilon, \tau$ will be determined later. Choose intervals $I_i = [R + \frac{2iL}{3} - L, R + \frac{2iL}{3}]$, for $0 \leq i \leq \infty$. These intervals are chosen so that $|I_i| = L$, $|I_i \cap I_{i+1}| = \frac{1}{3}L$ and $I_i \cap I_{i+2} = \emptyset$. We shall choose gauges on I_i inductively. We claim that there exist constant $c_2 = c_2(A_0, L)$, integer $2 < K \leq \infty$, Yang-Mills connections $A_i \in C^{k,\mu}$ in Coulomb gauge around A_0 and gauge g on $M \times \bigcup_{r=1}^K I_i$, such that $g(\mathbb{A})$ is in standard form around A_0 ,

$$|g(\mathbb{A}) - A_i|_{S^{k,\mu}(I_i)} \leq \varepsilon, \quad 0 \leq i \leq K, \quad (2.47)$$

$$|A_i - A_0|_{C^{k,\mu}} \leq \frac{\tau}{2}, \quad 0 \leq i \leq K, \quad (2.48)$$

and

$$|A(t) - A_0|_{C^{k,\mu}(I_K)} \geq c_2\tau, \quad (2.49)$$

In the following, we prove the above claim by induction. For $0 \leq i \leq 3$, we note that by Lemma 6, there exists gauge g on $[R - L, R + 2L] = \bigcup_{0 \leq i \leq 3} I_i$ so that $g(\mathbb{A})$ is in standard form around A_0 and

$$|g_0(\mathbb{A}) - A_0|_{S^{k,\mu}([R-L, R+2L])} \leq \varepsilon_2 \leq \varepsilon$$

Hence we may choose $A_i = A_0$, $0 \leq i \leq 3$ in above. It is quite obvious that if $\varepsilon \leq \frac{\tau}{2}$, then (2.47), (2.48) are satisfied for $0 \leq i \leq 3$.

In the proof following, constants C or c only depend on L and A_0 . Assume the claim is true for all integers j such that $j \leq i$ and $\mathbb{A} = A(t) + \beta(t)dt$ on $[-L, \frac{2i}{3}L]$. If $|A(t) - A_0|_{C^{k,\mu}(I_i)} \geq c_2\tau$ where $c_2 = c_2(A_0, L)$ is to be determined later, then we are done by setting $K = i$. So we assume

$$|A(t) - A_0|_{C^{k,\mu}(I_i)} \leq c_2\tau \quad (2.50)$$

We have by (2.47) and (2.50),

$$|A_i - A_0|_{C^{k,\mu}} \leq 2c_2\tau, \quad (2.51)$$

if $\varepsilon \leq c_2\tau$. Using the claim at the beginning, we may find tangent Yang-Mills connections \tilde{A}_{i+1} , and gauge transformations \tilde{g} on I_{i+1} such that

$$|\tilde{g}(\mathbb{A}) - \tilde{A}_{i+1}|_{S^{k,\mu}(I_{i+1})} \leq \varepsilon_2, \quad (2.52)$$

$$|\tilde{A}_{i+1}|_{C^{k,\mu}} \leq c_1, \quad (2.53)$$

Let $t_i = \frac{2i}{3}L - \frac{L}{6}$ be the middle point of $I = I_i \cap I_{i+1}$ and let $h = g(t_i) \cdot \tilde{g}(t_i)^{-1}$ be a gauge on M . By (2.47), (2.48), (2.52) and (2.53), we have

$$\begin{aligned} |\tilde{g}(t_i)(A(t_i))|_{C^{k,\mu}} &\leq |\tilde{A}_{i+1}|_{C^{k,\mu}} + C\varepsilon_2 \leq C, \\ |h(\tilde{g}(t_i)(A(t_i)))|_{C^{k,\mu}} &= |g(t_i)(A(t_i))|_{C^{k,\mu}} \leq |A_i|_{C^{k,\mu}} + C\varepsilon \leq C. \end{aligned}$$

Hence by remarks following Lemma 7, we have

$$|h|_{C^{k+1,\mu}} \leq C \quad (2.54)$$

Therefore,

$$\begin{aligned} |h(A_{i+1}) - A_i|_{C^{k,\mu}} &\leq |h(A_{i+1}) - h(\tilde{g}(t_i)(A(t_i)))|_{C^{k,\mu}} + |g(t_i)(A(t_i)) - A_i|_{C^{k,\mu}} \\ &\leq C\varepsilon_2 + C'\varepsilon \leq C\varepsilon \end{aligned} \quad (2.55)$$

We let $g' = h\tilde{g}$ on I_{i+1} , then

$$|g'(\mathbb{A}) - h(\tilde{A}_{i+1})|_{S^{k,\mu}(I_{i+1})} \leq |h(\tilde{g}(\mathbb{A}) - \tilde{A}_{i+1})|_{S^{k,\mu}(I_{i+1})} \leq C\varepsilon_2 \quad (2.56)$$

$$\begin{aligned} |h(\tilde{A}_{i+1}) - A_0|_{C^{k,\mu}} &\leq |h(\tilde{A}_{i+1}) - A_i|_{C^{k,\mu}} + |A_i - A_0|_{C^{k,\mu}} \\ &\text{by (2.51) and (2.52)} \leq C\varepsilon_2 + 2c_2\tau \leq Cc_2\tau \end{aligned} \quad (2.57)$$

if $\varepsilon \leq c_2\tau$.

We observe from (2.56) and (2.57) that g' and $h(\tilde{A}_{i+1})$ have already satisfied the required estimates for the claim except that they are not in standard form and Coulomb gauge around A_0 yet. Next we'd like to make a further transformation to make them so. We first prove the following lemma.

Lemma 12 *Assume I is an interval of length L , $0 < \varepsilon' \leq \tau' \leq \tau < 1$, τ is sufficiently small depending on A_0 and L , \mathbb{A} is a connection on $E \times I$. A_0, A_1 are Yang-Mills connections on M with*

$$|A_1 - A_0|_{C^{k,\mu}} \leq \tau' \quad (2.58)$$

$$|\mathbb{A} - A_1|_{S^{k,\mu}(I)} \leq \varepsilon' \quad (2.59)$$

Then there exist gauges $\bar{\eta} \in C^{(k,1),\mu}(I)$ and $\eta \in C^{k+1,\mu}(M)$ such that $\bar{\eta}(\mathbb{A})$, ηA_1 are respectively in standard form and Coulomb gauge around A_0 , and

$$\begin{aligned} |\eta(A_1) - A_0|_{C^{k,\mu}} &\leq C\tau' \\ |\bar{\eta}(\mathbb{A}) - \eta(A_1)|_{S^{k,\mu}(I)} &\leq C\varepsilon' \end{aligned}$$

Proof. We follow the two-step construction of standard gauges in the proof of Lemma 10. First, we have for each connection A on M close to A_0 in $C^{k,\mu}$ norm, a unique (though not canonically defined) gauge transformation σ_A (which is the first component of $\Psi(A - A_0)$ by the notations in Lemma 10) such that

$$d_{A_0}^*(\sigma_A(A) - A_0) = 0$$

and σ_A depend on A smoothly. Assume $\mathbb{A} = \tilde{A}(t) + \tilde{\beta}(t)dt$, we let

$$\begin{aligned}\bar{\eta}_0(t) &= \sigma_{\tilde{A}(t)} \\ \eta &= \sigma_{A_1}\end{aligned}$$

We have,

$$\begin{aligned}|\bar{\eta}_0(\mathbb{A}) - \eta(A_1)|_{\mathcal{S}^{k,\mu}(I)} &\leq |\sigma_{\tilde{A}(t)}(\mathbb{A} - A_1)|_{\mathcal{S}^{k,\mu}(I)} + |\sigma_{\tilde{A}(t)}(A_1) - \sigma_{A_1}A_1|_{\mathcal{S}^{k,\mu}(I)} \\ &\leq |\sigma_{\tilde{A}(t)}|_{C^{(k,1),\mu}(I)}|\mathbb{A} - A_1|_{\mathcal{S}^{k,\mu}(I)} + \left|\frac{\partial\sigma_{\tilde{A}(t)}}{\partial t}\sigma_{\tilde{A}(t)}^{-1}\right|_{C^{(k-1,1),\mu}(I)} \\ &\quad + |\sigma_{\tilde{A}(t)}^{-1}d\sigma_{\tilde{A}(t)} - \sigma_{A_1}^{-1}d\sigma_{A_1}|_{C^{(k,0),\mu}(I)} + |\sigma_{\tilde{A}(t)} - \sigma_{A_1}|_{C^{(k,1),\mu}(I)}|A_1|_{C^{k,\mu}} \\ &\leq C\varepsilon',\end{aligned}\tag{2.60}$$

Here we used the following

$$|\sigma_{\tilde{A}(t)}|_{C^{(k,1),\mu}(I)} \leq C \quad \text{by (2.59),}\tag{2.61}$$

$$\left|\frac{\partial\sigma_{\tilde{A}(t)}}{\partial t}\right|_{C^{(k-1,1),\mu}(I)} \leq C\varepsilon' \quad \text{by (2.59),}\tag{2.62}$$

$$|\sigma_{\tilde{A}(t)} - \sigma_{A_1}|_{C^{(k,1),\mu}(I)} \leq C\varepsilon' \quad \text{by smoothness of } \sigma \text{ and (2.59) .}\tag{2.63}$$

