# *INFINITESIMAL CHEREDNIK ALGEBRAS AS W-ALGEBRAS*

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# INFINITESIMAL CHEREDNIK ALGEBRAS AS $W$-ALGEBRAS

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*Dedicated to Evgeny Borisovich Dynkin on his 90th birthday*

**Abstract.** In this article we establish an isomorphism between universal infinitesimal Cherednik algebras and $W$-algebras for Lie algebras of the same type and 1-*block* nilpotent elements. As a consequence we obtain some fundamental results about infinitesimal Cherednik algebras.

## Introduction

This paper is aimed at the identification of two algebras of seemingly different nature. The first, finite $W$-algebras, are algebras constructed from a pair $(\mathfrak{g}, e)$, where $e$ is a nilpotent element of a finite dimensional simple Lie algebra $\mathfrak{g}$. Their theory has been extensively studied during the last decade. For the related references see, for example, reviews [L6], [W] and articles [BGK], [BK1], [BK2], [GG], [L1], [L2], [L3], [P1], [P2].

The second class of algebras we consider in this paper are the so called infinitesimal Cherednik algebras of type $\mathfrak{gl}_n$ and $\mathfrak{sp}_{2n}$, introduced in [EGG]. These are certain continuous analogues of the rational Cherednik algebras and in the case of $\mathfrak{gl}_n$ are deformations of the universal enveloping algebra $U(\mathfrak{sl}_{n+1})$. In both cases we call $n$ the *rank* of an algebra. The theory of those algebras is less developed, while the main references there are: [EGG], [T1], [T2], [DT].

This paper is organized in the following way:

• In Section 1, we recall the definitions of infinitesimal Cherednik algebras $H_\zeta(\mathfrak{gl}_n)$, $H_\zeta(\mathfrak{sp}_{2n})$, and introduce their modified versions, called the universal length $m$ infinitesimal Cherednik algebras. We also recall the definitions and basic results about the finite $W$-algebras $U(\mathfrak{g}, e)$.

• In Section 2, we prove our main result, establishing an abstract isomorphism

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of $W$-algebras $U(\mathfrak{sl}_{n+m}, e_m)$ (respectively $U(\mathfrak{sp}_{2n+2m}, e_m)$) with the universal infinitesimal Cherednik algebras $H_m(\mathfrak{gl}_n)$ (respectively $H_m(\mathfrak{sp}_{2n})$).

• In Section 3, we establish explicitly a Poisson analogue of the aforementioned isomorphism. As a result we deduce two claims needed to carry out the arguments of the previous section.

• In Section 4, we derive several important consequences about algebras $H_\zeta(\mathfrak{gl}_n)$, $H_\zeta(\mathfrak{sp}_{2n})$. This clarifies some lengthy computations from [T1], [T2], [DT] and proves new results. Using the results of [DT, Sect. 3], about the Casimir element of $H_\zeta(\mathfrak{gl}_n)$, we determine the aforementioned isomorphism $H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$ explicitly.

• In Section 5, we recall the machinery of completions of the graded deformations of Poisson algebras, developed by the first author in [L1]. This provides the decomposition theorem for the completions of infinitesimal Cherednik algebras. This is analogous to a result by Bezrukavnikov and Etingof ([BE, Thm. 3.2]) in the theory of rational Cherednik algebras.

• In the Appendix, we provide some computations.

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## 1. Basic definitions

### 1.1. Infinitesimal Cherednik algebras of $\mathfrak{gl}_n$

We recall the definition of the infinitesimal Cherednik algebras $H_\zeta(\mathfrak{gl}_n)$ following [EGG]. Let $V_n$ and $V_n^*$ be the basic representation of $\mathfrak{gl}_n$ and its dual. Choose a basis $\{y_i\}_{1\leq i\leq n}$ of $V_n$ and let $\{x_i\}_{1\leq i\leq n}$ denote the dual basis of $V_n^*$. For any $\mathfrak{gl}_n$-invariant pairing $\zeta : V_n \times V_n^* \to U(\mathfrak{gl}_n)$, define an algebra $H_\zeta(\mathfrak{gl}_n)$ as the quotient of the semi-direct product algebra $U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)$ by the relations $[y, x] = \zeta(y, x)$ and $[x, x'] = [y, y'] = 0$ for all $x, x' \in V_n^*$ and $y, y' \in V_n$. Consider an algebra filtration on $H_\zeta(\mathfrak{gl}_n)$ by setting $\deg(V_n) = \deg(V_n^*) = 1$ and $\deg(\mathfrak{gl}_n) = 0$.

**Definition 1.** We say that $H_\zeta(\mathfrak{gl}_n)$ *satisfies the PBW property* if the natural surjective map $U(\mathfrak{gl}_n) \ltimes S(V_n \oplus V_n^*) \twoheadrightarrow \mathrm{gr}H_\zeta(\mathfrak{gl}_n)$ is an isomorphism, where $S$ denotes the symmetric algebra. We call these $H_\zeta(\mathfrak{gl}_n)$ the *infinitesimal Cherednik algebras of* $\mathfrak{gl}_n$.

It was shown in [EGG, Thm. 4.2], that the PBW property holds for $H_\zeta(\mathfrak{gl}_n)$ if and only if $\zeta = \sum_{j=0}^k \zeta_j r_j$ for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_j(y, x) \in U(\mathfrak{gl}_n)$ is the symmetrization of $\alpha_j(y, x) \in S(\mathfrak{gl}_n) \simeq \mathbb{C}[\mathfrak{gl}_n]$ and $\alpha_j(y, x)$ is defined via the expansion

$$(x, (1-\tau A)^{-1}y)\det(1-\tau A)^{-1} = \sum_{j\geq 0} \alpha_j(y, x)(A)\tau^j, \quad A \in \mathfrak{gl}_n.$$

Let us define the *length* of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{\geq -1} \mid \zeta_{\geq m+1} = 0\}$.

**Example 1** (cf. [EGG, Example 4.7]). If $l(\zeta) = 1$ then $H_\zeta(\mathfrak{gl}_n) \cong U(\mathfrak{sl}_{n+1})$. Thus, for an arbitrary $\zeta$, we can regard $H_\zeta(\mathfrak{gl}_n)$ as a deformation of $U(\mathfrak{sl}_{n+1})$.

One interesting problem is to find deformation parameters $\zeta$ and $\zeta'$ of the above form with $H_\zeta(\mathfrak{gl}_n) \simeq H_{\zeta'}(\mathfrak{gl}_n)$. Even for $n = 1$ (when $H_\zeta(\mathfrak{gl}_1)$ are simply the *generalized Weyl algebras*), the answer to this question (given in [BJ]) is quite nontrivial. Instead, we will look only for the filtration preserving isomorphisms, where both algebras are endowed with the *Nth standard filtration* $\{\mathcal{F}^{(N)}_\bullet\}$. Those are induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)$ with $\deg(\mathfrak{gl}_n) = 2$ and $\deg(V_n \oplus V_n^*) = N$, where $N > l(\zeta)$. For $N \geq \max\{l(\zeta)+1, l(\zeta')+1, 3\}$ we have the following result (see Appendix A for a proof):

**Lemma 1.**

(a) *$N$-standardly filtered algebras $H_\zeta(\mathfrak{gl}_n)$ and $H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic if and only if there exist $\lambda \in \mathbb{C}, \theta \in \mathbb{C}^*, s \in \{\pm\}$ such that $\zeta' = \theta\varphi_\lambda(\zeta^s)$, where*
- *$\varphi_\lambda : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ is an isomorphism defined by $\varphi_\lambda(A) = A + \lambda \cdot \operatorname{tr} A$ for any $A \in \mathfrak{gl}_n$,*
- *for $\zeta = \zeta_0 r_0 + \zeta_1 r_1 + \zeta_2 r_2 + \cdots$ we define $\zeta^- := \zeta_0 r_0 - \zeta_1 r_1 + \zeta_2 r_2 - \cdots$, $\zeta^+ := \zeta$.*

(b) *For any length $m$ deformation $\zeta$, there is a length $m$ deformation $\zeta'$ with $\zeta'_m = 1$, $\zeta'_{m-1} = 0$, such that algebras $H_\zeta(\mathfrak{gl}_n)$ and $H_{\zeta'}(\mathfrak{gl}_n)$ are isomorphic as filtered algebras.*

### 1.2. Infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$

Let $V_{2n}$ be the standard $2n$-dimensional representation of $\mathfrak{sp}_{2n}$ with a symplectic form $\omega$. Given any $\mathfrak{sp}_{2n}$-invariant pairing $\zeta : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})$ we define an algebra $H_\zeta(\mathfrak{sp}_{2n}) := U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})/([x, y] - \zeta(x, y) \mid x, y \in V_{2n})$. It has a filtration induced from the grading $\deg(\mathfrak{sp}_{2n}) = 0,\ \deg(V_{2n}) = 1$ on $T(\mathfrak{sp}_{2n} \oplus V_{2n})$.

**Definition 2.** Algebra $H_\zeta(\mathfrak{sp}_{2n})$ is referred to as the *infinitesimal Cherednik algebra of* $\mathfrak{sp}_{2n}$ if it satisfies the *PBW property*: $U(\mathfrak{sp}_{2n}) \ltimes S(V_{2n}) \xrightarrow{\sim} \mathrm{gr} H_\zeta(\mathfrak{sp}_{2n})$.

It was shown in [EGG, Thm. 4.2], that $H_\zeta(\mathfrak{sp}_{2n})$ satisfies the PBW property if and only if $\zeta = \sum_{j=0}^{k} \zeta_j r_{2j}$ for some nonnegative integer $k$ and $\zeta_j \in \mathbb{C}$, where $r_{2j}(x, y) \in U(\mathfrak{sp}_{2n})$ is the symmetrization of $\beta_{2j}(x, y) \in S(\mathfrak{sp}_{2n}) \simeq \mathbb{C}[\mathfrak{sp}_{2n}]$ and $\beta_{2j}(x, y)$ is defined via the expansion

$$\omega(x, (1 - \tau^2 A^2)^{-1} y) \det(1 - \tau A)^{-1} = \sum_{j \geq 0} \beta_{2j}(x, y)(A)\tau^{2j}, \quad A \in \mathfrak{sp}_{2n}.$$

Similarly to the $\mathfrak{gl}_n$-case, we define the *length* of such $\zeta$ by $l(\zeta) := \min\{m \in \mathbb{Z}_{\geq -1} \mid \zeta_{\geq m+1} = 0\}$.

**Example 2** (cf. [EGG, Example 4.11])**.** For $\zeta_0 \neq 0$ we have

$$H_{\zeta_0 r_0}(\mathfrak{sp}_{2n}) \cong U(\mathfrak{sp}_{2n}) \ltimes W_n,$$

where $W_n$ is the $n$th Weyl algebra. Thus, $H_\zeta(\mathfrak{sp}_{2n})$ can be regarded as a deformation of $U(\mathfrak{sp}_{2n}) \ltimes W_n$.

For any $N > 2l(\zeta)$, we introduce the $N$th standard filtration $\{\mathcal{F}^{(N)}_\bullet\}$ on $H_\zeta(\mathfrak{sp}_{2n})$ by setting $\deg(\mathfrak{sp}_{2n}) = 2,\ \deg(V_{2n}) = N$. The following result is analogous to Lemma 1:

**Lemma 2.** *For $N \geq \max\{2l(\zeta)+1, 2l(\zeta')+1, 3\}$, the $N$-standardly filtered algebras $H_\zeta(\mathfrak{sp}_{2n})$ and $H_{\zeta'}(\mathfrak{sp}_{2n})$ are isomorphic if and only if there exists $\theta \in \mathbb{C}^*$ such that $\zeta' = \theta\zeta$.*

### 1.3. Universal algebras $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

It is natural to consider a version of those algebras with $\zeta_j$ being independent central variables. This motivates the following notion of the universal length $m$ infinitesimal Cherednik algebras.

**Definition 3.** The *universal length $m$ infinitesimal Cherednik algebra* $H_m(\mathfrak{gl}_n)$ is the quotient of $U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [A, x] = A(x), \quad [A, y] = A(y),$$
$$[y, x] = \sum_{j=0}^{m-2} \zeta_j r_j(y, x) + r_m(y, x),$$

where $x, x' \in V_n^*,\ y, y' \in V_n,\ A \in \mathfrak{gl}_n$ and $\{\zeta_j\}_{j=0}^{m-2}$ are central. The filtration is induced from the grading on $T(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ with $\deg(\mathfrak{gl}_n) = 2$, $\deg(V_n \oplus V_n^*) = m+1,\ \deg(\zeta_i) = 2(m-i)$ (the latter is chosen in such a way that $\deg(\zeta_j r_j) = 2m$ for all $j$).

Algebra $H_m(\mathfrak{gl}_n)$ is free over $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$ and $H_m(\mathfrak{gl}_n)/(\zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2})$ is the usual infinitesimal Cherednik algebra $H_{\zeta_c}(\mathfrak{gl}_n)$ with $\zeta_c = c_0 r_0 + \cdots + c_{m-2} r_{m-2} + r_m$. In fact, for odd $m$, $H_m(\mathfrak{gl}_n)$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{gl}_n$, while for even $m$, there is an action of $\mathbb{Z}/2\mathbb{Z}$ we should quotient by[1].

*Remark 1.* One can consider all possible quotients

$$U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]/I \ \text{ for}$$
$$I = ([x, x'], [y, y'], [A, x] - A(x), [A, y] - A(y), [y, x] - \eta(y, x)),$$

with a $\mathfrak{gl}_n$-invariant pairing $\eta : V_n \times V_n^* \to U(\mathfrak{gl}_n)[\zeta_0, \ldots, \zeta_{m-2}]$ such that the inequality $\deg(\eta(y, x)) \leq 2m$ holds. Such a quotient satisfies a PBW property if and only if $\eta(y, x) = \sum_{i=0}^{m} \eta_i(\zeta_0, \ldots, \zeta_{m-2}) r_i(y, x)$ with $\deg(\eta_i(\zeta_0, \ldots, \zeta_{m-2})) \leq 2(m-i)$ (this is completely analogous to [EGG, Thm. 4.2]).

We define the universal version of $H_\zeta(\mathfrak{sp}_{2n})$ in a similar way:

**Definition 4.** The *universal length $m$ infinitesimal Cherednik algebra* $H_m(\mathfrak{sp}_{2n})$ is defined as

$$H_m(\mathfrak{sp}_{2n}) := U(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]/J \ \text{ for}$$
$$J = ([A, x] - A(x), [x, y] - \sum_{j=0}^{m-1} \zeta_j r_{2j}(x, y) - r_{2m}(x, y)),$$

[1] This follows from our proof of Lemma 1.

where $A \in \mathfrak{sp}_{2n},\ x, y \in V_{2n}$ and $\{\zeta_i\}_{i=0}^{m-1}$ are central. The filtration is induced from the grading on $T(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$ with $\deg(\mathfrak{sp}_{2n}) = 2,\ \deg(V_{2n}) = 2m+1$ and $\deg(\zeta_i) = 4(m-i)$.

The algebra $H_m(\mathfrak{sp}_{2n})$ is free over the subalgebra $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-1}]$ and the algebra $H_m(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$ is the usual infinitesimal Cherednik algebra $H_{\zeta_c}(\mathfrak{sp}_{2n})$ for $\zeta_c = c_0 r_0 + \cdots + c_{m-1} r_{2(m-1)} + r_{2m}$. In fact, the algebra $H_m(\mathfrak{sp}_{2n})$ can be viewed as a universal family of length $m$ infinitesimal Cherednik algebras of $\mathfrak{sp}_{2n}$, due to Lemma 2.

*Remark 2.* Analogously to Remark 1, the result of [EGG, Thm. 4.2], generalizes straightforwardly to the case of $\mathfrak{sp}_{2n}$-invariant pairings $\eta : V_{2n} \times V_{2n} \to U(\mathfrak{sp}_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$.

### 1.4. Poisson counterparts of $H_\zeta(\mathfrak{g})$ and $H_m(\mathfrak{g})$

Following [DT], we introduce the Poisson algebras $H_m^{\mathrm{cl}}(\mathfrak{g})$ for $\mathfrak{g}$ being $\mathfrak{gl}_n$ or $\mathfrak{sp}_{2n}$.

As algebras these are $S(\mathfrak{gl}_n \oplus V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]$ (respectively $S(\mathfrak{sp}_{2n} \oplus V_{2n})[\zeta_0, \ldots, \zeta_{m-1}]$) with a Poisson bracket $\{\cdot, \cdot\}$ modeled after the commutator $[\cdot, \cdot]$ from the definition of $H_m(\mathfrak{g})$, so that $\{y, x\} = \alpha_m(y, x) + \sum_{j=0}^{m-2} \zeta_j \alpha_j(y, x)$ (respectively $\{x, y\} = \beta_{2m}(x, y) + \sum_{j=0}^{m-1} \zeta_j \beta_{2j}(x, y)$). Their quotients $H_m^{\mathrm{cl}}(\mathfrak{gl}_n)/(\zeta_0 - c_0, \ldots, \zeta_{m-2} - c_{m-2})$ and $H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})/(\zeta_0 - c_0, \ldots, \zeta_{m-1} - c_{m-1})$, are the Poisson infinitesimal Cherednik algebras $H_{\zeta_c}^{\mathrm{cl}}(\mathfrak{gl}_n)$ ($\zeta_c = c_0 \alpha_0 + \cdots + c_{m-2} \alpha_{m-2} + \alpha_m$) and $H_{\zeta_c}^{\mathrm{cl}}(\mathfrak{sp}_{2n})$ ($\zeta_c = c_0 \beta_0 + \cdots + c_{m-1} \beta_{2m-2} + \beta_{2m}$) from [DT, Sects. 5 and 7] respectively.