Next let $\bar{\eta}_1$ be the second step construction of gauge in Lemma 10 for the connection $\bar{\eta}_0(\mathbb{A}) = A'(t) + \beta'(t)dt$ on I . By (2.60), we have

$$|\beta'(t)|_{C^{(k-1,1),\mu}} \leq C\varepsilon'$$

And hence by (2.31), we have

$$|\bar{\eta}_1 - \text{Id}|_{C^{(k,1),\mu}} \leq C\varepsilon'\tag{2.64}$$

Now we let $\bar{\eta} = \bar{\eta}_1\bar{\eta}_0$, then $\bar{\eta}(\mathbb{A})$ and $\eta(A_1)$ are respectively in standard form and Coulomb gauge relative to A_0 , and

$$\begin{aligned}|\bar{\eta}(\mathbb{A}) - \eta(A_1)|_{\mathcal{S}^{k,\mu}(I)} &\leq |\bar{\eta}_1\bar{\eta}_0(\mathbb{A}) - \bar{\eta}_0(\mathbb{A})|_{\mathcal{S}^{k,\mu}(I)} + |\bar{\eta}_0(\mathbb{A}) - \eta(A_1)|_{\mathcal{S}^{k,\mu}(I)} \\ &\leq |d\bar{\eta}_1\bar{\eta}_1^{-1}|_{C^{(k,0),\mu}(I)} + \left|\frac{\partial\bar{\eta}_1}{\partial t}\bar{\eta}_1^{-1}\right|_{C^{(k-1,1),\mu}} \\ &\quad + |\bar{\eta}_1 - \text{Id}|_{C^{(k,1),\mu}(I)}|\bar{\eta}_0(\mathbb{A})|_{\mathcal{S}^{k,\mu}(I)} + C\varepsilon' \leq C\varepsilon', \text{ by (2.64) .} \\ |\eta(A_1) - A_0|_{C^{k,\mu}} &\leq C|A_1 - A_0|_{C^{k,\mu}} \leq C\tau'.\end{aligned}$$

This proves the lemma. \square

In view of (2.56) and (2.57), we can apply the lemma with \mathbb{A} , A_1 , I and ε' , τ' replaced by $g'(\mathbb{A})$, $h\tilde{A}_{i+1}$, I_{i+1} , ε_2 and $c_2\tau$. Thus there exists gauges $\bar{\eta}$ on I_{i+1} and η on M such that

$$\begin{aligned}|\bar{\eta}g'(\mathbb{A}) - \eta h\tilde{A}_{i+1}|_{\mathcal{S}^{k,\mu}(I_{i+1})} &\leq c_3\varepsilon_2 \\ |\eta h\tilde{A}_{i+1} - A_0|_{C^{k,\mu}} &\leq c_4c_2\tau\end{aligned}$$

Compare g and $\bar{\eta}g'$ on $I = I_{i+1} \cap I_i$, we see that they both make \mathbb{A} into standard form around A_0 , hence by Lemma 10, they must differ by a pull back of $\sigma \in \text{Stab}(A_0) \cap C^{k+1,\mu}(M)$, i.e. $g = \sigma\bar{\eta}g'$ on I . Therefore we can extend g to be over I_{i+1} by letting

$$g|_{I_{i+1}} = \sigma\bar{\eta}g'$$

Now we let $A_{i+1} = \sigma\eta h\tilde{A}_{i+1}$, then $g(\mathbb{A})$ and A_{i+1} are respectively in standard form and Coulomb gauge around A_0 , and

$$\begin{aligned} |g(\mathbb{A}) - A_{i+1}|_{S^{k,\mu}(I)} &\leq C|\bar{\eta}g'(\mathbb{A}) - \eta h\tilde{A}_{i+1}|_{S^{k,\mu}(I)} \leq c_5\varepsilon_2 \\ |A_{i+1} - A_0|_{C^{k,\mu}} &\leq C|\eta h\tilde{A}_{i+1} - A_0|_{C^{k,\mu}} \leq c_6c_2\tau \end{aligned}$$

Finally, if we take c_2, e_2 sufficiently small such that $c_6c_2 \leq \frac{1}{2}$, $c_5\varepsilon_2 \leq \varepsilon$ and $0 \ll \varepsilon \ll \tau \ll 1$ satisfying all the requirements in the proof and cited lemmas, then the induction step of the claim follows through. It is easy to see that the claim implies our proposition if we let $N = K$. \square

2.5 Growth and lower bounds of solutions at the end of existence interval

We are to prove the following Proposition, which states growth and bound conditions that the solution to the Yang-Mills evolution equation (2.1) and (2.2) must satisfy at the end of maximal existence interval of gauges as in Prop. 4.

Proposition 5 *Given $L > 0$ and $1 > \eta > 0$. If $0 < \varepsilon \ll \tau \ll 1$ depending on A_0, L and η , and $\mathbb{A} = A(t) + \beta(t)dt$ is a Yang-Mills connection with property (C) and in the gauge on $[R, R'] = [R, R + \frac{2}{3}NL]$ for constants ε, τ as in Prop. 4 with $N < \infty$ maximal, then*

$$\sup_{t \in [R'-L, R']} \|A(t) - A_0\| \leq (1 + \eta) \sup_{t \in [R'-2L, R'-L]} \|A(t) - A_0\|, \quad (2.65)$$

$$\sup_{t \in [R'-L, R']} \|A(t) - A_0\| \geq c\tau, \quad (2.66)$$

for some constant $c = c(L, \eta, A_0)$.

Proof. Since $N < \infty$, c) in Prop. 4 gives

$$|A(t) - A_0|_{C^{k,\mu}([R'-L, R'])} \geq c_2\tau \quad (2.67)$$

for $c_2 = c_2(A_0, L) > 0$. We first show

Lemma 13 *There exists a Yang-Mills connection A_1 with*

$$|A_1 - A(t)|_{C^{k,\mu}([R'-2L,R'])} \leq \delta(\varepsilon) \quad (2.68)$$

$$d_{A_0}^*(A_1 - A_0) = 0 \quad (2.69)$$

where δ is an increasing function with $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon) = 0$.

Proof. Assume the lemma is not true, then there exists a sequence $\varepsilon_j \rightarrow 0$ and a sequence of Yang-Mills connections $\mathbb{A}_j = A_j(t) + \beta_j(t)dt$ on $I = [R' - 2L, R']$ in standard form gauge around A_0 and satisfy conditions in Prop. 4 (b), i.e.

$$|\mathbb{A}_j - A_0|_{\mathcal{S}^{k,\mu}(I)} \leq \tau, \quad (2.70)$$

$$\left| \frac{\partial}{\partial t} \mathbb{A}_j \right|_{\mathcal{S}^{k-1,\mu}(I)} \leq \varepsilon_j, \quad (2.71)$$

and there exists $\varepsilon_0 > 0$ such that for any j there does not exist Yang-Mills connection A_1 such that (2.68) and (2.69) are true with ε and $A(t)$ replaced by ε_0 and $A_j(t)$. By compactness of \mathcal{S}^k in $\mathcal{S}^{k,\mu}$, by taking a subsequence, we may assume $\mathbb{A}_j \rightarrow \mathbb{A}' = A'(t) + \beta'(t)dt$ in $\mathcal{S}^k(I)$ and it follows that \mathbb{A}' is also a Yang-Mills connection in standard form around A_0 on I and

$$|\mathbb{A}'|_{\mathcal{S}^{k,\mu}(I)} \leq \liminf_{i \rightarrow \infty} |\mathbb{A}_j|_{\mathcal{S}^{k,\mu}(I)} \leq \tau$$

From (2.71), we have $\dot{A}'(t) = 0$, $\dot{\beta}'(t) = 0$. From Yang-Mills equations (2.1) and (2.2), we have

$$d_{A'}^* d_{A'} \beta' = 0$$

Since A' is close to A_0 in C^k norm, $d_{A'}^* d_{A'}$ would be invertible on the space $\text{Ker}(d_{A_0})$ where β' lies in, hence $\beta' \equiv 0$. From (2.1), we have A' is Yang-Mills, hence \mathbb{A}' is in fact the pullback of a Yang-Mills on E . Now (2.1) implies that

$$d_{A_j(t)}^* F_{A_j(t)} = d_1 + d_2 + d_3 \quad (2.72)$$

$$|d_1|_{C^{k-2,\mu}(I)} = |\ddot{A}_j(t) + \dot{A}_j(t)|_{C^{k-2,\mu}(I)} \leq 2\varepsilon_j \text{ by (2.71)}$$

$$|d_2|_{C^{k-2,\mu}(I)} = |d_{A_j} \dot{\beta}_j(t)|_{C^{k-2,\mu}(I)} \leq C\varepsilon_j \text{ by (2.71)}$$

$$\begin{aligned} |d_3|_{C^{k-2,\mu}(I)} &= |(n-4)d_{A_j} \beta_j(t) \pm *[\beta_j(t), *d_{A_j} \beta_j(t)]|_{C^{k-2,\mu}(I)} \\ &\leq C|\beta_j(t)|_{C^{k-1,\mu}(I)} := C\delta_j \end{aligned}$$

where $\delta_j = |\beta_j(t)|_{C^{k-1,\mu}(I)} \rightarrow 0$ since $\mathbb{A}_j \rightarrow \mathbb{A}'$ in $\mathcal{S}^k(I)$. The left hand side of (2.72) is a uniformly elliptic second order linear operator acting on $A_j(t) - A'$ with coefficients bounded in $C^{k,\mu}(M)$ uniformly in t . Hence we can apply elliptic estimates and bootstrapping to get

$$|A_j(t) - A'|_{C^{k,\mu}(I)} \leq C(\varepsilon_j + \delta_j + |A_j(t) - A'|_{C^{k-1,\mu}(I)}) \rightarrow 0$$