Let us describe the Poisson centers of the algebras $H_m^{\mathrm{cl}}(\mathfrak{gl}_n)$ and $H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})$.

For $\mathfrak{g} = \mathfrak{gl}_n$ and $1 \le k \le n$ we define an element $\tau_k \in H_m^{\mathrm{cl}}(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^n x_i \{\widetilde{Q}_k, y_i\}$, where $1 + \sum_{j=1}^n \widetilde{Q}_j z^j = \det(1 + zA)$. We set $\zeta(w) := \sum_{i=0}^{m-2} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{gl}_n)$ via

$$c(t) = 1 + \sum_{i=1}^{n} (-1)^i c_i t^i := \mathrm{Res}_{z=0} \zeta(z^{-1}) \frac{\det(1 - tA)}{\det(1 - zA)} \frac{z^{-1} dz}{1 - t^{-1} z}.$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $1 \le k \le n$ we define an element $\tau_k \in H_m^{\mathrm{cl}}(\mathfrak{g})$ by $\tau_k := \sum_{i=1}^{2n} \{\widetilde{Q}_k, y_i\} y_i^*$, where $1 + \sum_{j=1}^n \widetilde{Q}_j z^{2j} = \det(1 + zA)$, while $\{y_i\}_{i=1}^{2n}$ and $\{y_i^*\}_{i=1}^{2n}$ are the dual bases of $V_{2n}$, that is, $\omega(y_i, y_j^*) = 1$. We set $\zeta(w) := \sum_{i=0}^{m-1} \zeta_i w^i + w^m$ and define $c_i \in S(\mathfrak{sp}_{2n})$ via

$$c(t) = 1 + \sum_{i=1}^{n} c_i t^{2i} := 2\mathrm{Res}_{z=0} \zeta(z^{-2}) \frac{\det(1 - tA)}{\det(1 - zA)} \frac{z^{-1} dz}{1 - t^{-2} z^2}.$$

The following result is a straightforward generalization of [DT, Thms. 5.1 and 7.1]:

**Theorem 3.** *Let $\mathfrak{z}_{\mathrm{Pois}}(A)$ denote the Poisson center of the Poisson algebra $A$. We have:*

(a) *$\mathfrak{z}_{\mathrm{Pois}}(H_m^{\mathrm{cl}}(\mathfrak{gl}_n))$ is a polynomial algebra in free generators $\zeta_0, \ldots, \zeta_{m-2}, \tau_1 + c_1, \ldots, \tau_n + c_n$;*

(b) *$\mathfrak{z}_{\mathrm{Pois}}(H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n}))$ is a polynomial algebra in free generators $\zeta_0, \ldots, \zeta_{m-1}, \tau_1 + c_1, \ldots, \tau_n + c_n$.*

### 1.5. $W$-algebras

Here we recall finite $W$-algebras following [GG].

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $e \in \mathfrak{g}$ be a nonzero nilpotent element. We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form $( \ , \ )$. Let $\chi$ be the element of $\mathfrak{g}^*$ corresponding to $e$ and $\mathfrak{z}_\chi$ be the stabilizer of $\chi$ in $\mathfrak{g}$ (which is the same as the centralizer of $e$ in $\mathfrak{g}$). Fix an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}$. Then $\mathfrak{z}_\chi$ is $\mathrm{ad}(h)$–stable and the eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{z}_\chi$ are nonnegative integers.

Consider the $\mathrm{ad}(h)$–weight grading on $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$, that is, $\mathfrak{g}(i) := \{\xi \in \mathfrak{g} \mid [h, \xi] = i\xi\}$. Equip $\mathfrak{g}(-1)$ with the symplectic form $\omega_\chi(\xi, \eta) := \langle \chi, [\xi, \eta]\rangle$. Fix a Lagrangian subspace $l \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := \bigoplus_{i\leq -2} \mathfrak{g}(i) \oplus l \subset \mathfrak{g}$, $\mathfrak{m}' := \{\xi - \langle \chi, \xi\rangle, \xi \in \mathfrak{m}\} \subset U(\mathfrak{g})$.

**Definition 5** (cf. [P1], [GG]). By the $W$-algebra associated with $e$ (and $l$), we mean the algebra $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}')^{\mathrm{ad}\,\mathfrak{m}}$ with multiplication induced from $U(\mathfrak{g})$.

Let $\{F^{\mathrm{st}}_\bullet\}$ denote the PBW filtration on $U(\mathfrak{g})$, while $U(\mathfrak{g})(i) := \{x \in U(\mathfrak{g}) \mid [h, x] = ix\}$. Define $F_k U(\mathfrak{g}) = \sum_{i+2j\leq k} (F^{\mathrm{st}}_j U(\mathfrak{g}) \cap U(\mathfrak{g})(i))$ and equip $U(\mathfrak{g}, e)$ with the induced filtration, denoted $\{F_\bullet\}$ and referred to as the *Kazhdan* filtration.

One of the key results of [P1], [GG] is a description of the associated graded algebra $\mathrm{gr}_{F_\bullet} U(\mathfrak{g}, e)$. Recall that the affine subspace $\mathrm{S} := \chi + (\mathfrak{g}/[\mathfrak{g}, f])^* \subset \mathfrak{g}^*$ is called the *Slodowy slice*. As an affine subspace of $\mathfrak{g}$, the Slodowy slice S coincides with $e+\mathfrak{c}$, where $\mathfrak{c} = \mathrm{Ker}_\mathfrak{g}\, \mathrm{ad}(f)$. So we can identify $\mathbb{C}[\mathrm{S}] \cong \mathbb{C}[\mathfrak{c}]$ with the symmetric algebra $S(\mathfrak{z}_\chi)$. According to [GG, Sect. 3], algebra $\mathbb{C}[\mathrm{S}]$ inherits a Poisson structure from $\mathbb{C}[\mathfrak{g}^*]$ and is also graded with $\deg(\mathfrak{z}_\chi \cap \mathfrak{g}(i)) = i + 2$.

**Theorem 4** (cf. [GG, Thm. 4.1]). *The filtered algebra* $U(\mathfrak{g}, e)$ *does not depend on the choice of* $l$ *(up to a distinguished isomorphism) and* $\mathrm{gr}_{F_\bullet} U(\mathfrak{g}, e) \cong \mathbb{C}[\mathrm{S}]$ *as graded Poisson algebras.*

### 1.6. Additional properties of $W$-algebras

We want to describe some other properties of $U(\mathfrak{g}, e)$.

(a) Let $G$ be the adjoint group of $\mathfrak{g}$. There is a natural action of the group $Q := Z_G(e, h, f)$ on $U(\mathfrak{g}, e)$, due to [GG]. Let $\mathfrak{q}$ stand for the Lie algebra of $Q$. In [P2] Premet constructed a Lie algebra embedding $\mathfrak{q} \overset{\iota}{\hookrightarrow} U(\mathfrak{g}, e)$. The adjoint action of $\mathfrak{q}$ on $U(\mathfrak{g}, e)$ coincides with the differential of the aforementioned $Q$-action.

(b) Restricting the natural map $U(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{m}} \to U(\mathfrak{g}, e)$ to $Z(U(\mathfrak{g}))$, we get an algebra homomorphism $Z(U(\mathfrak{g})) \overset{\rho}{\to} Z(U(\mathfrak{g}, e))$, where $Z(A)$ stands for the center of an algebra $A$. According to the following theorem, $\rho$ is an isomorphism:

**Theorem 5.**

(a) [P1, Sect. 6.2] *The homomorphism* $\rho$ *is injective.*

(b) [P2, footnote to Quest. 5.1] *The homomorphism* $\rho$ *is surjective.*

## 2. Main theorem

Let us consider $\mathfrak{g} = \mathfrak{sl}_N$ or $\mathfrak{g} = \mathfrak{sp}_{2N}$, and let $e_m \in \mathfrak{g}$ be a 1-*block* nilpotent element of Jordan type $(1, \ldots, 1, m)$ or $(1, \ldots, 1, 2m)$, respectively. We make a particular choice for $e_m$:

- $e_m = E_{N-m+1,N-m+2} + \cdots + E_{N-1,N}$ in the case of $\mathfrak{sl}_N,\ 2 \le m \le N$,
- $e_m = E_{N-m+1,N-m+2} + \cdots + E_{N+m-1,N+m}$ in the case of $\mathfrak{sp}_{2N},\ 1 \le m \le N$.[2]

Recall the Lie algebra inclusion $\iota : \mathfrak{q} \hookrightarrow U(\mathfrak{g}, e)$ from Section 1.6. In our cases:

• For $(\mathfrak{g}, e) = (\mathfrak{sl}_{n+m}, e_m)$, we have $\mathfrak{q} \simeq \mathfrak{gl}_n$. Define $\overline{T} \in U(\mathfrak{sl}_{n+m}, e_m)$ to be the $\iota$-image of the identity matrix $I_n \in \mathfrak{gl}_n$, the latter being identified with

$$T_{n,m} = \mathrm{diag}(m/(n+m), \ldots, m/(n+m), -n/(n+m), \ldots, -n/(n+m))$$

under the inclusion $\mathfrak{q} \hookrightarrow \mathfrak{sl}_{n+m}$. Let Gr be the induced $\mathrm{ad}(\overline{T})$-weight grading on $U(\mathfrak{sl}_{n+m}, e_m)$, with the $j$th grading component denoted by $U(\mathfrak{sl}_{n+m}, e_m)_j$.

• For $(\mathfrak{g}, e) = (\mathfrak{sp}_{2n+2m}, e_m)$, we have $\mathfrak{q} \simeq \mathfrak{sp}_{2n}$. Define

$$\overline{T}' := \iota(I'_n) \in U(\mathfrak{sp}_{2n+2m}, e_m),$$

where $I'_n = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \in \mathfrak{sp}_{2n} \simeq \mathfrak{q}$. Let Gr be the induced $\mathrm{ad}(\overline{T}')$-weight grading on $U(\mathfrak{sp}_{2n+2m}, e_m) = \bigoplus_j U(\mathfrak{sp}_{2n+2m}, e_m)_j$.

**Lemma 6.** *There is a natural Lie algebra inclusion* $\Theta : \mathfrak{gl}_n \ltimes V_n \hookrightarrow U(\mathfrak{sl}_{n+m}, e_m)$ *such that* $\Theta\mid_{\mathfrak{gl}_n} = \iota\mid_{\mathfrak{gl}_n}$ *and* $\Theta(V_n) = F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$.

*Proof.* First, choose a Jacobson–Morozov $\mathfrak{sl}_2$-triple $(e_m, h_m, f_m) \subset \mathfrak{sl}_{n+m}$ in a standard way[3]. As a vector space, $\mathfrak{z}_\chi \cong \mathfrak{gl}_n \oplus V_n \oplus V_n^* \oplus \mathbb{C}^{m-1}$ with $\mathfrak{gl}_n = \mathfrak{z}_\chi(0) = \mathfrak{q}$, $V_n \oplus V_n^* \subset \mathfrak{z}_\chi(m-1)$, and $\xi_j \in \mathfrak{z}_\chi(2m-2j-2)$. Here $\mathbb{C}^{m-1}$ has a basis $\{\xi_{m-2-j} = E_{n+1,n+j+2} + \cdots + E_{n+m-j-1,n+m}\}_{j=0}^{m-2}$, $V_n \oplus V_n^*$ is embedded via $y_i \mapsto E_{i,n+m},\ x_i \mapsto E_{n+1,i}$, while $\mathfrak{gl}_n \cong \mathfrak{sl}_n \oplus \mathbb{C} \cdot I_n$ is embedded in the following way: $\mathfrak{sl}_n \hookrightarrow \mathfrak{sl}_{n+m}$ as a *left-up block*, while $I_n \mapsto T_{n,m}$.

Under the identification $\mathrm{gr}_{F_\bullet} U(\mathfrak{sl}_{n+m}, e_m) \simeq \mathbb{C}[\mathrm{S}] \simeq S(\mathfrak{z}_\chi)$, the induced grading $\mathrm{Gr}'$ on $S(\mathfrak{z}_\chi)$ is the $\mathrm{ad}(T_{n,m})$-weight grading. Together with the above description of $\mathrm{ad}(h_m)$-grading on $\mathfrak{z}_\chi$, this implies that $F_m U(\mathfrak{sl}_{n+m}, e_m)_1 = 0$ and that $F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$ coincides with the image of the composition $V_n \hookrightarrow \mathfrak{z}_\chi \hookrightarrow S(\mathfrak{z}_\chi)$. Let $\Theta(y) \in F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_1$ be the element whose image is identified with $y$. We also set $\Theta(A) := \iota(A)$ for $A \in \mathfrak{gl}_n$. Finally, we define $\Theta : \mathfrak{gl}_n \oplus V_n \hookrightarrow U(\mathfrak{sl}_{n+m}, e_m)$ by linearity.

We claim that $\Theta$ is a Lie algebra inclusion, that is,

$$[\Theta(A), \Theta(B)] = \Theta([A, B]), \quad [\Theta(y), \Theta(y')] = 0, \quad [\Theta(A), \Theta(y)] = \Theta(A(y)),$$
$$\forall\ A, B \in \mathfrak{gl}_n, y, y' \in V_n.$$

The first equality follows from $[\Theta(A), \Theta(B)] = [\iota(A), \iota(B)] = \iota([A, B]) = \Theta([A, B])$. The second one follows from the observation that $[\Theta(y), \Theta(y')] \in F_{2m}U(\mathfrak{g}, e_m)_2$ and the only such element is 0. Similarly, $[\Theta(A), \Theta(y)] \in F_{m+1}U(\mathfrak{g}, e_m)_1$, so that $[\Theta(A), \Theta(y)] = \Theta(y')$ for some $y' \in V_n$. Since $y' = \mathrm{gr}(\Theta(y')) = \mathrm{gr}([\Theta(A), \Theta(y)]) = [A, y] = A(y)$, we get $[\Theta(A), \Theta(y)] = \Theta(A(y))$. $\square$

Our main result is:

[2] We view $\mathfrak{sp}_{2N}$ as corresponding to the pair $(V_{2N}, \omega_{2N})$, where $\omega_{2N}$ is represented by the skew symmetric *antidiagonal* matrix $J = (J_{ij} := (-1)^j \delta_{i+j}^{2N+1})_{1 \le i,j \le 2N}$. In this presentation, $A = (a_{ij}) \in \mathfrak{sp}_{2N}$ if and only if $a_{2N+1-j,2N+1-i} = (-1)^{i+j+1} a_{ij}$ for any $1 \le i, j \le 2N$.

[3] That is, we set $h_m := \sum_{j=1}^{m} (m+1-2j)E_{n+j,n+j}$ and $f_m := \sum_{j=1}^{m-1} j(m-j)E_{n+j+1,n+j}$.

**Theorem 7.**

(a) *For $m \geq 2$, there is a unique isomorphism*

$$\overline{\Theta} : H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$$

*of filtered algebras, whose restriction to $\mathfrak{sl}_n \ltimes V_n \hookrightarrow H_m(\mathfrak{gl}_n)$ is equal to $\Theta$.*

(b) *For $m \geq 1$, there are exactly two isomorphisms*

$$\overline{\Theta}_{(1)},\ \overline{\Theta}_{(2)} : H_m(\mathfrak{sp}_{2n}) \xrightarrow{\sim} U(\mathfrak{sp}_{2n+2m}, e_m)$$

*of filtered algebras such that $\overline{\Theta}_{(i)}\ |_{\mathfrak{sp}_{2n}} = \iota\ |_{\mathfrak{sp}_{2n}}$; moreover, $\overline{\Theta}_{(2)} \circ \overline{\Theta}_{(1)}^{-1} : y \mapsto -y, A \mapsto A, \zeta_k \mapsto \zeta_k$.*

Let us point out that there is no explicit presentation of $W$-algebras in terms of generators and relations in general. Among the few known cases are: (a) $\mathfrak{g} = \mathfrak{gl}_n$, due to [BK1], (b) $\mathfrak{g} \ni e$, the minimal nilpotent, due to [P2, Sect. 6]. The latter corresponds to $(e_2, \mathfrak{sl}_N)$ and $(e_1, \mathfrak{sp}_{2N})$ in our notation. We establish the corresponding isomorphisms explicitly in Appendix B.

*Proof of Theorem* 7.

(a) Analogously to Lemma 6, we have an identification $F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1} \simeq V_n^*$. For any $x \in V_n^*$, let $\Theta(x) \in F_{m+1}U(\mathfrak{sl}_{n+m}, e_m)_{-1}$ be the element identified with $x \in V_n^*$. The same argument as in the proof of Lemma 6 implies $[\Theta(A), \Theta(x)] = \Theta(A(x))$.