But this is a contradiction to our assumption at beginning. \square

Let A_1 be as in Lemma 13. Let $t_0 \in [R' - L, R']$ such that $|A_0 - A(t_0)|_{C^{k,\mu}} = \sup_{t \in [R' - L, R']} |A_0 - A(t)|$, then by (2.67) and (2.68), we have

$$|A_1 - A_0|_{C^{k,\mu}} \geq |A_0 - A(t_0)|_{C^{k,\mu}} - |A(t_0) - A_1|_{C^{k,\mu}} \geq c_2\tau - \delta(\varepsilon) \geq \frac{1}{2}c_2\tau \quad (2.73)$$

if $\delta(\varepsilon) \leq \frac{1}{2}c_2\tau$. Now we have for any $t \in [R' - 2L, R']$,

$$|A(t) - A_0|_{C^{k,\mu}} \geq |A_1 - A_0|_{C^{k,\mu}} - |A_1 - A(t)|_{C^{k,\mu}} \geq \frac{1}{2}c_2\tau - \delta(\varepsilon) \geq \frac{1}{4}c_2\tau$$

if $\delta(\varepsilon) \leq \frac{1}{4}c_2\tau$. By a priori elliptic estimates for Yang-Mills connections in standard form (2.38), we have

$$\begin{aligned} \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\| &\geq C^{-1} |A(R' - \frac{3L}{2}) - A_0|_{C^{k,\mu}} \geq \frac{1}{4}C^{-1}c_2\tau = c_7\tau \\ \sup_{t \in [R' - L, R']} \|A(t) - A_0\| &\geq C^{-1} |A(R' - \frac{L}{2}) - A_0|_{C^{k,\mu}} \geq c_7\tau \\ \sup_{t \in [R' - L, R']} \|A(t) - A_0\| &\leq \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\| + 2CL \left| \frac{\partial}{\partial t} A(t) \right|_{C^0([R' - 2L, R'])} \\ &\leq (1 + \eta) \sup_{t \in [R' - 2L, R' - L]} \|A(t) - A_0\| \end{aligned}$$

if $4CL\varepsilon \leq \eta c_7\tau$. This implies that if $0 < \varepsilon \ll \tau \ll 1$ depending on A_0 , L and η , then (2.65) and (2.66) hold. \square

Chapter 3

An asymptotic convergence result for a class of evolution equations

By fixing the gauge as in last Chapter, we can reduce the main theorem into a problem quite similar to the case of Theorem 2 in [SL1]. However, the situation in [SL1] is somewhat different from what we have here. For example, here we have growth control over only on the $C^{k,\mu}$ norm on the “spatial” direction, not on the t direction and only at the end of our existence interval $[R, R']$, not on the whole interval. We also have to consider the Yang-Mills functional restricted to the slice $A_0 + \text{Ker}(d_{A_0}^*)$. On the other hand, we have the bound

$$\left| \frac{\partial^l}{\partial t^l} A(t) \right|_{C^{k-l,\mu}(I)} \leq \varepsilon$$

which allows us to compare norms of connections at different points on the time interval. We shall modify the proof in [SL1] for our case. Section 3.1 to 3.3 are statements of the theorem we needed here, some results in [SL1] and adaptations in our case. In Section 3.4, we give the proof of the theorem in Section 3.1. In Section 3.5, we finished the proof of uniqueness of tangent connection and the rate of convergence to the tangent connection as stated in the Theorem 2 .

3.1 A type of nonlinear evolution equations

Let E be a vector bundle on Riemannian manifold M and let \mathcal{E} be a functional of ‘energy type’ defined for sections $a \in C^1(M, E)$ by

$$\mathcal{E}(a) = \int_M E(x, u, \nabla u) \tag{3.1}$$

where $F = F(x, z, p)$ for $x \in M$, $z \in E$, $p \in T_x M \otimes E_x$ depend smoothly on (x, z, p) and F is uniformly convex in the p variable for $p \in T_x M \otimes E_x$ and $|z|, |p|$ small. We

also require that F has analytic dependence on $(z, p) \in E \times T_x M \otimes E_x$ with uniform bounds on F and its derivatives in z, p for sufficiently small $|z|, |p|$. By this we mean that there exists $c_0 > 0$ such that

$$F(x, z + \sum_i \lambda_{1,i} w_i, p + \sum_j \lambda_{2,j} q_j) = \sum_{|\alpha| \geq 0} F_\alpha(x, z, w_1, \dots, w_m, p, q_1, \dots, q_{m'}) \lambda^\alpha \quad (3.2)$$

where $0 \leq i \leq m, 0 \leq j \leq m', m$ and m' being the dimension of fibers of bundle E and $TM \otimes E$. $|z|, |w|, |p_i|, |q_j| \leq c_0, z, w \in E_x, p_i, q_j \in T_x M \otimes E_x, \lambda = (\lambda_{1,1}, \dots, \lambda_{1,m}, \lambda_{2,1}, \dots, \lambda_{2,m'}) \in \mathbb{R}^{m+m'}, |\lambda| \leq 1$. The above expansion and its derivatives (in terms of z and p only) should converge absolutely in the above domain of variables and

$$\sup_{|\alpha|+|\beta|=j, |z|, |p| \leq c_0} |D_z^\alpha D_p^\beta F(x, z, p)| \leq c(j) \quad (3.3)$$

The Euler-Lagrange operator for $\mathcal{E}(a)$, denoted by $\mathcal{M}(a)$, is uniquely characterized by

$$-(\mathcal{M}(a), b)_{L^2(M)} = \frac{d}{ds} \mathcal{E}(a + sb)|_{s=0} \quad (3.4)$$

(Hence $\mathcal{M}(a) = -\text{grad } \mathcal{E}(a)$). We also require that $\mathcal{M}(0) = 0$, i.e. 0 is a critical point of \mathcal{E} . By uniform convexity, we have that $\mathcal{M}(a)$ is a second order quasi-linear operator which is uniformly elliptic for $|a|_{C^1(M)}$ sufficiently small. Thus the linearization (the Jacobi operator at 0)

$$La := \frac{d}{ds} \mathcal{M}(su)|_{s=0} \quad (3.5)$$

is a second order linear elliptic self adjoint operator.

We fix $k \geq 4$ in this chapter. Assume $a = a(x, t) \in C^{k,\mu}(M \times I, E \times I)$ is a section of $E \times I$. We consider equation of the following form

$$\ddot{a} - \gamma \dot{a} + N(a) + G_1(\dot{a}) + G_2(\ddot{a}) = 0 \quad (3.6)$$

In (3.6), $\gamma > 0$ is a constant. $N(a)$ is a second-order quasi-linear differential operator for $a \in C^{2,\mu}(M)$ which is elliptic if $|a|_{C^1}$ is sufficiently small. And N approximates \mathcal{M} well in the following sense

$$\|\mathcal{M}(a) - N(a)\| \leq \frac{1}{4} \min\{\|N(a)\|, \|\mathcal{M}(a)\|\}, \quad (3.7)$$

for any $a \in C^{k,\mu}(M)$ with $|a|_{C^{k,\mu}} \leq \sigma$, where $\sigma = \sigma(\mathcal{E})$ is a constant. Let $G_0(a) := N(a) - L(a)$ (L as in (3.5)). We require that

$$G_i : C^{l,\mu}(M \times I) \rightarrow C^{l,\mu}(M \times I) \quad \text{for } 0 \leq i \leq 2, 0 \leq l \leq k-2 \quad (3.8)$$

are continuously differentiable linear maps and

$$D^\beta G_i : C^{l,\mu}(M \times I) \rightarrow C^{l,\mu}(M \times I) \quad (3.9)$$

are continuous linear maps for $0 \leq |\beta| \leq k-2$, $0 \leq l \leq k-|\beta|-2$ with the following bounds on operator norms:

$$\|D^\beta G_i(a)\| \leq C|a|_{C^{k,\mu}} \quad (3.10)$$

If $a = a(x, t) \in C^{k,\mu}(M \times [t_1, t_2])$ is a solution of (3.6) with $|a|_{C^{k,\mu}(I)} \leq c_0$, from standard Schauder theory and Sobolev theory (for example, in [GT] and [Mo]), that for $\sigma \in (0, \frac{1}{2}(t_2 - t_1))$ and $0 \leq l \leq k-2$,

$$|a|_{C^{l,\mu}([t_1+\sigma, t_2-\sigma])} \leq c_l \sigma^{-(l+\mu)} \|a\|_{[t_1, t_2]} \quad (3.11)$$

Remark. In [SL1], an equation similar to (3.6) is considered with a nonhomogeneous term which decays exponentially. This corresponds to the case of nonproduct metric on our cylindrical bundles and manifolds. For simplicity, we have assumed that the metric is always the product metric on cylinders and avoid such a nonhomogeneous term. As we said before, all our arguments hold through with only technical adjustments with such a fast-decreasing nonhomogeneous term.

Example. In our case, if $\mathbb{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$ is in standard gauge around Yang-Mills A_0 , we replace E by $\mathfrak{g}_E \otimes \Lambda^1 M$. Let $\mathcal{E}(a) = \text{YM}(A_0 + a)$ in this case. Then from (2.1) and (2.2), $a(x, t)$ satisfies an equation of the form of (3.6), where

$$\begin{aligned} \gamma &= (n-4) \\ N(a) &= -(d_{A_0+a}^* F_{A_0+a} + d_{A_0} d_{A_0}^* a) \\ L(a) &= -(d_{A_0} d_{A_0}^* + d_{A_0}^* d_{A_0} + (-1)^n * [a, *F_{A_0}]) \\ G_0(a) &= N(a) - L(a) = (-1)^n * [a, *(d_{A_0} a + a \wedge a)] + d_{A_0}^* (a \wedge a) \\ G_1(\dot{a}) &= (n-4)d_A \beta + *[\beta, *(d_A \beta - \dot{a})] - [\dot{a}, \beta] + d_A (D_A G(\dot{a}))(*[a, *\dot{a}]) + d_A G_A(*[\dot{a}, *\dot{a}]) \\ G_2(\ddot{a}) &= d_A G_A(*[a, *\ddot{a}]) \end{aligned}$$

where $\beta = G_A(*[a, *\dot{a}])$, $G_A = (\Delta_A)^{-1}$. We regard only one \dot{a} in each term of G_1 as the variable, others (dependent on a and \dot{a} will be seen as parts of coefficients of G_1 . Similarly, we only regard the \ddot{a} in G_2 as the variable. By the smoothing properties of the parametrix G_A , and (3.32) later, it is easy to see that these operators satisfy the stated properties if $|a|_{C^{k,\mu}(I)} \leq \tau$.

Let $\mu_1 \leq \mu_2 \leq \dots$ and ϕ_1, ϕ_2, \dots are the eigenvalues and corresponding orthonormal (in L^2 norm) eigenfunctions of the operator L on $\Gamma(E)$, we let

$$\lambda_i^\pm = \frac{1}{2}(\gamma \pm \sqrt{\gamma^2 - 4\mu_i}). \quad (3.12)$$

Every solution of the linear evolution equation

$$\mathcal{L}(a) = \ddot{a} - \gamma\dot{a} + L(a) = 0, \quad (3.13)$$

can be written as

$$\begin{aligned} a(x, t) = & \sum_{i \in I_1} (a_i \cos \alpha_i t - b_i \sin \alpha_i t) e^{\frac{\gamma t}{2}} \phi_i(x) \\ & + \sum_{i \in I_2} (a_i + b_i t) e^{\frac{\gamma t}{2}} \phi_i(x) + \sum_{i \in I_3} (a_i e^{\lambda_i^+ t} + b_i e^{\lambda_i^- t}) \phi_i(x) \end{aligned} \quad (3.14)$$

for suitable constants a_i, b_i , where

$$\begin{aligned} I_1 &= \{i : \mu_i < -\frac{\gamma}{4}\}, \quad \alpha_i = \text{Im } \lambda_i^+ \\ I_2 &= \{i : \mu_i = -\frac{\gamma}{4}\}, \\ I_3 &= \{i : \mu_i > -\frac{\gamma}{4}\}. \end{aligned}$$

And the L^2 norm square of $a(\cdot, t)$ can be written as

$$\begin{aligned} \|a(t)\|^2 = & \sum_{i \in I_1} (a_i \cos \alpha_i t - b_i \sin \alpha_i t)^2 e^{\gamma t} \\ & + \sum_{i \in I_2} (a_i + b_i t)^2 e^{\gamma t} + \sum_{i \in I_3} (a_i e^{\lambda_i^+ t} + b_i e^{\lambda_i^- t})^2. \end{aligned} \quad (3.15)$$

As a notation, we let

$$\delta_1 = \min\{s : s \in \{\text{Re } \lambda_i^\pm\}, s > 0\}, \quad \delta_2 = \min\{|s| : s \in \{\text{Re } \lambda_i^\pm\}, s < 0\} \quad (3.16)$$

Assume $\varepsilon \leq \tau, \eta, L$ are positive constants, $a = a(x, t) \in C^{k, \mu}(M \times [R, R'])$, $R' - R \geq 3L$ and possibly $R' = \infty$. We make the following definition.

Definition. We call $a = a(x, t)$ $(\varepsilon, \tau, \eta, L)$ -bounded on $[R, R']$ if there exists $0 < c = c(\eta, L) < 1$ such that

$$|a|_{C^{k, \mu}([R, R'])}^* \leq \tau \quad (3.17)$$

$$\left| \frac{\partial a}{\partial t} \right|_{C^{k-1, \mu}([R, R'])}^* \leq \varepsilon \quad (3.18)$$

$$|a|_{C^{k, \mu}([R, R+2L])}^* \leq \varepsilon \quad (3.19)$$

and if $R' < \infty$ then

$$\|a\|_{[R'-L, R']} \leq (1 + \eta) \|a\|_{[R'-2L, R'-L]}, \quad (3.20)$$

$$\sup_{[R'-L, R']} \|a(t)\| \geq c\tau. \quad (3.21)$$

We state our theorem about solutions to (3.6) as follows.

Theorem 3 Assume $L > 0$. There exists $\eta \in (0, 1)$ such that if $0 < \varepsilon \ll \tau \ll 1$ depending on E, L and η , $a(t)$ is an $(\varepsilon, \tau, \eta, L)$ -bounded solution to (3.6) and

$$\mathcal{E}(a(t)) - \mathcal{E}(0) \geq -c\varepsilon, \quad (3.22)$$

then $R' = \infty$, $|a|_{C^{k,\mu}([R,\infty))}^* \leq \tau$ and

$$\lim_{t \rightarrow \infty} |a(t) - w|_{C^{2,\mu}}^* = 0 \quad (3.23)$$

3.2 Growth estimates

We have the following result on growth of solutions to (3.6).

Lemma 14 Assume $a \in C^{k,\mu}(M \times [0, 3L])$ satisfy (3.6), where G_i satisfies properties above. For any given $L > 0, 1 > \eta > 0, \delta < \frac{1}{4} \min\{\delta_1, \delta_2\}$, where δ_1, δ_2 as in (3.16), there exists $\tau_0 = \tau_0(L, \eta) > 0$, such that if $|a|_{C^{k,\mu}([0,3L])}^* \leq \tau_0$, then the following are true. Denote $S(j) = \sup_{t \in [(j-1)L, jL]} \|a(t)\|$, we have

- (i) $S(2) \geq e^{\delta L/2} S(1) \Rightarrow S(3) \geq e^{(\delta_1 - \delta)L} S(2)$
- (ii) $S(2) \geq e^{-(\delta_2 - \delta)L} S(1) \Rightarrow S(3) \geq (1 - \eta) S(2)$
- (iii) $S(2) \geq e^{\delta L/2} S(3) \Rightarrow S(1) \geq e^{(\delta_2 - \delta)L} S(2)$
- (iv) $S(2) \geq e^{-(\delta_1 - \delta)L} S(3) \Rightarrow S(1) \geq (1 - \eta) S(2)$
- (v) If $S(2) \geq \max\{e^{-(\delta_1 - \delta)L} S(3), e^{-(\delta_2 - \delta)L} S(1)\}$,
then $S(2) \leq (1 + \eta) \inf_{t \in [L, 2L]} \|a(t)\|$, $\|\dot{a}(t)\|_{C^1} \leq \eta \|a(t)\|$, $t \in [L, 2L]$.

Proof. We give a sketch of the proof in [SL1]. First we can prove that (i)-(v) hold for solutions to the linear equation (3.13) by using the expression of the L^2 norm of (3.15). Then by a blow-up argument we can prove that for τ_0 sufficiently small, the lemma holds for solutions to (3.6). In the proof, we also need the regularity properties of solutions to (3.6). \square

The next proposition is a modified and simpler form of Theorem 4 in [SL1] (without considering a possible inhomogeneous term in (3.6)).

Proposition 6 Assume a on $[R, R + NL]$ satisfies (3.6) as above. For $0 < \eta < 1$, there exists $\tau_0 > 0$, such that if $0 < |a|_{C^{k,\mu}([R, R + NL])}^* < \tau_0$, then there exists integers $1 \leq k_1 \leq k_2 \leq N - 1$ such that the following hold, where $S(j) = \sup_{t \in [(j-1)L, jL]} \|a(t)\|$.

- (a) $S(j) \leq e^{-(\delta_2 - \delta)L} S(j - 1)$, for $1 \leq j \leq k_1 - 1$.
- (b) $\|a(t_1)\| \leq (1 + \eta) \|a(t_2)\|$, for $t_1, t_2 \in [k_1 L, (k_2 - 1)L]$, $|t_1 - t_2| \leq L$
 $\|\dot{a}(t)\| \leq \eta \|a(t)\|$, for $t \in [k_1 L, (k_2 - 1)L]$.
- (c) $S(j) \geq e^{-(\delta_2 - \delta)L} S(j - 1)$, for $k_2 + 1 \leq j \leq N - 1$.

Proof. The proposition is proved by repetitive use of previous lemma. We set

$$k_1 = \min_{1 \leq j \leq k-1} \{S(j) \geq e^{-(\delta_2 - \delta)L} S(j-1)\}$$

$$k_2 = \min_{k_1 \leq j \leq N-1} \{S(j+1) \geq e^{(\delta_1 - \delta)L} S(j)\}$$

And it is easy to check the theorem holds. \square

This proposition allow us to divide the existence interval into three parts according to the ‘growth rate’ of the L^2 norms of a .

3.3 Łojasiewicz inequalities

The inequalities in the following proposition are infinite dimensional generalizations to analytic functionals given by Leon Simon [SL1] of Łojasiewicz inequalities with regard to critical points of analytic functions. Let $\mathcal{E}(a)$ be an analytic elliptic functional for $a \in C^1(M)$ as in Section 3.1.