Let $\{\widetilde{F}_j\}_{j=2}^{n+m}$ be the standard degree $j$ generators of the algebra $\mathbb{C}[\mathfrak{sl}_{n+m}]^{\mathrm{SL}_{n+m}} \simeq S(\mathfrak{sl}_{n+m})^{\mathrm{SL}_{n+m}}$ (that is, $1 + \sum_{j=2}^{n+m} \widetilde{F}_j(A)z^j = \det(1 + zA)$ for $A \in \mathfrak{sl}_{n+m}$) and $F_j := \mathrm{Sym}(\widetilde{F}_j) \in U(\mathfrak{sl}_{n+m})$ be the free generators of $Z(U(\mathfrak{sl}_{n+m}))$. For all $0 \leq i \leq m-2$ we set $\Theta_i := \rho(F_{m-i}) \in Z(U(\mathfrak{sl}_{n+m}, e_m))$. Then $\mathrm{gr}(\Theta_k) = \widetilde{F}_{m-k|\mathrm{S}} \equiv \xi_k \mod S(\mathfrak{gl}_n \oplus \bigoplus_{l=k+1}^{m-2} \mathbb{C}\xi_l)$, where $\xi_k$ was defined in the proof of Lemma 6.

Let $U'$ be a subalgebra of $U(\mathfrak{sl}_{n+m}, e_m)$, generated by $\Theta(\mathfrak{gl}_n)$ and $\{\Theta_k\}_{k=0}^{m-2}$. For all $y \in V_n,\ x \in V_n^*$ we define $W(y, x) := [\Theta(y), \Theta(x)] \in F_{2m}U(\mathfrak{sl}_{n+m}, e_m)_0 \subset U'$. Let us point out that equalities $[\Theta(A), \Theta(x)] = \Theta([A, x]),\ [\Theta(A), \Theta(y)] = \Theta([A, y])$ (for all $A \in \mathfrak{gl}_n, y \in V_n, x \in V_n^*$) imply the $\mathfrak{gl}_n$-invariance of $W : V_n \times V_n^* \to U' \simeq U(\mathfrak{gl}_n)[\Theta_0, \ldots .\Theta_{m-2}]$.

By Theorem 4, $U(\mathfrak{sl}_{n+m}, e_m)$ has a basis formed by the ordered monomials in

$$\{\Theta(E_{ij}),\ \Theta(y_k),\ \Theta(x_l),\ \Theta_0, \ldots, \Theta_{m-2}\}.$$

In particular, $U(\mathfrak{sl}_{n+m}, e_m) \simeq U(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\Theta_0, \ldots, \Theta_{m-2}]/(y \otimes x - x \otimes y - W(y, x))$ satisfies the PBW property. According to Remark 1, there exist polynomials $\eta_i \in \mathbb{C}[\Theta_0, \ldots, \Theta_{m-2}]$, for $0 \leq i \leq m-2$, such that $W(y, x) = \sum \eta_j r_j(y, x)$ and $\deg(\eta_i(\Theta_0, \ldots, \Theta_{m-2})) \leq 2(m-i)$. As a consequence of the latter condition: $\eta_m, \eta_{m-1} \in \mathbb{C}$.

The following claim follows from the main result of the next section (Theorem 10):

**Claim 8.**

(i) *The constant* $\eta_m$ *is nonzero.*
(ii) *The polynomial* $\eta_i(\Theta_0,\ldots,\Theta_{m-2})$ *contains a nonzero multiple of* $\Theta_i$ *for any* $i\leq m-2$.

This claim implies the existence and uniqueness of the isomorphism $\overline{\Theta}:H_m(\mathfrak{gl}_n)\xrightarrow{\sim} U(\mathfrak{sl}_{n+m},e_m)$ with $\overline{\Theta}(y_k)=\Theta(y_k)$ and $\overline{\Theta}(A)=\Theta(A)$ for $A\in\mathfrak{sl}_n$.

Moreover, $\overline{\Theta}(x_k)=\eta_m^{-1}\Theta(x_k)$ and $\overline{\Theta}(I_n)=\Theta(I_n)-n\eta_{m-1}/(n+m)\eta_m$ [4], while $\overline{\Theta}(\zeta_k)\in\mathbb{C}[\Theta_k,\ldots,\Theta_{m-2}]$.

(b) Choose a Jacobson–Morozov $\mathfrak{sl}_2$-triple $(e_m,h_m,f_m)\subset\mathfrak{sp}_{2n+2m}$ in a standard way.[5] As a vector space, $\mathfrak{z}_\chi\cong\mathfrak{sp}_{2n}\oplus V_{2n}\oplus\mathbb{C}^m$ with $\mathfrak{sp}_{2n}=\mathfrak{z}_\chi(0)$, $V_{2n}=\mathfrak{z}_\chi(2m-1)$ and $\xi_j\in\mathfrak{z}_\chi(4m-4j-2)$. Here $\mathbb{C}^m$ has a basis $\{\xi_{m-k}=E_{n+1,n+2k}+\cdots+E_{n+2m-2k+1,n+2m}\}_{k=1}^m$, $V_{2n}$ is embedded via

$$y_i\mapsto E_{i,n+2m}+(-1)^{n+i+1}E_{n+1,2n+2m+1-i},$$
$$y_{n+i}\mapsto E_{n+2m+i,n+2m}+(-1)^{i+1}E_{n+1,n+1-i},\ i\leq n,$$

while $\mathfrak{q}=\mathfrak{z}_\chi(0)\simeq\mathfrak{sp}_{2n}$ is embedded in a natural way (via four $n\times n$ *corner blocks* of $\mathfrak{sp}_{2n+2m}$).

Recall the grading Gr on $U(\mathfrak{sp}_{2n+2m},e_m)$. The induced grading $\mathrm{Gr}'$ on the space $\mathrm{gr}\,U(\mathfrak{sp}_{2n+2m},e_m)$ is the $\mathrm{ad}(I'_n)$-weight grading on $S(\mathfrak{z}_\chi)$. The operator $\mathrm{ad}(I'_n)$ acts trivially on $\mathbb{C}^m$, with even eigenvalues on $\mathfrak{sp}_{2n}$ and with eigenvalues $\pm1$ on $V_{2n}^\pm$, where $V_{2n}^+$ is spanned by $\{y_i\}_{i\leq n}$, while $V_{2n}^-$ is spanned by $\{y_{n+i}\}_{i\leq n}$.

Analogously to Lemma 6, we get identifications of $F_{2m+1}U(\mathfrak{sp}_{2n+2m},e_m)_{\pm1}$ and $V_{2n}^\pm$. For $y\in V_{2n}^\pm$, let $\Theta(y)$ be the corresponding element of $F_{2m+1}U(\mathfrak{sp}_{2n+2m},e_m)_{\pm1}$, while for $A\in\mathfrak{sp}_{2n}$ we set $\Theta(A):=\iota(A)$. We define $\Theta:\mathfrak{sp}_{2n}\oplus V_{2n}\hookrightarrow U(\mathfrak{sp}_{2n+2m},e_m)$ by linearity. The same reasoning as in the $\mathfrak{gl}_n$-case proves that $[\Theta(A),\Theta(y)]=\Theta(A(y))$ for any $A\in\mathfrak{sp}_{2n},y\in V_{2n}$.

Finally, the argument involving the center goes along the same lines, so we can pick central generators $\{\Theta_k\}_{0\leq k\leq m-1}$ such that $\mathrm{gr}(\Theta_k)\equiv\xi_k \mod S(\mathfrak{sp}_{2n}\oplus\mathbb{C}\xi_{k+1}\oplus\ldots\oplus\mathbb{C}\xi_{m-1})$.

Let $U'$ be the subalgebra of $U(\mathfrak{sp}_{2n+2m},e_m)$, generated by $\Theta(\mathfrak{sp}_{2n})$ and $\{\Theta_k\}_{k=0}^{m-1}$. For $x,y\in V_{2n}$, we set $W(x,y):=[\Theta(x),\Theta(y)]\in F_{4m}U(\mathfrak{sp}_{2n+2m},e_m)_{\mathrm{even}}\subset U'$. The map

$$W:V_{2n}\times V_{2n}\to U'\simeq U(\mathfrak{sp}_{2n})[\Theta_0,\ldots,\Theta_{m-1}]$$

is $\mathfrak{sp}_{2n}$-invariant.

Since $U(\mathfrak{sp}_{2n+2m},e_m)\simeq U(\mathfrak{sp}_{2n})\ltimes T(V_{2n})[\Theta_0,\ldots,\Theta_{m-1}]/(x\otimes y-y\otimes x-W(x,y))$ satisfies the PBW property, there exist polynomials $\eta_i\in\mathbb{C}[\Theta_0,\ldots,\Theta_{m-1}]$, for $0\leq i\leq m-1$, such that $W(x,y)=\sum\eta_j r_{2j}(x,y)$ and $\deg(\eta_i(\Theta_0,\ldots,\Theta_{m-1}))\leq 4(m-i)$ (Remark 2).

The following result is analogous to Claim 8 and will follow from Theorem 10 as well:

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[4] The appearance of the constant $n\eta_{m-1}/(n+m)\eta_m$ is explained by the proof of Lemma 1(b).

[5] That is, $h_m:=\sum_{j=1}^{2m}(2m+1-2j)E_{n+j,n+j}$ and $f_m:=\sum_{j=1}^{2m-1}j(2m-j)E_{n+j+1,n+j}$.

**Claim 9.**

(i) *The constant* $\eta_m$ *is nonzero.*
(ii) *The polynomial* $\eta_i(\Theta_0,\ldots,\Theta_{m-1})$ *contains a nonzero multiple of* $\Theta_i$ *for any* $i\leq m-1$.

This claim implies Theorem 7(b), where $\overline{\Theta}_{(i)}(y)=\lambda_i\cdot\Theta(y)$ for all $y\in V_{2n}$ and $\lambda_i^2=\eta_m^{-1}$. $\square$

## 3. Poisson analogue of Theorem 7

To state the main result of this section, let us introduce more notation:

- In the contexts of $(\mathfrak{sl}_{n+m},e_m)$ and $(\mathfrak{sp}_{2n+2m},e_m)$, we use $\mathrm{S}_{n,m}$ and $\mathfrak{z}_{n,m}$ instead of S and $\mathfrak{z}_\chi$.
- Let $\overline{\iota}:\mathfrak{gl}_n\oplus V_n\oplus V_n^*\oplus\mathbb{C}^{m-1}\xrightarrow{\sim}\mathfrak{z}_{n,m}$ be the identification from the proof of Lemma 6.
- Let $\overline{\iota}:\mathfrak{sp}_{2n}\oplus V_{2n}\oplus\mathbb{C}^{m}\xrightarrow{\sim}\mathfrak{z}_{n,m}$ be the identification from the proof of Theorem 7(b).
- Define $\overline{\Theta}_k=\mathrm{gr}(\Theta_k)\in S(\mathfrak{z}_{n,m})$ $0\leq k\leq m-s$, where $s=1$ for $\mathfrak{sp}_{2N}$ and $s=2$ for $\mathfrak{sl}_N$.
- We consider the Poisson structure on $S(\mathfrak{z}_{n,m})$ arising from the identification

$$S(\mathfrak{z}_{n,m})\cong\mathbb{C}[\mathrm{S}_{n,m}].$$

The following theorem can be viewed as a Poisson analogue of Theorem 7:

**Theorem 10.**

(a) *The formulas*

$$\overline{\Theta}^{\mathrm{cl}}(A)=\overline{\iota}(A),\quad \overline{\Theta}^{\mathrm{cl}}(y)=\overline{\iota}(y),\quad \overline{\Theta}^{\mathrm{cl}}(x)=\overline{\iota}(x),\quad \overline{\Theta}^{\mathrm{cl}}(\zeta_k)=(-1)^{m-k}\overline{\Theta}_k$$

*define an isomorphism* $\overline{\Theta}^{\mathrm{cl}}:H_m^{\mathrm{cl}}(\mathfrak{gl}_n)\xrightarrow{\sim}S(\mathfrak{z}_{n,m})\simeq\mathbb{C}[\mathrm{S}_{n,m}]$ *of Poisson algebras.*

(b) *The formulas*

$$\overline{\Theta}^{\mathrm{cl}}(A)=\overline{\iota}(A),\quad \overline{\Theta}^{\mathrm{cl}}(y)=\overline{\iota}(y)/\sqrt{2},\quad \overline{\Theta}^{\mathrm{cl}}(\zeta_k)=\overline{\Theta}_k$$

*define an isomorphism* $\overline{\Theta}^{\mathrm{cl}}:H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})\xrightarrow{\sim}S(\mathfrak{z}_{n,m})\simeq\mathbb{C}[\mathrm{S}_{n,m}]$ *of Poisson algebras.*

Claims 8 and 9 follow from this theorem.

*Remark 3.* An alternative proof of Claims 8 and 9 is based on the recent result of [LNS] about the universal Poisson deformation of $\mathrm{S}\cap\mathcal{N}$ (here $\mathcal{N}$ denotes the nilpotent cone of the Lie algebra $\mathfrak{g}$). We find this argument a bit overkilling (besides, it does not provide precise formulas in the Poisson case).

*Proof of Theorem* 10.

(a) The Poisson algebra $S(\mathfrak{z}_{n,m})$ is equipped both with the Kazhdan grading and the internal grading $\mathrm{Gr}'$. In particular, the same reasoning as in the proof of Theorem 7(a) implies:

$$\{\overline{\iota}(A),\overline{\iota}(B)\}=\overline{\iota}([A,B]),\quad \{\overline{\iota}(A),\overline{\iota}(y)\}=\overline{\iota}(A(y)),\quad \{\overline{\iota}(A),\overline{\iota}(x)\}=\overline{\iota}(A(x)).$$

We set $\overline{W}(y,x):=\{\overline{\iota}(y),\overline{\iota}(x)\}$ for all $y\in V_n, x\in V_n^*$. Arguments analogous to those used in the proof of Theorem 7(a) imply an existence of polynomials $\overline{\eta}_i\in\mathbb{C}[\overline{\Theta}_0,\ldots,\overline{\Theta}_{m-2}]$, such that $\overline{W}(y,x)=\sum_j\overline{\eta}_j\alpha_j(y,x)$ and $\deg(\overline{\eta}_j(\overline{\Theta}_0,\ldots,\overline{\Theta}_{m-2}))=2(m-j)$.

Combining this with Theorem 3(a) one gets that

$$\tau_1'=\sum_i x_iy_i+\sum_j\overline{\eta}_j\,\mathrm{tr}\,S^{j+1}A$$

is a Poisson-central element of $S(\mathfrak{z}_{n,m})\cong\mathbb{C}[\mathrm{S}_{n,m}]$.

Let $\overline{\rho}:\mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathfrak{sl}_{n+m}])\to\mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathrm{S}_{n,m}])$ be the restriction homomorphism. The Poisson analogue of Theorem 5 (which is, actually, much simpler) states that $\overline{\rho}$ is an isomorphism. In particular, $\tau_1'=c\overline{\rho}(\widetilde{F}_{m+1})+p(\overline{\rho}(\widetilde{F}_2),\ldots,\overline{\rho}(\widetilde{F}_m))$ for some $c\in\mathbb{C}$ and a polynomial $p$.

Note that $\overline{\rho}(\widetilde{F}_i)=\overline{\Theta}_{m-i}$ for all $2\le i\le m$. Let us now express $\overline{\rho}(\widetilde{F}_{m+1})$ via the generators of $S(\mathfrak{z}_{n,m})$. First, we describe explicitly the slice $\mathrm{S}_{n,m}$. It consists of the following elements:

$$\Big\{e_m+\sum_{i,j\le n}x_{i,j}E_{i,j}+\sum_{i\le n}u_iE_{i,n+1}+\sum_{i\le n}v_iE_{n+m,i}+\sum_{k\le m-1}w_kf_m^k-\gamma_{n,m}\sum_{n<j\le n+m}E_{jj}\Big\},$$
$$\text{where}\ \ \gamma_{n,m}=\frac{1}{m}\sum_{i\le n}x_{ii}$$

which can also be explicitly depicted as follows:

$$\mathrm{S}_{n,m}=\left\{X=\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} & u_1 & 0 & 0 & \cdots & 0\\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} & u_2 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} & u_n & 0 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \lambda & 1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & \star & \lambda & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & \star & \star & \star & \cdots & 1\\ v_1 & v_2 & \cdots & v_n & \star & \star & \star & \cdots & \lambda \end{pmatrix}\right\}$$

For $X\in\mathfrak{sl}_{n+m}$ of the above form let us define $X_1\in\mathfrak{gl}_n,\ X_2\in\mathfrak{gl}_m$ by

$$X_1:=\sum_{i,j\le n}x_{i,j}E_{i,j},\quad X_2:=e_m+\sum_{k\le m-1}w_kf_m^k-\frac{x_{11}+\cdots+x_{nn}}{m}\sum_{n<j\le n+m}E_{jj},$$

that is, $X_1$ and $X_2$ are the left-up $n\times n$ and right-down $m\times m$ blocks of $X$, respectively.

The following result is straightforward:

**Lemma 11.** *Let $X, X_1, X_2$ be as above. Then:*

(i) *For* $2 \leq k \leq m$ : $\widetilde{F}_k(X) = \operatorname{tr}\Lambda^k(X_1) + \operatorname{tr}\Lambda^{k-1}(X_1)\operatorname{tr}\Lambda^1(X_2) + \cdots + \operatorname{tr}\Lambda^k(X_2)$.

(ii) *We have* $\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \operatorname{tr}\Lambda^{m+1}(X_1) + \operatorname{tr}\Lambda^m(X_1)\operatorname{tr}\Lambda^1(X_2) + \cdots + \operatorname{tr}\Lambda^{m+1}(X_2)$.