Proposition 7 *There exists constants $0 < \theta < \frac{1}{2}$, $2 \leq \gamma$, $0 < \sigma$ depending only on \mathcal{E} , such that if $|a|_{C^{k,\mu}} < \sigma$, then*

$$\|\mathcal{M}(u)\| \geq \left(\inf_{\zeta \in \mathcal{S}} \|a - \zeta\|\right)^\gamma \quad (3.24)$$

where $\mathcal{S} = \{\zeta \in C^{k,\mu}(E) : |\zeta|_{C^{k,\mu}(\Sigma)} < c_0, \mathcal{M}(\zeta) = 0\}$, and

$$\|\mathcal{M}(u)\| \geq |\mathcal{E}(a) - \mathcal{E}(0)|^{1-\theta} \quad (3.25)$$

Let σ, θ in this section be the same as in Prop. 7. Assume $a \in C^{k,\mu}(M \times [t_1, t_2])$ satisfies the following equation

$$\dot{a} = N(a) + R(a) \quad (3.26)$$

where $N(a)$ approximates well the gradient \mathcal{M} of \mathcal{E} in the sense of (3.7) and

$$\|R(a)(t)\| \leq \frac{1}{2} \|\dot{a}(t)\|, \quad \forall t \in [t_1, t_2] \quad (3.27)$$

Then we have the following stability lemma as in [SL1] for solutions to the above approximate flow equations. More precisely,

Lemma 15 *Suppose a satisfies $|a(t)|_{C^{k,\mu}([t_1, t_2])} \leq \sigma$, and suppose that for some constant $\varepsilon > 0$,*

$$\mathcal{E}(a(t)) > \mathcal{E}(0) - \varepsilon \quad \text{for all } t \in [t_1, t_2] \quad (3.28)$$

Then

$$\int_{t_1}^{t_2} \|\dot{a}(t)\| dt \leq C\theta^{-1}(|\mathcal{E}(a(t_1)) - \mathcal{E}(0)|^\theta + \varepsilon^\theta), \quad (3.29)$$

In particular

$$\sup_{t \in [t_1, t_2]} \|a(t) - a(t_1)\| \leq C\theta^{-1}(|\mathcal{E}(a(t_1)) - \mathcal{E}(0)|^\theta + \varepsilon^\theta). \quad (3.30)$$

Proof. The proof is elementary if we use (3.7) properly to control N and \mathcal{M} by each other in the estimates and use (3.25). (See the proof in [SL1] for example). \square

The next lemma is a gauge invariant form of the inequality (3.25) and is essentially proven in [MMR].

Lemma 16 *Let E be a vector bundle on M . A, B are $C^{k,\mu}$ connections on E and B is a Yang-Mills connection. There exists $\varepsilon_3 > 0$ and $\theta \in (0, \frac{1}{2})$, such that if $|A - B|_{C^{k,\mu}} < \varepsilon_3$, then the following inequality holds*

$$\left(\int_M |F_A|^2 - |F_B|^2 d\sigma \right)^{1-\theta} \leq 2 \|d_A^* F_A\| \quad (3.31)$$

Proof. Let $a = A - B$, if $|a|_{C^{k,\mu}} < \varepsilon_3$, by Lemma 10. we can find a smooth transformation σ , such that $\sigma(A) = B + a'$, $a' \in \text{Ker}(d_B^*) \subset \Omega^1(\text{End } E)$, with $|a'|_{C^{k,\mu}} \leq C|a|_{C^{k,\mu}}$. Since both sides of (3.31) are invariant under gauge transformations of A , we can assume that $A = B + a$ with $a \in \text{Ker}(d_B^*)$, and $|a|_{C^{k,\mu}} < C\varepsilon_3$. Let U be a small $C^{k,\mu}$ neighborhood of 0 in $\text{Ker}(d_B^*) \subset \Omega^1(\text{End } E|_M)$ and consider $\mathcal{E} : U \rightarrow \mathbb{R}$ by $\mathcal{E}(b) = \text{YM}(B + b)$, $\forall b \in U$. Let \mathcal{M} be the Euler-Lagrange operator of \mathcal{E} on U . We claim that if U is sufficiently small, $A = B + a$, $a \in U$, then

$$\frac{3}{4} \|\mathcal{M}(a)\| \leq \|d_A^* F_A\| \leq \frac{5}{4} \|\mathcal{M}(a)\| \quad (3.32)$$

Indeed, by $\langle \mathcal{M}(a), b \rangle_{L^2} = \langle d_A^* F_A, b \rangle_{L^2}$, $\forall b \in \text{Ker}(d_B^*)$ and $\mathcal{M}(a) \in \text{Ker}(d_B^*)$, it follows that $\mathcal{M}(a) = p_{\text{Ker}(d_B^*)}(d_A^* F_A)$, where p is L^2 projection. From proof of Lemma 10, we can easily see that

$$\begin{aligned} p_{\text{Ker}(d_B^*)}(d_A^* F_A) &= d_A^* F_A - (d_B^* d_B)^{-1} d_B^*(d_A^* F_A) \\ &= d_A^* F_A \pm G_B(*[a, *d_A^* F_A]), \end{aligned} \quad (3.33)$$

where we used the identity $d_A^* d_A^* F_A = 0$ (see (4.3)). From this it is easy to derive (3.32) if $|a|_{C^{k,\mu}}$ is small. Apply (3.25) to \mathcal{E} and use (3.32) (note that \mathcal{E} is only defined on $\text{Ker}(d_B^*)$, not on the space of all sections; however Prop. 7 is still true in this case), we obtain that there exists $\theta \in (0, \frac{1}{2})$, such that

$$\|\mathcal{E}(a) - \mathcal{E}(0)\|^{1-\theta} \leq 2 \|d_A^* F_A\|$$

This last inequality is exactly (3.31) which we want to prove. \square

3.4 Proof of Theorem 3

In the following we shall carry on the proof of Theorem 3 following mostly the methods of proving Theorem 1 in [SL1]. Assume $R' < \infty$. We may assume that $R' = R + NL$ for $N \geq 2$ by changing R within an amount of L if necessary. It is important to note that by differentiation of (2.1), we have \dot{a} satisfies an equation of (3.6) form on $[R, R']$, with the same τ (but we need to change k to $k - 1$ for \dot{a} ; since $k \geq 4$, this is still in the regular range). Assume $\tau < \tau_0$, where τ_0 is as in Prop. 6. Then by Prop. 6, we have $1 \leq j_1 \leq j_2 \leq N - 1$ for a and $1 \leq k_1 \leq k_2 \leq N - 1$ for \dot{a} such that the conclusions of Prop. 6 hold. We adopt the notation

$$S(j, a) = \sup_{t \in [R+(j-1)L, R+jL]} \|a(t)\| \quad (3.34)$$

By (3.20), we have

$$S(N, a) \leq (1 + \eta)S(N - 1, a)$$

Combined with the conclusions of Prop. 6, if η is sufficiently small such that $1 + \eta < e^{(\delta_1 - \delta)L}$, this implies that

$$j_2 = N - 1$$

We shall assume in the next that $\tau \geq \varepsilon^\alpha$ for the constant $\alpha = \frac{\theta}{8}$, where θ is the constant in Lemma 15. We claim that

$$\|a(t)\| \leq C\varepsilon^{2\alpha} \quad \text{for } t \in [R, R + NL] \quad (3.35)$$

For $t \in [R, R + (k_1 - 1)L]$, by using Prop. 6 (a), we have

$$\|\dot{a}(t)\| \leq S(\lceil t/L \rceil + 1, \dot{a}) \leq e^{-(\delta_2 - \delta)(t-L)} S(1, \dot{a}) \leq e^{-\delta(t-L)} \varepsilon \quad (3.36)$$

Hence

$$\|a(t)\| \leq \|a(R)\| + \int_0^t \|\dot{a}\| dt \leq \varepsilon + \delta^{-1} e^{\delta L} \varepsilon \leq \varepsilon^{\frac{1}{2}}, \quad \text{for } t \in [R, R + (k_1 - 1)L], \quad (3.37)$$

if $\varepsilon^{\frac{1}{2}} \leq (1 + \delta^{-1} e^{\delta L})^{-1}$.

Next we consider the case $t \in [R + (k_1 - 1)L, R + k_2 L]$. By using the fact that $|a(t)|_{C^{k, \mu}}^* \leq \tau \leq \varepsilon^\alpha$ and $\|\ddot{a}(t)\| \leq \eta \|\dot{a}(t)\|$ (which follows from (b) in Prop. 6) as well as (3.32), we can see that Lemma 15 applies to \dot{a} on $[R + k_1 L, R + (k_2 - 1)L]$. Therefore

$$\begin{aligned} \int_{R+k_1 L}^{R+(k_2-1)L} \|\dot{a}\| dt &\leq C(|\mathcal{E}(a(R + k_1 L)) - \mathcal{E}(0)|^\theta + \varepsilon^{\frac{\theta}{2}}) \\ &\leq C(|a(R + k_1 L)|_{C^1}^\theta + \varepsilon^{\frac{\theta}{2}}) \\ &\leq C\left(\sup_{t \in [R+k_1 L-1, R+k_1 L+1]} \|a(t)\|^\theta + \varepsilon^{\frac{\theta}{2}}\right) \\ &\leq C\varepsilon^{\frac{\theta}{2}} \leq C\varepsilon^{2\alpha} \end{aligned} \quad (3.38)$$

and hence for $\tau \in [R + (k_1 - 1)L, R + k_2L]$, by (3.18) $\|\ddot{a}\| \leq C\varepsilon$ and we have

$$\begin{aligned} \|\dot{a}(t)\| &\leq \|\dot{a}(R + k_1L)\| + C\varepsilon^{2\alpha} + C\varepsilon \\ &\leq \|\dot{a}(R + (k_1 - 1)L)\| + C\varepsilon + C\varepsilon^{2\alpha} \\ &\leq C\varepsilon^{\frac{1}{2}} + C\varepsilon + C\varepsilon^{2\alpha} \leq C\varepsilon^{2\alpha}. \end{aligned} \quad (3.39)$$

The case $t \in [R + k_2L, R + NL]$ is a little more intricate. However, we essentially just follow the proof in [SL1, p.556 –p.558] and use $|\frac{\partial a}{\partial t}|_{C^{k-1,\mu}([R,R'])}^* \leq \varepsilon$ and elliptic estimates where necessary, we could show the the claim (3.35) in this case.

Now (3.35) implies

$$\sup_{t \in [R+(N-1)L, R+NL]} \|a(t)\| \leq C\varepsilon^{2\alpha} \quad (3.40)$$

which gives a contradiction to our assumption (3.21) if ε is sufficiently small relative to τ . Therefore we must have $R' = \infty$. Hence by (3.17), $|a|_{C^{2,\mu}([R,\infty))}^* \leq \tau$. This implies that $\|\dot{a}(t)\| \leq C\varepsilon$ for $t \in [R, \infty)$ and this in turn implies that $k_2 = \infty$, otherwise $\|\dot{a}(t)\|$ will increase unboundedly by Prop. 6 (c). Hence by (3.38),

$$\int_0^\infty \|\dot{a}\| dt < \infty \quad (3.41)$$

(3.41) implies that $\lim_{t \rightarrow \infty} a(t)$ exists in L^2 norm, and hence relative to $C^{2,\mu}$ norms by estimates (3.11). In case $k_1 = \infty$, from Prop. 6 (a) and elliptic estimates we have

$$|\dot{a}(t)|_{C^{1,\mu}}^* \rightarrow 0. \quad (3.42)$$

In case $k_1 < \infty$, from elliptic estimates

$$|\dot{a}(t)|_{C^{1,\mu}} \leq c\|\dot{a}(t)\|_{[t-1,t+1]}, \quad t \geq R + k_1L \quad (3.43)$$

If there is a sequence $t_k \rightarrow \infty$ such that $\limsup_{k \rightarrow \infty} |\dot{a}|_{C^{1,\mu}}^* > 0$, then (3.43) implies $\limsup_{k \rightarrow \infty} \|\dot{a}\|_{[t_k-1,t_k+1]} > 0$, which contradicts (3.41). Hence (3.42) also holds. Thus by taking the limit as $t \rightarrow \infty$ in (3.6), we have $\mathcal{M}(w) = 0$. By using elliptic estimates, we can derive the convergence (3.23). This completes the proof of Theorem 3.

3.5 Completion of Proof of Theorem 2

In this section we apply Theorem 3 to complete our proof of Theorem 2. We shall use the monotonicity formula to show that the hypotheses of (3.1) are satisfied. And then use Lemma 16 to obtain an integral decay estimate for $\|\dot{a}(t)\|$ and prove the convergence results. The idea mainly comes from [SL2], where the case for energy minimizing harmonic maps with a tangent map which has isolated singularity is treated. As before, assume $\mathbb{A} = A(t) + \beta(t)dt = A_0 + a(t) + \beta(t)dt$, $t \in [R, R']$, a

connection on $S^{n-1} \times [R, R']$, is in the gauge given in Prop. 4, then by results from Section 2.5 and Prop. 4, we see that a is indeed a $(\varepsilon, \tau, \eta, L)$ -bounded solution of an equation in the form of (3.6) (we may replace the k in Prop. 4 by $k + 1$ and obtain bounds on $|\cdot|_{C^{k,\mu}}^*$ norms. Notice the growth estimates in Section 2.5 are stated in L^2 norms and hence is independent of k). Let $\mathbb{A}_1 = \phi^*(\mathbb{A})$ be the original connection on $B_1(0) \setminus \{0\}$. Since \mathbb{A}_1 is stationary, by monotonicity formula,

$$\begin{aligned}
& 4 \int_{B_\rho(0)} r^{4-n} \left| \frac{\partial}{\partial r} \right] F_{\mathbb{A}_1} \|^2 dx \\
& \leq \lim_{\sigma \rightarrow 0} \left\{ \int_{B_\rho(0)} \rho^{4-n} |F_{\mathbb{A}_1}|^2 dx - \int_{B_\sigma(0)} \sigma^{4-n} |F_{\mathbb{A}_1}|^2 dx \right\} \\
& = \lim_{\lambda_i \rightarrow 0} \left\{ \int_{B_\rho(0)} \rho^{4-n} |F_{\mathbb{A}_1}|^2 dx - \int_{B_{\lambda_i}(0)} \lambda_i^{4-n} |F_{\mathbb{A}_1}|^2 dx \right\} \\
& = \lim_{\lambda_i \rightarrow 0} \left\{ \int_{B_\rho(0)} \rho^{4-n} |F_{\mathbb{A}_1}|^2 dx - \int_{B_1(0)} |F_{\mathbb{A}_{1,\lambda_i}}|^2 dx \right\} \\
& = \int_{B_\rho(0)} \rho^{4-n} (|F_{\mathbb{A}_1}|^2 - |F_{A_0}|^2) dx \\
& \leq \frac{1}{n-4} \int_{\partial B_\rho(0)} \rho^{5-n} (|F_{\mathbb{A}_1}|^2 - |F_{A_0}|^2) d\sigma
\end{aligned} \tag{3.44}$$

where the last inequality follows from monotonicity formula (1.20) and the fact that A_0 is radially symmetric. We also used the fact that there is no loss of curvature energy in limiting to tangent Yang-Mills connections (From the compactness theorem and the assumption that blow-up locus has $H^{n-4} = 0$, see Prop. 2 (2)). Under our change of coordinates, let $\eta(t) = \dot{a}(t) - d_A \beta(t) dt$ and $T = -\log(\rho) \geq R$, (3.44) becomes, for any $T \in [R, R']$,

$$\begin{aligned}
\int_T^\infty \|\eta(t)\|^2 dt & \leq \frac{1}{n-4} \int_{S^{n-1}} |F_A(T) - \eta(T) dt|^2 - |F_{A_0}|^2 d\sigma \\
& = C \int_{S^{n-1}} |F_A(T)|^2 - |F_{A_0}|^2 d\sigma + C \|\eta(T)\|^2
\end{aligned} \tag{3.45}$$

At this point we notice that if we let $\mathcal{E}(a) = \text{YM}(A_0 + a)$, then (3.45), (3.60) below and (3.18) give

$$\mathcal{E}(a(T)) - \mathcal{E}(0) \geq -\|\eta(T)\|^2 \geq C \|\dot{a}\|^2 \geq -C\varepsilon^2 \geq -\varepsilon \tag{3.46}$$

if $0 < \varepsilon \ll \tau \ll 1$. We observe now all assumptions of Theorem 3 are satisfied for a , hence $N = \infty$ and there is a Yang-Mills connection A_1 with $|A_1 - A_0|_{C^{k,\mu}} \leq \tau$ and $d_{A_0}^*(A_1 - A_0) = 0$ such that

$$\lim_{t \rightarrow \infty} |A(t) - A_1|_{C^{k,\mu}}^* = 0 \tag{3.47}$$

(Note: By Theorem 3 we have $C^{2,\mu}$ convergence and then use (2.1), elliptic estimates and bootstrapping methods we can get $C^{k,\mu}$ convergence). Since A_0 is a tangent connection of \mathbb{A} , there exists a sequence $T_i \rightarrow \infty$ and gauge transformations $g_i \in \Gamma(\text{Aut } P \times [0, 1])$ such that

$$\lim_{i \rightarrow \infty} |g_i(\mathbb{A}_{T_i}) - A_0|_{C^{k,\mu}([0,1])}^* = 0, \quad (3.48)$$

where $\mathbb{A}_{T_i} = (T_i)^*(\mathbb{A})$ for $T_i : x \mapsto x + T_i$. From (3.47) and (3.48), we can derive that $|g_i|_{C^{k+1,\mu}([0,1])}^* \leq C$ are uniformly bounded, therefore we may assume, by taking a subsequence if necessary, that $g_i \rightarrow g_0$ in $C^{k+1}([0, 1])$. Then (3.47) and (3.48) implies that $g_0(A_1) = A_0$ and g_0 is independent of $t \in [0, 1]$. However, if τ is small enough, it is well known that the orbit of A_0 under gauge transformations intersects the τ -neighborhood (under $C^{k,\mu}$ -norm) of A_0 in $A_0 + \text{Ker}(d_{A_0}^*)$ only at A_0 . Therefore, $A_0 = A_1$. Assume A_2 is another tangent connection of \mathbb{A} , then from the same argument before, $A_2 = g_2(A_0)$ and therefore A_0 is the unique tangent connection of \mathbb{A} up to gauge equivalences. Hence A_0 is the unique tangent connection of \mathbb{A} and we have convergence (3.47) with A_1 replaced by A_0 . We shall show the convergence rate in the following.

Since $|A(t) - A_0|_{C^{k,\mu}} = |a(t)|_{C^{k,\mu}} \leq \tau$, we can apply Lemma 16 to the right hand side of (3.45) and obtain

$$\int_T^\infty \|\eta(t)\|^2 dt \leq C \|d_{A(T)}^* F_{A(T)}\|^{\frac{1}{1-\theta}} + C \|\eta(T)\|^2, \quad (3.49)$$

From (2.1), we have

$$\|d_{A(t)}^* F_{A(t)}\| \leq \|\ddot{a}(t)\| + \|\dot{a}(t)\| + \|d_A \dot{\beta}(t)\| + \|d_A \beta(t)\| + C\tau \|\beta(t)\| \quad (3.50)$$

Recall that

$$d_A^* d_A \beta = - * [a, * \dot{a}] \quad (3.51)$$

Therefore by elliptic estimates

$$|\beta(t)|_{C^{k+1,\mu}} \leq C |* [a, * \dot{a}]|_{C^{k-1,\mu}} \leq C\tau |\dot{a}|_{C^{k-1,\mu}} \quad (3.52)$$

$$|d_A \beta|_{C^{k,\mu}} \leq C\tau |\dot{a}|_{C^{k-1,\mu}} \quad (3.53)$$

By differentiating (3.51) with respect to t , we obtain

$$d_A^* d_A \dot{\beta} = \pm * [\dot{a}, d_A \beta] \pm d_A^* [\dot{a}, \beta] \pm (*[a, * \dot{a}])' \quad (3.54)$$

Notice that $\dot{\beta} \in \text{Ker}(\Delta_B)^\perp$, hence we can apply elliptic estimates again to get

$$|\dot{\beta}(t)|_{C^{k+1,\mu}} \leq C\tau |\dot{a}(t)|_{C^{k,\mu}} \quad (3.55)$$

$$|d_A \dot{\beta}(t)|_{C^{k,\mu}} \leq C\tau |\dot{a}(t)|_{C^{k,\mu}} \quad (3.56)$$