Combining both statements of this lemma with the standard equality

$$\sum_{0\leq j\leq l} (-1)^j \operatorname{tr} S^{l-j}(X_1)\operatorname{tr}\Lambda^j(X_1) = 0, \qquad \forall l \geq 1, \tag{1}$$

we obtain the following result:

**Lemma 12.** *For any $X \in \mathrm{S}_{n,m}$ we have:*

$$\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{2\leq j\leq m} (-1)^{m-j}\widetilde{F}_j(X)\operatorname{tr} S^{m+1-j}(X_1) + (-1)^m \operatorname{tr} S^{m+1}(X_1). \tag{2}$$

*Proof of Lemma* 12. Lemma 11(i) and equality (1) imply by induction on $k$:

$$\operatorname{tr}\Lambda^k(X_2) = \widetilde{F}_k(X) - \operatorname{tr} S^1(X_1)\widetilde{F}_{k-1}(X) + \operatorname{tr} S^2(X_1)\widetilde{F}_{k-2}(X) - \ldots + (-1)^k \operatorname{tr} S^k(X_1)\widetilde{F}_0(X),$$

for all $k \leq m$, where $\widetilde{F}_1(X) = 0$, $\widetilde{F}_0(X) = 1$.

Those equalities together with Lemma 11(ii) imply:

$$\widetilde{F}_{m+1}(X) = (-1)^m \sum u_i v_i + \sum_{0\leq j\leq m}\sum_{0\leq k< m+1-j} (-1)^k \operatorname{tr}\Lambda^{m+1-j-k}(X_1)\operatorname{tr} S^k(X_1)\widetilde{F}_j(X).$$

According to (1), we have

$$\sum_{0\leq k\leq m-j} (-1)^k \operatorname{tr}\Lambda^{m+1-j-k}(X_1)\operatorname{tr} S^k(X_1) = (-1)^{m-j}\operatorname{tr} S^{m+1-j}(X_1).$$

Recalling our convention $\widetilde{F}_1(X) := 0$, $\widetilde{F}_0(X) := 1$, we get (2). $\square$

Identifying $\mathbb{C}[\mathrm{S}_{n,m}]$ with $S(\mathfrak{z}_{n,m})$ we get

$$\overline{\rho}(\widetilde{F}_{m+1}) = (-1)^m\Big(\sum x_i y_i + \operatorname{tr} S^{m+1}A + \sum_{2\leq j\leq m}(-1)^j\overline{\Theta}_{m-j}\operatorname{tr} S^{m+1-j}A\Big). \tag{3}$$

Substituting this into $\tau_1' = c\overline{\rho}(\widetilde{F}_{m+1}) + p(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2})$ with $\overline{\Theta}_{m-1} := 0$, $\overline{\Theta}_m := 1$, we get

$$p(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2}) = (1 - (-1)^m c)\sum_i x_i y_i + \sum_{0\leq j\leq m}(\overline{\eta}_j(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2}) - (-1)^j c\overline{\Theta}_j)\operatorname{tr} S^{j+1}A.$$

Hence $c = (-1)^m$ and

$$p(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2}) = \sum_{0 \le j \le m} (\overline{\eta}_j(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2}) - (-1)^{m-j}\overline{\Theta}_j) \operatorname{tr} S^{j+1} A.$$

According to Remark 1, the last equality is equivalent to

$$\overline{\eta}_m = 1, \quad \overline{\eta}_{m-1} = 0, \quad \overline{\eta}_j(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-2}) = (-1)^{m-j}\overline{\Theta}_j, \quad \forall\, 0 \le j \le m-2, \ \ p = 0.$$

This implies the statement.

(b) Analogously to the previous case and the proof of Theorem 7(b) we have:

$$\{\overline{\iota}(A), \overline{\iota}(B)\} = \overline{\iota}([A,B]), \quad \{\overline{\iota}(A), \overline{\iota}(y)\} = \overline{\iota}(A(y)), \quad \{\overline{\iota}(x), \overline{\iota}(y)\} = \sum \overline{\eta}_j \beta_{2j}(x,y),$$

for some polynomials $\overline{\eta}_j \in \mathbb{C}[\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-1}]$, such that $\deg(\overline{\eta}_j(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-1})) = 4(m-j)$.

Due to Theorem 3(b), we get $\tau_1' := \sum_{i=1}^{2n} \{\widetilde{Q}_1, y_i\} y_i^* - 2\sum_j \overline{\eta}_j \operatorname{tr} S^{2j+2} A \in \mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{z}_{n,m}))$. In particular, $\tau_1' = c\overline{\rho}(\widetilde{F}_{m+1}) + p(\overline{\rho}(\widetilde{F}_1), \ldots, \overline{\rho}(\widetilde{F}_m))$ for some $c \in \mathbb{C}$ and a polynomial $p$.

Note that $\overline{\rho}(\widetilde{F}_k) = \overline{\Theta}_{m-k}$ for $1 \le k \le m$. Let us now express $\overline{\rho}(\widetilde{F}_{m+1})$ via the generators of $S(\mathfrak{z}_{n,m})$. First, we describe explicitly the slice $\mathrm{S}_{n,m}$. It consists of the following elements:

$$\Big\{ e_m + \overline{\iota}(X_1) + \sum_{i \le n} v_i U_{i,n+1} + \sum_{i \le n} v_{n+i} U_{n+2m+i,n+1} + \sum_{k \le m} w_k f_m^{2k-1} \ \Big|\ X_1 \in \mathfrak{sp}_{2n}, \ v_i, v_{n+i}, w_k \in \mathbb{C} \Big\},$$

where $U_{i,j} := E_{i,j} + (-1)^{i+j+1} E_{2n+2m+1-j, 2n+2m+1-i} \in \mathfrak{sp}_{2n+2m}$. For $X \in \mathfrak{sp}_{2n+2m}$ of the above form let us define $X_2 := e_m + \sum_{k \le m} w_k f_m^{2k-1} \in \mathfrak{sp}_{2m}$, viewed as the *centered* $2m \times 2m$ block of $X$.

Analogously to (3), we get the following formula:

$$\overline{\rho}(\widetilde{F}_{m+1}) = \frac{1}{4} \sum_{i=1}^{2n} \{\widetilde{Q}_1, y_i\} y_i^* - \operatorname{tr} S^{2m+2} A - \sum_{0 \le j \le m-1} \overline{\Theta}_j \operatorname{tr} S^{2j+2} A. \tag{4}$$

Comparing the above two formulas for $\tau_1'$, we get the equality:

$$\sum_{i=1}^{2n} \{\widetilde{Q}_1, y_i\} y_i^* - 2\sum_j \overline{\eta}_j \operatorname{tr} S^{2j+2} A = c \cdot \overline{\rho}(\widetilde{F}_{m+1}) + p(\overline{\Theta}_0, \ldots, \overline{\Theta}_{m-1}).$$

Arguments analogous to the one used in part (a) establish

$$c = 4, \ p = 0, \ \overline{\eta}_m = 2, \ \overline{\eta}_j = 2\overline{\Theta}_j, \quad \forall\, j < m.$$

Part (b) follows. $\square$

*Remark 4.* Recalling the standard convention $U(\mathfrak{g},0)=U(\mathfrak{g})$ and Example 1, we see that Theorem 7(a) (as well as Theorem 10(a)) obviously holds for $m=1$ with $e_1:=0\in\mathfrak{sl}_{n+1}$.

The results of Theorems 7 and 10 can be naturally generalized to the case of the universal infinitesimal Hecke algebras of $\mathfrak{so}_n$. However, this requires reproving some basic results about the latter algebras, similar to those of [EGG], [DT], and is discussed separately in [T].

## 4. Consequences

In this section we use Theorem 7 to get some new (and recover some old) results about the algebras of interest. On the $W$-algebra side, we get presentations of $U(\mathfrak{sl}_n,e_m)$ and $U(\mathfrak{sp}_{2n},e_m)$ via generators and relations (in the latter case there was no presentation known for $m>1$). We get many more results about the structure and the representation theory of infinitesimal Cherednik algebras using the corresponding results on $W$-algebras.

Also we determine the isomorphism from Theorem 7(a) basically explicitly.

### 4.1. Centers of $H_m(\mathfrak{gl}_n)$ and $H_m(\mathfrak{sp}_{2n})$

We set $s=2$ for $\mathfrak{g}=\mathfrak{sl}_N$ and $s=1$ for $\mathfrak{g}=\mathfrak{sp}_{2N}$. Recall the elements $\{\widetilde{F}_i\}_{i=s}^N$, where $\deg(\widetilde{F}_i)=(3-s)i$. These are the free generators of the Poisson center $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{g}))$. The Lie algebra $\mathfrak{q}=\mathfrak{z}_{\mathfrak{g}}(e,h,f)$ from Section 1.6 equals $\mathfrak{gl}_n$ for $(\mathfrak{g},e)=(\mathfrak{sl}_{n+m},e_m)$ and $\mathfrak{sp}_{2n}$ for $(\mathfrak{g},e)=(\mathfrak{sp}_{2n+2m},e_m)$. Thus $\{\widetilde{Q}_j\}$ from Section 1.4 are the free generators of $\mathfrak{z}_{\mathrm{Pois}}(S(\mathfrak{q}))$, and $Q_j:=\mathrm{Sym}(\widetilde{Q}_j)$ are the free generators of $Z(U(\mathfrak{q}))$.

The following result is a straightforward generalization of formulas (3) and (4):

**Proposition 13.** *There exist* $\{b_i\}_{i=1}^n\in S(\mathfrak{g})^{\mathrm{ad}\,\mathfrak{g}}[\overline{\rho}(\widetilde{F}_s),\ldots,\overline{\rho}(\widetilde{F}_m)]$*, such that:*

$$\overline{\rho}(\widetilde{F}_{m+i})\equiv s_{n,m}\tau_i+b_i \mod \mathbb{C}[\overline{\rho}(\widetilde{F}_s),\ldots,\overline{\rho}(\widetilde{F}_{m+i-1})],\quad \forall\ 1\le i\le n,$$

*where* $s_{n,m}=(-1)^m$ *for* $\mathfrak{g}=\mathfrak{gl}_n$ *and* $s_{n,m}=1/4$ *for* $\mathfrak{g}=\mathfrak{sp}_{2n}$.

Define $t_k\in H_m(\mathfrak{gl}_n)$ by $t_k:=\sum_{i=1}^n x_i[Q_k,y_i]$ and $t_k\in H_m(\mathfrak{sp}_{2n})$ by $t_k:=\sum_{i=1}^{2n}[Q_k,y_i]y_i^*$. Combining Proposition 13, Theorems 5, 7 with $\mathrm{gr}(Z(U(\mathfrak{g},e)))=\mathfrak{z}_{\mathrm{Pois}}(\mathbb{C}[\mathrm{S}])$ we get

**Corollary 14.** *For* $\mathfrak{g}$ *being either* $\mathfrak{gl}_n$ *or* $\mathfrak{sp}_{2n}$*, there exist*

$$C_1,\ldots,C_n\in Z(U(\mathfrak{g}))[\zeta_0,\ldots,\zeta_{m-s}],$$

*such that the center* $Z(H_m(\mathfrak{g}))$ *is a polynomial algebra in free generators* $\{\zeta_i\}\cup\{t_j+C_j\}_{j=1}^n$.

Considering the quotient of $H_m(\mathfrak{g})$ by the ideal $(\zeta_0-a_0,\ldots,\zeta_{m-s}-a_{m-s})$ for any $a_i\in\mathbb{C}$, we see that the center of the standard infinitesimal Cherednik algebra $H_a(\mathfrak{g})$ contains a polynomial subalgebra $\mathbb{C}[t_1+c_1,\ldots,t_n+c_n]$ for some $c_j\in Z(U(\mathfrak{g}))$.

Together with [DT, Thms. 5.1 and 7.1] this yields:

**Corollary 15.** *We actually have* $Z(H_a(\mathfrak{g})) = \mathbb{C}[t_1 + c_1, \dots, t_n + c_n]$.

For $\mathfrak{g} = \mathfrak{gl}_n$ this is [T1, Thm. 1.1], while for $\mathfrak{g} = \mathfrak{sp}_{2n}$ this is [DT, Conj. 7.1].

### 4.2. Symplectic leaves of Poisson infinitesimal Cherednik algebras

By Theorem 10, we get an identification of the full Poisson-central reductions of the algebras $\mathbb{C}[\mathrm{S}_{n,m}]$ and $H^{\mathrm{cl}}_m(\mathfrak{gl}_n)$ or $H^{\mathrm{cl}}_m(\mathfrak{sp}_{2n})$. As an immediate consequence we obtain the following proposition, which answers a question raised in [DT]:

**Proposition 16.** *Poisson varieties corresponding to arbitrary full central reductions of Poisson infinitesimal Cherednik algebras* $H^{\mathrm{cl}}_\varsigma(\mathfrak{g})$ *have finitely many symplectic leaves.*

### 4.3. Analogue of Kostant's theorem

As another immediate consequence of Theorem 7 and discussions from Section 4.1, we get a generalization of the following classical result:

**Proposition 17.**

(a) *The infinitesimal Cherednik algebras* $H_\varsigma(\mathfrak{g})$ *are free over their centers.*
(b) *The full central reductions of* $\operatorname{gr} H_\varsigma(\mathfrak{g})$ *are normal, complete intersection integral domains.*

For $\mathfrak{g} = \mathfrak{gl}_n$ this is [T2, Thm. 2.1], while for $\mathfrak{g} = \mathfrak{sp}_{2n}$ this is [DT, Thm. 8.1].

### 4.4. Category $\mathcal{O}$ and finite dimensional representations of $H_m(\mathfrak{sp}_{2n})$

The categories $\mathcal{O}$ for the finite $W$-algebras were first introduced in [BGK] and were further studied by the first author in [L3]. Namely, recall that we have an embedding $\mathfrak{q} \subset U(\mathfrak{g}, e)$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{q}$ and set $\mathfrak{g}_0 := \mathfrak{z}_\mathfrak{g}(\mathfrak{t})$. Pick an integral element $\theta \in \mathfrak{t}$ such that $\mathfrak{z}_\mathfrak{g}(\theta) = \mathfrak{g}_0$. By definition, the category $\mathcal{O}$ (for $\theta$) consists of all finitely generated $U(\mathfrak{g}, e)$-modules $M$, where the action of $\mathfrak{t}$ is diagonalizable with finite dimensional eigenspaces and, moreover, the set of weights is bounded from above in the sense that there are complex numbers $\alpha_1, \dots, \alpha_k$ such that for any weight $\lambda$ of $M$ there is $i$ with $\alpha_i - \langle \theta, \lambda \rangle \in \mathbb{Z}_{\leqslant 0}$. The category $\mathcal{O}$ has analogues of Verma modules, $\Delta(N^0)$. Here $N^0$ is an irreducible module over the $W$-algebra $U(\mathfrak{g}_0, e)$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}$. In the cases of interest $((\mathfrak{g}, e) = (\mathfrak{sl}_{n+m}, e_m), (\mathfrak{sp}_{2n+2m}, e_m))$, we have $\mathfrak{g}_0 = \mathfrak{gl}_n \times \mathbb{C}^{m-1}, \mathfrak{g}_0 = \mathfrak{sp}_{2n} \times \mathbb{C}^m$ and $e$ is principal in $\mathfrak{g}_0$. In this case, the $W$-algebra $U(\mathfrak{g}_0, e)$ coincides with the center of $U(\mathfrak{g}_0)$. Therefore $N^0$ is a one-dimensional space, and the set of all possible $N^0$ is identified, via the Harish-Chandra isomorphism, with the quotient $\mathfrak{h}^*/W_0$, where $\mathfrak{h}, W_0$ are a Cartan subalgebra and the Weyl group of $\mathfrak{g}_0$ (we take the quotient with respect to the dot-action of $W_0$ on $\mathfrak{h}^*$). As in the usual BGG category $\mathcal{O}$, each Verma module has a unique irreducible quotient, $L(N^0)$. Moreover, the map $N^0 \mapsto L(N^0)$ is a bijection between the set of finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules, $\mathfrak{h}^*/W_0$, in our case, and the set of irreducible objects in $\mathcal{O}$. We remark that all finite dimensional irreducible modules lie in $\mathcal{O}$.

One can define a formal character for a module $M \in \mathcal{O}$. The characters of Verma modules are easy to compute basically thanks to [BGK, Thm. 4.5(1)]. So to compute the characters of the simples, one needs to determine the multiplicities of the simples in the Vermas. This was done in [L3, Sect. 4] in the case when $e$ is principal in $\mathfrak{g}_0$. The multiplicities are given by values of certain Kazhdan-Lusztig polynomials at 1 and so are hard to compute, in general. In particular, one cannot classify finite dimensional irreducible modules just using those results.

When $\mathfrak{g} = \mathfrak{sl}_{n+m}$, a classification of the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules was obtained in [BK2]; this result is discussed in the next section. When $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, one can describe the finite dimensional irreducible representations using [L2, Thm. 1.2.2]. Namely, the centralizer of $e$ in $\mathrm{Ad}(\mathfrak{g})$ is connected. So, according to [L2], the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules are in one-to-one correspondence with the primitive ideals $\mathcal{J} \subset U(\mathfrak{g})$ such that the associated variety of $U(\mathfrak{g})/\mathcal{J}$ is $\overline{\mathbb{O}}$, where we write $\mathbb{O}$ for the adjoint orbit of $e$. The set of such primitive ideals is computable (for a fixed central character, those are in one-to-one correspondence with certain left cells in the corresponding integral Weyl group), but we will not need details on that.