By previous interior elliptic estimates,

$$|\dot{a}(t)|_{C^{k,\mu}} \leq C \|\dot{a}\|_{[t-1,t+1]} \quad (3.57)$$

Applying (3.52), (3.53), (3.56) and (3.57) to the right hand side of (3.50) gives us

$$\|d_{A(t)}^* F_{A(t)}\| \leq \|\ddot{a}(t)\| + \|\dot{a}(t)\| + C\tau |\dot{a}(t)|_{C^{k,\mu}} \leq C \|\dot{a}\|_{[t-1,t+1]} \quad (3.58)$$

We have also from elliptic estimates for Sobolev norms applied to (3.51), that

$$\|d_A \beta(t)\| \leq C\tau \|\dot{a}(t)\| \quad (3.59)$$

If τ is small, (3.59) implies

$$\frac{1}{2} \|\dot{a}(t)\| \leq \|\eta(t)\| \leq 2 \|\dot{a}(t)\| \quad (3.60)$$

Put together (3.58) and (3.60) and plug in both sides (3.49), we have

$$\int_T^\infty \|\dot{a}\|^2 ds \leq C \left(\int_{T-1}^{T+1} \|\dot{a}\|^2 \right)^{\frac{1}{2(1-\theta)}} \quad (3.61)$$

where $\theta \in (0, \frac{1}{2})$ depend only on A_0 . (3.61) gives an integral decay estimate for $\|\dot{a}\|$. Recall that $|a(t)|_{C^{k,\mu}} \leq \tau$, for $R < t < \infty$. Now it is easy to show (for example, as in [SL2]) that there exists $T_1 > 0$, $\alpha > 0$, such that

$$\int_t^\infty \|\dot{a}(s)\| ds < Ct^{-\alpha}, \quad \text{for } t \geq T_1 \quad (3.62)$$

Therefore

$$\|A(t) - A_0\| \leq Ct^{-\alpha}, \quad \text{for } t \geq T_1 \quad (3.63)$$

and by elliptic estimates,

$$|A(t) - A_0|_{C^{k,\mu}} \leq C(k)t^{-\alpha}, \quad \text{for } t \geq T_1. \quad (3.64)$$

The desired rate of convergence is obtained and the proof of Theorem 2 is finished.

Remark. We call a smooth Yang-Mills connection A_0 on S^{n-1} *integrable* if for every solution $a \in \Omega^1(\mathfrak{g}_E)$ of

$$La = \Delta_{A_0} a + (-1)^n * [a, *F_{A_0}] = 0, \quad (3.65)$$

where L is, as in Section 3.1, the linearization of $d_{A_0+a}^* F_{A_0+a}$ at 0, there exists a path of Yang-Mills connections $A(t)$, $t \in (-\varepsilon, \varepsilon)$ with $A(0) = A_0$ such that

$$\left. \frac{\partial}{\partial t} \right|_{t=0} A(t) = a. \quad (3.66)$$

This has the geometric meaning that A_0 has an integrable neighborhood in the moduli space of smooth Yang-Mills connections on S^{n-1} with tangent space at A_0 being given by Jacobi fields, the solutions to (3.65). If the tangent connection A_0 of \mathbb{A} in Theorem 1 is integrable, then following the methods of [AS] or [CT], we can prove that the convergence in Theorem 1 is of order r^α , or of order $e^{-\alpha t}$ since $t = -\log r$. This rate of convergence is like the rate of convergence of gradient flow to a nondegenerate critical point. We have this exponential convergence too if 0 is not an eigenvalue of L . In general, we don't think there is such a fast convergence to the tangent connection. We should be able to construct examples of slow (logarithmic) convergence if we have examples of non-integrable Yang-Mills connections, following the spirit of Adams and Simon [AS], in which they constructed examples of minimal submanifolds and harmonic maps with isolated singularities and slow convergence to tangent objects.

Chapter 4

A result for Yang-Mills flows

In this chapter we give an application of the previous methods to Yang-Mills flows. We shall show that a flow which starts from a connection sufficiently close (in smooth norms) to a smooth local minimizer of the Yang-Mills functional will converge asymptotically to a smooth Yang-Mills connection near the minimizer. Our method, like before, still consists of two steps, first we choose a suitable gauge, and then we adapt a result in [SL1] to our case .

4.1 Yang-Mills flows

Consider the the following Yang-Mills flow equation for connection on bundle E on Riemannian manifolds M

$$\frac{\partial}{\partial t} A(t) = -d_{A(t)}^* F_{A(t)} \quad (4.1)$$

Idealistically, if (4.1) has a solution $A(t)$ on $[0, \infty)$, the limit of $A(t)$ at ∞ should be a Yang-Mills connection. And this gives us a way to homotopically deform an arbitrary connection into a Yang-Mills connection and hopefully we can have a Morse theory suitably defined. However, the long-range existence of solutions of (4.1) as well as the existence and regularity of the limit in general are not at all obvious. Nonetheless, near a local minimizer of the Yang-Mills connection, we can solve the above difficulties.

We first note that (4.1) is not parabolic due to the fact that $d_A^* F_A$ is not elliptic in A . As before, we hope to use the Coulomb gauge to make the equation parabolic. We note that (4.1) actually implies

$$d_A^*(\dot{A}) = 0, \quad (4.2)$$

where $\dot{A} = \frac{\partial}{\partial t} A(t)$. This follows easily because

$$d_A^* d_A^* F_A = \{F_A, F_A\} = 0 \quad (4.3)$$

where $\{, \}$ is defined by Lie bracket on the bundle parts and Riemannian product on the form part and hence is skew-symmetric.

Assume A_0 is a fixed smooth Yang-Mills connection, we have the following theorem now. Fix l integer such that $H^l(S^{n-1}) \subset C^{3,\mu}(S^{n-1})$.

Theorem 4 *There exists $\varepsilon = \varepsilon(A_0) > 0$, $\alpha = \alpha(A_0) > 0$ such that for any given smooth $a_0 \in \Omega^1(\mathfrak{g}_E)$ with $\|a_0\|_{H^{l+2}} < \varepsilon$, there is a $T_* > 0$ and $A(t)$, a $C^\infty(M \times [0, T_*))$ solution of (4.1) satisfying $A(0) = A_0 + a_0$, $\sup_{[0, T_*)} \|A(t) - A_0\|_{H^l} < \varepsilon^\alpha$ and either*

$$T_* < \infty \text{ and } \lim_{t \uparrow T_*} \text{YM}(A(t)) \leq \text{YM}(A_0) - \varepsilon \quad (4.4)$$

or

$$T_* = \infty \text{ and } \lim_{t \rightarrow \infty} (|\dot{A}(t)|_{C^1} + |A(t) - A_1|_{C^2}) = 0 \quad (4.5)$$

where A_1 is a smooth Yang-Mills connection on M .

Assume I is an interval of possibly infinite length and $A(t) + \beta(t)dt$ is a smooth connection on $E \times I$, where $A(t)$ are connections on E and $\beta(t) \in \Gamma(\mathfrak{g}_E)$. We also assume that under a gauge transformation $g \in \Gamma(\text{Aut } P \times I)$,

$$g(A(t) + \beta(t)dt) = A_1(t) \quad (4.6)$$

where $A_1(t)$ are connections on E and satisfies (4.1) and hence (4.2) for $t \in I$. From (4.6), we obtain,

$$A_1 = gAg^{-1} - dg g^{-1} \quad (4.7)$$

$$\frac{\partial}{\partial t} g = g\beta \quad (4.8)$$

We observe that (4.7) implies that A_1 and A are gauge equivalent connections on M . Substitute (4.7) and (4.8) into (4.1) and (4.2), by straightforward computation, we obtain the following equations for A and β

$$\dot{A} = -d_A^* F_A + d_A \beta \quad (4.9)$$

$$d_A^*(\dot{A} - d_A \beta) = 0 \quad (4.10)$$

We shall view (4.9) and (4.10) (which is implied by (4.9)) as equations for the connection $A + \beta dt$ on $M \times I$. It is easy to see that they are gauge invariant equations, i.e., if $g(A + \beta dt) = A_1 + \beta_1 dt$ then $A_1 + \beta_1 dt$ also satisfies (4.9) and (4.10). This point of view enables us to consider as before a standard form of $A + \beta dt$ around a connection A_0 on E and make the system (4.9) parabolic in A and (4.10) elliptic in β . If $A + \beta dt = A_0 + a(t) + \beta(t)dt$ is under such a standard form, i.e. $d_{A_0}^* a = 0$ and $\beta \in \text{Ker}(d_{A_0})^\perp$, then we may rewrite (4.9) as

$$\dot{a} = -d_{A_0+a}^* F_{A_0+a} - d_{A_0} d_{A_0}^* a + d_A G_A(*[a, *a]) \quad (4.11)$$

where $G_A = (\Delta_A)^{-1} : \text{Im}(d_A^*) \rightarrow \text{Ker}(d_{A_0})^\perp$.

From above, we observe that instead of proving Theorem 4 in terms of (4.1), it suffices to prove the same conclusions hold for $A(t) = A_0 + a(t)$, $a(t) \in \text{Ker}(d_{A_0}^*)$ with $a(t)$ being solution to (4.11) with initial value a_0 (up to a gauge transformation, we may assume $d_{A_0}^* a_0 = 0$). For if we prove the latter, a solution to the former may be obtained by

$$\bar{A}(t) = g(A(t) + \beta(t)), \quad \frac{\partial}{\partial t} g = g\beta$$

where $\beta = G_A(*[a, * \dot{a}])$. We can show that $g(t) \rightarrow g_0$ for some g_0 and $\dot{g} \rightarrow 0$ in C^k as $t \rightarrow \infty$, therefore $\bar{A}(t)$ will have limit at infinity $g_0(A_1)$ if A_1 is the limit of $A(t)$ and the same conclusions hold for \bar{A} . We have the following obvious corollary from the theorem.

Corollary 1 *If A_0 is a smooth local minimizer of Yang-Mills functional on E . Then there exists $\varepsilon = \varepsilon(A_0) > 0$ and $\alpha = \alpha(A_0) > 0$ such that for any given smooth $a_0 \in \Omega^1(\mathfrak{g}_E)$ with $\|a_0\|_{H^{l+2}} < \varepsilon$, there is a $A(t)$, a $C^\infty(M \times [0, \infty))$ solution of (4.1) satisfying $A(0) = A_0 + a_0$, and*

$$\lim_{t \rightarrow \infty} (|\dot{A}(t)|_{C^1} + |A(t) - A_1|_{C^2}) = 0 \quad (4.12)$$

where A_1 is a smooth Yang-Mills connection on M with $|A_1|_{H^l} \leq \varepsilon^\alpha$.

4.2 Proof of Theorem 4

The following sketch of proof is taken from the proof of Theorem 2 in [SL1] with very slight adjustment to our case. For solutions to the following linear parabolic equation

$$\dot{a} - La = f \quad (4.13)$$

where L a uniformly elliptic operator with smooth coefficients, the following initial value estimates hold

$$\begin{aligned} & \sup_{[t_1, t_2]} (\|a(t)\|_{H^k} + \|\dot{a}\|_{H^k}) + \|a(t)\|_{H^{k+1}[t_1, t_2]} + \|\dot{a}\|_{H^{k+1}[t_1, t_2]} \\ & \leq c_1(k) (\|a(t_1)\|_{H^k} + \|\dot{a}(t_1)\|_{H^k} + \|f\|_{H^{k-1}([t_1, t_2])} + \|\dot{f}\|_{H^{k-1}([t_1, t_2])}) \end{aligned} \quad (4.14)$$

provided $|t_1 - t_2| \leq c_1(k)^{-1}$. By choosing suitable cutoff functions and using (4.14) we can derive the following interior estimates:

$$\begin{aligned} & \sup_{[t_1 + \sigma, t_2 - \sigma]} (\|a(t)\|_{H^k} + \|\dot{a}\|_{H^k}) + \|a(t)\|_{H^{k+1}[t_1 + \sigma, t_2 - \sigma]} + \|\dot{a}\|_{H^{k+1}[t_1 + \sigma, t_2 - \sigma]} \\ & \leq c_2(k) \sigma^{-k} (\|a(t)\|_{[t_1, t_2]} + \|f\|_{H^{k-1}([t_1, t_2])} + \|\dot{f}\|_{H^{k-1}([t_1, t_2])}) \end{aligned} \quad (4.15)$$

We note that when $\|a\|_{C^2([t_1, t_2])}$ is sufficiently small, (4.11) approximates (4.13) well and we can use (4.14) and (4.15) and contraction mapping theorem to obtain the following extension result. There is a constant $c_3 = c_3(A_0) \geq 1$, such that for any given solution $a(t)$ of (4.11) on $[0, t]$, with

$$\sup_{[0, T]} (\|a(t)\|_{H^l} + \|\dot{a}(t)\|_{H^l}) \leq \varepsilon_1 \leq c_3^{-1}, \quad (4.16)$$

extends (uniquely) to a solution \tilde{a} on $[0, T+d]$, $d \in (0, 1)$ depending only on A_0 , with

$$\sup_{[T, T+d]} (\|\tilde{a}(t)\|_{H^l} + \|\dot{\tilde{a}}(t)\|_{H^l}) \leq c_3 \varepsilon_1. \quad (4.17)$$

In particular, for any given $0 < \varepsilon_1 < c_3^{-1}$, we can extend the solution a to an interval $[0, T_*)$ (possibly with $T_* = \infty$) such that

$$\|a(t)\|_{H^l} + \|\dot{a}(t)\|_{H^l} \leq \varepsilon_1, \quad \text{on } [0, T_*) \quad (4.18)$$

and in case $T_* < \infty$,

$$\lim_{t \uparrow T_*} (\|a(t)\|_{H^l} + \|\dot{a}(t)\|_{H^l}) = \varepsilon_1 \quad (4.19)$$

On $t \in [0, T_*)$, we take the t derivative of (4.11), we find that \dot{a} satisfy the following equation

$$\dot{w} - Lw + Rw = 0 \quad (4.20)$$

where L is the linearization of $-(d_{A_0+a}^* F_{A_0+a} + d_{A_0} d_{A_0}^* a)$ at $a = 0$. R is a differentiable linear operator such that $D^i R : H^k \rightarrow H^k$ are continuous and $\|D^i R\| \leq c_4 \varepsilon_1$ for $0 \leq i \leq 2$ and $0 \leq k \leq 2 - i$. (The norms are operator norms for the above spaces, we use the fact that $H^l \subset C^{3, \mu}$). Notice that there are regularity estimates for solutions to (4.20).

We have the following claim

If $\delta \in (0, \min\{\mu_j : \mu_j > 0\})$, where μ_j are eigenvalues of L , there exists $\varepsilon_2 = \varepsilon_2(\delta, L) \in (0, 1)$, such that if $\varepsilon_1 < \varepsilon_2$ and if \dot{a} is a solution to (4.20) on $[0, T_*)$ with (4.18) holds, let $S(i) = \sup_{[T+i-1, T+i]} e^{\delta t} \|\dot{a}(t)\|$, then

$$S(2) \geq \max\{S(1), \varepsilon_1^{\frac{1}{2}}\} \Rightarrow S(3) \geq S(2) \quad (4.21)$$

This claim may be proved by the similar methods as in Lemma 14 we used before, i.e., first prove it for the linearize equation, in which case the proof is simple because the solution can be written as combinations of eigenfunctions and then prove it for the nonlinear perturbed equation (4.20) by contradiction using a blow-up argument to reduce to the linear case.

To summarize, there exists $\varepsilon_1 = \varepsilon_1(\delta, A_0) \in (0, \varepsilon_2)$ such that if $\varepsilon < \varepsilon_1$ and if $\|a_0\|_{H^{l+2}} \leq \varepsilon$, then there exists $a(t)$ solution of (4.11) on $[0, T_*)$, $a(0) = a_0$ and $T_1 < T_* - 1$, such that (4.18) and (4.19) hold and

$$\|\dot{a}(t)\| \leq \varepsilon^{\frac{1}{2}} e^{\delta t}, \quad t < T_1 \quad (4.22)$$

$$\|\dot{a}(t)\| \geq c_5^{-1} \varepsilon^{\frac{1}{2}} e^{\delta t}, \quad T_1 < t < T_* - 1 \quad (4.23)$$

We assume $\varepsilon_1 < \sigma$ (σ as in Lemma 15, choose ε_1 smaller if necessary) and $\varepsilon = \varepsilon_1^{4/\theta}$ (θ as in Lemma 15), i.e., we take $\alpha = \frac{\theta}{4}$ and $\varepsilon_1 = \varepsilon^\alpha$. We shall use the elliptic functional $\mathcal{E}(a) = \text{YM}(A_0 + a)$ for $a \in \text{Ker}(d_{A_0}^*)$.

If $\mathcal{E}(a(T_1)) < \mathcal{E}(0) - \varepsilon$, then (4.4) holds with T_* replaced by T_1 and we are done. We assume $\mathcal{E}(a(T_1)) > \mathcal{E}(0) - \varepsilon$ in the following. We have

$$\|a(t)\| \leq c\varepsilon^{\frac{1}{2}}, \quad 0 < t < T_1. \quad (4.24)$$

Since ε is small, we have that in (4.11) that

$$\|d_A G_A(*[a, *\dot{a}])\| \leq c_6 \varepsilon \|\dot{a}\| \leq \frac{1}{2} \|\dot{a}\|$$

Hence the assumption of Lemma 15 is satisfied by $a(t)$ for $t \in [0, T_*)$ and

$$\begin{aligned} \sup_{[T_1, T_*-1]} \|a(t)\| - \|a(T_1)\| &\leq \int_{T_1}^{T_*-1} \|\dot{a}(t)\| dt \\ &\leq c(\varepsilon^{\frac{\theta}{2}} + |\mathcal{E}(a(T_1)) - \mathcal{E}(0)|^\theta) \\ &\leq c(\varepsilon^{\frac{\theta}{2}} + \|a(T_1)\|_{H^1}^\theta) \leq c\varepsilon^{\frac{\theta}{2}} \leq c\varepsilon_1^2, \end{aligned} \quad (4.25)$$

where the estimates on $\|a(T_1)\|_{H^1}^\theta$ is obtained through bounds on $\|a(T_1)\|$, $\|\dot{a}(T_1)\|$ together with the estimates (4.14) and (4.15).

Now by interior estimates (4.15),

$$\|a(t)\|_{H^l} + \|\dot{a}(t)\|_{H^l} \leq c\varepsilon_1^2, \quad 1 \leq t < T_* - 2. \quad (4.26)$$

If ε_1 is sufficiently small, then by the previous extension results and estimates we have

$$\|a(t)\|_{H^l} + \|\dot{a}(t)\|_{H^l} < \frac{1}{2} \varepsilon_1, \quad 0 < t < T_*, \text{ if } T_* < \infty. \quad (4.27)$$

This is a contradiction to (4.19). Hence $T_* = \infty$ and from (4.25) we have

$$\int_0^\infty \|\dot{a}(t)\| dt \leq c\varepsilon_1^2 < \infty. \quad (4.28)$$

Hence there is L^2 limit for $a(t)$ as $t \rightarrow \infty$, from previous elliptic estimates we see that this convergence is actually smooth and the limit a_1 gives rise to a smooth Yang-Mills connection $A_1 = A_0 + a_1$. Theorem 4 is proved.

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