One can also describe all $N^0 \in \mathfrak{h}^*/W_0$ such that $\dim L(N^0) < \infty$ when $e$ is principal in $\mathfrak{g}_0$. This is done in [L4, 5.1]. Namely, choose a representative $\lambda \in \mathfrak{h}^*$ of $N^0$ that is, *antidominant* for $\mathfrak{g}_0$, meaning that $\langle \alpha^\vee, \lambda \rangle \notin \mathbb{Z}_{>0}$ for any positive root $\alpha$ of $\mathfrak{g}_0$. Then we can consider the irreducible highest weight module $L(\lambda)$ for $\mathfrak{g}$ with highest weight $\lambda - \rho$. Let $\mathcal{J}(\lambda)$ be its annihilator in $U(\mathfrak{g})$; this is a primitive ideal that depends only on $N^0$ and not on the choice of $\lambda$. Then $\dim L(N^0) < \infty$ if and only if the associated variety of $U(\mathfrak{g})/\mathcal{J}(\lambda)$ is $\overline{\mathbb{O}}$. The associated variety is computable thanks to results of [BV]; however, this computation requires quite a lot of combinatorics. It seems that one can still give a closed combinatorial answer for $(\mathfrak{sp}_{2n+2m}, e_m)$ similar to that for $(\mathfrak{sl}_{n+m}, e_m)$ but we are not going to elaborate on that.

Now let us discuss the infinitesimal Cherednik algebras. In the $\mathfrak{gl}_n$-case the category $\mathcal{O}$ was defined in [T1, Def. 4.1] (see also [EGG, Sect. 5.2]). Under the isomorphism of Theorem 7(a), that category $\mathcal{O}$ basically coincides with its $W$-algebra counterpart. The classification of finite dimensional irreducible modules and the character computation in that case was done in [DT], but the character formulas for more general simple modules were not known. For the algebras $H_m(\mathfrak{sp}_{2n})$, no category $\mathcal{O}$ was introduced, in general; the case $n = 1$ was discussed in [Kh]. The classification of finite dimensional irreducible modules was not known either.

### 4.5. Finite dimensional representations of $H_m(\mathfrak{gl}_n)$

Let us compare classifications of the finite dimensional irreducible representations of $U(\mathfrak{sl}_{n+m}, e_m)$ from [BK2] and $H_a(\mathfrak{gl}_n)$ from [DT].

In the notation of [BK2][6], a nilpotent element $e_m \in \mathfrak{gl}_{n+m}$ corresponds to the partition $(1, \ldots, 1, m)$ of $n + m$. Let $S_m$ act on $\mathbb{C}^{n+m}$ by permuting the last $m$ coordinates. According to [BK2, Thm. 7.9], there is a bijection between the irreducible finite dimensional representations of $U(\mathfrak{gl}_{n+m}, e_m)$ and the orbits of the $S_m$-action on $\mathbb{C}^{n+m}$ containing a strictly dominant representative. An element $\overline{\nu} = (\nu_1, \ldots, \nu_{n+m}) \in \mathbb{C}^{n+m}$ is called strictly dominant if $\nu_i - \nu_{i+1}$ is a positive integer for all $1 \leq i \leq n$. The corresponding irreducible $U(\mathfrak{gl}_{n+m}, e_m)$-representation is denoted $L_{\overline{\nu}}$. Viewed as a $\mathfrak{gl}_n$-module (since $\mathfrak{gl}_n = \mathfrak{q} \subset U(\mathfrak{gl}_{n+m}, e_m)$), $L_{\overline{\nu}} =$

[6] In the loc.cit. $\mathfrak{g} = \mathfrak{gl}_{n+m}$, rather then $\mathfrak{sl}_{n+m}$. Nevertheless, it is not very crucial since $\mathfrak{gl}_{n+m} = \mathfrak{sl}_{n+m} \oplus \mathbb{C}$.

$L'_{\overline{\nu}} \oplus \bigoplus_{i\in I} L'_{\eta_i}$, where $L'_\eta$ is the highest weight $\eta$ irreducible $\mathfrak{gl}_n$-module, $\overline{\nu} := (\nu_1, \ldots, \nu_n)$ and $I$ denotes some set of weights $\eta < \overline{\nu}$.

Let us now recall [DT, Thm. 4.1], which classifies all irreducible finite dimensional representations of the infinitesimal Cherednik algebra $H_a(\mathfrak{gl}_n)$. They turn out to be parameterized by strictly dominant $\mathfrak{gl}_n$-weights $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (that is, $\lambda_i - \lambda_{i+1}$ is a positive integer for every $1 \leq i < n$), for which there exists a positive integer $k$ satisfying $P(\overline{\lambda}) = P(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$. Here $P$ is a degree $m+1$ polynomial function on the Cartan subalgebra $\mathfrak{h}_n$ of all diagonal matrices of $\mathfrak{gl}_n$, introduced in [DT, Sect. 3.2]. According to [DT, Thm. 3.2] (see Theorem 18(b) below), we have $P = \sum_{j\geq 0} w_j h_{j+1}$, where both $w_j$ and $h_j$ are defined in the next section (see the notation preceding Theorem 18).

These two descriptions are intertwined by a natural bijection, sending $\overline{\nu} = (\nu_1, \ldots, \nu_{n+m})$ to $\overline{\lambda} := (\nu_1, \ldots, \nu_n)$, while $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is sent to the class of $\overline{\nu} = (\lambda_1, \ldots, \lambda_n, \nu_{n+1}, \ldots, \nu_{n+m})$ with $\{\nu_{n+1}, \ldots, \nu_{n+m}\} \cup \{\lambda_n\}$ being the set of roots of the polynomial $P(\lambda_1, \ldots, \lambda_{n-1}, t) - P(\overline{\lambda})$.

### 4.6. Explicit isomorphism in the case $\mathfrak{g} = \mathfrak{gl}_n$

We compute the images of particular central elements of $H_m(\mathfrak{gl}_n)$ and $U(\mathfrak{sl}_{n+m}, e_m)$ under the corresponding Harish-Chandra isomorphisms. Comparison of these images enables us to determine the isomorphism $\overline{\Theta}$ of Theorem 7(a) explicitly, in the same way as Theorem 10(a) was deduced.

Let us start from the following commutative diagram:

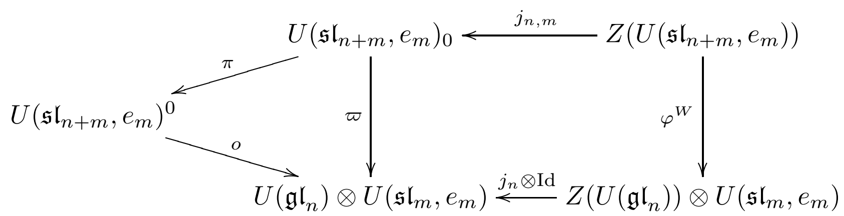


Diagram 1

In the above diagram:

• $U(\mathfrak{sl}_{n+m}, e_m)_0$ is the 0-weight component of $U(\mathfrak{sl}_{n+m}, e_m)$ with respect to the grading Gr.

• $U(\mathfrak{sl}_{n+m}, e_m)^0 := U(\mathfrak{sl}_{n+m}, e_m)_0 / I$, where

$$I = (U(\mathfrak{sl}_{n+m}, e_m)_0 \cap U(\mathfrak{sl}_{n+m}, e_m) U(\mathfrak{sl}_{n+m}, e_m)_{>0}).$$

• $\pi$ is the quotient map, while $o$ is an isomorphism constructed in [L3, Thm. 4.1].[7]

• The homomorphism $\varpi$ is defined as $\varpi := o \circ \pi$, making the triangle commutative.

• The homomorphisms $j_{n+m},\ j_n$ are the natural inclusions.

• The homomorphism $\varphi^W$ is the restriction of $\varpi$ to the center, making the square commutative.

[7] Here we actually use the fact that $U(\mathfrak{gl}_n) \otimes U(\mathfrak{sl}_m, e_m)$ is the finite $W$-algebra $U(\mathfrak{gl}_n \oplus \mathfrak{sl}_m, 0 \oplus e_m)$.

• $U(\mathfrak{sl}_m, e_m) \cong Z(U(\mathfrak{sl}_m, e_m)) \cong Z(U(\mathfrak{sl}_m))$ since $e_m$ is a principal nilpotent of $\mathfrak{sl}_m$.

We have an analogous diagram for the universal infinitesimal Cherednik algebra of $\mathfrak{gl}_n$:

$$\begin{array}{ccccc}
 & & H_m(\mathfrak{gl}_n)_0 & \xleftarrow{\;j'_{n,m}\;} & Z(H_m(\mathfrak{gl}_n)) \\
 & \swarrow^{\pi'} & & & \\
H_m(\mathfrak{gl}_n)^0 & & \downarrow{\scriptstyle \varpi'} & & \downarrow{\scriptstyle \varphi^H} \\
 & \searrow_{o'} & & & \\
 & & U(\mathfrak{gl}_n)\otimes\mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}] & \xleftarrow{\;j_n\otimes \mathrm{Id}\;} & Z(U(\mathfrak{gl}_n))\otimes\mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}]
\end{array}$$

DIAGRAM 2

In the above diagram:

• $H_m(\mathfrak{gl}_n)_0$ is the degree 0 component of $H_m(\mathfrak{gl}_n)$ with respect to the grading Gr, defined by setting $\deg(\mathfrak{gl}_n) = \deg(\zeta_0) = \ldots = \deg(\zeta_{m-2}) = 0,\ \deg(V_n) = 1,\ \deg(V_n^*) = -1$.

• $H_m(\mathfrak{gl}_n)^0$ is the quotient of $H_m(\mathfrak{gl}_n)_0$ by $H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0}$.[8]

• $\pi'$ denotes the quotient map, $o'$ is the natural isomorphism, $\varpi' := o' \circ \pi'$.

• The inclusion $j'_{n,m}$ is a natural inclusion of the center.

• The homomorphism $\varphi^H$ is the one induced by restricting $\varpi'$ to the center.

The isomorphism $\overline{\Theta}$ of Theorem 7(a) intertwines the gradings Gr, inducing an isomorphism $\overline{\Theta}^0 : H_m(\mathfrak{gl}_n)^0 \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)^0$. This provides the following commutative diagram:

$$\begin{array}{ccc}
Z(H_m(\mathfrak{gl}_n)) & \xrightarrow{\;\vartheta\;} & Z(U(\mathfrak{sl}_{n+m}, e_m)) \\
\downarrow{\scriptstyle \varphi^H} & & \downarrow{\scriptstyle \varphi^W} \\
Z(U(\mathfrak{gl}_n))\otimes\mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}] & \xrightarrow{\;\underline{\vartheta}\;} & Z(U(\mathfrak{gl}_n))\otimes Z(U(\mathfrak{sl}_m))
\end{array}$$

DIAGRAM 3

In the above diagram:

• The isomorphism $\vartheta$ is the restriction of the isomorphism $\overline{\Theta}$ to the center.

• The isomorphism $\underline{\vartheta}$ is the restriction of the isomorphism $\overline{\Theta}^0$ to the center.

Let $\mathrm{HC}_N$ denote the Harish-Chandra isomorphism

$$\mathrm{HC}_N : Z(U(\mathfrak{gl}_N)) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}_N^*]^{S_N,\bullet},$$

where $\mathfrak{h}_N \subset \mathfrak{gl}_N$ is the Cartan subalgebra consisting of the diagonal matrices and $(S_N, \bullet)$-action arises from the $\rho_N$-*shifted* $S_N$-action on $\mathfrak{h}_N^*$ with $\rho_N = ((N-1)/2, (N-3)/2, \ldots, (1-N)/2) \in \mathfrak{h}_N^*$. This isomorphism has the following property:

[8] It is easy to see that $H_m(\mathfrak{gl}_n)_0 \cap H_m(\mathfrak{gl}_n)H_m(\mathfrak{gl}_n)_{>0}$ is actually a two-sided ideal of $H_m(\mathfrak{gl}_n)_0$.

any central element $z \in Z(U(\mathfrak{gl}_N))$ acts on the Verma module $M_{\lambda-\rho_N}$ of $U(\mathfrak{gl}_N)$ via $\mathrm{HC}_N(z)(\lambda)$.

According to Corollary 14, the center $Z(H_m(\mathfrak{gl}_n))$ is the polynomial algebra in free generators $\{\zeta_0, \ldots, \zeta_{m-2}, t'_1, \ldots, t'_n\}$, where $t'_k = t_k + C_k$. In particular, any central element of Kazhdan degree $2(m+1)$ has the form $ct'_1 + p(\zeta_0, \ldots, \zeta_{m-2})$ for some $c \in \mathbb{C}$ and $p \in \mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$.

Following [DT], we call $t'_1 = t_1 + C_1$ the Casimir element[9]. An explicit formula for $\varphi^H(t'_1)$ is provided by [DT, Thm. 3.1], while for any $0 \le k \le m-2$ we have $\varphi^H(\zeta_k) = 1 \otimes \zeta_k$.

To formulate the main results about the Casimir element $t'_1$, we introduce:

• the generating series $\zeta(z) = \sum_{i=0}^{m-2} \zeta_i z^i + z^m$ (already introduced in Section 1.4),

• a unique degree $m+1$ polynomial $f(z)$ satisfying $f(z) - f(z-1) = \partial^n(z^n \zeta(z))$ and $f(0) = 0$,

• a unique degree $m+1$ polynomial $g(z) = \sum_{i=1}^{m+1} g_i z^i$ satisfying $\partial^{n-1}(z^{n-1} g(z)) = f(z)$,

• a unique degree $m$ polynomial $w(z) = \sum_{i=0}^{m} w_i z^i$ satisfying

$$f(z) = (2\sinh(\partial/2))^{n-1}(z^n w(z)),$$

• the symmetric polynomials $\sigma_i(\lambda_1, \ldots, \lambda_n)$ via

$$(u+\lambda_1)\cdots(u+\lambda_n) = \sum \sigma_i(\lambda_1, \ldots, \lambda_n) u^{n-i},$$

• the symmetric polynomials $h_j(\lambda_1, \ldots, \lambda_n)$ via

$$(1-u\lambda_1)^{-1}\cdots(1-u\lambda_n)^{-1} = \sum h_j(\lambda_1, \ldots, \lambda_n) u^j,$$

• the central element $H_j \in Z(U(\mathfrak{gl}_n))$ which is the symmetrization of $\operatorname{tr} S^j(\cdot) \in \mathbb{C}[\mathfrak{gl}_n] \cong S(\mathfrak{gl}_n)$.

The following theorem summarizes the main results of [DT, Sect. 3]:

**Theorem 18.**

(a) [DT, Thm. 3.1] $\varphi^H(t'_1) = \sum_{j=1}^{m+1} H_j \otimes g_j$ (*where* $g_j$ *are viewed as elements of* $\mathbb{C}[\zeta_0, \ldots, \zeta_{m-2}]$),

(b) [DT, Thm. 3.2] $(\mathrm{HC}_n \otimes \mathrm{Id}) \circ \varphi^H(t'_1) = \sum_{j=0}^{m} h_{j+1} \otimes w_j$.

Let $\mathrm{HC}'_N$ denote the Harish-Chandra isomorphism $Z(U(\mathfrak{sl}_N)) \xrightarrow{\sim} \mathbb{C}[\overline{\mathfrak{h}}_N^*]^{S_N, \bullet}$, where $\overline{\mathfrak{h}}_N$ is the Cartan subalgebra of $\mathfrak{sl}_N$, consisting of the diagonal matrices, which can be identified with $\{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid \sum z_i = 0\}$. The natural inclusion $\overline{\mathfrak{h}}_N \hookrightarrow \mathfrak{h}_N$ induces the map

$$\mathfrak{h}_N^* \to \overline{\mathfrak{h}}_N^* : \ (\lambda_1, \ldots, \lambda_N) \mapsto (\lambda_1 - \mu, \ldots, \lambda_N - \mu), \ \textit{where} \ \ \mu := \frac{\lambda_1 + \cdots + \lambda_N}{N}.$$

[9] The Casimir element is uniquely defined up to a constant.

The isomorphisms $\mathrm{HC}'_{n+m}, \mathrm{HC}'_m, \mathrm{HC}_n$ fit into the following commutative diagram:

$$\begin{array}{ccccc}
 & & Z(U(\mathfrak{sl}_{n+m})) & \xleftarrow{\mathrm{HC}'^{-1}_{n+m}} & \mathbb{C}[\mathbb{C}^{n+m-1}]^{S_{n+m},\bullet} \\
 & \swarrow^{\rho} & & & \\
Z(U(\mathfrak{sl}_{n+m}, e_m)) & & \Big\downarrow{\scriptstyle \overline{\varphi}^W} & & \Big\downarrow{\scriptstyle \varphi^C} \\
 & \searrow_{\varphi^W} & & & \\
 & & Z(U(\mathfrak{gl}_n)) \otimes Z(U(\mathfrak{sl}_m)) & \xleftarrow{\mathrm{HC}_n^{-1} \otimes \mathrm{HC}'^{-1}_m} & \mathbb{C}[\mathbb{C}^n]^{S_n,\bullet} \otimes \mathbb{C}[\mathbb{C}^{m-1}]^{S_m,\bullet}
\end{array}$$

Diagram 4

In the above diagram:

- $\rho$ is the isomorphism of Theorem 5.
- The homomorphism $\overline{\varphi}^W$ is defined as the composition $\overline{\varphi}^W := \varphi^W \circ \rho$.
- The homomorphism $\varphi^C$ arises from an identification $\mathbb{C}^n \times \mathbb{C}^{m-1} \cong \mathbb{C}^{n+m-1}$ defined by

$$(\lambda_1, \ldots, \lambda_n, \nu_1, \ldots, \nu_m) \mapsto \left( \lambda_1, \ldots, \lambda_n, \nu_1 - \frac{\lambda_1 + \cdots + \lambda_n}{m}, \ldots, \nu_m - \frac{\lambda_1 + \cdots + \lambda_n}{m} \right).$$

In particular, $\varphi^C$ is injective, so that $\varphi^W$ is injective and, hence, $\varphi^H$ is injective.

Define $\overline{\sigma}_k \in \mathbb{C}[\overline{\mathfrak{h}}_N^*]$ as the restriction of $\sigma_k$ to $\mathbb{C}^{N-1} \hookrightarrow \mathbb{C}^N$. According to Lemma 12,

$$\varphi^C(\overline{\sigma}_{m+1}) = (-1)^m h_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} h_{m+1-j} \otimes 1 \cdot \varphi^C(\overline{\sigma}_j). \tag{5}$$

Define $S_k \in Z(U(\mathfrak{sl}_{n+m}))$ by $S_k := (\mathrm{HC}'_{n+m})^{-1}(\overline{\sigma}_k)$ for all $0 \leq k \leq n+m$, so that $S_0 = 1,\ S_1 = 0$. Similarly, define $T_k \in Z(U(\mathfrak{gl}_n))$ as $T_k := \mathrm{HC}_n^{-1}(h_k)$ for all $k \geq 0$, so that $T_0 = 1$.

Equality (5) together with the commutativity of Diagram 4 imply

$$\overline{\varphi}^W(S_{m+1}) = (-1)^m T_{m+1} \otimes 1 + \sum_{j=2}^{m} (-1)^{m-j} T_{m+1-j} \otimes 1 \cdot \overline{\varphi}^W(S_j).$$

According to our proof of Theorem 7(a), we have $\overline{\Theta}(A) = \Theta(A) + s \operatorname{tr} A$ for all $A \in \mathfrak{gl}_n$, where $s = -\eta_{m-1}/(n+m)\eta_m$. In particular, $\underline{\vartheta}^{-1}(X \otimes 1) = \varphi_{-s}(X) \otimes 1$ for all $X \in Z(U(\mathfrak{gl}_n))$, where $\varphi_{-s}$ was defined in Lemma 1.

As a consequence, we get:

$$\begin{aligned} \underline{\vartheta}^{-1}(\overline{\varphi}^W(S_{m+1})) = (-1)^m \varphi_{-s}(T_{m+1}) \otimes 1 \\ + \sum_{j=2}^{m} (-1)^{m-j} \varphi_{-s}(T_{m+1-j}) \otimes 1 \cdot \underline{\vartheta}^{-1}(\overline{\varphi}^W(S_j)). \end{aligned} \tag{6}$$

The following identity is straightforward:

**Lemma 19.** *For any positive integer $i$ and any constant $\delta \in \mathbb{C}$ we have*

$$h_i(\lambda_1+\delta,\ldots,\lambda_n+\delta)=\sum_{j=0}^{i}\binom{n+i-1}{j}h_{i-j}(\lambda_1,\ldots,\lambda_n)\delta^j.$$

As a result, we get

$$\varphi_{-s}(T_i)=\sum_{j=0}^{i}\binom{n+i-1}{j}(-s)^jT_{i-j}. \tag{7}$$

Combining equations (6) and (7), we get:

$$\begin{aligned}\underline{\vartheta}^{-1}(\overline{\varphi}^W(S_{m+1}))&=(-1)^mT_{m+1}\otimes 1\\ &\quad+(-1)^{m+1}s(n+m)T_m\otimes 1+\sum_{l=-1}^{m-2}(-1)^lT_{l+1}\otimes 1\cdot\overline{V}_l,\end{aligned} \tag{8}$$

where $\overline{V}_l=\underline{\vartheta}^{-1}(\overline{\varphi}^W(V_l))$ and for $0\le l\le m-2$ we have

$$V_l=\sum_{0\le j\le m-l}s^{m-l-j}\binom{n+m-j}{m-l-j}S_j.$$

On the other hand, the commutativity of Diagram 3 implies

$$\underline{\vartheta}^{-1}(\overline{\varphi}^W(S_{m+1}))=\varphi^H(\vartheta^{-1}(\rho(S_{m+1}))).$$

Recall that there exist $c\in\mathbb{C}$, $p\in\mathbb{C}[\zeta_0,\ldots\zeta_{m-2}]$ such that $\vartheta^{-1}(\rho(S_{m+1}))=ct_1'+p$. As $\varphi^H(\zeta_i)=1\otimes\zeta_i$ and $\varphi^H(t_1')=\sum_{j=0}^m T_{j+1}\otimes w_j$ (by Theorem 18(b)), we get

$$\varphi^H(\vartheta^{-1}(\rho(S_{m+1})))=1\otimes p(\zeta_0,\ldots,\zeta_{m-2})+\sum_{0\le j\le m}T_{j+1}\otimes cw_j. \tag{9}$$

Recalling the equalities $w_m=1, w_{m-1}=(n+m)/2$, the comparison of (8) and (9) yields:

- The coefficients of $T_{m+1}$ must coincide, so that $(-1)^m=cw_m\Rightarrow c=(-1)^m$.
- The coefficients of $T_m$ must coincide, so that $cw_{m-1}=(-1)^{m+1}(n+m)s\Rightarrow s=-1/2$.
- The coefficients of $T_{j+1}$ must coincide for all $j\ge 0$, so that

$$w_j=(-1)^{m-j}\overline{V}_j\Rightarrow\vartheta(w_j)=(-1)^{m-j}\rho(V_j).$$

Recall that $\overline{\eta}_m=1$, and so $\eta_m=\overline{\eta}_m=1$. As a result $s=-\eta_{m-1}/(n+m)$, so that $\eta_{m-1}=(n+m)/2$.

The above discussion can be summarized as follows:

**Theorem 20.** *Let* $\overline{\Theta}: H_m(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{sl}_{n+m}, e_m)$ *be the isomorphism from Theorem 7(a). Then* $\overline{\Theta}(A) = \Theta(A) - \frac{1}{2}\operatorname{tr} A$, $\overline{\Theta}(y) = \Theta(y)$, $\overline{\Theta}(x) = \Theta(x)$, *while* $\overline{\Theta}\mid_{\mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}]}$ *is uniquely determined by* $\overline{\Theta}(w_j) = (-1)^{m-j}\rho(V_j)$ *for all* $0 \le j \le m-2$.

### 4.7. Higher central elements

It was conjectured in [DT, Rem. 6.1], that the action of central elements $t'_i = t_i + c_i \in Z(H_m(\mathfrak{gl}_n))$ on the Verma modules of $H_a(\mathfrak{gl}_n)$ should be obtained from the corresponding formulas at the Poisson level (see Theorem 3) via a *basis change* $\zeta(z) \rightsquigarrow w(z)$ and a $\rho_n$*-shift*. Actually, that is not true. However, we can choose another set of generators $u_i \in Z(H_m(\mathfrak{gl}_n))$, whose action is given by formulas similar to those of Theorem 3.

Let us define:

- central elements $u_i \in Z(H_m(\mathfrak{gl}_n))$ by $u_i := \underline{\vartheta}^{-1}(\rho(S_{m+i}))$ for all $0 \le i \le n$,
- the generating polynomial

$$\widetilde{u}(t) := \sum_{i=0}^{n} (-1)^i u_i t^i,$$

- the generating polynomial

$$S(z) := \sum_{i=0}^{n} (-1)^i \underline{\vartheta}^{-1}(\overline{\varphi}^W(S_{m-i})) z^i \in \mathbb{C}[\zeta_0,\ldots,\zeta_{m-2}; z].$$

The following result is proved using the arguments of Section 4.6:

**Theorem 21.** *We have:*

$$(\mathrm{HC}_n \otimes \mathrm{Id}) \circ \varphi^H(\widetilde{u}(t)) = (\varphi_{1/2} \otimes \mathrm{Id})\left(\mathrm{Res}_{z=0}\ S(z^{-1}) \prod_{1\le i\le n} \frac{1-t\lambda_i}{1-z\lambda_i}\frac{z^{-1}dz}{1-t^{-1}z}\right).$$

## 5. Completions

### 5.1. Completions of graded deformations of Poisson algebras

We first recall the machinery of completions, elaborated by the first author (our exposition follows [L7]). Let $Y$ be an affine Poisson scheme equipped with a $\mathbb{C}^*$-action, such that the Poisson bracket has degree $-2$. Let $\mathcal{A}_\hbar$ be an associative flat graded $\mathbb{C}[\hbar]$-algebra (where $\deg(\hbar) = 1$) such that $[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subset \hbar^2\mathcal{A}_\hbar$ and $\mathbb{C}[Y] = \mathcal{A}_\hbar/(\hbar)$ as a graded Poisson algebra. Pick a point $x \in Y$ and let $I_x \subset \mathbb{C}[Y]$ be the maximal ideal of $x$, while $\widetilde{I}_x$ will denote its inverse image in $\mathcal{A}_\hbar$.

**Definition 6.** The completion of $\mathcal{A}_\hbar$ at $x \in Y$ is by definition $\mathcal{A}_\hbar^{\wedge_x} := \varprojlim \mathcal{A}_\hbar/\widetilde{I}_x^n$.

This is a complete topological $\mathbb{C}[[\hbar]]$-algebra, flat over $\mathbb{C}[[\hbar]]$, such that $\mathcal{A}_\hbar^{\wedge_x}/(\hbar) = \mathbb{C}[Y]^{\wedge_x}$. Our main motivation for considering this construction is the decomposition theorem, generalizing the corresponding classical result at the Poisson level:

**Proposition 22** (cf. [K, Thm. 2.3])**.** *The formal completion $\widehat{Y}_x$ of $Y$ at $x \in Y$ admits a product decomposition $\widehat{Y}_x = \mathcal{Z}_x \times \widehat{Y}^s_x$, where $Y^s$ is the symplectic leaf of $Y$ containing $x$ and $\mathcal{Z}_x$ is a local formal Poisson scheme.*

Fix a maximal symplectic subspace $V \subset T^*_x Y$. One can choose an embedding $V \overset{i}{\hookrightarrow} \widetilde{I}_x^{\wedge_x}$ such that $[i(u), i(v)] = \hbar^2\omega(u,v)$ and composition $V \overset{i}{\hookrightarrow} \widetilde{I}_x^{\wedge_x} \twoheadrightarrow T^*_x Y$ is the identity map. Finally, we define $W_\hbar(V) := T(V)[h]/(u \otimes v - v \otimes u - \hbar^2\omega(u,v))$, which is graded by setting $\deg(V) = 1,\ \deg(\hbar) = 1$ (the *homogenized Weyl algebra*). Then we have:

**Theorem 23** ([L7, Sect. 2.1], Decomposition theorem)**.** *There is a splitting*

$$\mathcal{A}_\hbar^{\wedge_x} \cong W_\hbar(V)^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \underline{\mathcal{A}'_\hbar},$$

*where $\underline{\mathcal{A}'_\hbar}$ is the centralizer of $V$ in $\mathcal{A}_\hbar^{\wedge_x}$.*

*Remark 5.* Recall that a filtered algebra $\{F_i(B)\}_{i \geq 0}$ is called a *filtered deformation* of $Y$ if $\mathrm{gr}_{F_\bullet} B \cong \mathbb{C}[Y]$ as Poisson graded algebras. Given such $B$, we set $\mathcal{A}_\hbar := \mathrm{Rees}_\hbar(B)$ (the Rees algebra of the filtered algebra $B$), which naturally satisfies all the above conditions.

This remark provides the following interesting examples of $\mathcal{A}_\hbar$:

• *The homogenized Weyl algebra.*

Algebra $W_\hbar(V)$ from above is obtained via the Rees construction from the usual Weyl algebra. In the case $V = V_n \oplus V_n^*$ with a natural symplectic form, we denote $W_\hbar(V)$ just by $W_{\hbar,n}$.

• *The homogenized universal enveloping algebra.*

For any graded Lie algebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ with a Lie bracket of degree $-2$, we define

$$U_\hbar(\mathfrak{g}) := T(\mathfrak{g})[\hbar]/(x \otimes y - y \otimes x - \hbar^2[x,y] \mid x, y \in \mathfrak{g}),$$

graded by setting $\deg(\mathfrak{g}_i) = i,\ \deg(\hbar) = 1$.

• *The homogenized universal infinitesimal Cherednik algebra of $\mathfrak{gl}_n$.*

Define $H_{\hbar,m}(\mathfrak{gl}_n)$ as a quotient

$$H_{\hbar,m}(\mathfrak{gl}_n) := U_\hbar(\mathfrak{gl}_n) \ltimes T(V_n \oplus V_n^*)[\zeta_0, \ldots, \zeta_{m-2}]/J,$$

where

$$J = \Bigg( [x,x'], [y,y'], [A,x] - \hbar^2 A(x), [A,y] - \hbar^2 A(y), \\ [y,x] - \hbar^2 \left( \sum_{j=0}^{m-2} \zeta_j r_j(y,x) + r_m(y,x) \right) \Bigg).$$

This algebra is graded by setting $\deg(V_n \oplus V_n^*) = m+1,\ \deg(\zeta_i) = 2(m-i)$.

• *The homogenized universal infinitesimal Cherednik algebra of $\mathfrak{sp}_{2n}$.*
Define $H_{\hbar,m}(\mathfrak{sp}_{2n})$ as a quotient

$$H_{\hbar,m}(\mathfrak{sp}_{2n}) := U_\hbar(\mathfrak{sp}_{2n}) \ltimes T(V_{2n})[\zeta_0,\ldots,\zeta_{m-1}]/J,$$

where

$$J = \left([A,y]-\hbar^2 A(y), [x,y]-\hbar^2\left(\sum_{j=0}^{m-1}\zeta_j r_{2j}(x,y)+r_{2m}(x,y)\right)\right).$$

This algebra is graded by setting $\deg(V_{2n})=2m+1,\ \deg(\zeta_i)=4(m-i)$.

• *The homogenized $W$-algebra.*
The homogenized $W$-algebra, associated to $(\mathfrak{g},e)$, is defined by

$$U_\hbar(\mathfrak{g},e) := (U_\hbar(\mathfrak{g})/U_\hbar(\mathfrak{g})\mathfrak{m}')^{\operatorname{ad}\mathfrak{m}}.$$

There are many interesting contexts in which Theorem 23 proves to be a useful tool. Among such let us mention rational Cherednik algebras ([BE]), symplectic reflection algebras ([L5]) and $W$-algebras ([L1], [L7]).

Actually, combining results of [L7] with Theorem 7, we get isomorphisms

$$\Psi_m : H_{\hbar,m}(\mathfrak{gl}_n)^{\wedge_v} \xrightarrow{\sim} H_{\hbar,m+1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_v}, \quad (*)$$

$$\Upsilon_m : H_{\hbar,m}(\mathfrak{sp}_{2n})^{\wedge_v} \xrightarrow{\sim} H_{\hbar,m+1}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,2n}^{\wedge_v}, \quad (\spadesuit)$$

where $v\in V_n$ (respectively $v\in V_{2n}$) is a nonzero element and $m\geq 1$.

These decompositions can be viewed as *quantizations* of their Poisson versions:

$$\Psi_m^{\mathrm{cl}} : H_m^{\mathrm{cl}}(\mathfrak{gl}_n)^{\wedge_v} \xrightarrow{\sim} H_{m+1}^{\mathrm{cl}}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}}W_n^{\mathrm{cl},\wedge_v}, \quad (\star)$$

$$\Upsilon_m^{\mathrm{cl}} : H_m^{\mathrm{cl}}(\mathfrak{sp}_{2n})^{\wedge_v} \xrightarrow{\sim} H_{m+1}^{\mathrm{cl}}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}}W_{2n}^{\mathrm{cl},\wedge_v}, \quad (\heartsuit)$$

where $W_n^{\mathrm{cl}}\simeq\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ with $\{x_i,x_j\}=\{y_i,y_j\}=0, \{x_i,y_j\}=\delta_i^j$.

Isomorphisms (*) and ($\spadesuit$) are not unique and, what is worse, are inexplicit.

Let us point out that localizing at other points of $\mathfrak{gl}_n\times V_n\times V_n^*$ (respectively $\mathfrak{sp}_{2n}\times V_{2n}$) yields other decomposition isomorphisms. In particular, one gets [T3, Thm. 3.1][10] as follows:

*Remark 6.* For $n=1, m>0$, consider $e' := e_m + E_{1,2n+2}\in \mathrm{S}_{1,m}\subset\mathfrak{sp}_{2m+2}$, which is a subregular nilpotent element of $\mathfrak{sp}_{2m+2}$. The above arguments yield a decomposition isomorphism

$$H_{\hbar,m}(\mathfrak{sp}_2)^{\wedge_{E_{12}}} \xrightarrow{\sim} U_\hbar(\mathfrak{sp}_{2m+2},e')^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,1}^{\wedge_0}. \quad (\clubsuit)$$

The full central reduction of ($\clubsuit$) provides an isomorphism of [T3, Thm. 3.1].[11]

In Appendix C, we establish explicitly suitably modified versions of (*) and ($\spadesuit$) for the cases $m=-1,\ 0$, which do not follow from the above arguments. In particular, the reader will get a flavor of what the formulas look like.

[10] This result is stated in [T3]. However, its proof in the loc. cit. is wrong.

[11] We use an isomorphism of the $W$-algebra $U(\mathfrak{sp}_{2m+2},e')$ and the non-commutative deformation of Crawley-Boevey and Holland of type $\mathsf{D}_{m+2}$ Kleinian singularity.

## A. Proof of Lemmas 1, 2

*Proof of Lemma* 1(a). Let $\phi : H_\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H_{\zeta'}(\mathfrak{gl}_n)$ be a filtration preserving isomorphism. We have $\phi(1) = 1$, so that $\phi$ is the identity on the 0th level of the filtration.

Since $\mathcal{F}_2^{(N)}(H_\zeta(\mathfrak{gl}_n)) = \mathcal{F}_2^{(N)}(H_{\zeta'}(\mathfrak{gl}_n)) = U(\mathfrak{gl}_n)_{\leq 1}$, we have $\phi(A) = \psi(A) + \gamma(A)$, $\forall A \in \mathfrak{gl}_n$, with $\psi(A) \in \mathfrak{gl}_n, \gamma(A) \in \mathbb{C}$. Then $\phi([A,B]) = [\phi(A), \phi(B)]$, $\forall A, B \in \mathfrak{gl}_n$, if and only if $\gamma([A,B]) = 0$ and $\psi$ is an automorphism of the Lie algebra $\mathfrak{gl}_n$. Since $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$, we have $\gamma(A) = \lambda \cdot \operatorname{tr} A$ for some $\lambda \in \mathbb{C}$. For $n \geq 3$, $\operatorname{Aut}(\mathfrak{gl}_n) = \operatorname{Aut}(\mathfrak{sl}_n) \times \operatorname{Aut}(\mathbb{C}) = (\mu_2 \ltimes \mathrm{SL}(n)) \times \mathbb{C}^*$, where $-1 \in \mu_2$ acts on $\mathfrak{sl}_n$ via $\sigma\colon A \mapsto -A^t$. This determines $\phi$ up to the filtration level $N-1$.

Finally, $\mathcal{F}_N^{(N)}(H_\zeta(\mathfrak{gl}_n)) = \mathcal{F}_N^{(N)}(H_{\zeta'}(\mathfrak{gl}_n)) = V_n \oplus V_n^* \oplus U(\mathfrak{gl}_n)_{\leq N}$. As we just explained, $\phi_{|U(\mathfrak{gl}_n)}$ is parameterized by $(\epsilon, T, \nu, \lambda) \in (\mu_2 \ltimes \mathrm{SL}(n)) \times \mathbb{C}^* \times \mathbb{C}$ (no $\mu_2$ for $n = 1, 2$). Let $I_n \in \mathfrak{gl}_n$ be the identity matrix. Note that $[I_n, y] = y, [I_n, x] = -x, [I_n, A] = 0$ for any $y \in V_n, x \in V_n^*, A \in \mathfrak{gl}_n$.

Since $\phi(y) = \phi([I_n, y]) = [\nu \cdot I_n + n\lambda, \phi(y)] = \nu[I_n, \phi(y)]$, $\forall y \in V_n$, we get $\nu = \pm 1$.

*Case* 1: $\nu = 1$. Then $\phi(y) \in V_n,\ \phi(x) \in V_n^*$ $(\forall y \in V_n, x \in V_n^*)$. Since $V_n \not\simeq V_n^\sigma$ as $\mathfrak{sl}_n$-modules for $n \geq 3$ and $\operatorname{End}_{\mathfrak{sl}_n}(V_n) = \mathbb{C}^*$, we get $\epsilon = 1 \in \mu_2$ (so that $\phi(A) = TAT^{-1}$, $\forall A \in \mathfrak{sl}_n$) and there exist $\theta_1, \theta_2 \in \mathbb{C}^*$ such that $\phi(y) = \theta_1 \cdot T(y)$, $\phi(x) = \theta_2 \cdot T(x)$ $(\forall y \in V_n, x \in V_n^*)$. Hence, we get $\varphi(T,\lambda)(\zeta(y,x)) = \phi([y,x]) = [\phi(y), \phi(x)] = \theta\zeta'(T(y), T(x))$, where $\theta = \theta_1\theta_2$ and the isomorphism $\varphi(T,\lambda) : U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ is defined by $A \mapsto TAT^{-1} + \lambda \operatorname{tr} A$, $\forall A \in \mathfrak{gl}_n$. Thus, $\zeta' = \theta^{-1}\varphi_\lambda(\zeta^+)$ in that case.

*Case* 2: $\nu = -1$. Then $\phi(y) \in V_n^*,\ \phi(x) \in V_n$ $(\forall y \in V_n,\ x \in V_n^*)$. Similarly to the above reasoning we get $\epsilon = -1, \phi(A) = -TA^tT^{-1} + \lambda \operatorname{tr} A$ $(\forall A \in \mathfrak{gl}_n)$, so that there exist $\theta_1, \theta_2 \in \mathbb{C}^*$ such that $\phi(y_i) = \theta_1 \cdot T(x_i)$, $\phi(x_j) = \theta_2 \cdot T(y_j)$. Then $\phi(\zeta(y_i, x_j)) = -\theta_1\theta_2\zeta'(T(y_j), T(x_i))$. Hence, $\zeta' = -\theta_1^{-1}\theta_2^{-1}\varphi_{-\lambda}(\zeta^-)$ in that case.

Finally, the above arguments also provide isomorphisms $\phi_{\theta,\lambda,s} : H_\zeta(\mathfrak{gl}_n) \xrightarrow{\sim} H_{\theta\varphi_\lambda(\zeta^s)}(\mathfrak{gl}_n)$ for any deformation $\zeta$, constants $\lambda \in \mathbb{C}, \theta \in \mathbb{C}^*$ and $s \in \{\pm\}$. $\square$

*Proof of Lemma* 1(b). Let $\zeta$ be a length $m$ deformation. Since $(\theta\zeta)_m = \theta\zeta_m$, we can assume $\zeta_m = 1$. We claim that $\varphi_\lambda(\zeta)_{m-1} = 0$ for $\lambda = -\zeta_{m-1}/(n+m)$, which is equivalent to $\partial\alpha_m/\partial I_n = (n+m)\alpha_{m-1}$. This equality follows from comparing coefficients of $s\tau^m$ in the identity

$$\sum \alpha_i(y,x)(A + sI_n)\tau^i = (1 - s\tau)^{-n-1} \sum \alpha_i(y,x)(A)(\tau(1-s\tau)^{-1})^i. \quad \square$$

*Proof of Lemma* 2. Let $\phi : H_\zeta(\mathfrak{sp}_{2n}) \xrightarrow{\sim} H_{\zeta'}(\mathfrak{sp}_{2n})$ be a filtration preserving isomorphism. Being an isomorphism, we have $\phi(1) = 1$, so that $\phi$ is the identity on the 0th level of the filtration.

Since $\mathcal{F}_2^{(N)}(H_\zeta(\mathfrak{sp}_{2n})) = \mathcal{F}_2^{(N)}(H_{\zeta'}(\mathfrak{sp}_{2n})) = U(\mathfrak{sp}_{2n})_{\leq 1}$, we have $\phi(A) = \psi(A) + \gamma(A)$ for all $A \in \mathfrak{sp}_{2n}$, with $\psi(A) \in \mathfrak{sp}_{2n}, \gamma(A) \in \mathbb{C}$. Then $\phi([A,B]) = [\phi(A), \phi(B)]$, $\forall A, B \in \mathfrak{sp}_{2n}$, if and only if $\gamma([A,B]) = 0$ and $\psi$ is an automorphism of the Lie algebra $\mathfrak{sp}_{2n}$. Since $[\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n}] = \mathfrak{sp}_{2n}$, we have $\gamma \equiv 0$. Meanwhile, any automorphism of $\mathfrak{sp}_{2n}$ is inner, since $\mathfrak{sp}_{2n}$ is a simple Lie algebra whose Dynkin diagram has no automorphisms. This proves $\phi_{|U(\mathfrak{sp}_{2n})} = \operatorname{Ad}(T)$, $T \in \mathrm{Sp}_{2n}$. Composing with an automorphism $\phi'$ of $H_{\zeta'}(\mathfrak{sp}_{2n})$, defined by $\phi'(A)=\mathrm{Ad}(T^{-1})(A), \phi'(x)=T^{-1}(x)$ $(A\in\mathfrak{sp}_{2n}, x\in V_{2n})$, we can assume $\phi_{|U(\mathfrak{sp}_{2n})}=\mathrm{Id}$.

Recall the element $I'_n=\mathrm{diag}(1,\ldots,1,-1,\ldots,-1)\in\mathfrak{sp}_{2n}$. Since $\mathrm{ad}(I'_n)$ has only even eigenvalues on $U(\mathfrak{sp}_{2n})$ and eigenvalues $\pm1$ on $V_{2n}$, we actually have $\phi(V_{2n})\subset V_{2n}$. Together with $\mathrm{End}_{\mathfrak{sp}_{2n}}(V_{2n})=\mathbb{C}^*$ this implies the result.

The converse, that is, $H_\zeta(\mathfrak{sp}_{2n})\cong H_{\theta\zeta}(\mathfrak{sp}_{2n})$ for any $\zeta$ and $\theta\in\mathbb{C}^*$, is obvious. □

## B. Minimal nilpotent case

We compute the isomorphism of Theorem 7 explicitly for the case of $e\in\mathfrak{g}$ being the minimal nilpotent. This case has been considered in detail in [P2, Sect. 4].

To state the main result we introduce some more notation. Let $z_1,\ldots,z_{2s}$ be a Witt basis of $\mathfrak{g}(-1)$, i.e., $\omega_\chi(z_{i+s},z_j)=\delta_i^j$, $\omega_\chi(z_i,z_j)=\omega_\chi(z_{i+s},z_{j+s})=0$ for any $1\le i,j\le s$. We also define $\sharp:\mathfrak{g}(0)\to\mathfrak{g}(0)$ by $x^\sharp:=x-\frac{1}{2}(x,h)h$. Finally, we set $c_0:=-n(n+1)/4$ for $\mathfrak{g}=\mathfrak{sl}_{n+1}$ and $c_0:=-n(2n+1)/8$ for $\mathfrak{g}=\mathfrak{sp}_{2n}$. Then we have the following theorem:

**Theorem 24** (cf. [P2, Thm. 6.1])**.** *The algebra $U(\mathfrak{g},e)$ is generated by the Casimir element $C$ and the subspaces $\Theta(\mathfrak{z}_\chi(i))$ for $i=0,1$, subject to the following relations:*

(i) $[\Theta_x,\Theta_y]=\Theta_{[x,y]}$, $[\Theta_x,\Theta_u]=\Theta_{[x,u]}$ *for all $x,y\in\mathfrak{z}_\chi(0), u\in\mathfrak{z}_\chi(1)$;*
(ii) *$C$ is central in $U(\mathfrak{g},e)$;*
(iii) *for all $u,v\in\mathfrak{z}_\chi(1)$,*

$$\begin{aligned}[\Theta_u,\Theta_v]&=\tfrac{1}{2}(f,[u,v])(C-\Theta_{\mathrm{Cas}}-c_0)\\&\quad+\tfrac{1}{2}\sum_{1\le i\le 2s}(\Theta_{[u,z_i]^\sharp}\Theta_{[v,z_i^*]^\sharp}+\Theta_{[v,z_i^*]^\sharp}\Theta_{[u,z_i]^\sharp}),\end{aligned}$$

*where $\Theta_{\mathrm{Cas}}$ is a Casimir element of the Lie algebra $\Theta(\mathfrak{z}_\chi(0))$.*

Our goal is to construct explicitly isomorphisms of Theorem 7 for those two cases, that is, for $\mathfrak{g}=\mathfrak{sl}_{n+1},\ \mathfrak{sp}_{2n+2}$, and a minimal nilpotent $e\in\mathfrak{g}$.

**Lemma 25.** *Formulas*

$$\begin{gathered}\widetilde{\gamma}(\zeta_0)=\frac{c_0-C}{2},\quad \widetilde{\gamma}(y_i)=\Theta_{E_{i,n+1}},\quad \widetilde{\gamma}(x_i)=\Theta_{E_{n,i}},\\ \widetilde{\gamma}(A)=\Theta_A,\quad A\in\mathfrak{gl}_n\simeq\mathfrak{z}_\chi(0)\end{gathered}\tag{10}$$

*establish the isomorphism $H_2(\mathfrak{gl}_{n-1})\xrightarrow{\sim}U(\mathfrak{sl}_{n+1},E_{n,n+1})$ from Theorem* 7(a)*.*

*Proof.* Choose a natural $\mathfrak{sl}_2$-triple $(e,h,f)=(E_{n,n+1},E_{n,n}-E_{n+1,n+1},E_{n+1,n})$ in $\mathfrak{g}=\mathfrak{sl}_{n+1}$. Then $\{E_{i,n+1},E_{ni}\}_{1\le i\le n-1}$ form a basis of $\mathfrak{z}_\chi(1)$, while $\{E_{ij},E_{11}-E_{kk},\ T_{n-1,2}\}_{1\le i\ne j\le n-1}^{2\le k\le n-1}$ form a basis of $\mathfrak{z}_\chi(0)$. Identifying $\mathfrak{z}_\chi(1)$ with $V_{n-1}\oplus V_{n-1}^*$, we get an epimorphism of algebras $\gamma: U(\mathfrak{gl}_{n-1})\ltimes T(V_{n-1}\oplus V_{n-1}^*)[C]\twoheadrightarrow U(\mathfrak{sl}_{n+1},E_{n,n+1})$ defined by

$$\begin{gathered}\gamma(C)=C,\quad \gamma(y_i)=\Theta_{E_{i,n+1}},\quad \gamma(x_i)=\Theta_{E_{n,i}},\quad \gamma(I_{n-1})=\Theta_{T_{n-1,2}},\\ \gamma(A)=\Theta_A,\quad A\in\mathfrak{sl}_{n-1}\subset\mathfrak{sl}_{n+1}.\end{gathered}$$

According to Theorem 24, its kernel $\mathrm{Ker}(\gamma)$ is generated by

$$w\otimes w'-w'\otimes w-\tfrac{1}{2}\big(f,[\gamma(w),\gamma(w')]\big)\big(C-\gamma^{-1}(\Theta_{\mathrm{Cas}})-c_0\big) \\ -\gamma^{-1}\Bigg(\mathrm{Sym}\sum_{1\le i\le 2s}\Theta_{[w,z_i]^\sharp}\Theta_{[w',z_i^*]^\sharp}\Bigg),$$

with $w,w'\in V_{n-1}\oplus V_{n-1}^*$, $\gamma^{-1}(\Theta_\varsigma)\in\mathfrak{gl}_{n-1}\oplus V_{n-1}\oplus V_{n-1}^*$ well-defined for $\varsigma\in\mathfrak{z}_\chi(0)\oplus\mathfrak{z}_\chi(1)$.

Choose the Witt basis of $\mathfrak{g}(-1)$ as $z_i:=E_{i,n}, z_{i+s}:=E_{n+1,i},\ 1\le i\le n-1=:s$.

- For $w,w'\in V_{n-1}$ or $w,w'\in V_{n-1}^*$ we just get $w\otimes w'-w'\otimes w\in\mathrm{Ker}(\gamma)$.
- For $w=y_p\in V_{n-1}, w'=x_q\in V_{n-1}^*$ we get the following element of $\mathrm{Ker}(\gamma)$:

$$y_p\otimes x_q-x_q\otimes y_p+\frac{\delta_p^q}{2}\big(C-\gamma^{-1}(\Theta_{\mathrm{Cas}})-c_0\big) \\ -\gamma^{-1}\Bigg(\mathrm{Sym}\sum_{1\le i\le 2s}\Theta_{[E_{p,n+1},z_i]^\sharp}\Theta_{[E_{nq},z_i^*]^\sharp}\Bigg).$$

For $1\le i\le s$ we obviously have $[E_{p,n+1},z_i]=0$, while

$$[E_{p,n+1},z_{i+s}]=E_{pi}-\delta_p^iE_{n+1,n+1}\Rightarrow[E_{p,n+1},z_{i+s}]^\sharp=E_{pi}-\tfrac{1}{2}\delta_p^i(E_{nn}+E_{n+1,n+1}).$$

A similar argument implies

$$[E_{nq},z_{i+s}^*]=E_{iq}-\delta_q^iE_{nn}\Rightarrow[E_{nq},z_{i+s}^*]^\sharp=E_{iq}-\tfrac{1}{2}\delta_q^i(E_{nn}+E_{n+1,n+1}).$$

Thus

$$\Theta_{[E_{p,n+1},z_{i+s}]^\sharp}=\gamma(E_{pi})+\tfrac{1}{2}\delta_p^i\gamma(I_{n-1}),\ \Theta_{[E_{nq},z_{i+s}^*]^\sharp}=\gamma(E_{iq})+\tfrac{1}{2}\delta_q^i\gamma(I_{n-1}),$$

so that

$$\gamma^{-1}(\mathrm{Sym}\sum\Theta_{[E_{p,n+1},z_i]^\sharp}\Theta_{[E_{nq},z_i^*]^\sharp})=\mathrm{Sym}\Big(\sum E_{pi}E_{iq}\Big) \\ +\mathrm{Sym}(I_{n-1}\cdot E_{pq})+\frac{1}{4}\delta_p^qI_{n-1}^2.$$

On the other hand, since $\gamma^{-1}(\gamma(E_{lk})^*)=E_{kl}+\frac{1}{2}\delta_k^lI_{n-1}$, we get

$$\gamma^{-1}(\Theta_{\mathrm{Cas}})=\sum_{k\ne l}E_{kl}E_{lk}+\sum_kE_{kk}^2+\tfrac{1}{2}I_{n-1}^2.$$

Let $\widetilde{R}_{n-1}:=\sum E_{ii}^2+\frac{1}{2}\sum_{i\ne j}(E_{ii}E_{jj}+E_{ij}E_{ji})$. Then we get $y_p\otimes x_q-x_q\otimes y_p-\Bigg(\frac{c_0-C}{2}\cdot\underbrace{\delta_p^q}_{r_0(y_p,x_q)}+\underbrace{\mathrm{Sym}\Big(\sum E_{pi}E_{iq}+I_{n-1}\cdot E_{pq}+\delta_p^q\widetilde{R}_{n-1}\Big)}_{r_2(y_p,x_q)}\Bigg)\in\mathrm{Ker}(\gamma)$. This implies the statement of the lemma. $\square$

**Lemma 26.** *Formulas*

$$\widetilde{\gamma}(\xi_0)=\frac{c_0-C}{2},\quad \widetilde{\gamma}(y_i)=\frac{\Theta_{v_i}}{\sqrt{2}},\quad \widetilde{\gamma}(A)=\Theta_A,\quad A\in\mathfrak{sp}_{2n}\simeq\mathfrak{z}_\chi(0) \tag{11}$$

*establish the isomorphism* $H_1(\mathfrak{sp}_{2n})\xrightarrow{\sim}U(\mathfrak{sp}_{2n+2},E_{1,2n+2})$ *from Theorem* 7(b).

*Proof.* First, choose an $\mathfrak{sl}_2$-triple $(e,h,f)=(E_{1,2n+2},E_{11}-E_{2n+2,2n+2},E_{2n+2,1})$ in $\mathfrak{g}=\mathfrak{sp}_{2n+2}$. Then $\{v_k:=E_{k+1,2n+2}+(-1)^kE_{1,2n+2-k}\}_{1\le k\le 2n}$ form a basis of $\mathfrak{z}_\chi(1)$, while $\mathfrak{z}_\chi(0)\simeq\mathfrak{sp}_{2n}$. Identifying $\mathfrak{z}_\chi(1)$ with $V_{2n}$ via $y_k\mapsto v_k$, we get an algebra epimorphism

$$\gamma: U(\mathfrak{sp}_{2n})\ltimes T(V_{2n})[C]\twoheadrightarrow U(\mathfrak{sp}_{2n+2},E_{1,2n+2}),$$
$$C\mapsto C,\quad y_i\mapsto\Theta_{v_i},\quad A\mapsto\Theta_A\quad (A\in\mathfrak{sp}_{2n}).$$

According to Theorem 24, its kernel $\mathrm{Ker}(\gamma)$ is generated by $\{y_q\otimes y_p-y_p\otimes y_q-(\ldots)\}_{p,q\le 2n}$. Let us now compute the expression represented by the ellipsis.

Choose the Witt basis of $\mathfrak{g}(-1)$ with respect to the form $\omega_\chi$ as

$$z_i:=\frac{(-1)^{i+1}}{2}(E_{2n+2-i,1}+(-1)^iE_{2n+2,i+1}),$$
$$z_{i+s}:=E_{i+1,1}-(-1)^iE_{2n+2,2n+2-i},\quad 1\le i\le n=:s.$$

Since $(f,[v_q,v_p])=2(-1)^q\delta_{p+q}^{2n+1}$, the above expression in ellipsis equals to:

$$(-1)^q\delta_{p+q}^{2n+1}\big(C-\gamma^{-1}(\Theta_{\mathrm{Cas}})-c_0\big)+\gamma^{-1}\Bigg(\mathrm{Sym}\Bigg(\sum_{1\le i\le 2s}\Theta_{[v_q,z_i]^\sharp}\Theta_{[v_p,z_i^*]^\sharp}\Bigg)\Bigg),$$

where $\gamma^{-1}(\Theta_\varsigma)\in\mathfrak{sp}_{2n}\oplus V_{2n}$ is well-defined for any $\varsigma\in\mathfrak{z}_\chi(0)\oplus\mathfrak{z}_\chi(1)$, though $\gamma$ is not injective.

For any $1\le k,l\le 2n,\ 1\le j\le n$ it is easily verified that

$$[v_k,z_j]=-\tfrac{1}{2}(E_{k+1,j+1}-(-1)^{k+j}E_{2n+2-j,2n+2-k})-\tfrac{1}{2}\delta_k^j\cdot h,$$
$$[v_l,z_{j+s}]=(-1)^{j+1}(E_{l+1,2n+2-j}+(-1)^{l-j}E_{j+1,2n+2-l})+(-1)^l\delta_{l+j}^{2n+1}\cdot h,$$

so that

$$[v_k,z_j]^\sharp=\frac{(-1)^{k+j}E_{2n+2-j,2n+2-k}-E_{k+1,j+1}}{2},$$
$$[v_l,z_{j+s}]^\sharp=(-1)^{j+1}E_{l+1,2n+2-j}+(-1)^{l+1}E_{j+1,2n+2-l}.$$

We also have

$$\gamma^{-1}(\Theta_{\mathrm{Cas}})=\frac{1}{4}\sum_{i,j}(E_{j,i}+(-1)^{i+j+1}E_{2n+1-i,2n+1-j})$$
$$\times(E_{i,j}+(-1)^{i+j+1}E_{2n+1-j,2n+1-i}).$$

On the other hand, it is straightforward to check that

$$\begin{aligned} r_0(y_q,y_p) &= (-1)^p\delta_{p+q}^{2n+1}, \\ r_2(y_q,y_p) &= \frac{(-1)^{q+1}}{4}\operatorname{Sym}\sum_s \big(E_{s,2n+1-q}+(-1)^{s+q}E_{q,2n+1-s}\big) \\ &\quad\times\big(E_{p,s}+(-1)^{p+s+1}E_{2n+1-s,2n+1-p}\big) \\ &\quad+\frac{(-1)^p}{8}\delta_{p+q}^{2n+1}\operatorname{Sym}\sum_{i,j}\big(E_{i,j}+(-1)^{i+j+1}E_{2n+1-j,2n+1-i}\big) \\ &\quad\times\big(E_{j,i}+(-1)^{i+j+1}E_{2n+1-i,2n+1-j}\big). \end{aligned}$$

To summarize, the kernel of the epimorphism $\gamma$ is generated by the elements

$$\{y_q\otimes y_p-y_p\otimes y_q-(2r_2(y_q,y_p)+(c_0-C)r_0(y_q,y_p))\}_{p,q\leq 2n}.$$

This implies the statement of the lemma. □

## C. Decompositions (*) and (♠) for $m=-1,\ 0$

• Decomposition isomorphism $H_{\hbar,-1}(\mathfrak{gl}_n)^{\wedge_v}\cong H'_{\hbar,0}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_v}$

Here $H'_{\hbar,0}(\mathfrak{gl}_{n-1})$ is defined similarly to $H_{\hbar,0}(\mathfrak{gl}_{n-1})$ with an additional central parameter $\zeta_0$ and the main relation being $[y,x]=\hbar^2\zeta_0r_0(y,x)$, while $H_{\hbar,-1}(\mathfrak{gl}_n):=U_\hbar(\mathfrak{gl}_n\ltimes(V_n\oplus V_n^*))$.

*Notation:* We use $y_k,\ x_l,\ e_{k,l}$ when referring to the elements of $H_{\hbar,-1}(\mathfrak{gl}_n)$ and capital $Y_i,\ X_j,\ E_{i,j}$ when referring to the elements of $H'_{\hbar,0}(\mathfrak{gl}_{n-1})$. We also use indices $1\leq k,l\leq n$ and $1\leq i,j,i',j'<n$ to distinguish between $\leq n$ and $<n$. Finally, set $v_n:=(0,\dots,0,1)\in V_n$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 27.** *Formulas*

$$\Psi_{-1}(y_k)=z_k,\quad \Psi_{-1}(e_{n,k})=z_n\partial_k,$$
$$\Psi_{-1}(e_{i,j})=E_{i,j}+z_i\partial_j,\quad \Psi_{-1}(e_{i,n})=z_n^{-1}Y_i\ -\sum_{j<n}z_n^{-1}z_jE_{i,j}+z_i\partial_n,$$
$$\Psi_{-1}(x_j)=X_j,\quad \Psi_{-1}(x_n)=-z_n^{-1}\zeta_0-\sum_{p<n}z_n^{-1}z_pX_p$$

*define an isomorphism* $\Psi_{-1}:H_{\hbar,-1}(\mathfrak{gl}_n)^{\wedge_{v_n}}\xrightarrow{\sim}H'_{\hbar,0}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_{v_n}}$.

Its proof is straightforward and is left to an interested reader (most of the verifications are the same as those carried out in the proof of Lemma 28 below).

• Decomposition isomorphism $H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_v}\cong H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_v}$

Here $H'_{\hbar,1}(\mathfrak{gl}_{n-1})$ is an algebra defined similarly to $H_{\hbar,1}(\mathfrak{gl}_{n-1})$ with an additional central parameter $\zeta_0$ and the main relation being $[y,x]=\hbar^2(\zeta_0r_0(y,x)+r_1(y,x))$. We follow analogous conventions as for variables $y_k,\ x_l,\ e_{k,l},\ Y_i,\ X_j,\ E_{i,j}$ and indices $i,j,i',j',k,l$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 28.** *Formulas*

$$\Psi_0(y_k) = z_k, \quad \Psi_0(e_{n,k}) = z_n\partial_k,$$
$$\Psi_0(e_{i,j}) = E_{i,j} + z_i\partial_j, \quad \Psi_0(e_{i,n}) = z_n^{-1}Y_i \ - \sum_{j<n} z_n^{-1}z_jE_{i,j} + z_i\partial_n,$$
$$\Psi_0(x_j) = -\partial_j + X_j, \quad \Psi_0(x_n) = -\partial_n - \sum_{i<n} z_n^{-1}z_iX_i - z_n^{-1}\Big(\zeta_0 + \sum_{i<n} E_{i,i}\Big)$$

*define an isomorphism* $\Psi_0 : H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_{v_n}} \xrightarrow{\sim} H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_{v_n}}$.

*Proof.* These formulas provide a homomorphism

$$H_{\hbar,0}(\mathfrak{gl}_n)^{\wedge_{v_n}} \to H'_{\hbar,1}(\mathfrak{gl}_{n-1})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W_{\hbar,n}^{\wedge_{v_n}}$$

if and only if $\Psi_0$ preserves all the defining relations of $H_{\hbar,0}(\mathfrak{gl}_n)$. This is quite straightforward and we present only the most complicated verifications, leaving the rest to an interested reader.

∘ Verification of $[\Psi_0(e_{i,n}), \Psi_0(e_{i',j'})] = -\hbar^2\delta^i_{j'}\Psi_0(e_{i',n})$:

$$\begin{aligned}[\Psi_0(e_{i,n}), \Psi_0(e_{i',j'})] &= [z_n^{-1}Y_i \ - \sum_{p<n} z_n^{-1}z_pE_{i,p} + z_i\partial_n,\ E_{i',j'} + z_{i'}\partial_{j'}]\\ &= \hbar^2\Big(-\delta^i_{j'}z_n^{-1}Y_{i'} - z_n^{-1}z_{i'}E_{i,j'} + \delta^i_{j'}\sum_{p<n} z_n^{-1}z_pE_{i',p}\\ &\qquad + z_n^{-1}z_{i'}E_{i,j'} - \delta^i_{j'}z_{i'}\partial_n\Big)\\ &= -\hbar^2\delta^i_{j'}\Psi_0(e_{i',n}).\end{aligned}$$

∘ Verification of $[\Psi_0(e_{i,n}), \Psi_0(x_j)] = -\hbar^2\delta^j_i\Psi_0(x_n)$:

$$\begin{aligned}[\Psi_0(e_{i,n}), \Psi_0(x_j)] &= [z_n^{-1}Y_i - \sum_{1\le q\le n-1} z_n^{-1}z_qE_{i,q} + z_i\partial_n, -\partial_j + X_j]\\ &= -\hbar^2z_n^{-1}E_{i,j} + \delta^j_i\hbar^2\partial_n + \delta^j_i\hbar^2\sum_{q<n} z_n^{-1}z_qX_q + z_n^{-1}[Y_i, X_j]\\ &= -\hbar^2z_n^{-1}E_{i,j} + \delta^j_i\hbar^2\Big(\partial_n + \sum_{q<n} z_n^{-1}z_qX_q\Big)\\ &\quad + \hbar^2z_n^{-1}\Big(E_{i,j} + \delta^j_i\sum_{i<n} E_{i,i} + \delta^j_i\zeta_0\Big)\\ &= -\delta^j_i\hbar^2\Psi_0(x_n).\end{aligned}$$

∘ Verification of $[\Psi_0(e_{i,n}), \Psi_0(x_n)] = 0$:

$$\begin{aligned}[\Psi_0(e_{i,n}), \Psi_0(x_n)] &= [z_n^{-1}Y_i - \sum_{p<n} z_n^{-1}z_pE_{i,p} + z_i\partial_n,\\ &\qquad -\partial_n - \sum_{j<n} z_n^{-1}z_jX_j - z_n^{-1}\Big(\zeta_0 + \sum_{j<n} E_{j,j}\Big)]\\ &= \hbar^2\Big(\sum_{p<n} z_n^{-2}z_pE_{i,p} - z_n^{-2}Y_i + z_iz_n^{-2}\zeta_0 + z_iz_n^{-2}\sum_{j<n} E_{j,j}\\ &\qquad + z_n^{-2}Y_i - \sum_{j<n} z_jz_n^{-2}[Y_i, X_j]\Big) = 0.\end{aligned}$$

Once homomorphism $\Psi_0$ is established, it is easy to check that the map

$$z_k \mapsto y_k, \quad \partial_k \mapsto y_n^{-1}e_{n,k}, \quad E_{i,j} \mapsto e_{i,j} - y_i y_n^{-1} e_{n,j}, \quad \zeta_0 \mapsto -\sum_{k\le n} y_k x_k - \sum_{k\le n} e_{k,k},$$
$$X_j \mapsto x_j + y_n^{-1}e_{n,j}, \quad Y_i \mapsto \sum_{1\le q\le n} y_q(e_{i,q} - y_i y_n^{-1} e_{n,q})$$

provides the inverse to $\Psi_0$. This completes the proof of the lemma. □

• Decomposition isomorphism $H_{\hbar,-1}(\mathfrak{sp}_{2n})^{\wedge_v} \cong H'_{\hbar,0}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_v}_{\hbar,2n}$

Here $H'_{\hbar,0}(\mathfrak{sp}_{2n-2})$ is defined similarly to $H_{\hbar,0}(\mathfrak{sp}_{2n-2})$ with an additional central parameter $\zeta_0$ and the main relation being $[x,y]=\hbar^2\zeta_0 r_0(x,y)$, while $H_{\hbar,-1}(\mathfrak{sp}_{2n}){:=} U_\hbar(\mathfrak{sp}_{2n}\ltimes V_{2n})$.

*Notation:* We use $y_k,\ u_{k,l} := e_{k,l} + (-1)^{k+l+1}e_{2n+1-l,2n+1-k}$ when referring to the elements of $H_{\hbar,-1}(\mathfrak{sp}_{2n})$ and $Y_i,\ U_{i,j} := E_{i,j} + (-1)^{i+j+1}E_{2n-1-j,2n-1-i}$ when referring to the elements of $H'_{\hbar,0}(\mathfrak{sp}_{2n-2})$. Note that $\{u_{k,l}\}_{k,l\ge 1}^{k+l\le 2n+1}$ is a basis of $\mathfrak{sp}_{2n}$, while $\{U_{i,j}\}_{i,j\ge 1}^{i+j\le 2n-1}$ is a basis of $\mathfrak{sp}_{2n-2}$. We use indices $1\le k,l\le 2n$ and $1\le i,j\le 2n-2$. Finally, set $v_1 := (1,0,\ldots,0)\in V_{2n}$.

The following lemma establishes explicitly the aforementioned isomorphism:

**Lemma 29.** *Define $\psi_1(u_{k,l}) := z_k\partial_l + (-1)^{k+l+1}z_{2n+1-l}\partial_{2n+1-k}$ for all $k,l$. We also define*

$$\psi_0(u_{1,k}) = 0, \quad \psi_0(u_{i+1,1}) = Y_i, \quad \psi_0(u_{i+1,j+1}) = U_{i,j}, \quad \psi_0(u_{2n,1}) = \zeta_0.$$

*Formulas $\Upsilon_{-1}(y_k) = z_k$, $\Upsilon(u_{k,l}) = \psi_0(u_{k,l}) + \psi_1(u_{k,l})$ give rise to an isomorphism*

$$\Upsilon_{-1} : H_{\hbar,-1}(\mathfrak{sp}_{2n})^{\wedge_{v_1}} \xrightarrow{\sim} H'_{\hbar,0}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_{v_1}}_{\hbar,2n}.$$

The proof of this lemma is straightforward and is left to an interested reader.

• Finally, we have the case of $\mathfrak{g}=\mathfrak{sp}_{2n},\ m=0$.

There is also a decomposition isomorphism

$$\Upsilon_0 : H_{\hbar,0}(\mathfrak{sp}_{2n})^{\wedge_v} \xrightarrow{\sim} H_{\hbar,1}(\mathfrak{sp}_{2n-2})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}W^{\wedge_v}_{\hbar,2n}.$$

This isomorphism can be made explicit, but we find the formulas quite heavy and unrevealing, so we leave them to an interested reader.

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